Magnetic differential equations in stationary linear ideal MHD and their numerical solution

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Simple Example

We would like to solve an equation of the type

$$\dot{p}_n(s) + inp_n(s) = q_n(s) \tag{1}$$

for $p(s)$ with periodic boundary conditions at $s = 0 \ldots 2\pi$. With the ansatz $p_n = \sum_m p_{mn} e^{ims}$ and the same for q_n we obtain the analytical solution

$$p_{mn} = \frac{q_{mn}}{i(m+n)}. \tag{2}$$

The "natural" lowest order finite difference scheme using midpoint values where needed and with equidistant points in s yields

$$\frac{p_n^{(k+1)} - p_n^{(k)}}{\Delta s} + \frac{1}{2} in(p_n^{(k+1)} + p_n^{(k)}) = \frac{1}{2}(q_n^{(k+1)} + q_n^{(k)}). \tag{3}$$

A corresponding matrix with periodic boundary conditions is

$$\begin{pmatrix} in/2 - 1/\Delta s & in/2 + 1/\Delta s & & \\ & in/2 - 1/\Delta s & in/2 + 1/\Delta s & \\ & & \cdots & \\ in/2 + 1/\Delta s & & & in/2 - 1/\Delta s \end{pmatrix} \boldsymbol{p} = \begin{pmatrix} 1/2 & 1/2 & & \\ & 1/2 & 1/2 & \\ & & \cdots & \\ 1/2 & & & 1/2 \end{pmatrix} \boldsymbol{q}. \tag{4}$$

Magnetic differential equation for pressure

Magnetic differential equations arise from a number of problems in plasma physics. We consider for example the magnetohydrodynamic equilibrium

$$c\nabla p = \boldsymbol{J} \times \boldsymbol{B}, \tag{5}$$

with pressure p, current $\boldsymbol{J}$ and magnetic field $\boldsymbol{B}$. Scalar multiplication with $\boldsymbol{B}$ yields the homogenous magnetic differential equation

$$\boldsymbol{B} \cdot \nabla p = 0. \tag{6}$$

We consider an axisymmetric equilibrium field $\boldsymbol{B}_0$ given by

$$\boldsymbol{B}_0 = \nabla\psi \times \nabla\varphi + B_{0\varphi}\nabla\varphi. \tag{7}$$

Within linear perturbation theory where $\boldsymbol{B} = \boldsymbol{B}_0 + \delta\boldsymbol{B}$ and $p = p_0 + \delta p$, a source term enters the right-hand side with

$$\boldsymbol{B}_0 \cdot \nabla\delta p = -\delta\boldsymbol{B} \cdot \nabla p_0 = -\delta B^{\psi} p_0'(\psi). \tag{8}$$

For an axisymmetric plasma in a tokamak, we reduce the dimensionality by introducing cylindrical coordinates and an expansion in φ, such as

$$\delta p(R, Z, \varphi) = \sum_n p_n(R, Z) e^{in\varphi}. \tag{9}$$

The remaining equations in the poloidal RZ plane for each harmonic are

$$\boldsymbol{B}_0^{\mathrm{pol}} \cdot \nabla p_n + inB_0^{\varphi} p_n = -B_n^{\psi} p_0'(\psi), \tag{10}$$

where the superscript *pol* marks the projection into the RZ plane. This can also be written as

$$(\nabla\psi \times \nabla\varphi) \cdot \nabla p_n + \frac{in}{R^2} B_{0\varphi} p_n = -B_n^r p_0'(r), \tag{11}$$

where r is any flux surface label. In cylindrical coordinates we have

$$B_0^R = (\nabla\psi \times \nabla\varphi)^R = \frac{1}{R}\frac{\partial\psi}{\partial Z}, \tag{12}$$

$$B_0^Z = (\nabla\psi \times \nabla\varphi)^Z = -\frac{1}{R}\frac{\partial\psi}{\partial R}. \tag{13}$$

Finally, we write the form

$$R\boldsymbol{B}_0^{\mathrm{pol}} \cdot \nabla p_n + inB_{0(\varphi)} p_n = -RB_n^r p_0'(r). \tag{14}$$

Or normalized

$$R\boldsymbol{h}_0^{\mathrm{pol}} \cdot \nabla p_n + inh_{0(\varphi)} p_n = -Rh_n^r p_0'(r). \tag{15}$$

Finite difference method to treat pressure perturbation

To solve equations of the type of Eq. (10) we use the lowest order scheme with forward differences in the gradient, midpoint values for the remaining terms, and periodic boundary conditions.

$$\boldsymbol{h}_0^{\mathrm{pol}} \cdot \frac{\boldsymbol{r}^{(k+1)} - \boldsymbol{r}^{(k)}}{(\Delta r)^2}(p_n^{(k+1)} - p_n^{(k)}) + \frac{inh_0^{\varphi}}{2}(p_n^{(k+1)} + p_n^{(k)}) = \frac{p_0'(\psi)}{2}(h_n^{\psi(k+1)} + h_n^{\psi(k)}). \tag{16}$$

This equation has the general form

$$\left(a_k + \frac{b_k}{2}\right) p_n^{(k+1)} + \left(-a_k + \frac{b_k}{2}\right) p_n^{(k)} = \frac{c_k}{2}(h_n^{\psi(k+1)} + h_n^{\psi(k)}), \tag{17}$$

with

$$a_k = \boldsymbol{h}_0^{\mathrm{pol}} \cdot \frac{\boldsymbol{r}^{(k+1)} - \boldsymbol{r}^{(k)}}{(\Delta r)^2}, \tag{18}$$

$$b_k = inh_0^{\varphi}, \tag{19}$$

$$c_k = p_0'(\psi). \tag{20}$$

In our simple example of Eq. (3) we can identify

$$a_k = \frac{1}{\Delta s}, \tag{21}$$

$$b_k = in,$$

$$c_k = 1. \tag{22}$$

In general matrix form such a scheme is written as

$$A_{jk} p_n^{(k)} = M_{jk} h_n^{\psi(k)}, \tag{23}$$

where the elements of the stiffness matrix A are

$$A_{jk} = \left(a_j + \frac{b_j}{2}\right) \delta_{(j-1)k} + \left(-a_j + \frac{b_j}{2}\right) \delta_{jk}, \tag{24}$$

and elements of the mass matrix M are

$$M_{jk} = \frac{c_j}{2}(\delta_{(j-1)k} + \delta_{jk}). \tag{25}$$

At $k = N$, with N the number of rows, one should replace $\delta_{(j-1)k}$ by δ_{j1} for periodic boundary conditions.

Perturbation in current density

First variant: Use linear perturbation

$$\boldsymbol{j} \times \boldsymbol{B} \approx \boldsymbol{j}_0 \times \boldsymbol{B}_0 + \delta\boldsymbol{j} \times \boldsymbol{B}_0 + \boldsymbol{j}_0 \times \delta\boldsymbol{B} = c(\nabla p_0 + \nabla \delta p), \tag{26}$$

resulting in

$$\delta\boldsymbol{j} \times \boldsymbol{B}_0 = \delta\boldsymbol{j}_\perp \times \boldsymbol{B}_0 = c\nabla\delta p - \boldsymbol{j}_0 \times \delta\boldsymbol{B}. \tag{27}$$

Second variant: Use derived expresion for $\delta\boldsymbol{j}_\perp$ with

$$\delta\boldsymbol{j}_\perp = j_{0\|}\frac{\delta\boldsymbol{B}_\perp}{B_0} - \frac{c\boldsymbol{h}_0 \cdot \delta\boldsymbol{B}}{B_0^2}\boldsymbol{h}_0 \times \nabla p_0 + \frac{c}{B_0}\boldsymbol{h}_0 \times \nabla\delta p. \tag{28}$$

Take cross product with $\boldsymbol{B}_0$.

- First term

$$j_{0\|}\frac{\delta\boldsymbol{B}_\perp}{B_0} \times \boldsymbol{B}_0 = \delta\boldsymbol{B}_\perp \times (j_{0\|}\boldsymbol{h}_0) = \delta\boldsymbol{B}_\perp \times \boldsymbol{j}_{0\|}. \tag{29}$$

- Second term

$$-\frac{c\boldsymbol{h}_0\cdot\delta\boldsymbol{B}}{B_0^2}(\boldsymbol{h}_0\times\nabla p_0)\times\boldsymbol{B}_0 = -\frac{\delta B_\parallel}{B_0}(\boldsymbol{h}_0\times(\boldsymbol{j}_0\times\boldsymbol{B}_0))\times\boldsymbol{h}_0$$
$$= -\delta B_\parallel(\boldsymbol{j}_{0\perp}\times\boldsymbol{h}_0) = \delta\boldsymbol{B}_\parallel\times\boldsymbol{j}_{0\perp}. \tag{30}$$

- Third term

$$\frac{c}{B_0}(\boldsymbol{h}_0\times\nabla\delta p)\times\boldsymbol{B}_0 = c(\nabla\delta p-(\boldsymbol{h}_0\cdot\nabla\delta p)\boldsymbol{h}_0)$$

Summed up this yields

$$\delta\boldsymbol{j}_\perp\times\boldsymbol{B}_0 = \delta\boldsymbol{B}\times\boldsymbol{j}_0-\delta\boldsymbol{B}_\perp\times\boldsymbol{j}_{0\perp}+c\nabla\delta p-c(\boldsymbol{h}_0\cdot\nabla\delta p)\boldsymbol{h}_0. \tag{31}$$

If Eq. (27) is fulfilled, the two extra terms must cancel each other. Eq. (31) restricts only perpendicular components with

$$\delta\boldsymbol{j}_\perp\times\boldsymbol{B}_0 = (\delta\boldsymbol{B}\times\boldsymbol{j}_0)_\perp+c\nabla_\perp\delta p. \tag{32}$$

In addition, the restriction

$$(\delta\boldsymbol{B}\times\boldsymbol{j}_0)_\parallel = c\nabla_\parallel\delta p \tag{33}$$

follows from the linear perturbation in p made before.

Finite volume method to treat current density perturbation

Now we solve the linearised current balance in an FVM scheme. We write the conservative form

$$\nabla\cdot\boldsymbol{J}_{\parallel n}^{\mathrm{pol}}+inh_0^\varphi j_{\parallel n} = -\nabla\cdot\boldsymbol{j}_\perp^{\mathrm{pol}}-inj_{\perp n}^\varphi\,.$$

The divergence operator is defined via

$$\nabla\cdot\boldsymbol{u} = \frac{1}{R\sqrt{g_p}}\frac{\partial}{\partial x^k}(R\sqrt{g_p}u^k)\,.$$

When working with R, Z as coordinates in the poloidal plane, $\sqrt{g_p}=1$. We multiply by R to obtain

$$\frac{\partial}{\partial x^k}(Rh^k j_{\parallel n})+inRh^\varphi j_{\parallel n} = -\frac{\partial}{\partial x^k}(Rj_{\perp n}^k)-inRj_{\perp n}^\varphi\,.$$

Integration over a triangle yields

$$\oint Rj_{\parallel n}\boldsymbol{h}_0^{\mathrm{pol}}\cdot\boldsymbol{n}d\Gamma+in\int Rh_0^\varphi j_{\parallel n}d\Omega = -\oint R\boldsymbol{j}_{\perp n}^{\mathrm{pol}}\cdot\boldsymbol{n}d\Gamma-in\int Rj_{\perp n}^\varphi d\Omega \tag{34}$$

where scalar products with $\boldsymbol{n}$ pointing towards the outer normal vector of the edge are taken component-wise in R and Z. We assume a field-aligned mesh with $\boldsymbol{h}_0$ parallel to edge no. 3. In- and outflux of the parallel current are only over edges 1 and 2.

On edges 1 and 2 we sum up the flux from left- and right hand side of Eq. 34 to use the flux of the total perturbed current harmonic in the poloidal plane,

$$\boldsymbol{j}_n^{\mathrm{pol}} = j_{\|n}\boldsymbol{h}_0^{\mathrm{pol}} + \boldsymbol{j}_{\perp n}^{\mathrm{pol}},$$

as an unknown:

$$\int_{1,2} R\boldsymbol{j}_n^{\mathrm{pol}} \cdot \boldsymbol{n} d\Gamma + in \int R h_0^{\varphi} j_{\|n} d\Omega = -\int_3 R\boldsymbol{j}_{\perp n}^{\mathrm{pol}} \cdot \boldsymbol{n} d\Gamma - in \int R j_{\perp n}^{\varphi} d\Omega. \tag{35}$$

In addition, we add terms with $j_n^{\varphi} = h_0^{\varphi} j_{\|n} + j_{\perp n}^{\varphi}$ together again to obtain

$$\int_{1,2} R\boldsymbol{j}_n^{\mathrm{pol}} \cdot \boldsymbol{n} d\Gamma + \int_3 R\boldsymbol{j}_{\perp n}^{\mathrm{pol}} \cdot \boldsymbol{n} d\Gamma + in \int R j_n^{\varphi} d\Omega = 0. \tag{36}$$

To compute known quantities we use the linear equation of the perturbation given by

$$\boldsymbol{j}_n \times \boldsymbol{B}_0 = c(\nabla p_n + in\, p_n \nabla\varphi) - \boldsymbol{j}_0 \times \boldsymbol{B}_n. \tag{37}$$

Again $\boldsymbol{B}_n$ is known from the last iteration of the field solver, p_n has been computed in the earlier step and $\boldsymbol{j}_n$ is unknown.

Cross-field term on edge 3

Comparison to Sergei:

$$\boldsymbol{j}_{\perp n} = j_{0\|} \frac{\boldsymbol{B}_{\perp n}}{B_0} - \frac{cB_{\|n}}{B_0^2} \boldsymbol{h}_0 \times \nabla p_0 + \frac{c}{B_0} \boldsymbol{h}_0 \times (\nabla p_n + inp_n \nabla\varphi). \tag{38}$$

We take

$$\boldsymbol{j}_{\perp n} \cdot \nabla\psi = j_{0\|} \frac{\boldsymbol{B}_{\perp n}}{B_0} \cdot \nabla\psi + \frac{c}{B_0} (\boldsymbol{h}_0 \times (\nabla p_n + inp_n \nabla\varphi)) \cdot \nabla\psi. \tag{39}$$

We have

$$\nabla\psi \cdot (\boldsymbol{h}_0 \times \nabla\varphi) = -\boldsymbol{h}_0 \cdot (\nabla\psi \times \nabla\varphi) = -\boldsymbol{h}_0 \cdot \boldsymbol{B}_0^{\mathrm{pol}}, \tag{40}$$

and

$$\nabla\psi \cdot (\boldsymbol{h}_0 \times \nabla p_n) = \nabla p_n \cdot (\nabla\psi \times \boldsymbol{h}_0) = \nabla p_n \cdot (\nabla\psi \times (h_{0\varphi} \nabla\varphi + \boldsymbol{h}_0^{\mathrm{pol}})) \tag{41}$$

$$= h_{0\varphi} \nabla p_n \cdot (h_{0\varphi} \boldsymbol{B}_0^{\mathrm{pol}} + \nabla\psi \times \boldsymbol{h}_0^{\mathrm{pol}}) \tag{42}$$

Volumetric source term

For the computation of toroidal j_n^{φ} in the element volume we start again with

$$\boldsymbol{j}_n \times \boldsymbol{B}_0 = c(\nabla p_n + in\, p_n \nabla\varphi) - \boldsymbol{j}_0 \times \boldsymbol{B}_n. \tag{43}$$

with

$$\boldsymbol{B}_0 = \nabla\psi \times \nabla\varphi + B_{0\varphi} \nabla\varphi. \tag{44}$$

We use a local orthonormal coordinate system on each triangle edge with $\boldsymbol{e}_1$ the unit vector along the edge in counter-clockwise orientation, $\boldsymbol{e}_2 = \boldsymbol{n}$ the outward unit normal and $\boldsymbol{e}_3 = R\nabla\varphi$ pointing inside the plane. Taking a scalar product of $\boldsymbol{e}_1$ with Eq. (43) yields

$$\begin{aligned}
\boldsymbol{e}_1 \cdot (\boldsymbol{j}_n \times \boldsymbol{B}_0) &= \boldsymbol{e}_1 \cdot (\boldsymbol{j}_n \times (\nabla\psi \times \nabla\varphi + B_{0\varphi}\nabla\varphi)) \\
&= \boldsymbol{j}_n \cdot ((\nabla\psi \times \nabla\varphi) \times \boldsymbol{e}_1 + B_{0(\varphi)}\boldsymbol{e}_3 \times \boldsymbol{e}_1) \qquad (45) \\
&= \boldsymbol{j}_n \cdot \big((\boldsymbol{e}_1 \cdot \nabla\psi)\nabla\varphi + B_{0(\varphi)}\boldsymbol{e}_2\big) = (\boldsymbol{e}_1 \cdot \nabla\psi) j_n^{\varphi} + B_{0(\varphi)}\boldsymbol{j}_n^{\mathrm{pol}} \cdot \boldsymbol{n} \qquad (46)
\end{aligned}$$

The right-hand side yields:

$$\boldsymbol{e}_1 \cdot (\nabla p_n + in\, p_n \nabla\varphi) = \boldsymbol{e}_1 \cdot \nabla p_n \qquad (47)$$

$$\boldsymbol{e}_1 \cdot (\boldsymbol{j}_0 \times \boldsymbol{B}_n) = \boldsymbol{e}_1 \cdot (B_{n\varphi}\boldsymbol{j}_0^{\mathrm{pol}} \times \nabla\varphi - j_{0\varphi}\boldsymbol{B}_n^{\mathrm{pol}} \times \nabla\varphi). \qquad (48)$$

We use the fact that ∇p_0 is parallel to $\nabla\psi$, so the cross product in the equilibrium is purely radial,

$$\begin{aligned}
\boldsymbol{j}_0 \times \boldsymbol{B}_0 &= c\nabla p_0 \\
&= \boldsymbol{j}_0^{\mathrm{pol}} \times (B_{0\varphi}\nabla\varphi) + j_{0\varphi}\nabla\varphi \times (\nabla\psi \times \nabla\varphi) \\
&= \boldsymbol{j}_0^{\mathrm{pol}} \times (B_{0\varphi}\nabla\varphi) + \frac{j_{0\varphi}}{R^2}\nabla\psi. \qquad (49)
\end{aligned}$$

So

$$\boldsymbol{j}_0^{\mathrm{pol}} \times \nabla\varphi = \frac{1}{B_{0\varphi}}\left(c\nabla p_0 - \frac{j_{0\varphi}}{R^2}\nabla\psi\right), \qquad (50)$$

and

$$\begin{aligned}
\boldsymbol{e}_1 \cdot (\boldsymbol{j}_0 \times \boldsymbol{B}_n) &= \frac{B_{n\varphi}}{B_{0\varphi}}\boldsymbol{e}_1 \cdot \left(c\nabla p_0 - \frac{j_{0\varphi}}{R^2}\nabla\psi\right) - j_{0(\varphi)}\boldsymbol{e}_1 \cdot (\boldsymbol{B}_n^{\mathrm{pol}} \times \boldsymbol{e}_3) \\
&= \frac{B_{n(\varphi)}}{B_{0(\varphi)}}\boldsymbol{e}_1 \cdot \left(c\nabla p_0 - \frac{j_{0(\varphi)}}{R}\nabla\psi\right) - j_{0(\varphi)}\boldsymbol{B}_n^{\mathrm{pol}} \cdot \boldsymbol{n}. \qquad (51)
\end{aligned}$$

Thus, with the notation $\boldsymbol{e}_1 \cdot \nabla \equiv \partial_1$ we obtain on each edge

$$j_n^{\varphi}\partial_1\psi = -B_{0(\varphi)}\boldsymbol{j}_n^{\mathrm{pol}} \cdot \boldsymbol{n} + c\partial_1 p_n - \frac{B_{n(\varphi)}}{B_{0(\varphi)}}\left(c\partial_1 p_0 - \frac{j_{0(\varphi)}}{R}\partial_1\psi\right) + j_{0(\varphi)}\boldsymbol{B}_n^{\mathrm{pol}} \cdot \boldsymbol{n}. \qquad (52)$$

For the discretisation we take values such as $B_{0(\varphi)}$ constant on the whole triangle and compute the mean from edges 1 and 2 weighted by the edge length of the remaining quantities. The first term on the right-hand side will contribute to the vector of unknowns. Derivatives ∂_1 are then computed as differences between values on the two nodes of each edge.

Basics

With the convention of outwards pointing normals and counter-clockwise ordering of vertices in triangles, we have vectors along and across edges:

$$\boldsymbol{l} = l^R\boldsymbol{e}_R + l^Z\boldsymbol{e}_Z, \qquad (53)$$

$$\boldsymbol{n} = l^Z\boldsymbol{e}_R - l^R\boldsymbol{e}_Z = \hat{T}\boldsymbol{l}, \qquad (54)$$

with the transformation $\hat{T}$ of a 90 degree rotation to the right.

Implementation

We start with

$$\int_{1,2} \boldsymbol{j}_n^{\mathrm{pol}} \cdot \boldsymbol{n} d\Gamma + \int_3 \boldsymbol{j}_{\perp n}^{\mathrm{pol}} \cdot \boldsymbol{n} d\Gamma + in \int j_n^{\varphi} d\Omega = 0 \tag{55}$$

Taking $R \approx$ const. in one element.
Approximation of constant fluxes across each edge yields

$$I_1 + I_2 - \frac{inl_3}{|\nabla\psi|^2} \left(cp_n - \boldsymbol{j}_0^{\mathrm{pol}} \cdot \boldsymbol{A}_n \right) \boldsymbol{n}_3 \cdot \nabla\psi + \frac{inS_\Omega}{2} (j_{n1}^{\varphi} + j_{n2}^{\varphi}) = 0. \tag{56}$$

with

$$I_1 = \int_{1,2} \boldsymbol{j}_n^{\mathrm{pol}} \cdot \boldsymbol{n} d\Gamma.$$

Taking

$$j_n^{\varphi} = -\frac{B_{0(\varphi)}}{\partial_1 \psi} \boldsymbol{j}_n^{\mathrm{pol}} \cdot \boldsymbol{n} + c\frac{\partial_1 p_n}{\partial_1 \psi} - \frac{B_{n(\varphi)}}{B_{0(\varphi)}} \left(c\frac{\partial_1 p_0}{\partial_1 \psi} - \frac{j_{0(\varphi)}}{R} \right) + \frac{j_{0(\varphi)}}{\partial_1 \psi} \boldsymbol{B}_n^{\mathrm{pol}} \cdot \boldsymbol{n}, \tag{57}$$

we get on edge 1 (and similar on edge 2)

$$j_{n1}^{\varphi} = -\frac{B_{0(\varphi)}}{\Delta\psi} I_1 + c\frac{\Delta p_n}{\Delta\psi} - \frac{B_{n(\varphi)}}{B_{0(\varphi)}} \left(c\frac{\Delta p_0}{\Delta\psi} - \frac{j_{0(\varphi)}}{R} \right) + \frac{j_{0(\varphi)}}{\Delta\psi} l_1 \boldsymbol{B}_n^{\mathrm{pol}} \cdot \boldsymbol{n}. \tag{58}$$

Here $\Delta\psi = (\psi_{1e} - \psi_{1a}) = \boldsymbol{l}_1 \cdot \nabla\psi = l_1 \partial_1 \psi$ marks a difference between the two vertices, which is constant for flux surface quantities such as ψ or p_0.
Finally the equation in I_1 and I_2 as unknowns is

$$\left(1 - \frac{inS_\Omega}{2} \frac{B_{0(\varphi)}}{\Delta\psi} \right) I_1 + \left(1 - \frac{inS_\Omega}{2} \frac{B_{0(\varphi)}}{\Delta\psi} \right) I_2 = q, \tag{59}$$

with all the other terms moved to the right-hand side q.
In general matrix form, we call the ingoing current into triangle Nr. (k) counted in clockwise direction $I^{(k)}$. In triangle (k), this is equal to $I_1 = -I^{(k)}$ and $I_2 = I^{(k+1)}$. The matrix form of Eq. (59) is then

$$A_{jk} I^{(k)} = \boldsymbol{q}$$

where the elements of the stiffness matrix A are

$$A_{jk} = \left(1 - \frac{inS_{\Omega k}}{2} \frac{B_{0(\varphi)}}{\Delta\psi} \right) \delta_{(j-1)k} + \left(-1 - \frac{inS_{\Omega k}}{2} \frac{B_{0(\varphi)}}{\Delta\psi} \right) \delta_{jk}, \tag{60}$$

and the kth entry of vector $\boldsymbol{q}$ is

$$q_k = q_k^1 + q_k^2 + q_k^3$$

where

$$q_k^1 = \frac{in}{|\nabla\psi|^2}\left(cp_n - \boldsymbol{j}_0^{\mathrm{pol}} \cdot \boldsymbol{A}_n\right) l_3 \boldsymbol{n}_3 \cdot \nabla\psi \tag{61}$$

is evaluated on edge Nr. 3 (mean value for p_n there),

$$q_k^2 = \frac{B_{n(\varphi)}}{B_{0(\varphi)}}\left(c\frac{\Delta p_0}{\Delta\psi} - \frac{j_{0(\varphi)}}{R}\right) \tag{62}$$

is evaluated with constant values inside triangle k, and

$$q_k^3 = -c\frac{\Delta p_n}{\Delta\psi} - \frac{j_{0(\varphi)}}{\Delta\psi} l \boldsymbol{B}_n^{\mathrm{pol}} \cdot \boldsymbol{n}, \tag{63}$$

is an average over edge 1 and 2 of triangle k.
In the usual notation:

$$A_{jk} = \left(a_j + \frac{b_j}{2}\right)\delta_{(j-1)k} + \left(-a_j + \frac{b_j}{2}\right)\delta_{jk}, \tag{64}$$

$$a_j = 1 \tag{65}$$

$$b_j = -inS_{\Omega k}\frac{B_{0(\varphi)}}{\Delta\psi} \tag{66}$$

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or in components of coordinates x^k with $k = 1, 2$ in the poloidal plans,

$$B^k \frac{\partial}{\partial x^k} p_n + inB^\varphi p_n = q_n \,. \tag{67}$$

Generating $\boldsymbol{B}$ from the stream function $\psi = A_\varphi$ we obtain

$$B^1 = -\frac{1}{R\sqrt{g_p}}\frac{\partial\psi}{\partial x^2} \tag{68}$$

$$B^2 = \frac{1}{R\sqrt{g_p}}\frac{\partial\psi}{\partial x^1} \tag{69}$$

Here, g_p is the metric determinant of the 2D metric tensor in the poloidal plane. Using ψ as one coordinate x^1, and the distance s in the poloidal direction with $ds = \sqrt{dR^2 + dZ^2}$, we have an orthogonal system with

$$\hat{g}_P = \begin{pmatrix} g_{\psi\psi} & \\ & 1 \end{pmatrix} \tag{70}$$

This means that $\sqrt{g_p} = \sqrt{g_{\psi\psi}} = 1/|\nabla\psi|$. The transport law becomes

$$\frac{1}{R\sqrt{g_p}}\frac{\partial p_n}{\partial s} + inB^\varphi p_n = q_n \,. \tag{71}$$

This is a one-dimensional problem along the poloidal $\boldsymbol{B}$ direction.

Finite Difference Method

Multiplying by $-iR\sqrt{g_p}$ we obtain

$$nR\sqrt{g_p}B^\varphi p_n - i\dot{p}_n = -iR\sqrt{g_p}q_n\,. \tag{72}$$

Discretizing with a forward Euler method and evaluating averages at the midpoints we obtain

$$\frac{n}{2}\left(R^k\sqrt{g_p^k}B^{\varphi k}p_n^k + R^{k+1}\sqrt{g_p^{k+1}}B^{\varphi k+1}p_n^{k+1}\right) - i\frac{p_n^{k+1}-p_n^k}{\Delta s^k} \tag{73}$$

$$= -\frac{1}{2}i\left(R^k\sqrt{g_p^k}q_n^k + R^{k+1}\sqrt{g_p^{k+1}}q_n^{k+1}\right)\,. \tag{74}$$

Coefficients should be filled into a sparse matrix and the discrete equations solved e.g. by UMF-PACK.

Analytical solution

$$i(mB^\vartheta + nB^\varphi)p_{mn} = q_{mn}$$

$$p_{mn} = \frac{q_{mn}}{i(mB^\vartheta + nB^\varphi)}\,.$$

We take circular flux surfaces in the large aspect ratio limit, such that the scaling with R vanishes and as coordinates minor r and ϑ.
In this case

$$B^\vartheta = \frac{1}{R_0\sqrt{g_P}}\frac{\partial\psi}{\partial r}\,.$$

We set $\psi = r^2/4$ so $\frac{\partial\psi}{\partial r} = |\nabla\psi| = r/2$. Due to the circular flux surfaces, we have a orthogonal system and $\sqrt{g_P} = r$, so $B^\vartheta = 1/(2R_0)$.

Volumetric source term

By taking the inner product with $\boldsymbol{e}_\psi$ we obtain

$$\begin{aligned}\boldsymbol{e}_\psi\cdot(\boldsymbol{j}_n\times\boldsymbol{B}_0) &= \boldsymbol{e}_\psi\cdot(\boldsymbol{j}_n\times(\nabla\psi\times\nabla\varphi + B_{0\varphi}\nabla\varphi))\\ &= \boldsymbol{j}_n\cdot((\nabla\psi\times\nabla\varphi)\times\boldsymbol{e}_\psi + B_{0\varphi}\nabla\varphi\times\boldsymbol{e}_\psi) \qquad (75)\\ &= \boldsymbol{j}_n\cdot\left(\nabla\varphi + \frac{B_{0\varphi}}{|\nabla\psi|^2}\nabla s\right) = j_n^\varphi \pm \frac{B_{0\varphi}}{|\nabla\psi|^2}j_{n\parallel} \qquad (76)\end{aligned}$$

on the left-hand side. The right-hand side yields:

$$\boldsymbol{e}_\psi\cdot(\nabla p_n + in\,p_n\nabla\varphi) = \boldsymbol{e}_\psi\cdot\nabla p_n = \frac{\partial p_n}{\partial\psi} \tag{77}$$

$$\begin{aligned}\boldsymbol{e}_\psi\cdot(\boldsymbol{j}_0\times\boldsymbol{B}_n) &= \frac{\nabla\psi}{|\nabla\psi|^2}\cdot(B_{n\varphi}\boldsymbol{j}_0^{\mathrm{pol}}\times\nabla\varphi - j_{0\varphi}\boldsymbol{B}_n^{\mathrm{pol}}\times\nabla\varphi)\\ &= \frac{\boldsymbol{B}_0^{\mathrm{pol}}}{|\nabla\psi|^2}\cdot(j_{0\varphi}\boldsymbol{B}_n^{\mathrm{pol}} - B_{n\varphi}\boldsymbol{j}_0^{\mathrm{pol}}). \qquad (78)\end{aligned}$$

can use

$$\frac{\nabla\psi}{|\nabla\psi|^2}\cdot\left(B_{n\varphi}\boldsymbol{j}_0^{\mathrm{pol}}\times\nabla\varphi - j_{0\varphi}\boldsymbol{B}_n^{\mathrm{pol}}\times\nabla\varphi\right) = \frac{B_{n\varphi}}{B_{0\varphi}}\frac{\nabla\psi}{|\nabla\psi|^2}\cdot\left(c\nabla p_0 - j_{0\varphi}\nabla\psi\right) + j_{0\varphi}\frac{\left(B_n^{\mathrm{pol}}\right)^2}{|\nabla\psi|^2}$$
$$=$$

Take scalar product with $\nabla\psi$ to obtain

$$\nabla\psi\cdot\left(\boldsymbol{j}_0^{\mathrm{pol}}\times(B_{0\varphi}\nabla\varphi)\right) = -B_{0\varphi}\boldsymbol{j}_0^{\mathrm{pol}}\cdot(\nabla\psi\times\nabla\varphi) = -B_{0\varphi}\boldsymbol{j}_0^{\mathrm{pol}}\cdot\boldsymbol{B}_0^{\mathrm{pol}} \tag{79}$$

$$\nabla\psi\cdot$$

Cross product with $\boldsymbol{e}_\varphi$ yields

$$\begin{aligned} B_{0\varphi}\boldsymbol{e}_\varphi\times(\boldsymbol{j}_n\times\nabla\varphi) &= \boldsymbol{j}_n - j_{n\varphi}\nabla\varphi \qquad (80)\\ \boldsymbol{e}_\varphi\times(\boldsymbol{j}_n\times(\nabla\psi\times\nabla\varphi)) &= \boldsymbol{e}_\varphi\times(j_n^\varphi\nabla\psi - j_n^\psi\nabla\varphi) = j_n^\varphi\boldsymbol{e}_\varphi\times\nabla\psi\\ &= j_{n\varphi}\nabla\varphi\times\nabla\psi \end{aligned}$$

- Scalar product with $\boldsymbol{B}_0^{\mathrm{pol}} = \nabla\psi\times\nabla\varphi$ yields

$$\boldsymbol{B}_0^{\mathrm{pol}}\cdot(\boldsymbol{j}_n\times\boldsymbol{B}_0^{\mathrm{pol}}) = 0 \tag{81}$$
$$B_{0\varphi}(\nabla\psi\times\nabla\varphi)\cdot(\boldsymbol{j}_n\times\nabla\varphi) = \frac{B_{0\varphi}}{R^2}\boldsymbol{j}_n\cdot\nabla\psi \tag{82}$$

Right-hand side:

$$\begin{aligned} \boldsymbol{B}_0^{\mathrm{pol}}\cdot(\nabla p_n + in\,p_n\nabla\varphi) &= \boldsymbol{B}_0^{\mathrm{pol}}\cdot\nabla p_n\\ \boldsymbol{B}_0^{\mathrm{pol}}\cdot(\boldsymbol{j}_0\times\boldsymbol{B}_n) &= (\nabla\psi\times\nabla\varphi)\cdot(\boldsymbol{j}_0\times\boldsymbol{B}_n)\\ &= (\boldsymbol{j}_0\cdot\nabla\psi)(\boldsymbol{B}_n\cdot\nabla\varphi) - (\boldsymbol{B}_n\cdot\psi)(\boldsymbol{j}_0\cdot\nabla\varphi) \end{aligned}$$

We need to solve

$$\nabla\cdot\delta\boldsymbol{j} = 0, \tag{83}$$
$$\nabla\delta p = \frac{1}{c}\left(\delta\boldsymbol{j}\times\boldsymbol{B}_0 + \boldsymbol{j}_0\times\delta\boldsymbol{B}\right) \tag{84}$$

for $\delta\boldsymbol{j}$. Splitting into toroidal and poloidal parts for a single harmonic in φ we obtain

$$\nabla\cdot\boldsymbol{j}_n^{\mathrm{pol}} + in\,j_n^\varphi = 0. \tag{85}$$

The second equation reads

$$\boldsymbol{j}_n\times\boldsymbol{B}_0 = c(\nabla p_n + in\,p_n\nabla\varphi) - \boldsymbol{j}_0\times\boldsymbol{B}_n. \tag{86}$$

The toroidal part of the cross product is

$$(\boldsymbol{j}_n\times\boldsymbol{B}_0)_\varphi = R(j_n^Z B_0^R - j_n^R B_0^Z) = R\sqrt{g_P}j_n^\psi B_0^{\mathrm{pol}} \tag{87}$$
$$= in\,p_n - (\boldsymbol{j}_0\times\boldsymbol{B}_n)_\varphi \tag{88}$$
$$\Rightarrow j_n^\psi \text{on flux surface edge}$$

On one triangle edge

$$\begin{aligned} (\boldsymbol{j}_n\times\boldsymbol{B}_0)_\parallel &= R(j_{n\perp}B_0^\varphi - j_n^\varphi B_0^\perp)\\ &\approx c\frac{p_2 - p_1}{l} - (\boldsymbol{j}_0\times\boldsymbol{B}_n)_\parallel \end{aligned}$$

Finite Volume Method

The divergence operator is defined via

$$\nabla \cdot \boldsymbol{u} = \frac{1}{R\sqrt{g_p}} \frac{\partial}{\partial x^k}(R\sqrt{g_p} u^k) \,.$$

When working with R, Z as coordinates in the poloidal plane, $\sqrt{g_p} = 1$. We multiply by R to obtain

$$\frac{\partial}{\partial x^k}(Rh^k j_{\|n}) + inRh^\varphi j_{\|n} = -\frac{\partial}{\partial x^k}(Rj^k_{\perp n}) - inRj^\varphi_{\perp n} \,.$$

Integration over a triangle yields

$$\oint Rj_{\|n}\boldsymbol{h} \cdot \boldsymbol{n} d\Gamma + in \int Rh^\varphi j_{\|n} d\Omega = -\oint R\boldsymbol{j}^{\mathrm{pol}}_{\perp n} \cdot \boldsymbol{n} d\Gamma - in \int Rj^\varphi_{\perp n} d\Omega \tag{89}$$

where scalar products with $\boldsymbol{n}$ pointing towards the outer normal vector of the edge are taken component-wise in R and Z. We assume a field-aligned mesh with $\boldsymbol{h}$ parallel to edge no. 3. The in- and outflux are over edges 1 and 2.

General Finite Volume Method

Eq. (89) is of the form

$$\oint u\,\boldsymbol{h} \cdot \boldsymbol{n} d\Gamma + in \int u\,h^\varphi d\Omega = -\oint \boldsymbol{v} \cdot \boldsymbol{n} d\Gamma - in \int w\, d\Omega$$

with $u = Rj_{\|n}$, $\boldsymbol{v} = R\boldsymbol{j}^{\mathrm{pol}}_{\perp n}$ and $w = Rj^\varphi_{\perp n}$. We approximate the flux by a flux value times edge length. Since the mesh is field-aligned, only two of the three triangle edges play a role for fluxes and we can write

$$\oint u\,\boldsymbol{h} \cdot \boldsymbol{n} d\Gamma \approx U_1 + U_2 = u_1 l_1 + u_2 l_2, \tag{90}$$

where

$$U_1 = \int_1 u\boldsymbol{h} \cdot \boldsymbol{n} \mathrm{d}\Gamma_1. \tag{91}$$

and so on. In the second term we can use

$$\int u\,h^\varphi d\Omega \approx \frac{u_1 + u_2}{2} h^\varphi S$$

where S is the surface of the triangle.

For $\boldsymbol{v}$ the normal components to the edges (fluxes through edges) are required via

$$\begin{aligned}\oint \boldsymbol{v} \cdot \boldsymbol{n} d\Gamma &\approx V_1 + V_2 + V_3 \\ &= \boldsymbol{v}_1 \cdot \boldsymbol{n}_1 l_1 + \boldsymbol{v}_2 \cdot \boldsymbol{n}_2 l_2 + \boldsymbol{v}_3 \cdot \boldsymbol{n}_3 l_3\end{aligned} \tag{92}$$

Central difference scheme:

$$\dot{p}_H = \frac{\Delta s^{k-1}}{\Delta s^k(\Delta s^k + \Delta s^{k-1})}p^{k+1} + \frac{\Delta s^k - \Delta s^{k-1}}{\Delta s^k \Delta s^{k-1}}p^k - \frac{\Delta s^k}{\Delta s^{k-1}(\Delta s^k + \Delta s^{k-1})}p^{k-1}$$

In harmonics in φ this becomes

$$\boldsymbol{B} \cdot \nabla_{RZ} p_n + inB^\varphi p_n = s_n \,. \tag{93}$$

If we take no toroidicity and harmonic RHS term with poloidal harmonic m we obtain

$$\begin{aligned} i(mB^\vartheta + nB^\varphi)p_{mn} &= s_{mn} \\ p_{mn} &= \frac{s_{mn}}{i(mB^\vartheta + nB^\varphi)} \,. \end{aligned}$$

Without toroidicity:

$$\begin{aligned} \sqrt{g} &= r \\ B^\vartheta &= \frac{1}{r}\partial_r A_\varphi \end{aligned}$$

So for $A_\varphi = r^2/4$ we get $B^\vartheta = 1/2$. Furthermore, we choose $B^\varphi = 1$. We have

$$\begin{aligned} R &= R_0 + r\cos\vartheta \\ Z &= r\sin\vartheta \end{aligned}$$

$$\begin{aligned} \frac{\partial R}{\partial r} &= (R - R_0)/r \\ \frac{\partial Z}{\partial r} &= Z/r \end{aligned}$$

$$\begin{aligned} \frac{\partial R}{\partial \vartheta} &= -z \\ \frac{\partial Z}{\partial \vartheta} &= R - R_0 \end{aligned}$$

$$\begin{aligned} B^R &= \frac{\partial R}{\partial \vartheta}B^\vartheta = -z/2 \\ B^Z &= \frac{\partial Z}{\partial \vartheta}B^\vartheta = (R - R_0)/2 \end{aligned}$$

In real and imaginary parts this is

$$\boldsymbol{B} \cdot \nabla_{RZ}(\Re p_n + i\Im p_n) + inB^\varphi(\Re p_n + i\Im p_n) = (\Re s_n + i\Im s_n) \tag{94}$$

$$\boldsymbol{B} \cdot \nabla_{RZ}\Re p_n - nB^\varphi \Im p_n = \Re s_n \tag{95}$$

$$\boldsymbol{B} \cdot \nabla_{RZ}\Im p_n + nB^\varphi \Re p_n = \Im s_n \tag{96}$$

Combining

$$\begin{aligned} \boldsymbol{B} \cdot \nabla_{RZ}(\boldsymbol{B} \cdot \nabla_{RZ}\Re p_n) - nB^\varphi(\Im s_n - nB^\varphi \Re p_n) &= \boldsymbol{B} \cdot \nabla_{RZ} \cdot \Re s_n \\ \boldsymbol{B} \cdot \nabla_{RZ}(\boldsymbol{B} \cdot \nabla_{RZ}\Im p_n) + nB^\varphi(\Re s_n + nB^\varphi \Im p_n) &= \boldsymbol{B} \cdot \nabla_{RZ} \cdot \Im s_n \end{aligned}$$

In Flat space:

$$\begin{aligned}
B^R\partial_R(\boldsymbol{B}\cdot\nabla_{RZ}\Re p_n) + B^Z\partial_Z(\boldsymbol{B}\cdot\nabla_{RZ}\Re p_n)\\
=(B^R\partial_R + B^Z\partial_Z)(B^R\partial_R + B^Z\partial_Z)\Re p_n\\
=\left((B^R)^2\partial_R^2 + 2B^RB^Z\partial_R\partial_Z + \left(B^Z\right)^2\partial_Z^2\right)\Re p_n
\end{aligned}$$

This equation is parabolic and not, as such, suited for FEM.
New:

$$\begin{aligned}
\boldsymbol{B}\cdot\nabla_{RZ}(\boldsymbol{B}\cdot\nabla_{RZ}\Re p_n) &= \nabla_{RZ}\cdot(\boldsymbol{B}(\boldsymbol{B}\cdot\nabla_{RZ}\Re p_n))\\
&= \nabla_{RZ}\cdot(\boldsymbol{B}\nabla_{RZ}\cdot(\boldsymbol{B}\Re p_n))
\end{aligned}$$

In real and imaginary parts this is

$$\begin{aligned}
i(mB^\vartheta + nB^\varphi)(\Re p_{mn} + i\Im p_{mn}) &= (\Re s_{mn} + i\Im s_{mn})\\
\Im p_{mn} &= -\Re s_{mn}/(mB^\vartheta + nB^\varphi)\\
\Re p_{mn} &= \Im s_{mn}/(mB^\vartheta + nB^\varphi)
\end{aligned}$$

We have

$$s = \sum_n s_n(\vartheta)e^{in\varphi} = \sum_{mn} s_{mn}e^{i(m\vartheta+n\varphi)}$$

$$\begin{aligned}
&= \sum_n (\Re s_n + i\Im s_n)(\cos n\varphi + i\sin n\varphi)\\
&= \sum_n (\Re s_n\cos n\varphi - \Im s_n\sin n\varphi) + i(\Re s_n\sin n\varphi + \Im s_n\cos n\varphi)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{mn}(\Re s_{mn} + i\Im s_{mn})(\cos(m\vartheta+n\varphi) + i\sin(m\vartheta+n\varphi))\\
&= \sum_{mn}(\Re s_{mn}\cos(m\vartheta+n\varphi) - \Im s_{mn}\sin(m\vartheta+n\varphi))\\
&+ i(\Re s_{mn}\sin(m\vartheta+n\varphi) + \Im s_{mn}\cos(m\vartheta+n\varphi))
\end{aligned}$$

$$\begin{aligned}
s_n &= s_{mn}e^{im\vartheta} = (\Re s_{mn} + i\Im s_{mn})(\cos m\vartheta + i\sin m\vartheta)\\
&= \Re s_{mn}\cos m\vartheta - \Im s_{mn}\sin m\vartheta + i(\Re s_{mn}\sin m\vartheta + \Im s_{mn}\cos m\vartheta)
\end{aligned}$$

Test:

$$\begin{aligned}
s &= \Im s_{mn}(\cos(m\vartheta+n\varphi) - i\sin(m\vartheta+n\varphi))\\
s_n &= s_{mn}e^{im\vartheta} = \Im s_{mn}(-\sin m\vartheta + i\cos m\vartheta)\\
&=
\end{aligned}$$

$$\Re p_{mn} = \Im s_{mn}/(mB^\vartheta + nB^\varphi)$$

$$p_n = \Re p_{mn}(\cos m\vartheta + i\sin m\vartheta)$$

Pseudotoroidal coordinates

$$\begin{aligned} R &= R_0 + r\cos\vartheta \\ Z &= r\sin\vartheta \end{aligned}$$

$$\begin{aligned} \boldsymbol{e}_r &= \frac{\partial R}{\partial r}\boldsymbol{e}_R + \frac{\partial Z}{\partial r}\boldsymbol{e}_Z \\ &= \boldsymbol{e}_R\cos\vartheta + \boldsymbol{e}_Z\sin\vartheta \end{aligned}$$

$$\begin{aligned} \boldsymbol{e}_\vartheta &= \frac{\partial R}{\partial \vartheta}\boldsymbol{e}_R + \frac{\partial Z}{\partial \vartheta}\boldsymbol{e}_Z \\ &= -\boldsymbol{e}_R r\sin\vartheta + \boldsymbol{e}_Z r\cos\vartheta \end{aligned}$$