Invertible Time *Logprod* Function for Realized Volatility

An Empirical Application

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April 5, 2025

In time-series modelling, it can be convenient to use both positive and negative time indices. Suppose we consider time as a discrete quantity on the number line:

$$T = \{-t, -t+1, \ldots, -1, 0, 1, \ldots, t-1, t\} \tag{1}$$

Let r_t be the directly observed return of a price process p_t.

Recognize the identity:

$$r_{\delta t} = \frac{p_t}{p_{t-\delta}} - 1 \overset{\delta \to 0}{\approx} \log p_t - \log p_{t-\delta} \tag{2}$$

We can advance on the concept of realized volatility using a backward-looking (negative-index) time window:

$$\begin{align} RV_{\text{today}} &= \text{logprod}(r_{\{-M,-M+1,\ldots,-1,0\}}) \quad \forall \quad t \in T < 0 \tag{3} \\ &= \sum_{i=-M}^{0} r_i^2 \tag{4} \\ &= \prod_{i=-M}^{0} e^{r_i^2} \tag{5} \end{align}$$

We formally define an exponential realized volatility metric as:

$$\operatorname{logprod}(r_{[-M,0]}) = \prod_{i=-M}^{0} e^{r_i^2} = \exp\left(\sum_{i=-M}^{0} r_i^2\right) \tag{6}$$

Hence,

$$\log\Big(\operatorname{logprod}(r_{[-M,0]})\Big) = \sum_{i=-M}^{0} r_i^2 = RV_{\text{standard}} \tag{7}$$

This formulation shows that we can express standard realized volatility either additively (as a sum of squared returns) or multiplicatively (as a product of exponentials), with the two being interchangeable via logarithmic transformation. This invertibility opens potential use cases in models where multiplicative structure is more natural or efficient.

The pricinpal goal of the ***logprod*** formulation is two-fold. The use of negative time indices is employed to simplify notation compared to the traditional index-minus-lag method for indicating data points in the past relative to the time of interest. It further reduces algebraic strain by creating an exponential function which has the realized volatility as an exponent, allowing for multiplicative operations.