1 Theory and Implementation

Hamiltonian perturbation

$$\tilde{H} = \frac{mv^2}{2}\frac{2 - B_0\eta}{B_0}\tilde{B}.$$

The factor

$$2 - B_0\eta = 2 - \frac{v_\perp^2}{v^2} = 1 + \frac{v_\parallel^2}{v^2} \approx 1$$

depends weakly on ϑ. If harmonics $b_{\mathbf{m}}$ of $\tilde{b} = \tilde{B}/B_0$ are used, the Fourier components translate like

$$H_{\mathbf{m}} = \frac{mv^2}{2}\left(2 - B_0\eta\right) b_{\mathbf{m}}.$$

Bounce average

The code calculates bounce averages via an ODE integration over bounce orbits. The time derivative of ϑ is given by

$$\dot{\vartheta} = v_\parallel h^\vartheta$$

and the one of the parallel velocity is

$$\begin{aligned}
\dot{v}_\parallel &= -\sigma v \frac{\eta \dot{B}}{2\sqrt{1 - \eta B}} \\
&= -\frac{v^2\eta}{2v_\parallel}\frac{\partial B}{\partial \vartheta}\dot{\vartheta} \\
&= -\frac{v^2\eta}{2}h^\vartheta \frac{\partial B}{\partial \vartheta}.
\end{aligned}$$

Bounce integration of other quantities A is done by adding them as $\dot{z} = A$ in the time step for the ODE solver. For bounce averages they are divided by the bounce time. Its value and the stopping condition results from a root finding method where the angle ϑ passes $\vartheta = 0$ from negative to positive after one turn.

Toroidal rotation

The canonical toroidal rotation frequency Ω^φ is the bounce averaged sum of electric and magnetic drift

$$\begin{aligned}
\Omega_{tE}^c &= -\frac{c}{B^\vartheta\sqrt{g}}\frac{\partial \Phi}{\partial r_\varphi} \\
\Omega_{tB}^c &= -\frac{v^2(2 - \eta B)}{2\omega_c B^\vartheta \sqrt{g}}\frac{\partial B}{\partial r_\varphi} + \frac{v^2(1 - \eta B)}{\omega_c\sqrt{g}B}\left(\frac{\partial B_\vartheta}{\partial r_\varphi} + q\frac{\partial B_\varphi}{\partial r_\varphi} + B_\varphi \frac{dq}{dr_\varphi}\right) \\
&= \frac{v^2}{2\omega_c\sqrt{g}h^\vartheta B}\left(-(2 - \eta B)\frac{\partial B}{\partial r_\varphi} + 2(1 - \eta B)h^\vartheta\left(\frac{\partial B_\vartheta}{\partial r_\varphi} + q\frac{\partial B_\varphi}{\partial r_\varphi} + B_\varphi\frac{dq}{dr_\varphi}\right)\right).
\end{aligned}$$

Flux surface average

The flux surface average is defined by

$$\langle A \rangle = \left(\int_0^{2\pi} d\vartheta \int_0^{2\pi} d\varphi \sqrt{g} \right)^{-1} \int_0^{2\pi} d\vartheta \int_0^{2\pi} d\varphi \sqrt{g} A.$$

The normalisation quantity is denoted by

$$S_r := \int_0^{2\pi} d\vartheta \int_0^{2\pi} d\varphi \sqrt{g}.$$

It coincides with the physical flux surface area

$$\begin{aligned} S &= \int_0^{2\pi} d\vartheta \int_0^{2\pi} d\varphi |\nabla r| \sqrt{g} \\ &= \langle |\nabla r| \rangle S_r \end{aligned}$$

for the choice

$$r = r_{\mathrm{eff}}$$

with

$$\langle |\nabla r_{\mathrm{eff}}| \rangle = 1.$$

Otherwise,

$$\langle |\nabla r| \rangle = \frac{dr}{dr_{\mathrm{eff}}}.$$

Torque density

The flux averaged torque density $\left\langle S_{p_\varphi} \right\rangle$ is given by a functional

$$\mathcal{F}_S^r [g] := -\frac{\pi}{2} \frac{(2\pi)^2}{S_r} m_\varphi^2 \langle |\nabla r| \rangle \int d\vartheta \int dJ_\perp dJ_\vartheta |H_{\mathbf{m}}|^2 \delta(m_\varphi \Omega^\varphi + m_\vartheta \Omega^\vartheta) g$$

applied to the radial derivative of f_0,

$$\left\langle S_{p_\varphi} \right\rangle = \mathcal{F}_S^r \left[\frac{\partial f_0}{\partial r} \right].$$

The result is independent from the choice of the radial variable r.

Flux

The particle flux through a flux surface is

$$\begin{aligned} \Gamma &= -\frac{1}{m \omega_c \sqrt{g_{\mathrm{eff}}} h^\vartheta} \left\langle S_{p_\varphi} \right\rangle \\ &= -\frac{1}{\langle |\nabla r| \rangle} \frac{1}{m \omega_c \sqrt{g} h^\vartheta} \mathcal{F}_S^r \left[\frac{\partial f_0}{\partial r} \right]. \end{aligned}$$

Diffusion coefficients

Thermodynamic forces are

$$A_1 = \frac{n'}{n} - \frac{e\Phi'}{T} - \frac{3}{2}\frac{T'}{T},$$
$$A_2 = \frac{T'}{T}$$

with derivatives over r_{eff}. The radial derivative of a Maxwellian is

$$\frac{\partial f_0(H,r)}{\partial r} = \frac{1}{\langle|\nabla r|\rangle}\frac{\partial f_0(H,r_{\mathrm{eff}})}{\partial r_{\mathrm{eff}}} = \frac{1}{\langle|\nabla r|\rangle}\left(A_1 + \frac{mv^2}{2T}A_2\right) f_0$$

The first two diffusion coefficients are defined by

$$\begin{aligned}
-n\left(D_{11}A_1 + D_{12}A_2\right) &= \Gamma \\
&= -\frac{1}{\langle|\nabla r|\rangle}\frac{1}{m\omega_c\sqrt{g}h^{\vartheta}}\mathcal{F}_S^r\left[\frac{\partial f_0}{\partial r}\right] \\
&= -\frac{1}{\langle|\nabla r|\rangle^2}\frac{1}{m\omega_c\sqrt{g}h^{\vartheta}}\mathcal{F}_S^r\left[\left(A_1 + \frac{mv^2}{2T}A_2\right) f_0\right]
\end{aligned}$$

Comparing the coefficients of A_i, we have

$$D_{11} = \frac{1}{n\langle|\nabla r|\rangle^2}\frac{1}{m\omega_c\sqrt{g}h^{\vartheta}}\mathcal{F}_S^r\left[f_0\right]$$
$$D_{12} = \frac{1}{n\langle|\nabla r|\rangle^2}\frac{1}{m\omega_c\sqrt{g}h^{\vartheta}}\mathcal{F}_S^r\left[\frac{mv^2}{2T}f_0\right]$$

Useful formulas

$$v_{\parallel} = \sigma v\sqrt{1-\eta B}$$

$$v_{\perp} = v\sqrt{\eta B}$$

$$\begin{aligned}
B &\approx \frac{B_0}{1+\varepsilon\cos\vartheta} \\
\frac{B(\pi)}{B(0)} &\approx \frac{1+\varepsilon}{1-\varepsilon} \\
\frac{B(\pi)}{B(0)} - \varepsilon\frac{B(\pi)}{B(0)} &= 1+\varepsilon \\
\varepsilon &\approx \frac{\frac{B(\pi)}{B(0)} - 1}{\frac{B(\pi)}{B(0)} + 1}
\end{aligned}$$

2 Tests

test_magfie

Magnetic field quantities to compare with superbanana. **OK**

test_bounce

Orbits at trapped-passing boundary to compare with superbanana. **OK**

test_torfreq

Toroidal frequencies to compare with superbanana. **TBD**