Numerical solution of magnetic differential equations

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Simple Example

We would like to solve an equation of the type

$$\dot{p}_n + inp_n = q_n \tag{1}$$

for $p(s)$ with periodic boundary conditions at $s = 0 \dots 2\pi$. With the ansatz $p_n = \sum_m p_{mn} e^{ims}$ and the same for q_n we obtain the analytical solution

$$p_{mn} = \frac{q_{mn}}{i(m+n)}. \tag{2}$$

A finite difference scheme yields

$$\frac{p_n^{k+1} - p_n^k}{\Delta s} + \frac{1}{2} in(p_n^{k+1} + p_n^k) = q_n^k. \tag{3}$$

A corresponding matrix with periodic boundary conditions is

$$\begin{pmatrix} in/2 - 1/\Delta s & in/2 + 1/\Delta s & & \\ & in/2 - 1/\Delta s & in/2 + 1/\Delta s & \\ & & \dots & \\ in/2 + 1/\Delta s & & & in/2 - 1/\Delta s \end{pmatrix} \boldsymbol{p} = \boldsymbol{q}$$

$$\begin{pmatrix} in/2 - 1/\Delta s & in/2 + 1/\Delta s \\ in/2 + 1/\Delta s & in/2 - 1/\Delta s \end{pmatrix} \boldsymbol{p} = \boldsymbol{q}$$

Magnetic Differential Equation

Magnetic differential equations arise from a number of problems in plasma physics. We consider for example the magnetohydrodynamic equilibrium

$$\nabla p = \boldsymbol{J} \times \boldsymbol{B}, \tag{4}$$

with pressure p, current $\boldsymbol{J}$ and magnetic field $\boldsymbol{B}$. Scalar multiplication with $\boldsymbol{B}$ yields the homogenous magnetic differential equation

$$\boldsymbol{B} \cdot \nabla p = 0\,. \tag{5}$$

In the linear perturbation theory, a source term enters the right-hand side with

$$\boldsymbol{B} \cdot \nabla p = q \,. \tag{6}$$

For an axisymmetric plasma in a tokamak, we reduce the dimensionality by introducing cylindrical coordinates and an expansion in φ,

$$p(R, Z, \varphi) = \sum_n p_n(R, Z) e^{in\varphi} \,. \tag{7}$$

The remaining equation in the poloidal RZ plane for each harmonic are

$$\boldsymbol{B} \cdot \nabla_{RZ} p_n + inB^{\varphi} p_n = q_n \,. \tag{8}$$

or in components of coordinates x^k with $k = 1, 2$ in the poloidal plans,

$$B^k \frac{\partial}{\partial x^k} p_n + inB^{\varphi} p_n = q_n \,. \tag{9}$$

Generating $\boldsymbol{B}$ from the stream function $\psi = A_\varphi$ we obtain

$$B^1 = -\frac{1}{R\sqrt{g_p}} \frac{\partial \psi}{\partial x^2} \tag{10}$$

$$B^2 = \frac{1}{R\sqrt{g_p}} \frac{\partial \psi}{\partial x^1} \tag{11}$$

Here, g_p is the metric determinant of the 2D metric tensor in the poloidal plane. Using ψ as one coordinate x^1, and the distance s in the poloidal direction with $ds = \sqrt{dR^2 + dZ^2}$, we have an orthogonal system with

$$\hat{g}_P = \begin{pmatrix} g_{\psi\psi} & \\ & 1 \end{pmatrix} \tag{12}$$

This means that $\sqrt{g_p} = \sqrt{g_{\psi\psi}} = 1/|\nabla \psi|$. The transport law becomes

$$\frac{1}{R\sqrt{g_p}} \frac{\partial p_n}{\partial s} + inB^{\varphi} p_n = q_n \,. \tag{13}$$

This is a one-dimensional problem along the poloidal $\boldsymbol{B}$ direction.

Finite Difference Method

Multiplying by $-iR\sqrt{g_p}$ we obtain

$$nR\sqrt{g_p} B^{\varphi} p_n - i\dot{p}_n = -iR\sqrt{g_p} q_n \,. \tag{14}$$

Discretizing with a forward Euler method and evaluating averages at the midpoints we obtain

$$\frac{n}{2} \left(R^k \sqrt{g_p^k} B^{\varphi k} p_n^k + R^{k+1} \sqrt{g_p^{k+1}} B^{\varphi k+1} p_n^{k+1} \right) - i \frac{p_n^{k+1} - p_n^k}{\Delta s^k} \tag{15}$$

$$= -\frac{1}{2} i \left(R^k \sqrt{g_p^k} q_n^k + R^{k+1} \sqrt{g_p^{k+1}} q_n^{k+1} \right) \,. \tag{16}$$

Coefficients should be filled into a sparse matrix and the discrete equations solved e.g. by UMF-PACK.

Analytical solution

$$i(mB^\vartheta + nB^\varphi)p_{mn} = q_{mn}$$
$$p_{mn} = \frac{q_{mn}}{i(mB^\vartheta + nB^\varphi)}\,.$$

We take circular flux surfaces in the large aspect ratio limit, such that the scaling with R vanishes and as coordinates minor r and ϑ.

In this case

$$B^\vartheta = \frac{1}{R_0\sqrt{g_P}}\frac{\partial\psi}{\partial r}\,.$$

We set $\psi = r^2/4$ so $\frac{\partial\psi}{\partial r} = |\nabla\psi| = r/2$. Due to the circular flux surfaces, we have a orthogonal system and $\sqrt{g_P} = r$, so $B^\vartheta = 1/(2R_0)$.

Perturbation in current density

First variant: Use linear perturbation

$$\boldsymbol{j}\times\boldsymbol{B} \approx \boldsymbol{j}_0\times\boldsymbol{B}_0 + \delta\boldsymbol{j}\times\boldsymbol{B}_0 + \boldsymbol{j}_0\times\delta\boldsymbol{B} = c(\nabla p_0 + \nabla\delta p), \tag{17}$$

resulting in

$$\delta\boldsymbol{j}\times\boldsymbol{B}_0 = \delta\boldsymbol{j}_\perp\times\boldsymbol{B}_0 = c\nabla\delta p - \boldsymbol{j}_0\times\delta\boldsymbol{B}. \tag{18}$$

Second variant: Use derived expresion for $\delta\boldsymbol{j}_\perp$ with

$$\delta\boldsymbol{j}_\perp = j_{0\parallel}\frac{\delta\boldsymbol{B}_\perp}{B_0} - \frac{c\boldsymbol{h}_0\cdot\delta\boldsymbol{B}}{B_0^2}\boldsymbol{h}_0\times\nabla p_0 + \frac{c}{B_0}\boldsymbol{h}_0\times\nabla\delta p. \tag{19}$$

Take cross product with $\boldsymbol{B}_0$.

- First term

$$j_{0\parallel}\frac{\delta\boldsymbol{B}_\perp}{B_0}\times\boldsymbol{B}_0 = \delta\boldsymbol{B}_\perp\times(j_{0\parallel}\boldsymbol{h}_0) = \delta\boldsymbol{B}_\perp\times\boldsymbol{j}_{0\parallel}. \tag{20}$$

- Second term

$$\begin{aligned}-\frac{c\boldsymbol{h}_0\cdot\delta\boldsymbol{B}}{B_0^2}(\boldsymbol{h}_0\times\nabla p_0)\times\boldsymbol{B}_0 &= -\frac{\delta B_\parallel}{B_0}(\boldsymbol{h}_0\times(\boldsymbol{j}_0\times\boldsymbol{B}_0))\times\boldsymbol{h}_0\\ &= -\delta B_\parallel(\boldsymbol{j}_{0\perp}\times\boldsymbol{h}_0) = \delta\boldsymbol{B}_\parallel\times\boldsymbol{j}_{0\perp}.\end{aligned} \tag{21}$$

- Third term

$$\frac{c}{B_0}(\boldsymbol{h}_0\times\nabla\delta p)\times\boldsymbol{B}_0 = c(\nabla\delta p - (\boldsymbol{h}_0\cdot\nabla\delta p)\boldsymbol{h}_0)$$

Summed up this yields

$$\delta\boldsymbol{j}_\perp\times\boldsymbol{B}_0 = c\nabla\delta p - c(\boldsymbol{h}_0\cdot\nabla\delta p)\boldsymbol{h}_0 - \boldsymbol{j}_0\times\delta\boldsymbol{B}, \tag{22}$$

which contains an extra term $c(\boldsymbol{h}_0\cdot\nabla\delta p)\boldsymbol{h}_0$ compared to Eq. (18). **TODO: WHY?**

Finite Volume Methode

Now we a similar problem using a FVM scheme. We write the conservative form

$$\nabla \cdot (\boldsymbol{h}_0 j_{\|n}) + inh_0^\varphi j_{\|n} = -\nabla \cdot \boldsymbol{j}_\perp^{\mathrm{pol}} - inj_{\perp n}^\varphi \,.$$

The divergence operator is defined via

$$\nabla \cdot \boldsymbol{u} = \frac{1}{R\sqrt{g_p}} \frac{\partial}{\partial x^k} (R\sqrt{g_p} u^k) \,.$$

When working with R, Z as coordinates in the poloidal plane, $\sqrt{g_p} = 1$. We multiply by R to obtain

$$\frac{\partial}{\partial x^k} (Rh^k j_{\|n}) + inRh^\varphi j_{\|n} = -\frac{\partial}{\partial x^k} (Rj_{\perp n}^k) - inRj_{\perp n}^\varphi \,.$$

Integration over a triangle yields

$$\oint Rj_{\|n} \boldsymbol{h}_0^{\mathrm{pol}} \cdot \boldsymbol{n} d\Gamma + in \int Rh_0^\varphi j_{\|n} d\Omega = -\oint R\boldsymbol{j}_{\perp n}^{\mathrm{pol}} \cdot \boldsymbol{n} d\Gamma - in \int Rj_{\perp n}^\varphi d\Omega \tag{23}$$

where scalar products with $\boldsymbol{n}$ pointing towards the outer normal vector of the edge are taken component-wise in R and Z. We assume a field-aligned mesh with $\boldsymbol{h}_0$ parallel to edge no. 3. In- and outflux of the parallel current are only over edges 1 and 2.

On edges 1 and 2 we sum up the flux from left- and right hand side of Eq. 23 to use the flux of the total perturbed current harmonic in the poloidal plane,

$$\boldsymbol{j}_n^{\mathrm{pol}} = j_{\|n} \boldsymbol{h}_0^{\mathrm{pol}} + \boldsymbol{j}_{\perp n}^{\mathrm{pol}},$$

as an unknown:

$$\int_{1,2} R\boldsymbol{j}_n^{\mathrm{pol}} \cdot \boldsymbol{n} d\Gamma + in \int Rh_0^\varphi j_{\|n} d\Omega = -\int_3 R\boldsymbol{j}_{\perp n}^{\mathrm{pol}} \cdot \boldsymbol{n} d\Gamma - in \int Rj_{\perp n}^\varphi d\Omega. \tag{24}$$

In addition, we add terms with $j_n^\varphi = h_0^\varphi j_{\|n} + j_{\perp n}^\varphi$ together again to obtain

$$\int_{1,2} R\boldsymbol{j}_n^{\mathrm{pol}} \cdot \boldsymbol{n} d\Gamma + in \int Rj_n^\varphi d\Omega = -\int_3 R\boldsymbol{j}_{\perp n}^{\mathrm{pol}} \cdot \boldsymbol{n} d\Gamma. \tag{25}$$

On edge 3 we use

$$\boldsymbol{j}_n \times \boldsymbol{B}_0 = c(\nabla p_n + in\, p_n \nabla\varphi) - \boldsymbol{j}_0 \times \boldsymbol{B}_n. \tag{26}$$

Scalar multiplication by $\boldsymbol{e}_\varphi = \frac{\partial \boldsymbol{R}}{\partial \varphi}$ yields

$$\boldsymbol{e}_\varphi \cdot (\boldsymbol{j}_n \times \boldsymbol{B}_0) = \boldsymbol{j}_n \cdot (\boldsymbol{B}_0 \times \boldsymbol{e}_\varphi) = RB_0^{\mathrm{pol}} \boldsymbol{j}_n \cdot \boldsymbol{n} \tag{27}$$

This results in

$$\begin{aligned} \boldsymbol{j}_n \cdot \boldsymbol{n} &= \frac{1}{RB_0^{\mathrm{pol}}} \left(in\, p_n - \boldsymbol{j}_0 \cdot (\boldsymbol{B}_n \times \boldsymbol{e}_\varphi) \right) \\ &= \frac{1}{RB_0^{\mathrm{pol}}} \left(in\, p_n + R\boldsymbol{j}_0 \cdot \boldsymbol{B}_n^{\mathrm{pol}} \right) . \end{aligned}$$

TODO:

- Unit vector:

$$\boldsymbol{h} \approx \boldsymbol{h}_0 + \frac{\delta \boldsymbol{B}_\perp}{B_0}$$
$$\delta \boldsymbol{B}_\perp = \delta \boldsymbol{B} - \boldsymbol{h}_0(\boldsymbol{h}_0 \cdot \delta \boldsymbol{B})$$

- Current density:

$$\begin{aligned} \boldsymbol{j} - \boldsymbol{j}_0 &= (j_{0\|} + \delta j'_\|)\boldsymbol{h} + \boldsymbol{j}_\perp - j_{0\|}\boldsymbol{h}_0 - \boldsymbol{j}_{0\perp} \\ &= \delta j'_\| \boldsymbol{h}_0 + j_{0\|}\frac{\delta \boldsymbol{B}_\perp}{B_0} + \delta \boldsymbol{j}'_\perp \equiv \delta \boldsymbol{j}. \end{aligned} \tag{28}$$

- More:
  - Take from above
  - More:

$$(\boldsymbol{h}_0 \times (\boldsymbol{j}_0 \times \boldsymbol{B}_0)) \times \boldsymbol{B}_0 = B_0(\boldsymbol{j}_0 \times \boldsymbol{B}_0)$$
$$(\boldsymbol{h}_0 \times \nabla \delta p) \times \boldsymbol{B}_0 = B_0 \nabla \delta p - (\boldsymbol{B}_0 \cdot \nabla \delta p)\boldsymbol{h}_0$$

  - Put together:

$$\delta \boldsymbol{j}_\perp \times \boldsymbol{B}_0 = c\nabla \delta p - (\boldsymbol{h}_0 \cdot \nabla \delta p)\boldsymbol{h}_0 - \boldsymbol{j}_0 \times \delta \boldsymbol{B}$$

- Currents

$$\boldsymbol{j}_0 \times \boldsymbol{B}_0 + \boldsymbol{j}_0 \times \delta \boldsymbol{B} + \delta \boldsymbol{j} \times \boldsymbol{B}_0 = c(\nabla p_0 + \nabla p_1) \tag{29}$$

$$\delta \boldsymbol{B} \cdot (\boldsymbol{j}_0 \times \boldsymbol{B}_0) = c(\delta \boldsymbol{B} \cdot \nabla p_0 + \delta \boldsymbol{B} \cdot \nabla p_1) \tag{30}$$

$$\delta \boldsymbol{B} \cdot (\boldsymbol{j}_0 \times \boldsymbol{B}_0) = c(\delta \boldsymbol{B} \cdot \nabla p_0 + \delta \boldsymbol{B} \cdot \nabla p_1) \tag{31}$$

On edge 3 we evaluate

$$\boldsymbol{j}^{\mathrm{pol}}_{\perp n} \cdot \boldsymbol{n} = j_{0\|}\frac{B^{\mathrm{pol}}_{\perp n}}{B_0} + \frac{c}{B_0}\boldsymbol{n} \cdot (\boldsymbol{h}_0 \times (\nabla p_n + inp_n \nabla \varphi)). \tag{32}$$

Rotating the triple product and using the fact that both, $\boldsymbol{n}$ and ∇p_n are in the poloidal plane, yields

$$\boldsymbol{n} \cdot \boldsymbol{h}_0 \times (\nabla p_n + inp_n \nabla \varphi) = inp_n \boldsymbol{h}_0 \cdot \nabla \varphi \times \boldsymbol{n} = \frac{in}{R} p_n \tag{33}$$

on edge 3.

Old

We need to solve

$$\nabla \cdot \delta \boldsymbol{j} = 0, \tag{34}$$
$$\nabla \delta p = \frac{1}{c}\left(\delta \boldsymbol{j} \times \boldsymbol{B}_0 + \boldsymbol{j}_0 \times \delta \boldsymbol{B}\right) \tag{35}$$

for $\delta \boldsymbol{j}$. Splitting into toroidal and poloidal parts for a single harmonic in φ we obtain

$$\nabla \cdot \boldsymbol{j}_n^{\mathrm{pol}} + in\, j_n^{\varphi} = 0. \tag{36}$$

The second equation reads

$$\boldsymbol{j}_n \times \boldsymbol{B}_0 = c(\nabla p_n + in\, p_n \nabla \varphi) - \boldsymbol{j}_0 \times \boldsymbol{B}_n. \tag{37}$$

The toroidal part of the cross product is

$$\begin{aligned}
(\boldsymbol{j}_n \times \boldsymbol{B}_0)_\varphi &= R(j_n^Z B_0^R - j_n^R B_0^Z) = R\sqrt{g_P} j_n^\psi B_0^{\mathrm{pol}} && (38)\\
&= in\, p_n - (\boldsymbol{j}_0 \times \boldsymbol{B}_n)_\varphi && (39)\\
&\Rightarrow j_n^\psi \text{on flux surface edge}
\end{aligned}$$

On one triangle edge

$$\begin{aligned}
(\boldsymbol{j}_n \times \boldsymbol{B}_0)_\parallel &= R(j_{n\perp} B_0^\varphi - j_n^\varphi B_0^\perp)\\
&\approx c\frac{p_2 - p_1}{l} - (\boldsymbol{j}_0 \times \boldsymbol{B}_n)_\parallel
\end{aligned}$$

Finite Volume Method

The divergence operator is defined via

$$\nabla \cdot \boldsymbol{u} = \frac{1}{R\sqrt{g_p}} \frac{\partial}{\partial x^k}(R\sqrt{g_p} u^k)\,.$$

When working with R, Z as coordinates in the poloidal plane, $\sqrt{g_p} = 1$. We multiply by R to obtain

$$\frac{\partial}{\partial x^k}(R h^k j_{\parallel n}) + inR h^\varphi j_{\parallel n} = -\frac{\partial}{\partial x^k}(R j_{\perp n}^k) - inR j_{\perp n}^\varphi\,.$$

Integration over a triangle yields

$$\oint R j_{\parallel n} \boldsymbol{h} \cdot \boldsymbol{n} d\Gamma + in \int R h^\varphi j_{\parallel n} d\Omega = -\oint R \boldsymbol{j}_{\perp n}^{\mathrm{pol}} \cdot \boldsymbol{n} d\Gamma - in \int R j_{\perp n}^\varphi d\Omega \tag{40}$$

where scalar products with $\boldsymbol{n}$ pointing towards the outer normal vector of the edge are taken component-wise in R and Z. We assume a field-aligned mesh with $\boldsymbol{h}$ parallel to edge no. 3. The in- and outflux are over edges 1 and 2.

General Finite Volume Method

Eq. (40) is of the form

$$\oint u\, \boldsymbol{h} \cdot \boldsymbol{n} d\Gamma + in \int u\, h^\varphi d\Omega = -\oint \boldsymbol{v} \cdot \boldsymbol{n} d\Gamma - in \int w\, d\Omega$$

with $u = Rj_{\|n}$, $\boldsymbol{v} = R\boldsymbol{j}_{\perp n}^{\mathrm{pol}}$ and $w = Rj_{\perp n}^{\varphi}$. We approximate the flux by a flux value times edge length. Since the mesh is field-aligned, only two of the three triangle edges play a role for fluxes and we can write

$$\oint u\,\boldsymbol{h}\cdot\boldsymbol{n}d\Gamma \approx U_1 + U_2 = u_1 l_1 + u_2 l_2, \tag{41}$$

where

$$U_1 = \int_1 u\boldsymbol{h}\cdot\boldsymbol{n}\mathrm{d}\Gamma_1. \tag{42}$$

and so on. In the second term we can use

$$\int u\,h^{\varphi}d\Omega \approx \frac{u_1+u_2}{2}h^{\varphi}S$$

where S is the surface of the triangle.

For $\boldsymbol{v}$ the normal components to the edges (fluxes through edges) are required via

$$\begin{aligned}\oint \boldsymbol{v}\cdot\boldsymbol{n}d\Gamma &\approx V_1 + V_2 + V_3\\ &= \boldsymbol{v}_1\cdot\boldsymbol{n}_1 l_1 + \boldsymbol{v}_2\cdot\boldsymbol{n}_2 l_2 + \boldsymbol{v}_3\cdot\boldsymbol{n}_3 l_3\end{aligned} \tag{43}$$

Central difference scheme:

$$\dot{p}_H = \frac{\Delta s^{k-1}}{\Delta s^k(\Delta s^k + \Delta s^{k-1})}p^{k+1} + \frac{\Delta s^k - \Delta s^{k-1}}{\Delta s^k \Delta s^{k-1}}p^k - \frac{\Delta s^k}{\Delta s^{k-1}(\Delta s^k + \Delta s^{k-1})}p^{k-1}$$

In harmonics in φ this becomes

$$\boldsymbol{B}\cdot\nabla_{RZ}p_n + inB^{\varphi}p_n = s_n\,. \tag{44}$$

If we take no toroidicity and harmonic RHS term with poloidal harmonic m we obtain

$$\begin{aligned}i(mB^{\vartheta} + nB^{\varphi})p_{mn} &= s_{mn}\\ p_{mn} &= \frac{s_{mn}}{i(mB^{\vartheta} + nB^{\varphi})}\,.\end{aligned}$$

Without toroidicity:

$$\begin{aligned}\sqrt{g} &= r\\ B^{\vartheta} &= \frac{1}{r}\partial_r A_{\varphi}\end{aligned}$$

So for $A_\varphi = r^2/4$ we get $B^\vartheta = 1/2$. Furthermore, we choose $B^\varphi = 1$. We have

$$\begin{aligned} R &= R_0 + r\cos\vartheta \\ Z &= r\sin\vartheta \end{aligned}$$

$$\begin{aligned} \frac{\partial R}{\partial r} &= (R - R_0)/r \\ \frac{\partial Z}{\partial r} &= Z/r \end{aligned}$$

$$\begin{aligned} \frac{\partial R}{\partial \vartheta} &= -z \\ \frac{\partial Z}{\partial \vartheta} &= R - R_0 \end{aligned}$$

$$\begin{aligned} B^R &= \frac{\partial R}{\partial \vartheta} B^\vartheta = -z/2 \\ B^Z &= \frac{\partial Z}{\partial \vartheta} B^\vartheta = (R - R_0)/2 \end{aligned}$$

In real and imaginary parts this is

$$\boldsymbol{B} \cdot \nabla_{RZ}(\Re p_n + i\Im p_n) + inB^\varphi(\Re p_n + i\Im p_n) = (\Re s_n + i\Im s_n) \tag{45}$$

$$\boldsymbol{B} \cdot \nabla_{RZ}\Re p_n - nB^\varphi \Im p_n = \Re s_n \tag{46}$$

$$\boldsymbol{B} \cdot \nabla_{RZ}\Im p_n + nB^\varphi \Re p_n = \Im s_n \tag{47}$$

Combining

$$\begin{aligned} \boldsymbol{B} \cdot \nabla_{RZ}(\boldsymbol{B} \cdot \nabla_{RZ}\Re p_n) - nB^\varphi(\Im s_n - nB^\varphi \Re p_n) &= \boldsymbol{B} \cdot \nabla_{RZ} \cdot \Re s_n \\ \boldsymbol{B} \cdot \nabla_{RZ}(\boldsymbol{B} \cdot \nabla_{RZ}\Im p_n) + nB^\varphi(\Re s_n + nB^\varphi \Im p_n) &= \boldsymbol{B} \cdot \nabla_{RZ} \cdot \Im s_n \end{aligned}$$

In Flat space:

$$\begin{aligned} B^R\partial_R(\boldsymbol{B} \cdot \nabla_{RZ}\Re p_n) &+ B^Z\partial_Z(\boldsymbol{B} \cdot \nabla_{RZ}\Re p_n) \\ &= (B^R\partial_R + B^Z\partial_Z)(B^R\partial_R + B^Z\partial_Z)\Re p_n \\ = \Big((B^R)^2\partial_R^2 &+ 2B^R B^Z \partial_R\partial_Z + \left(B^Z\right)^2 \partial_Z^2\Big)\Re p_n \end{aligned}$$

This equation is parabolic and not, as such, suited for FEM.

New:

$$\begin{aligned} \boldsymbol{B} \cdot \nabla_{RZ}(\boldsymbol{B} \cdot \nabla_{RZ}\Re p_n) &= \nabla_{RZ} \cdot (\boldsymbol{B}(\boldsymbol{B} \cdot \nabla_{RZ}\Re p_n)) \\ &= \nabla_{RZ} \cdot (\boldsymbol{B}\nabla_{RZ} \cdot (\boldsymbol{B}\Re p_n)) \end{aligned}$$

In real and imaginary parts this is

$$\begin{aligned} i(mB^\vartheta + nB^\varphi)(\Re p_{mn} + i\Im p_{mn}) &= (\Re s_{mn} + i\Im s_{mn}) \\ \Im p_{mn} &= -\Re s_{mn}/(mB^\vartheta + nB^\varphi) \\ \Re p_{mn} &= \Im s_{mn}/(mB^\vartheta + nB^\varphi) \end{aligned}$$

We have

$$s = \sum_n s_n(\vartheta)e^{in\varphi} = \sum_{mn} s_{mn}e^{i(m\vartheta+n\varphi)}$$

$$\begin{aligned}
&= \sum_n (\Re s_n + i\Im s_n)(\cos n\varphi + i\sin n\varphi) \\
&= \sum_n (\Re s_n \cos n\varphi - \Im s_n \sin n\varphi) + i(\Re s_n \sin n\varphi + \Im s_n \cos n\varphi)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{mn} (\Re s_{mn} + i\Im s_{mn})(\cos(m\vartheta + n\varphi) + i\sin(m\vartheta + n\varphi)) \\
&= \sum_{mn} (\Re s_{mn} \cos(m\vartheta + n\varphi) - \Im s_{mn} \sin(m\vartheta + n\varphi)) \\
&+ i(\Re s_{mn} \sin(m\vartheta + n\varphi) + \Im s_{mn} \cos(m\vartheta + n\varphi))
\end{aligned}$$

$$\begin{aligned}
s_n &= s_{mn}e^{im\vartheta} = (\Re s_{mn} + i\Im s_{mn})(\cos m\vartheta + i\sin m\vartheta) \\
&= \Re s_{mn} \cos m\vartheta - \Im s_{mn} \sin m\vartheta + i(\Re s_{mn} \sin m\vartheta + \Im s_{mn} \cos m\vartheta)
\end{aligned}$$

Test:

$$\begin{aligned}
s &= \Im s_{mn}(\cos(m\vartheta + n\varphi) - i\sin(m\vartheta + n\varphi)) \\
s_n &= s_{mn}e^{im\vartheta} = \Im s_{mn}(-\sin m\vartheta + i\cos m\vartheta) \\
&=
\end{aligned}$$

$$\Re p_{mn} = \Im s_{mn}/(mB^\vartheta + nB^\varphi)$$

$$p_n = \Re p_{mn}(\cos m\vartheta + i\sin m\vartheta)$$

Pseudotoroidal coordinates

$$\begin{aligned}
R &= R_0 + r\cos\vartheta \\
Z &= r\sin\vartheta
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{e}_r &= \frac{\partial R}{\partial r}\boldsymbol{e}_R + \frac{\partial Z}{\partial r}\boldsymbol{e}_Z \\
&= \boldsymbol{e}_R \cos\vartheta + \boldsymbol{e}_Z \sin\vartheta
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{e}_\vartheta &= \frac{\partial R}{\partial \vartheta}\boldsymbol{e}_R + \frac{\partial Z}{\partial \vartheta}\boldsymbol{e}_Z \\
&= -\boldsymbol{e}_R r\sin\vartheta + \boldsymbol{e}_Z r\cos\vartheta
\end{aligned}$$