Numerical solution of magnetic differential equations

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Magnetic differential equations arise from a number of problems in plasma physics. We consider for example the magnetohydrodynamic equilibrium

$$\nabla p = \boldsymbol{J} \times \boldsymbol{B}, \tag{1}$$

with pressure p, current $\boldsymbol{J}$ and magnetic field $\boldsymbol{B}$. Scalar multiplication with $\boldsymbol{B}$ yields the homogenous magnetic differential equation

$$\boldsymbol{B} \cdot \nabla p = 0\,. \tag{2}$$

In the linear perturbation theory, a source term enters the right-hand side with

$$\boldsymbol{B} \cdot \nabla p = q\,. \tag{3}$$

For an axisymmetric plasma in a tokamak, we reduce the dimensionality by introducing cylindrical coordinates and an expansion in φ,

$$p(R, Z, \varphi) = \sum_n p_n(R, Z) e^{in\varphi}\,. \tag{4}$$

The remaining equation in the poloidal RZ plane for each harmonic are

$$\boldsymbol{B} \cdot \nabla_{RZ} p_n + inB^{\varphi} p_n = q_n\,. \tag{5}$$

Finite Difference Method

or in components of coordinates x^k with $k = 1, 2$ in the poloidal plans,

$$B^k \frac{\partial}{\partial x^k} p_n + inB^{\varphi} p_n = q_n\,. \tag{6}$$

Generating B from the stream function $\psi = A_\varphi$ we obtain

$$B^1 = -\frac{1}{R\sqrt{g_p}} \frac{\partial \psi}{\partial x^2} \tag{7}$$

$$B^2 = \frac{1}{R\sqrt{g_p}} \frac{\partial \psi}{\partial x^1} \tag{8}$$

Here, g_p is the metric determinant of the 2D metric tensor in the poloidal plane. Using ψ as one coordinate x^1, and the distance s in the poloidal direction with $ds = \sqrt{dR^2 + dZ^2}$ yields

$$\frac{1}{R\sqrt{g_p}} \frac{\partial p_n}{\partial s} + inB^{\varphi} p_n = q_n\,. \tag{9}$$

This is a one-dimensional problem along the poloidal $\boldsymbol{B}$ direction. Multplying by $-iR\sqrt{g_p}$ and using (TODO $\sqrt{g_p}$) the central difference formula around the k-th point p_n^k,

$$nR\sqrt{g_p} B^{\varphi k} p_n^k - ip'_H = -iR\sqrt{g_p} q_n^k \tag{10}$$

we use the finite difference scheme

$$p'_H = \frac{\Delta s^{k-1}}{\Delta s^k(\Delta s^k + \Delta s^{k-1})} p^{k+1} + \frac{\Delta s^k - \Delta s^{k-1}}{\Delta s^k \Delta s^{k-1}} p^k - \frac{\Delta s^k}{\Delta s^{k-1}(\Delta s^k + \Delta s^{k-1})} p^{k-1}$$

We need to solve

$$A\boldsymbol{p} = \boldsymbol{q} \tag{11}$$

Vectors $\boldsymbol{p}$ and $\boldsymbol{q}$ contain values p_n^k and q_n^k at the nodes. With our choice of variables, we have an orthogonal system with

$$\hat{g}_P = \begin{pmatrix} g_{\psi\psi} & \\ & 1 \end{pmatrix}$$

This means that $\sqrt{g_p} = \sqrt{g_{\psi\psi}} = 1/|\nabla\psi|$.

Analytical solution

$$\begin{aligned} i(mB^\vartheta + nB^\varphi)p_{mn} &= q_{mn} \\ p_{mn} &= \frac{q_{mn}}{i(mB^\vartheta + nB^\varphi)} \,. \end{aligned}$$

We take circular flux surfaces in the large aspect ratio limit, such that the scaling with R vanishes and as coordinates minor r and ϑ.

In this case

$$B^\vartheta = \frac{1}{R_0\sqrt{g_P}} \frac{\partial\psi}{\partial r} \,.$$

We set $\psi = r^2/4$ so $\frac{\partial\psi}{\partial r} = |\nabla\psi| = r/2$. Due to the circular flux surfaces, we have a orthogonal system and $\sqrt{g_P} = r$, so $B^\vartheta = 1/(2R_0)$.

Finite Volume Method

Now we solve the same problem using a FVM scheme. We write the conservative form

$$\nabla \cdot (\boldsymbol{h} j_{\parallel n}) + inh^\varphi j_{\parallel n} = -\nabla \cdot \boldsymbol{j}_\perp^{\mathrm{pol}} - inj_{\perp n}^\varphi \,.$$

The divergence operator is defined via

$$\nabla \cdot \boldsymbol{u} = \frac{1}{R\sqrt{g_p}} \frac{\partial}{\partial x^k}(R\sqrt{g_p} u^k) \,.$$

When working with R, Z as coordinates in the poloidal plane, $\sqrt{g_p} = 1$. We multiply by R to obtain

$$\frac{\partial}{\partial x^k}(Rh^k j_{\parallel n}) + inRh^\varphi j_{\parallel n} = -\frac{\partial}{\partial x^k}(Rj_{\perp n}^k) - inRj_{\perp n}^\varphi \,.$$

Integration over a triangle yields

$$\oint Rj_{\parallel n}\boldsymbol{h} \cdot d\boldsymbol{\Gamma} + in\int Rh^\varphi j_{\parallel n} d\Omega = -\oint R\boldsymbol{j}_{\perp n}^{\mathrm{pol}} \cdot d\boldsymbol{\Gamma} - in\int Rh^\varphi j_{\perp n} d\Omega \tag{12}$$

where scalar products with $d\boldsymbol{\Gamma}$ pointing towards the outer normal vector of the edge are taken component-wise in R and Z. We assume a field-aligned mesh with $\boldsymbol{h}$ parallel to edge no. 3. The in- and outflux are over edges 1 and 2.

General Finite Volume Method

Eq. (12) is of the form

$$\oint u\,\boldsymbol{h}\cdot d\boldsymbol{\Gamma} + in\int u\,h^\varphi d\Omega = -\oint \boldsymbol{v}\cdot d\boldsymbol{\Gamma} - in\int v\,h^\varphi d\Omega$$

with $u = Rj_{\|n}$, $\boldsymbol{v} = R\boldsymbol{j}_{\perp n}^{\mathrm{pol}}$ and $v = j_{\perp n}$.

Old

In harmonics in φ this becomes

$$\boldsymbol{B}\cdot\nabla_{RZ}p_n + inB^\varphi p_n = s_n\,. \tag{13}$$

If we take no toroidicity and harmonic RHS term with poloidal harmonic m we obtain

$$\begin{aligned} i(mB^\vartheta + nB^\varphi)p_{mn} &= s_{mn} \\ p_{mn} &= \frac{s_{mn}}{i(mB^\vartheta + nB^\varphi)}\,. \end{aligned}$$

Without toroidicity:

$$\begin{aligned} \sqrt{g} &= r \\ B^\vartheta &= \frac{1}{r}\partial_r A_\varphi \end{aligned}$$

So for $A_\varphi = r^2/4$ we get $B^\vartheta = 1/2$. Furthermore, we choose $B^\varphi = 1$. We have

$$\begin{aligned} R &= R_0 + r\cos\vartheta \\ Z &= r\sin\vartheta \end{aligned}$$

$$\begin{aligned} \frac{\partial R}{\partial r} &= (R-R_0)/r \\ \frac{\partial Z}{\partial r} &= Z/r \end{aligned}$$

$$\begin{aligned} \frac{\partial R}{\partial \vartheta} &= -z \\ \frac{\partial Z}{\partial \vartheta} &= R-R_0 \end{aligned}$$

$$\begin{aligned} B^R &= \frac{\partial R}{\partial \vartheta}B^\vartheta = -z/2 \\ B^Z &= \frac{\partial Z}{\partial \vartheta}B^\vartheta = (R-R_0)/2 \end{aligned}$$

In real and imaginary parts this is

$$\boldsymbol{B}\cdot\nabla_{RZ}(\Re p_n + i\Im p_n) + inB^\varphi(\Re p_n + i\Im p_n) = (\Re s_n + i\Im s_n) \tag{14}$$

$$\boldsymbol{B}\cdot\nabla_{RZ}\Re p_n - nB^\varphi\Im p_n = \Re s_n \tag{15}$$

$$\boldsymbol{B}\cdot\nabla_{RZ}\Im p_n + nB^\varphi\Re p_n = \Im s_n \tag{16}$$

Combining

$$\begin{aligned} \boldsymbol{B}\cdot\nabla_{RZ}(\boldsymbol{B}\cdot\nabla_{RZ}\Re p_n) - nB^\varphi(\Im s_n - nB^\varphi\Re p_n) &= \boldsymbol{B}\cdot\nabla_{RZ}\cdot\Re s_n \\ \boldsymbol{B}\cdot\nabla_{RZ}(\boldsymbol{B}\cdot\nabla_{RZ}\Im p_n) + nB^\varphi(\Re s_n + nB^\varphi\Im p_n) &= \boldsymbol{B}\cdot\nabla_{RZ}\cdot\Im s_n \end{aligned}$$

In Flat space:

$$
\begin{aligned}
B^R\partial_R(\boldsymbol{B}\cdot\nabla_{RZ}\Re p_n) + B^Z\partial_Z(\boldsymbol{B}\cdot\nabla_{RZ}\Re p_n)&\\
= (B^R\partial_R + B^Z\partial_Z)(B^R\partial_R + B^Z\partial_Z)\Re p_n&\\
= \left((B^R)^2\partial_R^2 + 2B^RB^Z\partial_R\partial_Z + \left(B^Z\right)^2\partial_Z^2\right)\Re p_n&
\end{aligned}
$$

This equation is parabolic and not, as such, suited for FEM.

New:

$$
\begin{aligned}
\boldsymbol{B}\cdot\nabla_{RZ}(\boldsymbol{B}\cdot\nabla_{RZ}\Re p_n) &= \nabla_{RZ}\cdot(\boldsymbol{B}(\boldsymbol{B}\cdot\nabla_{RZ}\Re p_n))\\
&= \nabla_{RZ}\cdot(\boldsymbol{B}\nabla_{RZ}\cdot(\boldsymbol{B}\Re p_n))
\end{aligned}
$$

In real and imaginary parts this is

$$
\begin{aligned}
i(mB^\vartheta + nB^\varphi)(\Re p_{mn} + i\Im p_{mn}) &= (\Re s_{mn} + i\Im s_{mn})\\
\Im p_{mn} &= -\Re s_{mn}/(mB^\vartheta + nB^\varphi)\\
\Re p_{mn} &= \Im s_{mn}/(mB^\vartheta + nB^\varphi)
\end{aligned}
$$

We have

$$
s = \sum_n s_n(\vartheta)e^{in\varphi} = \sum_{mn} s_{mn}e^{i(m\vartheta+n\varphi)}
$$

$$
\begin{aligned}
&= \sum_n(\Re s_n + i\Im s_n)(\cos n\varphi + i\sin n\varphi)\\
&= \sum_n(\Re s_n\cos n\varphi - \Im s_n\sin n\varphi) + i(\Re s_n\sin n\varphi + \Im s_n\cos n\varphi)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{mn}(\Re s_{mn} + i\Im s_{mn})(\cos(m\vartheta+n\varphi) + i\sin(m\vartheta+n\varphi))\\
&= \sum_{mn}(\Re s_{mn}\cos(m\vartheta+n\varphi) - \Im s_{mn}\sin(m\vartheta+n\varphi))\\
&+ i(\Re s_{mn}\sin(m\vartheta+n\varphi) + \Im s_{mn}\cos(m\vartheta+n\varphi))
\end{aligned}
$$

$$
\begin{aligned}
s_n &= s_{mn}e^{im\vartheta} = (\Re s_{mn} + i\Im s_{mn})(\cos m\vartheta + i\sin m\vartheta)\\
&= \Re s_{mn}\cos m\vartheta - \Im s_{mn}\sin m\vartheta + i(\Re s_{mn}\sin m\vartheta + \Im s_{mn}\cos m\vartheta)
\end{aligned}
$$

Test:

$$
\begin{aligned}
s &= \Im s_{mn}(\cos(m\vartheta+n\varphi) - i\sin(m\vartheta+n\varphi))\\
s_n &= s_{mn}e^{im\vartheta} = \Im s_{mn}(-\sin m\vartheta + i\cos m\vartheta)\\
&=
\end{aligned}
$$

$$
\Re p_{mn} = \Im s_{mn}/(mB^\vartheta + nB^\varphi)
$$

$$
p_n = \Re p_{mn}(\cos m\vartheta + i\sin m\vartheta)
$$

Pseudotoroidal coordinates

$$
\begin{aligned}
R &= R_0 + r\cos\vartheta \\
Z &= r\sin\vartheta
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{e}_r &= \frac{\partial R}{\partial r}\boldsymbol{e}_R + \frac{\partial Z}{\partial r}\boldsymbol{e}_Z \\
&= \boldsymbol{e}_R\cos\vartheta + \boldsymbol{e}_Z\sin\vartheta
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{e}_\vartheta &= \frac{\partial R}{\partial \vartheta}\boldsymbol{e}_R + \frac{\partial Z}{\partial \vartheta}\boldsymbol{e}_Z \\
&= -\boldsymbol{e}_R r\sin\vartheta + \boldsymbol{e}_Z r\cos\vartheta
\end{aligned}
$$