CHAPTER THREE

BINARY OPERATION

Operation rules in ordinary Algebra:

These operational rules are

1) Closure:

- A statement is opened when no limitation is placed on it, and closed when a limitation is placed on it.
- For example Kofi is a boy is an opened statement, but Kofi is a boy in the class is a closed statement.
- Also *x* is a number is an opened statement, but *x* is a number less than 10 is a closed one.

2) Commutative law:

a) If a + b = b + a, then the given operation which is + or addition is commutative.

b) If a × b = b × a, then the given operation which is × or multiplication is commutative.

c) Lastly if a $\Delta\ b$ = b $\Delta\ a$, then the given operation which is Δ is commutative.

Q1) Using the numbers 3 and 4, determine whether or not addition is commutative.

N/B L.H.S = Left hand side and R.H.S= Right hand side.

Soln.

For addition (+) to be commutative, then 3+ 4 = 4 + 3.

Consider the L.H.S i.e 3 + 4 =7.

Consider the R.H.S i.e 4+3 =7.

Since L.H.S = R.H.S, then addition is commutative.

Q2) Using the numbers 3 and 4, determine whether or not subtraction is commutative.

Soln.

If subtraction is commutative, then $3 - 4 = 4 - 3$.

Considering the L.H.S, $3 - 4 = -1$.

Considering the R.H.S, $4 - 3 = 1$.

Since the L.H.S $\neq$ R.H.S

i.e L.H.S is not equal to the R.H.S, then subtraction is not commutative.

Q3) Using the numbers 3 and 4, determine whether or not multiplication is commutative.

Soln.

For multiplication to be commutative, then $3\times4 = 4\times3$.

L.H.S = $3\times4 = 12$.

R.H.S = $4\times3 = 12$.

Since L.H.S = R.H.S, then the operation which multiplication is commutative.

3) Associative Law:

a) If $(a + b) + c = a + (b + c)$, then addition is associative.
b) If $(a \times b) \times c = a \times (b \times c)$, then multiplication is associative.
c) If $(a * b) * c = a * (b * c)$, then the operation which is $*$, is associative.

Q1) Using the numbers 2, 3 and 5, determine whether or not addition is associative.

Soln.

If + (addition) is associative, then $(2+3) + 5 = 2 + (3+5)$.

L.H.S = $(2+3) + 5 = 5+5 = 10$

R.H.S = $2 + (3+5) = 2+8 = 10$

Since the L.H.S = R.H.S, then addition is associative.

Q2) Using 2, 3 and 5, determine whether or not multiplication is associative.

Soln.

If multiplication is associative, then $(2\times3)\times 5 = 2\times(3\times5)$.

L.H.S = $(2\times3)\times5 = 6\times5 = 30$

R.H.S = $2\times(3\times5) = 2\times15 = 30$

Since L.H.S = R.H.S => × (multiplication) is associative.

Q3) Using 2,3 and 5, determine whether or not subtraction is associative.

Soln

If subtraction (-) is associative, then $(5 - 3) - 2 = 5 - (3 - 2)$

L.H.S = $(5-3) - 2 = 2 - 2 = 0$

R.H.S = $5 - (3 - 2) = 5 - 1 = 4$

Since R.H.S ≠ L.H.S, then (–) or subtraction is not associative.

NB: $a\times(b + c)$ is the same as $a(b + c)$

4) Distributive Law:

- If $a(b + c)$ or $a \times (b + c) = ab + ac$, then multiplication is said to be distributive over addition (+), or multiplication is said to be distributive with respect to addition.

- Also if $a(b\,\Delta\,c) = ab\,\Delta\,ac$, then multiplication is distributive over Δ, or multiplication is distributive with respect to the operation Δ.

An operation:

* An operation is a symbol, with a given meaning.

* For example if $a*b = 2a+b$, then the symbol * becomes an operation, and $a*b$ means that take twice of a and add it to b.

* Also given that a Δ b = $a^2 + b^2$, then the symbol Δ becomes an operation, and a Δ b means add a squared to b squared.

* Other examples of operation which we are familiar with are addition (+), subtraction (-), division (÷) and multiplication (×).

* Lastly any symbol can be used to represent an operation, provided its meaning is given.

The Identity Element:

The identity element of a given operation has no effect on that given operation.

* For example the identity element of addition is 0 (zero), since any number added to zero gives us the same number. (i.e zero has no effect on addition).

* For examples are

3 + 0 = 3

5 + 0 = 5

2 + 0 =2

* The identity element of multiplication is one, since one has no effect on multiplication.

i.e 2×1=2

4×1= 4

5×1= 5

* Therefore assume Δ to be a given operation, and if i.e = the identity element of Δ, then

2 Δ i.e = 2

4 Δ i.e = 4

1 Δ i.e = 6

Q1)

Δ	1	2	3	4
1	4	1	7	2
2	6	2	3	3
3	5	3	4	5
4	1	4	1	4

The given table is that for the operation Δ. By making a careful study of it, determine the identity element of the given operation.

Soln.

From the table

$1 \Delta 2 = 1$

$2 \Delta 2 = 2$

$3 \Delta 2 = 3$

$4 \Delta 2 = 4$

=> Any number Δ 2 = that number

=> 2 has no effect on the given operation, and as such it is the identity element.

Q2)

*	2	3	5	7
2	3	1	2	4
3	1	4	3	2

5	7	3	5	6
7	6	2	7	4

The given table is that drawn for a certain operation, which is represented by the symbol *. By careful analysis, determine the identity element for the given operation.

<u>Soln</u>.

A careful study indicates the following:

2 * 5 = 6

3 * 5 = 3

5 * 5 = 5

7 * 5 = 7

- This implies that 5 had no effect on the given operation. Therefore the identity element = 5

Q3)

m	2	3	4	5
2	4	6	8	10
3	6	9	12	15
4	8	12	16	20
5	10	15	20	25

The given table is that drawn for a certain operation which is represented by the symbol m. Find the identity element.

<u>Soln.</u>

A study of the table, reveals the following facts:

1). 2 $\textcircled{m}$ 2 = 4

But 2 × 2 = 4

2). 3 $\textcircled{m}$ 4 = 12

But 3 × 4 = 12

3). 4 $\textcircled{m}$ 4 = 16

But 4 × 4 = 16

This trend is reflected throughout the whole table.

From this evidence, then the symbol which the given operation represent is x or the multiplication sign, and since the identity element for multiplication is 1 => the required identity element is 1.

Binary operation:

Q1) The binary operation ▽is defined on the set of natural numbers by a ▽ b = a + b.

Determine whether or not ▽ is

i. commutative.
ii. associative.
iii. Is multiplication distributive, with respect to the given operation▽.

Soln.

1. For the operation ▽ to be commutative, then

a ▽ b = b ▽ a

Consider L.H.S:

a ▽ b = a + b, => L.H.S = a + b

Consider the R.H.S. Since a $\triangledown$ b = a + b

=> b $\triangledown$ a = b + a, which can also be written as a + b.

=> R.H.S = a + b

Since the R.H.S = L.H.S, then the given operation is commutative.

ii) Let a, b and c $\in$ N i.e. be member of the set of natural numbers. Then for $\triangledown$ to be associative, $(a\triangledown b)\triangledown c = a\triangledown(b\triangledown c)$.

Consider the L.H.S

i.e. $(a\triangledown b)\triangledown c$, solve what is inside the bracket first.

i.e. $(a\triangledown b) = a + b$

=> $(a\triangledown b)\triangledown c = (a + b)\triangledown c$

But $a\triangledown b = a + b$

=> $(a + b)\triangledown c = (a + b) + c = a + b + c$

$\therefore$ L.H.S = a + b + c

Consider the R.H.S i.e.

$a\triangledown(b\triangledown c)$

Solve what is inside the bracket first => $(b\triangledown c) = b + c$

=> $a\triangledown(b\triangledown c) = a\triangledown (b + c)$

But $a\triangledown b = a + b$

=> $a\triangledown(b + c) = a + (b + c)$

= a + b + c => R.H.S = a + b +c

Since L.H.S = R.H.S, then $\triangledown$ is associative.

iii) For multiplication to be distributive with respect to $\triangledown$, then $a(b\triangledown c) = ab\triangledown ac$

Consider the L.H.S i.e. $a\,(b \triangledown c)$; solve what is inside the bracket first.

i.e. $(b \triangledown c) = b + c$

$\Rightarrow a\,(b \triangledown c) = a\,(b + c) = ab + ac$

$\Rightarrow$ L.H.S $= ab + ac$

Consider the R.H.S

i.e. $ab \triangledown ac$.

Since $a \triangledown b = a + b$

$\Rightarrow ab \triangledown ac = ab + ac$

$\Rightarrow$ R.H.S $= ab + ac$

Since L.H.S = R.H.S, then multiplication is distributive with respect to $\triangledown$.

Q2)

$\top$	2	3	5	6
2	4	5	7	8
3	5	6	8	9
5	7	8	10	11
6	8	9	11	12

The given table is that for the binary operation $\top$. Find the identity element.

<u>Soln.</u>

A careful study of the given table revealed these facts:

1. $3 \top 2 = 5$

But $3 + 2 = 5$

$\top$

2. 3   5 = 8

But 3 + 5 = 8

3. 5 T 2 = 7

But 5 + 2 = 7

4. 3 T 6 = 9

But 3 + 6 = 9

The trend can be noticed throughout the table.

For this reason, then the binary operation given is most likely to represent the addition symbol. Since the identity element of addition is 0, then the identity element for the given table is 0.

<u>**The Inverse:**</u>

The inverse of an item with respect to a given operation, acts on that particular item, to give us the identity element of that given operation.

Q1) Find the inverse of 3, with respect to the addition operation.

<u>Soln.</u>

Since the identity element of addition is 0, then 3 + inverse = 0

=> Inverse = 0 – 3 = -3

Q2) Find the inverse of 7 with reference to addition.

<u>Soln.</u>

7 + inverse = 0

=> inverse = 0 – 7 = -7

Q3) Find the inverse of 3 with respect to multiplication.

Soln.

Since the identity element of multiplication is 1, then 3 × inverse = 1

=> inverse = $\frac{1}{3}$.

Q4) Determine the inverse of 5 with respect to multiplication.

Soln.

5 × inverse = 1

=> inverse = $\frac{1}{5}$.

Q5) The binary operation ◊ is defined on the set N of natural numbers by a ◊ b = 2a + b.

Determine whether or not the given operation ◊

i. is commutative.

ii. is associative.

iii. Determine whether multiplication is distributive over ◊.

Soln.

i. Let a, b and c ∈ N.

If ◊ is commutative, then

a ◊ b = b ◊ a.

Consider the L.H.S

a ◊ b = 2a + b

=> L.H.S = 2a + b.

Consider the R.H.S

i.e. b ◊ a = ? From a ◊ b = 2a + b

=> b ◊ a = 2b + a

=> R.H.S = 2b + a

Since L.H.S ≠ R.H.S i.e. L.H.S is not equal to the R.H.S, then the given operation which is ◊ is not commutative.

ii. If ◊is associative, then

(a ◊ b) ◊ c = a ◊ (b ◊ c)

Consider the L.H.S

i.e. (a ◊ b) ◊ c, solve what is inside the bracket first

=> (a ◊ b) = 2a + b

=> (a ◊ b) ◊ c = (2a + b) ◊ c.

Since a ◊ b = 2a + b

Then (2a + b) ◊b = 2 (2a + b) + b

= 4a + 2b + b = 4a + 3b

=> L.H.S = 4a + 3b

Consider the R.H.S

i.e. a ◊ (b ◊ c). Solve what is inside the bracket first.

Since a ◊ b = 2a + b, then

(b ◊ c) = 2b + c

=> a ◊ (b ◊ c) = a ◊ (2b + c).

Since a ◊ b = 2a + b

Then a ◊ (2b + c) = 2a + (2b + c)

= 2a + 2b + c => R.H.S = 2a + 2b + c. Since the L.H.S ≠ R.H.S, then the given operation ◊ is not association.

iii. If multiplication is distributive over ◊, then a ◊ (b ◊ c) = ab ◊ ac.

Consider the L.H.S i.e. a (b ◊ c).

Solve what is inside the bracket first.

Since a ◊ b = 2a + b, then

(b ◊ c) = 2b + c

=> a (b ◊ c) = a (2b + c)

= 2ab + ac => L.H.S = 2ab + ac

Consider the R.H.S i.e. ab ◊ ac

Since a ◊ b = 2a + b, then ab ◊ ac = 2ab + ac

Since L.H.S = R.H.S => multiplication is distributive over ◊.

Q6) A binary operation is defined on the set R of real numbers by a Δ b = ab – 2.

Determine whether Δ is

a. commutative.

b. associative.

c. Is multiplication distributive over Δ?

d. Find the identity element.

<u>Soln.</u>

Let a, b and c ∈ R.

a. If Δ is commutative, then

$a \Delta b = b \Delta a$

Consider the L.H.S

i.e. $a \Delta b = ab - 2$

=> L.H.S $= ab - 2$

Consider the R.H.S i.e. $b \Delta a$

Since $a \Delta b = ab - 2$, then

$b \Delta a = ba - 2 = ab - 2$ => R.H.S $= ab - 2$

Since L.H.S = R.H.S, then Δ is commutative.

b. If Δ is associative, then

$(a \Delta b)\Delta c = a \Delta (b \Delta c)$

Consider the L.H.S

i.e. $(a \Delta b) \Delta c$,

Since $a \Delta b = ab - 2$

=> $(a \Delta b) \Delta c = (ab - 2) \Delta c$

Since $a \Delta b = ab - 2$, then

$(ab - 2) \Delta c = (ab - 2) c - 2 = abc - 2c -2$

=> L.H.S $= abc - 2c - 2$.

Consider the R.H.S i.e.

$a \Delta (b \Delta c)$,

Since $a \Delta b = ab - 2$, then

$b \Delta c = bc - 2$ => $a \Delta (b \Delta c)$

$= a \Delta (bc - 2)$.

Since $a \Delta b = ab - 2$, then

$a \Delta (bc - 2) = a(bc - 2) - 2$

$= abc - 2a - 2 =$ R.H.S.

Since L.H.S ≠ R.H.S => Δ is not associative.

c. If multiplication is distributive over Δ, then $a(b \Delta c) = ab \Delta ac$.

Consider the L.H.S i.e. $a(b \Delta c)$.

Since $a \Delta b = ab - 2$, then

$b \Delta c = bc - 2$

=> $a(b \Delta c) = a(bc - 2) = abc - 2a$

=> L.H.S $= abc - 2a$.

Consider the R.H.S i.e. $ab \Delta ac$

Since $a \Delta b = ab - 2$, then $ab \Delta ac = (ab)(ac) - 2 = a^2bc - 2$

Since L.H.S ≠ R.H.S, then multiplication is not distributive over Δ.

d. Let e = the identity element, then

$a \Delta e = a$ ……… eqn①

Since $a \Delta b = ab - 2$, then $a \Delta e = ae - 2$ and since from eqn, ① $a \Delta e = a$, => $ae - 2 = a$ => $ae = a + 2$ => $e = \frac{a+2}{a}$.

Q7) The binary operation ⊗ is defined on the set of real numbers by $a \otimes b = a + b + 1$. Determine whether or not ⊗ is

a. commutative.

b. associative.

c. Is multiplication distributive with respect to ⊗?

d. Find the identity element.

Soln.

Let a, b and c $\in$ R.

If $\otimes$ is commutative, then a $\otimes$ b = b $\otimes$ a

Consider the L.H.S i.e. a $\otimes$ b

a $\otimes$ b = a + b + 1= L.H.S

Consider the R.H.S i.e. b $\otimes$ a

Since a $\otimes$ b = a + b + 1, then b $\otimes$ a = b + a + 1 = a + b + 1

=> R.H.S = a + b + 1.

Since L.H.S = R.H.S => $\otimes$ is commutative.

b. If $\otimes$ is associative, then

(a $\otimes$ b) $\otimes$ c = a$\otimes$ (b $\otimes$ c).

Consider the L.H.S i.e. (a $\otimes$ b) $\otimes$ c,

Since a $\otimes$ b = a + b + 1, => (a $\otimes$ b)$\otimes$ c = (a + b + 1)$\otimes$ c.

From a $\otimes$ b = a + b + 1,

=> (a + b+ 1)$\otimes$ b = (a + b +1)+ b +1 = a + b + 1+ b + 1 = a + 2b + 2, => L.H.S = a + 2b +2.

Consider the R.H.S i.e. a $\otimes$ (b $\otimes$ c)

From a $\otimes$ b = a + b + 1, then b $\otimes$ c = b + c + 1 => a $\otimes$ (b $\otimes$ c) = a $\otimes$ (b + c +1).

From a $\otimes$ b = a + b +1, then a $\otimes$ (b + c +1) = a + (b +c +1) + 1

= a + b + c +1+ 1 = a + b + c +2

=> R.H.S = a + b + c + 2.

Since the LHS ≠ R.H.S => ⊗ is not associative.

c. If multiplication is distributive with respect to⊗, then a (b ⊗ c) = ab ⊗ ac

Consider the L.H.S i.e. a (b ⊗ c)

Since a ⊗ b = a + b +1, then b ⊗ c = b + c + 1, => a (b ⊗ c) = a (b + c +1) = ab + ac + a => L.H.S = ab + ac +a

Consider the R.H.S i.e. ab ⊗ ac.

Since a ⊗ b = a + b +1, then ab ⊗ ac = ab + ac + 1 = R.H.S.

Since L.H.S ≠ R.H.S, then multiplication is not distributive with respect to⊗.

d. Let e = the identity element

=> a ⊗ e = a ……….. eqn ①

But since a ⊗ b = a + b +1, then a ⊗ e = a + e +1. From eqn ①, a ⊗ e = a

=> a + e +1 = a => e + 1 = a – a => e + 1 = 0 => e = -1.

Q8) (a) The binary operation $\odot$ is defined on the set R, of real numbers, by

$\mathrm{x} \odot \mathrm{y} = 2\mathrm{x} + 2\mathrm{y}$, where x and y $\in$ R. If 3 and 5 $\in$ R, determine whether or not the given operation is commutative.

(b) Given also that 8 ∈ R, determine whether or not multiplication is distributive with respect to the given operation.

Soln.

(a) $\mathrm{x} \odot \mathrm{y} = 2\mathrm{x} + 2\mathrm{y}$ where x and y are real numbers.

If 3 and 5 are real numbers and the operation is commutative, then 3⊙ 5 = 5⊙ 3.

Consider the L.H.S i.e. 3 ⊙ 5.

Since $\mathrm{x} \odot \mathrm{y} = 2\mathrm{x} + 2\mathrm{y}$, then 3 ⊙ 5 = 2(3) + 2(5) = 6 + 10 = 16

=> L.H.S = 16 .

Consider the R.H.S i.e $5 \odot 3$.

Since $x \odot y = 2x + 2y$,

then $5 \odot 3 = 2(5) + 2(3) = 10 + 6 = 16$,

=>R.H.S = 16 .

Since L.H.S = R.H.S, then the given operation is commutative.

b)If 3, 5 and 8 ∈ R or are real numbers, and the given operation is assoiciative, then

$(3 \odot 5) \odot 8 = 3 \odot (5 \odot 8)$.

Consider the L.H.S

i.e $(3 \odot 5) \odot 8$.

Solve what is inside the bracket first. i.e

$3 \odot 5$

Since $x \odot y = 2x + 2y$, then $3 \odot 5 = (2(3) + 2(5) = 6 + 10 = 16$.

$\therefore (3 \odot 5) \odot 8 = 16 \odot 8$.

Since $x \odot y = 2x + 2y$, then

$16 \odot 8 = 2(16) + 2(8)$

= 32 + 16 = 48, => L.H.S = 48.

Consider the R.H.S i.e $3 \odot (5 \odot 8)$.

Solve what is inside the bracket first.

From $x \odot y = 2x + 2y$,

=> $5 \odot 8 = 2(5) + 2(8) = 10 + 16 = 26$.

$\therefore 3 \odot (5 \odot 8) = 3 \odot 26$.

Since $x \odot y = 2x + 2y$, then

$3 \odot 26 = 2(3) + 2(26) = 6 + 52 = 58$,

=> R.H.S = 58.

=> Since L.H.S ≠ R.H.S, then the given operation is not associative.

(c) If 3, 5 and 8 ϵ R or are real numbers and multiplication is distributive with respect to $\odot$, then $3(5 \odot 8) = 15 \odot 24$.

N/B: $3\times 5 = 15$ and $3 \times 8 = 24$.

Consider L.H.S i.e $3(5 \odot 8)$.

Since $x \odot y = 2x + 2y$, then

$5 \odot 8 = 2(5) + 2(8) = 10 + 16 = 26$,

=>$3(5 \odot 8) = 3(26) = 78$.

L.H.S = 78.

Consider the R.H.S i.e $15 \odot 24$.

Since $x \odot y = 2x + 2y$, then

$15 \odot 24 = 2(15) + 2(24)$

= 30 + 48 = 78, => R.H.S = 78.

Since L.H.S = R.H.S, then multiplication is distributive with respect to the given equation.

Q9) (a) The binary operation Δ is defined on the set N of natural numbers, by $a \Delta b = 2ab + b$, where a and b ∈ N.

If 3 and 6 ∈ N, show whether or not Δ $is\ commutative$.

(b)Given also that 10 ∈ N, determine whether or not multiplication is distributive over the given operation as well as whether the given operation is associative.

Soln.

(a) a Δ b = 2ab + b where a and b ∈ N,

i.e are members of the set of natural numbers. Also 3 and 6 ∈ N.

If the given operation is commutative, then 3 Δ 6 = 6 Δ 3.

Consider the L.H.S i.e 3 Δ 6.

Since a Δ b = 2ab + b, then

3 Δ 6 = 2(3)(6) + 6 = 36 + 6 = 42

=> L.H.S = 42.

Consider the R.H.S i.e 6 Δ 3.

Since a Δ b = 2ab + b, then

6Δ 3 = 2(6)(3) + 3 = 36 +3 = 39

=> R.H.S = 39.

Since L.H.S ≠ R.H.S, then the given operation is not commutative.

b) If 3, 6 and 10 ∈ N and the given operation is associative, then (3 Δ 6) Δ 10 = 3 Δ $(6 \Delta 10)$.

Consider the L.H.S i.e (3 Δ $6) \Delta 10$.

Solve what is inside the bracket first i.e 3 Δ 6.

From a Δ b = 2ab + b

=>3 Δ 6 = 2 (3)(6) + 6 = 36 + 6 = 42

∴ $(3 \Delta 6) \Delta 10$ = 42 Δ 10.

Since a Δ b = 2ab + b, then

42 Δ 10 = 2(42)(10) + 10 = 840 + 10 = 850

=> L.H.S = 850.

=> Consider the R.H.S i.e 3 Δ $(6\Delta 10)$.

Solve what is inside the bracket first

i.e 6 Δ 10.

From a Δ b = 2ab + b, then

6 Δ 10 = 2(6)(10) + 10

= 120 + 10 = 130.

∴3 Δ (6 $\Delta 10)$ = 3 Δ 130.

Since a Δ b = 2ab + b, then

3 Δ 130 = 2(3)(130) + 130 = 780 + 130

= 910, => R.H.S = 910. Since L.H.S ≠ R.H.S, then the given operation is not associative.

b). If multiplication is distributive over Δ, and $3, 6$ and 10 ∈ N, then 3 (6 $\Delta 10) = 18\ \Delta\ 30$.

Consider the L.H.S i.e3(6 Δ 10).

Solve what is inside the bracket first i.e 6 Δ 10.

Since a Δ b = 2ab + b, then

$6\ \Delta 10 = 2(6)(10) + \ 10 = 120 + 10$

= 130.

∴ L.H.S = 130

Consider R.H.S i.e 18 Δ 30.

From a Δ b = 2ab + b

=> 18 Δ 30 = 2(18)(30) + 30

= 1080 + 30 = 1110.

∴ R.H.S = 1110.

Since L.H.S ≠ R.H.S, then multiplication is not distributive over the given operation.

Q10) The binary operation ◊ is defined on the set of natural numbers by x ◊ y = $\frac{x-y}{y}$, where y ≠ 0.

a. Evaluate the following:

i. 8 ◊ 2 ii. 2 ◊ (4 ◊ 1) iii. 3 ◊ (5 ◊ 2)

iv. (6 ◊ 2) ◊ 4 v. (6 ◊ 3) ◊ (4 ◊ 2).

b. Given that 8 ◊ k = 12 ◊ 3, find the value of k.

c. If m ◊ 8 = 12 ◊ 3, determine the value of m

Soln

a). x ◊ y = $\frac{x-y}{y}$ => 8 ◊ 2 = $\frac{8-2}{2}$ = $\frac{6}{2}$ = 3

ii. To solve 2 ◊ (4 ◊ 1), solve what is inside the bracket first. i.e. 4 ◊ 1.

Since x ◊ y = $\frac{x-y}{y}$, then

4 ◊ 1 = $\frac{4-1}{1}$ = $\frac{3}{1}$ = 3

∴ 2 ◊ (4 ◊ 1) = 2 ◊ 3

But since x ◊ y = $\frac{x-y}{y}$

2 ◊ 3 = $\frac{2-3}{3}$ = - $\frac{1}{3}$

iii. To solve 3 ◊ (5 ◊ 2), solve what is inside the bracket first. i.e. 5 ◊ 2.

Since x ◊ y = $\frac{x - y}{y}$

then 5 ◊ 2 = $\frac{5 - 2}{2} = \frac{3}{2}$ = 1.5

∴3 ◊ (5 ◊ 2) = 3 ◊ 1.5

But since x ◊ y = $\frac{x - y}{y}$, then

3 ◊ 1.5 = $\frac{3 - 1.5}{1.5} = \frac{1.5}{1.5}$ = 1

iv. To solve (6 ◊ 2) ◊ 4, solve what is inside the bracket first. i.e 6 ◊ 2

Since x ◊ y = $\frac{x - y}{y}$,

then 6 ◊ 2 = $\frac{6 - 2}{2} = \frac{4}{2}$ = 2

∴ (6 ◊ 2) ◊ 4 = 2 ◊ 4

But since x ◊ y = $\frac{x - y}{y}$, then

2 ◊ 4 = $\frac{2 - 4}{4} = \frac{-2}{4} = \frac{-1}{2}$ = - 0.5

v. To solve (6 ◊ 3) ◊ (4 ◊ 2)

Since x ◊ y = $\frac{x - y}{y}$ =>

6 ◊ 3 = $\frac{6 - 3}{3} = \frac{3}{3}$ = 1

Also 4 ◊ 2 = $\frac{4 - 2}{2} = \frac{2}{2}$ = 1

(6 ◊ 3) ◊ (4 ◊ 2) = 1 ◊ 1

From x ◊ y = $\frac{x - y}{y}$ => 1 ◊ 1 = $\frac{1 - 1}{1} = \frac{0}{1}$ = 0.

b). We are given that 8 ◊ k = 12 ◊ 3.

First evaluate both the L.H.S and the R.H.S, equate them and then solve for k.

Consider the L.H.S i.e. 8 ◊ k.

Since x ◊ y = $\frac{x-y}{y}$ => 8 ◊ k = $\frac{8-k}{k}$ => L.H.S = $\frac{8-k}{k}$.

Consider the R.H.S i.e. 12 ◊ 3

Since x ◊ y = $\frac{x-y}{y}$ => 12 ◊ 3 = $\frac{12-3}{3}$ = $\frac{9}{3}$ = 3

=> R.H.S = 3.

Equating L.H.S to the R.H.S

=> $\frac{8-k}{k}$ = 3 => 3k = 8 – k

=> 3k + k = 8 => 4k = 8, => k = $\frac{8}{4}$ => k = 2.

c).m ◊ 8 = 12 ◊ 3

Solve first the L.H.S i.e m ◊ 8

Since x ◊ y = $\frac{x-y}{y}$ => m ◊ 8 = $\frac{m-8}{8}$ => L.H.S = $\frac{m-8}{8}$.

Consider the R.H.S i.e. 12 ◊ 3

12 ◊ 3 = $\frac{12-3}{3}$ = $\frac{9}{3}$ = 3

Equating L.H.S to the R.H.S

$\frac{m-8}{8}$ = 3 => m – 8 = 8 × 3

=> m – 8 = 24 => m = 24 + 8 = 32

Modulo Arithmetic:

- To convert a given number into a given modulo, we first divide that given number by the modulo, and consider only the remainder.

- But if a given number is less than a given modulo, we maintain the number.

Q1) Convert the following numbers into modulo 5:

a. 4 b. 2 c. 1 d. 3 e. 8 f. 7 g. 10 h. 11

Soln.

a. 4 when changed into modulo 5 still remains 4, since 4 is less than 5.

b. 2 changed into modulo 5 remains 2, since 2 is less than 5.

c. 1 still remains 1.

d. 3 still remains 3, since it is less than 5.

e. To convert 8 into modulo 5, we divide the 8 by the 5 and consider only the remainder.

i.e. $\frac{8}{5}$ = 1 remainder 3

=> 8 in modulo 5 = 3.

f. To change 7 into modulo 5, we divide the 7, by the 5 i.e. $\frac{7}{5}$ = 1 remainder 2

=> 7 in modulo 5 = 2.

g. To change 10 into modulo 5, divide the 10 by the 5 and consider only the remainder

i.e. $\frac{10}{5}$ = 2 remainder 0

=> 10 in modulo 5 = 0.

h. 11 in modulo 5 => $\frac{11}{5}$ = 2 remainder 1

=> 11 in modulo 5 = 1.

Q2) Convert the following numbers into modulo 3:

a. 8 b. 6 c. 2 d. 3 e. 12 f. 13 g. 7 h. 20

Soln.

a. $\frac{8}{3}$ = 2 remainder 2

=> 8 in modulo 3 = 2.

b. $\frac{6}{3}$ = 2 remainder o

=> 6 in modulo 3 = 0.

c. 2 in modulo 3 = 2 since 2 is less than 3.

d. $\frac{3}{3}$ = 1 remainder = 0

=> 3in modulo 3 = 0.

e. $\frac{12}{3}$ = 4 remainder 0.

=> 12 in modulo 3 = 0.

f. $\frac{13}{3}$ = 4 remainder 1.

=> 13 in modulo 3 = 1

g. $\frac{7}{3}$ = 2 remainder 1.

=> 7 in modulo 3 = 1

h. $\frac{20}{3}$ = 6 remainder 2

=> 20 in modulo 3 = 2.

Q3) Draw a table for addition or + in modulo 4 on the set P = {0, 1, 2, 3}.

Soln.

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

1. 0 + 0 = 0 2. 0 + 1 = 1

3. 0 + 2 = 2 4. 0 + 3 = 3

5. 1 + 0 = 1 6. 1 + 1 = 2

7. 1 + 2 = 3 8. 1 + 3 = 4 = 0

9. 4 + 2 = 6 = 2 10. 2 + 1 = 3

11. 2 + 2 = 4 = 0 12. 2 + 3 = 5 = 1

13. 3 + 0 = 3 14. 3 + 1 = 4 = 0

15. 3 + 2 = 5 = 1 16. 3 + 3 = 6 = 2.

NB: Since we are to construct the table in modulo 4, then number from 4 upwards must be converted into modulo 4.

Q4) Draw a table for addition (+) in modulo 7 on the set Y = {2, 3, 4, 6}

Soln.

+	2	3	4	6
2	4	5	6	1
3	5	6	0	2
4	6	0	1	3

6	1	2	3	5

1. $2 + 2 = 4$
2. $2 + 3 = 5$
3. $2 + 4 = 6$
4. $2 + 6 = 8 = 1$
5. $3 + 2 = 5$
6. $3 + 3 = 6$
7. $3 + 4 = 7 = 0$
8. $3 + 6 = 9 = 2$
9. $4 + 2 = 6$
10. $4 + 3 = 7 = 0$
11. $4 + 4 = 8 = 1$
12. $4 + 6 = 10 = 3$
13. $6 + 6 = 12 = 5$

Q5) Construct a table for multiplication in modulo 4.

<u>Soln.</u>

- The set of numbers under modulo 4 = {0, 1, 2, 3}
- This is the set of numbers used to construct the table.

×	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

1. $1 \times 1 = 1$
2. $1 \times 2 = 2$
3. $1 \times 3 = 3$
4. $2 \times 2 = 4 = 0$
5. $2 \times 3 = 6 = 2$
6. $3 \times 1 = 3$

7. $3 \times 2 = 6 = 2$ 8. $3 \times 3 = 9 = 1$

Q6) a). Draw a table for multiplication, × in modulo 5.

b). Using your table, find n if

i). $2 \times n = 4$ ii). $3 \times n = 2$

iii). $n \times 2 = 4$ iv. $n \times n = 4$

c). Use your table to evaluate the following:

i). $(2 \times 3) + (1 \times 4)$. ii. $(3 \times 1) + (3 \times 4)$

<u>Soln.</u>

The set of numbers under modulo 5 = {0, 1, 2, 3, 4}.

We use these numbers to construct our table.

×	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

1. $1 \times 1 = 1$ 2. $1 \times 2 = 2$

3. $1 \times 3 = 3$ 4. $1 \times 4 = 4$

5. $2 \times 2 = 4$ 6. $2 \times 3 = 6 = 1$

7. $2 \times 4 = 8 = 3$ 8. $3 \times 4 = 12 = 2$

9. $4 \times 2 = 8 = 3$ 10. $4 \times 4 = 16 = 1$

(a). i). $2 \times n = 4$

From the table $2 \times 2 = 4 \Rightarrow n = 2$

ii). $3 \times n = 2$

from the table $3 \times 4 = 2 \Rightarrow n = 4$

iii). $n \times 2 = 4$.

From the table $2 \times 2 = 4 \Rightarrow n = 2$.iv). $n \times n = 4$, but since $2 \times 2 = 4 \Rightarrow n = 2$.

(c). i). $(2 \times 3) + (1 \times 4) =?$

$2 \times 3 = 1$ and $1 \times 4 = 4$

$\Rightarrow (2 \times 3) + (1 \times 4) = 1 + 4 = 5 = 0$

ii). $(3 \times 1) + (3 \times 4) =?$

$3 \times 1 = 3$ and $3 \times 4 = 2$

$\Rightarrow (3 \times 1) + (3 \times 4) = 3 + 2 = 5= 0$

Q7. a. Draw tables for addition $\oplus$ and multiplication $\otimes$ for arithmetic modulo 6.

b. Use your tables to find the truth set of

i. $5 \otimes n = 1$ ii. $2 \oplus n = 0$

iii. $n \otimes 1 = 3$ iv. $n \oplus 4 = 1$

c. Use your table to evaluate the following:

i. $(3 \oplus 4) \oplus (4 \oplus 2)$

ii. $(2 \oplus 1) \otimes (1 \oplus 4)$

iii. $(2 \otimes 1) \oplus (4 \oplus 2)$

NB: Two tables should be drawn. One for the addition and the other for the multiplication.

<u>Soln.</u>

The set of numbers under modulo 6 = {0, 1, 2, 3, 4, 5}

$\oplus$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

1. 0 + 1 = 1
2. 0 + 4 = 4
3. 0 + 5 = 5
4. 2 + 4 = 6 = 0
5. 2 + 1 = 3
6. 2 + 5 = 7 = 1
7. 3 + 3 = 6 = 0
8. 1 + 5 = 6 = 0
9. 5 + 5 = 10 = 4
10. 5 + 4 = 9 = 3
11. 4 + 4 = 8 = 2

$\otimes$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4

3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

1.0 × 4 = 0 2. 1 × 1 = 1

3. 1 × 5 = 5 4. 2 × 2 = 4

5. 2 × 3 = 6 = 0 6. 2 × 4 = 8 = 2

7. 2 × 5 = 10 = 4 8. 3 × 3 = 9 = 3

9. 3 × 4 = 12 = 0 10. 4 × 4 = 16 = 4

11. 5 × 4 = 20 = 2 12. 5 × 5 = 25 = 1

NB: For some of the tables, only certain selected values will be computed as an illustration.

b. i. 5 ⨂ n = 1

From the table since 5 ⨂ 5 = 1, => n = 5

ii. 2 ⊕ n = 0.

From the table since 2 ⊕ 4 = 0, => n = 4. iii. n ⨂ 1 = 3

From the multiplication table,

3 ⨂ 1= 3 => n = 3.

iv. n ⊕ 4 = 1

From the addition table, since

3 + 4 = 1 => n = 3

c. i. (3 ⊕ 4) ⊕ (4 ⊕ 2) = ?

$3 \oplus 4 = 1$ and $4 \oplus 2 = 0$

=> $(3 \oplus 4) + (4 \oplus 2) = 1 \oplus 0 = 1$

ii. $(2 \oplus 1) \otimes (1 \oplus 4) = ?$

Since $2 \oplus 1 = 3$ and $1 \oplus 4 = 5$,

=> $(2 \oplus 1) \otimes (1 \oplus 4) = 3 \otimes 5$, and from the multiplication table $3 \otimes 5 = 3$.

iii. $(2 \otimes 1) + (4 \oplus 2) = ?$

Since $2 \otimes 1 = 2$ and $4 \oplus 2 = 0$,=> $(2 \otimes 1) + (4 \oplus 2) = 2 \oplus 0 = 2$.

Q8)

$\otimes$	2	4	6	8
2	4	8	2	6
4	8	6	4	2
6	2	4	6	8
8	6	2	8	4

The given table is that for the binary operation $\otimes$ in modulo 10. Determine the identity element.

Soln.

A careful analysis of the given table came out with these facts:

1. $2 \otimes 2 = 4$

But $2 \times 2 = 4$

2. $4 \otimes 2 = 8$

But $4 \times 2 = 8$

3. $4 \otimes 4 = 6$

But $4 \times 4 = 16 = 6$, in modulo 10

4. $6 \otimes 6 = 6$

But $6 \times 6 = 36 = 6$, in modulo 10

5. $4 \otimes 8 = 2$

But $4 \times 8 = 32 = 2$, in modulo 10

From these facts, we can conclude that the binary operation $\otimes$ *is most likely to* represent the multiplication symbol.

- For this reason the identity element is 1, which is the identity element for multiplication.

Q9. The operations * and Δ are defined in modulo 5 as $M * N = M + N + 1$ and $M \Delta N = 3MN$, where $M, N \in S = \{0,1,2,3,4\}$.

(a) Form tables for * and Δ on the set S.

b. Use your table to evaluate the following:

1. $2 * n = 3$ 2. $n \Delta n = 2$

3. $(1 \Delta 2) * 4 =$ 4. $(4 \Delta 1) \Delta (2 \Delta 4) =$

5. $(1 \Delta 3) * (2 \Delta 1) =$ 6. $(2 \Delta 1) * (3 * 1) =$

NB: We must first construct two tables, one for the operation * and another for the operation Δ.

<u>Soln</u>

*	0	1	2	3	4

0	1	2	3	4	0
1	2	3	4	0	1
2	3	4	0	1	2
3	4	0	1	2	3
4	0	1	2	3	4

M * N = M + N + 1

1. 0 * 0 = 0 + 0 + 1 = 0 + 1 = 1

2. 0 * 1 = 0 + 1 + 1 = 2

3. 0 * 2 = 0 + 2 + 1 = 3

4. 0 * 3 = 0 + 3 + 1 = 4

5. 0 * 4 = 0 + 4 + 1 = 5 = 0

6. 1 * 0 = 1 + 0 + 1 = 2

7. 1 * 1 = 1 + 1 = 1 = 3

8. 1 * 2 = 1 + 2 + 1 = 4

9. 1 * 3 = 1 + 3 + 1 = 5 = 0

10. 1 * 4 = 1 + 4 + 1 = 6 = 1

11. 2 * 0 = 2 + 0 + 1 = 3

12. 2 * 1 = 2 + 1 + 1 = 4

13. 2 * 2 = 2 + 2 + 1 = 5 = 0

14. 2 * 3 = 2 + 3 + 1 = 6 = 1

15.2 * 4 = 2 + 4 + 1 = 7 = 2

16. 3 * 0 = 3 + 0 + 1 = 4

17. 3 * 1 = 3 + 1 + 1 = 5 = 0

18. 3 * 2 = 3 + 2 + 1 = 6 = 1

19. 3 * 3 = 3 + 3 + 1 = 7 = 2

20. 3 * 4 = 3 + 4 + 1 = 8 = 3

21. 4 * 1 = 4 + 1 + 1 = 6 = 1

22. 4 * 0 = 4 + 0 + 1 = 5 = 0

23. 4 * 1 = 4 + 1 + 1 = 6 = 1

24. 4 * 2 = 4 + 2 + 1 = 7 = 2

25. 4 * 3 = 4 + 3 + 1 = 8 = 3

26. 4 * 4 = 4 + 4 + 1 = 9 = 4

Table 2

Δ	0	1	2	3	4
0	0	0	0	0	0
1	0	3	1	4	2
2	0	1	2	3	4

3	0	4	3	2	1
4	0	2	4	1	3

M Δ N = 3 MN

1. 0 Δ 0 = 3 (0) (0) = 0

2. 0 Δ 1 = 3 (0) (1) = 0

3. 0 Δ 2 = 3 (0) (2) = 0

4. 0 Δ 3 = 3 (0) (3) = 0

5. 0 Δ 4 = 3 (0) (4) = 0

6. 1 Δ 0 = 3 (1) (0) = 0

7. 1 Δ 1 = 3 (1) (1) = 3

8. 1 Δ 2 = 3 (1) (2) = 6 = 1

9. 1 Δ 3 = 3 (1) (3) = 9 = 4

10. 1 Δ 4 = 3 (1) (4) = 12 = 2

11. 2 Δ 0 = 3 (2) (0) = 0

12. 2 Δ 1 = 3 (2) (1) = 6 = 1

13. 2 Δ 2 = 3 (2) (2) = 12 = 2

14. 2 Δ 3 = 3 (2) (3) = 18 = 3

15. 2 Δ 4 = 3 (2) (4) = 24 = 4

16. 3 Δ 0 = 3 (3) (0) = 0

17. 3 Δ 1 = 3 (3) (1) = 9 = 4

18. 3 Δ 2 = 3 (3) (2) = 18 = 3

19. 3 Δ 3 = 3 (3) (3) = 27 = 2

20. 3 Δ 4 = 3 (3) (4) = 36 = 1

21. 4 Δ 0 = 3 (4) (0) = 0

22. 4 Δ 1 = 3 (4) (1) = 12 = 2

23. 4 Δ 2 = 3 (4) (2) = 24 = 4

24. 4 Δ 3 = 3 (4) (3) = 36 = 1

25. 4 Δ 4 = 3 (4) (4) = 48 = 3

b. i. 2 * n = 3

From the first table, 2 * 0 = 3 => n = 0

2. n Δ n = 2

From the second table,

2 Δ 2 = 2 => n = 2

Also 3 Δ 3 = 2 => n = 3.

∴ n = 2 or 3.

3. (1 Δ 2) * 4 = ?

From the second table, 1 Δ 2 = 1

=> (1 Δ 2) * 4 = 1 * 4.

From the fist table, 1 * 4 = 1

4. (4 Δ 1) Δ (2 Δ 4) = ?

From the second table,

4 Δ 1 = 2 and 2 Δ 4 = 4

=> (4 Δ 1) Δ (2 Δ 4) = 2 Δ 4 = 4

5. (1 Δ 3) * (2 Δ 1) = ?

1 Δ 3 = 4 and 2 Δ 1 = 1

=> (1 Δ 3) * (2 Δ 1) = 4 * 1 = 1

6. (2 * 1) Δ (3 * 4) = ?

2 * 1 = 4 and 3 * 4 = 3

=> (2 * 1) Δ (3 * 4) = 4 Δ 3 = 1

7. (2 Δ 1) * (3 * 1) = ?

2 Δ 1 = 1 and 3 * 1 = 0

=> (2 Δ 1) * (3 * 1) = 1 * 0 = 2.

NB: 3MN = 3 × M × N => 3 (2) (4) = 3 × 2 × 4 = 24

Questions

Q1) The operation ⊡ is defined by x ⊡ y = x + y, on the set R, of real numbers. Determine whether or not the given operation is

a. commutative Ans: Yes

b. associative Ans: Yes

c. Is multiplication distributive over ⊡ Ans: Yes

Q2) The binary operation ⨁ is defined on the set N of natural numbers by x ⨁ y = x× y + 2.

i. Is the given operation commutative Ans: Yes

ii. is the operation associative Ans: No

iii. Is multiplication distributive over ⨁ Ans: No

Q3) A binary operation ◊ is defined on the set of real numbers by x ◊ y = x + 2y + 1. Determine whether or not the given operation is (i) commutative. Ans: No

ii). associative Ans: No

Q4) The operation * is defined on the set N of natural numbers by a * b = ab – 2. Determine whether

i. * is commutative. Ans: Yes

ii. * is associative. Ans: No

iii. multiplication is distributive with respect to * .
Ans: No

Q5) The binary operation ▽ is defined on the set R of real numbers by a ▽ b = 3ab + 1. Evaluate

1. 2 ▽ 4 Ans: 25

2. 1 ▽ 3 Ans: 10

3. -2 ▽ 3 Ans: -17

4. -2 ▽ -3 Ans: 19

5. $\sqrt{2} \triangledown \sqrt{5}$ Ans: $3\sqrt{10} + 1$

6. $2\sqrt{4} \triangledown 3\sqrt{2}$ Ans: $36\sqrt{2} + 1$

Q6)

⊗	2	5	7	9
2	4	2	2	1
5	6	7	5	6
7	5	3	7	5
9	1	2	9	1

The given table is that drawn for the operation ⊗, in an unknown modulo. Determine the identity element for the given operation. Ans: 7

Q7)

✧	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	0	2
3	0	3	0	3	0
4	0	4	2	0	4

The table drawn is that for a binary operation in modulo 6. Find the identity element. Ans: 1

Q8)

	0	1	2	3	4

‡					
0	0	0	0	0	0
1	0	3	1	4	2
2	0	1	2	3	4
3	0	4	3	2	1
4	0	2	4	1	3

The given table is that for the given binary operation in modulo 5: Find the identity element Ans = 2

Q9.

⍓	2	3	4	6
2	4	5	6	1
3	5	6	0	2
4	6	0	1	3
6	1	2	3	5

The given table has been drawn in arithmetic modulo 7, for the operation⍓. Find the identity element. Ans: 0

Q10)

⑥	2	3	4	5

2	4	0	2	4
3	0	3	0	3
4	2	0	4	2
5	4	3	2	1

The table is in modulo 6 and is that for the binary operation ⑥. Determine the identity element. Ans: 1

Q11) a. Construct a table for multiplication in modulo 4, on the set P = {1, 3, 5, 7}.

Ans:

×	1	3	5	7
1	1	3	1	3
3	3	1	3	1
5	1	3	1	3
7	3	1	3	1

b. Use your table to evaluate

i. 3 × 5 Ans: 3

ii. 5 × 1 Ans: 1

iii. (3 × 1) + (7 × 7) Ans: 4

iv. (5 × 1) × (7 × 3) Ans: 1

Q12) Construct a table for addition in arithmetic modulo 4.

Ans:

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

b. Use your table to evaluate

i. 1 + 3 Ans: 0

ii. (2 + 2) + (1 + 3) Ans: 0

iii. (2 + 3) × (3 + 0) Ans: 3

Q13) a. A binary operation ⊗ is defined on the set R of real numbers by a ⊗ b = 2ab – 3. Construct a table for ⊗ in modulo 5.

Soln.

⊗	0	1	2	3	4
0	-3	-3	-3	-3	-3
1	-3	-1	1	3	0
2	-3	1	0	4	3
3	-3	3	4	0	1
4	-3	3	3	1	4

b. Determine the truth set of the following:

i. $n \otimes n = 0$ Ans: $n = 2$ or 3

ii. $3 \otimes n = 4$ Ans: $n = 2$ or 4

iii. $(2 \otimes 4) \otimes (3 \otimes 3)$ Ans: 3