Gyrokinetic equilibrium

Abstract

I. FLUID VELOCITY IN GYROKINETICS

Pressure tensor of strongly magnetized plasma component ($\rho_L \ll L$) in the lowest order over Larmor radius is given by Chew-Goldberger-Low (CGL) formula, see Eq.(20) of Ref. 1,

$$\mathbf{P} \approx \mathbf{P}_{\rm CGL} = p_\perp \mathbf{I} + (p_\| - p_\perp)\mathbf{h}\mathbf{h}, \tag{1}$$

where $\mathbf{I}$ is a unit tensor, $\mathbf{h} = \mathbf{B}/B$, and $p_\perp$ and $p_\|$ are perpndicular and parallel pressure, respectively. Formula (1) follows immediately from gyrophase-independent distribuion function in guiding center varables $f = f(\mathbf{r}_g, v_\perp, v_\|)$ if one ignores there a difference between the actual position $\mathbf{r}$ and the guiding center position $\mathbf{r}_g = \mathbf{r} - \boldsymbol{\rho}_L$,

$$\mathbf{P} = m\int \mathrm{d}^3v(\mathbf{v}-\mathbf{V})(\mathbf{v}-\mathbf{V})f(\mathbf{r}-\boldsymbol{\rho}_L, v_\perp, v_\|) \approx m\int \mathrm{d}^3v(\mathbf{v}-\mathbf{V}_0)(\mathbf{v}-\mathbf{V}_0)f(\mathbf{r}, v_\perp, v_\|) = \mathbf{P}_{\rm CGL}, \tag{2}$$

with

$$p_\perp = \frac{m}{2}\int \mathrm{d}^3v v_\perp^2 f, \qquad p_\| = m\int \mathrm{d}^3v v_\|^2 f, \tag{3}$$

and $\mathbf{V}$ being the lowest order fluid velocity

$$\mathbf{V}_0 = V_\|\mathbf{h}, \qquad nV_\| = \int \mathrm{d}^3v v_\| f(\mathbf{r}, v_\perp, v_\|). \tag{4}$$

Let us compute perpendicular fluid velocity in the first order over Larmor radius from the stationary momentum equation,

$$\nabla\cdot(mn\mathbf{V}\mathbf{V} + \mathbf{P}) = en\left(\mathbf{E} + \frac{1}{c}\mathbf{V}\times\mathbf{B}\right), \tag{5}$$

assuming in the l.h.s. $\mathbf{V} = \mathbf{V}_0$ and $\mathbf{P} = \mathbf{P}_{\rm CGL}$,

$$\begin{aligned}\mathbf{V}_\perp &= \frac{c\mathbf{E}\times\mathbf{h}}{B} + \frac{c}{enB}\mathbf{h}\times\left(\nabla\cdot(mn\mathbf{V}_0\mathbf{V}_0 + \mathbf{P}_{\rm CGL})\right)\\ &= \frac{c\mathbf{E}\times\mathbf{h}}{B} + \frac{c}{enB}\left(\mathbf{h}\times\nabla p_\perp + \left(mnV_\|^2 + p_\| - p_\perp\right)\mathbf{h}\times(\mathbf{h}\cdot\nabla)\,\mathbf{h}\right).\end{aligned} \tag{6}$$

Let us compute now fluid velocity from the gyrokinetic distribution $f(\mathbf{r}_g, v_\perp, v_\|)$ using

$$\mathbf{v} = \mathbf{v}_L + \mathbf{v}_g, \qquad \mathbf{r} = \mathbf{r}_g + \boldsymbol{\rho}_L, \tag{7}$$

where

$$\mathbf{v}_L = \mathbf{v}_L(\mathbf{r}_g) = v_\perp\left(\mathbf{n}(\mathbf{r}_g)\cos\phi + \mathbf{h}(\mathbf{r}_g)\times\mathbf{n}(\mathbf{r}_g)\sin\phi\right), \qquad \boldsymbol{\rho}_L = \boldsymbol{\rho}_L(\mathbf{r}_g) = \frac{\mathbf{h}(\mathbf{r}_g)\times\mathbf{v}_L}{\omega_c(\mathbf{r}_g)}, \tag{8}$$

where $\mathbf{n}$ is unit vector orthogonal to $\mathbf{h}$, ϕ is a gyrophase and the guiding center velocity is

$$\mathbf{v}_g = \mathbf{v}_g(\mathbf{r}_g) = v_\|\mathbf{h} + \frac{c\mathbf{E}\times\mathbf{h}}{B} + \frac{v_\perp^2}{2\omega_c}\frac{\mathbf{h}\times\nabla\mathbf{B}}{B} + \frac{v_\|^2}{\omega_c}\mathbf{h}\times(\mathbf{h}\cdot\nabla)\,\mathbf{h} \tag{9}$$

with all functions of the coordinates evaluated at the guiding center position. Fluid velocity is

$$
\begin{aligned}
\mathbf{V} &= \int \mathrm{d}^3 r_g \int \mathrm{d}^3 v (\mathbf{v}_L(\mathbf{r}_g) + \mathbf{v}_g(\mathbf{r}_g)) f(\mathbf{r}_g, v_\perp, v_\parallel) \delta(\mathbf{r} - \mathbf{r}_g - \boldsymbol{\rho}_L(\mathbf{r}_g)) \\
&\approx \int \mathrm{d}^3 r_g \int \mathrm{d}^3 v (\mathbf{v}_L(\mathbf{r}_g) + \mathbf{v}_g(\mathbf{r}_g)) f(\mathbf{r}_g, v_\perp, v_\parallel) \left(\delta(\mathbf{r} - \mathbf{r}_g) - \boldsymbol{\rho}_L(\mathbf{r}_g) \cdot \nabla \delta(\mathbf{r} - \mathbf{r}_g)\right) \\
&= \int \mathrm{d}^3 v (\mathbf{v}_L(\mathbf{r}) + \mathbf{v}_g(\mathbf{r})) f(\mathbf{r}, v_\perp, v_\parallel) - \nabla \cdot \int \mathrm{d}^3 v \boldsymbol{\rho}_L(\mathbf{r}) (\mathbf{v}_L(\mathbf{r}) + \mathbf{v}_g(\mathbf{r})) f(\mathbf{r}, v_\perp, v_\parallel) \\
&= \int \mathrm{d}^3 v \mathbf{v}_g(\mathbf{r}) f(\mathbf{r}, v_\perp, v_\parallel) - \nabla \cdot \int \mathrm{d}^3 v \boldsymbol{\rho}_L(\mathbf{r}) \mathbf{v}_L(\mathbf{r}) f(\mathbf{r}, v_\perp, v_\parallel) \\
&\equiv n\mathbf{V}_g + n\mathbf{V}_L.
\end{aligned} \tag{10}
$$

This difference between the fluid velocity $\mathbf{V}$ and flow velocity of guiding centers $\mathbf{V}_g$ is a well known "paradox" (see §5 of Ref. 2). Substituting (9) explicitly we get for the flow velocity of guiding centers

$$
\mathbf{V}_g \equiv \frac{1}{n} \int \mathrm{d}^3 v \mathbf{v}_g(\mathbf{r}) f(\mathbf{r}, v_\perp, v_\parallel) = V_\parallel \mathbf{h} + \frac{c\mathbf{E} \times \mathbf{h}}{B} + \frac{p_\perp}{mn\omega_c} \frac{\mathbf{h} \times \nabla \mathbf{B}}{B} + \frac{mnV_\parallel^2 + p_\parallel}{mn\omega_c} \mathbf{h} \times (\mathbf{h} \cdot \nabla) \mathbf{h}, \tag{11}
$$

and for the Larmor gyration flow velocity

$$
\mathbf{V}_L \equiv -\frac{1}{n} \nabla \cdot \int \mathrm{d}^3 v \mathbf{v}_L(\mathbf{r}) f(\mathbf{r}, v_\perp, v_\parallel) = -\frac{1}{mn} \nabla \times \frac{p_\perp \mathbf{h}}{\omega_c}. \tag{12}
$$

For the sum we get

$$
\mathbf{V}_g + \mathbf{V}_L = \frac{c\mathbf{E} \times \mathbf{h}}{B} + \frac{\mathbf{h} \times \nabla p_\perp}{mn\omega_c} + \frac{mnV_\parallel^2 + p_\parallel - p_\perp}{mn\omega_c} \mathbf{h} \times (\mathbf{h} \cdot \nabla) \mathbf{h} + \mathbf{h} \left(V_\parallel + \frac{p_\perp}{mn\omega_c} \mathbf{h} \cdot \nabla \times \mathbf{h} \right) \tag{13}
$$

where we used

$$
\nabla \times \mathbf{h} = \mathbf{h} \times (\mathbf{h} \cdot \nabla) \mathbf{h} + \mathbf{h}\mathbf{h} \cdot \nabla \times \mathbf{h}.
$$

Expression (13) differs from $\mathbf{V}_\perp + \mathbf{V}_0$ with $\mathbf{V}_\perp$ given by (6) by re-definition of the parallel velocity $V_\parallel$ in (4) by the term linear in Larmor radius. Difference of quantity $V_\parallel$ defined by (4) from actual parallel fluid velocity is because the guding center vaiable $v_\parallel$ consides with parallel velocity only in the leading order. Since Larmor gyration flow is divergence free, $\nabla \cdot (n\mathbf{V}_L) = 0$, a steady state guiding center flow is divergence free too, $\nabla \cdot (n\mathbf{V}_g) = 0$, what follows also directly from the steady state gyrokinetic equation.

II. IDEAL MHD EQUILIBRIUM

Ideal force balance equation is obtained from (5) ignoring there the inertial term $(m \to 0)$ what coresponds to slow rotations, setting in te CLG tensor (1) $p_\perp = p_\parallel = p$ and suming up the species assuming qusineutrality,

$$
\sum_{species} en = 0, \qquad \sum_{species} en\mathbf{V} = \mathbf{j}, \qquad \sum_{species} p \to p,
$$

resulting in

$$\nabla p = \frac{1}{c}\mathbf{j} \times \mathbf{B}. \tag{14}$$

Perpendicular current density is obtained directly from (14) or summing up the sepcies in (6) under the above assumtions,

$$\mathbf{j}_\perp = \sum_{species} en\mathbf{V}_\perp = \frac{c}{B}\mathbf{h} \times \nabla p. \tag{15}$$

Since scalar product of (14) with $\mathbf{h}$ is $\mathbf{h} \cdot \nabla p = 0$, pressure is a flux function, $p = p(\psi)$, and profile of the pressure is one of the inputs in equilibrium comutations. Parallel curent is obtained from the steady state condition

$$\nabla \cdot \mathbf{j} = \nabla \cdot \left(\mathbf{j}_\perp + j_\parallel \mathbf{h}\right), \tag{16}$$

which results in magneic differential equation for $j_\parallel$,

$$\mathbf{B} \cdot \nabla \frac{j_\parallel}{B} = -\nabla \cdot \frac{c}{B}\mathbf{h} \times \nabla p. \tag{17}$$

Solution to this equation, $j_\parallel/B$ is determined up to a free constant,

$$j_\parallel = j_\parallel^{\mathrm{PS}} + B\frac{\langle j_\parallel B\rangle}{\langle B^2\rangle}, \qquad \langle j_\parallel^{\mathrm{PS}} B\rangle = 0, \tag{18}$$

where the profile of flux function $\langle j_\parallel B\rangle$ is another input provided from the neoclassical computation.

A. Linearized ideal MHD equilibrium

We assume a slightly perturbed field $\mathbf{B} = \mathbf{B}_0 + \delta\mathbf{B}$. Pertubed pressure $p = p_0 + \delta p$ is constant along its field lines what results for linear order pressure perturbation in magnetic differential equation (MDE),

$$0 = \mathbf{B} \cdot \nabla p \approx \mathbf{B}_0 \cdot \nabla \delta p + \delta\mathbf{B} \cdot \nabla p_0 \quad => \quad \mathbf{B}_0 \cdot \nabla \delta p = -\delta\mathbf{B} \cdot \nabla p_0, \tag{19}$$

where we used $\mathbf{B}_0 \cdot \nabla p_0 = 0$. For the harmonic perturbation,

$$\delta\mathbf{B} = \mathrm{Re}\left(\mathbf{B}_n \mathrm{e}^{in\varphi}\right), \qquad \delta p = \mathrm{Re}\left(p_n \mathrm{e}^{in\varphi}\right), \tag{20}$$

we get MDE as

$$\mathbf{B}_0^{\mathrm{pol}} \cdot \nabla p_n + inB_0^\varphi p_n = \nabla \cdot \left(\mathbf{B}_0^{\mathrm{pol}} p_n\right) + inB_0^\varphi p_n = B_0^\vartheta \frac{\partial p_n}{\partial \vartheta} + inB_0^\varphi p_n = -\mathbf{B}_n \cdot \nabla p_0 = -p_0' B_n^{\psi_0}, \tag{21}$$

where we used the property that poloidal field is divergence free in the axisymmeric system, $\nabla \cdot \mathbf{B}_0^{\mathrm{pol}} = 0$. Here poloidal and toroidal part of any vector $\mathbf{A}$ is

$$\mathbf{A}^{\mathrm{pol}} = \mathbf{A} - \frac{\partial \mathbf{r}}{\partial \varphi} A^\varphi = \mathbf{A} - \frac{1}{R^2} A^\varphi \nabla \varphi, \qquad A^\varphi = \mathbf{A} \cdot \nabla \varphi. \tag{22}$$

Note that $B_0^\varphi = B_{0\varphi}/R^2$ where $B_{0\varphi} = B_{0\varphi}(\psi_0)$ is a flux function. For $n \neq 0$ MDE (21) has a unique periodic solution. Form of MDE (21) suitable for the *FreeFEM* is form of a conservation law,

$$\nabla \cdot \left(\mathbf{B}_0^{\mathrm{pol}} p_n \right) + in B_0^\varphi p_n = -p_0' B_n^{\psi_0}, \tag{23}$$

where $p_0' = \mathrm{d}p_0/\mathrm{d}\psi_0 = p_0'(\psi_0)$ and normal component $B_n^{\psi_0}$ is already an input for the MC code. Flux function $B_{0\varphi}$ is also available in the grid data and the poloidal field components in cylindrical coordinates,

$$B_0^R = -\frac{1}{R}\frac{\partial A_{0\varphi}}{\partial Z}, \qquad B_0^Z = \frac{1}{R}\frac{\partial A_{0\varphi}}{\partial R}, \tag{24}$$

can be easily computed from $\psi_{\mathrm{pol}} = -A_{0\varphi}$ stored in the grid nodes (exact first and second derivatives of this quantity are also available in the magetic code generating the grid input). For comutations of current density we lineaize first the unit vector along $\mathbf{B}$,

$$\mathbf{h} = \frac{\mathbf{B}_0 + \delta\mathbf{B}}{\sqrt{B_0^2 + 2\mathbf{B}_0 \cdot \delta\mathbf{B} + \delta\mathbf{B}^2}} \approx \mathbf{h}_0 + \frac{\delta\mathbf{B}_\perp}{B_0}, \qquad \mathbf{h}_0 = \frac{\mathbf{B}_0}{B_0}, \qquad \delta\mathbf{B}_\perp = \delta\mathbf{B} - \mathbf{h}_0\mathbf{h}_0 \cdot \delta\mathbf{B}. \tag{25}$$

Now we linearize current density,

$$\mathbf{j} - \mathbf{j}_0 = (j_{0\|} + \delta j_\|')\mathbf{h} + \mathbf{j}_\perp - j_{0\|}\mathbf{h}_0 - \mathbf{j}_{0\perp} \approx \delta j_\|'\mathbf{h}_0 + j_{0\|}\frac{\delta\mathbf{B}_\perp}{B_0} + \delta\mathbf{j}_\perp' \equiv \delta\mathbf{j}, \tag{26}$$

where

$$\delta\mathbf{j}_\perp' = \frac{c}{B_0^2}\delta\mathbf{B}_\perp \times \nabla p_0 - \frac{c\mathbf{h}_0 \cdot \delta\mathbf{B}}{B_0^2}\mathbf{h}_0 \times \nabla p_0 + \frac{c}{B_0}\mathbf{h}_0 \times \nabla\delta p. \tag{27}$$

Here prime on components of the current does not mean the radial derivative (in contrast to p_0' in Eqs. (21) and (23)) but denotes that subscripts $\perp$ and $\|$ do not mean here that respective primed quantity is perpendicular or parallel to the unperturbed field $\mathbf{B}_0$, i.e. $\delta j_\|' \neq \mathbf{h}_0 \cdot \delta\mathbf{j}$ and $\delta\mathbf{j}_\perp' \neq \delta\mathbf{j} - \mathbf{h}_0\mathbf{h}_0 \cdot \delta\mathbf{j}$. It is convenient to re-define components of the current so that they would really be perpendicular and parallel to $\mathbf{B}_0$,

$$\delta\mathbf{j} = \delta j_\|\mathbf{h}_0 + \delta\mathbf{j}_\perp, \tag{28}$$

$$\delta j_\| = \delta j_\|' + \mathbf{h}_0 \cdot \delta\mathbf{j}_\perp' \tag{29}$$

$$\begin{aligned} \delta\mathbf{j}_\perp &= j_{0\|}\frac{\delta\mathbf{B}_\perp}{B_0} + \delta\mathbf{j}_\perp' - \mathbf{h}_0\mathbf{h}_0 \cdot \delta\mathbf{j}_\perp' \\ &= j_{0\|}\frac{\delta\mathbf{B}_\perp}{B_0} - \frac{c\mathbf{h}_0 \cdot \delta\mathbf{B}}{B_0^2}\mathbf{h}_0 \times \nabla p_0 + \frac{c}{B_0}\mathbf{h}_0 \times \nabla\delta p. \end{aligned} \tag{30}$$

Last expression results from the fact that the first term in (27) is purely parallel to $\mathbf{h}_0$ while the rest two are purely perpendicular.

The unknown $\delta j_\|$ is, as usually, obtained from $\nabla \cdot \delta\mathbf{j} = 0$,

$$\mathbf{B}_0 \cdot \nabla\frac{\delta j_\|}{B_0} = -\nabla \cdot \delta\mathbf{j}_\perp. \tag{31}$$

Looking again for the harmonic perturbation,

$$\delta j_\parallel = \mathrm{Re}\left(j_{\parallel n}\mathrm{e}^{in\varphi}\right), \qquad \delta \mathbf{j}_\perp = \mathrm{Re}\left(\mathbf{j}_{\perp n}\mathrm{e}^{in\varphi}\right), \tag{32}$$

where

$$\mathbf{j}_{\perp n} = j_{0\parallel}\frac{\mathbf{B}_{\perp n}}{B_0} - \frac{cB_{\parallel n}}{B_0^2}\mathbf{h}_0 \times \nabla p_0 + \frac{c}{B_0}\mathbf{h}_0 \times \left(\nabla p_n + inp_n\nabla\varphi\right), \tag{33}$$

$$\mathbf{B}_{\perp n} = \left(\mathbf{B}_n\right)_\perp = \mathbf{B}_n - \mathbf{h}_0\mathbf{h}_0\cdot\mathbf{B}_n, \qquad B_{\parallel n} = \left(\mathbf{B}_n\right)_\parallel = \mathbf{h}_0\cdot\mathbf{B}_n, \tag{34}$$

magnetic differential equation (31) in the form of the coservation law is

$$\nabla\cdot\left(\mathbf{h}_0^{\mathrm{pol}} j_{\parallel n}\right) + inh_0^\varphi j_{\parallel n} = -\nabla\cdot\mathbf{j}_{\perp n}^{\mathrm{pol}} - inj_{\perp n}^\varphi. \tag{35}$$

Here poloidal and toroidal components of various vectors are defined in (22).

B. Straight cylinder geometry

An example where magnetic differential equations are algebraic and there is no mode coupling is straight cylinder geometry (generaly MDE are algebraic in straight field line flux coordinates but there is a mode coupling). Formally cylindrical variables (r,ϑ,z) associated with cylinder axis (therefore z is small here in order not to confuse with Z variable of cylindrical variables associated with main axis of te torus) are expressed via flux coordinates (r,ϑ,φ) relaing $z = R_0\varphi$ so that perturbations are periodic with period $2\pi R_0$ (here R_0 is a constant parameter). Metric determinant of such flux coordinates is simply $\sqrt{g} = rR_0$ (like in quasitoroidal coordinate system with the only difference that $R_0 = const$). In this case equilibrium quantities are functions of $\psi_0 = r$ only, and dependence of perturbed quntities on the polidal angle is harmonic, $\propto \exp(im\vartheta)$. Then,

$$\nabla\cdot\left(\mathbf{B}_0^{\mathrm{pol}}A\right) = imB_0^\vartheta A,$$

and the solution to first MDE, Eq. (23), is

$$p_n = \frac{ip_0'B_n^r}{B_0^\vartheta(m+nq)}. \tag{36}$$

MDE (35) can be simplified if one takes into account the zero order equilibrium,

$$\mathbf{h}_0\cdot\nabla B_0 = 0, \qquad \mathbf{h}_0\cdot\nabla\frac{j_{0\parallel}}{B_0} = 0, \qquad \nabla\cdot\left(\mathbf{B}_0\times\nabla p_0\right) = \frac{4\pi}{c}\mathbf{j}_0\cdot\nabla p_0 = 0, \qquad \nabla r\cdot\nabla\times\frac{\mathbf{h}_0}{B_0} = 0 \tag{37}$$

where first two relations and the last relation are valid for the cylinder only,

$$\begin{aligned} i(m+nq)h_0^\vartheta\left(j_{\parallel n} - j_{0\parallel}\frac{B_{\parallel n}}{B_0}\right) &= -B_n^r\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{j_{0\parallel}}{B_0}\right) + \frac{icB_{\parallel n}}{B_0^2}\mathbf{h}_0\times\nabla p_0\cdot\left(m\nabla\vartheta + n\nabla\varphi\right) \\ &\quad - icp_n\left(m\nabla\vartheta + n\nabla\varphi\right)\cdot\nabla\times\frac{\mathbf{h}_0}{B_0} \qquad (38) \\ &= -B_n^r\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{j_{0\parallel}}{B_0}\right) + \frac{icB_{\parallel n}p_0'}{rR_0B_0^2}\left(mh_{0\varphi} - nh_{0\vartheta}\right) + \frac{icp_n}{rR_0}\frac{\mathrm{d}}{\mathrm{d}r}\frac{mh_{0\varphi} - nh_{0\vartheta}}{B_0}. \end{aligned}$$

We see that parallel curent density $j_{\parallel n}$ is determned by B^r_n and $B_{\parallel n}$ via an algebraic relation. Radial component of the perturbed current density follows from (33) as

$$j^r_n = B^r_n \frac{j_{0\parallel}}{B_0} - \frac{icp_n}{rR_0B_0}\left(mh_{0\varphi} - nh_{0\vartheta}\right), \tag{39}$$

i.e. it is proportional to B^r_n.

If there is not mistake, Maxwell equations should reduce to a single second order ODE for the component B^r_n, i.e. to the equation of Furth[3] (this equation is also given by (5) and (6) in Ref. 4). We need only three Maxwell equations,

$$\frac{4\pi}{c} j^r_n = \left(\nabla \times \delta\mathbf{B}\right)_n \cdot \nabla r, \qquad \frac{4\pi}{c} j_{\parallel n} = \left(\nabla \times \delta\mathbf{B}\right)_n \cdot \mathbf{h}_0, \qquad \left(\nabla \cdot \delta\mathbf{B}\right)_n = 0 \tag{40}$$

because the thrid component of the curl is redundant - it means just $\nabla \cdot \delta\mathbf{j} = 0$ what has aleady been used in our derivation. Currents in these equations are given by (39) and (38). Explicitly Eqs. (40) are

$$\begin{aligned}
\frac{4\pi r R_0}{c} j^r_n &= i\left(mB_{\varphi n} - nB_{\vartheta n}\right), \\
\frac{4\pi r R_0}{c} j_{\parallel n} &= i(nh_{0\vartheta} - mh_{0\varphi})B^r_n - h_{0\vartheta}\frac{\partial}{\partial r}B_{\varphi n} + h_{0\varphi}\frac{\partial}{\partial r}B_{\vartheta n} \\
&= i(nh_{0\vartheta} - mh_{0\varphi})B^r_n + \frac{\partial}{\partial r}\left(h_{0\varphi}B_{\vartheta n} - h_{0\vartheta}B_{\varphi n}\right) + h'_{0\vartheta}B_{\varphi n} - h'_{0\varphi}B_{\vartheta n}, \\
0 &= \frac{1}{r}\frac{\partial}{\partial r} r B^r_n + i\left(\frac{m}{r^2}B_{\vartheta n} + \frac{n}{R_0^2}B_{\varphi n}\right).
\end{aligned} \tag{41}$$

It is helpful to expess poloidal and toroidal components of perturbation field as follows

$$\begin{aligned}
B_{\vartheta n} &= h_{0\vartheta}B_{\parallel n} + h^\varphi_0\left(h_{0\varphi}B_{\vartheta n} - h_{0\vartheta}B_{\varphi n}\right), \\
B_{\varphi n} &= h_{0\varphi}B_{\parallel n} - h^\vartheta_0\left(h_{0\varphi}B_{\vartheta n} - h_{0\vartheta}B_{\varphi n}\right),
\end{aligned} \tag{42}$$

and first eliminate combination $h_{0\varphi}B_{\vartheta n} - h_{0\vartheta}B_{\varphi n} \equiv rR_0\left(\mathbf{B}_n \times \mathbf{h}_0\right)^r$ which does not enter current components used in the set. Note that all expressions here can be easily transformed from flux coordinates to the associated cylindrical coordinates using the notation

$$n = k_z R_0, \qquad A_\varphi = R_0 A_z, \qquad A^\varphi = \frac{1}{R_0}A_z. \tag{43}$$

With such a re-notation quantity R_0 vanishes from equations.

Introducing vector $\mathbf{k}$ which is tangential to the flux surface,

$$\mathbf{k} = m\nabla\vartheta + n\nabla\varphi = m\nabla\vartheta + k_z\nabla z, \qquad k_z = \frac{n}{R_0}, \tag{44}$$

we can denote

$$\mathbf{h}_0 \cdot \mathbf{k} = mh^\vartheta_0 + nh^\varphi_0 = (m + nq)h^\vartheta_0, \qquad \left(\mathbf{h}_0 \times \mathbf{k}\right)^r = \frac{nh_{0\vartheta} - mh_{0\varphi}}{rR_0}. \tag{45}$$

Since radial component is the only non-zero component of $\mathbf{h}_0 \times \mathbf{k}$ we have also

$$(\mathbf{h}_0 \cdot \mathbf{k})^2 + ((\mathbf{h}_0 \times \mathbf{k})^r)^2 = k^2 = \frac{m^2}{r^2} + k_z^2. \tag{46}$$

In this notation (36) and (39) take the form

$$\begin{aligned} p_n &= \frac{i p_0' B_n^r}{B_0 \mathbf{h}_0 \cdot \mathbf{k}}, \\ j_n^r &= B_n^r \frac{j_{0\|}}{B_0} + \frac{i c p_n}{B_0} (\mathbf{h}_0 \times \mathbf{k})^r = \left(j_{0\|} - \frac{c p_0'}{B_0} \frac{(\mathbf{h}_0 \times \mathbf{k})^r}{\mathbf{h}_0 \cdot \mathbf{k}} \right) \frac{B_n^r}{B_0} \\ &= \frac{\mathbf{j}_0 \cdot \mathbf{k}}{\mathbf{h}_0 \cdot \mathbf{k}} \frac{B_n^r}{B_0}, \end{aligned} \tag{47}$$

where

$$\mathbf{j}_0 = j_{0\|} \mathbf{h}_0 + \frac{c}{B_0} \mathbf{h}_0 \times \nabla p_0 \tag{48}$$

is the total equilibrium current density. First and last Maxwell equations (41) take the form

$$\begin{aligned} \frac{4\pi i}{c} j_n^r &= \mathbf{h}_0 \cdot \mathbf{k} \ (\mathbf{B}_n \times \mathbf{h}_0)^r + (\mathbf{h}_0 \times \mathbf{k})^r B_{\|n}, \\ \frac{i}{r} \frac{\partial}{\partial r} r B_n^r &= - (\mathbf{h}_0 \times \mathbf{k})^r (\mathbf{B}_n \times \mathbf{h}_0)^r + \mathbf{h}_0 \cdot \mathbf{k} \ B_{\|n}, \end{aligned} \tag{49}$$

which allows to express $B_{\|n}$ and $(\mathbf{B}_n \times \mathbf{h}_0)^r$ via B_n^r and its derivative (system determinant is given by (46)):

$$\begin{aligned} (\mathbf{B}_n \times \mathbf{h}_0)^r &= \frac{1}{k^2} \left(\frac{4\pi i}{c} \mathbf{h}_0 \cdot \mathbf{k} \ j_n^r - i (\mathbf{h}_0 \times \mathbf{k})^r \frac{1}{r} \frac{\partial}{\partial r} r B_n^r \right), \\ B_{\|n} &= \frac{1}{k^2} \left(\frac{4\pi i}{c} (\mathbf{h}_0 \times \mathbf{k})^r j_n^r + i \mathbf{h}_0 \cdot \mathbf{k} \frac{1}{r} \frac{\partial}{\partial r} r B_n^r \right). \end{aligned} \tag{50}$$

Equation (38) for the parallel current density takes the form

$$\mathbf{h}_0 \cdot \mathbf{k} \left(j_{\|n} - j_{0\|} \frac{B_{\|n}}{B_0} \right) = i B_n^r \frac{\mathrm{d}}{\mathrm{d}r} \left(\frac{j_{0\|}}{B_0} \right) - \frac{c B_{\|n} p_0'}{B_0^2} (\mathbf{h}_0 \times \mathbf{k})^r - \frac{c p_n}{r} \frac{\mathrm{d}}{\mathrm{d}r} \frac{r (\mathbf{h}_0 \times \mathbf{k})^r}{B_0}. \tag{51}$$

Finally, second of Maxwell equations (41) is

$$\begin{aligned} \frac{4\pi}{c} j_{\|n} &= i (\mathbf{h}_0 \times \mathbf{k})^r B_n^r + \frac{1}{r} \frac{\partial}{\partial r} r (\mathbf{B}_n \times \mathbf{h}_0)^r + \frac{h_{0\vartheta}' h_{0\varphi} - h_{0\varphi}' h_{0\vartheta}}{r R_0} B_{\|n} - \left(h_{0\vartheta}' h_0^\vartheta + h_{0\varphi}' h_0^\varphi \right) (\mathbf{B}_n \times \mathbf{h}_0)^r, \\ &= i (\mathbf{h}_0 \times \mathbf{k})^r B_n^r + \frac{1}{r} \frac{\partial}{\partial r} r (\mathbf{B}_n \times \mathbf{h}_0)^r + \frac{4\pi j_{0\|}}{c} \frac{B_{\|n}}{B_0} - \frac{1}{r} \hat{h}_{0\vartheta}^2 (\mathbf{B}_n \times \mathbf{h}_0)^r \\ &= i (\mathbf{h}_0 \times \mathbf{k})^r B_n^r + \frac{\partial}{\partial r} (\mathbf{B}_n \times \mathbf{h}_0)^r + \frac{4\pi j_{0\|}}{c} \frac{B_{\|n}}{B_0} + \frac{1}{r} h_{0z}^2 (\mathbf{B}_n \times \mathbf{h}_0)^r, \end{aligned} \tag{52}$$

where $\hat{h}_{0\vartheta}$ means the physical component, $h_{0z} = \hat{h}_{0z}$, and we used Maxwell equations for the equilibrium field,

$$\frac{h_{0\vartheta}' h_{0\varphi} - h_{0\varphi}' h_{0\vartheta}}{r R_0} = \frac{B_{0\vartheta}' h_{0\varphi} - B_{0\varphi}' h_{0\vartheta}}{r R_0 B_0} = \frac{4\pi}{c B_0} (j_0^\varphi h_{0\varphi} + j_0^\vartheta h_{0\vartheta}) = \frac{4\pi}{c B_0} j_{0\|}.$$

Thus (52) is

$$\frac{4\pi}{c}\left(j_{\|n}-j_{0\|}\frac{B_{\|n}}{B_0}\right)=i\left(\mathbf{h}_0\times\mathbf{k}\right)^r B_n^r+\frac{\partial}{\partial r}\left(\mathbf{B}_n\times\mathbf{h}_0\right)^r+\frac{1}{r}h_{0z}^2\left(\mathbf{B}_n\times\mathbf{h}_0\right)^r. \tag{53}$$

Substituting here $j_{\|n}$ from (51) where p_n has been substituted from (47) we get

$$\begin{aligned}&\frac{4\pi}{c}\left(iB_n^r\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{j_{0\|}}{B_0}\right)-\frac{cB_{\|n}p_0'}{B_0^2}\left(\mathbf{h}_0\times\mathbf{k}\right)^r-\frac{icB_n^r p_0'}{B_0}\frac{1}{\mathbf{h}_0\cdot\mathbf{k}}\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\frac{r\left(\mathbf{h}_0\times\mathbf{k}\right)^r}{B_0}\right)\\&=\mathbf{h}_0\cdot\mathbf{k}\left(i\left(\mathbf{h}_0\times\mathbf{k}\right)^r B_n^r+\frac{\partial}{\partial r}\left(\mathbf{B}_n\times\mathbf{h}_0\right)^r+\frac{1}{r}h_{0z}^2\left(\mathbf{B}_n\times\mathbf{h}_0\right)^r\right).\end{aligned} \tag{54}$$

Substituting here tangential components of the magnetic field (50) and using explicit expression for the radial current (47) we get a second order ODE for B_n^r.

$$\begin{aligned}&\frac{4\pi}{c\mathbf{h}_0\cdot\mathbf{k}}\left[B_n^r\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{j_{0\|}}{B_0}\right)-\frac{cp_0'}{B_0^2}\frac{\left(\mathbf{h}_0\times\mathbf{k}\right)^r}{k^2}\left(\frac{4\pi}{c}\left(\mathbf{h}_0\times\mathbf{k}\right)^r\frac{\mathbf{j}_0\cdot\mathbf{k}}{\mathbf{h}_0\cdot\mathbf{k}}\frac{B_n^r}{B_0}+\mathbf{h}_0\cdot\mathbf{k}\frac{1}{r}\frac{\partial}{\partial r}rB_n^r\right)\right.\\&\left.-\frac{cB_n^r p_0'}{B_0}\frac{1}{\mathbf{h}_0\cdot\mathbf{k}}\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\frac{r\left(\mathbf{h}_0\times\mathbf{k}\right)^r}{B_0}\right]-\left(\mathbf{h}_0\times\mathbf{k}\right)^r B_n^r-\frac{\partial}{\partial r}\frac{1}{k^2}\left(\frac{4\pi}{c}\mathbf{j}_0\cdot\mathbf{k}\frac{B_n^r}{B_0}-\left(\mathbf{h}_0\times\mathbf{k}\right)^r\frac{1}{r}\frac{\partial}{\partial r}rB_n^r\right)\\&-\frac{h_{0z}^2}{rk^2}\left(\frac{4\pi}{c}\mathbf{j}_0\cdot\mathbf{k}\frac{B_n^r}{B_0}-\left(\mathbf{h}_0\times\mathbf{k}\right)^r\frac{1}{r}\frac{\partial}{\partial r}rB_n^r\right)=0.\end{aligned} \tag{55}$$

We regroup the terms in the following way,

$$\left(\mathbf{h}_0\times\mathbf{k}\right)^r\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{r}{k^2}\frac{\partial B_n^r}{\partial r}\right)+\frac{1}{k^2}C_1\frac{\partial B_n^r}{\partial r}+C_0B_n^r=0, \tag{56}$$

where

$$\begin{aligned}C_1&=\frac{\partial}{\partial r}\left(\mathbf{h}_0\times\mathbf{k}\right)^r-\frac{4\pi}{cB_0}\mathbf{j}_0\cdot\mathbf{k}+\frac{h_{0z}^2}{r}\left(\mathbf{h}_0\times\mathbf{k}\right)^r-\frac{4\pi}{c}\frac{cp_0'}{B_0^2}\left(\mathbf{h}_0\times\mathbf{k}\right)^r\\&=\frac{\partial}{\partial r}\left(\mathbf{h}_0\times\mathbf{k}\right)^r+\frac{h_{0z}^2}{r}\left(\mathbf{h}_0\times\mathbf{k}\right)^r-\frac{4\pi}{cB_0}j_{\|0}\mathbf{h}_0\cdot\mathbf{k}\\&=\frac{\partial}{\partial r}\left(\mathbf{h}_0\times\mathbf{k}\right)^r+\frac{h_{0z}^2}{r}\left(\mathbf{h}_0\times\mathbf{k}\right)^r-\mathbf{h}_0\cdot\mathbf{k}\,\mathbf{h}_0\cdot\nabla\times\mathbf{h}_0=0,\end{aligned} \tag{57}$$

where we used explicit expression (48) for $\mathbf{j}_0$ and Maxwell equation for $j_{\|0}$, and the last equality can be checked in components noting that vector $\mathbf{k}$ is a gradient and using $|\mathbf{h}_0|=1$. For the constant C_0 we get

$$\begin{aligned}C_0=&\frac{4\pi}{c\mathbf{h}_0\cdot\mathbf{k}}\left[\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{j_{0\|}}{B_0}\right)-\frac{cp_0'}{B_0^2}\frac{\left(\mathbf{h}_0\times\mathbf{k}\right)^r}{k^2}\left(\frac{4\pi}{c}\left(\mathbf{h}_0\times\mathbf{k}\right)^r\frac{\mathbf{j}_0\cdot\mathbf{k}}{\mathbf{B}_0\cdot\mathbf{k}}+\frac{1}{r}\mathbf{h}_0\cdot\mathbf{k}\right)\right.\\&\left.-\frac{cp_0'}{\mathbf{B}_0\cdot\mathbf{k}}\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\frac{r\left(\mathbf{h}_0\times\mathbf{k}\right)^r}{B_0}\right]-\left(\mathbf{h}_0\times\mathbf{k}\right)^r-\frac{\partial}{\partial r}\frac{1}{k^2}\left(\frac{4\pi}{c}\mathbf{j}_0\cdot\mathbf{k}\frac{1}{B_0}-\left(\mathbf{h}_0\times\mathbf{k}\right)^r\frac{1}{r}\right)\\&-\frac{h_{0z}^2}{rk^2}\left(\frac{4\pi}{c}\mathbf{j}_0\cdot\mathbf{k}\frac{1}{B_0}-\left(\mathbf{h}_0\times\mathbf{k}\right)^r\frac{1}{r}\right).\end{aligned} \tag{58}$$

Since $\mathbf{k}$ is a gradient, we can write

$$\frac{4\pi}{c}\mathbf{j}_0\cdot\mathbf{k}=\mathbf{k}\cdot\nabla\times\mathbf{B}_0=\nabla\cdot\mathbf{B}_0\times\mathbf{k}=\frac{1}{r}\frac{\partial}{\partial r}r\left(\mathbf{B}_0\times\mathbf{k}\right)^r. \tag{59}$$

Therefore

$$C_0 = \frac{4\pi}{c\mathbf{h}_0\cdot\mathbf{k}}\left[\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{j_{0\|}}{B_0}\right) - \frac{cp_0'}{B_0^2}\frac{(\mathbf{h}_0\times\mathbf{k})^r}{k^2}\left(\frac{4\pi}{c}(\mathbf{h}_0\times\mathbf{k})^r\frac{\mathbf{j}_0\cdot\mathbf{k}}{\mathbf{B}_0\cdot\mathbf{k}} + \frac{1}{r}\mathbf{h}_0\cdot\mathbf{k}\right)\right.$$
$$\left. - \frac{cp_0'}{\mathbf{B}_0\cdot\mathbf{k}}\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\frac{r(\mathbf{h}_0\times\mathbf{k})^r}{B_0}\right] - (\mathbf{h}_0\times\mathbf{k})^r - \frac{\partial}{\partial r}\frac{1}{k^2B_0}\frac{\partial}{\partial r}(\mathbf{B}_0\times\mathbf{k})^r$$
$$- \frac{h_{0z}^2}{rk^2B_0}\frac{\partial}{\partial r}(\mathbf{B}_0\times\mathbf{k})^r. \tag{60}$$

In order that Eq. (56) is the same with Eq. (5) of Ref. 4 (which is a slightly re-notated equation of Furth et al[3]) constant C_0 must be the same with

$$C_0^F = (\mathbf{h}_0\times\mathbf{k})^r\left[\frac{\partial}{\partial r}\frac{1}{rk^2} - 1 - \frac{r}{\mathbf{k}\cdot\mathbf{B}_0}\left(\frac{\partial}{\partial r}\frac{1}{rk^2}\frac{4\pi}{c}(\mathbf{k}\times\mathbf{j}_0)^r + \frac{2}{r^2k^2}\left(k_zB_{0z}' + \frac{4\pi k_z^2}{\mathbf{k}\cdot\mathbf{B}_0}p_0'\right)\right)\right] \tag{61}$$

III. LINEAR δf METHOD

A. Full-f method, definitions

We present the kinetic equation in the form

$$\frac{\partial f(t,\mathbf{z})}{\partial t} + V^i(\mathbf{z})\frac{\partial f(t,\mathbf{z})}{\partial z^i} - \hat{L}_c f(t,\mathbf{z}) = 0, \tag{62}$$

where $\hat{L}_c$ is the linearized collision operator, $V^i(\mathbf{z})$ is the phase space velocity which obeys the Liouville's theorem

$$\frac{\partial}{\partial z^i}J(\mathbf{z})V^i(\mathbf{z}) = 0, \tag{63}$$

and $J(\mathbf{z})$ is the phase space Jacobian. For the Monte Carlo modelling, we re-write (62) in the form of the conservation law replacing the scalar distribution function $f(t,\mathbf{z})$ with a pseudo-scalar distribution $F(t,\mathbf{z}) = J(\mathbf{z})f(t,\mathbf{z})$,

$$\frac{\partial}{\partial t}F(t,\mathbf{z}) + \frac{\partial}{\partial z^i}V^i(\mathbf{z})F(t,\mathbf{z}) - J(\mathbf{z})\hat{L}_c\frac{1}{J(\mathbf{z})}F(t,\mathbf{z}) = 0. \tag{64}$$

Introducing the stochastic orbit $Z^i(\mathbf{z}_0,t)$ originating at $\mathbf{z}_0$, i.e. $Z^i(\mathbf{z}_0,0) = z_0^i$, solution to (64) satisfying the initial condition $F(0,\mathbf{z}) = F_0(\mathbf{z})$ is given by the expectation value

$$F(t,\mathbf{z}) = W_0\overline{\delta(\mathbf{z} - \mathbf{Z}(\tilde{\mathbf{z}}_0,t))}, \tag{65}$$

where

$$\delta(\mathbf{z}) \equiv \delta(z^1)\delta(z^2)\ldots\delta(z^N), \tag{66}$$

N is the phase space dimension and the initial orbit position $\tilde{\mathbf{z}}_0$ is also random distributed with probability density $F_0(\mathbf{z})/W_0$,

$$W_0\overline{\delta(\mathbf{z}-\tilde{\mathbf{z}}_0)} = F_0(\mathbf{z}), \qquad W_0 = \int \mathrm{d}^N z F_0(\mathbf{z}). \tag{67}$$

General procedure to construct the solution (65) can be found in Ref. 5, and here it is demonstrative to check explicitly only the collisonless case where orbits $Z^i(\mathbf{z}_0, t)$ are not stochastic and are the solutions to guiding center equations

$$\frac{\partial}{\partial t} Z^i(\mathbf{z}_0, t) = V^i(\mathbf{Z}(\mathbf{z}_0, t)), \qquad Z^i(\mathbf{z}_0, 0) = z_0^i. \tag{68}$$

Without loss of generality we can set $W_0 = 1$ and assume that initial position $\mathbf{z}_0$ is not random too, so that $F(t, \mathbf{z}) = \delta\left(\mathbf{z} - \mathbf{Z}(\mathbf{z}_0, t)\right)$. Denoting the result of substitution of this solution in the collisionless equation as

$$a(t, \mathbf{z}) = \frac{\partial}{\partial t} F(t, \mathbf{z}) + \frac{\partial}{\partial z^i} V^i(\mathbf{z}) F(t, \mathbf{z})$$

we obtain

$$a(t, \mathbf{z}) = \delta\left(\mathbf{z} - \mathbf{Z}(\mathbf{z}_0, t)\right) \frac{\partial}{\partial z^i} V^i(\mathbf{z}) + \left(V^i(\mathbf{z}) - V^i\left(\mathbf{Z}(\mathbf{z}_0, t)\right)\right) \frac{\partial}{\partial z^i} \delta\left(\mathbf{z} - \mathbf{Z}(\mathbf{z}_0, t)\right).$$

We can see that all phase space moments of this function are zero integrating by parts the product of $a(t, \mathbf{z})$ with arbitrary function $b(\mathbf{z})$

$$\int \mathrm{d}^N z a(t, \mathbf{z}) b(\mathbf{z}) = \int \mathrm{d}^N z \delta\left(\mathbf{z} - \mathbf{Z}(\mathbf{z}_0, t)\right) \left(b(\mathbf{z}) \frac{\partial}{\partial z^i} V^i(\mathbf{z}) - \frac{\partial}{\partial z^i} b(\mathbf{z}) \left(V^i(\mathbf{z}) - V^i\left(\mathbf{Z}(\mathbf{z}_0, t)\right)\right) \right) = 0.$$

B. Linear δf mehod

We assume that stationary solution f_0 of the kinetic equation is known for the unperturbed field,

$$V_0^i(\mathbf{z}) \frac{\partial f_0(\mathbf{z})}{\partial z^i} - \hat{L}_c f_0(\mathbf{z}) = 0. \tag{69}$$

Perturbation of the distribution function δf satisfies in the lowest order to the linearized kinetic equation

$$\frac{\partial \delta f}{\partial t} + V_0^i \frac{\partial \delta f}{\partial z^i} - \hat{L}_c \delta f = -\delta V^i \frac{\partial f_0}{\partial z^i} \equiv \dot{w} f_0, \tag{70}$$

where $f = f_0 + \delta f$ and $V^i = V_0^i + \delta V^i$, and

$$\dot{w}(\mathbf{z}) \equiv -\frac{\delta V^i(\mathbf{z})}{f_0(\mathbf{z})} \frac{\partial f_0(\mathbf{z})}{\partial z^i}. \tag{71}$$

Intruducing pseudo-scalar distrbutions $\delta F = J_0 \delta f$ and $\delta F_0 = J_0 f_0$ where J_0 is the Jacobian of phase space in the unperturbed field Eq. (70) is re-written as

$$\hat{L} \delta F(t, \mathbf{z}) \equiv \frac{\partial}{\partial t} \delta F(t, \mathbf{z}) + \frac{\partial}{\partial z^i} V_0^i(\mathbf{z}) \delta F(t, \mathbf{z}) - J_0(\mathbf{z}) \hat{L}_c \frac{1}{J_0(\mathbf{z})} \delta F(t, \mathbf{z}) = \dot{w}(\mathbf{z}) F_0(\mathbf{z}). \tag{72}$$

Unperturbed pseudoscalar distribution F_0 satisfies

$$\hat{L}F_0(\mathbf{z}) = 0, \tag{73}$$

which is the same with (69) in a steady state. According to Ref. 6 the desired δf algorithm is formulated using the extended phase space $(\mathbf{z}, w)$. Extended distribution function in this space, $F_{\text{ext}}(t, \mathbf{z}, w)$ desribes both, F_0 and δF as follows

$$F_0(t, \mathbf{z}) = \int_{-\infty}^{\infty} \mathrm{d}w \; F_{\text{ext}}(t, \mathbf{z}, w), \tag{74}$$

$$\delta F(t, \mathbf{z}) = \int\limits_{-\infty}^{\infty} \mathrm{d}w \; F_{\text{ext}}(t, \mathbf{z}, w) \; w. \tag{75}$$

Kinetic equation in the extended phase space is

$$\hat{L}F_{\text{ext}}(t, \mathbf{z}, w) + \frac{\partial}{\partial w} V^w(\mathbf{z}) F_{\text{ext}}(t, \mathbf{z}, w) = 0. \tag{76}$$

Multiplying this equation with w and integrating it over w one gets with help of (74) and (75)

$$\hat{L}\delta F(t, \mathbf{z}) = V^w(\mathbf{z}) F_0(t, \mathbf{z}), \tag{77}$$

what results using (72) in

$$V^w(\mathbf{z}) = \dot{w}(\mathbf{z}). \tag{78}$$

Extemding the orbits to $(\mathbf{Z}(\mathbf{z}_0, t), W(\mathbf{z}_0, w_0, t))$ where $\mathbf{Z}(\mathbf{z}_0, t)$ corresponds to the unperturbed field, and extra function W is determined by

$$\frac{\partial}{\partial t} W(\mathbf{z}_0, w_0, t) = \dot{w}\left(\mathbf{Z}(\mathbf{z}_0, t)\right), \qquad W(\mathbf{z}_0, w_0, 0) = w_0, \tag{79}$$

extended distribution is obtained in the same way as (65),

$$F_{\text{ext}}(t, \mathbf{z}) = W_0 \overline{\delta\left(\mathbf{z} - \mathbf{Z}(\tilde{\mathbf{z}}_0, t)\right) \delta\left(w - W(\tilde{\mathbf{z}}_0, w_0, t)\right)}. \tag{80}$$

Integrating it according to (74) we recover (65) and integral (75) gives the desired expression for δF,

$$\delta F(t, \mathbf{z}) = W_0 \overline{\delta\left(\mathbf{z} - \mathbf{Z}(\tilde{\mathbf{z}}_0, t)\right) W(\tilde{\mathbf{z}}_0, w_0, t)}, \tag{81}$$

where $w_0 = 0$ in order not to re-define the unperturbed distribution so that

$$\int \mathrm{d}^N z \; \delta F(t, \mathbf{z}) = 0. \tag{82}$$

Contition (82) is fulfilled due to

$$\int \mathrm{d}^N z \; \dot{w}(\mathbf{z}) F_0(\mathbf{z}_0) = 0.$$

Note that in case $\dot{w}$ is complex real and imaginary parts of δF are treated independently and then are combined in Eq. (79) so that this equation and expession (81) retain their forms.

C. Time scales, regularization, time averaging

Denoting the fundamental solution to (73) (Green's function) with $G(t, \mathbf{z}, \mathbf{z}_0)$ where

$$\hat{L}G(t, \mathbf{z}, \mathbf{z}_0) = 0, \qquad G(0, \mathbf{z}, \mathbf{z}_0) = \delta(\mathbf{z} - \mathbf{z}_0), \tag{83}$$

formal steady state solution to (72) is similar to Eq.(7) of Ref. 5 up to the notation,

$$\delta F(\mathbf{z}) = \int\limits_0^\infty \mathrm{d}t \int \mathrm{d}^N z_0 G(t, \mathbf{z}, \mathbf{z}_0) \dot{w}(\mathbf{z}_0) F_0(\mathbf{z}_0). \tag{84}$$

Infinite time limit here is not necessary in approximate computations because Green's function becomes axisymmetric for $t \gg \tau_{\mathrm{surf}}$, i.e. it becomes independent of both, φ and φ_0, and for our non-axisymmetric source $\dot{w}$ such that

$$\int\limits_{-\pi}^{\pi} \mathrm{d}\varphi \; \dot{w}(\mathbf{z}) = 0$$

contribution of times $t \gg \tau_{\mathrm{surf}}$ tends to zero due to toroidal averaging in (84). Therefore the upper limit (particle tracing time) should be limited to $t_0 \gg \tau_{\mathrm{surf}}$. On the other hand, t_0 should be much smaller than profile relaxation time due to the neoclassical transport τ_{tr} in order that one can use for f_0 prescribed profiles of density and temperature which are not necessarily the same with neoclassical steady state, $t_0 \ll \tau_{\mathrm{tr}}$. Since $\tau_{\mathrm{tr}} \gg \tau_{\mathrm{surf}}$ integration time t_0 can be chosen rather large however this is not good in Monte Carlo method because contribution of $t \gg \tau_{\mathrm{surf}}$ tends o accumulate in particle weight causing the numerical noise which is compensated only statistically. In order to prevent this accumulation in case of rather large t_0 we modify equation for the weight evolution (79) as follows,

$$\frac{\partial}{\partial t} W(\mathbf{z}_0, w_0, t) = \dot{w}\left(\mathbf{Z}(\mathbf{z}_0, t)\right) - \nu_0 W(\mathbf{z}_0, w_0, t), \tag{85}$$

where $\nu_0 \ll 1/\tau_{\mathrm{surf}}$ is the regularization sink rate.

For statistical reasons we one has to compute time average of the solution (81) instead of solution itself. Due to the steady this average is the same with the solution itself in the limit of infinite statistics, but it reduces the CPU cost. Namely, each test orbit in (81) must be followed for $t \gg \tau_{\mathrm{surf}}$ before it is scored and then after evolving for the time interval larger that τ_{surf} orbit can be ragarded as an orbit of a different test particle, i.e. statistically contribution of time average over interval $t_0 \gg \tau_{\mathrm{surf}}$ is of the order of contribution of $t_0/\tau_{\mathrm{surf}} \gg 1$ test particles. Even more important is that time averaging better represents the steady state rather than evolution of initial distribution what gives even much larger factor $t_0/\Delta t$ where Δt is time of traversing the grid cell. Thus we use instead of (81)

$$\delta F(t, \mathbf{z}) = \frac{W_0}{t_0} \int\limits_0^{t_0} \mathrm{d}t \overline{\delta\left(\mathbf{z} - \mathbf{Z}(\tilde{\mathbf{z}}_0, t)\right) W(\tilde{\mathbf{z}}_0, w_0, t)}. \tag{86}$$

D. Harmonic perturbations

In our case of harmonic perturbation

$$\dot{w}(\mathbf{z}) = \dot{w}_A(\mathbf{z}_{\rm pol})\mathrm{e}^{in\varphi}, \tag{87}$$

where $\mathbf{z}_{\rm pol}$ includes all variables except φ, we are interested in Fourier amplitude of the distribution function,

$$\delta F_n(t, \mathbf{z}_{\rm pol}) = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} \mathrm{d}\varphi\ \delta F(t, \mathbf{z})\mathrm{e}^{-in\varphi}. \tag{88}$$

Denoting with $\mathbf{Z}_{\rm pol}$ the "poloidal" projection of the orbit (projection to the $\mathbf{z}_{\rm pol}$ subspace) and with Z^φ the toroidal projection so that $\mathbf{Z} = (\mathbf{Z}_{\rm pol}, Z^\varphi)$ we notice that our unperturbed orits corresponding to the axisymmetric field have the properties

$$\mathbf{Z}_{\rm pol} = \mathbf{Z}_{\rm pol}(\mathbf{z}_{\rm pol}, t), \tag{89}$$

$$Z^\varphi = Z_0^\varphi(\mathbf{z}_{\rm pol}, t) + \varphi, \tag{90}$$

where Z_0 is the orbit starting at $\varphi = 0$. Respectively the weight has the property

$$W(\mathbf{z}, w_0, t) = W_{\rm pol}(\mathbf{z}_{\rm pol}, t)\mathrm{e}^{in\varphi} + w_0\mathrm{e}^{-\nu_0 t}, \tag{91}$$

where $W_{\rm pol}$ corresponds to the orbit starting at $\varphi = 0$ with initial weight $w_0 = 0$. Substituting (86) in (88) Fourier amlitude is obtained as

$$\delta F_n(t, \mathbf{z}_{\rm pol}) = \frac{W_0}{2\pi t_0}\int\limits_0^{t_0} \mathrm{d}t\ \overline{\delta\left(\mathbf{z}_{\rm pol} - \mathbf{Z}_{\rm pol}(\tilde{\mathbf{z}}_{0\rm pol}, t)\right) W_{\rm pol}(\tilde{\mathbf{z}}_{0\rm pol}, t)\exp\left(-inZ_0^\varphi(\tilde{\mathbf{z}}_{0\rm pol}, t)\right)}. \tag{92}$$

Quantities Z_0^φ and $W_{\rm pol}$ have been introduced here to show that the result is independent of the starting toroidal position. Actually one can use here the original Z^φ and W. In case $w_0 = 0$ the result is identical to (92) and will differ otherwise by the statistical noise due to statistical computation of Fourier amplitude of w_0 (which is zero anyway).

E. Collisionless limit

In the collisonless limit, the poloidal projection of the $\mathbf{Z}_{\rm pol}$ is periodic and the toroidal one is quasi-periodic

$$\begin{aligned} Z^i_{\rm pol}(\mathbf{z}_{\rm pol}, t + \tau_b(\mathbf{z}_{0\rm pol})) &= Z^i_{\rm pol}(\mathbf{z}_{\rm pol}, t)), \\ Z_0^\varphi(\mathbf{z}_{\rm pol}, t + \tau_b(\mathbf{z}_{\rm pol})) &= Z_0^\varphi(\mathbf{z}_{\rm pol}, t)) + \Delta\varphi_b(\mathbf{z}_{\rm pol}), \end{aligned} \tag{93}$$

where $\tau_b(\mathbf{z}_{0\mathrm{pol}})$ is bounce time and $\Delta\varphi_b(\mathbf{z}_{\mathrm{pol}})$ is the toroidal displacement during this time. Since collisionless orbits are confined absolutely, transport time τ_{tr} is infinte, and we can take t_0 in (92) as large as we want. Formal solution to Eq. (85) for $\varphi_0 = w_0 = 0$ (i.e. W_{pol}) is

$$W_{\mathrm{pol}}(\mathbf{z}_{0\mathrm{pol}}, t) = \int\limits_0^t \mathrm{d}t' \dot{w}_A \left(\mathbf{Z}_{\mathrm{pol}}(\mathbf{z}_{0\mathrm{pol}}, t')\right) \exp\left(inZ_0^{\varphi}(\mathbf{z}_{0\mathrm{pol}}, t') + \nu_0(t'-t)\right) \equiv \int\limits_0^t \mathrm{d}t' \dot{W}(t'), \tag{94}$$

where we denoted the subintegrand with $\dot{W}$ for brevity. Due to (93) we notice that

$$\dot{W}(t' + \tau_b) = \dot{W}(t') \exp\left(in\Delta\varphi_b + \nu_0\tau_b\right). \tag{95}$$

Thus we split te integral into $K = [t/\tau_b]$ complete bounce periods and a residual incomplete time interval and compute the sum

$$\begin{aligned}
W_{\mathrm{pol}}(\mathbf{z}_{0\mathrm{pol}}, t) &= \int\limits_{t-\tau_b}^{t} \mathrm{d}t' \dot{W}(t') + \int\limits_{t-2\tau_b}^{t-\tau_b} \mathrm{d}t' \dot{W}(t') + \cdots + \int\limits_{t-K\tau_b}^{t-(K-1)\tau_b} \mathrm{d}t' \dot{W}(t') + \int\limits_{0}^{t-K\tau_b} \mathrm{d}t' \dot{W}(t') \\
&= \left(1 + \exp\left(-in\Delta\varphi_b - \nu_0\tau_b\right) + \cdots + \exp\left(-i(K-1)n\Delta\varphi_b - (K-1)\nu_0\tau_b\right)\right) \int\limits_{t-\tau_b}^{t} \mathrm{d}t' \dot{W}(t') \\
&+ \exp\left(-iKn\Delta\varphi_b - K\nu_0\tau_b\right) \int\limits_{K\tau_b}^{t} \mathrm{d}t' \dot{W}(t') \\
&= \frac{1 - \exp\left(-iKn\Delta\varphi_b - K\nu_0\tau_b\right)}{1 - \exp\left(-in\Delta\varphi_b - \nu_0\tau_b\right)} \int\limits_{t-\tau_b}^{t} \mathrm{d}t' \dot{W}(t') + \exp\left(-iKn\Delta\varphi_b - K\nu_0\tau_b\right) \int\limits_{K\tau_b}^{t} \mathrm{d}t' \dot{W}(t').
\end{aligned} \tag{96}$$

For large t such that $K\nu_0\tau_b \approx \nu_0 t \gg 1$ respective exponents $\exp\left(-iKn\Delta\varphi_b - K\nu_0\tau_b\right)$ can be ignored,

$$\begin{aligned}
W_{\mathrm{pol}}(\mathbf{z}_{0\mathrm{pol}}, t) &\approx \frac{1}{1 - \exp\left(-in\Delta\varphi_b - \nu_0\tau_b\right)} \int\limits_{t-\tau_b}^{t} \mathrm{d}t' \dot{W}(t') = \frac{1}{\exp\left(in\Delta\varphi_b + \nu_0\tau_b\right) - 1} \int\limits_{t}^{t+\tau_b} \mathrm{d}t' \dot{W}(t') \\
&= \int\limits_0^t \mathrm{d}t' \dot{W}(t') + \frac{1}{\exp\left(in\Delta\varphi_b + \nu_0\tau_b\right) - 1} \int\limits_0^{\tau_b} \mathrm{d}t' \dot{W}(t').
\end{aligned} \tag{97}$$

Due to the property (95) of $\dot{W}$ and $\dot{W} \propto \exp(-\nu_0 t)$ one can check that last approximate form (96) has the following periodicity,

$$W_{\mathrm{pol}}(\mathbf{z}_{0\mathrm{pol}}, t + \tau_b) = W_{\mathrm{pol}}(\mathbf{z}_{0\mathrm{pol}}, t) \exp\left(in\Delta\varphi_b\right). \tag{98}$$

Aperiodic exponential factor in (98) is cancelled in the subintegrand of (92) by the aperiodic factor in the last exponent in (98). Therefore, the subintegrand in (98) is a periodic function of time for large enough t. Since we can choose t_0 as large as we want, we can use the approximate asymptotic form of (96) everywhere, exchange the order of statistical average (which is needed because of random starting positions) and time average and reduce time average to a single bounce time average what is valid for a periodic subintegrand,

$$\delta F_n(t,\mathbf{z}_{\rm pol}) = \overline{\frac{W_0}{2\pi}\frac{1}{\tau_b}\int\limits_0^{\tau_b} \mathrm{d}t\; \delta\left(\mathbf{z}_{\rm pol} - \mathbf{Z}_{\rm pol}(\tilde{\mathbf{z}}_{0\rm pol},t)\right) W_{\rm pol}(\tilde{\mathbf{z}}_{0\rm pol},t) \exp\left(-inZ_0^\varphi(\tilde{\mathbf{z}}_{0\rm pol},t)\right)}. \tag{99}$$

Using for $W_{\rm pol}$ its explicit form (97) with explicit subsitution of $\dot{W}$ one gets

$$\begin{aligned} W_{\rm pol}(\tilde{\mathbf{z}}_{0\rm pol},t) = \mathrm{e}^{-\nu_0 t}\Bigg(&\int\limits_0^t \mathrm{d}t'\dot{w}_A\left(\mathbf{Z}_{\rm pol}(\tilde{\mathbf{z}}_{0\rm pol},t')\right)\exp\left(inZ_0^\varphi(\tilde{\mathbf{z}}_{0\rm pol},t')+\nu_0 t'\right) \\ &+ \frac{1}{\exp\left(in\Delta\varphi_b+\nu_0\tau_b\right)-1}\int\limits_0^{\tau_b}\mathrm{d}t'\dot{w}_A\left(\mathbf{Z}_{\rm pol}(\tilde{\mathbf{z}}_{0\rm pol},t')\right)\exp\left(inZ_0^\varphi(\tilde{\mathbf{z}}_{0\rm pol},t')+\nu_0 t'\right)\Bigg). \end{aligned} \tag{100}$$

For the numerical evaluation it is more convenient to solve the differential equation rather than evaluate integrals in (100). Actually, function $W_{\rm pol}$ given by (100) satisfies the original equation (85) for the orbit starting at $\varphi = 0$. However, Eq. (100) does contradict our original definition of $W_{\rm pol}$ which assumes that $W_{\rm pol}(\mathbf{z}_{\rm pol},0) = 0$ (see next line after Eq. (91)). In order to remove this contradiction we introduce more general function $W^G_{\rm pol}(\mathbf{z}_{\rm pol},w_0,t)$ which satisfies (85) for the orbit starting at $\varphi = 0$ but has generally arbitrary initial value, $W^G_{\rm pol}(\mathbf{z}_{\rm pol},w_0,0) = w_0$. Then function (100) is defined as follows

$$W_{\rm pol}(\mathbf{z}_{\rm pol},t) = W^G_{\rm pol}(\mathbf{z}_{\rm pol},w_b,t), \qquad w_b = \frac{\exp\left(\nu_0\tau_b\right)}{\exp\left(in\Delta\varphi_b+\nu_0\tau_b\right)-1}W^G_{\rm pol}(\mathbf{z}_{\rm pol},0,\tau_b). \tag{101}$$

Obviously, in case of collisionless orbits orbit must be integrated over bounce time twice. During the first integration one obtains $\tau_b(\mathbf{z}_{\rm pol})$, $\Delta\varphi_b(\mathbf{z}_{\rm pol})$ and the starting weight $w_b(\mathbf{z}_{\rm pol})$ defined by the second of (101). Then orbit is restarted at the same point for evaluation of the perturbed distribution function (99).
One can see that the only random quantities in (99) are the initial values of poloidal phase space coordinates $\tilde{\mathbf{z}}_{0\rm pol}$, and calculation of the expectation value (99) can in principle be replaced with usual phase space integration over starting positions weighted with probability density F_0/W_0. In case of 4D integration over "polidal" variables $(R, Z, v_\perp, v_\parallel)$ this can still be an option which avoids the numerical noise.

F. Moments of the distribution function

We are not interested here in distribution function itself but in its moments, namely, in Fourier amplitudes of the perturbed density n_n, perturbed perpendicular pressure $p_{\perp n}$, and perturbed poloidal current density components j^R_n and j^Z_n. In practice, various densities are approximated by box averages. In our case, boxes are triangles in cylindrical variables. Let us denote with $\Theta_\Delta(R,Z)$ a function which is equal to 1 if (R,Z)

are inside the given triangle and 0 otherwise. We evaluate the following integral of the perturbed density δn,

$$\Delta N_n = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} \mathrm{d}\varphi \int\limits_{0}^{\infty} \mathrm{d}R \int\limits_{-\infty}^{\infty} \mathrm{d}Z \mathrm{e}^{-in\varphi}\Theta_\Delta(R,Z)\ \delta n(R,\varphi,Z). \tag{102}$$

Denoting with S_Δ the triangle area, Fourier amplitude of the density is obtained in the limit of zero triangle size as follows,

$$n_n = \lim_{\Delta\to 0}\frac{\Delta N_n}{S_\Delta}. \tag{103}$$

Denoting the velocity space Jacobian with J_y such that $J = RJ_y$ and

$$\delta n = \int \mathrm{d}^{N-3}y\ J_y \delta f = \frac{1}{R}\int \mathrm{d}^{N-3}y\ \delta F, \tag{104}$$

where $\mathbf{y}$ is the subset of phase space variables of dimension $N-3$ corresponding to the velocity space (we intendendly do not specify the phase space dimension N which can be 6 or 5 depending on whether the gyrophase is or is not included, since the result does not depend on this snce in the second case J_y includes an extra factor 2π), Fourier amplitude of the density integrated over the triangle area (102) takes the form

$$\Delta N_n = \frac{1}{2\pi}\int \mathrm{d}^N z\ \mathrm{e}^{-in\varphi}\Theta_\Delta(R,Z)\ \frac{1}{R}\delta F(\mathbf{z}) = \int \mathrm{d}^{N-1}z_\mathrm{pol}\frac{1}{R}\Theta_\Delta(R,Z)\delta F_n(\mathbf{z}_\mathrm{pol}). \tag{105}$$

Substituting here δF_n from the general collisional expression (92) we get

$$\begin{aligned}\Delta N_n &= \frac{W_0}{2\pi t_0}\int\limits_0^{t_0}\mathrm{d}t\ \overline{\frac{\Theta_\Delta\left(Z^R(\tilde{\mathbf{z}}_{0\mathrm{pol}},t), Z^Z(\tilde{\mathbf{z}}_{0\mathrm{pol}},t)\right)}{Z^R(\tilde{\mathbf{z}}_{0\mathrm{pol}},t)}W_\mathrm{pol}(\tilde{\mathbf{z}}_{0\mathrm{pol}},t)\exp\left(-inZ_0^\varphi(\tilde{\mathbf{z}}_{0\mathrm{pol}},t)\right)} \\ &= \frac{W_0}{2\pi t_0}\overline{\sum_k\int_{\Delta t_k}\mathrm{d}t\ \frac{1}{Z^R(\tilde{\mathbf{z}}_{0\mathrm{pol}},t)}W_\mathrm{pol}(\tilde{\mathbf{z}}_{0\mathrm{pol}},t)\exp\left(-inZ_0^\varphi(\tilde{\mathbf{z}}_{0\mathrm{pol}},t)\right)},\end{aligned} \tag{106}$$

where the last expression means integration over time spent by the orbit within the triangle (summation means that triangle can be visited several times during the orbit integration time $0 < t < t_0$). Here, Z^R and Z^Z denotes orbit components over R and Z variables respectively.

For the integral over the triangle area of the Fourier amplitude of the perturbed pressure,

$$\Delta P_{\perp n} = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} \mathrm{d}\varphi \int\limits_{0}^{\infty} \mathrm{d}R \int\limits_{-\infty}^{\infty} \mathrm{d}Z \mathrm{e}^{-in\varphi}\Theta_\Delta(R,Z)\ \delta p_\perp(R,\varphi,Z), \tag{107}$$

we obtain using the definition (3) in analogy to (106)

$$\Delta P_{\perp n} = \frac{W_0}{2\pi t_0}\overline{\sum_k\int_{\Delta t_k}\mathrm{d}t\ \frac{m\left(Z^{v_\perp}(\tilde{\mathbf{z}}_{0\mathrm{pol}},t)\right)^2}{2Z^R(\tilde{\mathbf{z}}_{0\mathrm{pol}},t)}W_\mathrm{pol}(\tilde{\mathbf{z}}_{0\mathrm{pol}},t)\exp\left(-inZ_0^\varphi(\tilde{\mathbf{z}}_{0\mathrm{pol}},t)\right)}, \tag{108}$$

where $Z^{v_\perp}(\tilde{\mathbf{z}}_{0\mathrm{pol}},t)$ denotes the value of perpendicular velocity on the orbit.

Instead of curent density components we compute currents through triangle edges.

$$\Delta I_n = \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} \mathrm{d}\varphi \int \mathrm{d}l \ \mathrm{e}^{-in\varphi} \mathbf{n} \cdot \delta \mathbf{j}(R, \varphi, Z), \tag{109}$$

where l is the length counted along the edge and $\mathbf{n}$ is the normal to the edge. Let $a(R, Z) = 0$ be the equation for the line containing the edge and Ω denotes the sub-surface in the (R, Z) plane containing the edge. Then we can write

$$\int \mathrm{d}l \ \mathbf{n} = \iint\limits_{\Omega} \mathrm{d}R\mathrm{d}Z \ \delta(a) \nabla a = \iint\limits_{\Omega} \mathrm{d}R\mathrm{d}Z \ \nabla\theta(a), \tag{110}$$

where $\theta(a)$ is Riemann's theta-function. Using this expression in (109) together with

$$\delta\mathbf{j} = e \int \mathrm{d}^3 v \ \delta f \mathbf{v} = e \int \mathrm{d}^3 v \ \delta f \dot{\mathbf{r}}, \tag{111}$$

we get

$$\begin{aligned} \Delta I_n &= \frac{e}{2\pi} \int\limits_{-\pi}^{\pi} \mathrm{d}\varphi \iint\limits_{\Omega} \mathrm{d}R\mathrm{d}Z \ \mathrm{e}^{-in\varphi} \int \mathrm{d}^3 v \ \delta f \dot{\mathbf{r}} \cdot \nabla\theta(a), \\ &= \frac{e}{2\pi} \int\limits_{-\pi}^{\pi} \mathrm{d}\varphi \iint\limits_{\Omega} \mathrm{d}R\mathrm{d}Z \ \mathrm{e}^{-in\varphi} \int \mathrm{d}^3 v \ \delta f \frac{\mathrm{d}}{\mathrm{d}t}\theta(a), \\ &= \frac{e}{2\pi} \int_{\Omega} \mathrm{d}^N z \mathrm{e}^{-in\varphi} \frac{1}{R} \delta F \frac{\mathrm{d}}{\mathrm{d}t}\theta(a), \\ &= e \int_{\Omega} \mathrm{d}^{N-1} z_{\mathrm{pol}} \frac{1}{R} \delta F_n \frac{\mathrm{d}}{\mathrm{d}t}\theta(a). \end{aligned} \tag{112}$$

Substituting here δF_n from the general collisional expression (92) we get

$$\Delta I_n = \frac{W_0 e}{2\pi t_0} \overline{\sum_k \frac{\sigma_k}{Z^R(\tilde{\mathbf{z}}_{0\mathrm{pol}}, t_k)} W_{\mathrm{pol}}(\tilde{\mathbf{z}}_{0\mathrm{pol}}, t_k) \exp\left(-in Z_0^{\varphi}(\tilde{\mathbf{z}}_{0\mathrm{pol}}, t_k)\right)}, \tag{113}$$

where k enumerates time moments t_k of edge crossing by the orbit and $\sigma_k = \pm 1$ denotes directions of such crossings.

In the collisionless case in formulas (106), (108) and (113) one should replace t_0 with τ_b (the latter one then should be moved uder the expectation value sign) and all crossings of triangle must be registered only during the bounce time. E.g., (113) is modified as follows,

$$\Delta I_n = \frac{W_0 e}{2\pi} \overline{\frac{1}{\tau_b(\tilde{\mathbf{z}}_{0\mathrm{pol}})} \sum_k \frac{\sigma_k}{Z^R(\tilde{\mathbf{z}}_{0\mathrm{pol}}, t_k)} W_{\mathrm{pol}}(\tilde{\mathbf{z}}_{0\mathrm{pol}}, t_k) \exp\left(-in Z_0^{\varphi}(\tilde{\mathbf{z}}_{0\mathrm{pol}}, t_k)\right)}. \tag{114}$$

G. Larmor radius ordering, antithetic variates

So far formulas in this section are for arbitrary set of phase space guiding center variables. Here velocity space variables are specified as $(v_\perp, v_\|)$ while coordinate space variables x^i are kept general.

$$\begin{aligned} V^i &= \dot{\mathbf{R}} \cdot \nabla x^i, \quad i = 1,2,3; \qquad V^4 = \dot{v}_\perp = \frac{v_\perp}{2B} \dot{\mathbf{R}} \cdot \nabla B, \\ V^5 &= \dot{v}_\| = -\frac{v_\perp^2}{2Bv_\|} \dot{\mathbf{R}} \cdot \nabla B - \frac{e}{mv_\|} \dot{\mathbf{R}} \cdot \nabla \Phi. \end{aligned} \tag{115}$$

In these expressions it is convenient to present $\dot{\mathbf{R}} = v_\| \mathbf{h} + \mathbf{v}_d$ where $\mathbf{v}_d$ is the cross-field guiding center drift velocity which is of the first order in ρ_L with respect to parallel motion. This explicit form is required for the derivation of the source term in (70). The unperturbed axisymmetric distribution function f_0 up to linear order in Larmor radius is

$$f_0 = f_M(r, v^2)\left(1 + g_0(\mathbf{z}_{\rm pol})\right), \tag{116}$$

where $r = r(\psi)$ is an effective radius, f_M is a local Maxwellian and g_0 is a neoclassical correction of the linear order in ρ_L. Then weight increment (71) up to the linear order in ρ_L is

$$\dot{w} = -\frac{\delta V^i}{f_0}\frac{\partial f_0}{\partial z^i} \approx -\frac{\delta V^i}{f_M}\frac{\partial f_M}{\partial z^i} - \delta V^i \frac{\partial g_0}{\partial z^i}. \tag{117}$$

In the first term guiding center velocity is taked up to the linear order in ρ_L

$$\frac{\delta V^i}{f_M}\frac{\partial f_M}{\partial z^i} = \left(A_1 + A_2 \frac{mv^2}{2T_0}\right) \delta\dot{\mathbf{R}} \cdot \nabla r + \frac{e}{T_0} \dot{\mathbf{R}}_0 \cdot \nabla \delta\Phi, \tag{118}$$

where n_0 and T_0 are the uperturbed density and temperature characterising the local Maxwellian, respectively, $\Phi = \Phi_0 + \delta\Phi$,

$$\dot{\mathbf{R}} = \dot{\mathbf{R}}_0 + \delta\dot{\mathbf{R}} = v_\| \mathbf{h}_0 + \mathbf{v}_{d0} + v_\| \delta\mathbf{h} + \delta\mathbf{v}_d, \tag{119}$$

and thermodynamic forces are

$$A_1 = \frac{1}{n_0}\frac{\partial n_0}{\partial T} + \frac{e}{T_0}\frac{\partial \Phi_0}{\partial r} - \frac{3}{2T_0}\frac{\partial T_0}{\partial r}, \qquad A_2 = \frac{1}{T_0}\frac{\partial T_0}{\partial r}. \tag{120}$$

Phase space velocity V^i is needed in the second term in (117) only in zero order over ρ_L,

$$\delta V^i \frac{\partial g_0}{\partial z^i} \approx v_\| \delta\mathbf{h} \cdot \nabla g_0 + \frac{v_\perp}{2}\left(v_\| \frac{\partial g_0}{\partial v_\perp} - v_\perp \frac{\partial g_0}{\partial v_\|}\right) \delta\left(\frac{\mathbf{h} \cdot \nabla B}{B}\right) - \frac{e}{m}\frac{\partial g_0}{\partial v_\|} \delta\left(\mathbf{h} \cdot \nabla \Phi\right). \tag{121}$$

Separating in the weight increment (117) zero and first order terms in Larmor radius, $\dot{w} = \dot{w}_0 + \dot{w}_1$ we get for the zero order increment

$$\dot{w}_0 = -v_\| \left(A_1 + A_2 \frac{mv^2}{2T_0}\right) \delta\mathbf{h} \cdot \nabla r - v_\| \frac{e}{T_0} \mathbf{h}_0 \cdot \nabla \delta\Phi, \tag{122}$$

and for the linear order increment

$$\dot{w}_1 = -\left(A_1 + A_2\frac{mv^2}{2T_0}\right)\delta\mathbf{v}_d\cdot\nabla r - \frac{e}{T_0}\mathbf{v}_{d0}\cdot\nabla\delta\Phi - \delta V^i\frac{\partial g_0}{\partial z^i}, \tag{123}$$

with the last term given by (121).

In the case of ideal perturbations such that electrostatic potential remains constant on the perturbed flux surfaces, $\mathbf{h}\cdot\nabla\Phi = 0$, perturbation of the potential satisfies magnetic differential equation

$$\mathbf{h}_0\cdot\nabla\delta\Phi + \delta\mathbf{h}\cdot\nabla\Phi_0 = 0,$$

and, as a result, terms with Φ_0 and $\delta\Phi$ cancel each other in (122), and $\dot{w}_0$ is determined only by density and temeperature gradients.

Let us check that in zero order over ρ_L there are no perturbed currents in the steady state in absence of collisions. In this approximation we ignore collision term and time derivative in (70), take there V_0^i also in zero order over ρ_L

$$V_0^i\frac{\partial\delta f}{\partial z^i} \approx v_\parallel\mathbf{h}_0\cdot\nabla\delta f + \frac{v_\perp}{2B_0}\left(v_\parallel\frac{\partial\delta f}{\partial v_\perp} - v_\perp\frac{\partial\delta f}{\partial v_\parallel}\right)\mathbf{h}\cdot\nabla B_0, \tag{124}$$

and set the source term to $\dot{w}_0 f_M$. Removing the common factor $v_\parallel$ one gets the perturbed kinetic equation in zero order over ρ_L

$$\mathbf{h}_0\cdot\nabla\delta f + \frac{v_\perp}{2B_0}\left(\frac{\partial\delta f}{\partial v_\perp} - \frac{v_\perp}{v_\parallel}\frac{\partial\delta f}{\partial v_\parallel}\right)\mathbf{h}_0\cdot\nabla B_0 = -\left[\left(A_1 + A_2\frac{mv^2}{2T_0}\right)\delta\mathbf{h}\cdot\nabla r + \frac{e}{T_0}\mathbf{h}_0\cdot\nabla\delta\Phi\right]f_M. \tag{125}$$

It can be seen that δf is an even function of $v_\parallel$ and produces no currents (since perpendicular currents are ignored in the zero order over ρ_L). If δf is computed with a straightforward Monte Carlo method, symmetry of this function is ensured only for infinite statistics. Because of statistical cancellation of parallel velocities, noise in the currents, which are given only by the next order in ρ_L is huge. In turn, perturbations of density and pressure which are determined by the even part of δf are not noisy. For this moments next order terms in ρ_L provide only a corection. In order to heal the noise problem in case of collisionless approximation where random numbers are used only for generation of initial positions of markers in the phase space, we use the method of "antithetic variates". Since the pseudo-scalar distribution of initial marker positions $F_0 = J_0 f_0 = J_0 f_M(1+g_0)$ differs from symmetric function only in the next order in ρ_L due to the correction g_0 we can take this assymetry into account with help of an extra weight, $w \to w w_{extra}$ where $w_{extra} = 1 + g_0(\tilde{\mathbf{z}}_0)$. This makes the initial distribution of markers strictly symmetric. Then, the initial position of each second particle can be taken the same with the position of previous particle but with opposite $v_\parallel$ sign. In this case initial distribution of markers is symmetric, and current in zero order over ρ_L is zero independent of statistics.

It should be noted that currents in our previous test model did not correspond to this ideal zero-order model because we retained drift in the unperturbed orbits (but neglected it in the source term where we set $\dot{w} = \dot{w}_0$). Thus, currents were of the same order as in the correct model what was sufficient for the convergence test.

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