

CHAPTER THIRTEEN

SURDS

When a square roots sign is squared or raised to the second power, the square root sign disappears.

$$(1) \quad (\sqrt{2})^2 = 2 \qquad (2)(\sqrt{3})^2 = 3$$

$$(3) \quad (\sqrt{a})^2 = a \qquad (4)(\sqrt{b})^2 = b$$

$$(5) \quad 2(\sqrt{3})^2 = 2(3) = 6$$

$$(6) \quad 5(\sqrt{2})^2 = 5(2) = 10$$

$$(7) \quad 2(\sqrt{a})^2 = 2(a) = 2a$$

$$(8) \quad (2\sqrt{3})^2 = 2^2 (\sqrt{3})^2 = 4(3) = 12$$

$$(9) \quad (5\sqrt{2})^2 = 5^2 (\sqrt{2})^2 = 25(2) = 50$$

$$(10) \quad (a\sqrt{b})^2 = a^2 (\sqrt{b})^2 = a^2(b) = a^2b$$

The Perfect Squares:

The Perfect squares are:

$$(1) \quad 2 \times 2 = 4 \qquad (2) 3 \times 3 = 9$$

$$(3) \quad 4 \times 4 = 16 \qquad (4) 5 \times 5 = 25$$

$$(5) \quad 6 \times 6 = 36 \qquad (6) 7 \times 7 = 49$$

$$(7) \quad 8 \times 8 = 64 \qquad (8) 9 \times 9 = 81$$

$$(9) \quad 10 \times 10 = 100$$

-- In surd manipulation, a number which is a multiple of a perfect square must be converted into the multiple of that perfect square

Examples:

$$1. \quad \sqrt{8} = \sqrt{4 \times 2} = \sqrt{4} \times \sqrt{2} = 2 \times \sqrt{2} = 2\sqrt{2}.$$

2. $\sqrt{12} = \sqrt{4 \times 3} = \sqrt{4} \times \sqrt{3} = 2 \times \sqrt{3} = 2\sqrt{3}.$
3. $\sqrt{32} = \sqrt{16 \times 2} = \sqrt{16} \times \sqrt{2}$
 $= 4 \times \sqrt{2} = 4\sqrt{2}$
4. $\sqrt{27} = \sqrt{9 \times 3} = \sqrt{9} \times \sqrt{3} = 3\sqrt{3}$
5. $\sqrt{69} = \sqrt{9 \times 7} = \sqrt{9} \times \sqrt{7} = 3\sqrt{7}$
6. $\sqrt{80} = \sqrt{16 \times 5} = \sqrt{16} \times \sqrt{5} = 4\sqrt{5}$
- (7) $\sqrt{50} = \sqrt{25 \times 2} = \sqrt{25} \times \sqrt{2} = 5\sqrt{2}$
- (8) $\sqrt{147} = \sqrt{49 \times 3} = \sqrt{49} \times \sqrt{3} = 7\sqrt{3}$
- (9) $\sqrt{125} = \sqrt{25 \times 5} = \sqrt{25} \times \sqrt{5} = 5\sqrt{5}$

How to determine whether a number is a multiple of a perfect square:

- The perfect squares 4, 9, 16, 25, 36, 49, 64, 81 and 100 are what we shall make use of.
- When a number is given and we want to know whether it is a multiple of a perfect square, we start dividing the numbers with the perfect squares in turn, starting with the highest which is 100.
- If we get an answer which is a whole number but not a decimal, then that particular number is a multiple of a perfect square.
- For example if we are given $\sqrt{448}$ and we want to know whether or not we can break it down, we first divide 448 by 100 which gives us 4.48.
- Since this is a decimal, then we try the next perfect square which is 81.
- Dividing 448 by 81 gives us 5.5 which is also a decimal. We then try the next perfect square which is 64.
- Dividing 448 by 64 gives us 7, which is a whole number.
- This implies that $\sqrt{448}$
 $= \sqrt{64 \times 7} = \sqrt{64} \times \sqrt{7} = 8\sqrt{7}.$

Example (2) Let us now determine whether $\sqrt{294}$ can be simplified or broken down.

- Dividing 294 by 100 gives us 2.94 which is a decimal.

- Dividing 294 by 81 gives us 3.6 which is a decimal.
- Dividing 294 by 64 gives us 4.59 which is also a decimal.
- Dividing 294 by 49 gives us 6 which is a whole number
- This implies that $\sqrt{294}$
 $= \sqrt{49 \times 6} = \sqrt{49} \times \sqrt{6} = 7\sqrt{6}$.

Examples (3) Now let us determine whether $\sqrt{150}$ can be simplified or broken down.

- Dividing 150 by 100 gives us 1.50 which is a decimal
- Dividing 150 by 81 gives us 1.85 which is a decimal.
- Dividing 150 by 49 gives us 3.06 which is a decimal.
- Dividing 150 by 36 gives us 4.2 which is a decimal.
- Dividing 150 by 25 gives us 6 which is a whole number.
- This implies that $\sqrt{150}$
 $= \sqrt{25 \times 6} = \sqrt{25} \times \sqrt{6} = 5\sqrt{6}$.

NB: If we divide a given number by all the perfect Squares we are supposed to use, and in each case get a decimal, then that number cannot be simplified or broken down.

For example, assume that we want to know whether or not $\sqrt{271}$ can be simplified or broken down.

- Dividing 271 by 100 gives us 2.71 which is a decimal.
- Dividing 271 by 81 gives us 3.34 which is a decimal
- Dividing 271 by 64 gives us 4.2 which is decimal.
- Dividing 271 by 49 gives us 5.5 which is decimal.
- Dividing 271 by 36 gives us 7.5 which is a decimal.
- Dividing 271 by 25 gives us 10.8 which is decimal.
- Dividing 271 by 16 gives us 16.9 which is decimal.
- Dividing 271 by 9 gives us 30.1 which is a decimal.

-- Dividing 271 by 4 gives us 67.7 which is also a decimal.

Since when all the perfect squares concerned, when used to divide the given number gave us decimals as our answer, then $\sqrt{271}$ cannot be simplified and should be left as $\sqrt{271}$.

Addition of surds:

-- In surd addition, we can only add if the numbers under the square root signs are the same.

-- If they are not the same, then nothing can be done.

Examples

$$(1) \quad a\sqrt{b} + c\sqrt{b} = (a + c)\sqrt{b}$$

$$(2) \quad 5\sqrt{2} + 4\sqrt{2} = (5 + 4)\sqrt{2} = 9\sqrt{2}$$

$$(3) \quad 6\sqrt{7} + 2\sqrt{7} = (6 + 2)\sqrt{7} = 8\sqrt{7}$$

$$(4) \quad 3\sqrt{5} + 2\sqrt{5} = (3 + 2)\sqrt{5} = 5\sqrt{5}$$

$$(5) \quad 2\sqrt{7} + 3\sqrt{7} = (2 + 3)\sqrt{7} = 5\sqrt{7}$$

$$(6) \quad 2\sqrt{3} + \sqrt{3} = 2\sqrt{3} + 1\sqrt{3} = (2 + 1)\sqrt{3} = 3\sqrt{3}$$

$$(7) \quad 5\sqrt{2} + \sqrt{2} = 5\sqrt{2} + 1\sqrt{2} = (5 + 1)\sqrt{2} = 6\sqrt{2}$$

$$(8) \quad 2\sqrt{5} + 4\sqrt{5} + 3\sqrt{5} = (2 + 4 + 3)\sqrt{5} = 9\sqrt{5}$$

$$(9) \quad 5\sqrt{2} + 2\sqrt{2} + 4\sqrt{2} = (5 + 2 + 4)\sqrt{2} = 11\sqrt{2}$$

$$(10) \quad \sqrt{3} + 4\sqrt{3} + 6\sqrt{3} = 1\sqrt{3} + 4\sqrt{3} + 6\sqrt{3} = (1 + 4 + 6)\sqrt{3} = 11\sqrt{3}$$

$$(11) \quad 2\sqrt{3} + 4\sqrt{2} = 2\sqrt{3} + 4\sqrt{2}$$

$$(12) \quad 5\sqrt{7} + 2\sqrt{6} = 5\sqrt{7} + 2\sqrt{6}$$

$$(13) \quad 4\sqrt{2} + 3\sqrt{5} = 4\sqrt{2} + 3\sqrt{5}$$

$$(14) \quad 5\sqrt{2} + 3\sqrt{3} + 2\sqrt{7} = 5\sqrt{2} + 3\sqrt{3} + 2\sqrt{7}$$

Simplify each of the following:

$$(Q1) \quad 2 + 3\sqrt{5} + 6 + 4\sqrt{5}$$

Solution

$$2 + 3\sqrt{5} + 6 + 4\sqrt{5} = 2 + 6 + 3\sqrt{5} + 4\sqrt{5}$$

$$= 8 + (3 + 4)\sqrt{5} = 8 + 7\sqrt{5}$$

$$(Q2) \quad 6\sqrt{7} + 1 + 4\sqrt{7} + 3 + 2\sqrt{7}$$

Solution

$$\begin{aligned} 6\sqrt{7} + 1 + 4\sqrt{7} + 3 + 2\sqrt{7} &= 6\sqrt{7} + 4\sqrt{7} + 2\sqrt{7} + 1 + 3 \\ &= (6 + 4 + 2)\sqrt{7} + 4 = 12\sqrt{7} + 4 \end{aligned}$$

$$(Q3) \quad 2\sqrt{3} + 5\sqrt{7} + 5\sqrt{3} + 2$$

Solution

$$\begin{aligned} 2\sqrt{3} + 5\sqrt{7} + 5\sqrt{3} + 2 &= 2\sqrt{3} + 5\sqrt{3} + 5\sqrt{7} + 2 \\ &= (2 + 5)\sqrt{3} + 5\sqrt{7} + 2 = 7\sqrt{3} + 5\sqrt{7} + 2 \end{aligned}$$

$$(Q4) \quad 2 + 5\sqrt{2} + \sqrt{3} + 6\sqrt{2} + 7\sqrt{3} + 6$$

Solution

$$\begin{aligned} 2 + 5\sqrt{2} + \sqrt{3} + 6\sqrt{2} + 7\sqrt{3} + 6 &= 2 + 5\sqrt{2} + 1\sqrt{3} + 6\sqrt{2} + 7\sqrt{3} + 6 \\ &= 2 + 6 + 5\sqrt{2} + 6\sqrt{2} + 1\sqrt{3} + 7\sqrt{3} \\ &= 8 + (5 + 6)\sqrt{2} + (1 + 7)\sqrt{3} \\ &= 8 + 11\sqrt{2} + 8\sqrt{3} \end{aligned}$$

$$(Q5) \quad 2\sqrt{8} + 3\sqrt{3} + 1$$

Solution

$$\begin{aligned} 2\sqrt{8} + 3\sqrt{3} + 1 &= 2\sqrt{4 \times 2} + 3\sqrt{3} + 1 \\ &= 2 \times \sqrt{4} \times \sqrt{2} + 3\sqrt{3} + 1 \\ &= 2 \times 2 \times \sqrt{2} + 3\sqrt{3} + 1 = 4\sqrt{2} + 3\sqrt{3} + 1 \end{aligned}$$

$$(Q6) \quad 3\sqrt{2} + 2 + 2\sqrt{8} + 4\sqrt{2} + 6$$

Solution

$$\begin{aligned} 3\sqrt{2} + 2 + 2\sqrt{8} + 4\sqrt{2} + 6 &= 3\sqrt{2} + 2 + 2\sqrt{4 \times 2} + 4\sqrt{2} + 6 \\ &= 3\sqrt{2} + 2 + 2 \times \sqrt{4} \times \sqrt{2} + 4\sqrt{2} + 6 \\ &= 3\sqrt{2} + 2 + 2 \times 2 \times \sqrt{2} + 4\sqrt{2} + 6 \\ &= 3\sqrt{2} + 4\sqrt{2} + 4\sqrt{2} + 6 + 2 \\ &= (3 + 4 + 4)\sqrt{2} + 8 \\ &= 11\sqrt{2} + 8 \end{aligned}$$

$$(Q7) \quad 5 + 3\sqrt{27} + 2 + 6\sqrt{3} + 2\sqrt{2} + \sqrt{12} + 1$$

Solution

$$\begin{aligned} 5 + 3\sqrt{27} + 2 + 6\sqrt{3} + 2\sqrt{2} + \sqrt{12} + 1 &= 5 + 3\sqrt{27} + 2 + 6\sqrt{3} + 2\sqrt{2} + \sqrt{12} + 1 \\ &= 5 + 3 \times \sqrt{9 \times 3} + 2 + 6\sqrt{3} + 2\sqrt{2} + \sqrt{4 \times 3} + 1 \\ &= 5 + 3 \times 3 \times \sqrt{3} + 2 + 6\sqrt{3} + 2\sqrt{2} + 2 \times \sqrt{3} + 1 \\ &= 5 + 9\sqrt{3} + 2 + 6\sqrt{3} + 2\sqrt{2} + 2\sqrt{3} + 1 \\ &= 5 + 2 + 1 + 9\sqrt{3} + 6\sqrt{3} + 2\sqrt{3} + 2\sqrt{2} \\ &= 8 + (9 + 6 + 2)\sqrt{3} + 2\sqrt{2} \\ &= 8 + 17\sqrt{3} + 2\sqrt{2} \end{aligned}$$

$$(Q8) \quad 4 + 2\sqrt{32} + 3\sqrt{2} + 1 + 2\sqrt{50}$$

Solution

$$\begin{aligned} & 4 + 2\sqrt{32} + 3\sqrt{2} + 1 + 2\sqrt{50} \\ &= 4 + 2\sqrt{16 \times 2} + 3\sqrt{2} + 1 + 2\sqrt{25 \times 2} \\ &= 4 + 2 \times \sqrt{16} \times \sqrt{2} + 3\sqrt{2} + 1 + 2 \times \sqrt{25} \times \sqrt{2} \\ &= 4 + 2 \times 4 \times \sqrt{2} + 3\sqrt{2} + 1 + 2 \times 5 \times \sqrt{2} \\ &= 4 + 8\sqrt{2} + 3\sqrt{2} + 1 + 10\sqrt{2} \\ &= 4 + 1 + 8\sqrt{2} + 3\sqrt{2} + 10\sqrt{2} \\ &= 5 + (8 + 3 + 10)\sqrt{2} \\ &= 5 + 21\sqrt{2} \end{aligned}$$

(Q9) $3\sqrt{7} + 5 + 2\sqrt{7} + 3\sqrt{16} + 2\sqrt{25} + 4\sqrt{128}$

Solution

$$\begin{aligned} & 3\sqrt{7} + 5 + 2\sqrt{7} + 3\sqrt{16} + 2\sqrt{25} + 4\sqrt{128} \\ &= 3\sqrt{7} + 5 + 2\sqrt{7} + 3(4) + 2(5) + 4\sqrt{64 \times 2} \\ &= 3\sqrt{7} + 5 + 2\sqrt{7} + 12 + 10 + 4 \times \sqrt{64} \times \sqrt{2} \\ &= 3\sqrt{7} + 2\sqrt{7} + 5 + 12 + 10 + 4 \times 8 \times \sqrt{2} \\ &= (3 + 2)\sqrt{7} + 27 + 32\sqrt{2} \\ &= 5\sqrt{7} + 27 + 32\sqrt{2} = 5\sqrt{7} + 32\sqrt{2} + 27 \end{aligned}$$

Subtraction of Surds:

In surds subtraction, we can only subtract when the numbers under the square root signs are the same.

Examples:

1. $5\sqrt{2} - 3\sqrt{2} = (5 - 3)\sqrt{2} = 2\sqrt{2}$
2. $8\sqrt{3} - 2\sqrt{3} = (8 - 2)\sqrt{3} = 6\sqrt{3}$
3. $5\sqrt{6} - 9\sqrt{6} = (5 - 9)\sqrt{6} = -4\sqrt{6}$
4. $8\sqrt{2} - 2\sqrt{2} - 3\sqrt{2} = (8 - 2 - 3)\sqrt{2} = 3\sqrt{2}$
5. $4\sqrt{3} - 2\sqrt{5} = 4\sqrt{3} - 2\sqrt{5}$
6. $6\sqrt{2} - 7\sqrt{7} = 6\sqrt{2} - 7\sqrt{7}$
7. $4\sqrt{5} - 2\sqrt{3} = 4\sqrt{5} - 2\sqrt{3}$

Simplify each of the following:

(Q1) $2\sqrt{75} - 4\sqrt{3} - 2$

Solution

$$\begin{aligned} & 2\sqrt{75} - 4\sqrt{3} - 2 \\ &= 2\sqrt{25 \times 3} - 4\sqrt{3} - 2 \\ &= 2 \times \sqrt{25} \times \sqrt{3} - 4\sqrt{3} - 2 \\ &= 2 \times 5 \times \sqrt{3} - 4\sqrt{3} - 2 \\ &= 10\sqrt{3} - 4\sqrt{3} - 2 \\ &= (10 - 4)\sqrt{3} - 2 = 6\sqrt{3} - 2 \end{aligned}$$

$$(Q2) \quad 5\sqrt{8} - 2 - 2\sqrt{2} - 3\sqrt{3}$$

Solution

$$\begin{aligned} & 5\sqrt{8} - 2 - 2\sqrt{2} - 3\sqrt{3} \\ &= 5\sqrt{4 \times 2} - 2 - 2\sqrt{2} - 3\sqrt{3} \\ &= 5 \times \sqrt{4} \times \sqrt{2} - 2 - 2\sqrt{2} - 3\sqrt{3} = 5 \times 2 \times \sqrt{2} - 2 - 2\sqrt{2} - 3\sqrt{3} \\ &= 10\sqrt{2} - 2 - 2\sqrt{2} - 3\sqrt{3} \\ &= 10\sqrt{2} - 2\sqrt{2} - 3\sqrt{3} - 2 \\ &= (10 - 2)\sqrt{2} - 3\sqrt{3} - 2 \\ &= 8\sqrt{2} - 3\sqrt{3} - 2 \end{aligned}$$

$$(Q3) \quad -4\sqrt{7} - 3\sqrt{36} - 2 - 2\sqrt{7} - 3\sqrt{72}$$

Solution

$$\begin{aligned} & -4\sqrt{7} - 3\sqrt{36} - 2 - 2\sqrt{7} - 3\sqrt{72} \\ &= -4\sqrt{7} - 3\sqrt{9 \times 4} - 2 - 2\sqrt{7} - 3\sqrt{36 \times 2} \\ &= -4\sqrt{7} - 3 \times \sqrt{9} \times \sqrt{4} - 2 - 2\sqrt{7} - 3 \times \sqrt{36} \times \sqrt{2} \\ &= -4\sqrt{7} - 3 \times 3 \times 2 - 2 - 2\sqrt{7} - 3 \times 6 \times \sqrt{2} \\ &= -4\sqrt{7} - 18 - 2 - 2\sqrt{7} - 18\sqrt{2} \\ &= -4\sqrt{7} - 2\sqrt{7} - 18 - 2 - 18\sqrt{2} \\ &= (-4 - 2)\sqrt{7} - 20 - 18\sqrt{2} \\ &= -6\sqrt{7} - 18\sqrt{2} - 20 \end{aligned}$$