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27 2025 .

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( ) —  $\{B(t)\}_{t \geq 0}$ ,

$$\frac{dB(t)}{dt} = r(t) B(t), \quad r(t) \text{ — } .$$

$$B(0) = 1.$$

$$B(t) = \exp\left(\int_0^t r(s) ds\right).$$

$$r(t) \equiv r \quad :$$

$$B(t) = e^{rt}.$$

( )  $B(t)$ , ,  
 $r$ . :

- -
- -
- **Numeraire** —

$[0, T]$

$\Delta t$ :

$$0, \Delta t, 2\Delta t, \dots, n\Delta t = T, \quad \Delta t = \frac{T}{n}.$$

,

$$B(t + \Delta t) = B(t)(1 + r\Delta t).$$

$n$

$$B(T) = B(0) (1 + r\Delta t)^n.$$

$$\Delta t \rightarrow 0 \quad (n \rightarrow \infty),$$

$$\lim_{\Delta t \rightarrow 0} (1 + r\Delta t)^{T/\Delta t} = e^{rT}.$$

,

$$\frac{dB(t)}{dt} = \lim_{\Delta t \rightarrow 0} \frac{B(t + \Delta t) - B(t)}{\Delta t} = rB(t).$$

$r = 16,5\% = 0.165$ .  $B(0) = 100\,000$  ,  $:$   
 $(T = 1)$  .

1.  $B_1(1)$  ( $n = 1$ ):

$$B_1(1) = 100\,000 \left(1 + \frac{0.16}{1}\right)^1 = 100\,000 \cdot 1.16 = \mathbf{116\,000} \quad .$$

2.  $B_{12}(1)$  ( $n = 12$ ):

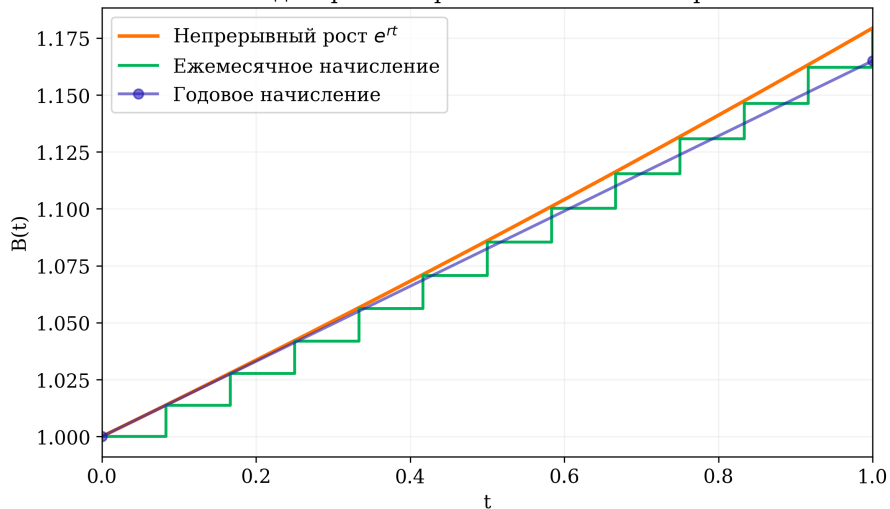
$$B_{12}(1) = 100\,000 \left(1 + \frac{0.16}{12}\right)^{12} \approx 100\,000 \cdot 1.17180 \approx \mathbf{117\,180} \quad .$$

3.  $B_{\text{cont}}(1)$  ( $n \rightarrow \infty$ ):

$$B_{\text{cont}}(1) = 100\,000 e^{0.16} \approx 100\,000 \cdot 1.17351 \approx \mathbf{117\,351} \quad .$$



### Сближение дискретного роста к экспоненте при $r = 16.5\%$



$$\overbrace{116\,000}^{B_1(1)} < \overbrace{117\,180}^{B_{12}(1)} < \overbrace{117\,351}^{B_{\text{cont}}(1)}$$

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:  $e^r$ .

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**1.**

, , 100 .  
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, , 100 .  
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: 99, 101, 98.5, 105 — , .

, , 100 .  
?

: 99, 101, 98.5, 105 — , .  
— , — , .



$$2. \quad (\Omega, \mathcal{F}, \mathbb{P})$$



- $(\Omega, \mathcal{F}, \mathbb{P})$ ,
1.  $\Omega$  — ( );
  2.  $\mathcal{F}$  —  $\sigma$ - ,  $\Omega$ , ;
  3.  $\mathbb{P}$  — ,  $A \in \mathcal{F}$   $\mathbb{P}(A)$   $[0, 1]$  .

( )

:

1. ( ):

$$\Omega = \{\text{up}, \text{down}, \text{flat}\}.$$

2. — :

$$\mathcal{F} = \sigma(\{\text{up}\}, \{\text{down}\}, \{\text{flat}\}).$$

3. :

$$P(\text{up}) = 0.3, \quad P(\text{down}) = 0.4, \quad P(\text{flat}) = 0.3.$$

$$(\Omega, \mathcal{F}, P).$$

**3.**

■  $\frac{1}{2} \frac{d}{dt} \left( \frac{1}{2} \frac{d}{dt} \right)$

■  $\frac{1}{2} \frac{d}{dt} \left( \frac{1}{2} \frac{d}{dt} \right)$

■  $\frac{1}{2} \frac{d}{dt} \left( \frac{1}{2} \frac{d}{dt} \right)$

■  $\frac{1}{2} \frac{d}{dt} \left( \frac{1}{2} \frac{d}{dt} \right)$

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P&L.

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P&L

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■ (K) — ,



- (K) — ,
- (T) — , ( ).  $T$

- $(K)$  — ,
- $(T)$  — , ( ).  $T$

( )

—  $K$   $T$ ,  
 $S_T$ .

**Call-** —  $K$ .

- ,  $S_T > K$ ;

- ,  $S_T \leq K$ .

**Put-** —  $K$ .

- ,  $S_T < K$ ;

- ,  $S_T \geq K$ .

- Long Call: . .
- Short Call: . .
- Long Put: . .
- Short Put: . .

(!!!) : long short — , .

- Long Call: . .
- Short Call: . .
- Long Put: . .
- Short Put: . .

(!!!) : long short — ,

- (long) .
- (short) .

( ... )

call-  $K$ .  $S_T$ :

$$\begin{cases} S_T \leq K, & \Rightarrow 0, \\ S_T > K, & K \quad S_T \Rightarrow S_T - K. \end{cases}$$

,

$$\text{Payoff}_{\text{call}} = \max(S_T - K, 0).$$

put

$K$ :

$$\left\{ \begin{array}{ll} S_T \geq K, & \Rightarrow 0, \\ S_T < K, & K - S_T \Rightarrow K - S_T. \end{array} \right.$$

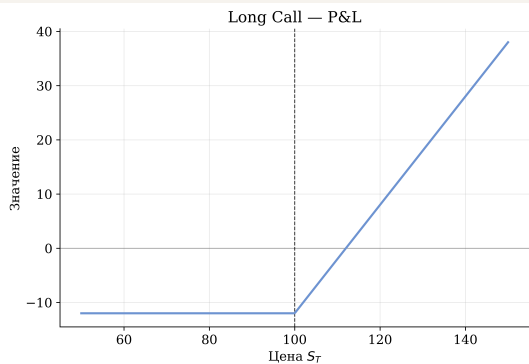
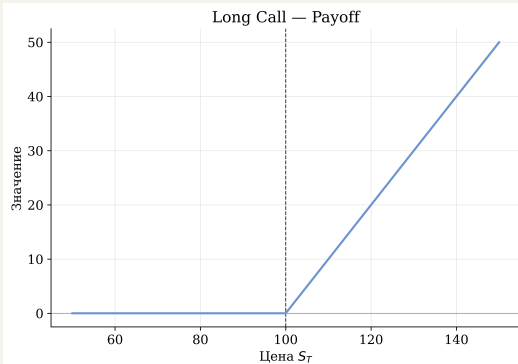
:

$$\text{Payoff}_{\text{put}} = \max(K - S_T, 0).$$

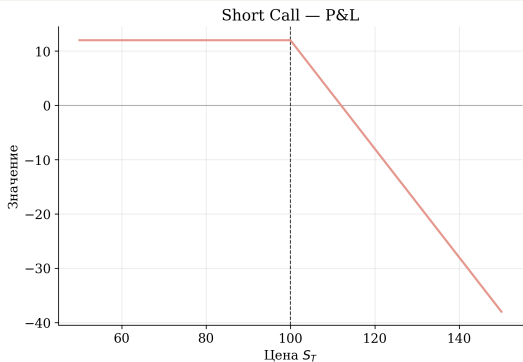
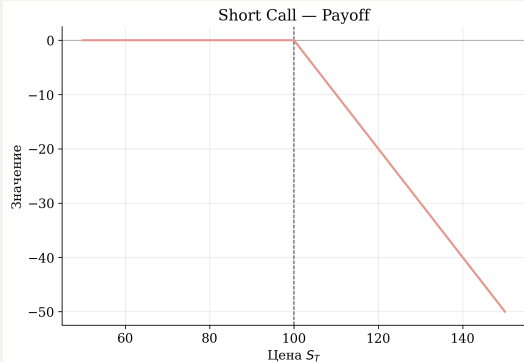


$$\begin{aligned}
 & : \\
 & \left\{ \begin{array}{ll} \text{Payoff}_{\text{Long Call}} & = \max(S_T - K, 0), \\ \text{Payoff}_{\text{Short Call}} & = -\max(S_T - K, 0), \\ \text{Payoff}_{\text{Long Put}} & = \max(K - S_T, 0), \\ \text{Payoff}_{\text{Short Put}} & = -\max(K - S_T, 0). \end{array} \right.
 \end{aligned}$$

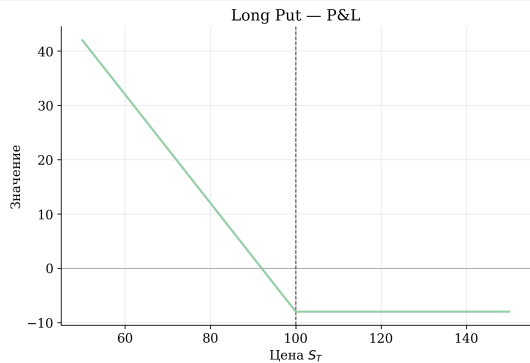
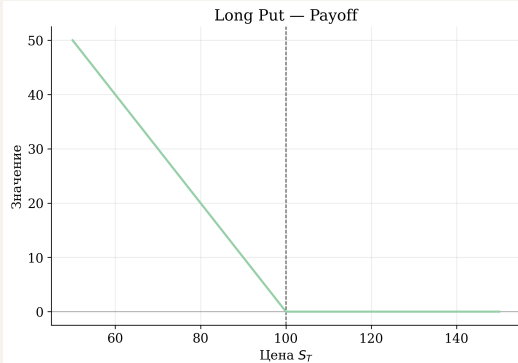
## Payoff - Long Call $\max(S_T - K, 0)$



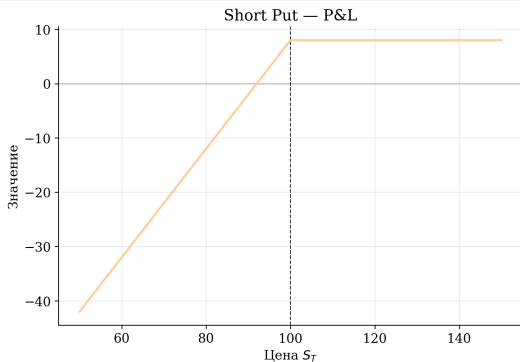
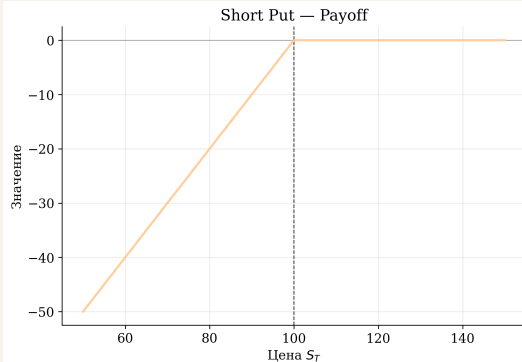
# Payoff - Short Call — $\max(S_T - K, 0)$



# Payoff - Long Put $\max(K - S_T, 0)$



# Payoff - Short Put — $\max(K - S_T, 0)$



:

$$S_T \in \{80, 100, 130\}.$$

$$K = 100.$$

:

$$S_T$$

:

Long Call (LC), Short Call (SC), Long Put (LP), Short Put (SP).

# CALL-OPTION

1. Long Call :

$$\text{Payoff}_{LC}(S_T) = \max(S_T - K, 0).$$

$$S_T = 80 : \quad \max(80 - 100, 0) = 0,$$

$$S_T = 100 : \quad \max(100 - 100, 0) = 0,$$

$$S_T = 130 : \quad \max(130 - 100, 0) = 30.$$

2. Short Call - call :

$$\text{Payoff}_{SC}(S_T) = -\max(S_T - K, 0).$$

$$S_T = 80 : \quad 0,$$

$$S_T = 100 : \quad 0,$$

$$S_T = 130 : \quad -30.$$

## 3. Long Put - :

$$\text{Payoff}_{LP}(S_T) = \max(K - S_T, 0).$$

$$S_T = 80 : \quad \max(100 - 80, 0) = 20,$$

$$S_T = 100 : \quad \max(100 - 100, 0) = 0,$$

$$S_T = 130 : \quad \max(100 - 130, 0) = 0.$$

## 4. Short Put - put « »:

$$\text{Payoff}_{SP}(S_T) = -\max(K - S_T, 0).$$

$$S_T = 80 : \quad -20,$$

$$S_T = 100 : \quad 0,$$

$$S_T = 130 : \quad 0.$$



$S_T$	$LC$	$SC$	$LP$	$SP$
80	0	0	20	-20
100	0	0	0	0
130	30	-30	0	0

- $(S_T = 80)$  put: 100 20.
- $(S_T = 100)$  — payoff .
- $(S_T = 130)$  call: 100 130 30.



- . ?

:

( )

— ,  $(a + b)^n$ .

■  $n$  —  $a$   $b$ ;

■  $a^k b^{n-k}$ ;

■ ,  $k$   $a$   $n$  ;

■  $\binom{n}{k}$ .

:  $(a + b)^n$  —  $a$   $b$   $n$  .

:

:

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k.$$

:

$$\binom{n}{k} = \frac{n!}{k! (n - k)!}.$$

:

— . :

$$\Omega = \{ \quad , \quad \}.$$

. , .

- 10 5 ;
  - 1000 — 500;
  - .
- ,

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— « ». , :

$$\xi : \Omega \rightarrow \mathbb{R}.$$

$\omega \in \Omega$   $\xi(\omega)$ . :

■ 5 : — H/T. :

$$X(\omega) = \omega.$$

■  $S_T$  — :

$$S_T(\omega) = T.$$

— , .

$\xi$  is a random variable with values in  $\mathbb{R}$ ,  
 $\xi$  is a random variable with values in  $\mathbb{R}$ .

$$P_\xi(A) = P(\xi^{-1}(A)).$$

- $\xi$  is a random variable with values in  $\mathbb{R}$ ,  $\omega$  is a random variable with values in  $\mathbb{R}$ .
- $\xi$  is a random variable with values in  $\mathbb{R}$ .



:

. , 5 3 ? :

- :  $n = 5$ ;
- « »:  $k = 3$ ;
- :  $p = \frac{1}{2}$ ;
- :  $q = 1 - p = \frac{1}{2}$ .

:

$$P(X = k) = \binom{n}{k} p^k q^{n-k}.$$

:

$$P(X = 3) = \binom{5}{3} \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^2.$$

:

:

$$\binom{5}{3} = \frac{5!}{3!2!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(3 \cdot 2 \cdot 1)(2 \cdot 1)} = 10.$$

:

$$P(X = 3) = 10 \cdot \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^2 = 10 \cdot \frac{1}{8} \cdot \frac{1}{4} = \frac{10}{32} = \frac{5}{16}.$$

$$\therefore \frac{5}{16}.$$

$S_0$  , . :

$S_T$  ,  $S_T$

— :

■  $\Leftrightarrow$  ;

■  $\Leftrightarrow$  .

100 .

:

■ — , 102;

■ — , 98.

, . :

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:

$$S_0 \rightarrow \begin{cases} uS_0 & (\quad) \\ dS_0 & (\quad) \end{cases}$$

:

$$B_1 = B_0(1 + r\Delta t)$$

, :

■  $S \leq B,$

■ ,

■ .

$\vdots$   
 $u \leq 1 + r\Delta t$

$\rightarrow$

$\vdots$   
 $d \geq 1 + r\Delta t$

$\rightarrow$

$r$

,  $\vdots$

$$d < 1 + r\Delta t < u.$$

$\vdots$

■

,

■

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:

$$\mathbb{E}^{\mathbb{Q}}[S_1] = S_0(1 + r\Delta t).$$

$$q = \frac{(1 + r\Delta t) - d}{u - d}.$$

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$$q = \frac{(1 + r\Delta t) - d}{u - d}$$

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$$C_u = \max(uS_0 - K, 0), \quad C_d = \max(dS_0 - K, 0).$$

$$C_0 = \frac{1}{1 + r\Delta t} (qC_u + (1 - q)C_d) .$$

$$S_0 \rightarrow \begin{cases} uS_0 & (C_u) \\ dS_0 & (C_d) \end{cases}$$
$$C_0 = \frac{qC_u + (1 - q)C_d}{1 + r\Delta t}$$

$$\Pi = \Delta S_0 - B_0,$$

$$\Pi_u = \Delta u S_0 - B_0(1 + r\Delta t), \quad \Pi_d = \Delta d S_0 - B_0(1 + r\Delta t).$$

$$\Pi_u = C_u, \quad \Pi_d = C_d.$$

$$(\Delta, B_0).$$

1:

:

$$\Delta u S_0 - B_0(1 + r\Delta t) = C_u,$$

$$\Delta d S_0 - B_0(1 + r\Delta t) = C_d.$$

—

$$\Delta B_0.$$

2:

:

$$(\Delta u S_0 - B_0(1 + r\Delta t)) - (\Delta d S_0 - B_0(1 + r\Delta t)) = C_u - C_d.$$

$$B_0 \quad :$$

$$\Delta S_0(u - d) = C_u - C_d.$$

:

$$\Delta = \frac{C_u - C_d}{(u - d)S_0}.$$



3:  $B_0$

$$\Delta = \frac{C_u - C_d}{(u - d)S_0}$$

, :

$$\Delta u S_0 - B_0(1 + r\Delta t) = C_u.$$

:

$$B_0(1 + r\Delta t) = \Delta u S_0 - C_u.$$

:

$$B_0 = \frac{\Delta u S_0 - C_u}{1 + r\Delta t}.$$

4:

(  $t = 0$ ):

$$\Pi_0 = \Delta S_0 - B_0.$$

, :

$$C_0 = \Pi_0.$$

:

$$C_0 = \Delta S_0 - B_0.$$

5:

:

$$\Delta S_0 = \frac{C_u - C_d}{u - d},$$

$$B_0 = \frac{\Delta u S_0 - C_u}{1 + r\Delta t}.$$

:

$$C_0 = \frac{C_u - C_d}{u - d} - \frac{\Delta u S_0 - C_u}{1 + r\Delta t}.$$

6:

$$\Delta S_0 = \frac{C_u - C_d}{u - d} :$$

$$\Delta u S_0 = u \cdot \frac{C_u - C_d}{u - d}.$$

:

$$C_0 = \frac{C_u - C_d}{u - d} - \frac{u(C_u - C_d)/(u - d) - C_u}{1 + r\Delta t}.$$
$$(u - d)(1 + r\Delta t).$$

:

$$C_0 = \frac{1}{1 + r\Delta t} \left[ \frac{(1 + r\Delta t) - d}{u - d} C_u + \frac{u - (1 + r\Delta t)}{u - d} C_d \right].$$

7: -

:

$$q = \frac{(1 + r\Delta t) - d}{u - d}, \quad 1 - q = \frac{u - (1 + r\Delta t)}{u - d}.$$

:

$$C_0 = \frac{qC_u + (1 - q)C_d}{1 + r\Delta t}$$

call .

$$C_0 = \frac{qC_u + (1 - q)C_d}{1 + r\Delta t}$$

:

- $q$  — ;
- , ;
- $q$  — ;
- = payoff.

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