FIT3139 Assignment 2 Q 2

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Part A:

Transition Matrix for $n = 4$:

We have a set of six possible arrangements for our cyclic arrangement:

$$\begin{Bmatrix} 1,1,0,0 \\ 1,0,1,0 \\ 1,0,0,1 \\ 0,1,1,0 \\ 0,1,0,1 \\ 0,0,1,1 \end{Bmatrix}$$

Given that only rows/arrangements 2 and 5 are transient and can each move only to absorbing states (and any of those 4 at that) we get a transition matrix as such:

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1/6 & 1/3 & 1/6 & 1/6 & 0 & 1/6 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1/6 & 0 & 1/6 & 1/6 & 1/3 & 1/6 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Canonical Form Matrix for $n = 4$:

Through 'swapping' the 5th column with the first column and the 5th row with the 1st row, we get a canonical form matrix as such:

$$P = \begin{bmatrix} 1/3 & 0 & 1/6 & 1/6 & 1/6 & 1/6 \\ 0 & 1/3 & 1/6 & 1/6 & 1/6 & 1/6 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

This can be associated with having states in the order:

$$\begin{Bmatrix} 0,1,0,1 \\ 1,0,1,0 \\ 1,0,0,1 \\ 0,1,1,0 \\ 1,1,0,0 \\ 0,0,1,1 \end{Bmatrix}$$

Montecarlo simulations of absorption times for $n = 4,\ 5,\ ...,\ 10$:

Below is a graph of the average simulated absorption times from 10000 random trials of Schelling models for the appropriate n values via Montecarlo Simulation:

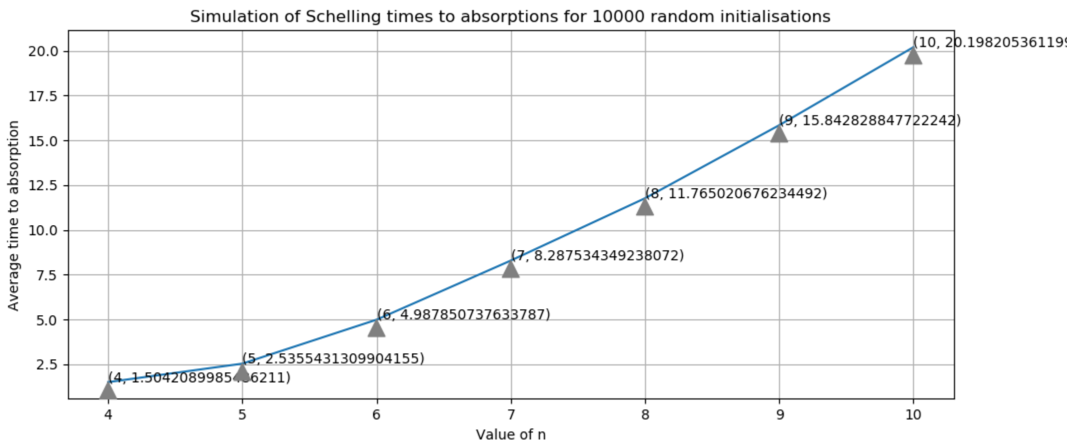

Fig 2.a.1: Montecarlo Simulated Absorption Times

Numerical Absorption Times for $n = 4$ and $n = 5$:

Case $n = 4$:

To calculate the average time to absorption, we need to find the t vector. We can find this via our canonical form matrix above, through the equations

$$t = N \cdot O$$

where O is a vector of ones, and

$$N = (1 - Q)^{-1}$$

Recalling that:

$$Q = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/3 \end{bmatrix}$$

We can find that:

$$N = \begin{bmatrix} 3/2 & 0 \\ 0 & 3/2 \end{bmatrix}$$

Which gives a t vector for the transient states of:

$$t = \begin{bmatrix} 3/2 \\ 3/2 \end{bmatrix}$$

As all the other states are absorbing and as such have $t = 0$ to absorption, we get an average absorption time of:

$$t_{ave} = \frac{\frac{3}{2} + \frac{3}{2}}{2} = \frac{3}{2} = 1.5$$

Which agrees with my simulated solution of $\tilde{t}_{ave} = 1.5042$

Case $n = 5$:

In the case of $n = 5$, there ten possible cyclical arrangements for our problem, which may be listed below:

$$\begin{Bmatrix} 0,1,0,1,1 \\ 0,1,1,0,1 \\ 1,0,1,1,0 \\ 1,1,0,1,0 \\ 1,0,1,0,1 \\ 1,1,1,0,0 \\ 0,1,1,1,0 \\ 0,0,1,1,1 \\ 1,0,0,1,1 \\ 1,1,0,0,1 \end{Bmatrix}$$

Of these, the final 5 are absorbing where all agents are happy, whereas the first five have only two happy agents in each. Each of these transient can access four of the five absorbing states in a single move with a probability 1/10, and otherwise will not move as to move from transient state to transient state violates the question. Hence, we end up with a Q matrix as such:

$$Q = \begin{bmatrix} 3/5 & 0 & 0 & 0 & 0 \\ 0 & 3/5 & 0 & 0 & 0 \\ 0 & 0 & 3/5 & 0 & 0 \\ 0 & 0 & 0 & 3/5 & 0 \\ 0 & 0 & 0 & 0 & 3/5 \end{bmatrix}$$

Following the same process described above, we get:

$$N = \begin{bmatrix} 5/2 & 0 & 0 & 0 & 0 \\ 0 & 5/2 & 0 & 0 & 0 \\ 0 & 0 & 5/2 & 0 & 0 \\ 0 & 0 & 0 & 5/2 & 0 \\ 0 & 0 & 0 & 0 & 5/2 \end{bmatrix}$$

Hence:

$$t = \begin{bmatrix} 5/2 \\ 5/2 \\ 5/2 \\ 5/2 \\ 5/2 \end{bmatrix}$$

Resulting in:

$$t_{ave} = \frac{5 \cdot \frac{5}{2}}{5} = \frac{25}{10} = 2.5$$

Which agrees with my simulated solution of $\tilde{t}_{ave} = 2.53554$

Part B:

Mistake Enabled $n = 4$ Stationary distribution:

Theoretical:

To find our probability matrix for a situation with mistakes, we must first find our mistake probability matrix. Maintaining the order we found above, being:

$$\begin{Bmatrix} 0,1,0,1 \\ 1,0,1,0 \\ 1,0,0,1 \\ 0,1,1,0 \\ 1,1,0,0 \\ 0,0,1,1 \end{Bmatrix}$$

we can quickly find a mistake matrix:

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1/6 & 1/6 & 1/3 & 0 & 1/6 & 1/6 \\ 1/6 & 1/6 & 0 & 1/3 & 1/6 & 1/6 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/3 & 0 \\ 1/6 & 1/6 & 1/6 & 1/6 & 0 & 1/3 \end{bmatrix}$$

This has the interesting property of causing the transitive and intransitive states to essentially flip in this format. Anyway. We can find the main probability matrix for our mistake chain via the equation $P' = (1-\epsilon) \cdot P + \epsilon \cdot M$, getting a Markov chain governed by:

$$P' = \begin{bmatrix} \epsilon + 1/3(1-\epsilon) & 0 & 1/6(1-\epsilon) & 1/6(1-\epsilon) & 1/6(1-\epsilon) & 1/6(1-\epsilon) \\ 0 & \epsilon + 1/3(1-\epsilon) & 1/6(1-\epsilon) & 1/6(1-\epsilon) & 1/6(1-\epsilon) & 1/6(1-\epsilon) \\ \epsilon/6 & \epsilon/6 & \epsilon/3 + (1-\epsilon) & 0 & \epsilon/6 & \epsilon/6 \\ \epsilon/6 & \epsilon/6 & 0 & \epsilon/3 + (1-\epsilon) & \epsilon/6 & \epsilon/6 \\ \epsilon/6 & \epsilon/6 & \epsilon/6 & \epsilon/6 & \epsilon/3 + (1-\epsilon) & 0 \\ \epsilon/6 & \epsilon/6 & \epsilon/6 & \epsilon/6 & 0 & \epsilon/3 + (1-\epsilon) \end{bmatrix}$$

To find that stationary distribution of this Markov chain, we can find its eigenvalues and from that the eigenvector associated with the eigenvalue that equals 1. From here on, I have solved with a value of $\epsilon = 0.01$, or a 1% chance of a mistake. The eigenvalues for this matrix are:

$$eigenvalues = \{1,\ 0.34,\ 0.33666667,\ 0.99333333,\ 0.99,\ 0.99333333\}$$

Clearly, we have an appropriate eigenvalue. Its eigenvector is:

$$eigenvector = [0.00251256,\ 0.00251256,\ 0.24874372,\ 0.24874372,\ 0.24874372,\ 0.24874372]$$

This is the stationary distribution. It is also worth noting we can find the stationary distribution given above by putting P' to a high power and taking any row. This yields the same result.

Montecarlo Simulated:

The results for the Montecarlo simulation are displayed below:

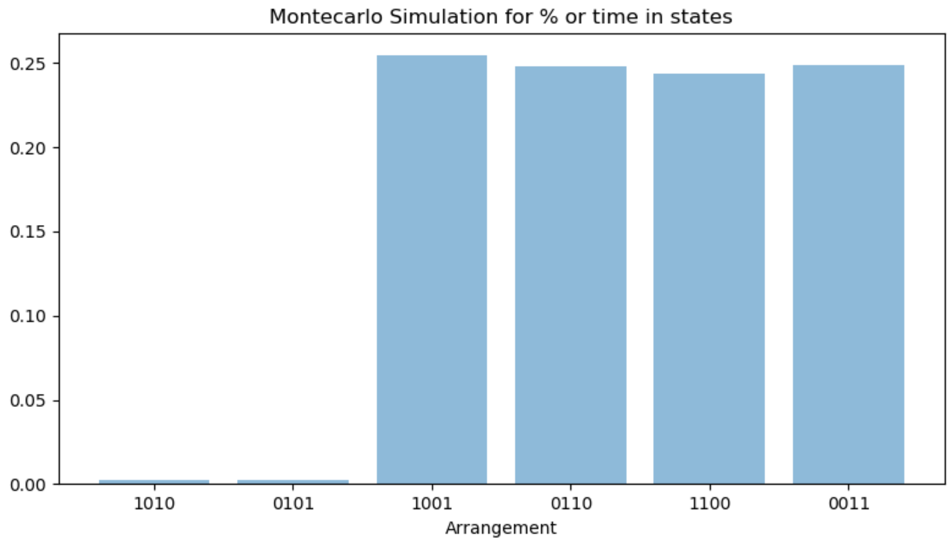

Fig 2.b.1: Montecarlo Simulated stationary distribution for $50,000,000$ swaps

As a vector, this simulated distribution is:

$$[0.0025216, 0.0025426, 0.254659, 0.2478816, 0.2436742, 0.248721]$$

This is evidently within any reasonable bounds of uncertainty for what we found theoretically.

Based on this, we can conclude that our model's stationary distribution heavily favours absorbing states, which take up about 99% of the sample distribution. It is worth noting that the mistakes distribution we found from theory would also be the distribution for perfect segregation of the agents, leading to the scenario where our ergodic chain is between a segregated and unsegregated model dictated by the value of ϵ.

Non-cyclic $n = 4$:

Theoretical:

In this case, when we examine our list of possible arrangements for size $n = 4$, which are:

$$\begin{Bmatrix} 0,1,0,1 \\ 1,0,1,0 \\ 1,0,0,1 \\ 0,1,1,0 \\ 1,1,0,0 \\ 0,0,1,1 \end{Bmatrix}$$

we find that only two cases are absorbing (the bottom two) and four transient cases. If we examine these transient cases, we further find two cases have no happy agents, and two have one happy agent each. From this we may create a 'correct' move probability matrix of:

$$C = \begin{bmatrix} 1/3 & 0 & 1/6 & 1/6 & 1/6 & 1/6 \\ 0 & 1/3 & 1/6 & 1/6 & 1/6 & 1/6 \\ 0 & 0 & 2/3 & 0 & 1/6 & 1/6 \\ 0 & 0 & 0 & 2/3 & 1/6 & 1/6 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

And, likewise, an incorrect probability matrix of:

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1/6 & 1/6 & 2/3 & 0 & 0 & 0 \\ 1/6 & 1/6 & 0 & 2/3 & 0 & 0 \\ 1/6 & 1/6 & 1/6 & 1/6 & 1/3 & 0 \\ 1/6 & 1/6 & 1/6 & 1/6 & 0 & 1/3 \end{bmatrix}$$

This, combined as we did in the previous section, gives:

$$P' = \begin{bmatrix} \epsilon + 1/3(1-\epsilon) & 0 & 1/6(1-\epsilon) & 1/6(1-\epsilon) & 1/6(1-\epsilon) & 1/6(1-\epsilon) \\ 0 & \epsilon + 1/3(1-\epsilon) & 1/6(1-\epsilon) & 1/6(1-\epsilon) & 1/6(1-\epsilon) & 1/6(1-\epsilon) \\ \epsilon/6 & \epsilon/6 & 2\epsilon/3 + 2(1-\epsilon)/3 & 0 & (1-\epsilon)/6 & (1-\epsilon)/6 \\ \epsilon/6 & \epsilon/6 & 0 & 2\epsilon/3 + 2(1-\epsilon)/3 & (1-\epsilon)/6 & (1-\epsilon)/6 \\ \epsilon/6 & \epsilon/6 & \epsilon/6 & \epsilon/6 & \epsilon/3 + (1-\epsilon) & 0 \\ \epsilon/6 & \epsilon/6 & \epsilon/6 & \epsilon/6 & 0 & \epsilon/3 + (1-\epsilon) \end{bmatrix}$$

To find that stationary distribution of this Markov chain, we can find its eigenvalues and from that the eigenvector associated with the eigenvalue that equals 1. From here on, I have solved with a value of $\epsilon = 0.01$, or a 1% chance of a mistake. The eigenvalues for this matrix are:

$$eigenvalues = \{1,\ 0.33666667,\ 0.34,\ 0.66333333,\ 0.66666667,\ 0.99333333\}$$

Clearly, we have an appropriate eigenvalue. Its eigenvector is:

$$eigenvector = [0.00251256,\ 0.00251256,\ 0.00738843,\ 0.00738843,\ 0.49009901,\ 0.4900990]$$

This is the stationary distribution. It is also worth noting we can find the stationary distribution given above by putting P' to a high power and taking any row. This yields the same result.

Montecarlo Simulated:

The results for the Montecarlo simulation are displayed below:

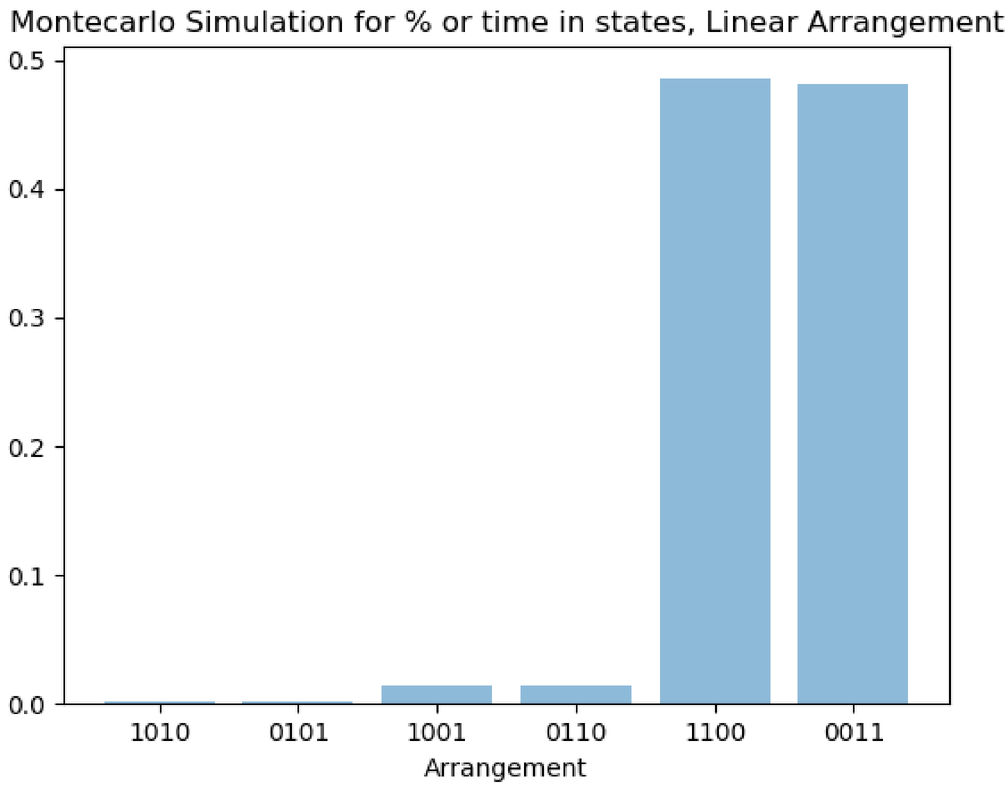

Fig 2.b.2: Montecarlo Simulated stationary distribution for $50,000,000$ swaps

As a vector, this simulated distribution is:

$$[0.002509, 0.0024644, 0.0071524, 0.0072226, 0.4926528, 0.4879988]$$

This is evidently within any reasonable bounds of uncertainty for what we found theoretically.

Based on this, we can conclude that our model's stationary distribution heavily favours absorbing states, though the absorbing states with two happy agents is significantly more represented than the other transient state. Nevertheless, the absorbing states still make up about 98% of the sample distribution.

Extensions of the Model:
Below are some potential extensions to the model:

- **Non-linear/cyclic arrangement** In the real world, people don't live on a circle or line. Therefore arranging a model where people live in at least a 2D world where neighbours exist in four possible directions would make the model more realistic. This can further be extended to have agents look at the density of other agents around them, making the locations agents choose continuous rather than discrete. This makes the model significantly more complex, but also more reticent of reality, which makes its conclusions more valuable.

- **Empty Spaces** It is possible for a house to be vacant in reality, so having more places than agents would lead to a more realistic change.

- **Variable 'happiness' criteria** People aren't just happy or unhappy. In reality, happiness is the product of many factors, so happiness should be a function of various characteristics of where a person lives, including the number or proportion of agents of the same category. Other factors could include the agent's age (and the ages of other agents near that agent), physical characteristics of the location (e.g.: weather, or other things that may make one slot more desirable for an agent over other. This would be hard to include in a Markov chain implementation (if possible at all)), or population density (relative percentage of empty locations).

- **Variable Tribalism** Not all people link happiness to being surrounded by people of the same group. The impact of tribalism should vary from person to person. This should be the same for the other factors that could contribute to happiness linked above as well, with - for example - some people favouring low population density (i.e.: country living) and others high (i.e.: city living). To extend this even further, one could make these agent preferences vary over time, as in reality, people's opinions change. By this stage we would no longer be able to model our scenario with markov chains however.

- **Birth/Death of agents** Movement of populations occurs over a protracted period of time, so deaths and births of agents would make it more realistic.

- **Family Ties** In reality, people like to live near family/friends, so agents should gain greater happiness from proximity to family members.

It is worth noting that these extensions become substantially more complex towards the end, and we can no longer model this situation as a simple Markov chain as easily, if at all. That being said, the first several ideas - in particular a 3D arrangement and having empty spaces - could be implemented using Markov Chains without undue effort, even if it would scale poorly.

In conclusion, while this model gives some insight into why segregation of communities can occur naturally, it is also highly simplistic. More factors need to be taken into account to achieve a holistic understanding of an issue as nuanced as segregation.