— -01-24

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0. ,

?

( ) — $\{B(t)\}_{t\geq 0}$,

$$\frac{dB(t)}{dt} = r(t)\,B(t), \quad r(t) \text{ —} \qquad .$$

$$B(0) = 1.$$

$$B(t) = \exp\left(\int_0^t r(s)\,ds\right).$$

$r(t) \equiv r$:

$$B(t) = e^{rt}.$$

( ) $B(t)$, ,

r. :

- -
- -
- **Numeraire —**

$[0, T]$ Δt:

$$0,\ \Delta t,\ 2\Delta t,\ \dots,\ n\Delta t = T, \qquad \Delta t = \frac{T}{n}.$$

,

$$B(t + \Delta t) = B(t)\big(1 + r\Delta t\big).$$

n

$$B(T) = B(0)\,(1 + r\Delta t)^n.$$

$\Delta t \to 0$ ($n \to \infty$),

$$\lim_{\Delta t \to 0} (1 + r\Delta t)^{T/\Delta t} = e^{rT}.$$

,

$$\frac{dB(t)}{dt} = \lim_{\Delta t \to 0} \frac{B(t + \Delta t) - B(t)}{\Delta t} = rB(t).$$

$B(0) = 100\,000$, :

$r = 16,5\% = 0.165.$

$(T = 1)$.

1. 1 ( ):

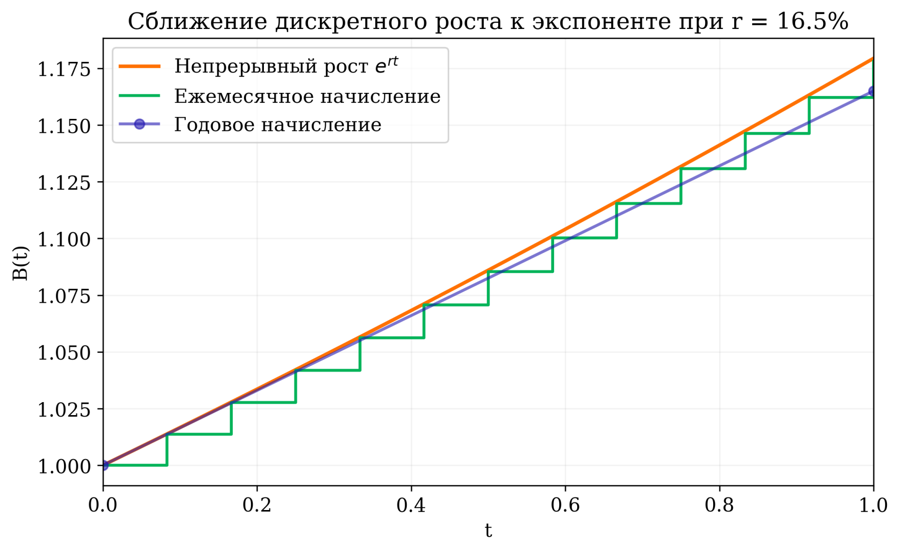
$$B_1(1) = 100\,000 \left(1 + \frac{0.16}{1}\right)^1 = 100\,000 \cdot 1.16 = \mathbf{116\,000} \quad .$$

2. $(n = 12)$:

$$B_{12}(1) = 100\,000 \left(1 + \frac{0.16}{12}\right)^{12} \approx 100\,000 \cdot 1.17180 \approx \mathbf{117\,180} \quad .$$

3. $(n \to \infty)$:

$$B_{\mathsf{cont}}(1) = 100\,000\, e^{0.16} \approx 100\,000 \cdot 1.17351 \approx \mathbf{117\,351} \quad .$$


Сближение дискретного роста к экспоненте при r = 16.5%
Непрерывный рост e^{rt}
Ежемесячное начисление
Годовое начисление
B(t)
t
1.000
1.025
1.050
1.075
1.100
1.125
1.150
1.175
0.0
0.2
0.4
0.6
0.8
1.0

$$\overbrace{116\,000}^{B_1(1)} < \overbrace{117\,180}^{B_{12}(1)} < \overbrace{117\,351}^{B_{\mathsf{cont}}(1)}$$

- — .
- , .
- : e^r.

— .

1.

, , 100 .
?

, , 100 .
?

: 99, 101, 98.5, 105 — , .

, , 100 .
?

: 99, 101, 98.5, 105 — , .
— , — , .

- , .
- $\rightarrow$.
- , , .
- , :
  - ;
  - ;
  - .
- (. . -) — .

2. $$(\Omega, \mathcal{F}, \mathbb{P})$$

— $(\Omega, \mathcal{F}, \mathbb{P})$,

1. Ω — ( );
2. $\mathcal{F}$ — σ- , Ω, ;
3. $\mathbb{P}$ — , $A \in \mathcal{F}$ $\mathbb{P}(A)$ $[0, 1]$

   .

( )

:

1. ( ):

$$\Omega = \{\text{up}, \text{down}, \text{flat}\}.$$

2. — :

$$\mathcal{F} = \sigma(\{\text{up}\}, \{\text{down}\}, \{\text{flat}\}).$$

3. :

$$P(\text{up}) = 0.3, \quad P(\text{down}) = 0.4, \quad P(\text{flat}) = 0.3.$$

$(\Omega, \mathcal{F}, P)$.

3.

- :
  - ,
  - ,
  - .
- P&L.
- .
- — P&L .

- — .
- , — **P&L**.
- .
- : .

4.

— , , ,

.

- ,
- .

:

- , .
- , , .

, .

- (K) — ,

- **(K)** — ,
- **(T)** — , ( ). T
  .

- **(K)** — ,
- **(T)** — , ( ). T
.

( )

— K T,
S_T.

Call- — K.

- , $S_T > K$;
- , $S_T \le K$.

Put- — K.

- , $S_T < K$;
- , $S_T \ge K$.

- **Long Call**: . .
- **Short Call**: . .
- **Long Put**: . .
- **Short Put**: . .

(!!!) : long short — , .

- **Long Call**: . .
- **Short Call**: . .
- **Long Put**: . .
- **Short Put**: . .

(!!!) : long short — ,
.

- (long) .
- (short) .

$$\left(\qquad \cdots \qquad \right)$$

Call

call- K. S_T:

$$\begin{cases} S_T \leq K, & \Rightarrow \quad 0, \\ S_T > K, & K \quad S_T \Rightarrow S_T - K. \end{cases}$$

,

$$\text{Payoff}_{\text{call}} = \max(S_T - K, 0).$$

Put

put K:

$$\begin{cases} S_T \geq K, & \Rightarrow \quad 0, \\ S_T < K, \quad K \quad , & S_T \Rightarrow K - S_T. \end{cases}$$

:

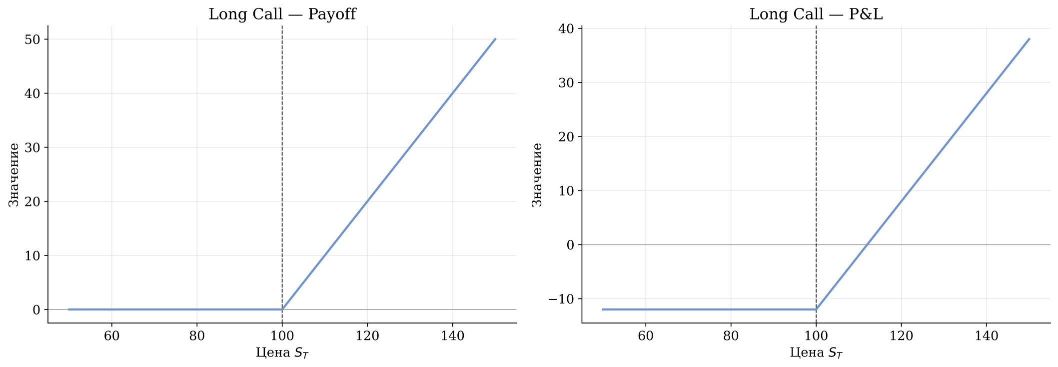
$$\text{Payoff}_{\text{put}} = \max(K - S_T, 0).$$

:

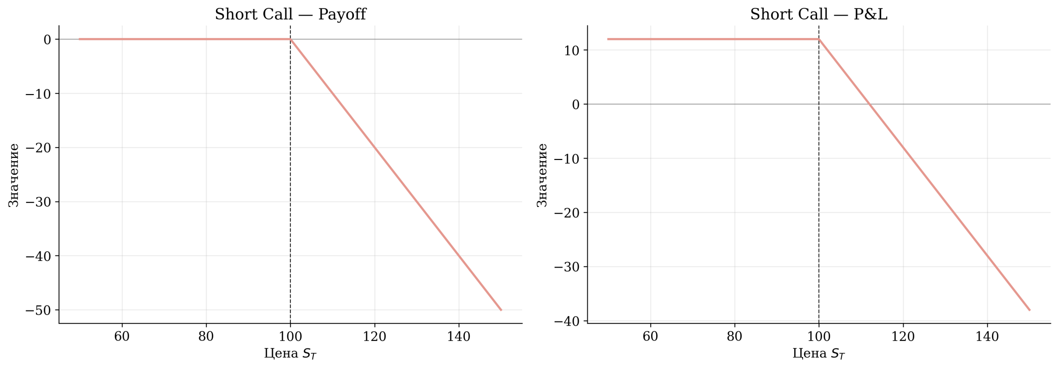
$$\begin{cases} \text{Payoff}_{\text{Long Call}} &= \max(S_T - K, 0), \\ \text{Payoff}_{\text{Short Call}} &= -\max(S_T - K, 0), \\ \text{Payoff}_{\text{Long Put}} &= \max(K - S_T, 0), \\ \text{Payoff}_{\text{Short Put}} &= -\max(K - S_T, 0). \end{cases}$$

, .

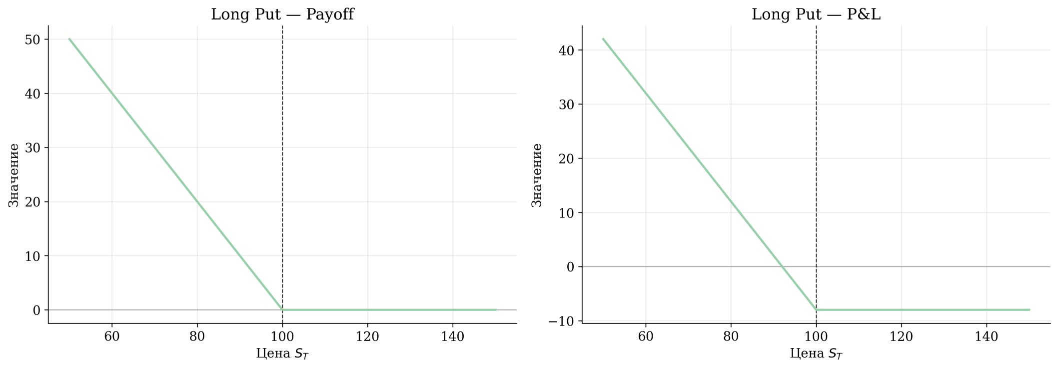
Payoff - Long Call $\max(S_T - K, 0)$



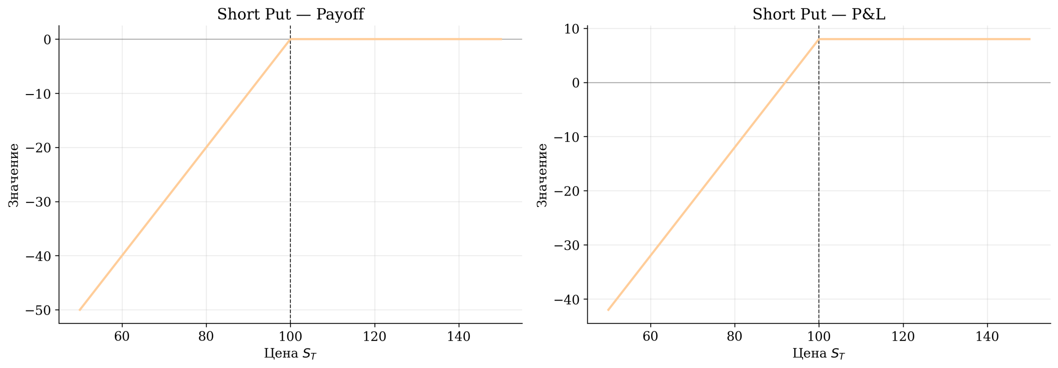
Payoff - Short Call $-\max(S_T - K, 0)$



Payoff - Long Put $\max(K - S_T, 0)$



Payoff - Short Put $-\max(K - S_T, 0)$



:

$$S_T \in \{80,\ 100,\ 130\}.$$

$K = 100$. :

S_T :

Long Call (LC), Short Call (SC), Long Put (LP), Short Put (SP).

CALL-OPTION

1. Long Call :

$$\text{Payoff}_{LC}(S_T) = \max(S_T - K, 0).$$

$$S_T = 80 : \quad \max(80 - 100, 0) = 0,$$

$$S_T = 100 : \quad \max(100 - 100, 0) = 0,$$

$$S_T = 130 : \quad \max(130 - 100, 0) = 30.$$

2. Short Call - call :

$$\text{Payoff}_{SC}(S_T) = -\max(S_T - K, 0).$$

$$S_T = 80 : \quad 0,$$

$$S_T = 100 : \quad 0,$$

$$S_T = 130 : \quad -30.$$

PUT-OPTION

3. Long Put - :

$$\text{Payoff}_{LP}(S_T) = \max(K - S_T, 0).$$

$$S_T = 80: \quad \max(100 - 80, 0) = 20,$$
$$S_T = 100: \quad \max(100 - 100, 0) = 0,$$
$$S_T = 130: \quad \max(100 - 130, 0) = 0.$$

4. Short Put - put « »:

$$\text{Payoff}_{SP}(S_T) = -\max(K - S_T, 0).$$

$$S_T = 80: \quad -20,$$
$$S_T = 100: \quad 0,$$
$$S_T = 130: \quad 0.$$

S_T	LC	SC	LP	SP
80	0	0	20	-20
100	0	0	0	0
130	30	-30	0	0

- $(S_T = 80)$ put: 100 20.
- $(S_T = 100)$ — payoff .
- $(S_T = 130)$ call: 100 130 30.

- . ?

: ( )

— , $(a+b)^n$.

- n — a b;
- $a^k b^{n-k}$;
- , k a n ;
- $\binom{n}{k}$.

: $(a+b)^n$ — a b n .

:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

:

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}.$$

— . :

$$\Omega = \{ \quad , \quad \}.$$

. , .

- 10 5 ;
- 1000 — 500;
- .

, .

( )

— « ». , :

$$\xi : \Omega \to \mathbb{R}.$$

$\omega \in \Omega$ $\xi(\omega)$. :

- 5 : — H/T. :

$$X(\omega) = \qquad \omega.$$

- S_T — :

$$S_T(\omega) = \qquad T.$$

— , .

: — , .

: — $\mathbb{R}$, ξ:

$$P_\xi(A) = P\big(\xi^{-1}(A)\big).$$

- , $\xi \in A$, ω, .
- .

. , 5 3 ? :

- : $n = 5$;
- « »: $k = 3$;
- : $p = \frac{1}{2}$;
- : $q = 1 - p = \frac{1}{2}$.

:

$$P(X = k) = \binom{n}{k} p^k q^{n-k}.$$

:

$$P(X = 3) = \binom{5}{3} \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^2.$$

:

:

$$\binom{5}{3} = \frac{5!}{3!\,2!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(3 \cdot 2 \cdot 1)(2 \cdot 1)} = 10.$$

:

$$P(X = 3) = 10 \cdot \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^2 = 10 \cdot \frac{1}{8} \cdot \frac{1}{4} = \frac{10}{32} = \frac{5}{16}.$$

: $\frac{5}{16}$.

, .

S_0. :

$$S_T \quad , \quad S_T$$

.

— :

- $\iff$;
- $\iff$.

100 .

:

- — , 102;
- — , 98.

, . : , .

:

- ,
- ,
- ,
- — — .

?

:

- (, ,);
- .

, .

, .

— , . :

$$S_0 \to \begin{cases} uS_0 & (\quad) \\ dS_0 & (\qquad) \end{cases}$$

:

$$B_1 = B_0(1 + r\Delta t)$$

, :

- S B,
- ,
- .

.

:

$$u \leq 1 + r\Delta t$$

$\rightarrow$.

:

$$d \geq 1 + r\Delta t$$

$\rightarrow$ r .

.

, :

$$d < 1 + r\Delta t < u.$$

:

- ,
- .

.

, :

$$\mathbb{E}^{\mathbb{Q}}[S_1] = S_0(1 + r\Delta t).$$

$$q = \frac{(1 + r\Delta t) - d}{u - d}.$$

, .

- :

$$q = \frac{(1 + r\Delta t) - d}{u - d}$$

- ;
- ;
- ;
- , — .

$$C_u = \max(uS_0 - K, 0), \qquad C_d = \max(dS_0 - K, 0).$$

$$C_0 = \frac{1}{1 + r\Delta t}\left(qC_u + (1-q)C_d\right).$$

$$S_0 \to \begin{cases} uS_0 & (C_u) \\ dS_0 & (C_d) \end{cases}$$

$$C_0 = \frac{qC_u + (1-q)C_d}{1 + r\Delta t}$$

:

$$\Pi = \Delta S_0 - B_0,$$

, :

$$\Pi_u = \Delta\, u S_0 - B_0(1 + r\Delta t), \qquad \Pi_d = \Delta\, d S_0 - B_0(1 + r\Delta t).$$

:

$$\Pi_u = C_u, \qquad \Pi_d = C_d.$$

:

$$(\Delta, B_0).$$

1:

:

$$\Delta u S_0 - B_0(1 + r\Delta t) = C_u,$$

$$\Delta d S_0 - B_0(1 + r\Delta t) = C_d.$$

— Δ B_0.

2:

:

$$(\Delta u S_0 - B_0(1 + r\Delta t)) - (\Delta d S_0 - B_0(1 + r\Delta t)) = C_u - C_d.$$

B_0 :

$$\Delta S_0(u - d) = C_u - C_d.$$

:

$$\Delta = \frac{C_u - C_d}{(u - d)S_0}.$$

3: B_0

$$\Delta = \frac{C_u - C_d}{(u-d)S_0}$$

, :

$$\Delta u S_0 - B_0(1 + r\Delta t) = C_u.$$

:

$$B_0(1 + r\Delta t) = \Delta u S_0 - C_u.$$

:

$$B_0 = \frac{\Delta u S_0 - C_u}{1 + r\Delta t}.$$

4:

($t = 0$):

$$\Pi_0 = \Delta S_0 - B_0.$$

, :

$$C_0 = \Pi_0.$$

:

$$C_0 = \Delta S_0 - B_0.$$

5:

:

$$\Delta S_0 = \frac{C_u - C_d}{u - d},$$

$$B_0 = \frac{\Delta u S_0 - C_u}{1 + r\Delta t}.$$

:

$$C_0 = \frac{C_u - C_d}{u - d} - \frac{\Delta u S_0 - C_u}{1 + r\Delta t}.$$

6:

$\Delta S_0 = \frac{C_u - C_d}{u-d}$:

$$\Delta u S_0 = u \cdot \frac{C_u - C_d}{u - d}.$$

:

$$C_0 = \frac{C_u - C_d}{u - d} - \frac{u(C_u - C_d)/(u - d) - C_u}{1 + r\Delta t}.$$

$$(u - d)(1 + r\Delta t).$$

:

$$C_0 = \frac{1}{1 + r\Delta t}\left[\frac{(1 + r\Delta t) - d}{u - d}C_u + \frac{u - (1 + r\Delta t)}{u - d}C_d\right].$$

:

$$q = \frac{(1 + r\Delta t) - d}{u - d}, \qquad 1 - q = \frac{u - (1 + r\Delta t)}{u - d}.$$

:

$$\boxed{C_0 = \frac{qC_u + (1 - q)C_d}{1 + r\Delta t}}$$

call .

$$C_0 = \frac{qC_u + (1-q)C_d}{1 + r\Delta t}$$

:

- q — ;
- , ;
- q — ;
- $=$ payoff.

.