Magnetic differential equations for stationary linear ideal MHD and their numerical solution

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Contents

1 Stationary linear perturbation of ideal MHD equilibrium

For the intended application on stationary (compared to MHD mode eigenfrequencies) non-axisymmetric magnetic perturbations by external coils, we consider a perturbed ideal MHD equilibrium for pressure p, currents $\boldsymbol{j}$ and magnetic field $\boldsymbol{B}$ fulfilling

$$\nabla p = \frac{1}{c}\boldsymbol{j} \times \boldsymbol{B}, \tag{1}$$

$$\nabla \times \boldsymbol{B} = \frac{4\pi}{c}\boldsymbol{j}, \tag{2}$$

$$\nabla \cdot \boldsymbol{B} = 0. \tag{3}$$

Starting with a given MHD equilibrium fulfilling eqs. (1) and (3) denoted by subscripts "0", linear order equations for an external magnetic perturbation (denoted by δ) split into a vacuum and a plasma part (subscript v and p,

respectively) are

$$\nabla \delta p = \frac{1}{c} \left(\boldsymbol{j}_0 \times \delta \boldsymbol{B} + \delta \boldsymbol{j} \times \boldsymbol{B}_0 \right), \tag{4}$$

$$\delta \boldsymbol{B} = \delta \boldsymbol{B}_{\mathrm{v}} + \delta \boldsymbol{B}_{\mathrm{p}}, \tag{5}$$

$$\delta \boldsymbol{B}_{\mathrm{v}} = \frac{1}{c} \oint \frac{I_{\mathrm{c}}(\boldsymbol{r}')\, \mathrm{d}\boldsymbol{l}' \times \boldsymbol{r}}{|\boldsymbol{r} - \boldsymbol{r}'|^3}, \tag{6}$$

$$\delta \boldsymbol{B}_{\mathrm{p}} = \nabla \times \delta \boldsymbol{A}, \tag{7}$$

$$\nabla \times (\nabla \times \delta \boldsymbol{A}) = \frac{4\pi}{c} \delta \boldsymbol{j}, \tag{8}$$

$$\Rightarrow \nabla \cdot \delta \boldsymbol{B} = \nabla \cdot \delta \boldsymbol{j} = 0. \tag{9}$$

Here the perturbation field in vacuum, $\delta \boldsymbol{B}_{\mathrm{v}}$, is pre-evaluated by a Biot-Savart integral over external coil currents $I_{\mathrm{c}}(\boldsymbol{r}')$ and the perturbation field in plasma $\delta \boldsymbol{B}_{\mathrm{p}}$ is computed from Ampère's law (8) from the plasma current density $\delta \boldsymbol{j}$ using a vector potential formulation. To find a consistent solution for the system, eq. (4) and eq. (8) are treated individually in an iterative way. The linearized force balance equation (4) is used to compute $\delta \boldsymbol{j}$ for given $\delta \boldsymbol{B}$ whereas eq. (8) yields $\delta \boldsymbol{B}_{\mathrm{p}}$ for given $\delta \boldsymbol{j}$. In the first iteration, $\delta \boldsymbol{B}$ is set equal to $\delta \boldsymbol{B}_{\mathrm{v}}$ in eq. (4). Then, eq. (8) and eq. (4) are solved in an alternating way until convergence is reached. In addition a preconditioner is used to enhance convergence. Here we limit the analysis to an axisymmetric unperturbed equilibrium and a single toroidal perturbation harmonic $\delta \boldsymbol{B} = \mathrm{Re}(\boldsymbol{B}_n \mathrm{e}^{\mathrm{i}n\varphi})$ with a similar notation for other perturbed quantities. As all equations are linear, a superposition of multiple harmonics is easily possible. Note that $n \neq 0$; such a perturbation is necessarily small and considered part of the axisymmetric equilibrium.

2   Linearized MHD force balance

The solution of eq. (4) can further be split into two steps: First the pressure perturbation δp is found, and then the plasma current density $\delta \boldsymbol{j}$ is computed using the condition $\nabla \cdot \delta \boldsymbol{j} = 0$. For an unperturbed equilibrium with nested flux surfaces, both steps can be performed in a radially local manner if a field-aligned computational grid is used, which will become clear below. Radial coupling happens by the combination of the two individual steps since their effective radial locations of computation are shifted by a half-step in radial grid distance.

2.1   Geometrical considerations

In axisymmetric coordinate systems, such as cylindrical (R, φ, Z), the equations to solve for harmonics in the toroidal angle φ are

$$\nabla p_n + \mathrm{i}n p_n \nabla \varphi = \frac{1}{c} \left(\boldsymbol{j}_0 \times \boldsymbol{B}_n + \boldsymbol{j}_n \times \boldsymbol{B}_0 \right), \tag{10}$$

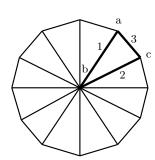$$\nabla \cdot \boldsymbol{j}_n^{\mathrm{pol}} + \mathrm{i}n j_n^{\varphi} = 0. \tag{11}$$

now with a 2D ∇ operator acting in the poloidal (RZ) plane. The divergence operator is defined via

$$\nabla \cdot \boldsymbol{u} = \frac{1}{R\sqrt{g_{\mathrm{pol}}}} \frac{\partial}{\partial x^k} (R\sqrt{g_{\mathrm{pol}}} u^k),$$
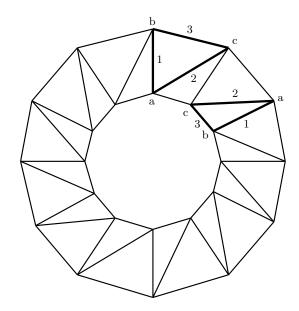
where $\sqrt{g_{\mathrm{pol}}}$ is the metric tensor of the coordinates in the poloidal plane, which is equal to 1 for cylindrical coordinates.

The representation of equilibrium field $\boldsymbol{B}_0$ is given by

$$\boldsymbol{B}_0 = \boldsymbol{B}_0^{\mathrm{pol}} + \boldsymbol{B}_0^{\mathrm{tor}}, \tag{12}$$

(a) The innermost loop of the grid with the magnetic axis at its center. Edge f lies on the flux surface in the infinitesimal limit.

(b) One of the outer loops of the grid with two alternating kinds of triangles with edge f lying on the inner and outer flux surface respectively.

Figure 1: The 2D mesh is given by a triangulation of poloidal cross-sections of the nested flux surfaces, resulting in "loops". The cross-sections are assumed to be circular for illustration purposes.

where

$$\boldsymbol{B}_0^{\mathrm{pol}} = \nabla\psi \times \nabla\varphi, \tag{13}$$

$$\boldsymbol{B}_0^{\mathrm{tor}} = B_{0\varphi}\nabla\varphi. \tag{14}$$

This leads to the following contravariant components for $\boldsymbol{B}_0^{\mathrm{pol}}$ in cylindrical coordinates:

$$B_0^R = (\nabla\psi \times \nabla\varphi)^R = -\frac{1}{R}\frac{\partial\psi}{\partial Z}, \tag{15}$$

$$B_0^Z = (\nabla\psi \times \nabla\varphi)^Z = \frac{1}{R}\frac{\partial\psi}{\partial R}. \tag{16}$$

2.1.1 Coordinate system on each edge

For each edge (length l) we use a local orthogonal coordinate system on each triangle edge with $\boldsymbol{l}$ the vector of length l along the edge in counter-clockwise orientation, $\boldsymbol{n}$ the outward normal of length l and $\nabla\varphi$ pointing inside the plane. We obtain relations

$$\boldsymbol{l} \times \boldsymbol{n} = l^2 R\nabla\varphi, \tag{17}$$

$$\boldsymbol{n} \times R\nabla\varphi = \boldsymbol{l}, \tag{18}$$

$$R\nabla\varphi \times \boldsymbol{l} = \boldsymbol{n}. \tag{19}$$

2.2 Pressure perturbation

Multiplying eq. (10) by $\boldsymbol{B}_0$ yields

$$c\boldsymbol{B}_0 \cdot \nabla p_n + \mathrm{i}ncp_n\boldsymbol{B}_0 \cdot \nabla\varphi = -\boldsymbol{B}_n \cdot (\boldsymbol{j}_0 \times \boldsymbol{B}_0) = -c\boldsymbol{B}_n \cdot \nabla p_0$$

$$\boldsymbol{B}_0^{\mathrm{pol}} \cdot \nabla p_n + \mathrm{i}np_nB_0^\varphi = -B_n^\psi p_0'(\psi). \tag{20}$$

To solve this equation on one flux surface, we use a lowest-order finite difference method. Nodes are indexed by superscript (k) and $\boldsymbol{r}^{(k)}$ is the position of node (k). ∇p_n is approximated at the midpoint of edge Γ_k as finite difference of p_n at nodes (k) and $(k+1)$. p_n is accordingly approximated at the midpoint as the arithmetic mean of the values at these nodes. This results in

$$\boldsymbol{B}_0^{\mathrm{pol}} \cdot \frac{\boldsymbol{r}^{(k+1)} - \boldsymbol{r}^{(k)}}{\left\|\boldsymbol{r}^{(k+1)} - \boldsymbol{r}^{(k)}\right\|} \frac{p_n^{(k+1)} - p_n^{(k)}}{\left\|\boldsymbol{r}^{(k+1)} - \boldsymbol{r}^{(k)}\right\|} + \mathrm{i}nB_0^{\varphi} \frac{p_n^{(k+1)} + p_n^{(k)}}{2} = -p_0'(\psi)B_n^{\psi}|_{\Gamma_k},$$

where a unit vector along the edge is used to get the correct sign for the gradient in the direction of the poloidal magnetic field; with a shorthand Δr for the distance between nodes we get

$$\boldsymbol{B}_0^{\mathrm{pol}} \cdot \frac{\boldsymbol{r}^{(k+1)} - \boldsymbol{r}^{(k)}}{(\Delta r)^2} \left(p_n^{(k+1)} - p_n^{(k)}\right) + \mathrm{i}nB_0^{\varphi} \frac{p_n^{(k+1)} + p_n^{(k)}}{2} = -p_0'(\psi)B_n^{\psi}|_{\Gamma_k}. \tag{21}$$

Reordering in terms of the unknowns yields

$$(b_k + a_k)p_n^{(k+1)} + (b_k - a_k)p_n^{(k)} = c_k \tag{22}$$

with

$$a_k = \boldsymbol{B}_0^{\mathrm{pol}} \cdot \frac{\boldsymbol{r}^{(k+1)} - \boldsymbol{r}^{(k)}}{(\Delta r)^2}, \tag{23}$$

$$b_k = \frac{\mathrm{i}nB_0^{\varphi}}{2}, \tag{24}$$

$$c_k = -p_0'(\psi)B_n^{\psi}|_{\Gamma_k}. \tag{25}$$

In matrix form this scheme is written as

$$A_{jk}p_n^k = c_j, \tag{26}$$

where the elements of the stiffness matrix are

$$A_{jk} = (b_j + a_j)\delta_{j-1,k} + (b_j - a_j)\delta_{jk}. \tag{27}$$

Note that for N nodes with periodic boundary conditions $p_n^{(N+1)} = p_n^{(1)}$, indices are taken *modulo N*, resulting in the following shape for the stiffness matrix:

$$A = \begin{pmatrix} b_1 - a_1 & b_1 + a_1 & 0 & \dots & 0 \\ 0 & b_2 - a_2 & b_2 + a_2 & \dots & 0 \\ 0 & 0 & b_3 - a_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_N + a_N & 0 & 0 & \dots & b_N - a_N \end{pmatrix}.$$

2.2.1 Analytical solution for very large aspect ratios

Using flux coordinates (r, ϑ, φ), a term in eq. (20) can be simplified:

$$\boldsymbol{B}_0^{\mathrm{pol}} \cdot \nabla p_n = B_0^{\vartheta}\boldsymbol{e}_{\vartheta} \cdot \nabla p_n = B_0^{\vartheta}\frac{\partial p_n}{\partial \vartheta}. \tag{28}$$

Now when we expand the perturbed quantities as series in ϑ as well as φ so that

$$\delta p = \sum_n p_n(r, \vartheta)\mathrm{e}^{\mathrm{i}n\varphi} = \sum_{m,n} p_{mn}(r)\mathrm{e}^{\mathrm{i}m\vartheta + \mathrm{i}n\varphi}, \tag{29}$$

eq. (20) reduces to

$$\mathrm{i}mp_{mn}B_0^{\vartheta} + \mathrm{i}np_{mn}B_0^{\varphi} = -B_{mn}^r p_0'(r) =: -q_{mn} \tag{30}$$

with a source term q_{mn}. This allows a simple algebraic solution where all quantities depend only on r:

$$p_{mn} = \frac{\mathrm{i}q_{mn}}{mB_0^\vartheta + nB_0^\varphi}. \tag{31}$$

Now an approximation can be made by switching from contravariant components to covariant components which are (for some reason) assumed to be constant. The factor R appearing in the metric components can also be considered constant over the domain in question – the toroidal geometry is effectively transformed to cylindrical geometry with a very thin and elongated cylinder and periodic boundary conditions.

2.3   Current perturbation

Multiplying eq. (11) by R yields

$$\frac{\partial}{\partial x^k}(Rj_n^k) + \mathrm{i}nRj_n^\varphi = 0. \tag{32}$$

Using the divergence theorem this can also be written in integral form in a specific triangular mesh element $\Omega^{(i)}$ as

$$\oint_{\partial\Omega^{(i)}} \mathrm{d}l\, R\boldsymbol{j}_n \cdot \hat{\boldsymbol{n}} + \mathrm{i}n \int_{\Omega^{(i)}} \mathrm{d}R\,\mathrm{d}Z\, Rj_n^\varphi = 0. \tag{33}$$

Here the first integral is performed over the 1-dimensional element boundary $\partial\Omega^{(i)} = \Gamma^{(i)}$. The first term is split into three contributions,

$$\oint_{\partial\Omega^{(i)}} \mathrm{d}l\, R\boldsymbol{j}_n \cdot \hat{\boldsymbol{n}} = \int_{\Gamma_\mathrm{i}^{(i)}, \Gamma_\mathrm{o}^{(i)}} \mathrm{d}l\, R\boldsymbol{j}_n \cdot \hat{\boldsymbol{n}} + \int_{\Gamma_\mathrm{f}^{(i)}} \mathrm{d}l\, R\boldsymbol{j}_n \cdot \hat{\boldsymbol{n}}, \tag{34}$$

where edge f is tangential to an adjacent flux surface and edges i and o are not. For the grid it is important to distinguish two types of triangles, starting from the second loop outside the magnetic axis, see fig. 1.

2.3.1   Toroidal unperturbed current

Since $\boldsymbol{B}_0$ and p_0 are directly available as input data, but $\boldsymbol{j}_0$ is not, the latter will be derived below from eq. (1), the condition of divergence-freeness and symmetry considerations.

We take a cross-product of eq. (1) by $\boldsymbol{B}_0$:

$$\begin{aligned}
\boldsymbol{B}_0 \times (\boldsymbol{j}_0 \times \boldsymbol{B}_0) &= B_0^2 \boldsymbol{j}_0 - (\boldsymbol{B}_0 \cdot \boldsymbol{j}_0)\boldsymbol{B}_0 \\
&= B_0^2(\boldsymbol{j}_0 - j_{0\|}\boldsymbol{h}_0) \\
&= B_0^2 \boldsymbol{j}_{0\perp}.
\end{aligned} \tag{35}$$

Therefore

$$\boldsymbol{j}_{0\perp} = \frac{-c\nabla p_0 \times \boldsymbol{B}_0}{B_0^2} \tag{36}$$

$$= \frac{-cp_0'(\psi)\nabla\psi \times (\nabla\psi \times \nabla\varphi + B_{0\varphi}\nabla\varphi)}{B_0^2}$$

$$= \frac{-cp_0'(\psi)}{B_0^2}\left(B_{0\varphi}\boldsymbol{B}_0^\mathrm{pol} - |\nabla\psi|^2\nabla\varphi\right). \tag{37}$$

which is the diamagnetic current density. For the parallel current density we use

$$\begin{aligned}
0 = \nabla \cdot \boldsymbol{j}_0 &= \nabla \cdot \boldsymbol{j}_{0\perp} + \nabla \cdot (j_{0\|}\boldsymbol{h}_0) \\
&= -c\nabla \cdot \left(\frac{\nabla p_0 \times \boldsymbol{B}_0}{B_0^2}\right) + \boldsymbol{B}_0 \cdot \nabla\left(\frac{j_{0\|}}{B_0}\right).
\end{aligned} \tag{38}$$

In straight-field line magnetic flux coordinates (r, ϑ, φ) with Jacobian $\sqrt{g}$ and the divergence of the diamagnetic current is

$$
\begin{aligned}
\nabla \cdot \boldsymbol{j}_{0\perp} &= -\frac{c}{\sqrt{g}} \frac{\partial}{\partial x^k} \left[\frac{\sqrt{g}}{B_0^2} \left(\nabla p_0 \times \boldsymbol{B}_0 \right)^k \right] \\
&= -\frac{c}{\sqrt{g}} \frac{\partial}{\partial x^k} \left(\frac{\sqrt{g}}{B_0^2} \frac{\varepsilon^{ijk}}{\sqrt{g}} \frac{\partial p_0}{\partial x^i} B_{0j} \right) \\
&= \frac{cp_0'(r)B_{0\varphi}}{\sqrt{g}} \frac{\partial}{\partial \vartheta} \left(\frac{1}{B_0^2} \right),
\end{aligned}
\tag{39}
$$

since p_0 and $B_{0\varphi}$ are constant on a flux surface, $\frac{\partial p_0}{\partial \vartheta} = 0$ and due to axisymmetry $\frac{\partial}{\partial \varphi} = 0$. The divergence of the parallel current is

$$
\nabla \cdot (j_{0\parallel}\boldsymbol{h}_0) = B_0^\vartheta \frac{\partial}{\partial \vartheta} \left(\frac{j_{0\parallel}}{B_0} \right).
\tag{40}
$$

With $\sqrt{g}B_0^\vartheta = \psi'(r)$ as a flux surface quantity, there are no dependencies of ϑ in front of the derivatives. Direct integration and a change of variables/notation as in

$$
\frac{p_0'(r)}{\psi'(r)} = \frac{\frac{\partial p_0}{\partial r}}{\frac{\partial \psi}{\partial r}} = \frac{\partial p_0}{\partial \psi} = p_0'(\psi)
\tag{41}
$$

yields

$$
j_{0\parallel} = -\frac{cp_0'(\psi)B_{0\varphi}}{B_0} + C(\psi)B_0.
\tag{42}
$$

With the extra condition of the flux-surface average $\left\langle j_{0\parallel}B_0 \right\rangle = 0$ for testing without bootstrap current and the assumption $\left\langle B_{0\varphi} \right\rangle = B_{0\varphi}$, we obtain

$$
\begin{aligned}
0 &= -cp_0'(\psi)B_{0\varphi} + C(\psi) \left\langle B_0^2 \right\rangle \\
\Rightarrow C(\psi) &= \frac{cp_0'(\psi)B_{0\varphi}}{\left\langle B_0^2 \right\rangle}.
\end{aligned}
\tag{43}
$$

In general,

$$
C(\psi) = \frac{cp_0'(\psi)B_{0\varphi}}{\left\langle B_0^2 \right\rangle} D(\psi),
\tag{44}
$$

$$
j_{0\parallel} = -\frac{cp_0'(\psi)B_{0\varphi}}{B_0} \left(\frac{B_0^2}{\left\langle B_0^2 \right\rangle} D(\psi) - 1 \right).
\tag{45}
$$

with $D(\psi)$ set to 1 for now and modified for the more general case $\left\langle j_{0\parallel}B_0 \right\rangle \not\equiv 0$.

For the unperturbed toroidal current density we have

$$
j_0^\varphi = j_{0\parallel}h_0^\varphi + \boldsymbol{j}_{0\perp} \cdot \nabla \varphi,
\tag{46}
$$

where

$$
\begin{aligned}
j_{0\parallel}h_0^\varphi &= -\frac{1}{B_0} cp_0'(\psi)B_{0\varphi} \frac{B_0^\varphi}{B_0} \left(\frac{B_0^2}{\left\langle B_0^2 \right\rangle} D(\psi) - 1 \right) \\
&= -cp_0'(\psi) \frac{(B_0^{\text{tor}})^2}{B_0^2} \left(\frac{B_0^2}{\left\langle B_0^2 \right\rangle} D(\psi) - 1 \right) \\
&= -cp_0'(\psi) \left(B_0^{\text{tor}} \right)^2 \left(\frac{D(\psi)}{\left\langle B_0^2 \right\rangle} - \frac{1}{B_0^2} \right).
\end{aligned}
\tag{47}
$$

and

$$
\begin{aligned}
\boldsymbol{j}_{0\perp} \cdot \nabla\varphi &= \frac{-cp_0'(\psi)\nabla\varphi \cdot (\nabla\psi \times \boldsymbol{B}_0)}{B_0^2} \\
&= \frac{-cp_0'(\psi)\boldsymbol{B}_0 \cdot (\nabla\varphi \times \nabla\psi)}{B_0^2} \tag{48} \\
&= \frac{cp_0'(\psi)}{B_0^2}\boldsymbol{B}_0^{\mathrm{pol}} \cdot \boldsymbol{B}_0 \\
&= cp_0'(\psi)\frac{\left(B_0^{\mathrm{pol}}\right)^2}{B_0^2}. \tag{49}
\end{aligned}
$$

It follows from

$$
\left(B_0^{\mathrm{pol}}\right)^2 + (B_0^{\mathrm{tor}})^2 = B_0^2 \tag{50}
$$

that

$$
j_0^\varphi = cp_0'(\psi)\left(1 - \frac{(B_0^{\mathrm{tor}})^2}{\langle B_0^2 \rangle}D(\psi)\right). \tag{51}
$$

2.3.2 Toroidal current perturbation

For the computation of toroidal j_n^φ in the element volume we start from eq. (10) with

$$
\boldsymbol{j}_n \times \boldsymbol{B}_0 = c(\nabla p_n + \mathrm{i}np_n\nabla\varphi) - \boldsymbol{j}_0 \times \boldsymbol{B}_n \tag{52}
$$

and

$$
\boldsymbol{B}_0 = \nabla\psi \times \nabla\varphi + B_{0\varphi}\nabla\varphi. \tag{53}
$$

Taking a scalar product of $\boldsymbol{l}$ with eq. (52) on some edge yields

$$
\begin{aligned}
\boldsymbol{l} \cdot (\boldsymbol{j}_n \times \boldsymbol{B}_0) &= \boldsymbol{l} \cdot (\boldsymbol{j}_n \times (\nabla\psi \times \nabla\varphi + B_{0\varphi}\nabla\varphi)) \\
&= \boldsymbol{j}_n \cdot ((\nabla\psi \times \nabla\varphi) \times \boldsymbol{l} + B_{0\varphi}\nabla\varphi \times \boldsymbol{l}) \\
&= \boldsymbol{j}_n \cdot ((\nabla\psi \times \nabla\varphi) \times \boldsymbol{l} + B_{0\varphi}\boldsymbol{n}/R) \\
&= \boldsymbol{j}_n \cdot \left((\boldsymbol{l} \cdot \nabla\psi)\nabla\varphi + B_{0(\varphi)}\boldsymbol{n}\right) \\
&= (\boldsymbol{l} \cdot \nabla\psi)j_n^\varphi + B_{0(\varphi)}\boldsymbol{j}_n^{\mathrm{pol}} \cdot \boldsymbol{n}. \tag{54}
\end{aligned}
$$

Further,

$$
\boldsymbol{l} \cdot \nabla\psi = R\nabla\psi \cdot (\boldsymbol{n} \times \nabla\varphi) = -R\boldsymbol{n} \cdot \boldsymbol{B}_0^{\mathrm{pol}}. \tag{55}
$$

Finally,

$$
\boldsymbol{l} \cdot (\boldsymbol{j}_n \times \boldsymbol{B}_0) = B_{0(\varphi)}\boldsymbol{j}_n^{\mathrm{pol}} \cdot \boldsymbol{n} - Rj_n^\varphi\boldsymbol{n} \cdot \boldsymbol{B}_0^{\mathrm{pol}}. \tag{56}
$$

The right-hand side of eq. (52) yields:

$$
\boldsymbol{l} \cdot (\nabla p_n + \mathrm{i}np_n\nabla\varphi) = \boldsymbol{l} \cdot \nabla p_n \tag{57}
$$

$$
\boldsymbol{l} \cdot (\boldsymbol{j}_0 \times \boldsymbol{B}_n) = \boldsymbol{l} \cdot (B_{n\varphi}\boldsymbol{j}_0^{\mathrm{pol}} \times \nabla\varphi + j_{0\varphi}\nabla\varphi \times \boldsymbol{B}_n^{\mathrm{pol}}). \tag{58}
$$

We use the fact that ∇p_0 is parallel to $\nabla\psi$, so the cross product in the equilibrium is purely radial,

$$
\begin{aligned}
\boldsymbol{j}_0 \times \boldsymbol{B}_0 &= c\nabla p_0 \\
&= \boldsymbol{j}_0^{\mathrm{pol}} \times (B_{0\varphi}\nabla\varphi) + j_{0\varphi}\nabla\varphi \times (\nabla\psi \times \nabla\varphi) \\
&= \boldsymbol{j}_0^{\mathrm{pol}} \times (B_{0\varphi}\nabla\varphi) + \frac{j_{0\varphi}}{R^2}\nabla\psi. \tag{59}
\end{aligned}
$$

Thus

$$\boldsymbol{j}_0^{\mathrm{pol}} \times \nabla\varphi = \frac{1}{B_{0\varphi}} \left(c\nabla p_0 - \frac{j_{0\varphi}}{R^2} \nabla\psi \right). \tag{60}$$

Also

$$\boldsymbol{l} \cdot (\nabla\varphi \times \boldsymbol{B}_n^{\mathrm{pol}}) = \boldsymbol{B}_n^{\mathrm{pol}} \cdot (\boldsymbol{l} \times \nabla\varphi)$$

$$= -\frac{1}{R} \boldsymbol{B}_n^{\mathrm{pol}} \cdot \boldsymbol{n}. \tag{61}$$

Finally

$$(\boldsymbol{l} \cdot \nabla\psi)j_n^\varphi + B_{0(\varphi)}\boldsymbol{j}_n^{\mathrm{pol}} \cdot \boldsymbol{n} = c\boldsymbol{l} \cdot \nabla p_n - \frac{B_{n\varphi}}{B_{0\varphi}} \left(c\boldsymbol{l} \cdot \nabla p_0 - \frac{j_{0\varphi}}{R^2} \boldsymbol{l} \cdot \nabla\psi \right) + \frac{j_{0\varphi}}{R} \boldsymbol{B}_n^{\mathrm{pol}} \cdot \boldsymbol{n} \tag{62}$$

This term is only meaningful to obtain j_n^φ on edges where $\boldsymbol{l} \cdot \nabla\psi \neq 0$ (i and o in fig. 1) and we obtain

$$Rj_n^\varphi = j_{n(\varphi)} = -\frac{B_{0\varphi}}{\boldsymbol{l} \cdot \nabla\psi}\boldsymbol{j}_n^{\mathrm{pol}} \cdot \boldsymbol{n} + \frac{cR}{\boldsymbol{l} \cdot \nabla\psi} \left(\boldsymbol{l} \cdot \nabla p_n - \frac{B_{n\varphi}}{B_{0\varphi}}\boldsymbol{l} \cdot \nabla p_0 \right) + j_{0\varphi} \left(\frac{B_{n\varphi}}{RB_{0\varphi}} + \frac{\boldsymbol{B}_n^{\mathrm{pol}} \cdot \boldsymbol{n}}{\boldsymbol{l} \cdot \nabla\psi} \right). \tag{63}$$

In the implementation, an average over edge i and o will be taken for this quantity.

On edge f, no connection between j_n^φ and $\boldsymbol{j}_n^{\mathrm{pol}} \cdot \boldsymbol{n}$ can be made, but the latter expression can be given in terms of already knwown quantities:

$$R\boldsymbol{j}_n^{\mathrm{pol}} \cdot \boldsymbol{n} = \frac{cR}{B_{0(\varphi)}}\boldsymbol{l} \cdot \nabla p_n + \frac{j_{0\varphi}}{B_{0(\varphi)}}\boldsymbol{B}_n^{\mathrm{pol}} \cdot \boldsymbol{n}. \tag{64}$$

2.3.3 Alternative formulation for the current through flux surface edges

As an alternative to eq. (64) it can be shown (see writeup on gyrokinetics) that

$$\boldsymbol{j}_{n\perp} = \boldsymbol{j}_n - (\boldsymbol{j}_n \cdot \boldsymbol{h}_0)\boldsymbol{h}_0$$

$$= j_{0\parallel}\frac{\boldsymbol{B}_{n\perp}}{B_0} - \frac{cB_{n\parallel}}{B_0^2}\boldsymbol{h}_0 \times \nabla p_0 + \frac{c}{B_0}\boldsymbol{h}_0 \times (\nabla p_n + \mathrm{i}np_n\nabla\varphi). \tag{65}$$

Scalar multiplication with $\boldsymbol{n}_{\mathrm{f}} \parallel \nabla p_0 \parallel \nabla\psi \perp \boldsymbol{h}_0$ – the first relation is true for infinitesimally small triangles – yields

$$\boldsymbol{j}_{n\perp} \cdot \boldsymbol{n}_{\mathrm{f}} = \boldsymbol{j}_{n\perp}^{\mathrm{pol}} \cdot \boldsymbol{n}_{\mathrm{f}} = j_{0\parallel}\frac{\boldsymbol{B}_n \cdot \boldsymbol{n}_{\mathrm{f}}}{B_0} + \frac{c}{B_0}\boldsymbol{n}_{\mathrm{f}} \cdot (\boldsymbol{h}_0 \times (\nabla p_n + \mathrm{i}np_n\nabla\varphi))$$

$$= j_{0\parallel}\frac{\boldsymbol{B}_n \cdot \boldsymbol{n}_{\mathrm{f}}}{B_0} + \frac{c}{B_0}\boldsymbol{n}_{\mathrm{f}} \cdot (h_{0\varphi}\nabla\varphi \times \nabla p_n + \mathrm{i}np_n\boldsymbol{h}_0^{\mathrm{pol}} \times \nabla\varphi)$$

$$= j_{0\parallel}\frac{\boldsymbol{B}_n \cdot \boldsymbol{n}_{\mathrm{f}}}{B_0} + \frac{c}{B_0}(h_{0\varphi}\nabla p_n \cdot (\boldsymbol{n}_{\mathrm{f}} \times \nabla\varphi) + \mathrm{i}np_n\boldsymbol{h}_0^{\mathrm{pol}} \cdot (\nabla\varphi \times \boldsymbol{n}_{\mathrm{f}}))$$

$$= j_{0\parallel}\frac{\boldsymbol{B}_n \cdot \boldsymbol{n}_{\mathrm{f}}}{B_0} + \frac{c}{RB_0}\boldsymbol{l}_{\mathrm{f}} \cdot (h_{0\varphi}\nabla p_n - \mathrm{i}np_n\boldsymbol{h}_0^{\mathrm{pol}})$$

$$= j_{0\parallel}\frac{\boldsymbol{B}_n \cdot \boldsymbol{n}_{\mathrm{f}}}{B_0} + \frac{c}{B_0^2}\boldsymbol{l}_{\mathrm{f}} \cdot \left(B_{0(\varphi)}\nabla p_n - \frac{\mathrm{i}n}{R}p_n\boldsymbol{B}_0^{\mathrm{pol}} \right). \tag{66}$$

2.3.4 Implementation

We start with

$$I_{\mathrm{i}} + I_{\mathrm{o}} + \mathrm{i}n \int_\Omega Rj_n^\varphi \mathrm{d}S = -I_{\mathrm{f}}, \tag{67}$$

where the notation for currents through edges, weighted by R, is

$$I_k = \int_{\Gamma_k} R\boldsymbol{j}_n^{\text{pol}} \cdot \hat{\boldsymbol{n}}\, \mathrm{d}l \approx R_k \boldsymbol{j}_n^{\text{pol}} \cdot \boldsymbol{n}_k, \tag{68}$$

where values are taken at the midpoint of edge Γ_k and assumed to be constant along that edge. I_{f} is already known from eq. (64) or eq. (66) and therefore acts as a source on the right-hand side. The remaining currents I_{i} and I_{o} are taken as unknowns and appear also in the last term of the left-hand side as

$$\mathrm{i}n\int_{\Omega} R j_n^{\varphi}\, \mathrm{d}S \approx \mathrm{i}nS_{\Omega}\frac{R_{\text{i}}j_{n\text{i}}^{\varphi} + R_{\text{o}}j_{n\text{o}}^{\varphi}}{2}$$
$$= -\frac{\mathrm{i}nS_{\Omega}}{2}\left(\frac{B_{0(\varphi),\text{i}}}{\boldsymbol{l}_{\text{i}}\cdot\nabla\psi}I_{\text{i}} + \frac{B_{0(\varphi),\text{o}}}{\boldsymbol{l}_{\text{o}}\cdot\nabla\psi}I_{\text{o}} + \cdots\right), \tag{69}$$

where S_{Ω} is the triangle surface area. In the approximation above, a term $R_{\text{f}}j_{n\text{f}}^{\varphi}$ is neglected because within one loop of triangles in fig. 1b, edge f alternates between the inner and outer flux surface and this "oscillation" of sample points would carry over to the approximation values. Furthermore, $R_{\text{f}}j_{n\text{f}}^{\varphi}$ can't be reformulated in terms of $R_{\text{f}}j_{n\text{f}}^{\varphi}$, introducing another unknown. This would lead to an overdetermined set of equations, possibly violating divergence-freeness. Instead, $j_{n(\varphi)}$ is approximated for each triangle by

$$\mathrm{i}n\int_{\Omega} R j_n^{\varphi} \approx \mathrm{i}nI_{(\varphi)}. \tag{70}$$

The remaining terms are moved to the right-hand-side as sources q, so the discretized equation in each triangle Ω is

$$\left(1 - \frac{\mathrm{i}nS_{\Omega}}{2}\frac{B_{0(\varphi)}|_{\Gamma_{\text{i}}}}{\boldsymbol{l}_{\text{i}}\cdot\nabla\psi}\right)I_{\text{i}} + \left(1 - \frac{\mathrm{i}nS_{\Omega}}{2}\frac{B_{0(\varphi)}|_{\Gamma_{\text{o}}}}{\boldsymbol{l}_{\text{o}}\cdot\nabla\psi}\right)I_{\text{o}} = q. \tag{71}$$

The source term is given by

$$q = -I_{\text{f}} - \frac{\mathrm{i}nS_{\Omega}}{2}\sum_{k=\text{i,o}}\frac{cR}{\boldsymbol{l}_k\cdot\nabla\psi}\left(\boldsymbol{l}_k\cdot\nabla p_n - \frac{B_{n\varphi}}{B_{0\varphi}}\boldsymbol{l}_k\cdot\nabla p_0\right) + j_{0\varphi}\left(\frac{B_{n\varphi}}{RB_{0\varphi}} + \frac{\boldsymbol{B}_n^{\text{pol}}\cdot\boldsymbol{n}}{\boldsymbol{l}_k\cdot\nabla\psi}\right). \tag{72}$$

The directional derivatives $\boldsymbol{l}_k\cdot\nabla\psi$ are approximated by a difference quotient with values taken at the nodes (for indexing see fig. 1),

$$\boldsymbol{l}_{\text{i}}\cdot\nabla\psi = l_{\text{i}}\hat{\boldsymbol{l}}_{\text{i}}\cdot\nabla\psi = l_{\text{i}}\frac{\partial\psi}{\partial\boldsymbol{l}_{\text{i}}} \approx l_{\text{i}}\frac{\Delta\psi}{l_{\text{i}}} = \Delta\psi, \tag{73}$$

$$\boldsymbol{l}_{\text{o}}\cdot\nabla\psi = l_{\text{o}}\hat{\boldsymbol{l}}_{\text{o}}\cdot\nabla\psi = l_{\text{o}}\frac{\partial\psi}{\partial\boldsymbol{l}_{\text{o}}} \approx l_{\text{o}}\frac{-\Delta\psi}{l_{\text{o}}} = -\Delta\psi. \tag{74}$$

Note that $\psi|_{\text{F}} \neq \psi|_{\text{I}} = \psi|_{\text{O}}$ for each triangle because values are taken from the nodes at adjacent flux surfaces. Thus $\Delta\psi$ is identical for the entire loop, but signs change depending on the direction of $\boldsymbol{l}_k$, alternating between triangles. It takes a positive value for $\boldsymbol{l}_k$ pointing in the radially outward direction. The same logic applies to $\boldsymbol{l}\cdot\nabla p_0$ terms, albeit with reversed sign because p_0 decreases radially outwards.

For the global indexing scheme, we call the ingoing current into triangle (i) counted in clockwise direction $I^{(i)}$. In triangle (i), this is equal to $I_{\text{i}} = -I^{(i)}$ and $I_{\text{o}} = I^{(i+1)}$. The matrix form of eq. (71) is then

$$A_{ij}I^{(j)} = q_i, \tag{75}$$

where the elements of the stiffness matrix A are

$$A_{ij} = \left(-1 + \frac{\mathrm{i}nS_{\Omega^{(i)}}}{2}\frac{B_{0(\varphi)}|_{\Gamma_{\text{i}}^{(i)}}}{\Delta\psi}\right)\delta_{ij} + \left(1 + \frac{\mathrm{i}nS_{\Omega^{(i)}}}{2}\frac{B_{0(\varphi)}|_{\Gamma_{\text{o}}^{(i)}}}{\Delta\psi}\right)\delta_{i+1,j}. \tag{76}$$

2.4   Test field

We can take

$$B_n^\varphi = B_n^\psi F(\psi)$$

where

$$F(\psi) = \frac{\mathrm{i}R_0}{nq}\frac{\partial q}{\partial \psi}$$

and (reminder) the normal component has been chosen as

$$B_n^\psi = \frac{B_{0\varphi}R_0}{R^2}.$$

For derivation one uses the Jacobian of symmetry flux coordinates $(\psi, \vartheta, \varphi)$

$$\sqrt{g} = \frac{1}{B_0^\vartheta} = \frac{q}{B_0^\varphi} = \frac{q}{R^2 B_{0\varphi}}.$$

2.5   Safety factor

Toroidal flux is

$$\psi_{\mathrm{tor}} = \int \mathrm{d}V\, \boldsymbol{B} \cdot \nabla\varphi$$

$$= 2\pi \int \mathrm{d}R\,\mathrm{d}Z\, RB^\varphi = \int \mathrm{d}R\,\mathrm{d}Z\, B_{(\varphi)}. \tag{77}$$

2.6   Current perturbation – alternative

We split the current into parallel and perpendicular current:

$$\boldsymbol{j}_n = j_{n\parallel}\boldsymbol{h}_0 + \boldsymbol{j}_{n\perp}. \tag{78}$$

We write the divergence-freeness condition as

$$\nabla \cdot (\boldsymbol{h}_0^{\mathrm{pol}} j_{n\parallel}) + \mathrm{i}n h_0^\varphi j_{n\parallel} = -\nabla \cdot \boldsymbol{j}_{n\perp}^{\mathrm{pol}} - \mathrm{i}n j_{n\perp}^\varphi. \tag{79}$$

In integral form, multiplied by R:

$$\oint R j_{n\parallel}\boldsymbol{h}_0^{\mathrm{pol}} \cdot \boldsymbol{n}\,\mathrm{d}l + \mathrm{i}n\int R h_0^\varphi j_{n\parallel}\,\mathrm{d}S = -\oint R\boldsymbol{j}_{n\perp}^{\mathrm{pol}} \cdot \boldsymbol{n}\,\mathrm{d}l - \mathrm{i}n\int R j_{n\perp}^\varphi\,\mathrm{d}S. \tag{80}$$

Approximate

$$R^1 j_{n\parallel}^1 \boldsymbol{h}_0^{\mathrm{pol}} \cdot \boldsymbol{n}^1 + R^2 j_{n\parallel}^2 \boldsymbol{h}_0^{\mathrm{pol}} \cdot \boldsymbol{n}^2 + \mathrm{i}n S_\Omega \frac{(h_{0(\varphi)}^1 j_{n\parallel}^1 + h_{0(\varphi)}^2 j_{n\parallel}^2)}{2} =$$
$$-R^1 \boldsymbol{j}_{n\perp}^{\mathrm{pol},1} \cdot \boldsymbol{n}^1 - R^2 \boldsymbol{j}_{n\perp}^{\mathrm{pol},2} \cdot \boldsymbol{n}^2 - R^3 \boldsymbol{j}_{n\perp}^{\mathrm{pol},3} \cdot \boldsymbol{n}^3 - \mathrm{i}n S_\Omega R j_{n\perp}^\varphi. \tag{81}$$

We use

$$j_{n\perp}^\varphi = \mathrm{TODO} - \frac{cB_{n\parallel}}{B_0^2}\nabla\varphi \cdot (\boldsymbol{h}_0 \times \nabla p_0) + \frac{c}{B_0}\nabla\varphi \cdot (\boldsymbol{h}_0 \times \nabla p_n) \tag{82}$$

$$= -\frac{cB_{n\parallel}}{B_0^3}\nabla p_0 \cdot ((\nabla\psi \times \nabla\varphi) \times \nabla\varphi) + \frac{c}{B_0^2}\nabla p_n \cdot ((\nabla\psi \times \nabla\varphi) \times \nabla\varphi) \tag{83}$$

$$= \frac{cB_{n\parallel}}{R^2 B_0^3}\nabla p_0 \cdot \nabla\psi - \frac{c}{R^2 B_0^2}\nabla p_n \cdot \nabla\psi \tag{84}$$

$$= \frac{cB_{n\parallel}}{R^2 B_0^3}p_0'(\psi)|\nabla\psi|^2 - \frac{c}{R^2 B_0^2}\nabla p_n \cdot \nabla\psi. \tag{85}$$

We would like to represent

$$\nabla\psi = (\nabla\psi)_1 \boldsymbol{l}^1 + (\nabla\psi)_2 \boldsymbol{l}^2. \tag{86}$$

We construct the system

$$\boldsymbol{l}^1 \cdot \nabla\psi = |l^1|^2(\nabla\psi)_1 + \boldsymbol{l}^1 \cdot \boldsymbol{l}^2(\nabla\psi)_2$$
$$\boldsymbol{l}^2 \cdot \nabla\psi = -(\boldsymbol{l}^1 \cdot \nabla\psi) = \boldsymbol{l}^1 \cdot \boldsymbol{l}^2(\nabla\psi)_1 + |l^2|^2(\nabla\psi)_2$$

or generally

$$v_1 = \frac{|l^2|^2 + (\boldsymbol{l}^1 \cdot \boldsymbol{l}^2)}{|l^1|^2|l^2|^2 - (\boldsymbol{l}^1 \cdot \boldsymbol{l}^2)^2}(\boldsymbol{l}^1 \cdot \boldsymbol{v})$$
$$v_2 = \frac{|l^1|^2 + (\boldsymbol{l}^1 \cdot \boldsymbol{l}^2)}{|l^1|^2|l^2|^2 - (\boldsymbol{l}^1 \cdot \boldsymbol{l}^2)^2}(\boldsymbol{l}^2 \cdot \boldsymbol{v}).$$

Finally

$$\boldsymbol{v} = \frac{|l^2|^2 + (\boldsymbol{l}^1 \cdot \boldsymbol{l}^2)}{|l^1|^2|l^2|^2 - (\boldsymbol{l}^1 \cdot \boldsymbol{l}^2)^2}(\boldsymbol{l}^1 \cdot \boldsymbol{v})\boldsymbol{l}^1 + \frac{|l^1|^2 + (\boldsymbol{l}^1 \cdot \boldsymbol{l}^2)}{|l^1|^2|l^2|^2 - (\boldsymbol{l}^1 \cdot \boldsymbol{l}^2)^2}(\boldsymbol{l}^2 \cdot \boldsymbol{v})\boldsymbol{l}^2 \tag{87}$$
$$= c_1(\boldsymbol{l}^1 \cdot \boldsymbol{v})\boldsymbol{l}^1 + c_2(\boldsymbol{l}^2 \cdot \boldsymbol{v})\boldsymbol{l}^2. \tag{88}$$

Flux

$$B_{n\parallel} = \boldsymbol{B}_n \cdot \boldsymbol{h} = \boldsymbol{B}_n^{\mathrm{pol}} \cdot \boldsymbol{h}^{\mathrm{pol}} + B_{n\varphi}h^\varphi$$
$$= c_1(\boldsymbol{n}^1 \cdot \boldsymbol{B}_n^{\mathrm{pol}})\boldsymbol{n}^1 \cdot \boldsymbol{h}^{\mathrm{pol}} + c_2(\boldsymbol{n}^2 \cdot \boldsymbol{B}_n^{\mathrm{pol}})\boldsymbol{n}^2 \cdot \boldsymbol{h}^{\mathrm{pol}} + B_{n(\varphi)}h_{(\varphi)}. \tag{89}$$

A   Overview of used vector quantities

$$\nabla\psi \parallel \nabla p_0 \parallel \boldsymbol{n}_3$$
$$\nabla\varphi \perp \left(\nabla\psi, ..., \boldsymbol{l}_3, \boldsymbol{B}_\bullet^{\mathrm{pol}}, \boldsymbol{h}_0^{\mathrm{pol}}, \boldsymbol{j}_\bullet^{\mathrm{pol}}, \nabla p_n\right)$$