Algebraic Type Theory

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Outline

1. Strictifying Homotopical Models
2. Natural Models of Type Theory
3. Some Type Formers
4. A Polynomial Monad

1. Strictifying Homotopical Models

- A **homotopical model** of (homotopy) type theory was defined to be a Quillen model category $\mathcal{E}$, with the Frobenius property and a fibrant, univalent universe $\dot{\mathsf{U}} \twoheadrightarrow \mathsf{U}$.
- We can extract a **strict model** of (homotopy) type theory from a homotopical one using some ideas of Voevodsky and Lumsdaine-Warren.
- The resulting structure is a **category with families**, which is a quite strict notion of a model of dependent type theory. (A related construction gives a **contextual category**.)
- I will do the cases of Σ and Π types, but one can add the Id-types and a universe U.

1. Dependent type theory

The system to be modelled has:

Basic types and terms: $A, B, \ldots, x{:}A, b{:}B, \ldots$

Dependent types and terms: $x{:}A \vdash b(x) : B(x), \ldots$

Contexts: $(x{:}A, y{:}B(x), \ldots), \Gamma, \Delta, \ldots$

Substitutions: $\sigma{:}\Delta \to \Gamma, \ldots$

Type forming operations: $\sum_{x:A} B(x), \prod_{x:A} B(x), \ldots$

Equations between terms: $\Gamma \vdash s = t : A$

1. Dependent type theory: Rules

Contexts:

$$\frac{x : A \vdash B(x)}{x : A,\ y : B(x) \vdash} \qquad \frac{\Gamma \vdash C}{\Gamma, z : C \vdash}$$

Sums:

$$\frac{x : A \vdash B(x)}{\sum_{x:A} B(x)} \qquad \frac{a : A \qquad b : B(a)}{\langle a, b \rangle : \sum_{x:A} B(x)}$$

$$\frac{c : \sum_{x:A} B(x)}{\mathrm{fst}\, c : A} \qquad \frac{c : \sum_{x:A} B(x)}{\mathrm{snd}\, c : B(\mathrm{fst}\, c)}$$

$$\mathrm{fst}\langle a, b \rangle = a : A \qquad \mathrm{snd}\langle a, b \rangle = b : B$$

$$\langle \mathrm{fst}\, c, \mathrm{snd}\, c \rangle = c : \sum_{x:A} B(x)$$

1. Dependent type theory: Rules

Products:

$$\frac{x : A \vdash B(x)}{\prod_{x:A} B(x)} \qquad \frac{x : A \vdash b : B(x)}{\lambda x.b : \prod_{x:A} B(x)}$$

$$\frac{a : A \qquad f : \prod_{x:A} B(x)}{fa : B(a)}$$

$$x : A \vdash (\lambda x.b)x = b : B(x)$$

$$\lambda x.fx = f : \prod_{x:A} B(x)$$

Substitution:

$$\frac{\sigma : \Delta \to \Gamma \qquad \Gamma \vdash a : A}{\Delta \vdash a[\sigma] : A[\sigma]}$$

2. Natural Models of Type Theory

Definition

A natural transformation $f : Y \to X$ of presheaves on a category $\mathbb{C}$ is called **representable** if its pullback along any $\mathsf{y}C \to X$ is represented:

$$
\begin{tikzcd}
\mathsf{y}D \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "f"] \\
\mathsf{y}C \arrow[r] & X
\end{tikzcd}
$$

Proposition

A representable natural transformation is the same thing as a **category with families** *in the sense of Dybjer.*

2. Natural Models as CwFs

Write the objects and arrows of $\mathbb{C}$ as $\sigma : \Delta \to \Gamma$, giving the **category of contexts and substitutions**.

A CwF is usually defined as a presheaf of **types in context**,

$$\mathsf{Ty} : \mathbb{C}^{\mathrm{op}} \to \mathsf{Set}\,,$$

together with a presheaf of **typed terms**,

$$\mathsf{Tm} : (\textstyle\int_{\mathbb{C}} \mathsf{Ty})^{\mathrm{op}} \to \mathsf{Set}\,.$$

But we will reformulate this notion using the equivalence

$$\mathsf{Set}^{(\int_{\mathbb{C}} \mathsf{Ty})^{\mathrm{op}}} \;\simeq\; \mathsf{Set}^{\mathbb{C}^{\mathrm{op}}}/\mathsf{Ty}\,.$$

So we will instead have a map $p : \mathsf{Tm} \to \mathsf{Ty}$.

2. Natural Models as CwFs

Let $p : \mathsf{Tm} \to \mathsf{Ty}$ be a **representable** map of presheaves on $\mathbb{C}$.

Then Ty is again the **presheaf of types in context**, and now Tm is the **presheaf of terms in context**, and p gives the **typing of terms**.

Formally, we **interpret**:

$$\begin{array}{rcl} \Gamma \vdash A & \approx & A \in \mathsf{Ty}(\Gamma) \\ \Gamma \vdash a : A & \approx & a \in \mathsf{Tm}(\Gamma) \end{array}$$

where $A = p \circ a$.

$$\begin{tikzcd} & \mathsf{Tm} \arrow[d, "p"] \\ \mathsf{y}\Gamma \arrow[ur, "a"] \arrow[r, "A"'] & \mathsf{Ty} \end{tikzcd}$$

NB: we will now just write Γ rather than $\mathsf{y}\Gamma$ for the representables.

2. Natural Models as CwFs

Naturality of $p : \mathsf{Tm} \to \mathsf{Ty}$ means that for any **substitution** $\sigma : \Delta \to \Gamma$, we have the required action on types and terms:

$$\begin{array}{rcl} \Gamma \vdash A & \Rightarrow & \Delta \vdash A[\sigma] \\ \Gamma \vdash a : A & \Rightarrow & \Delta \vdash a[\sigma] : A[\sigma] \end{array}$$

$$\begin{tikzcd} & & \mathsf{Tm} \arrow[d, "p"] \\ \Delta \arrow[r, "\sigma"'] \arrow[urr, bend left, "a{[\sigma]}"] \arrow[rr, bend right, "A{[\sigma]}"'] & \Gamma \arrow[ur, "a"] \arrow[r, "A"'] & \mathsf{Ty} \end{tikzcd}$$

2. Natural Models as CwFs

Given any further $\tau : \Delta' \to \Delta$ we clearly have

$$A[\sigma][\tau] = A[\sigma \circ \tau] \qquad\qquad a[\sigma][\tau] = a[\sigma \circ \tau]$$

and for the identity substitution $1 : \Gamma \to \Gamma$ we have

$$A[1] = A \qquad\qquad a[1] = a\,.$$

This is the **basic structure** of a CwF.

The remaining operation of **context extension**

$$\frac{\Gamma \vdash A}{\Gamma, x\!:\!A \vdash}$$

is given by the representability of $p : \mathsf{Tm} \to \mathsf{Ty}$ as follows.

2. Natural Models: Context Extension

Given $\Gamma \vdash A$ we need a new context $\Gamma.A$ together with a substitution $p_A : \Gamma.A \to A$ and a term

$$\Gamma.A \vdash q_A : A[p_A]\,.$$

Let $p_A : \Gamma.A \to \Gamma$ be the pullback of p along A.

$$\begin{tikzcd}
\Gamma.A \arrow[r, "q_A"] \arrow[d, "p_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Tm} \arrow[d, "p"] \\
\Gamma \arrow[r, "A"'] & \mathrm{Ty}
\end{tikzcd}$$

The map $q_A : \Gamma.A \to \mathrm{Tm}$ gives the required term $\Gamma.A \vdash q_A : A[p_A]$.

2. Natural Models: Context Extension

$$\begin{tikzcd}
\Delta \arrow[dr, dotted, "{(\sigma,a)}"] \arrow[drr, bend left, "a"] \arrow[ddr, bend right, "\sigma"'] & & \\
& \Gamma.A \arrow[r, "q_A"'] \arrow[d, "p_A"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Tm} \arrow[d, "p"] \\
& \Gamma \arrow[r, "A"'] & \mathrm{Ty}
\end{tikzcd}$$

The pullback means that given any substitution $\sigma : \Delta \to \Gamma$ and term $\Delta \vdash a : A[\sigma]$ there is a map

$$(\sigma, a) : \Delta \to \Gamma.A$$

satisfying

$$p_A(\sigma, a) = \sigma$$
$$q_A[\sigma, a] = a.$$

2. Natural Models: Context Extension

$$\begin{tikzcd}
\Delta \arrow[dr, dotted, "{(\sigma,a)}"] \arrow[drr, bend left, "a"] \arrow[ddr, bend right, "\sigma"'] & & \\
& \Gamma.A \arrow[r, "q_A"'] \arrow[d, "p_A"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Tm} \arrow[d, "p"] \\
& \Gamma \arrow[r, "A"'] & \mathrm{Ty}
\end{tikzcd}$$

By the uniqueness of (σ, a), we also have

$$(\sigma, a) \circ \tau \;=\; (\sigma \circ \tau, a[\tau]) \qquad \text{for any } \tau : \Delta' \to \Delta$$

and

$$(p_A, q_A) = 1.$$

These are precisely the **laws of a CwF**, under the equivalence

$$\mathsf{Set}^{(\int_{\mathbb{C}} \mathsf{Ty})^{\mathrm{op}}} \simeq \mathsf{Set}^{\mathbb{C}^{\mathrm{op}}}/\mathsf{Ty}$$

$\square$

2. Natural Models and Initiality

- The notion of a natural model is **essentially algebraic** (generalized algebraic, dependently typed algebraic, clan algebraic, finite limit theory, ...).
- The algebraic homomorphisms correspond exactly to **syntactic translations**.
- There are **initial algebras**, as well as **free algebras** over basic types and terms.
- The rules of type theory can be seen as a procedure for **generating the free algebras**.

2. Natural Models and Clans

Let $p : \dot{\mathsf{U}} \to \mathsf{U}$ be a natural model.

The fibration $\mathcal{F}_p \to \mathbb{C}$ of all pullbacks

$$A^* p : \Gamma.A \to \Gamma \qquad\qquad \text{for all } A : \Gamma \to \mathsf{U}$$

form a **display map category** (=: **pre-clan**).

Conversely, given any pre-clan $(\mathbb{C}, \mathcal{F})$, there is a natural model $p_{\mathcal{F}} : \dot{\mathsf{U}}_{\mathcal{F}} \to \mathsf{U}_{\mathcal{F}}$ over $\mathbb{C}$,

$$p_{\mathcal{F}} = \coprod_{f \in \mathcal{F}} \mathsf{y} f : \coprod_{f \in \mathcal{F}} \mathsf{y}\mathrm{dom}(f) \to \coprod_{f \in \mathcal{F}} \mathsf{y}\mathrm{cod}(f)\,.$$

There is an adjunction $p \dashv \mathcal{F}$

$$\begin{tikzcd} \mathsf{PreClan} \arrow[r, bend right, "p"'] & \mathsf{NatMod}\,. \arrow[l, bend right, "\mathcal{F}"'] \end{tikzcd}$$

3. Modeling the Type Formers

A natural model $p : \dot{\mathsf{U}} \to \mathsf{U}$ determines a **polynomial endofunctor**

$$P : \mathsf{Set}^{\mathbb{C}^{\mathsf{op}}} \to \mathsf{Set}^{\mathbb{C}^{\mathsf{op}}},$$

defined for every $X : \mathbb{C}^{\mathrm{op}} \to \mathsf{Set}$ by

$$P(X) \;=\; \sum_{A\,:\,\mathsf{U}} X^{[A]}\,,$$

where $[A] = p^{-1}(A)$ is the fiber of $p : \dot{\mathsf{U}} \to \mathsf{U}$ at $A : \mathsf{U}$.

3. Modeling the Type Formers: Polynomials

The construction of $P(X)$ can be described as follows.

$$
\begin{tikzcd}
\mathsf{Set}^{\mathbb{C}^{\mathrm{op}}} \arrow[r, "P"] \arrow[d, "\dot{\mathsf{U}}^*"'] & \mathsf{Set}^{\mathbb{C}^{\mathrm{op}}} \\
\mathsf{Set}^{\mathbb{C}^{\mathrm{op}}}/\dot{\mathsf{U}} \arrow[r, "\Pi_p"'] & \mathsf{Set}^{\mathbb{C}^{\mathrm{op}}}/U \arrow[u, "\Sigma_{\mathsf{U}}"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
X & X \times \dot{\mathsf{U}} \arrow[l] \arrow[d] & & P(X) \arrow[d] \\
& \dot{\mathsf{U}} \arrow[rr, "p"'] & & \mathsf{U}
\end{tikzcd}
$$

3. Modeling the Type Formers

Lemma

Maps $\Gamma \to P(X)$ *correspond naturally to pairs* (A, B) *where:*

$$\begin{tikzcd}
X & \Gamma.A \arrow[l, "B"'] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathsf{U}} \arrow[d, "p"] \\
 & \Gamma \arrow[r, "A"'] & \mathsf{U}
\end{tikzcd}$$

3. Modeling the Type Formers

Applying P to U itself therefore gives an object

$$P(\mathsf{U}) = \sum_{A:\mathsf{U}} \mathsf{U}^{[A]}$$

such that maps $\Gamma \to P(\mathsf{U})$ correspond naturally to **types in an extended context** $\Gamma.A \vdash B$

$$\begin{tikzcd}
\mathsf{U} & \Gamma.A \arrow[l, "B"'] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathsf{U}} \arrow[d, "p"] \\
 & \Gamma \arrow[r, "A"'] & \mathsf{U}
\end{tikzcd}$$

3. Modeling the Type Formers: Π

Proposition

A natural model $p : \dot{\mathsf{U}} \to \mathsf{U}$ models the rules for Π-types just if there are maps λ, Π making the following a pullback.

$$
\begin{tikzcd}
P(\dot{\mathsf{U}}) \arrow[r, "\lambda"] \arrow[d, "P(p)"'] & \dot{\mathsf{U}} \arrow[d, "p"] \\
P(\mathsf{U}) \arrow[r, "\Pi"'] & \mathsf{U}
\end{tikzcd}
$$

3. Modeling the Type Formers: Π

Proposition

A natural model $p : \dot{\mathsf{U}} \to \mathsf{U}$ models the rules for Π-types just if there are maps λ, Π making the following a pullback.

Proof:

$$\sum_{A:\mathsf{U}} \mathsf{U}^{[A]} \qquad \begin{tikzcd} P(\dot{\mathsf{U}}) \arrow[r, "\lambda"] \arrow[d] & \dot{\mathsf{U}} \arrow[d, "p"] \\ P(\mathsf{U}) \arrow[r, "\Pi"'] & \mathsf{U} \end{tikzcd}$$

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P(\mathsf{U}) \arrow[r, "\Pi"'] & \mathsf{U}
\end{tikzcd}$$

$$\sum_{A:\mathsf{U}} \mathsf{U}^{[A]}$$

$A \vdash B$ $\qquad\qquad$ $\Pi_A B$

3. Modeling the Type Formers: Π

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Proof:

$$\begin{tikzcd}
A \vdash b : B & \sum_{A:\mathsf{U}} \dot{\mathsf{U}}^{[A]} & P(\dot{\mathsf{U}}) \arrow[r, "\lambda"] \arrow[d] & \dot{\mathsf{U}} \arrow[d, "p"] & \lambda_A b \\
A \vdash B & \sum_{A:\mathsf{U}} \mathsf{U}^{[A]} & P(\mathsf{U}) \arrow[r, "\Pi"'] & \mathsf{U} & \Pi_A B
\end{tikzcd}$$

3. Modeling the Type Formers: Π

Proposition

A natural model $p : \dot{\mathsf{U}} \to \mathsf{U}$ models the rules for Π-types just if there are maps λ, Π making the following a pullback.

Proof:

$$\begin{tikzcd}
 & & f \\
\sum_{A:\mathsf{U}} \dot{\mathsf{U}}^{[A]} & P(\dot{\mathsf{U}}) \arrow[r, "\lambda"] \arrow[d] & \dot{\mathsf{U}} \arrow[d, "p"] \\
\sum_{A:\mathsf{U}} \mathsf{U}^{[A]} & P(\mathsf{U}) \arrow[r, "\Pi"'] & \mathsf{U} \\
A \vdash B & & \Pi_A B
\end{tikzcd}$$

3. Modeling the Type Formers: Π

Proposition

A natural model $p : \dot{\mathsf{U}} \to \mathsf{U}$ models the rules for Π-types just if there are maps λ, Π making the following a pullback.

Proof:

$A \vdash f(x) : B$ $\qquad\qquad$ $\lambda_A f(x) = f$

$$\begin{tikzcd}
\sum_{A:\mathsf{U}} \dot{\mathsf{U}}^{[A]} & P(\dot{\mathsf{U}}) \arrow[r, "\lambda"] \arrow[d] & \dot{\mathsf{U}} \arrow[d, "p"] \\
\sum_{A:\mathsf{U}} \mathsf{U}^{[A]} & P(\mathsf{U}) \arrow[r, "\Pi"'] & \mathsf{U}
\end{tikzcd}$$

$A \vdash B$ $\qquad\qquad$ $\Pi_A B$

3. Modeling the Type Formers: Σ

Proposition

A natural model $p : \dot{\mathsf{U}} \to \mathsf{U}$ models the rules for Σ-types just if there are maps (pair, Σ) making the following a pullback

$$\begin{tikzcd} Q \arrow[r, "\mathsf{pair}"] \arrow[d, "q"'] & \dot{\mathsf{U}} \arrow[d, "p"] \\ P(\mathsf{U}) \arrow[r, "\Sigma"'] & \mathsf{U} \end{tikzcd}$$

where $q : Q \to P(\mathsf{U})$ is the polynomial composition $P_q = P \circ P$.

Explicitly:

$$Q \;=\; \sum_{A:\mathsf{U}} \sum_{B:\mathsf{U}^A} \sum_{x:A} B(x)$$

3. Modeling the Type Formers: Strictification

Theorem

Given any Π-**tribe** $(\mathbb{C}, \mathcal{F})$, *for example a Quillen model category with the Frobenius property, the associated natural model* $p_{\mathcal{F}} : \dot{\mathsf{U}}_{\mathcal{F}} \to \mathsf{U}_{\mathcal{F}}$ *under the adjunction*

$$\begin{tikzcd} \mathsf{PreClan} \arrow[rr, bend right, "p"'] & & \mathsf{NatMod} \arrow[ll, bend right, "\mathcal{F}"'] \end{tikzcd} .$$

has Σ *and* Π *types (as well as* Id*-types).*

The natural model $p_{\mathcal{F}} : \dot{\mathsf{U}}_{\mathcal{F}} \to \mathsf{U}_{\mathcal{F}}$ is thus a **strictification** of the homotopical model $(\mathbb{C}, \mathcal{F})$.

4. A Polynomial Monad

Consider the rules for a terminal type T.

$$\frac{}{\vdash \mathsf{T}} \qquad\qquad \frac{}{\vdash * : \mathsf{T}} \qquad\qquad \frac{}{x : \mathsf{T} \vdash x = * : \mathsf{T}}$$

Proposition

A natural model $p : \dot{\mathsf{U}} \to \mathsf{U}$ models the rules for a terminal type just if there are maps $(*, \mathsf{T})$ making the following a pullback.

$$\begin{tikzcd}
1 \arrow[r, "*"] \arrow[d] & \dot{\mathsf{U}} \arrow[d, "p"] \\
1 \arrow[r, "\mathsf{T}"'] & \mathsf{U}
\end{tikzcd}$$

4. A Polynomial Monad

Consider the pullback squares for T and Σ.

$$
\begin{tikzcd}
1 \arrow[r, "*"] \arrow[d] & \dot{\mathsf{U}} \arrow[d, "p"] \\
1 \arrow[r, "\mathsf{T}"'] & \mathsf{U}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
Q \arrow[r, "\mathsf{pair}"] \arrow[d, "q"'] & \dot{\mathsf{U}} \arrow[d, "p"] \\
P(\mathsf{U}) \arrow[r, "\Sigma"'] & \mathsf{U}
\end{tikzcd}
$$

These determine cartesian natural transformations between the corresponding polynomial endofunctors.

$$\tau : 1 \Rightarrow P \qquad\qquad \sigma : P \circ P \Rightarrow P$$

4. A Polynomial Monad

Theorem (A-Newstead)

A natural model $p : \dot{\mathsf{U}} \to \mathsf{U}$ models T *and* Σ *types just if the associated polynomial endofunctor P has the structure of a cartesian monad.*

$$\tau : 1 \Rightarrow P \qquad\qquad \sigma : P \circ P \Rightarrow P$$

4. A Polynomial Monad

The monad laws correspond to the following type isomorphisms.

$\sigma \circ P\sigma = \sigma \circ \sigma_P$	$\sum_{a:A} \sum_{b:B(a)} C(a,b) \;\cong\; \sum_{(a,b):\sum_{a:A} B(a)} C(a,b)$
$\sigma \circ P\tau = 1$	$\sum_{a:A} 1 \;\cong\; A$
$\sigma \circ \tau_P = 1$	$\sum_{x:1} A \;\cong\; A$

4. A Polynomial Monad

The pullback square for Π

$$\begin{tikzcd} P(\dot{\mathsf{U}}) \arrow[r, "\lambda"] \arrow[d, "P(p)"'] & \dot{\mathsf{U}} \arrow[d, "p"] \\ P(\mathsf{U}) \arrow[r, "\Pi"'] & \mathsf{U} \end{tikzcd}$$

determines a cartesian natural transformation

$$\pi : P^{\mathbf{2}}(p) \Rightarrow p$$

where $P^{\mathbf{2}} : \hat{\mathbb{C}}^{\mathbf{2}} \to \hat{\mathbb{C}}^{\mathbf{2}}$ is the lift of P to the arrow category.

4. A Polynomial Monad

So a natural model $p : \dot{\mathsf{U}} \to \mathsf{U}$ models Π types just if it has an algebra structure for the lifted endofunctor $P^{\mathbf{2}}$.

$$\pi : P^{\mathbf{2}}(p) \Rightarrow p$$

The algebra laws correspond to the following type isomorphisms.

$\pi \circ P\pi \ = \ \pi \circ \sigma$	$\prod_{a:A} \prod_{b:B(a)} C(a,b) \ \cong \ \prod_{(a,b):\sum_{a:A} B(a)} C(a,b)$
$\pi \circ \tau \ = \ 1$	$\prod_{x:1} A \ \cong \ A$

Summary: Strictification

Theorem

A homotopical model of (homotopy) type theory determines a natural model $p : \dot{\mathsf{U}} \to \mathsf{U}$ which

(i) presents a polynomial monad, and an algebra for it, and

(ii) strictly models the Σ, Π and Id type formers.

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