Magnetic differential equations for stationary linear ideal MHD and their numerical solution

20th March 2018

Stationary linear perturbation of ideal MHD equilibrium

For the intended application on stationary (compared to MHD mode eigenfrequencies) non-axisymmetric magnetic perturbations by external coils, we consider a perturbed ideal MHD equilibrium for pressure p, currents $\boldsymbol{J}$ and magnetic field $\boldsymbol{B}$ fulfilling

$$\nabla p = \frac{1}{c}\boldsymbol{j} \times \boldsymbol{B}, \tag{1}$$

$$\nabla \times \boldsymbol{B} = \frac{4\pi}{c}\boldsymbol{j}, \tag{2}$$

$$\nabla \cdot \boldsymbol{B} = 0. \tag{3}$$

Starting with a given MHD equilibrium fulfilling Eqs. (1-3) denoted by subscripts "0", linear order equations for an external magnetic perturbation (denoted by δ) split into a vacuum and a plasma part (subscript v and p, respectively) are

$$\nabla \delta p = \frac{1}{c}\left(\boldsymbol{j}_0 \times \delta\boldsymbol{B} + \delta\boldsymbol{j} \times \boldsymbol{B}_0\right), \tag{4}$$

$$\delta\boldsymbol{B} = \delta\boldsymbol{B}_v + \delta\boldsymbol{B}_p, \tag{5}$$

$$\delta\boldsymbol{B}_v = \frac{1}{c}\oint \frac{I_c(\boldsymbol{r}')\mathrm{d}\boldsymbol{l}' \times \boldsymbol{r}}{|\boldsymbol{r} - \boldsymbol{r}'|^3}, \tag{6}$$

$$\delta\boldsymbol{B}_p = \nabla \times \delta\boldsymbol{A}, \tag{7}$$

$$\nabla \times (\nabla \times \delta\boldsymbol{A}) = \frac{4\pi}{c}\delta\boldsymbol{j}, \tag{8}$$

$$\Rightarrow \nabla \cdot \delta\boldsymbol{B} = \nabla \cdot \delta\boldsymbol{j} = 0. \tag{9}$$

Here the perturbation field in vacuum, $\delta\boldsymbol{B}_v$, is pre-evaluated by a Biot-Savart integral over external coil currents $I_c(\boldsymbol{r}')$ and the perturbation field in plasma $\delta\boldsymbol{B}_p$ is computed from Ampère's law (8) from the plasma current density $\delta\boldsymbol{j}$ using a vector potential formulation. To find a consistent solution for the system, Eq. (4) and Eq. (8) are treated individually in an iterative way. The linearized force balance equation (4) is used to compute $\delta\boldsymbol{j}$ for given $\delta\boldsymbol{B}$ whereas Eq. (8) yields $\delta\boldsymbol{B}_p$ for given $\delta\boldsymbol{j}$. In the first iteration, $\delta\boldsymbol{B}$ is set equal to $\delta\boldsymbol{B}_v$ in Eq. (4). Then, Eq. (8) and Eq. (4) are solved in an alternating way until convergence is reached. In addition a preconditioner is used to enhance convergence. Here we limit the analysis to an axisymmetric unperturbed equilibrium and a single toroidal perturbation harmonic $\delta\boldsymbol{B} = \mathrm{Re}(\boldsymbol{B}_n e^{in\varphi})$ with a similar notation for other perturbed quantities. As all equations are linear, a superposition of multiple harmonics is easily possible.

Linearized MHD force balance

The solution of Eq. (4) can further be split into two steps: First the pressure perturbation δp is found, and then the plasma current density $\delta \boldsymbol{j}$ is computed using the condition $\nabla \cdot \delta \boldsymbol{j} = 0$. For an unperturbed equilibrium with nested flux surfaces, both steps can be performed in a radially local manner if a field-aligned computational grid is used, what will become clear below. Radial coupling happens by the combination of the two individual steps since their effective radial locations of computation are shifted by a half-step in radial grid distance.

In axisymmetric coordinate systems, such as cylindrical (R, φ, Z), the equations to solve for harmonics in the toroidal angle φ are

$$\nabla p_n + inp_n \nabla \varphi = \frac{1}{c} \left(\boldsymbol{j}_0 \times \boldsymbol{B}_n + \boldsymbol{j}_n \times \boldsymbol{B}_0 \right), \tag{10}$$

$$\nabla \cdot \boldsymbol{j}_n^{\mathrm{pol}} + inj_n^{\varphi} = 0. \tag{11}$$

now with a 2D ∇ operator acting in the poloidal (RZ) plane. The divergence operator is defined via

$$\nabla \cdot \boldsymbol{u} = \frac{1}{R\sqrt{g_p}} \frac{\partial}{\partial x^k} (R\sqrt{g_p} u^k),$$

where $\sqrt{g_p}$ is the metric tensor of the coordinates in the poloidal plane, which is equal to 1 for cylindrical coordinates.

Representation of equilibrium field

$\boldsymbol{B}_0$ is given by

$$\boldsymbol{B}_0 = \boldsymbol{B}_0^{\mathrm{pol}} + \boldsymbol{B}_0^{\mathrm{tor}}, \tag{12}$$

where

$$\boldsymbol{B}_0^{\mathrm{pol}} = \nabla \psi \times \nabla \varphi, \tag{13}$$

$$\boldsymbol{B}_0^{\mathrm{tor}} = B_{0\varphi} \nabla \varphi. \tag{14}$$

Pressure perturbation

Already working, TODO: update description.

Current perturbation

Multiplying Eq. (11) by R yields

$$\frac{\partial}{\partial x^k} (Rj_n^k) + inRj_n^{\varphi} = 0. \tag{15}$$

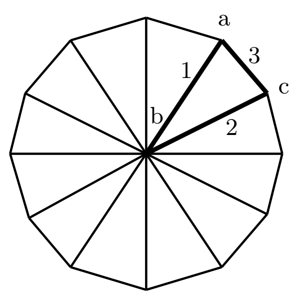
Using the divergence theorem this can also be written in integral form in a specific triangular mesh element Ω_i as

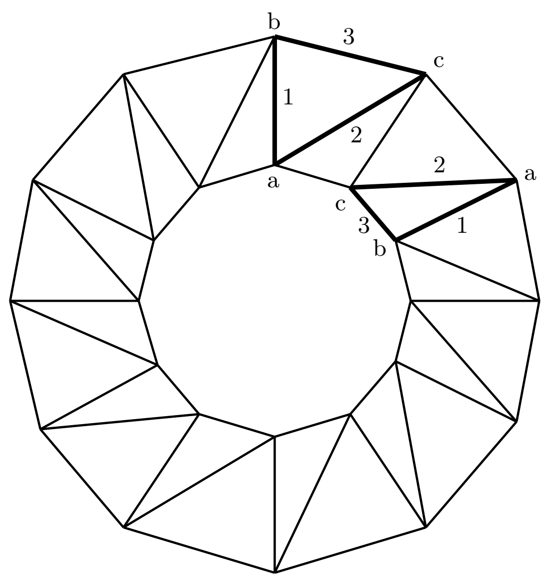
$$\oint_{\partial\Omega_i} \mathrm{d}l \, R\boldsymbol{j}_n \cdot \boldsymbol{n} + in \int_{\Omega_i} \mathrm{d}R\mathrm{d}Z \, Rj_n^{\varphi} = 0. \tag{16}$$

Here the first integral is performed over the 1-dimensional element boundary $\partial\Omega_i$. The first term is split into three contributions,

$$\oint_{\partial\Omega_i} \mathrm{d}l \, R\boldsymbol{j}_n \cdot \boldsymbol{n} = \int_{1,2} \mathrm{d}l \, R\boldsymbol{j}_n \cdot \boldsymbol{n} + \int_{3} \mathrm{d}l \, R\boldsymbol{j}_n \cdot \boldsymbol{n}, \tag{17}$$

where edge 3 is tangential to an adjacent flux surface and edges 1 and 2 are not. For the grid it is important to distinguish two types of triangles, starting from the second loop outside the magnetic axis:





Coordinate system on each edge

For each edge (length l) we use a local orthogonal coordinate system on each triangle edge with $\boldsymbol{l}$ the vector of length l along the edge in counter-clockwise orientation, $\boldsymbol{n}$ the outward normal (length l, in contrast to $\boldsymbol{n}$ under the integrals) and $\nabla\varphi$ pointing inside the plane. We obtain relations

$$\boldsymbol{l} \times \boldsymbol{n} = l^2 R \nabla\varphi. \tag{18}$$

$$\boldsymbol{n} \times \nabla\varphi = \boldsymbol{l}/R, \tag{19}$$

$$\nabla\varphi \times \boldsymbol{l} = \boldsymbol{n}/R. \tag{20}$$

Cross-field currents on edge 3

It can be shown (writeup gyrokinetics.pdf) that

$$\begin{aligned}\boldsymbol{j}_{\perp n} &= \boldsymbol{j}_n - (\boldsymbol{j}_n \cdot \boldsymbol{h}_0)\boldsymbol{h}_0 \\ &= j_{0\|}\frac{\boldsymbol{B}_{\perp n}}{B_0} - \frac{cB_{\|n}}{B_0^2}\boldsymbol{h}_0 \times \nabla p_0 + \frac{c}{B_0}\boldsymbol{h}_0 \times (\nabla p_n + inp_n\nabla\varphi). \end{aligned} \tag{21}$$

Scalar multiplication with $\boldsymbol{n}_3 \parallel \nabla p_0 \parallel \nabla\psi \perp \boldsymbol{h}_0$ yields

$$\begin{aligned}\boldsymbol{j}_{\perp n} \cdot \boldsymbol{n}_3 = \boldsymbol{j}_{\perp n}^{\text{pol}} \cdot \boldsymbol{n}_3 &= j_{0\|}\frac{\boldsymbol{B}_n \cdot \boldsymbol{n}_3}{B_0} + \frac{c}{B_0}\boldsymbol{n}_3 \cdot (\boldsymbol{h}_0 \times (\nabla p_n + inp_n\nabla\varphi)) \\ &= j_{0\|}\frac{\boldsymbol{B}_n \cdot \boldsymbol{n}_3}{B_0} + \frac{c}{B_0}\boldsymbol{n}_3 \cdot (h_{0\varphi}\nabla\varphi \times \nabla p_n + inp_n\boldsymbol{h}_0^{\text{pol}} \times \nabla\varphi) \\ &= j_{0\|}\frac{\boldsymbol{B}_n \cdot \boldsymbol{n}_3}{B_0} + \frac{c}{B_0}(h_{0\varphi}\nabla p_n \cdot (\boldsymbol{n}_3 \times \nabla\varphi) + inp_n\boldsymbol{h}_0^{\text{pol}}(\nabla\varphi \times \boldsymbol{n}_3)) \\ &= j_{0\|}\frac{\boldsymbol{B}_n \cdot \boldsymbol{n}_3}{B_0} + \frac{c}{RB_0}\boldsymbol{l}_3 \cdot (h_{0\varphi}\nabla p_n - inp_n\boldsymbol{h}_0^{\text{pol}}) \\ &= j_{0\|}\frac{\boldsymbol{B}_n \cdot \boldsymbol{n}_3}{B_0} + \frac{c}{B_0^2}\boldsymbol{l}_3 \cdot \left(B_{0(\varphi)}\nabla p_n - \frac{in}{R}p_n\boldsymbol{B}_0^{\text{pol}}\right). \end{aligned} \tag{22}$$

Toroidal unperturbed current

We take a cross-product of

$$\boldsymbol{j}_0 \times \boldsymbol{B}_0 = c\nabla p_0 \tag{23}$$

by $\boldsymbol{B}_0$:

$$\begin{aligned}\boldsymbol{B}_0 \times (\boldsymbol{j}_0 \times \boldsymbol{B}_0) &= B_0^2\boldsymbol{j}_0 - (\boldsymbol{B}_0 \cdot \boldsymbol{j}_0)\boldsymbol{B}_0 \\ &= B_0^2(\boldsymbol{j}_0 - j_{0\|}\boldsymbol{h}_0) \\ &= B_0^2\boldsymbol{j}_{0\perp}. \end{aligned} \tag{24}$$

Therefore

$$\boldsymbol{j}_{0\perp} = \frac{-c\nabla p_0 \times \boldsymbol{B}_0}{B_0^2} \tag{25}$$

$$\begin{aligned} &= \frac{-cp_0'(\psi)\nabla\psi \times (\nabla\psi \times \nabla\varphi + B_{0\varphi}\nabla\varphi)}{B_0^2} \\ &= \frac{-cp_0'(\psi)}{B_0^2}\left(B_{0\varphi}\boldsymbol{B}_0^{\text{pol}} - |\nabla\psi|^2\nabla\varphi\right). \end{aligned} \tag{26}$$

which is the diamagnetic current density. For the parallel current density we use

$$\begin{aligned}0 = \nabla \cdot \boldsymbol{j}_0 &= \nabla \cdot \boldsymbol{j}_{0\perp} + \nabla \cdot (j_{0\|}\boldsymbol{h}_0) \\ &= -c\nabla \cdot \left(\frac{\nabla p_0 \times \boldsymbol{B}_0}{B_0^2}\right) + \boldsymbol{B}_0 \cdot \nabla\left(\frac{j_{0\|}}{B_0}\right). \end{aligned} \tag{27}$$

In straight-field line magnetic flux coordinates (r, ϑ, φ) with Jacobian $\sqrt{g}$ and the divergence of the diamagnetic current is

$$\begin{aligned}\nabla \cdot \boldsymbol{j}_{0\perp} &= -\frac{c}{\sqrt{g}} \frac{\partial}{\partial x^k} \left[\frac{\sqrt{g}}{B_0^2} \left(\nabla p_0 \times \boldsymbol{B}_0\right)^k \right] \\ &= -\frac{c}{\sqrt{g}} \frac{\partial}{\partial x^k} \left(\frac{\sqrt{g}}{B_0^2} \frac{\varepsilon^{ijk}}{\sqrt{g}} \frac{\partial p_0}{\partial x^i} B_{0j} \right) \\ &= \frac{c p_0'(r) B_{0\varphi}}{\sqrt{g}} \frac{\partial}{\partial \vartheta} \left(\frac{1}{B_0^2} \right), \end{aligned} \tag{28}$$

since p_0 and $B_{0\varphi}$ are constant on a flux surface, $\frac{\partial p_0}{\partial \vartheta} = 0$ and due to axisymmetry $\frac{\partial}{\partial \varphi} = 0$. The divergence of the parallel current is

$$\nabla \cdot (j_{0\parallel} \boldsymbol{h}_0) = B_0^\vartheta \frac{\partial}{\partial \vartheta} \left(\frac{j_{0\parallel}}{B_0} \right). \tag{29}$$

With $\sqrt{g} B_0^\vartheta = \psi'(r)$ as a flux surface quantity, there are no dependencies of ϑ in front of the derivatives and direct integration yields

$$j_{0\parallel} = -\frac{c p_0'(\psi) B_{0\varphi}}{B_0} + C(\psi) B_0. \tag{30}$$

With the extra condition of the flux-surface average $\left\langle j_{0\parallel} B_0 \right\rangle = 0$ for testing without bootstrap current, we obtain

$$\begin{aligned} 0 &= c p_0'(\psi) B_{0\varphi} + C(\psi) \left\langle B_0^2 \right\rangle \\ \Rightarrow C(\psi) &= \frac{c p_0'(\psi) B_{0\varphi}}{\langle B_0^2 \rangle}. \end{aligned} \tag{31}$$

In general,

$$C(\psi) = \frac{c p_0'(\psi) B_{0\varphi}}{\langle B_0^2 \rangle} D(\psi), \tag{32}$$

$$j_{0\parallel} = -\frac{c p_0'(\psi) B_{0\varphi}}{B_0} \left(\frac{1}{B_0^2} - \frac{D(\psi)}{\langle B_0^2 \rangle} \right). \tag{33}$$

with $D(\psi)$ set to 1 for now.

For the unperturbed toroidal current density we have

$$j_0^\varphi = j_{0\parallel} h_0^\varphi + \boldsymbol{j}_{0\perp} \cdot \nabla \varphi, \tag{34}$$

where

$$\begin{aligned} j_{0\parallel} h_0^\varphi &= -B_0 c p_0'(\psi) B_{0\varphi} \frac{B_0^\varphi}{B_0} \left(\frac{1}{B_0^2} - \frac{D(\psi)}{\langle B_0^2 \rangle} \right) \\ &= -c p_0'(\psi) \left(B_0^{\mathrm{tor}}\right)^2 \left(\frac{1}{B_0^2} - \frac{D(\psi)}{\langle B_0^2 \rangle} \right). \end{aligned} \tag{35}$$

and

$$
\begin{aligned}
\boldsymbol{j}_{0\perp} \cdot \nabla\varphi &= \frac{-cp_0'(\psi)\nabla\varphi \cdot (\nabla\psi \times \boldsymbol{B}_0)}{B_0^2} \\
&= \frac{-cp_0'(\psi)\boldsymbol{B}_0 \cdot (\nabla\varphi \times \nabla\psi)}{B_0^2} \qquad (36) \\
&= \frac{cp_0'(\psi)}{B_0^2}\boldsymbol{B}_0^{\mathrm{pol}} \cdot \boldsymbol{B}_0 \\
&= cp_0'(\psi)\frac{\left(B_0^{\mathrm{pol}}\right)^2}{B_0^2}. \qquad (37)
\end{aligned}
$$

Toroidal current perturbation

For the computation of toroidal j_n^φ in the element volume we start with

$$\boldsymbol{j}_n \times \boldsymbol{B}_0 = c(\nabla p_n + in\, p_n \nabla\varphi) - \boldsymbol{j}_0 \times \boldsymbol{B}_n. \qquad (38)$$

with

$$\boldsymbol{B}_0 = \nabla\psi \times \nabla\varphi + B_{0\varphi}\nabla\varphi. \qquad (39)$$

Taking a scalar product of $\boldsymbol{l}$ with Eq. (38) on some edge yields

$$
\begin{aligned}
\boldsymbol{l} \cdot (\boldsymbol{j}_n \times \boldsymbol{B}_0) &= \boldsymbol{l} \cdot (\boldsymbol{j}_n \times (\nabla\psi \times \nabla\varphi + B_{0\varphi}\nabla\varphi)) \\
&= \boldsymbol{j}_n \cdot ((\nabla\psi \times \nabla\varphi) \times \boldsymbol{l} + B_{0\varphi}\nabla\varphi \times \boldsymbol{l}) \\
&= \boldsymbol{j}_n \cdot ((\nabla\psi \times \nabla\varphi) \times \boldsymbol{l} + B_{0\varphi}\boldsymbol{n}/R) \\
&= \boldsymbol{j}_n \cdot \big((\boldsymbol{l} \cdot \nabla\psi)\nabla\varphi + B_{0(\varphi)}\boldsymbol{n}\big) \\
&= (\boldsymbol{l} \cdot \nabla\psi) j_n^\varphi + B_{0(\varphi)}\boldsymbol{j}_n^{\mathrm{pol}} \cdot \boldsymbol{n}. \qquad (40)
\end{aligned}
$$

Further,

$$\boldsymbol{l} \cdot \nabla\psi = R\nabla\psi \cdot (\boldsymbol{n} \times \nabla\varphi) = -R\boldsymbol{n} \cdot \boldsymbol{B}_0^{\mathrm{pol}}. \qquad (41)$$

Finally,

$$\boldsymbol{l} \cdot (\boldsymbol{j}_n \times \boldsymbol{B}_0) = B_{0(\varphi)}\boldsymbol{j}_n^{\mathrm{pol}} \cdot \boldsymbol{n} - Rj_n^\varphi \boldsymbol{n} \cdot \boldsymbol{B}_0^{\mathrm{pol}}. \qquad (42)$$

The right-hand side of Eq. (38) yields:

$$
\begin{aligned}
\boldsymbol{l} \cdot (\nabla p_n + in\, p_n \nabla\varphi) &= \boldsymbol{l} \cdot \nabla p_n \qquad (43) \\
\boldsymbol{l} \cdot (\boldsymbol{j}_0 \times \boldsymbol{B}_n) &= \boldsymbol{l} \cdot (B_{n\varphi}\boldsymbol{j}_0^{\mathrm{pol}} \times \nabla\varphi + j_{0\varphi}\nabla\varphi \times \boldsymbol{B}_n^{\mathrm{pol}}). \qquad (44)
\end{aligned}
$$

We use the fact that ∇p_0 is parallel to $\nabla\psi$, so the cross product in the equilibrium is purely radial,

$$
\begin{aligned}
\boldsymbol{j}_0 \times \boldsymbol{B}_0 &= c\nabla p_0 \\
&= \boldsymbol{j}_0^{\mathrm{pol}} \times (B_{0\varphi}\nabla\varphi) + j_{0\varphi}\nabla\varphi \times (\nabla\psi \times \nabla\varphi) \\
&= \boldsymbol{j}_0^{\mathrm{pol}} \times (B_{0\varphi}\nabla\varphi) + \frac{j_{0\varphi}}{R^2}\nabla\psi. \qquad (45)
\end{aligned}
$$

Thus

$$\boldsymbol{j}_0^{\mathrm{pol}} \times \nabla\varphi = \frac{1}{B_{0\varphi}} \left(c\nabla p_0 - \frac{j_{0\varphi}}{R^2}\nabla\psi \right). \tag{46}$$

Also

$$\begin{aligned} \boldsymbol{l} \cdot (\nabla\varphi \times \boldsymbol{B}_n^{\mathrm{pol}}) &= \boldsymbol{B}_n^{\mathrm{pol}} \cdot (\boldsymbol{l} \times \nabla\varphi) \\ &= -\frac{1}{R}\boldsymbol{B}_n^{\mathrm{pol}} \cdot \boldsymbol{n}. \end{aligned} \tag{47}$$

Finally

$$(\boldsymbol{l} \cdot \nabla\psi) j_n^{\varphi} + B_{0(\varphi)}\boldsymbol{j}_n^{\mathrm{pol}} \cdot \boldsymbol{n} = c\boldsymbol{l} \cdot \nabla p_n - \frac{B_{n\varphi}}{B_{0\varphi}} \left(c\boldsymbol{l} \cdot \nabla p_0 - \frac{j_{0\varphi}}{R^2}\boldsymbol{l} \cdot \nabla\psi \right) + \frac{j_{0\varphi}}{R}\boldsymbol{B}_n^{\mathrm{pol}} \cdot \boldsymbol{n} \tag{48}$$

This term is only meaningful to obtain j_n^{φ} on edges 1 and 2 where $\boldsymbol{l} \cdot \nabla\psi \neq 0$ and we obtain

$$Rj_n^{\varphi} = j_{n(\varphi)} = -\frac{B_{0(\varphi)}}{(\boldsymbol{l} \cdot \nabla\psi)} R\boldsymbol{j}_n^{\mathrm{pol}} \cdot \boldsymbol{n} + \frac{cR}{(\boldsymbol{l} \cdot \nabla\psi)} \left(\boldsymbol{l} \cdot \nabla p_n - \frac{B_{n\varphi}}{B_{0\varphi}}\boldsymbol{l} \cdot \nabla p_0 \right) + j_{0\varphi} \left(\frac{B_{n\varphi}}{RB_{0\varphi}} + \frac{\boldsymbol{n} \cdot \boldsymbol{B}_n^{\mathrm{pol}}}{\boldsymbol{l} \cdot \nabla\psi} \right). \tag{49}$$

In the implementation, an average over edge 1 and 2 will be taken for this quantity.

Implementation

We start with

$$I_1 + I_2 + in \int Rj_n^{\varphi} dS = -I_3, \tag{50}$$

where the notation for currents through edges, weighted by R, is

$$I_k = \int_k R\,\boldsymbol{j}_n^{\mathrm{pol}} \cdot \boldsymbol{n} dl \approx R_k\,\boldsymbol{j}_n^{\mathrm{pol}} \cdot \boldsymbol{n}_k, \tag{51}$$

where values are taken constant along the edge at the midpoint. I_3 is already known with

$$I_3 = \frac{c}{B_0^2}\boldsymbol{l}_3 \cdot \left(R_3 B_{0(\varphi)} \nabla p_n - inp_n \boldsymbol{B}_0^{\mathrm{pol}} \right), \tag{52}$$

and therefore acts as a source on the right-hand side. The remaining currents I_1 and I_2 are taken as unknowns and appear also in the last term of the left-hand side as

$$\begin{aligned} in \int Rj_n^{\varphi} d\Omega &\approx inS_\Omega (R_1 j_{n1}^{\varphi} + R_2 j_{n2}^{\varphi})/2 \\ &= -\frac{inS_\Omega}{2} \left(\frac{B_{0(\varphi),1}}{(\boldsymbol{l}_1 \cdot \nabla\psi)} I_1 + \frac{B_{0(\varphi),2}}{(\boldsymbol{l}_2 \cdot \nabla\psi)} I_2 + \dots \right), \end{aligned} \tag{53}$$

where S_Ω is the triangle surface area and the remaining terms are moved to the right-hand-side as sources q. In each triangle Ω the discretized equation is thus

$$\left(1 - \frac{inS_\Omega}{2} \frac{B_{0(\varphi),1}}{(\boldsymbol{l}_1 \cdot \nabla\psi)} \right) I_1 + \left(1 - \frac{inS_\Omega}{2} \frac{B_{0(\varphi),2}}{(\boldsymbol{l}_2 \cdot \nabla\psi)} \right) I_2 = q. \tag{54}$$

In general matrix form, we call the ingoing current into triangle Nr. (k) counted in clockwise direction $I^{(k)}$. In triangle (k), this is equal to $I_1 = -I^{(k)}$ and $I_2 = I^{(k+1)}$. The matrix form of Eq. (54) is then

$$A_{jk} I^{(k)} = \boldsymbol{q}, \tag{55}$$

where the elements of the stiffness matrix A are

$$A_{jk} = \left(1 - \frac{inS_{\Omega k}}{2} \frac{B_{0(\varphi)k}}{\Delta\psi}\right) \delta_{(j-1)k} + \left(-1 - \frac{inS_{\Omega k}}{2} \frac{B_{0(\varphi)k}}{\Delta\psi}\right) \delta_{jk}. \tag{56}$$

Here, $\delta_{(j-1)k}$ are the outgoing currents into the next triangle in counter-clockwise direction, and $\Delta\psi$ is the difference in ψ counting in the outwards direction. $\boldsymbol{q}$ consists of entries $q^k = -I_3^k - inS_{\Omega k} w^k$ (no summation over k) where

$$w^k = \frac{cR}{\Delta\psi}\left(\Delta p_n - \frac{B_{n\varphi}}{B_{0\varphi}} \Delta p_0\right) + R j_{0(\varphi)} \left(\frac{B_{n\varphi}}{B_{0\varphi}} + \frac{R\boldsymbol{n} \cdot \boldsymbol{B}_n^{\mathrm{pol}}}{\boldsymbol{l} \cdot \nabla\psi}\right). \tag{57}$$

Generating a non-resonant test field

For derivation we use symmetry flux coordinates $(\psi, \vartheta, \varphi)$ with Jacobian

$$\sqrt{g} = \frac{1}{B_0^{\vartheta}} = \frac{q}{B_0^{\varphi}} = \frac{qR^2}{B_{0\varphi}}. \tag{58}$$

We would like to have a completely non-resonant field with $B_n^{\vartheta} = 0$. As an ansatz we take

$$B_n^{\varphi} = B_n^{\psi} F(\psi). \tag{59}$$

and the normal component of the test field

$$B_n^{\psi} = \frac{B_{0\varphi} R_0}{R^2}. \tag{60}$$

Divergence-freeness of $\boldsymbol{B}_n$ yields

$$\begin{aligned}
0 &= \frac{\partial}{\partial\psi}(\sqrt{g} B_n^{\psi}) + in\sqrt{g} B_n^{\psi} F(\psi) \\
&= \frac{\partial}{\partial\psi}\left(\frac{qR^2}{B_{0\varphi}} \frac{B_{0\varphi} R_0}{R^2}\right) + inF(\psi) \frac{qR^2}{B_{0\varphi}} \frac{B_{0\varphi} R_0}{R^2} \\
&= R_0 \left(\frac{\partial q}{\partial\psi} + inF(\psi) q\right).
\end{aligned} \tag{61}$$

To fulfill this, we take

$$F(\psi) = \frac{i}{nq} \frac{\partial q}{\partial\psi}. \tag{62}$$

With known B_n^{ψ} and B_n^{φ} we now proceed to find fluxes through triangle edges. Divergence-freeness in cylindrical coordinates gives

$$\int_{1,2} \mathrm{d}l \, R\boldsymbol{B}_n^{\mathrm{pol}} \cdot \boldsymbol{n} = -\int_3 \mathrm{d}l \, R\boldsymbol{B}_n^{\mathrm{pol}} \cdot \boldsymbol{n} - in \int_{\Omega_i} \mathrm{d}R \mathrm{d}Z \, R B_n^{\varphi}. \tag{63}$$

As for currents we use the notation for weighted magnetic fluxes through edges,

$$\Psi_k = \int_k R\, \boldsymbol{B}_n^{\mathrm{pol}} \cdot \boldsymbol{n} dl \approx R_k\, \boldsymbol{B}_n^{\mathrm{pol}} \cdot \boldsymbol{n}_k. \tag{64}$$

The term through edge l_3 orthogonal to $n_3 \parallel \nabla\psi$ is

$$-\Psi_3 \approx \mp R_3 l_3 \frac{B_n^\psi}{|\nabla\psi|}$$

with sign depending on edge orientation. The second term yields

$$-\Psi_\varphi = -in \int_{\Omega_i} \mathrm{d}R\mathrm{d}Z\, RB_n^\varphi \approx -inS_{\Omega k}\, R_k B_{nk}^\varphi.$$

The equation to solve is

$$A_{jk} I^{(k)} = \boldsymbol{q}, \tag{65}$$

with

$$A_{jk} = \delta_{(j-1)k} - \delta_{jk}, \tag{66}$$

and

$$q^k = \mp R_3 l_3 \frac{B_n^\psi}{|\nabla\psi|} - inS_{\Omega k}\, R_k B_{nk}^\varphi. \tag{67}$$

Here, a compatibility condition needs to be fulfilled - the flux through one flux surface shell, so the sum over all triangles k must vanish with

$$\sum_k (\Psi_\varphi^{(k)} + \Psi_3^{(k)}) = 0. \tag{68}$$

Safety factor

Toroidal flux is

$$\begin{aligned}\psi_{\mathrm{tor}} &= \int \mathrm{d}V\, \boldsymbol{B} \cdot \nabla\varphi \\ &= 2\pi \int \mathrm{d}R\mathrm{d}Z\, RB^\varphi = \int \mathrm{d}R\mathrm{d}Z B_{(\varphi)}.\end{aligned} \tag{69}$$

Current perturbation - alternative

We split the current into parallel and perpendicular current:

$$\boldsymbol{j}_n = j_{\parallel n} \boldsymbol{h}_0 + \boldsymbol{j}_{\perp n}. \tag{70}$$

We write the divergence-freeness condition as

$$\nabla \cdot (\boldsymbol{h}_0^{\mathrm{pol}} j_{\parallel n}) + inh_0^\varphi j_{\parallel n} = -\nabla \cdot \boldsymbol{j}_{\perp n}^{\mathrm{pol}} - inj_{\perp n}^\varphi. \tag{71}$$

In integral form, multiplied by R:

$$\oint R j_{\parallel n} \boldsymbol{h}_0^{\mathrm{pol}} \cdot \boldsymbol{n} \mathrm{d}l + in \int R h_0^{\varphi} j_{\parallel n} \mathrm{d}S = -\oint R \boldsymbol{j}_{\perp n}^{\mathrm{pol}} \cdot \boldsymbol{n} \mathrm{d}l - in \int R j_{\perp n}^{\varphi} \mathrm{d}S. \tag{72}$$

Approximate

$$\begin{aligned} &R^1 j_{\parallel n}^1 \boldsymbol{h}_0^{\mathrm{pol}} \cdot \boldsymbol{n}^1 + R^2 j_{\parallel n}^2 \boldsymbol{h}_0^{\mathrm{pol}} \cdot \boldsymbol{n}^2 + in S_\Omega \frac{(h_{0(\varphi)}^1 j_{\parallel n}^1 + h_{0(\varphi)}^2 j_{\parallel n}^2)}{2} = \\ &-R^1 \boldsymbol{j}_{\perp n}^{\mathrm{pol}\,1} \cdot \boldsymbol{n}^1 - R^2 \boldsymbol{j}_{\perp n}^{\mathrm{pol}\,2} \cdot \boldsymbol{n}^2 - R^3 \boldsymbol{j}_{\perp n}^{\mathrm{pol}\,3} \cdot \boldsymbol{n}^3 - in S_\Omega R j_{\perp n}^{\varphi}. \end{aligned} \tag{73}$$

We use

$$j_{\perp n}^{\varphi} = TODO - \frac{cB_{\parallel n}}{B_0^2} \nabla\varphi \cdot (\boldsymbol{h}_0 \times \nabla p_0) + \frac{c}{B_0} \nabla\varphi \cdot (\boldsymbol{h}_0 \times \nabla p_n) \tag{74}$$

$$= -\frac{cB_{\parallel n}}{B_0^3} \nabla p_0 \cdot ((\nabla\psi \times \nabla\varphi) \times \nabla\varphi) + \frac{c}{B_0^2} \nabla p_n \cdot ((\nabla\psi \times \nabla\varphi) \times \nabla\varphi) \tag{75}$$

$$= \frac{cB_{\parallel n}}{R^2 B_0^3} \nabla p_0 \cdot \nabla\psi - \frac{c}{R^2 B_0^2} \nabla p_n \cdot \nabla\psi \tag{76}$$

$$= \frac{cB_{\parallel n}}{R^2 B_0^3} p_0'(\psi) |\nabla\psi|^2 - \frac{c}{R^2 B_0^2} \nabla p_n \cdot \nabla\psi \tag{77}$$

We would like to represent

$$\nabla\psi = (\nabla\psi)_1 \boldsymbol{l}^1 + (\nabla\psi)_2 \boldsymbol{l}^2. \tag{78}$$

We construct the system

$$\begin{aligned} \boldsymbol{l}^1 \cdot \nabla\psi &= |l^1|^2 (\nabla\psi)_1 + \boldsymbol{l}^1 \cdot \boldsymbol{l}^2 (\nabla\psi)_2 \\ \boldsymbol{l}^2 \cdot \nabla\psi = -(\boldsymbol{l}^1 \cdot \nabla\psi) &= \boldsymbol{l}^1 \cdot \boldsymbol{l}^2 (\nabla\psi)_1 + |l^2|^2 (\nabla\psi)_2 \end{aligned}$$

or generally

$$\begin{aligned} v_1 &= \frac{|l^2|^2 + (\boldsymbol{l}^1 \cdot \boldsymbol{l}^2)}{|l^1|^2 |l^2|^2 - (\boldsymbol{l}^1 \cdot \boldsymbol{l}^2)^2} (\boldsymbol{l}^1 \cdot \boldsymbol{v}) \\ v_2 &= \frac{|l^1|^2 + (\boldsymbol{l}^1 \cdot \boldsymbol{l}^2)}{|l^1|^2 |l^2|^2 - (\boldsymbol{l}^1 \cdot \boldsymbol{l}^2)^2} (\boldsymbol{l}^2 \cdot \boldsymbol{v}) \end{aligned}$$

Finally

$$\boldsymbol{v} = \frac{|l^2|^2 + (\boldsymbol{l}^1 \cdot \boldsymbol{l}^2)}{|l^1|^2 |l^2|^2 - (\boldsymbol{l}^1 \cdot \boldsymbol{l}^2)^2} (\boldsymbol{l}^1 \cdot \boldsymbol{v}) \boldsymbol{l}^1 + \frac{|l^1|^2 + (\boldsymbol{l}^1 \cdot \boldsymbol{l}^2)}{|l^1|^2 |l^2|^2 - (\boldsymbol{l}^1 \cdot \boldsymbol{l}^2)^2} (\boldsymbol{l}^2 \cdot \boldsymbol{v}) \boldsymbol{l}^2 \tag{79}$$

$$= c_1 (\boldsymbol{l}^1 \cdot \boldsymbol{v}) \boldsymbol{l}^1 + c_2 (\boldsymbol{l}^2 \cdot \boldsymbol{v}) \boldsymbol{l}^2. \tag{80}$$

Flux

$$\begin{aligned} B_{\parallel n} &= \boldsymbol{B}_n \cdot \boldsymbol{h} = \boldsymbol{B}_n^{\mathrm{pol}} \cdot \boldsymbol{h}^{\mathrm{pol}} + B_{n\varphi} h^{\varphi} \\ &= c_1 (\boldsymbol{n}^1 \cdot \boldsymbol{B}_n^{\mathrm{pol}}) \boldsymbol{n}^1 \cdot \boldsymbol{h}^{\mathrm{pol}} + c_2 (\boldsymbol{n}^2 \cdot \boldsymbol{B}_n^{\mathrm{pol}}) \boldsymbol{n}^2 \cdot \boldsymbol{h}^{\mathrm{pol}} + B_{n(\varphi)} h_{(\varphi)}. \end{aligned} \tag{81}$$