Human Progress in the Discovery of the Roots of Polynomials

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What is progress? What does this term mean in terms of human beings and our societal structures? This is a very open question with many area's that one can delve deep into in an attempt to answer. In Peter Schouls' paper "Descartes and the Idea of Progress" we are told that the famed Descartes believed that only that "which is an application of Scientific knowledge can be a candidate for an instance of progress." Schouls continues to analyze Descartes definitions of "Scientific Knowledge". The term "Scientific Knowledge" had much different connotations to it than it does today. At its basic form we are defining "progress" to be "the acquiring of knowledge that can be put to some use" as "Scientific Knowledge" is knowledge that can be put to use. Then, when we refer to progress, we will be referring to this process of acquiring knowledge. Since we are speaking about "human" or "societal" progress we are referring to this process of acquiring knowledge in societal groups. This definition of human progress necessarily includes the idea that knowledge once acquired must be shared. One may make progress on one's own, but unless the knowledge acquired is shared then human progress has not taken place.

Finding the Roots of Polynomials

Figuring out how to solve the roots of polynomials fits with this idea of progress. Figuring the roots of polynomials leads to a huge amount of knowledge that can be put to use. They allow us to analyze shapes and motion and predict the future. Solving quadratics allows us to analyze motion with Newtonian Mechanics, which has been an underpinning of all of our technological human progress for several lifetimes. Interestingly, the efficiency of this progress has been frustrated by different societal hurdles that have been put in place that make talented mathematicians prefer not to share their knowledge. There are many ways a society can structure itself to either help or hinder the efficiency of human progress. In the early 1500's a process for finding the roots of polynomials of degree 3 and degree 4 was found. This discovery was shared with the world, and our definition of human progress was satisfied. However, when we dive into the history of this discovery, and then into the details of this discovery we find that the building up of this useful scientific knowledge could have been much more efficient. The lack of the proper sharing of knowledge that is demanded by our very definition of human progress hindered the process and caused progress to be achieved much more slowly than it otherwise could have been.

Progress on Polynomials

Making human progress on solving the roots of polynomials took us well over a millennium. According to Sir. Thomas Heath, Diophantus of Alexandria first formulated the idea of a polynomials with an unknown value. With Diophantus' notation it became possible to actually write down polynomials. Over 5 centuries later Al-Khwarizmi published the first algebraic solutions to both linear and quadratic equations. According to Elizabeth Roger's 2008 paper on Islamic Mathematics later Arabic scholars made more progress on polynomials, including Al-Karaji formulating the idea of a "root" of a polynomial and the idea of the unknown being handled as a square and a cube. However, it was not until the early 1500's that it was figured out how to solve the roots of polynomials of degree greater than 2.

Scipione Del Ferro

In the 1520's Italian Mathematician Scipione Del Ferro discovered a method to solve the roots of a depressed cubic. But he did not share this method publicly. As we will shortly learn, it turns out the method he found for solving the depressed cubic was necessary knowledge to have for one to solve the higher orders of polynomials as well. But Del Ferro never released this work publicly! Del Ferro made progress, but his lack of sharing his acquisition of knowledge led to a dearth of human progress, compared to what may have been.

At this time in Italy (depending on which city) prestigious academic positions were often obtained via a type of intellectual dual. A challenger would present a list of mathematical problems to an established academic and would receive a similar list back themselves. Whoever answers more of his set of questions would win. In winning, the prestige of the challenger could rise to such a point that they would be able to take the prestigious academic position from the challenged party. James D. Stein tells us in *How Math Explains the World* that “This, however, was an era in which Machiavelli was writing of the importance of subterfuge—and subterfuge, in Italian academe, was often how one survived.” (Stein p. 83) Del Ferro considered the depressed cubic problem to be his secret weapon. If challenged, he could use his secret solution to win. As a result, Del Ferro never publicly released his work. He kept it secret until his death. It was his method to survive. While Del Ferro's discovery would eventually lead to human progress, that progress would take many more years to achieve.

Del Ferro’s Discovery.

Del Ferro Figured out a general solution for finding the roots of a depressed cubic. A depressed cubic is a polynomial of degree 3 that is missing its degree 2 unknown value I.E:

$$\boldsymbol{x^3 + Cx + D = 0}$$

While trying different strategies, Del Ferro broke the unknown **x** into two different values, allowing for a “new degree of freedom in his expressions.” (Stein p. 85) He decomposed **x** into a difference of two other values **s** and **t**:

$$\boldsymbol{(s - t)^3 + C(s - t) + D = 0}$$

This can be algebraically expanded to:

$$(\boldsymbol{s^3 - t^3 + D}) + (\boldsymbol{C - 3st})(\boldsymbol{s - t}) = \boldsymbol{0}$$

If you can find a way to make $(\boldsymbol{s^3 - t^3 + D}) = \boldsymbol{0}$ and $(\boldsymbol{C - 3st}) = \boldsymbol{0}$ then the equation becomes $\boldsymbol{0 + 0}\ (\boldsymbol{s - t}) = \boldsymbol{0}$. Which means $\boldsymbol{s - t}$, which is **x**, would be the root of the cubic. With some algebra we find that if $\boldsymbol{3st = C}$ and $\boldsymbol{t^3 - s^3 = D}$ will make the first two terms $\boldsymbol{0}$ as we were looking for.

At this point we need to find an **s** and a **t** that satisfy $\boldsymbol{3st = C}$ and $\boldsymbol{t^3 - s^3 = D}$. We solve $\boldsymbol{3st = C}$ for **t** and have $\boldsymbol{t = \frac{C}{3s}}$, which we substitute into $\boldsymbol{t^3 - s^3 = D}$ and we find the result is:

$$\boldsymbol{\frac{C^3}{27s^3} - s^3 = D}$$

Which can be simplified to $\boldsymbol{s^6 + Ds^3 - \frac{c^3}{27} = 0}$. We can view $\boldsymbol{s^6}$ as $(\boldsymbol{s^3})^{\boldsymbol{2}}$, and we have:

$$(\boldsymbol{s^3})^{\boldsymbol{2}} + \boldsymbol{Ds^3 - \frac{C^3}{27} = 0}$$

But now we can view $\boldsymbol{s^3}$ as our new unknown $\boldsymbol{x_1}$, and by substituting we find we now have a quadratic equation:

$$\boldsymbol{x_1^2 + Dx_1 - \frac{C^3}{27} = 0}$$

This can now be easily solved by the quadratic formula which was established over 700 years before this simple change of variable was figured out to turn a depressed cubic into a quadratic. The idea presented here is very simple, [1] but somehow it took 700 years to figure out. And then, because of the lack of knowledge sharing, several more decades until this discovery progressed into the realm of human progress.

Antonio Fior vs Tartaglia

According to William Dunham in *Journey through Genius*, Del Ferro, on his deathbed, passed the secret of the depressed cubic to his student Antonio Fior. (Dunham p. 135) Antonio held onto this secret for another nine years before he used it to level a challenge at the well-known mathematician Niccolo Fontana, who was also known as “Tartaglia” because of his stutter. Fior challenged Tartaglia with a list of 30 Depressed Cubic’s. It became apparent to Tartaglia that if he wanted to keep his prestigious position that he was going to have to crack the secret behind the depressed cubic. After a “frantic round-the-clock attack on the depressed cubic” (Dunham p. 135) he solved it. He presented his solutions, won the challenge, and his challenger Fior faded from our picture of history. Again, like Del Ferro, Tartaglia kept his solution a secret from the world. He was more interested in using it to protect himself than he was in using it to benefit the progress of man.

Cardano’s Oath

Cardano, a famed mathematician, philosopher who was respected throughout all of Europe contacted Tartaglia and convinced him to share his secret in 1535. (Berlinghoff p. 111) Convincing Tartaglia was not easy, the depressed cubic was his ace-in-the-hole like it was for Del Ferro. Tartaglia did not want to share. Cardano convinced him with a holy oath: “I swear to you, by God’s holy Gospels, and as a true man of honor, not only never to publish your discoveries, if you teach me them, but I also promise you, and I pledge my faith as a true Christian, to note them down in code, so that after my death no one will be able to understand them.” (Gray p. 250) We can see that the focus of Tartaglia’s genius was not devoted to human progress, but a much baser form of individual progress. Tartaglia wanted his secret to benefit him, not anyone else.

Cardano and Ferrari's Progress

With Tartaglia's secret in his possession Cardano set to work to solve the general cubic. After six years he succeeded by finding a way to turn all general cubic equations into depressed cubic equations. (Stein p. 88) By sharing his progress with his student Lodovico Ferrari he broke his oath to Tartaglia. However, doing so allowed Ferrari to use a similar technique to transform a quintic equation into a cubic, which could then use Cardano's discovery to turn it into a depressed cubic, which could now be solved.

Cardano and Ferrari had made a huge amount of progress on solving the roots of polynomials, but they were unable to share this progress with the world. They were unable to convert their individual progress into human progress because of the oath Cardano swore to Tartaglia. His discoveries where all based on Tartaglia's if he were to share these discoveries with the world, he would by necessity need to disclose Tartaglia's secret method to the world. Cardano knows that Tartaglia was challenged by Fior, which means that Fior had to also know the secret. Cardano finds Del Ferro's original notes about the depressed cubic. In 1545 Cardano published *Ars Magna* in which he revealed the secret of the depressed cubic and attributed its discovery to both Tartaglia and Del Ferro.

Human Progress

Finally, we find that our definition of human progress has been achieved. Because the knowledge of the depressed cubic was shared with Cardano he was able to make progress on it. Cardano shared this knowledge, and he made more progress on it. They both published their work, sharing their knowledge with the world at large. Now mankind has the power needed to solve quintic equations, something that was not possible previously. But looking back on the long torturous maze of years it took for this progress to be made deserves our attention. Had Del

Ferro shared his knowledge immediately, perhaps a book like *Ars Magna* could have been published decades before it was. Many more human minds would have had the opportunity to dedicated themselves to furthering human progress even more. With the proper sharing of knowledge there is no reason why the progress that took a decade in this story could conceivably only taken ten minutes to achieve. The idea's that were shared were elementary, just difficult to see. The sharing of knowledge is instrumental in achieving progress for humanity.

Footnotes

[1] . The Mathematical Sequence Presented here was taken entirely from (Stein 87)

References

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