FIT3139 Assignment 1 Q 2

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May 7, 2020

Part A:

Question: Steady-state solution
Answer: In steady-state we know that $x_{x+1} = x_t = x^*$. Thus we can show that:

$$\begin{aligned} x^* &= mx^* + c \\ (1-m)x^* &= c \\ \therefore x^* &= \frac{c}{1-m} \end{aligned}$$

It is worth noting that when $m = 1$, there exists no steady-states or every point is a steady-state. Therefore a full answer for the steady state solution would be:

If $m \neq 1$, a steady-state equilibrium exists for x such that:

$$x^* = \frac{c}{1-m}$$

If $m = 0$ and $c = 0$, steady-state equilibria exist at every point in the domain.
If $m = 0$ and $c \neq 0$, $x^* = x^* + c$, so no steady-state equilibria exist.

Part B:

Question: Explicit solution
Answer: Expansion of the function to x_0 gives:

$$\begin{aligned} x_t &= mx_{t-1} + c \\ &= m(mx_{t-2} + c) + c = m^2 x_{t-2} + mc + c \\ &= m^2(mx_{t-3} + c) + mc + c = m^3 x_{t-3} + m^2 c + mc + c \\ &... \\ \therefore x_t &= m^t x_0 + \sum_{i=1}^{t} m^{i-1} \cdot c \end{aligned}$$

To put this formally, we get an explicit solution of:

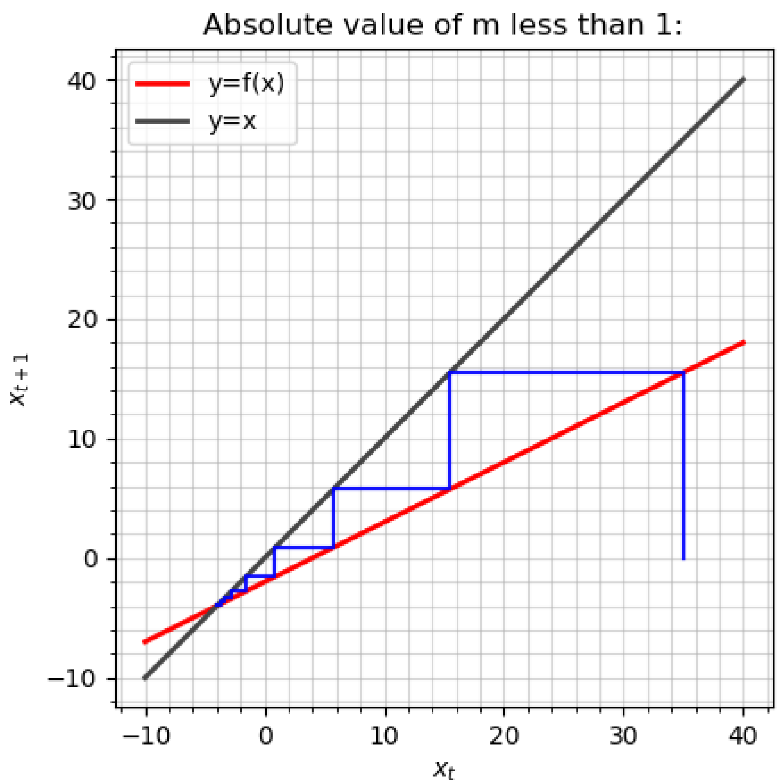
$$x_t = \begin{cases} x_0 & \text{if } t = 0 \\ m^t x_0 + \sum_{i=1}^{t} m^{i-1} \cdot c & \text{else} \end{cases}$$

Part C:

Question: Cobwebs and behaviours
Answer:

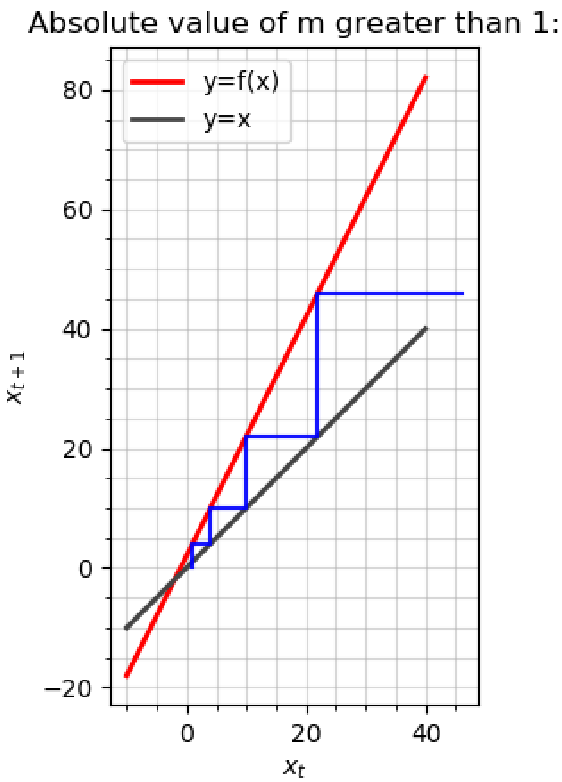
Case 1:

Note: Cobwebbing is in Blue



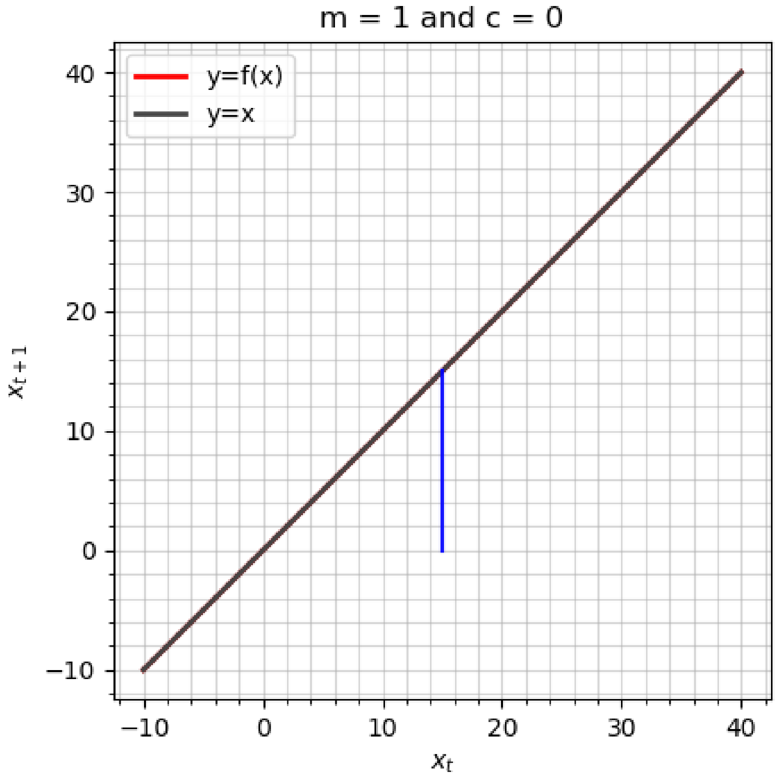
$x_0 = 35$
For this case, every point in the domain converges to the equilibrium. This is because the derivative is $|f'(x)| = |m| < 1$ and according to information provided in the tutorials, this makes the equilibrium $x^* = \dfrac{c}{1-m}$ locally stable. Hence, this case is convergent across its entire domain to x^*, the steady-state.

Case 2:



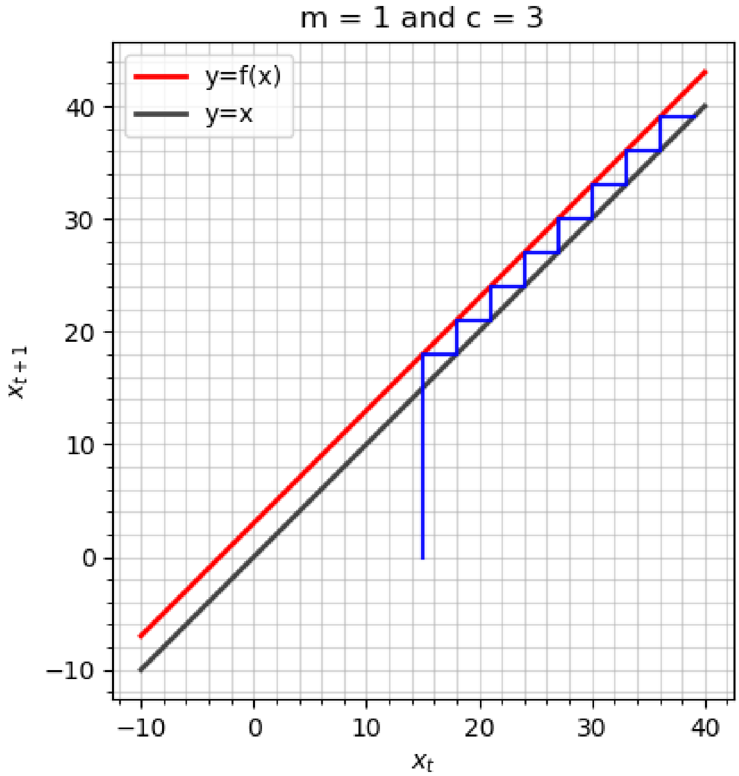
$x_0 = 2$
For this case, with the exception of the input $x_0 = x^*$ where x^* is the steady-state solution, all inputs diverge. Unlike Case 1, $|f'(x)| = |m| > 1$ so x^* is not locally stable, so only the input $x_0 = x^* = \dfrac{c}{1-m}$ does not diverge.

Case 3:



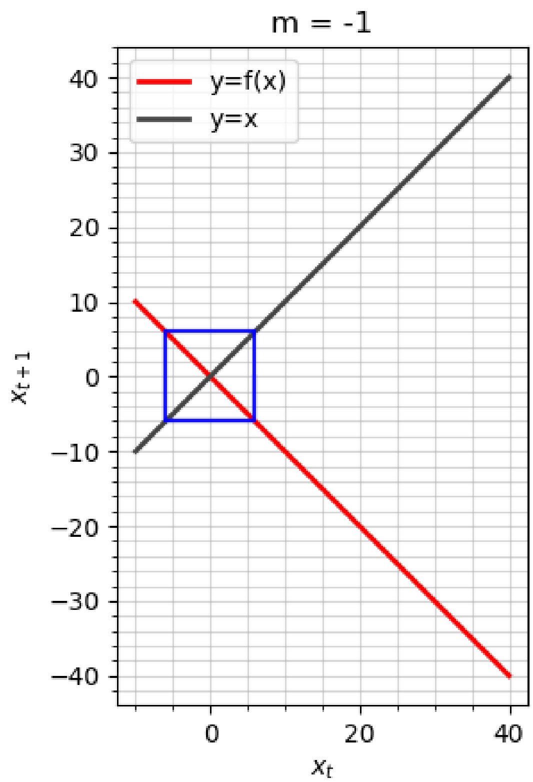
$x_0 = 15$
For this case, every input is an equilibrium, so the system is universally stable for all inputs.

Case 4:



$x_0 = 15$
This system has no equilibriums, and therefore all inputs x_0 will diverge to $+\infty$ if $c > 0$ or $-\infty$ if $c < 0$

Case 5:



$x_0 = 6$

This system is similar to Case 2 in that only the input $x_0 = x^* = \dfrac{c}{2}$ converges. All other inputs fail to converge as they oscillate between x_0 and $-x_0$.

Part D:

Question: Linearisation and stability conditions
Answer: If we take the Taylor Series expansion of $f(x)$ around x^* and truncate it at the 2nd derivative term we get:

$$f(x_t) \approx f(x^*) + f'(x^*) \cdot (x_t - x^*) + R_n$$

We can use this to find the linearisation of the system as such:

$$\begin{aligned} x_{t+1} &= f(x^*) + f'(x^*) \cdot (x_t - x^*) \\ &= f'(x^*) \cdot x_t + (f(x^*) - f'(x^*)x^*) \\ \therefore x_{t+1} &= m \cdot x_t + c \end{aligned}$$

where $m = f'(x^*)$ and $c = f(x^*) - f'(x^*)x^*$.
From Part C, we know of one scenario where x^* is locally stable in this form, that being Case 1. In this case we have that $|m| < 1$. Hence, for our given scenario, we require that:

$$f'(x^*) \in (-1, 1)$$

for local stability.