# Algebraic Type Theory

Steve Awodey

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### Outline

- 1. Strictifying Homotopical Models
- 2. Natural Models of Type Theory
- 3. Some Type Formers
- 4. A Polynomial Monad

# 1. Strictifying Homotopical Models

- A homotopical model of (homotopy) type theory was defined to be a Quillen model category E, with the Frobenius property and a fibrant, univalent universe U → U.
- We can extract a strict model of (homotopy) type theory from a homotopical one using some ideas of Voevodsky and Lumsdaine-Warren.
- The resulting structure is a category with families, which is a quite strict notion of a model of dependent type theory. (A related construction gives a contextual category.)
- I will do the cases of  $\Sigma$  and  $\Pi$  types, but one can add the Id-types and a universe U.

# 1. Dependent type theory

The system to be modelled has:

**Basic types and terms**:  $A, B, \ldots, x: A, b: B, \ldots$ 

**Dependent types and terms**:  $x:A \vdash b(x):B(x), \ldots$ 

**Contexts**:  $(x:A, y:B(x),...), \Gamma, \Delta, ...$ 

**Substitutions**:  $\sigma: \Delta \to \Gamma$ ,...

Type forming operations:  $\sum_{x:A} B(x)$ ,  $\prod_{x:A} B(x)$ , ...

**Equations between terms:**  $\Gamma \vdash s = t : A$ 

## 1. Dependent type theory: Rules

#### Contexts:

$$\frac{x:A \vdash B(x)}{x:A, \ y:B(x) \vdash} \qquad \frac{\Gamma \vdash C}{\Gamma, z:C \vdash}$$

#### Sums:

$$\frac{x : A \vdash B(x)}{\sum_{x : A} B(x)} \qquad \frac{a : A \qquad b : B(a)}{\langle a, b \rangle : \sum_{x : A} B(x)}$$

$$\frac{c : \sum_{x : A} B(x)}{\text{fst } c : A} \qquad \frac{c : \sum_{x : A} B(x)}{\text{snd } c : B(\text{fst } c)}$$

$$\text{fst}\langle a, b \rangle = a : A \qquad \text{snd}\langle a, b \rangle = b : B$$

$$\langle \text{fst } c, \text{snd } c \rangle = c : \sum_{x : A} B(x)$$

# 1. Dependent type theory: Rules

#### **Products**:

$$\frac{x : A \vdash B(x)}{\prod_{x : A} B(x)} \qquad \frac{x : A \vdash b : B(x)}{\lambda x . b : \prod_{x : A} B(x)}$$

$$\frac{a : A \qquad f : \prod_{x : A} B(x)}{fa : B(a)}$$

$$x : A \vdash (\lambda x . b) x = b : B(x)$$

$$\lambda x . fx = f : \prod_{x : A} B(x)$$

Substitution:

$$\frac{\sigma:\Delta\to\Gamma\qquad\Gamma\vdash a:A}{\Delta\vdash a[\sigma]:A[\sigma]}$$

# 2. Natural Models of Type Theory

#### Definition

A natural transformation  $f: Y \to X$  of presheaves on a category  $\mathbb C$  is called **representable** if its pullback along any  $y C \to X$  is represented:



### **Proposition**

A representable natural transformation is the same thing as a category with families in the sense of Dybjer.

Write the objects and arrows of  $\mathbb C$  as  $\sigma:\Delta\to\Gamma$ , giving the category of contexts and substitutions.

A CwF is usually defined as a presheaf of types in context,

$$\mathsf{Ty}:\mathbb{C}^{\mathrm{op}}\to\mathsf{Set}\,,$$

together with a presheaf of typed terms,

$$\mathsf{Tm}: (\int_{\mathbb{C}} \mathsf{Ty})^{\mathrm{op}} \to \mathsf{Set}\,.$$

But we will reformulate this notion using the equivalence

$$\mathsf{Set}^{\left(\int_{\mathbb{C}}\mathsf{Ty}\right)^{\mathrm{op}}}\ \simeq\ \mathsf{Set}^{\mathbb{C}^{\mathrm{op}}}/\mathsf{Ty}\,.$$

So we will instead have a map  $p : Tm \rightarrow Ty$ .

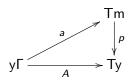
Let  $p: \mathsf{Tm} \to \mathsf{Ty}$  be a **representable** map of presheaves on  $\mathbb{C}$ .

Then Ty is again the **presheaf of types in context**, and now Tm is the **presheaf of terms in context**, and p gives the **typing of terms**.

Formally, we **interpret**:

$$\Gamma \vdash A \approx A \in Ty(\Gamma)$$
  
 $\Gamma \vdash a : A \approx a \in Tm(\Gamma)$ 

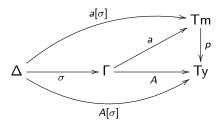
where  $A = p \circ a$ .



**NB:** we will now just write  $\Gamma$  rather than  $y\Gamma$  for the representables.

**Naturality** of  $p: \mathsf{Tm} \to \mathsf{Ty}$  means that for any **substitution**  $\sigma: \Delta \to \Gamma$ , we have the required action on types and terms:

$$\begin{array}{ccc} \Gamma \vdash A & \Rightarrow & \Delta \vdash A[\sigma] \\ \Gamma \vdash a : A & \Rightarrow & \Delta \vdash a[\sigma] : A[\sigma] \end{array}$$



Given any further  $\tau: \Delta' \to \Delta$  we clearly have

$$A[\sigma][\tau] = A[\sigma \circ \tau]$$
  $a[\sigma][\tau] = a[\sigma \circ \tau]$ 

and for the identity substitution  $1:\Gamma\to\Gamma$  we have

$$A[1] = A$$
  $a[1] = a$ .

This is the **basic structure** of a CwF.

The remaining operation of **context extension** 

$$\frac{\Gamma \vdash A}{\Gamma, x : A \vdash}$$

is given by the representability of  $p : Tm \rightarrow Ty$  as follows.

## 2. Natural Models: Context Extension

Given  $\Gamma \vdash A$  we need a new context  $\Gamma.A$  together with a substitution  $p_A : \Gamma.A \to A$  and a term

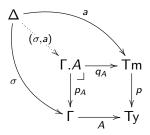
$$\Gamma.A \vdash q_A : A[p_A]$$
.

Let  $p_A : \Gamma.A \to \Gamma$  be the pullback of p along A.

$$\begin{array}{ccc}
\Gamma.A & \xrightarrow{q_A} & \mathsf{Tm} \\
\downarrow^{p_A} & & \downarrow^{p} \\
\Gamma & \xrightarrow{A} & \mathsf{Ty}
\end{array}$$

The map  $q_A : \Gamma.A \to \mathsf{Tm}$  gives the required term  $\Gamma.A \vdash q_A : A[p_A]$ .

## 2. Natural Models: Context Extension



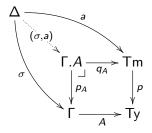
The pullback means that given any substitution  $\sigma: \Delta \to \Gamma$  and term  $\Delta \vdash a: A[\sigma]$  there is a map

$$(\sigma, a) : \Delta \rightarrow \Gamma.A$$

satisfying

$$egin{aligned} oldsymbol{p}_{\mathcal{A}}(\sigma, oldsymbol{a}) &= \sigma \ oldsymbol{q}_{\mathcal{A}}[\sigma, oldsymbol{a}] &= oldsymbol{a} \end{aligned}$$

### 2. Natural Models: Context Extension



By the uniqueness of  $(\sigma, a)$ , we also have

$$(\sigma,a)\circ \tau \ = \ (\sigma\circ \tau,a[ au]) \qquad \text{for any } \tau:\Delta' o \Delta$$

and

$$(p_A,q_A)=1.$$

These are precisely the laws of a CwF, under the equivalence

$$\mathsf{Set}^{\left(\int_{\mathbb{C}}\mathsf{Ty}\right)^{\mathrm{op}}}\simeq\,\mathsf{Set}^{\mathbb{C}^{\mathrm{op}}}\!/\mathsf{Ty}$$

# 2. Natural Models and Initiality

- The notion of a natural model is essentially algebraic (generalized algebraic, dependently typed algebraic, clan algebraic, finite limit theory, ...).
- The algebraic homomorphisms correspond exactly to syntactic translations.
- There are initial algebras, as well as free algebras over basic types and terms.
- The rules of type theory can be seen as a procedure for generating the free algebras.

## 2. Natural Models and Clans

Let  $p: \dot{U} \to U$  be a natural model.

The fibration  $\mathcal{F}_p o \mathbb{C}$  of all pullbacks

$$A^*p:\Gamma.A\to\Gamma$$
 for all  $A:\Gamma\to U$ 

form a display map category (=: pre-clan).

Conversely, given any pre-clan  $(\mathbb{C}, \mathcal{F})$ , there is a natural model  $p_{\mathcal{F}} : \dot{\mathsf{U}}_{\mathcal{F}} \to \mathsf{U}_{\mathcal{F}}$  over  $\mathbb{C}$ ,

$$p_{\mathcal{F}} = \coprod_{f \in \mathcal{F}} \mathsf{y} f : \coprod_{f \in \mathcal{F}} \mathsf{ydom}(f) \to \coprod_{f \in \mathcal{F}} \mathsf{ycod}(f).$$

There is an adjunction  $p \dashv \mathcal{F}$ 

$$\begin{array}{c} \mathcal{F} \\ \text{NatMod} \ . \end{array}$$

A natural model  $p: \dot{U} \rightarrow U$  determines a **polynomial endofunctor** 

$$P: \mathsf{Set}^{\mathbb{C}^{\mathsf{op}}} \to \mathsf{Set}^{\mathbb{C}^{\mathsf{op}}},$$

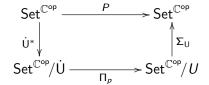
defined for every  $X:\mathbb{C}^{\mathrm{op}} o \mathsf{Set}$  by

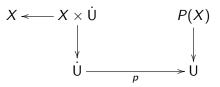
$$P(X) = \sum_{A \in \Pi} X^{[A]},$$

where  $[A] = p^{-1}(A)$  is the fiber of  $p : \dot{U} \to U$  at A : U.

# 3. Modeling the Type Formers: Polynomials

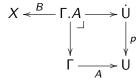
The construction of P(X) can be described as follows.





#### Lemma

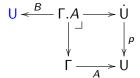
Maps  $\Gamma \to P(X)$  correspond naturally to pairs (A, B) where:



Applying P to U itself therefore gives an object

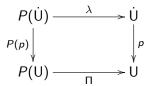
$$P(\mathsf{U}) = \sum_{\mathsf{A} \cdot \mathsf{U}} \mathsf{U}^{[\mathsf{A}]}$$

such that maps  $\Gamma \to P(U)$  correspond naturally to **types in an extended context**  $\Gamma.A \vdash B$ 



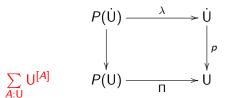
### Proposition

A natural model  $p:\dot{U}\to U$  models the rules for  $\Pi$ -types just if there are maps  $\lambda,\Pi$  making the following a pullback.



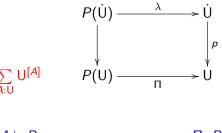
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$$A \vdash B$$

$$\Pi_A B$$

## Proposition

A natural model  $p: U \to U$  models the rules for  $\Pi$ -types just if there are maps  $\lambda, \Pi$  making the following a pullback.

$$A \vdash b : B \qquad \lambda_{A}b$$

$$\sum_{A:U} \dot{U}^{[A]} \qquad P(\dot{U}) \xrightarrow{\lambda} \dot{U}$$

$$\downarrow \qquad \qquad \downarrow^{p}$$

$$\sum_{A:U} U^{[A]} \qquad P(U) \xrightarrow{\Pi} U$$

$$A \vdash B \qquad \Pi_{A}B$$

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Proof:

 $\sum_{A:U} \dot{\mathbf{U}}[A] \qquad P(\dot{\mathbf{U}}) \xrightarrow{\lambda} \dot{\mathbf{U}} \qquad \downarrow^{p}$   $\sum_{A:U} \mathbf{U}[A] \qquad P(\mathbf{U}) \xrightarrow{\Pi} \mathbf{U}$   $A \vdash B \qquad \Pi_{A}B$ 

### Proposition

A natural model  $p: \dot{U} \to U$  models the rules for  $\Pi$ -types just if there are maps  $\lambda, \Pi$  making the following a pullback.

$$A \vdash f(x) : B \qquad \qquad \lambda_{A}f(x) = f$$

$$\sum_{A:U} \dot{U}^{[A]} \qquad P(\dot{U}) \xrightarrow{\lambda} \dot{U}$$

$$\downarrow \qquad \qquad \downarrow^{p}$$

$$\sum_{A:U} U^{[A]} \qquad P(U) \xrightarrow{\Pi} \dot{U}$$

$$A \vdash B \qquad \qquad \Pi_{A}B$$

## Proposition

A natural model  $p:\dot{U}\to U$  models the rules for  $\Sigma$ -types just if there are maps  $(pair,\Sigma)$  making the following a pullback

$$Q \xrightarrow{\text{pair}} \Rightarrow \dot{U}$$

$$\downarrow q \qquad \qquad \downarrow p$$

$$P(U) \xrightarrow{\Sigma} \qquad U$$

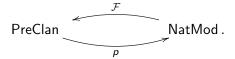
where  $q:Q\to P(U)$  is the polynomial composition  $P_q=P\circ P$ . Explicitly:

$$Q = \sum_{\Delta : \Pi} \sum_{B : \Pi A} \sum_{x : \Delta} B(x)$$

# 3. Modeling the Type Formers: Strictification

#### Theorem

Given any  $\Pi$ -tribe  $(\mathbb{C},\mathcal{F})$ , for example a Quillen model category with the Frobenius property, the associated natural model  $p_{\mathcal{F}}:\dot{U}_{\mathcal{F}}\to U_{\mathcal{F}}$  under the adjunction



has  $\Sigma$  and  $\Pi$  types (as well as Id-types).

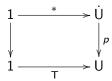
The natural model  $p_{\mathcal{F}}: \dot{\mathsf{U}}_{\mathcal{F}} \to \mathsf{U}_{\mathcal{F}}$  is thus a **strictification** of the homotopical model  $(\mathbb{C}, \mathcal{F})$ .

Consider the rules for a terminal type T.

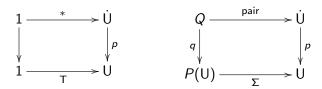
 $\overline{\vdash} T$   $\overline{\vdash} * : T$   $\overline{x} : T \vdash x = * : T$ 

## Proposition

A natural model  $p: U \to U$  models the rules for a terminal type just if there are maps (\*,T) making the following a pullback.



Consider the pullback squares for T and  $\Sigma$ .



These determine cartesian natural transformations between the corresponding polynomial endofunctors.

$$\tau: 1 \Rightarrow P$$
  $\sigma: P \circ P \Rightarrow P$ 

## Theorem (A-Newstead)

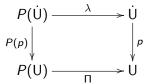
A natural model  $p:\dot{U}\to U$  models T and  $\Sigma$  types just if the associated polynomial endofunctor P has the structure of a cartesian monad.

$$\tau: 1 \Rightarrow P$$
  $\sigma: P \circ P \Rightarrow P$ 

The monad laws correspond to the following type isomorphisms.

$\sigma \circ P \sigma = \sigma \circ \sigma_P$	$\sum_{a:A} \sum_{b:B(a)} C(a,b) \cong \sum_{(a,b):\sum_{a:A} B(a)} C(a,b)$
$\sigma \circ P  au = 1$	$\sum_{a:A} 1 \cong A$
$\sigma \circ  au_{ extsf{ extsf{P}}} = 1$	$\sum_{x:1} A \cong A$

The pullback square for  $\Pi$ 



determines a cartesian natural transformation

$$\pi: P^2(p) \Rightarrow p$$

where  $P^2: \hat{\mathbb{C}}^2 \to \hat{\mathbb{C}}^2$  is the lift of P to the arrow category.

So a natural model  $p: \dot{\mathsf{U}} \to \mathsf{U}$  models  $\Pi$  types just if it has an algebra structure for the lifted endofunctor  $P^2$ .

$$\pi: P^2(p) \Rightarrow p$$

The algebra laws correspond to the following type isomorphisms.

$\pi \circ P\pi = \pi \circ \sigma$	$\prod_{a:A} \prod_{b:B(a)} C(a,b) \cong \prod_{(a,b):\sum_{a:A} B(a)} C(a,b)$
$\pi \circ \tau = 1$	$\prod_{x:1} A \cong A$

# Summary: Strictification

#### **Theorem**

A homotopical model of (homotopy) type theory determines a natural model  $p:\dot{U}\to U$  which

(i) presents a polynomial monad, and an algebra for it, and (ii) strictly models the  $\Sigma, \Pi$  and  $\operatorname{Id}$  type formers.

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