Magnetic differential equations for stationary linear ideal MHD and their numerical solution

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Stationary linear perturbation of ideal MHD equilibrium

For the intended application on stationary (compared to MHD mode eigenfrequencies) non-axisymmetric magnetic perturbations by external coils, we consider a perturbed ideal MHD equilibrium for pressure p, currents $\boldsymbol{J}$ and magnetic field $\boldsymbol{B}$ fulfilling

$$\nabla p = \frac{1}{c}\boldsymbol{j} \times \boldsymbol{B}, \tag{1}$$

$$\nabla \times \boldsymbol{B} = \frac{4\pi}{c}\boldsymbol{j}, \tag{2}$$

$$\nabla \cdot \boldsymbol{B} = 0. \tag{3}$$

Starting with a given MHD equilibrium fulfilling Eqs. (1-3) denoted by subscripts "0", linear order equations for an external magnetic perturbation (denoted by δ) split into a vacuum and a plasma part (subscript v and p, respectively) are

$$\nabla \delta p = \frac{1}{c}\left(\boldsymbol{j}_0 \times \delta\boldsymbol{B} + \delta\boldsymbol{j} \times \boldsymbol{B}_0\right), \tag{4}$$

$$\delta\boldsymbol{B} = \delta\boldsymbol{B}_v + \delta\boldsymbol{B}_p, \tag{5}$$

$$\delta\boldsymbol{B}_v = \frac{1}{c}\oint \frac{I_c(\boldsymbol{r}')\mathrm{d}\boldsymbol{l}' \times \boldsymbol{r}}{|\boldsymbol{r}-\boldsymbol{r}'|^3}, \tag{6}$$

$$\delta\boldsymbol{B}_p = \nabla \times \delta\boldsymbol{A}, \tag{7}$$

$$\nabla \times (\nabla \times \delta\boldsymbol{A}) = \frac{4\pi}{c}\delta\boldsymbol{j}, \tag{8}$$

$$\Rightarrow \nabla \cdot \delta\boldsymbol{B} = \nabla \cdot \delta\boldsymbol{j} = 0. \tag{9}$$

Here the perturbation field in vacuum, $\delta\boldsymbol{B}_v$, is pre-evaluated by a Biot-Savart integral over external coil currents $I_c(\boldsymbol{r}')$ and the perturbation field in plasma $\delta\boldsymbol{B}_p$ is computed from Ampère's law (8) from the plasma current density $\delta\boldsymbol{j}$ using a vector potential formulation. To find a consistent solution for the system, Eq. (4) and Eq. (8) are treated individually in an iterative way. The linearized force balance equation (4) is used to compute $\delta\boldsymbol{j}$ for given $\delta\boldsymbol{B}$ whereas Eq. (8) yields $\delta\boldsymbol{B}_p$ for given $\delta\boldsymbol{j}$. In the first iteration, $\delta\boldsymbol{B}$ is set equal to $\delta\boldsymbol{B}_v$ in Eq. (4). Then, Eq. (8) and Eq. (4) are solved in an alternating way until convergence is reached. In addition a preconditioner is used to enhance convergence. Here we limit the analysis to an axisymmetric unperturbed equilibrium and a single toroidal perturbation harmonic $\delta\boldsymbol{B} = \mathrm{Re}(\boldsymbol{B}_n e^{in\varphi})$ with a similar notation for other perturbed quantities. As all equations are linear, a superposition of multiple harmonics is easily possible.

Linearized MHD force balance

The solution of Eq. (4) can further be split into two steps: First the pressure perturbation δp is found, and then the plasma current density $\delta\boldsymbol{j}$ is computed using the condition $\nabla \cdot \delta\boldsymbol{j} = 0$. For an unperturbed equilibrium with nested flux surfaces, both steps can be performed in a radially local manner if a field-aligned computational grid is used, what will become clear below. Radial coupling happens by the combination of the two individual steps since their effective radial locations of computation are shifted by a half-step in radial grid distance.

In axisymmetric coordinate systems, such as cylindrical (R, φ, Z), the equations to solve for harmonics in the toroidal angle φ are

$$\nabla p_n + inp_n \nabla\varphi = \frac{1}{c}\left(\boldsymbol{j}_0 \times \boldsymbol{B}_n + \boldsymbol{j}_n \times \boldsymbol{B}_0\right), \tag{10}$$

$$\nabla \cdot \boldsymbol{j}_n^{\mathrm{pol}} + inj_n^{\varphi} = 0. \tag{11}$$

now with a 2D ∇ operator acting in the poloidal (RZ) plane. The divergence operator is defined via

$$\nabla \cdot \boldsymbol{u} = \frac{1}{R\sqrt{g_p}} \frac{\partial}{\partial x^k}(R\sqrt{g_p}u^k),$$

where $\sqrt{g_p}$ is the metric tensor of the coordinates in the poloidal plane, which is equal to one for cylindrical coordinates (RZ).

Representation of equilibrium field

$\boldsymbol{B}_0$ is given by

$$\boldsymbol{B}_0 = \boldsymbol{B}_0^{\mathrm{pol}} + \boldsymbol{B}_0^{\mathrm{tor}}, \tag{12}$$

where

$$\boldsymbol{B}_0^{\mathrm{pol}} = \nabla\psi \times \nabla\varphi, \tag{13}$$

$$\boldsymbol{B}_0^{\mathrm{tor}} = B_{0\varphi}\nabla\varphi. \tag{14}$$

In particular

$$|\boldsymbol{B}_0^{\mathrm{pol}}| = \frac{|\nabla\psi|}{R}, \tag{15}$$

and the poloidal unit vector

$$\boldsymbol{h}_0^{\mathrm{pol}} = \frac{\boldsymbol{B}_0^{\mathrm{pol}}}{|\boldsymbol{B}_0^{\mathrm{pol}}|} = R\frac{\boldsymbol{B}_0^{\mathrm{pol}}}{|\nabla\psi|}. \tag{16}$$

Pressure perturbation

Taking an inner product of Eq. TODO

Current perturbation

Multiplying Eq. (11) by R yields

$$\frac{\partial}{\partial x^k}(Rj_n^k) + inRj_n^{\varphi} = 0. \tag{17}$$

Using the divergence theorem this can also be written in integral form in a specific triangular mesh element Ω_i as

$$\oint_{\partial\Omega_i} \mathrm{d}l\, R\boldsymbol{j}_n \cdot \boldsymbol{n} + in\int_{\Omega_i} \mathrm{d}R\mathrm{d}Z\, Rj_n^{\varphi} = 0. \tag{18}$$

Here the first integral is performed over the 1-dimensional element boundary $\partial\Omega_i$. The first term is split into three contributions,

$$\oint_{\partial\Omega_i} \mathrm{d}l\, R\boldsymbol{j}_n \cdot \boldsymbol{n} = \int_{1,2} \mathrm{d}l\, R\boldsymbol{j}_n \cdot \boldsymbol{n} + \int_3 \mathrm{d}l\, R\boldsymbol{j}_n \cdot \boldsymbol{n}, \tag{19}$$

where edge 3 is tangential to an adjacent flux surface and edges 1 and 2 are not.

Cross-field currents on edge 3

It can be shown (writeup gyrokinetics.pdf) that

$$\begin{aligned}\boldsymbol{j}_{\perp n} &= \boldsymbol{j}_n - (\boldsymbol{j}_n \cdot \boldsymbol{h}_0)\boldsymbol{h}_0 \\ &= j_{0\|}\frac{\boldsymbol{B}_{\perp n}}{B_0} - \frac{cB_{\|n}}{B_0^2}\boldsymbol{h}_0 \times \nabla p_0 + \frac{c}{B_0}\boldsymbol{h}_0 \times (\nabla p_n + inp_n \nabla\varphi). \end{aligned} \tag{20}$$

Scalar multiplication with $\boldsymbol{n}_3 \parallel \nabla p_0 \parallel \nabla\psi \perp \boldsymbol{h}_0$ and setting $j_{0\|} \approx 0$ yields

$$\begin{aligned}\boldsymbol{j}_{\perp n} \cdot \boldsymbol{n}_3 &= \frac{c}{B_0}\boldsymbol{n}_3 \cdot (\boldsymbol{h}_0 \times (\nabla p_n + inp_n \nabla\varphi)) \\ &= \frac{c}{B_0}\boldsymbol{n}_3 \cdot (h_{0\varphi}\nabla\varphi \times \nabla p_n + inp_n \boldsymbol{h}_0^{\mathrm{pol}} \times \nabla\varphi) \\ &= \frac{c}{B_0}(h_{0\varphi}\nabla p_n \cdot (\boldsymbol{n}_3 \times \nabla\varphi) + inp_n \boldsymbol{h}_0^{\mathrm{pol}}(\nabla\varphi \times \boldsymbol{n}_3)) \\ &= \frac{c}{RB_0}\boldsymbol{l}_3 \cdot (h_{0\varphi}\nabla p_n - inp_n \boldsymbol{h}_0^{\mathrm{pol}}) \\ &= \frac{c}{B_0^2}\boldsymbol{l}_3 \cdot \left(B_{0(\varphi)}\nabla p_n - \frac{in}{R}p_n \boldsymbol{B}_0^{\mathrm{pol}}\right). \end{aligned} \tag{21}$$

Details

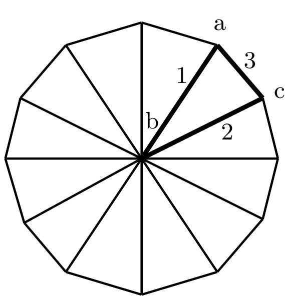
For each edge

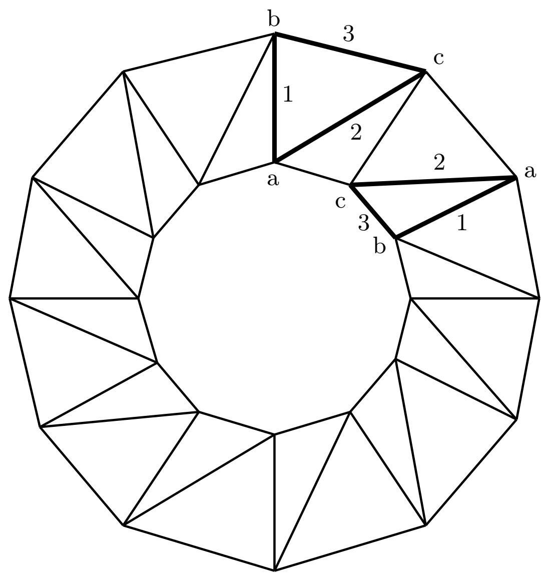
$$\boldsymbol{l} \times \boldsymbol{n} = l^2 R \nabla\varphi. \tag{22}$$

$$\boldsymbol{n} \times \nabla\varphi = \boldsymbol{l}/R, \tag{23}$$

$$\nabla\varphi \times \boldsymbol{l} = \boldsymbol{n}/R. \tag{24}$$

Implementation

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