29. The base of a certain solid is the circle $x^2 + y^2 = a^2$. Each cross-section of the solid cut out by a plane perpendicular to the x-axis is an isosceles right triangle with one of the equal sides in the base of the solid. Find its volume.

30. The base of a certain solid is the region between the x-axis and the curve $y = \cos x$ between $x = 0$ and $x = \pi/2$. Each cross-section of the solid cut out by a plane perpendicular to the x-axis is an equilateral triangle with one side in the plane of the solid. Find its volume.

In problems **31** to **36**, find the volume of the solid of revolution generated by revolving the specified region about the given axis.

31. Region bounded by $y = \cos x$, $y = 0$ from $x = 0$ to $x = \pi/2$ about the x-axis.

32. Region bounded by $y = \sqrt{x}$, $y = 0$ from $x = 0$ to $x = 4$ about the x-axis.

33. Region bounded by $y = \sqrt{x}$, $y = 0$ from $x = 0$ to $x = 4$ about the line $y = 2$.

34. Region bounded by $y = x^2 + 1$ and $y = 3 - x$ about the x-axis.

35. Region bounded by $x = a \sin^3 t$, $y = a \cos^3 t$, $0 \leq t \leq \pi/2$, $x = 0$, $y = 0$ about the x-axis.

36. Region bounded by $x = 2t + 1$, $y = 4t^2 - 1$, $-1/2 \leq t \leq 0$, $y = 0$ about the line $x = 1$.

In problems **37** to **41**, use the method of cylindrical shells to find the volume of the solid generated by revolving the specified region about the given axis.

37. Region bounded by $y = x$, $y = 2$ and $x = 0$ about the y-axis.

38. Region bounded by $y = 2x - x^2$ and $y = x$ about the y-axis.

39. Region bounded by $y = x^2$ and $y = x$ about the y-axis.

40. Region inside the triangle with vertices at $(0, 0)$, $(a, 0)$ and $(0, b)$ about the y-axis.

41. Region inside the circle $x^2 + y^2 = a^2$ about the line $y = b$, $b > a > 0$.

In problems **42** to **50**, find the surface area of the solid generated by revolving the curve C about the given line.

42. $(x - b)^2 + y^2 = a^2$, $b \geq a$ about the y-axis.

43. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, $a \geq b$, $y \geq 0$ about the x-axis.

44. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, $a \geq b$, $x \geq 0$ about the y-axis.

45. $y = \dfrac{x^4}{4} + \dfrac{1}{8x^2}$, $1 \leq x \leq 2$ about the line $y = -1$.

46. $x = \dfrac{y^3}{3} + \dfrac{1}{4y}$, $1 \leq y \leq 2$ about the line $x = -1$.

47. $x = a(1t - \sin t)$, $y = a(1 - \cos t)$, $0 \leq t \leq 2\pi$ about the x-axis.

48. $x = a \cos^3 t$, $y = a \sin^3 t$, $0 \leq t \leq \pi/2$ about the x-axis.

49. $x = e^t \cos t$, $y = e^t \sin t$, $0 \leq t \leq \pi/2$ about the y-axis.

50. $r = a(1 + \cos \theta)$, $0 \leq \theta \leq \pi$ about the initial line.

1.4 Improper Integrals

While defining the definite integral $\int_a^b f(x)\,dx$, we had assumed that

(i) a and b are finite constants.

(ii) $f(x)$ is bounded for all x in $[a, b]$.

If in the above integral, (i) a or b or both a and b are infinite, or (ii) a, b are finite but $f(x)$ becomes infinite at $x = a$ or $x = b$ or at one or more points within the interval (a, b), then the definite integral is respectively called

(i) *improper integral of the first kind.*

(ii) *improper integral of the second kind.*

To define the improper integrals, we assume the following:

(i) The integrand $f(x)$ is of the same sign within its range of integration. Without any loss of generality, we assume that $f(x) \geq 0$ (when $f(x) \leq 0$, we can write $g(x) = -f(x)$ so that $g(x) \geq 0$). We shall discuss later, the case when $f(x)$ changes sign within its range of integration.

(ii) $f(x)$ is continuous over each finite subinterval $[\alpha, \beta]$ contained in the range of integration. Hence, there exists a positive constant K independent of α and β such that

$$\int_{\alpha}^{\beta} f(x)\, dx < K.$$

The improper integrals are evaluated by a limiting process.

1.4.1 Improper Integrals of the First Kind (Range of Integration is Infinite)

We shall now discuss methods to evaluate improper integrals of the form

$$\text{(i) } \int_{a}^{\infty} f(x)\, dx, \qquad \text{(ii) } \int_{-\infty}^{b} f(x)\, dx, \quad \text{and} \qquad \text{(iii) } \int_{-\infty}^{\infty} f(x)\, dx$$

if they exist. We define these improper integrals as follows:

(i)
$$\int_{a}^{\infty} f(x)\, dx = \lim_{p \to \infty} \int_{a}^{p} f(x)\, dx. \tag{1.61}$$

If the limit exists and is finite, say equal to l_1, then the improper integral converges and has the value l_1. Otherwise, the improper integral diverges.

(ii)
$$\int_{-\infty}^{b} f(x)\, dx = \lim_{p \to -\infty} \int_{p}^{b} f(x)\, dx. \tag{1.62}$$

If the limit exists and is finite, say equal to l_2, then the improper integral converges and has the value l_2. Otherwise, the improper integral diverges.

(iii)
$$\int_{-\infty}^{\infty} f(x) dx = \lim_{a \to -\infty} \int_{a}^{c} f(x) dx + \lim_{b \to \infty} \int_{c}^{b} f(x) dx \tag{1.63}$$

where c is any finite constant including zero. If both the limits on the right hand side exist separately and are finite, say equal to l_3 and l_4 respectively, then the improper integral converges and has the value $l_3 + l_4$. If one or both the limits do not exist or are infinite, then the improper integral diverges.

Example 1.35 Evaluate the following improper integrals, if they exist.

$$\text{(i) } \int_{0}^{\infty} \frac{dx}{a^2 + x^2}, \quad a > 0, \qquad \text{(ii) } \int_{-\infty}^{0} e^{x}\, dx,$$

(iii) $\int_0^{\infty} x \sin x \, dx$, (iv) $\int_0^{\infty} e^{-ax} \cos px \, dx$, $a > 0$, p constant.

Solution

(i) $$\int_0^{\infty} \frac{dx}{a^2 + x^2} = \lim_{b\to\infty} \int_0^b \frac{dx}{a^2 + x^2} = \lim_{b\to\infty} \left[\frac{1}{a} \tan^{-1}\left(\frac{b}{a}\right)\right] = \frac{\pi}{2a}.$$

Therefore, the improper integral converges to $\pi/(2a)$.

(ii) $$\int_{-\infty}^{0} e^x dx = \int_0^{\infty} e^{-x} dx = \lim_{b\to\infty} \int_0^b e^{-x} dx = \lim_{b\to\infty} (1 - e^{-b}) = 1.$$

Therefore, the improper integral converges to 1.

(iii) $$\int_0^{\infty} x \sin x \, dx = \lim_{b\to\infty} \int_0^b x \sin x \, dx = \lim_{b\to\infty} (\sin b - b \cos b).$$

Since this limit does not exist, the improper integral diverges.

(iv) Using the result

$$\int e^{-ax} \cos px \, dx = \frac{e^{-ax}}{a^2 + p^2} (p \sin px - a \cos px),$$

we obtain $$\int_0^b e^{-ax} \cos px \, dx = \left[\frac{e^{-ax}}{a^2 + p^2} (p \sin px - a \cos px)\right]_0^b$$

$$= \frac{1}{a^2 + p^2} [e^{-ab} (p \sin bp - a \cos bp) + a].$$

Now, $\sin bp$ and $\cos bp$ have finite values and $\lim_{b\to\infty} e^{-ab} = 0$. Hence,

$$\int_0^{\infty} e^{-ax} \cos px \, dx = \lim_{b\to\infty} \int_0^b e^{-ax} \cos px \, dx = \frac{a}{a^2 + p^2}.$$

Therefore, the improper integral converges to $a/(a^2 + p^2)$.

Example 1.36 Discuss the convergence of the improper integral $\int_1^{\infty} \frac{dx}{x^p}$.

Solution We have

$$\int_1^b \frac{dx}{x^p} = \frac{1}{1-p} \left[x^{1-p}\right]_1^b = \frac{1}{1-p} [b^{1-p} - 1]$$

Now, $$\lim_{b\to\infty} [b^{1-p}] = \begin{cases} \infty, & \text{if } p < 1 \\ 0, & \text{if } p > 1. \end{cases}$$

Therefore, the improper integral converges if $p > 1$ and diverges if $p < 1$.

For $p = 1$, we have

$$\int_1^{\infty} \frac{dx}{x} = \lim_{b\to\infty} \int_1^{b} \frac{dx}{x} = \lim_{b\to\infty} \left[\ln x\right]_1^b = \lim_{b\to\infty} \ln b.$$

Since the limit does not exist, the improper integral diverges. Hence, the given improper integral converges to $1/(p-1)$ for $p > 1$ and diverges for $p \le 1$.

Example 1.37 Discuss the convergence of the integral $\int_{-\infty}^{\infty} xe^{-x^2}\, dx$.

Solution We write

$$I = \int_{-\infty}^{\infty} xe^{-x^2}\, dx = \int_{-\infty}^{c} xe^{-x^2}\, dx + \int_{c}^{\infty} xe^{-x^2}\, dx$$

where c is any finite constant. We have

$$I = \lim_{a\to-\infty} \int_a^c xe^{-x^2}\, dx + \lim_{b\to\infty} \int_c^b xe^{-x^2}\, dx$$

$$= \lim_{a\to-\infty} \left[\frac{1}{2}\left(e^{-a^2} - e^{-c^2}\right)\right] + \lim_{b\to\infty} \left[\frac{1}{2}\left(e^{-c^2} - e^{-b^2}\right)\right]$$

$$= \frac{1}{2}\left(-e^{-c^2} + e^{c^2}\right) = 0.$$

Therefore, the given improper integral converges to 0.

It is not always possible to study the convergence/divergence of an improper integral by evaluating it as was done in the previous examples. A simple example is the integral $\int_0^{\infty} e^{-x^2}\, dx$ which cannot be evaluated directly. We now present some results which can be used to discuss the convergence or divergence of improper integrals. In this case, we cannot find the value of the improper integral, that is the value to which it converges. However, we may be able to find a bound of the integral.

Theorem 1.8 (Comparison Test 1) If $0 \le f(x) \le g(x)$ for all x, then

(i) $\int_a^{\infty} f(x)\, dx$ converges if $\int_a^{\infty} g(x)\, dx$ converges.

(ii) $\int_a^{\infty} g(x)\, dx$ diverges if $\int_a^{\infty} f(x)\, dx$ diverges.

Theorem 1.9 (Comparison Test 2) Suppose that $f(x)$ and $g(x)$ are positive functions and let

$$\lim_{x\to\infty} \left[\frac{f(x)}{g(x)}\right] = L, \quad 0 < L < \infty. \qquad (1.64)$$

Then, the improper integrals $\int_a^{\infty} f(x)\, dx$ and $\int_a^{\infty} g(x)\, dx$ converge or diverge together.

Example 1.38 Discuss the convergence of the following improper integrals

(i) $\int_1^\infty e^{-x^2}\,dx.$ (ii) $\int_1^\infty \frac{dx}{(e^{-x}+1)x^2}$, (iii) $\int_2^\infty \frac{dx}{\ln x}$,

(iv) $\int_2^\infty \frac{dx}{x(\ln x)^p}$, (v) $\int_1^\infty \frac{x\tan^{-1}x}{\sqrt{4+x^3}}\,dx.$

Solution

(i) We have $e^{-x^2} < e^{-x}$ for all $x \ge 1$. Consider the improper integral $\int_1^\infty e^{-x}\,dx$.

We have
$$\int_1^\infty e^{-x}\,dx = \lim_{b\to\infty}\int_1^b e^{-x}\,dx = \lim_{b\to\infty}[1-e^{-b}] = 1.$$

Therefore, the integral $\int_1^\infty e^{-x}\,dx$ is convergent. By Comparison Test 1 (i), the given integral is also convergent. Further, its value is less than 1.

(ii) Let $f(x) = \dfrac{1}{(e^{-x}+1)x^2}$ and $g(x) = \dfrac{1}{x^2}$.

Now
$$\lim_{x\to\infty}\left[\frac{f(x)}{g(x)}\right] = \lim_{x\to\infty}\left[\frac{1}{(e^{-x}+1)x^2}\right]\left[\frac{x^2}{1}\right] = \lim_{x\to\infty}\frac{1}{e^{-x}+1} = 1.$$

Also, $\int_1^\infty g(x)dx = \int_1^\infty \frac{dx}{x^2}$ converges to 1 (see Example 1.36). Therefore, by Comparison Test 2, the given improper integral is also convergent. Its value is less than 1.

Alternative We have $\dfrac{1}{(e^{-x}+1)x^2} < \dfrac{1}{x^2}$ for all $x \ge 1$. The improper integral $\int_1^\infty \frac{dx}{x^2}$ is convergent. Therefore, by Comparison Test 1 (i), the given improper integral converges.

(iii) We have $\ln x < x$ for all $x > 0$. Hence,
$$\frac{1}{\ln x} > \frac{1}{x} \text{ and } \int_2^\infty \frac{dx}{\ln x} > \int_2^\infty \frac{dx}{x}.$$

Let $g(x) = 1/(\ln x)$ and $f(x) = 1/x$. We have $g(x) > f(x)$. Now, the integral
$$\int_2^\infty f(x)\,dx = \int_2^\infty \frac{dx}{x} \text{ is divergent (see Example 1.36).}$$

Therefore, by Comparison Test 1 (ii), the integral $\int_2^\infty g(x)\,dx = \int_2^\infty \frac{dx}{\ln x}$ is also divergent.

(iv) Substitute $\ln x = t$. We get
$$I = \int_2^\infty \frac{dx}{x(\ln x)^p} = \int_{\ln 2}^\infty \frac{dt}{t^p}$$

which is convergent for $p > 1$ and divergent for $p \le 1$ (see Example 1.36).

(v) Let $$f(x) = \frac{x \tan^{-1} x}{\sqrt{4 + x^3}} = \frac{\tan^{-1} x}{\sqrt{x}\sqrt{1+4x^{-3}}} \text{ and } g(x) = \frac{1}{\sqrt{x}}.$$

We find that $$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\tan^{-1} x}{\sqrt{1+4x^{-3}}} = \frac{\pi}{2}.$$

Hence, by Comparison Test 2, the integrals $\int_1^\infty f(x)dx$ and $\int_1^\infty g(x)dx$ converge or diverge together. Now, $\int_1^\infty g(x)\,dx$ is divergent. Therefore, $\int_1^\infty f(x)\,dx$ is also divergent.

1.4.2 Improper Integral of the Second Kind

Now consider an improper integral of the form $\int_a^b f(x)\,dx$, where a, b are finite constants, $f(x)$ is continuous in (a, b) and has infinite discontinuity (becomes infinite) at (i) $x = a$, or (ii) $x = b$, or (iii) $x = a$ and $x = b$, or (iv) $f(x)$ is continuous in (a, b) except at $x = c$, $a < c < b$, where $f(x)$ has an infinite discontinuity.

If $f(x)$ has a finite number of points of discontinuity, $c_1, c_2, \ldots, c_m$, $a \le c_1 < c_2 \ldots < c_m \le b$, then we write the integral as

$$\int_a^b f(x)dx = \int_a^{c_1} f(x)dx + \int_{c_1}^{c_2} f(x)dx + \ldots + \int_{c_m}^b f(x)dx \tag{1.65}$$

and consider each integral on the right hand side separately.

Infinite discontinuity at $x = a$ Since the function $f(x)$ is continuous at all points except at $x = a$, the integral $\int_{a+\varepsilon}^b f(x)dx$ is a proper integral and exists for every ε, $0 < \varepsilon < b - a$.

We evaluate the improper integral as

$$\int_a^b f(x)dx = \lim_{\varepsilon\to 0} \int_{a+\varepsilon}^b f(x)dx.$$

If this limit exists and is finite, say equal to l_1, then the improper integral converges to l_1. Otherwise, it diverges.

Infinite discontinuity at $x = b$ Since the function $f(x)$ is continuous at all points except at $x = b$, the integral $\int_a^{b-\varepsilon} f(x)dx$ is a proper integral and exists for every ε, $0 < \varepsilon < b - a$.

We evaluate the improper integral as

$$\int_a^b f(x)dx = \lim_{\varepsilon\to 0} \int_a^{b-\varepsilon} f(x)dx.$$

If this limit exists and in finite, say equal to l_2, then the improper integral converges to l_2, otherwise it diverges.

Infinite discontinuity at $x = a$ and $x = b$. We write the improper integral as

$$\int_a^b f(x)dx = \int_a^{\alpha} f(x)dx + \int_{\alpha}^b f(x)dx$$

where α is any finite constant between a and b at which f is defined. We evaluate the improper integral as

$$\int_a^b f(x)dx = \lim_{\varepsilon \to 0} \int_{a+\varepsilon}^{\alpha} f(x)dx + \lim_{\xi \to 0} \int_{\alpha}^{b-\xi} f(x)\, dx.$$

If both the limits exist and are finite, then the improper integral converges. Otherwise, it diverges.

Infinite discontinuity at $x = c$, $a < c < b$. We write the improper integral as

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx = \lim_{\varepsilon \to 0} \int_a^{c-\varepsilon} f(x)dx + \lim_{\xi \to 0} \int_{c+\xi}^b f(x)dx.$$

The given improper integral converges, if both the integrals on the right hand side converge. If one or both the integrals on the right hand side diverge, then the given improper integral diverges.

The following tests can be used to discuss the convergence or divergence of the above improper integrals. In this case, we cannot find the value of the improper integral, that is the value to which it converges. However, we may be able to find a bound of the integral.

Theorem 1.10 (Comparison Test 3) If $0 \le f(x) \le g(x)$ for all x in $[a, b]$, then

(i) $\int_a^b f(x)\, dx$ converges if $\int_a^b g(x)\, dx$ converges.

(ii) $\int_a^b g(x)\, dx$ diverges if $\int_a^b f(x)\, dx$ diverges.

Theorem 1.11 (Comparison Test 4) If $f(x)$ and $g(x)$ are two positive functions and

(i) a is a point of infinite discontinuity such that

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{h \to 0} \frac{f(a+h)}{g(a+h)} = l_1, \quad 0 < l_1 < \infty,$$

or (ii) b is a point of infinite discontinuity such that

$$\lim_{x \to b^-} \frac{f(x)}{g(x)} = \lim_{h \to 0} \frac{f(b-h)}{g(b-h)} = l_2, \quad 0 < l_2 < \infty,$$

then, the improper integrals $\int_a^b f(x)dx$ and $\int_a^b g(x)\, dx$ converge or diverge together.

Example 1.39 Evaluate the following improper integrals, if they exist:

(i) $\displaystyle\int_0^4 \frac{dx}{\sqrt{x}}$, (ii) $\displaystyle\int_0^2 \frac{dx}{\sqrt{4-x^2}}$,

(iii) $\int_{-1}^{1} \sqrt{\frac{1+x}{1-x}}\, dx$, (iv) $\int_0^3 \frac{dx}{3x - x^2}$,

(v) $\int_{-1}^{1} \frac{dx}{x^2}$, (vi) $\int_0^3 \frac{dx}{x^2 - 3x + 2}$,

(vii) $\int_1^\infty \frac{dx}{x\sqrt{1-x^2}}$.

Solution

(i) $$\int_0^4 \frac{dx}{\sqrt{x}} = \lim_{\varepsilon \to 0} \int_\varepsilon^4 \frac{dx}{\sqrt{x}} = 2 \lim_{\varepsilon \to 0} [2 - \sqrt{\varepsilon}] = 4.$$

Therefore, the improper integral converges to 4.

(ii) $$\int_0^2 \frac{dx}{\sqrt{4-x^2}} = \lim_{\varepsilon \to 0} \int_0^{2-\varepsilon} \frac{dx}{\sqrt{4-x^2}} = \lim_{\varepsilon \to 0} \sin^{-1}\left(1 - \frac{\varepsilon}{2}\right) = \sin^{-1} 1 = \frac{\pi}{2}.$$

Therefore, the improper integral converges to $\pi/2$.

(iii) $$\int_{-1}^{1} \sqrt{\frac{1+x}{1-x}}\, dx = \lim_{\varepsilon \to 0} \int_{-1}^{1-\varepsilon} \sqrt{\frac{1+x}{1-x}}\, dx = \lim_{\varepsilon \to 0} \left[\int_{-1}^{1-\varepsilon} \frac{dx}{\sqrt{1-x^2}} - \frac{1}{2} \int_{-1}^{1-\varepsilon} \frac{-2x}{\sqrt{1-x^2}}\, dx \right]$$

$$= \lim_{\varepsilon \to 0} \left[\left\{ \sin^{-1}(1-\varepsilon) - \sin^{-1}(-1) \right\} - \left\{ \sqrt{1-(1-\varepsilon)^2} - \sqrt{1-1} \right\} \right]$$

$$= \sin^{-1}(1) - \sin^{-1}(-1) = 2 \sin^{-1}(1) = \pi.$$

Therefore, the improper integral converges to π.

(iv) Here, the integrand $f(x)$ has infinite discontinuity, at both the end points $x = 0$ and $x = 3$. We take any point, say $x = c$, inside the interval of integration, at which $f(x)$ is defined. We write

$$\int_0^3 \frac{dx}{3x - x^2} = \int_0^c \frac{dx}{3x - x^2} + \int_c^3 \frac{dx}{3x - x^2} = \lim_{\varepsilon \to 0} \int_\varepsilon^c \frac{dx}{x(3-x)} + \lim_{\xi \to 0} \int_c^{3-\xi} \frac{dx}{x(3-x)}$$

$$= \frac{1}{3} \lim_{\varepsilon \to 0} \left[\ln\left(\frac{x}{3-x}\right) \right]_\varepsilon^c + \frac{1}{3} \lim_{\xi \to 0} \left[\ln\left(\frac{x}{3-x}\right) \right]_c^{3-\xi}$$

$$= \frac{1}{3} \lim_{\varepsilon \to 0} \left[\ln\left(\frac{c}{3-c}\right) - \ln\left(\frac{\varepsilon}{3-\varepsilon}\right) \right] + \frac{1}{3} \lim_{\xi \to 0} \left[\ln\left(\frac{3-\xi}{\xi}\right) - \ln\left(\frac{c}{3-c}\right) \right].$$

Since the limits do not exist, the improper integral diverges.

(v) The integrand has infinite discontinuity at $x = 0$ which lies inside the interval of integration. We write

$$\int_{-1}^{1}\frac{dx}{x^2}=\int_{-1}^{0}\frac{dx}{x^2}+\int_{0}^{1}\frac{dx}{x^2}=\lim_{\varepsilon\to 0}\int_{-1}^{-\varepsilon}\frac{dx}{x^2}+\lim_{\xi\to 0}\int_{\xi}^{1}\frac{dx}{x^2}$$

$$=\lim_{\varepsilon\to 0}\left[\frac{1}{\varepsilon}-1\right]+\lim_{\xi\to 0}\left[\frac{1}{\xi}-1\right]\to\infty.$$

Therefore, the improper integral diverges.

(vi) The integrand has infinite discontinuities at $x = 1$ and $x = 2$, both of which lie inside the interval of integration. We write

$$\int_0^3\frac{dx}{x^2-3x+2}=\int_0^1\frac{dx}{(x-1)(x-2)}+\int_1^2\frac{dx}{(x-1)(x-2)}+\int_2^3\frac{dx}{(x-1)(x-2)}.$$

$$=I_1+I_2+I_3.$$

We find that

(a) the integrand in I_1 has infinite discontinuity at $x = 1$,

(b) the integrand $f(x)$ in I_2 has infinite discontinuity at both the end points $x = 1$ and $x = 2$. We take any point, say $x = c$ inside the limits of integration, at which $f(x)$ is defined. We also find that $f(x) < 0$ when $1 < x < 2$. We write $g(x) = -f(x)$ so that $g(x) > 0$ when $1 < x < 2$. Therefore, we can write

$$I_2=-\int_1^c\frac{dx}{(x-1)(2-x)}-\int_c^2\frac{dx}{(x-1)(2-x)},$$

(c) the integrand in I_3 has infinite discontinuity at $x = 2$.

Hence, we can write

$$\int_0^3\frac{dx}{x^2-3x+2}=\lim_{\varepsilon_1\to 0}\int_0^{1-\varepsilon_1}\frac{dx}{(x-1)(x-2)}-\lim_{\varepsilon_2\to 0}\int_{1+\varepsilon_2}^{c}\frac{dx}{(x+1)(2-x)}$$

$$-\lim_{\varepsilon_3\to 0}\int_c^{2-\varepsilon_3}\frac{dx}{(x-1)(2-x)}+\lim_{\varepsilon_4\to 0}\int_{2+\varepsilon_4}^{3}\frac{dx}{(x-1)(x-2)}$$

$$=\lim_{\varepsilon_1\to 0}\left[\ln\left(\frac{\varepsilon_1+1}{\varepsilon_1}\right)-\ln 2\right]-\lim_{\varepsilon_2\to 0}\left[\ln\left(\frac{c-1}{2-c}\right)-\ln\left(\frac{\varepsilon_2}{1-\varepsilon_2}\right)\right]$$

$$-\lim_{\varepsilon_3\to 0}\left[\ln\left(\frac{1-\varepsilon_3}{\varepsilon_3}\right)-\ln\left(\frac{c-1}{2-c}\right)\right]+\lim_{\varepsilon_4\to 0}\left[\ln\left(\frac{1}{2}\right)-\ln\left(\frac{\varepsilon_4}{\varepsilon_4+1}\right)\right].$$

Since the limits do not exist, the improper integral diverges.

Note that the improper integral I_1 diverges. We could have concluded that the improper integral diverges without discussing the convergence/divergence of I_2 and I_3.

(vii) $$\int_1^{\infty}\frac{dx}{x\sqrt{x^2-1}}=\int_1^c\frac{dx}{x\sqrt{x^2-1}}+\int_c^{\infty}\frac{dx}{x\sqrt{x^2-1}}.$$

$$= \lim_{\varepsilon\to 0}\int_{1+\varepsilon}^{c}\frac{dx}{x\sqrt{x^2-1}} + \lim_{b\to\infty}\int_{c}^{b}\frac{dx}{x\sqrt{x^2-1}} = \lim_{\varepsilon\to 0}[\sec^{-1}x]_{1+\varepsilon}^{c} + \lim_{b\to\infty}[\sec^{-1}x]_{c}^{b}$$

$$= \lim_{\varepsilon\to 0}[\sec^{-1}c - \sec^{-1}(1+\varepsilon)] + \lim_{b\to\infty}[\sec^{-1}b - \sec^{-1}c]$$

$$= \sec^{-1}c - \sec^{-1}1 + \frac{\pi}{2} - \sec^{-1}c = \frac{\pi}{2}$$

Therefore, the improper integral converges to $\pi/2$.

Example 1.40 Discuss the convergence of the improper integral $\int_a^b \frac{dx}{(x-a)^p}$, $p>0$.

Solution The integrand has infinite discontinuity at $x=a$. We write

$$\int_a^b \frac{dx}{(x-a)^p} = \lim_{\varepsilon\to 0}\int_{a+\varepsilon}^{b}\frac{dx}{(x-a)^p} = \frac{1}{1-p}\lim_{\varepsilon\to 0}\left[\frac{1}{(b-a)^{p-1}} - \frac{1}{\varepsilon^{p-1}}\right]$$

$$= \begin{cases} 1/\left[(1-p)(b-a)^{p-1}\right] & \text{if } p<1 \\ \infty, & \text{if } p>1. \end{cases}$$

For $p=1$, we get

$$\int_c^b \frac{dx}{x-a} = \lim_{\varepsilon\to 0}\int_{a+\varepsilon}^{b}\frac{dx}{x-a} = \lim_{\varepsilon\to 0}\ln\left[\frac{b-a}{\varepsilon}\right] = \infty.$$

Therefore, the improper integral converges for $p<1$ and diverges for $p\ge 1$.

Example 1.41 Show that the improper integral $\int_{-\pi/2}^{\pi/2}\tan x\,dx$ is divergent.

Solution The integrand has infinite discontinuity at $x=\pm\pi/2$. We write

$$\int_{-\pi/2}^{\pi/2}\tan x\,dx = \lim_{\varepsilon\to 0}\int_{-(\pi/2)+\varepsilon}^{c}\tan x\,dx + \lim_{\xi\to 0}\int_{c}^{(\pi/2)-\xi}\tan x\,dx$$

$$= \lim_{\varepsilon\to 0}\left[-\ln(\cos x)\right]_{-(\pi/2)+\varepsilon}^{c} + \lim_{\xi\to 0}\left[-\ln(\cos x)\right]_{c}^{(\pi/2)-\xi}$$

$$= \lim_{\varepsilon\to 0}\left\{\ln\left[\cos\left(-\frac{\pi}{2}+\varepsilon\right)\right] - \ln\left[\cos(c)\right]\right\}$$

$$- \lim_{\xi\to 0}\left\{\ln\left[\cos\left(\frac{\pi}{2}-\xi\right)\right] - \ln\left[\cos(c)\right]\right\}.$$

Since the limits do not exist, the improper integral diverges.

Note that if we write

$$\int_{-\pi/2}^{\pi/2}\tan x\,dx = \lim_{\varepsilon\to 0}\int_{-(\pi/2)+\varepsilon}^{(\pi/2)-\varepsilon}\tan x\,dx$$

we get $\int_{-\pi/2}^{\pi/2}\tan x\,dx = 0$, which is not the correct solution.

Example 1.42 Discuss the convergence of the following improper integrals

(i) $\int_1^2 \frac{\sqrt{x}}{\ln x}\, dx$, (ii) $\int_0^{\pi/2} \frac{\sin x}{x\sqrt{x}}\, dx$.

Solution

(i) We have $f(x) = (\sqrt{x}/\ln x) \geq 0$, $1 < x \leq 2$. The point $x = 1$ is the only point of infinite discontinuity.

Let $g(x) = 1/(x \ln x)$. Then, we have

$$\lim_{x\to 1^+} \frac{f(x)}{g(x)} = \lim_{h\to 0} \frac{f(1+h)}{g(1+h)} = \lim_{h\to 0} \left[\frac{\sqrt{1+h}}{\ln(1+h)}\right] [(1+h)\ln(1+h)]$$

$$= \lim_{h\to 0} (1+h)^{3/2} = 1.$$

Therefore, both the integrals $\int_1^2 f(x)\, dx$ and $\int_1^2 g(x)\, dx$ converge or diverge together.

Now,
$$\int_1^2 g(x)\, dx = \int_1^2 \frac{dx}{x\ln x} = \lim_{\varepsilon\to 0} \int_{1+\varepsilon}^2 \frac{dx}{x\ln x} = \lim_{\varepsilon\to 0} [\ln(\ln x)]_{1+\varepsilon}^2$$

$$= \lim_{\varepsilon\to 0} [\ln(\ln 2) - \ln(\ln(1+\varepsilon))] \to \infty.$$

Since $\int_1^2 g(x)dx$ is divergent, the given integral $\int_1^2 f(x)dx$ is also divergent by Comparison Test 4.

(ii) We have $f(x) = \frac{\sin x}{x\sqrt{x}} = \left(\frac{\sin x}{x}\right)\left(\frac{1}{\sqrt{x}}\right) \leq \frac{1}{\sqrt{x}}$, since $\sin x/x$ is bounded and $(\sin x/x) \leq 1$, $0 \leq x \leq \pi/2$. Let $g(x) = 1/\sqrt{x}$. We have $f(x) \leq g(x)$, $0 < x < \pi/2$.

Now, $g(x)$ has a point of discontinuity at $x = 0$. Hence

$$\int_0^{\pi/2} g(x)dx = \int_0^{\pi/2} \frac{dx}{\sqrt{x}} = \lim_{\varepsilon\to 0} \int_\varepsilon^{\pi/2} \frac{dx}{\sqrt{x}} = \lim_{\varepsilon\to 0}\left[\sqrt{2\pi} - 2\sqrt{\varepsilon}\right] = \sqrt{2\pi}.$$

Since $\int_0^{\pi/2} g(x)dx$ is convergent, the given integral $\int_0^{\pi/2} f(x)dx$ is also convergent by Comparison Test 3 (i).

Example 1.43 Show that the improper integral $\int_0^{\pi/2} \frac{\cos^m x}{x^n}\, dx$ converges when $n < 1$.

Solution We have $f(x) = \frac{\cos^m x}{x^n} < \frac{1}{x^n}$, $0 < x < \pi/2$. $x = 0$ is the point of infinite discontinuity of $f(x)$. Let $g(x) = 1/x^n$. Then $f(x) < g(x)$.

Since the integral $\int_0^{\pi/2} g(x)\,dx = \int_0^{\pi/2} \frac{dx}{x^n}$ is convergent for $n < 1$ (see Example 1.40), the given integral is also convergent for $n < 1$ by Comparison Test 3(i).

1.4.3 Absolute Convergence of Improper Integrals

In the previous sections, we had assumed that $f(x)$ is of the same sign throughout the interval of integration. Now, assume that $f(x)$ changes sign within the interval of integration. In this case, we consider absolute convergence of the improper integral.

Absolute convergence The improper integral $\int_a^b f(x)dx$ is said to be *absolutely convergent* if $\int_a^b |f(x)|\,dx$ is convergent.

Theorem 1.12 An absolutely convergent improper integral is convergent, that is if $\int_a^b |f(x)|\,dx$ converges, then $\int_a^b f(x)dx$ converges.

Since, $|f|$ is always positive within the interval of integration, all the comparison tests can be used to discuss the absolute convergence of the given improper integral.

Example 1.44 Show that the improper integral $\int_0^1 \frac{\sin(1/x)}{x^p}\,dx$ converges absolutely for $p < 1$.

Solution The integrand changes sign within the interval of integration. Hence, we consider the absolute convergence of the given integral. The function $f(x) = \sin(1/x)/x^p$ has a point of infinite discontinuity at $x = 0$. We have

$$|f(x)| = \left|\frac{\sin(1/x)}{x^p}\right| \le \frac{1}{x^p}.$$

Since $\int_0^1 \frac{1}{x^p}\,dx$ converges for $p < 1$, the given improper integral converges absolutely for $p < 1$.

Example 1.45 Show that the improper integral $\int_{-\infty}^{\infty} \frac{\sin x}{1+x^2}\,dx$ converges.

Solution The integrand $f(x)$ changes sign within the interval of integration. Hence, we consider the absolute convergence of the given integral. We have

$$|I| = \left|\int_{-\infty}^{\infty} \frac{\sin x}{1+x^2}\,dx\right| \le \int_{-\infty}^{\infty} \left|\frac{\sin x}{1+x^2}\right| dx$$

$$= \lim_{a\to-\infty} \int_a^c \left|\frac{\sin x}{1+x^2}\right| dx + \lim_{b\to\infty} \int_c^b \left|\frac{\sin x}{1+x^2}\right| dx = I_1 + I_2.$$

Now,

$$I_1 = \lim_{a\to-\infty} \int_a^c \left|\frac{\sin x}{1+x^2}\right| dx \le \lim_{a\to-\infty} \int_a^c \frac{dx}{1+x^2} = \lim_{a\to-\infty} [\tan^{-1}c - \tan^{-1}a] = \tan^{-1}c + \frac{\pi}{2}.$$

$$I_2 = \lim_{b\to\infty}\int_c^b \left|\frac{\sin x}{1+x^2}\right| dx \le \lim_{b\to\infty}\int_c^b \frac{dx}{1+x^2} = \lim_{b\to\infty}[\tan^{-1}b - \tan^{-1}c] = \frac{\pi}{2} - \tan^{-1}c.$$

Hence, $|I| \le I_1 + I_2 \le \pi$. Therefore, the given improper integral converges.

1.4.4 Beta and Gamma Functions

Beta and Gamma functions are improper integrals which are commonly encountered in many science and engineering applications. These functions are used in evaluating many definite integrals.

Gamma function

Consider the improper integral $$I(\alpha) = \int_0^\infty x^{\alpha-1}\, e^{-x}\, dx, \quad \alpha > 0. \tag{1.66}$$

We write the integral as $$I(\alpha) = \int_0^c x^{\alpha-1} e^{-x}\, dx + \int_c^\infty x^{\alpha-1}\, e^{-x}\, dx = I_1 + I_2, \quad 0 < c < \infty.$$

The integral I_1 is an improper integral of the second kind as the integrand has a point of discontinuity at $x = 0$, whenever $0 < \alpha < 1$. For $\alpha \ge 1$, it is a proper integral. The integral I_2 is an improper integral of the first kind as its upper limit is infinite. We consider the two integrals separately.

Convergence at $x = 0$, $0 < \alpha < 1$, of the first integral I_1

In the integral I_1, let $f(x) = x^{\alpha-1}e^{-x}$ and $g(x) = x^{\alpha-1}$. Now, $\lim_{x\to 0^+}\dfrac{f(x)}{g(x)} = \lim_{x\to 0^+}\dfrac{x^{\alpha-1}\, e^{-x}}{x^{\alpha-1}} = 1.$

Since $\int_0^c g(x)\, dx = \int_0^c \dfrac{dx}{x^{1-\alpha}}$ converges when $1 - \alpha < 1$, or $\alpha > 0$, the improper integral I_1 is convergent for all $\alpha > 0$.

Convergence at ∞, of the second integral I_2

Without loss of generality, let $c \ge 1$. Otherwise, the integral can be written as the sum of two integrals with the intervals $(c, 1)$, $(1, \infty)$. The first integral is a proper integral.

Let n be a positive integer such that $n > \alpha - 1$, $\alpha > 0$. Then,

$$\alpha - 1 < n, \quad x^{\alpha-1} < x^n \quad \text{and} \quad x^{\alpha-1}e^{-x} < x^n e^{-x}, \quad 1 < x < \infty.$$

Therefore, $$\int_c^\infty x^{\alpha-1}e^{-x}\, dx < \int_c^\infty x^n e^{-x}\, dx = \lim_{b\to\infty}\int_c^b x^n e^{-x}\, dx$$

$$= \lim_{b\to\infty}[e^{-x}\{\text{polynomial of degree } n \text{ in } x,\ P_n(x)\}]_c^b$$

$$= \lim_{b\to\infty}[e^{-b}P_n(b) - e^{-c}P_n(c)] = -e^{-c}P_n(c)$$

since $\lim_{b\to\infty}[b^k/e^b] = 0$ for fixed k.

The limit exists and the integral I_2 converges for $\alpha > 0$.

Hence the given improper integral (Eq. (1.66)) converges when $\alpha > 0$.

This improper integral is called the *Gamma function* and is denoted by $\Gamma(\alpha)$. Therefore,

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} dx, \quad \alpha > 0. \tag{1.67}$$

Some identities of Gamma functions

1. $\Gamma(1) = \int_0^{\infty} e^{-x}\, dx = 1.$ (1.68)

2. $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha).$ (1.69)

   Integrating Eq. (1.66) by parts, we get

$$\Gamma(\alpha+1) = \int_0^{\infty} x^{\alpha} e^{-x}\, dx = -\left[x^{\alpha} e^{-x}\right]_0^{\infty} + \alpha \int_0^{\infty} x^{\alpha-1} e^{-x}\, dx = \alpha\Gamma(\alpha).$$

   If α is negative and not an integer, then we write $\Gamma(\alpha) = \dfrac{1}{\alpha}\Gamma(\alpha + 1).$

3. $\Gamma(m + 1) = m!$, for any position integer m. (1.70)

   We have $\Gamma(m + 1) = m\Gamma(m) = m(m-1)\,\Gamma(m-1) = \ldots = m(m-1)\ldots 1\Gamma(1) = m!.$

4. $\Gamma(1/2) = \sqrt{\pi}.$ (1.71)

   We have $\Gamma(1/2) = \int_0^{\infty} x^{-1/2} e^{-x}\, dx = 2\int_0^{\infty} e^{-u^2}\, du.$ (set $x = u^2$).

   We write

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = \left[2\int_0^{\infty} e^{-u^2}\, du\right]\left[2\int_0^{\infty} e^{-v^2}\, dv\right] = 4\int_0^{\infty}\int_0^{\infty} e^{-(u^2+v^2)}\, du\, dv.$$

   Changing to polar coordinates $u = r\cos\theta$, $v = r\sin\theta$, we obtain $du dv = r\, dr\, d\theta$ and

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 4\int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} re^{-r^2}\, dr\, d\theta = 2\pi\int_0^{\infty} re^{-r^2}\, dr = -\pi\left[e^{-r^2}\right]_0^{\infty} = \pi.$$

   Hence, $$\Gamma(1/2) = \sqrt{\pi}.$$

   (In Chapter 2, we shall discuss evaluation of double integrals and change of variables.)

5. $\Gamma(-1/2) = -2\sqrt{\pi}.$ (1.72)

   We have $\Gamma(\alpha) = [\Gamma(\alpha + 1)]/\alpha$. Substituting $\alpha = -1/2$, we get

$$\Gamma\left(-\frac{1}{2}\right) = \frac{\Gamma(1/2)}{(-1/2)} = -2\sqrt{\pi}.$$

Beta function

Consider the improper integral

$$I = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx, \quad 0 < m < 1, \quad 0 < n < 1. \tag{1.73}$$

Note that I is a proper integral for $m \geq 1$ and $n \geq 1$. The improper integral has points of infinite discontinuity at (i) $x = 0$, when $m < 1$ and (ii) $x = 1$, when $n < 1$. When $m < 1$ and $n < 1$, we take a number, say c between 0 and 1 and write the improper integral as

$$I = \int_0^c x^{m-1}(1-x)^{n-1}\,dx + \int_c^1 x^{m-1}(1-x)^{n-1}\,dx = I_1 + I_2$$

where
$$I_1 = \int_0^c x^{m-1}(1-x)^{n-1}\,dx \quad \text{and} \quad I_2 = \int_c^1 x^{m-1}(1-x)^{n-1}\,dx.$$

I_1 is an improper integral, since $x = 0$ is a point of infinite discontinuity, while I_2 is an improper integral, since $x = 1$ is a point of infinite discontinuity. We consider these two integrals separately.

Convergence at $x = 0$, $0 < m < 1$, of the integral I_1

In the integral I_1, let $f(x) = x^{m-1}(1-x)^{n-1}$ and $g(x) = x^{m-1}$.

Now,
$$\lim_{x\to 0^+} \frac{f(x)}{g(x)} = \lim_{x\to 0^+} \frac{x^{m-1}(1-x)^{n-1}}{x^{m-1}} = 1$$

and $\int_0^c g(x)\,dx = \int_0^c \frac{dx}{x^{1-m}}$ is convergent only when $1 - m < 1$, or $m > 0$.

Therefore, the improper integral I_1 converges when $m > 0$.

Convergence at $x = 1$, $0 < n < 1$, of the integral I_2

In the integral I_2, let $f(x) = x^{m-1}(1-x)^{n-1}$ and $g(x) = (1-x)^{n-1}$.

Now,
$$\lim_{x\to 1} \frac{f(x)}{g(x)} = \lim_{x\to 1} \frac{x^{m-1}(1-x)^{n-1}}{(1-x)^{n-1}} = 1$$

and
$$\int_c^1 g(x)\,dx = \int_c^1 \frac{dx}{(1-x)^{1-n}}$$ converges when $1 - n < 1$, or $n > 0$.

Therefore, the improper integral I_2 converges when $n > 0$. Combining the two results, we deduce that the given improper integral (Eq. (1.73)) converges when $m > 0$ and $n > 0$. This improper integral is called the *Beta function* and is denoted by $\beta(m, n)$. Therefore,

$$\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx, \quad m > 0, n > 0. \tag{1.74}$$

Some identities of Beta functions

1. $\beta(m, n) = \beta(n, m)$ (1.75)

   (substitute $x = 1 - t$ in Eq. (1.74) and simplify).

2. $\beta(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}(\theta)\cos^{2n-1}(\theta)\,d\theta = 2\int_0^{\pi/2} \sin^{2n-1}(\theta)\cos^{2m-1}(\theta)\,d\theta.$ (1.76)

   (substitute $x = \sin^2\theta$ in Eq. (1.74) and simplify).

3. $\beta(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx$ (1.77)

(substitute $x = t/(1 + t)$ in Eq. (1.74) and simplify).

4. $\beta(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$. (1.78)

We can prove this result using double integrals and change of variables. We have

$$\Gamma(m) = \int_0^\infty x^{m-1}\, e^{-x}\, dx = 2\int_0^\infty u^{2m-1}\, e^{-u^2}\, du, \quad (\text{set } x = u^2)$$

$$\Gamma(n) = \int_0^\infty x^{n-1}\, e^{-x}\, dx = 2\int_0^\infty v^{2n-1}\, e^{-v^2}\, dv, \quad (\text{set } x = v^2)$$

$$\Gamma(m)\Gamma(n) = 4\int_0^\infty \int_0^\infty u^{2m-1}\, v^{2n-1}\, e^{-(u^2+v^2)}\, du\, dv.$$

Changing to polar coordinates, $u = r \cos \theta$, $v = r \sin \theta$, we get

$$\Gamma(m)\,\Gamma(n) = 4\int_{\theta=0}^{\pi/2} \int_0^\infty \cos^{2m-1}(\theta)\sin^{2n-1}(\theta)\, r^{2m+2n-1}\, e^{-r^2}\, dr\, d\theta$$

$$= 4\left[\int_0^\infty r^{2m+2n-1}\, e^{-r^2}\, dr\right]\left[\int_0^{\pi/2} \cos^{2m-1}(\theta)\sin^{2n-1}(\theta)\, d\theta\right]$$

$$= 2\beta(m, n)\int_0^\infty r^{2m+2n-1}\, e^{-r^2}\, dr, \quad (\text{using Eq. (1.76)}).$$

We also have

$$\Gamma(m + n) = \int_0^\infty x^{m+n-1}\, e^{-x}\, dx = 2\int_0^\infty r^{2m+2n-1}\, e^{-r^2}\, dr, \quad (\text{set } x = r^2).$$

Combining the two results, we obtain

$$\Gamma(m)\,\Gamma(n) = \beta(m, n)\,\Gamma(m + n), \quad \text{or} \quad \beta(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$$

5. $\beta(m, n) = \beta(m + 1, n) + \beta(m, n + 1)$.

We have

$$\beta(m + 1, n) = 2\int_0^{\pi/2} \sin^{2m+1}(\theta)\cos^{2n-1}(\theta)\, d\theta = 2\int_0^{\pi/2} \sin^{2m-1}(\theta)\sin^2\theta\cos^{2n-1}(\theta)\, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}(\theta)\cos^{2n-1}(\theta)\,(1 - \cos^2\theta)\, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}(\theta)\cos^{2n-1}(\theta)\, d\theta - 2\int_0^{\pi/2} \sin^{2m-1}(\theta)\cos^{2n+1}(\theta)\, d\theta$$

$$= \beta(m, n) - \beta(m, n+1).$$

Therefore, $\beta(m, n) = \beta(m+1, n) + \beta(m, n+1).$

Example 1.46 Given that $\int_0^\infty \frac{x^{p-1}}{1+x} = \frac{\pi}{\sin p\pi}$, show that $\Gamma(p)\,\Gamma(1-p) = \frac{\pi}{\sin p\pi}$.

Solution Let $\frac{x}{1+x} = y$. Solving for x, we get $x = \frac{y}{1-y}$ and $dx = \frac{1}{(1-y)^2}\, dy$.

Then, $I = \int_0^\infty \frac{x^{p-1}}{1+x}\, dx = \int_0^1 y^{p-1}(1-y)^{-p}\, dy = \beta(p, 1-p) = \frac{\Gamma(p)\Gamma(1-p)}{\Gamma(1)} = \Gamma(p)\,\Gamma(1-p).$

Hence, the result.

Example 1.47 Evaluate the following improper integrals

(i) $\int_0^\infty \sqrt{x}\, e^{-x^2}\, dx$, (ii) $\int_0^\infty e^{-x^3}\, dx$

in terms of Gamma functions.

Solution

(i) Substitute $x = \sqrt{t}$. We get $dx = dt/(2\sqrt{t})$ and

$$I = \int_0^\infty \sqrt{x}\, e^{-x^2}\, dx = \frac{1}{2}\int_0^\infty t^{-1/4}\, e^{-t}\, dt = \frac{1}{2}\int_0^\infty t^{(3/4)-1}\, e^{-t}\, dt = \frac{1}{2}\Gamma\left(\frac{3}{4}\right).$$

(ii) Substitute $x = t^{1/3}$. We get $dx = \frac{1}{3}\, t^{-2/3}\, dt$ and

$$I = \int_0^\infty e^{-x^3}\, dx = \frac{1}{3}\int_0^\infty t^{-2/3}\, e^{-t}\, dt = \frac{1}{3}\int_0^\infty t^{(1/3)-1}\, e^{-t}\, dt = \frac{1}{3}\Gamma\left(\frac{1}{3}\right).$$

Example 1.48 Using Beta and Gamma functions, evaluate the integral

$$I = \int_{-1}^1 (1-x^2)^n\, dx, \text{ where } n \text{ is a positive integer.}$$

Solution We have $I = \int_{-1}^1 (1+x)^n (1-x)^n\, dx.$

Let $1+x = 2t$. Then, $dx = 2dt$ and $1-x = 2(1-t)$. We obtain

$$I = 2^{2n+1}\int_0^1 t^n(1-t)^n\, dt = 2^{2n+1}\beta(n+1, n+1)$$

$$= 2^{2n+1}\frac{\Gamma(n+1)\Gamma(n+1)}{\Gamma(2n+2)} = \frac{2^{2n+1}(n!)^2}{(2n+1)!}.$$

Example 1.49 Express $\int_0^1 x^m(1-x^p)^n\, dx$ in terms of Beta function and hence evaluate the integral $\int_0^1 x^{3/2}(1-\sqrt{x})^{1/2}\, dx$.

Solution Let $x^p = y$. Then $px^{p-1}\, dx = dy$. We obtain

$$I = \int_0^1 x^m(1-x^p)^n\, dx = \frac{1}{p}\int_0^1 y^{(m-p+1)/p}(1-y)^n\, dy$$

$$= \frac{1}{p}\int_0^1 y^{[(m+1)/p-1]}(1-y)^n\, dy = \frac{1}{p}\beta\left(\frac{m+1}{p}, n+1\right)$$

Now, comparing the integral $\int_0^1 x^{3/2}(1-\sqrt{x})^{1/2}\, dx$ with the given integral, we find that $m = 3/2$, $p = 1/2$ and $n = 1/2$. Therefore,

$$\int_0^1 x^{3/2}(1-\sqrt{x})^{1/2}\, dx = 2\beta\left(5, \frac{3}{2}\right) = \frac{2\Gamma(5)\Gamma(3/2)}{\Gamma(13/2)}.$$

Now, $$\Gamma(5) = 4! = 24,\quad \Gamma\left(\frac{13}{2}\right) = \frac{11}{2}\cdot\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{10395}{32}\Gamma\left(\frac{3}{2}\right).$$

Hence, $$I = \frac{2(24)(32)\,\Gamma(3/2)}{10395\,\Gamma(3/2)} = \frac{1536}{10395} = \frac{512}{3465}.$$

Example 1.50 Using Beta and Gamma functions, show that for any positive integer m

(i) $$\int_0^{\pi/2} \sin^{2m-1}(\theta)\, d\theta = \frac{(2m-2)(2m-4)\dots 2}{(2m-1)(2m-3)\dots 3}.$$

(ii) $$\int_0^{\pi/2} \sin^{2m}(\theta)\, d\theta = \frac{(2m-1)(2m-3)\dots 1}{(2m)(2m-2)\dots 2}\,\frac{\pi}{2}.$$

Solution From Eq. (1.76), we obtain

$$\beta\left(m, \frac{1}{2}\right) = 2\int_0^{\pi/2} \sin^{2m-1}(\theta)\, d\theta \quad \text{and} \quad \beta\left(m+\frac{1}{2}, \frac{1}{2}\right) = 2\int_0^{\pi/2} \sin^{2m}(\theta)\, d\theta.$$

(i) $$I = \int_0^{\pi/2} \sin^{(2m-1)}(\theta)\, d\theta = \frac{1}{2}\beta\left(m, \frac{1}{2}\right) = \frac{\Gamma(m)\Gamma(1/2)}{2\Gamma(m+1/2)}.$$

We have $\Gamma(m) = (m-1)!$, and

$$\Gamma\left(m+\frac{1}{2}\right) = \left(m-\frac{1}{2}\right)\left(m-\frac{3}{2}\right)\dots\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{1}{2^m}\,[(2m-1)\,(2m-3)\dots 3.1]\,\Gamma\left(\frac{1}{2}\right).$$

Therefore,

$$I = \frac{(m-1)!2^m\,\Gamma(1/2)}{2(2m-1)(2m-3)\ldots 3\cdot 1\cdot\Gamma(1/2)} = \frac{2^{m-1}[(m-1)(m-2)\ldots 2\cdot 1]}{(2m-1)(2m-3)\ldots 3\cdot 1}$$

$$= \frac{(2m-2)(2m-4)\ldots 4\cdot 2}{(2m-1)(2m-3)\ldots 3\cdot 1}.$$

(ii) $$I = \int_0^{\pi/2} \sin^{2m}(\theta)\,d\theta = \frac{1}{2}\beta\left(m+\frac{1}{2},\frac{1}{2}\right) = \frac{\Gamma(m+1/2)\Gamma(1/2)}{2\Gamma(m+1)}$$

$$= \frac{1}{2(m!)}\left[\frac{(2m-1)(2m-3)\ldots 3\cdot 1}{2^m}\right](\sqrt{\pi})^2 = \frac{(2m-1)(2m-3)\ldots 3\cdot 1}{2^{m+1}[m(m-1)\ldots 2\cdot 1]}(\pi)$$

$$= \frac{(2m-1)(2m-3)\ldots 3\cdot 1}{(2m)(2m-2)\ldots 4\cdot 2}\,\frac{\pi}{2}.$$

Example 1.51 Evaluate $\int_0^\infty 2^{-9x^2}\,dx$ using the Gamma function.

Solution We write

$$I = \int_0^\infty 2^{-9x^2}\,dx = \int_0^\infty e^{-9x^2 \ln 2}\,dx$$

Substitute $9x^2 \ln 2 = y$. Then, $x = \dfrac{\sqrt{y}}{3\sqrt{\ln 2}}$ and $dx = \dfrac{y^{-1/2}\,dy}{6\sqrt{\ln 2}}$.

Therefore,

$$I = \frac{1}{6\sqrt{\ln 2}}\int_0^\infty y^{-1/2}\,e^{-y}\,dy = \frac{1}{6\sqrt{\ln 2}}\int_0^\infty y^{(1/2)-1}\,e^{-y}\,dy$$

$$= \frac{\Gamma(1/2)}{6\sqrt{\ln 2}} = \frac{1}{6}\sqrt{\frac{\pi}{\ln 2}}.$$

Example 1.52 Show that

$$\Gamma(2n) = \frac{2^{2n-1}}{\sqrt{\pi}}\Gamma\left(n+\frac{1}{2}\right)\Gamma(n) \tag{1.79}$$

and

$$\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \pi\sqrt{2}. \tag{1.80}$$

Solution From Eq. (1.78), we have

$$\frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)} = \beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}(\theta)\cos^{2n-1}(\theta)\,d\theta. \tag{1.81}$$

Setting $m = n$, we get

$$\frac{[\Gamma(n)]^2}{\Gamma(2n)} = \beta(n, n) = 2\int_0^{\pi/2} \sin^{2n-1}(\theta)\cos^{2n-1}(\theta)\, d\theta$$

$$= \frac{1}{2^{2n-2}}\int_0^{\pi/2} \sin^{2n-1}(2\theta)\, d\theta.$$

Substituting, $2\theta = \frac{\pi}{2} - \phi$, we get $d\theta = -\frac{1}{2}\, d\phi$. Hence, we obtain

$$\frac{[\Gamma(n)]^2}{\Gamma(2n)} = \frac{-1}{2^{2n-1}}\int_{\pi/2}^{-\pi/2} \cos^{2n-1}(\phi)\, d\phi$$

$$= \frac{1}{2^{2n-1}}\int_{-\pi/2}^{\pi/2} \cos^{2n-1}\phi\, d\phi = \frac{2}{2^{2n-1}}\int_0^{\pi/2} \cos^{2n-1}(\theta)\, d\theta \tag{1.82}$$

since $\cos\theta$ is an even function.

Setting $m = 1/2$ in Eq. (1.81), we obtain

$$\frac{\Gamma(n)\Gamma(1/2)}{\Gamma(n+1/2)} = 2\int_0^{\pi/2} \cos^{2n-1}(\theta)\, d\theta. \tag{1.83}$$

Comparing Eqs. (1.82) and (1.83), we have

$$\frac{[\Gamma(n)]^2}{\Gamma(2n)} = \frac{1}{2^{2n-1}}\left[\frac{\Gamma(n)\Gamma(1/2)}{\Gamma(n+1/2)}\right]$$

or

$$\Gamma(2n) = \frac{2^{2n-1}}{\sqrt{\pi}}\Gamma(n)\Gamma(n+1/2), \quad \left(\text{since } \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right)$$

which is the required result.

Setting $n = 1/4$ in Eq. (1.79), we obtain

$$\Gamma\left(\frac{1}{2}\right) = \frac{2^{-1/2}}{\sqrt{\pi}}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$$

$$\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \pi\sqrt{2}.$$

Example 1.53 Show that $\int_0^{\pi/2} \sqrt{\tan x}\, dx = \frac{\pi}{\sqrt{2}}$.

Solution We have

$$I = \int_0^{\pi/2} \sqrt{\tan x}\, dx = \int_0^{\pi/2} \sin^{1/2} x \cos^{-1/2} x\, dx = \frac{1}{2}\beta\left(\frac{3}{4}, \frac{1}{4}\right)$$

$$= \frac{1}{2}\frac{\Gamma(1/4)\Gamma(3/4)}{\Gamma(1)} = \frac{\pi\sqrt{2}}{2} = \frac{\pi}{\sqrt{2}} \quad \text{(using Eq. (1.80)).}$$

1.4.5 Improper Integrals Involving a Parameter

Often, we come across integrals of the form

$$\phi(\alpha) = \int_{a(\alpha)}^{b(\alpha)} f(x, \alpha)\, dx \tag{1.84}$$

where α is a parameter and the integrand f is such that the integral cannot be evaluated by standard methods. We can evaluate some of these integrals by differentiating the integral with respect to the parameter, that is first obtain $\phi'(\alpha)$, evaluate the integral (that is integrate with respect to x) and then integrate $\phi'(\alpha)$ with respect to α. Note that f is a function of two variables x and α. When we differentiate f with respect to α, we treat x as a constant and denote the derivative *as* $\partial f/\partial \alpha$ (partial derivative of f with respect to α, Chapter 2 discusses partial derivatives in detail). We assume that f, $\partial f/\partial \alpha$, $a(\alpha)$ and $b(\alpha)$ are continuous functions of α.

We now present the formula which gives the derivative of $\phi(\alpha)$.

Theorem 1.16 (Leibniz formula) If $a(\alpha)$, $b(\alpha)$, $f(x, \alpha)$ and $\partial f/\partial \alpha$ are continuous functions of α, then

$$\frac{d\phi}{d\alpha} = \int_{a(\alpha)}^{b(\alpha)} \frac{\partial f}{\partial \alpha}(x, \alpha)\, dx + f(b, \alpha)\frac{db}{d\alpha} - f(a, \alpha)\frac{da}{d\alpha}. \tag{1.85}$$

Proof Let $\Delta\alpha$ be an increment in α and Δa, Δb be the corresponding increments in a and b. We have

$$\Delta\phi = \phi(\alpha + \Delta\alpha) - \phi(\alpha) = \int_{a+\Delta a}^{b+\Delta b} f(x, \alpha + \Delta\alpha)\, dx - \int_a^b f(x, \alpha)\, dx$$

$$= \int_{a+\Delta a}^{a} f(x, \alpha + \Delta\alpha)\, dx + \int_a^b f(x, \alpha + \Delta\alpha)\, dx + \int_b^{b+\Delta b} f(x, \alpha + \Delta\alpha)\, dx - \int_a^b f(x, \alpha)\, dx$$

or
$$\frac{\Delta\phi}{\Delta\alpha} = \int_{a+\Delta a}^{a} \frac{1}{\Delta\alpha} f(x, \alpha + \Delta\alpha) dx + \int_a^b \frac{1}{\Delta\alpha}[f(x, \alpha + \Delta\alpha) - f(x, \alpha)]\, dx$$

$$+ \int_b^{b+\Delta b} \frac{1}{\Delta\alpha} f(x, \alpha + \Delta\alpha) dx. \tag{1.86}$$

Using the mean value theorem of integrals

$$\int_{x_0}^{x_1} f(x)\, dx = (x_1 - x_0)\, f(\xi), \quad x_0 < \xi < x_1$$

we get
$$\int_{a+\Delta a}^{a} f(x, \alpha + \Delta\alpha)\, dx = -\Delta a\, f(\xi_1, \alpha + \Delta\alpha), \quad a < \xi_1 < a + \Delta a \tag{1.87}$$

and
$$\int_b^{b+\Delta b} f(x, \alpha + \Delta\alpha) dx = \Delta b\, f(\xi_2, \alpha + \Delta\alpha), \quad b < \xi_2 < b + \Delta b. \tag{1.88}$$

Using the Lagrange mean value theorem, we get

$$f(x, \alpha + \Delta\alpha) - f(x, \alpha) = \Delta\alpha \frac{\partial f}{\partial \alpha}(x, \xi_3), \quad \alpha < \xi_3 < \alpha + \Delta\alpha. \tag{1.89}$$

We note that

$$\lim_{\Delta\alpha\to 0}\xi_1=a,\ \lim_{\Delta\alpha\to 0}\xi_2=b \quad\text{and}\quad \lim_{\Delta\alpha\to 0}\xi_3=\alpha. \tag{1.90}$$

Taking limits as $\Delta\alpha\to 0$ on both sides of Eq. (1.86) and using the results in Eqs. (1.87) to (1.90), we obtain

$$\frac{d\phi}{d\alpha}=\int_a^b \frac{\partial f}{\partial\alpha}(x,\alpha)\,dx+f(b,\alpha)\frac{db}{d\alpha}-f(a,\alpha)\frac{da}{d\alpha}.$$

Remark 6

(a) If the limits $a(\alpha)$ and $b(\alpha)$ are constants, then we obtain from Eq. (1.85)

$$\frac{d\phi}{d\alpha}=\int_a^b \frac{\partial f}{\partial\alpha}(x,\alpha)\,dx. \tag{1.90}$$

(b) If the integrand f is independent of α, then we obtain from Eq. (1.85)

$$\frac{d\phi}{d\alpha}=f(b)\frac{db}{d\alpha}-f(a)\frac{da}{d\alpha}. \tag{1.92}$$

Example 1.54 Evaluate the integral $\int_0^\infty \frac{e^{-\alpha x}\sin x}{x}\,dx$, $\alpha>0$ and deduce that

(i) $\int_0^\infty \frac{\sin x}{x}\,dx=\frac{\pi}{2}$, (ii) $\int_0^\infty \frac{\sin ax}{x}\,dx=\frac{\pi}{2}$, $a>0$.

Solution Let $$\phi(\alpha)=\int_0^\infty \frac{e^{-\alpha x}\sin x}{x}\,dx. \tag{1.93}$$

The limits of integration are independent of the parameter α. we obtain

$$\frac{d\phi}{d\alpha}=\int_0^\infty \frac{\partial}{\partial\alpha}\left[\frac{e^{-\alpha x}\sin x}{x}\right]dx=-\int_0^\infty \frac{xe^{-\alpha x}\sin x}{x}\,dx=-\int_0^\infty e^{-\alpha x}\sin x\,dx.$$

Using the result $\int e^{-\alpha x}\sin x\,dx=\frac{e^{-\alpha x}}{1+\alpha^2}(\alpha\sin x+\cos x)$, we obtain

$$\frac{d\phi}{d\alpha}=\left[\frac{e^{-\alpha x}}{1+\alpha^2}(\alpha\sin x+\cos x)\right]_0^\infty=-\frac{1}{1+\alpha^2}.$$

Integrating with respect to α, we get

$\phi(\alpha)=-\tan^{-1}\alpha+c$, where c is the constant of integration.

From Eq. (1.93), we get the condition $\phi(\infty)=0$. Hence,

$$\phi(\infty)=0=-\tan^{-1}\infty+c,\quad\text{or}\quad c=\pi/2.$$

Therefore, $$\phi(\alpha)=\int_0^\infty \frac{e^{-\alpha x}\sin x}{x}\,dx=\frac{\pi}{2}-\tan^{-1}\alpha.$$

(i) Setting $\alpha=0$, we obtain $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx=\frac{\pi}{2}$. (1.94)

(ii) Substituting $x=ay$ on the left hand side of Eq. (1.94), we obtain

$$\int_0^\infty \frac{\sin x}{x}=\int_0^\infty \frac{\sin ay}{y}\,dy=\frac{\pi}{2}.$$

Example 1.55 Using the result $\displaystyle\int_0^\infty e^{-x^2}\,dx=\frac{\sqrt{\pi}}{2}$, evaluate the integral $\displaystyle\int_0^\infty e^{-x^2}\cos(2\alpha x)\,dx$.

Solution Let $\phi(\alpha)=\displaystyle\int_0^\infty e^{-x^2}\cos(2\alpha x)\,dx$. (1.95)

The limits of integration are independent of the parameter α. Hence,

$$\frac{d\phi}{d\alpha}=\int_0^\infty \frac{\partial}{\partial\alpha}[e^{-x^2}\cos(2\alpha x)]\,dx=\int_0^\infty(-2x)\,e^{-x^2}\sin(2\alpha x)\,dx$$

$$=[e^{-x^2}\sin(2\alpha x)]_0^\infty-2\alpha\int_0^\infty e^{-x^2}\cos(2\alpha x)dx=-2\alpha\phi.$$

Integrating the differential equation $\dfrac{d\phi}{d\alpha}+2\alpha\phi=0$, we obtain $\phi(\alpha)=ce^{-\alpha^2}$.

From Eq. (1.95), we get the condition $\phi(0)=\displaystyle\int_0^\infty e^{-x^2}\,dx=\frac{\sqrt{\pi}}{2}$.

Using this condition, we obtain $\phi(0)=\dfrac{\sqrt{\pi}}{2}=c$.

Therefore, $\phi(\alpha)=\displaystyle\int_0^\infty e^{-x^2}\cos(2\alpha x)\,dx=\frac{\sqrt{\pi}}{2}e^{-\alpha^2}$.

Example 1.56 Evaluate the integral $\displaystyle\int_0^\infty \frac{\tan^{-1}(ax)}{x(1+x^2)}\,dx,\ a>0$ and $a\neq 1$.

Solution Let $\phi(a)=\displaystyle\int_0^\infty \frac{\tan^{-1}(ax)}{x(1+x^2)}\,dx$. (1.96)

We have

$$\frac{d\phi}{da}=\int_0^\infty \frac{\partial}{\partial a}\left[\frac{\tan^{-1}(ax)}{x(1+x^2)}\right]dx=\int_0^\infty \frac{dx}{(1+x^2)(1+a^2x^2)}$$

$$=\frac{1}{a^2-1}\int_0^\infty\left[\frac{a^2}{a^2x^2+1}-\frac{1}{1+x^2}\right]dx$$

$$= \frac{1}{a^2 - 1}\left[\left\{a \tan^{-1}(ax)\right\}_0^{\infty} - \left\{\tan^{-1}(x)\right\}_0^{\infty}\right] = \frac{\pi}{2}\left[\frac{a-1}{a^2-1}\right] = \frac{\pi}{2(a+1)}.$$

Integrating with respect to a, we obtain

$$\phi(a) = \frac{\pi}{2} \ln(a+1) + c.$$

From Eq. (1.96), we get the condition $\phi(0) = 0$. Using this condition, we obtain $\phi(0) = 0 = c$.

Therefore, $\phi(a) = \int_0^{\infty} \frac{\tan^{-1}(ax)}{x(1+x^2)}\, dx = \frac{\pi}{2} \ln(a+1)$.

1.4.6 Error Functions

Error functions arise in the theory of probability and solution of certain types of partial differential equations (see section 8.7).

Let us first consider the following function that arises in defining the normal probability distribution (the case when mean $= \mu = 0$ and variance $= \sigma^2 = 1$)

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}. \tag{1.97}$$

This function is also called the *Gaussian function*. The bell shaped *normal curve* defined by Eq. (1.97) is given in Fig. 1.20. The area under the curve and above the x-axis is given by

$$I = \int_{-\infty}^{\infty} f(x)\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}\, dx. \tag{1.98}$$

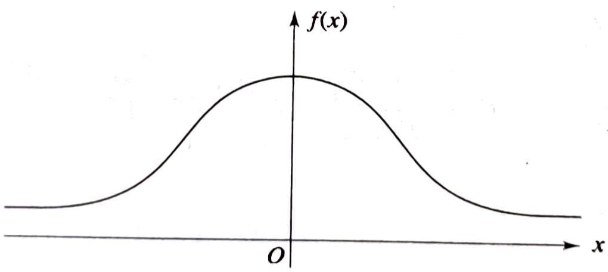

Fig. 1.20. Normal curve.

Setting $u = x/\sqrt{2}$, we get

$$I = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2}\, du.$$

It was shown in equation (1.71), that

$$\int_0^{\infty} e^{-u^2}\, du = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}.$$

Hence, $$I = \int_{-\infty}^{\infty} f(x)\,dx = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2}\,du = \frac{1}{\sqrt{\pi}}(2)\left(\frac{\sqrt{\pi}}{2}\right) = 1,$$

that is, the total area under the normal curve is 1. Since the area is symmetric about the y-axis, we get

$$\int_{-\infty}^{0} f(x)\,dx = \int_{0}^{\infty} f(x)dx = \frac{1}{2}.$$

The area under the curve, from $-\infty$ to any point z, is given by

$$\phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-x^2/2}\,dx. \tag{1.99}$$

Hence, by definition $\phi(0) = 1/2$. The function $\phi(z)$ is called the distribution function of the normal distribution with mean 0 and variance 1. Setting $x = -y$ in Eq. (1.99), we get $dx = -dy$ and.

$$\begin{aligned}\phi(z) &= -\frac{1}{\sqrt{2\pi}} \int_{\infty}^{-z} e^{-y^2/2}\,dy = \frac{1}{\sqrt{2\pi}} \int_{-z}^{\infty} e^{-y^2/2}\,dy \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2}\,dy - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-z} e^{-y^2/2}\,dy \\ &= 1 - \phi(-z)\end{aligned}$$

or $$\phi(-z) = 1 - \phi(z). \tag{1.100}$$

Values of the distribution function are tabulated for various values of z. Further, the area under the curve from $x = 0$ to $x = z$ is given by

$$I(z) = \frac{1}{\sqrt{2\pi}} \int_{0}^{z} e^{-x^2/2}\,dx = \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{z} e^{-x^2/2}\,dx - \int_{-\infty}^{0} e^{-x^2/2}\,dx\right] = \phi(z) - \frac{1}{2} \tag{1.101}$$

or $$\phi(z) = \frac{1}{2} + I(z).$$

Error function $erf(x)$

The *error function* is also called the *error integral function*. It is defined by

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-t^2}\,dt. \tag{1.102}$$

Let $t^2 = u$. Then, $dt = \frac{1}{2t}\,du = \frac{1}{2\sqrt{u}}\,du$, and

$$erf(x) = \frac{1}{\sqrt{\pi}} \int_{0}^{x^2} u^{-1/2} e^{-u}\,du. \tag{1.103}$$

This is another form of the error function. Using this definition, we obtain

$$erf(\infty) = \frac{1}{\sqrt{\pi}} \int_0^{\infty} u^{-1/2} e^{-u} du = \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}\right) = \frac{1}{\sqrt{\pi}}(\sqrt{\pi}) = 1. \tag{1.104}$$

Let $t^2 = u^2/2$ in Eq. (1.102). Then,

$$2t\, dt = u\, du, \quad dt = \frac{u}{2t} du = \frac{du}{\sqrt{2}}, \text{ and}$$

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{2}x} e^{-u^2/2} \frac{du}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^{\sqrt{2}x} e^{-u^2/2} du. \tag{1.105}$$

Using Eq. (1.101), we can write

$$I(\sqrt{2}\,x) = \frac{1}{\sqrt{2\pi}} \int_0^{\sqrt{2}x} e^{-x^2/2} dx = \phi(\sqrt{2}\,x) - \frac{1}{2}.$$

Therefore, $$erf(x) = 2I(\sqrt{2}\,x) = 2\phi(\sqrt{2}\,x) - 1. \tag{1.106}$$

Hence, the error function can be evaluated using this relation.

Complementary error function $erfc(x)$

Using the definition of the error function given in Eqs. (1.103) and (1.104), we write

$$erf(x) = \frac{1}{\sqrt{\pi}} \int_0^{x^2} u^{-1/2} e^{-u} du = \frac{1}{\sqrt{\pi}} \int_0^{\infty} u^{-1/2} e^{-u} du - \frac{1}{\sqrt{\pi}} \int_{x^2}^{\infty} u^{-1/2} e^{-u} du$$

$$= 1 - \frac{1}{\sqrt{\pi}} \int_{x^2}^{\infty} u^{-1/2} e^{-u} du = 1 - erfc(x) \tag{1.107}$$

where we define

$$erfc\,(x) = \frac{1}{\sqrt{\pi}} \int_{x^2}^{\infty} u^{-1/2} e^{-u} du. \tag{1.108}$$

The function $erfc\,(x)$ is called the complementary error function.

Using Eqs. (1.107), (1.102) and (1.104), we can write

$$erfc(x) = 1 - erf(x) = 1 - \frac{2}{\sqrt{\pi}} \int_0^{x} e^{-t^2} dt$$

$$= \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-t^2} dt - \frac{2}{\sqrt{\pi}} \int_0^{x} e^{-t^2} dt = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt. \tag{1.109}$$

Eqs. (1.102) and (1.109) are the commonly used definitions of error function and complementary error function respectively. The graphs of $erf(x)$ and $erfc(x)$ are given in Fig. 1.21.

Some properties of error functions

1. $erf(-x) = -erf(x)$. (1.110)

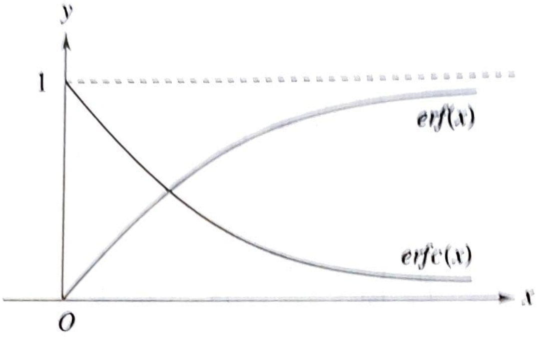

Fig. 1.21. Error function and complementary error function.

Using the definition given in Eq. (1.102), we get

$$erf(-x) = \frac{2}{\sqrt{\pi}} \int_0^{-x} e^{-t^2}\, dt = \frac{2}{\sqrt{\pi}} \int_0^{x} e^{-u^2}\, (-du) \quad \text{(setting } t = -u\text{)}$$

$$= -erf(x). \tag{1.111}$$

2. $erfc\,(-x) = 1 + erf(x) = 2 - erfc\,(x)$.

Using Eq. (1.107), we get

$$erfc\,(-x) = 1 - erf(-x) = 1 + erf(x)$$

$$= 1 + [1 - erfc\,(x)] = 2 - erfc\,(x).$$

3. *Derivative of error function:* We have

$$\frac{d}{dx}[erf(\alpha x)] = \frac{2\alpha}{\sqrt{\pi}} e^{-\alpha^2 x^2}. \tag{1.112}$$

From the definition, we have

$$erf(\alpha x) = \frac{2}{\sqrt{\pi}} \int_0^{\alpha x} e^{-t^2}\, dt. \tag{1.113}$$

Consider x as a parameter. Comparing Eq. (1.113) with Eq. (1.84)

$$\phi(a) = \int_{a(\alpha)}^{b(\alpha)} f(x, \alpha)\, d\alpha, \quad \text{where } \alpha \text{ is a parameter} \tag{1.114}$$

we get $f(t, x) = \frac{2}{\sqrt{\pi}} e^{-t^2}$, $b(x) = \alpha x$, $a(x) = 0$, $\phi(x) = erf(\alpha x)$.

Using Eq. (1.85)

$$\frac{d\phi}{d\alpha} = \int_{a(\alpha)}^{b(\alpha)} \frac{\partial f}{\partial \alpha}(x, \alpha)\, dx + f(b, \alpha)\frac{db}{d\alpha} - f(a, \alpha)\frac{da}{d\alpha} \tag{1.115}$$

we obtain

$$\frac{d}{dx}[erf(\alpha x)] = \frac{2}{\sqrt{\pi}}\int_0^{\alpha x}\frac{\partial}{\partial x}(e^{-t^2})dt + f(\alpha x, x)\frac{d}{dx}(\alpha x) - 0$$

$$= \frac{2\alpha}{\sqrt{\pi}}e^{-\alpha^2 x^2}.$$

4. *Integral of error function:* We have

$$\int_0^t erf(\alpha x)dx = t\,erf(\alpha t) + \frac{1}{\alpha\sqrt{\pi}}[e^{-\alpha^2 t^2} - 1]. \tag{1.116}$$

Integrating the left hand side by parts, we obtain

$$\int_0^t 1\cdot erf(\alpha x)\,dx = [x\,erf(\alpha x)]_0^t - \int_0^t x\frac{d}{dx}[erf(\alpha x)]dx$$

$$= t\,erf(\alpha t) - \frac{2\alpha}{\sqrt{\pi}}\int_0^t x e^{-\alpha^2 x^2}\,dx$$

using Eq. (1.112). Let $\alpha^2 x^2 = u$. Then, $2\alpha^2 x\,dx = du$ or $x\,dx = du/(2\alpha^2)$. Hence,

$$\int_0^t erf(\alpha x)\,dx = t\,erf(\alpha t) - \left(\frac{2\alpha}{\sqrt{\pi}}\right)\left(\frac{1}{2\alpha^2}\right)\int_0^{\alpha^2 t^2} e^{-u}\,du$$

$$= t\,erf(\alpha t) + \frac{1}{\alpha\sqrt{\pi}}\left[e^{-u}\right]_0^{\alpha^2 t^2}$$

$$= t\,erf(\alpha t) + \frac{1}{\alpha\sqrt{\pi}}\left[e^{-\alpha^2 t^2} - 1\right].$$

Exercise 1.3

In problems **1** to **25**, discus the convergence or divergence of the given improper integral. Find its value if it exists.

1. $\int_0^\infty \frac{dx}{4+x}$.

2. $\int_2^\infty \frac{\ln x}{x}\,dx$.

3. $\int_3^\infty \frac{dx}{x^2+2x}$.

4. $\int_0^\infty \frac{x\,dx}{x^4+1}$.

5. $\int_0^\infty x^2 e^{-ax}\,dx,\ a > 0$.

6. $\int_0^\infty e^{-ax}\sin bx\,dx,\ a > 0$.

7. $\int_1^\infty x e^{-x^2}\,dx$.

8. $\int_{-\infty}^\infty \frac{dx}{x^2+2x+2}$.

9. $\int_{-\infty}^\infty \frac{dx}{e^x+e^{-x}}$.

10. $\int_0^\infty e^{-x}\,dx$.

11. $\int_0^\infty \frac{x^{p-1}}{1+x}\,dx,\ 0 < p < 1$.

12. $\int_1^\infty \frac{x+4}{x^{3/2}}\,dx$.

13. $\int_0^\infty \frac{dx}{x^3+1}$.

14. $\int_1^3 \frac{dx}{x\ln x}$.

15. $\int_0^4 \frac{dx}{x^2-2x-8}$.

16. $\int_0^{\pi/2} \frac{dx}{\cos x}$. 17. $\int_0^{2} \ln x \, dx$. 18. $\int_0^1 \frac{x}{\sqrt{1-x^2}} dx$.

19. $\int_{-1}^{1} \frac{dx}{x^4}$. 20. $\int_0^1 \frac{dx}{\sqrt{x+x^3}}$. 21. $\int_1^3 \frac{\sqrt{x}}{\ln x} dx$.

22. $\int_0^{\pi/2} \frac{\sin^n x}{x^m} dx$. 23. $\int_2^{\infty} \frac{\sin x}{x(\ln x)^2} dx$. 24. $\int_0^{\pi} \frac{\cos x}{\sqrt{x}} dx$.

25. $\int_0^{\infty} \frac{x^p}{1+x^q} dx$, (i) $q \ge 0$, (ii) $q < 0$.

In problems **27** to **40**, evaluate the integrals using the Beta and Gamma functions.

26. $\int_0^{\pi/2} \frac{dx}{\sqrt{\sin x}}$. 27. $\int_0^{\pi/2} \sin^2\theta \cos^4\theta \, d\theta$. 28. $\int_0^{\pi/2} \sin^3\theta \cos^5\theta \, d\theta$.

29. $\int_0^{\pi/2} \cos^m \theta \, d\theta$, m integer. 30. $\int_0^{a} x\sqrt{a^3 - x^3} \, dx$. 31. $\int_0^1 \frac{dx}{\sqrt[3]{1-x^3}}$.

32. $\int_0^1 x^n (\ln x)^m \, dx$. 33. $\int_0^a \frac{x^{3/2}}{\sqrt{a^2 - x^2}} dx$. 34. $\int_0^1 \frac{dx}{\sqrt{-\ln x}}$.

35. $\int_0^1 x^k (1-x)^{n-k} dx, k > 0$. 36. $\int_0^{\infty} \frac{dx}{1+x^4}$. 37. $\int_0^{\infty} \frac{x^a}{a^x} dx$, $a > 1$.

38. $\int_0^{\infty} t^k e^{-st} dt, s > 0, k > 0$. 39. $\int_0^{\infty} t^4 e^{-2t^2} dt$. 40. $\int_0^{\infty} x^{1/3} e^{-x^2} dx$.

Establish the following results.

41. $\int_0^p x^m (p^q - x^q)^n dx = \left(\frac{p^{m+nq+1}}{q}\right) \beta\left(n+1, \frac{m+1}{q}\right)$, m, n, p, q are positive constants.

42. $\int_a^b (x-a)^{m-1} (b-x)^{n-1} dx = (b-a)^{m+n-1} \beta(m, n)$, m, n, a, b are positive constants.

43. $\int_{-\infty}^{\infty} \frac{e^{mx}}{ae^{nx} + b} dx = \frac{\pi}{n}\left(\frac{b}{a}\right)^{m/n} \left[\frac{1}{b \sin(m\pi/n)}\right]$, a, b, m, n are positive constants.

44. $\int_{-1}^{1} (1-x^2)^n dx = \frac{2^{2n+1}(n!)^2}{(2n+1)!}$, n is positive integer.

45. $\int_0^m x^n \left(1 - \frac{x}{m}\right)^{m-1} dx = m^{n+1} \beta(m, n+1)$, m, n, are positive constants.

46. $\int_0^{\infty} x^m e^{-\alpha x^n} dx = \frac{1}{n\alpha^{(m+1)/n}} \Gamma\left(\frac{m+1}{n}\right)$, m, n, α are positive constants.

47. $\int_0^{\infty} e^{-mx}(1-e^{-x})^n\,dx = \beta\,(m, n+1)$, m, n, are positive constants.

48. $\int_0^1 x^m(\ln x)^n\,dx = \dfrac{(-1)^n\,n!}{(m+1)^{n+1}}$, $m > 1$ and n is a positive integer.

49. $\int_0^{\infty} \dfrac{\sin x}{x^p}\,dx = \dfrac{p}{2\Gamma(p)\sin(p\pi/2)}$, $0 < p < 1$, $\left(\text{given that } \int_0^{\infty} \dfrac{x^{p-1}}{1+x}\,dx = \dfrac{\pi}{\sin p\pi}\right)$.

50. For large n, $n! \approx \sqrt{2\pi n}\; n^n e^{-n}$ (*Stirling's formula*).

51. $\int_0^1 \dfrac{x^{m-1}(1-x)^{n-1}}{(a+bx)^{m+n}}\,dx = \dfrac{1}{a^n(a+b)^m}\,\beta(m, n)$.

Using the concept of differentiation of integrals (assuming that the differentiation is valid) evaluate the following integrals:

52. $\int_0^{\infty} \dfrac{e^{-ax}-e^{-bx}}{x}\,dx$, $a > 0$, $b > 0$.

53. $\int_0^1 \dfrac{x^a - x^b}{\log x}\,dx$, $a > b > -1$.

54. $\int_0^1 x^n(\log x)^k\,dx$, n any integer > -1.

55. $\int_0^{\pi/2\alpha} \alpha \sin \alpha x\,dx$.

56. $\int_{-\infty}^{\infty} x^2 e^{-x^2}\,dx$, where $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$

57. $\int_0^{\alpha^3} \cot^{-1}(x/\alpha^3)\,dx$.

58. $\int_0^{\pi} \dfrac{\cos x}{(a+b\cos x)^3}\,dx$, given that $\int_0^{\pi} \dfrac{dx}{a+b\cos x} = \dfrac{\pi}{\sqrt{a^2-b^2}}$, $a > b > 0$,

59. $\int_0^{\infty} e^{-x^2-(a^2/x^2)}\,dx$, $a > 0$, given that $\int_0^{\infty} e^{-x^2}\,dx = \dfrac{\sqrt{\pi}}{2}$,

60. $\int_0^{\pi/2} \log(1-\alpha^2\sin^2 x)\,dx$, $|x| < 1$.

61. $\int_0^{\infty} \dfrac{dx}{(x^2+1)^{n+1}}$, n any positive integer.

62. Show that $\dfrac{d}{dx}[erfc\,(\alpha x)] = -\dfrac{2\alpha}{\sqrt{\pi}}\,e^{-\alpha^2x^2}$.

63. Show that $erf\,(x) = \dfrac{2}{\sqrt{\pi}}\left[x - \dfrac{x^3}{3(1!)} + \dfrac{x^5}{5(2!)} - \dfrac{x^7}{7(3!)} + \ldots\right]$.

64. Show that $\int_0^t erfc\,(\alpha x)\,dx = t\,erfc\,(\alpha t) - \dfrac{1}{\alpha\sqrt{\pi}}\,[e^{-\alpha^2t^2} - 1]$.

65. Show that $\int_0^{\infty} e^{-t^2-2\alpha t}\,dt = \dfrac{\sqrt{\pi}}{2}\,e^{\alpha^2}\,[1 - erf\,(\alpha)]$.

66. The relation $\int_0^{\infty} e^{-x^2}\cos(2bx)\,dx = \dfrac{\sqrt{\pi}}{2}\,e^{-b^2}$ is given (see Example 13.40). Deduce the result for $\int_0^{\infty} e^{-\alpha^2x^2}\cos(px)\,dx$. Integrate this result with respect to p, taken as a parameter, from $p = 0$ to $p = s$ and show that $\int_0^{\infty} e^{-\alpha^2x^2}\left(\dfrac{\sin(sx)}{x}\right)dx = \dfrac{\pi}{2}\,erf\left(\dfrac{s}{2\alpha}\right)$.

Exercise 1.2

1. 29/6
2. 1/6.
3. 1/6.
4. $4ab \tan^{-1}(a/b)$.
5. 1/6.
6. $(e^2 - 5)/(4e)$.
7. 4/3.
8. $3\pi ab/8$.
9. $9\pi a^2/2$.
10. 7/120.
11. $a^2(4 - \pi)/2$.
12. $a^2[3\sqrt{3} - \pi]/3$.
13. $a^2(8 - \pi)/4$.
14. $(13\sqrt{13} - 8)/3$.
15. $3a/2$.
16. 123/32.
17. $2a\pi/3$.
18. 2.
19. $\ln(e + e^{-1})$.
20. $4a$.
21. $8a$.
22. $\sqrt{5}(e^2 - 1)/2$.
23. $\ln(1 + \sqrt{2})$.
24. 8.
25. $[f(r_2) - f(r_1)]/a$, where $f(r) = \frac{r}{2}\sqrt{a^2 + r^2} + \frac{a^2}{2} \ln [r + \sqrt{a^2 + r^2}]$.
26. $(r_2 - r_1)\sqrt{1 + b^2}/b$.
27. For any given x, the cross-section is an ellipse with semi-manor and semiminor axes as $(b/a)\sqrt{a^2 - x^2}$ and $(c/a)\sqrt{a^2 - x^2}$. Area of cross-section is $\pi bc(a^2 - x^2)/a^2$; volume $= 4\pi abc/3$.
28. Area of cross-section is $4(a^2 - x^2)$; volume $= 16a^3/3$.
29. Area of cross-section is $2(a^2 - x^2)$; volume $= 8a^3/3$.
30. Area of cross-section is $\sqrt{3}\cos^2 x/4$; volume $= \pi\sqrt{3}/16$.
31. $\pi^2/4$.
32. 8π.
33. $40\pi/3$.
34. $117\pi/5$.
35. $16\pi a^3/105$.
36. $\pi/2$.
37. $8\pi/3$.
38. $\pi/6$.
39. $\pi/6$.
40. $\pi ba^2/3$.
41. $2\pi^2 a^2 b$.
42. $4\pi^2 ab$.
43. $2\pi b^2 + \frac{2\pi ba^2}{\sqrt{a^2 - b^2}} \sin^{-1}\left[\frac{\sqrt{a^2 - b^2}}{a}\right]$.
44. $2\pi a^2 + \frac{2\pi ab^2}{\sqrt{a^2 - b^2}} \ln\left[\frac{a + \sqrt{a^2 - b^2}}{b}\right]$.
45. $24783\pi/1024$.
46. $2489\pi/192$.
47. $64\pi a^2/3$.
48. $6\pi a^2/5$.
49. $2\sqrt{2}\,\pi(e^{\pi} - 2)/5$.
50. $32\pi a^2/5$.

Exercise 1.3

1. Diverges.
2. Diverges.
3. Converges, $[\ln(5/3)]/2$.
4. Converges, $\pi/4$.
5. Converges, $2/a^3$.
6. Converges, $b/(a^2 + b^2)$.
7. Converges, $1/(2e)$.
8. Converges, π.
9. Converges, $\pi/2$.
10. Converges, 1.
11. Converges, $0 < p < 1$.
12. Diverges.
13. Converges.
14. Diverges.
15. Diverges.
16. Diverges.
17. Converges, $2(\ln 2 - 1)$.
18. Converges, 1.
19. Diverges.
20. Converges.
21. Let $g(x) = 1/(x - 1)$, diverges.
22. Let $g(x) = 1/x^{m-n}$, converges for $m < n + 1$.
23. Absolutely convergent.
24. Absolutely convergent.

25. (i) For arbitrarily large x, the integrand behaves like x^{p-q}, the integral will converge if $p - q + 1 < 0$. In the neighborhood of $x = 0$, the integrand behaves like x^p, the integral will converge if $p + 1 > 0$. Therefore, for convergence $-1 < p < q - 1$, $q > 0$.

(ii) At ∞, the integrand behaves like x^p, the integral will converge if $p + 1 < 0$. At 0, the integrand behaves like x^{p-q}. The integral will converge if $p - q + 1 > 0$. Therefore, for convergence $q - 1 < p < 1$, $q < 0$.

26. $\frac{1}{2}\beta\left(\frac{1}{4}, \frac{1}{2}\right)$. **27.** $\frac{\pi}{32}$. **28.** $\frac{1}{24}$. **29.** $\frac{1}{2}\beta\left(\frac{1}{2}, \frac{m+1}{2}\right)$.

30. Let $x = a \sin^{2/3} \theta$, $\frac{1}{3} a^{7/2} \beta\left(\frac{2}{3}, \frac{3}{2}\right)$. **31.** $x = \sin^{2/3} \theta$, $\frac{1}{3}\beta\left(\frac{1}{3}, \frac{2}{3}\right)$.

32. Let $x = e^{-t}$, $\frac{(-1)^m}{(n+1)^{m+1}} \Gamma(m + 1)$. 33. Let $x = a \sin \theta$, $\frac{1}{2} a^{3/2} \beta\left(\frac{5}{4}, \frac{1}{2}\right)$.

34. $x = e^{-u}$, $\sqrt{\pi}$. **35.** Let $x = \sin^2\theta$, $\beta(k + 1, n - k + 1)$.

36. Let $x^2 = \tan \theta$, $\frac{1}{4}\beta\left(\frac{1}{4}, \frac{3}{4}\right) = \frac{1}{4}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \frac{\pi}{2\sqrt{2}}$.

37. Write $a^x = e^{x \ln a}$ and let $x \ln a = t$, $\frac{\Gamma(a+1)}{(\ln a)^{n+1}}$.

38. Let $st = T$, $\frac{\Gamma(k+1)}{s^{k+1}}$. **39.** Let $2t^2 = T$, $3\sqrt{2\pi}/64$. **40.** Let $x = \sqrt{t}$, $\frac{1}{2}\Gamma\left(\frac{2}{3}\right)$

41. Let $x^q = p^q y$. **42.** Let $x = b \sin^2\theta + a \cos^2 \theta$. **43.** Let $ae^{mx} = bt$.

44. Let $x = \sin \theta$ and use Eq. (1.79). **45.** Let $x = my$.

46. Let $\alpha x^n = t$. **47.** Let $1 - e^{-x} = t$. **48.** Let $x = e^{-y}$, then let $(m + 1) y = t$.

49. We have $\frac{1}{x^p} = \frac{1}{\Gamma(p)}\int_0^\infty y^{p-1} e^{-xy}\, dy$. Substitute for $1/x^p$ and integrate $\int_0^\infty \sin x\, e^{-xy}\, dx$. Then, let $y = \sqrt{t}$.

50. We have $\Gamma(n + 1) = \int_0^\infty x^n e^{-x}\, dx = \int_0^\infty e^{n\ln x - x}\, dx$. The function $n \ln x - x$ has maximum at $x = n$. Write $x = n + y$. We obtain

$$\Gamma(n + 1) = n^n e^{-n} \int_{-n}^{\infty} e^{n\ln[1+(y/n)]-y}\, dy$$

Expand $\ln [1 + (y/n)]$, $|y/n| < 1$, approximate to first term and let $y = \sqrt{2n}\, u$.

51. Let $x/(a + bx) = z/(a + b)$.

In problems **52** to **61**, let the given integral be denoted by ϕ, unless mentioned otherwise.

52. Find $d\phi/da$, integrate with respect to a, use $\phi(b) = 0$. We get $\phi(a) = \ln (b/a)$.

53. Find $d\phi/da$, integrate with respect to a, use $\phi(b) = 0$. We get $\phi(a) = \ln [(a + 1)/(b + 1)]$.

54. Use $\int_0^1 x^n dx = 1/(n+1)$ and differentiate it k times with respect to the parameter n.

We get $\phi = [(-1)^k k!]/(n+1)^{k+1}$.

55. Use the Leibniz rule to get $\phi'(\alpha) = 0$. Integrate and use $\phi(0) = 0$. We obtain $\phi(\alpha) = 0$.

56. In $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$, let $x = \alpha y$ and define $\phi(\alpha) = \int_{-\infty}^{\infty} e^{-\alpha^2 y^2} dy = \dfrac{\sqrt{\pi}}{\alpha}$. Differentiate with respect to α and set $\alpha = 1$. We get $\phi = \sqrt{\pi}/2$.

57. Find $d\phi/d\alpha$, integrate with respect to α, use $\phi(1) = (\pi/4) + (\ln 2)/2$. We get $\phi(\alpha) = [(\pi + 2\ln 2)\,\alpha^2]/4$.

58. Differentiating $\int_0^{\pi} \dfrac{dx}{a + b\cos x} = \dfrac{\pi}{\sqrt{a^2 - b^2}}$ with respect to b and readjusting the terms, we get

$$\int_0^{\pi} \frac{dx}{(a + b\cos x)^2} = \frac{\pi a}{(a^2 - b^2)^{3/2}}.$$

Differentiate again with respect to b. We get $\phi = -3\pi ab/[2(a^2 - b^2)^{5/2}]$.

59. Find $\dfrac{d\phi}{da}$ and substitute $x = \dfrac{a}{y}$. We get $\dfrac{d\phi}{da} = -2\phi$. Use $\phi(0) = \int_0^{\infty} e^{-x^2} dx = \dfrac{\sqrt{\pi}}{2}$. We obtain

$$\phi(a) = \frac{\sqrt{\pi}}{2} e^{-2a}.$$

60. Find $\dfrac{d\phi}{d\alpha}$. Integrate and use $\phi(0) = 0$. We get $\phi(\alpha) = \pi \ln [\{1 + \sqrt{1 - \alpha^2}\}/2]$.

61. Consider $\phi(\alpha) = \int_0^{\infty} \dfrac{dx}{x^2 + \alpha^2} = \dfrac{\pi}{2\sqrt{\alpha}}$. Differentiate n times with respect to α and set $\alpha = 1$. We get

$$\phi(\alpha) = \frac{\pi}{2}\left[\frac{(2n)!}{2^{2n}(n!)^2}\right].$$

62. Use Eqs. (1.114) and (1.115). With $f(t, x) = (2/\sqrt{\pi})e^{-t^2}$ we get

$$\frac{d}{dx}[erfc(\alpha x)] = 0 + 0 - f(\alpha x, x)\frac{d}{dx}(\alpha x) = -\frac{2\alpha}{\sqrt{\pi}} e^{-\alpha^2 x^2}.$$

63. $$erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2} dt = \frac{2}{\sqrt{\pi}}\int_0^x \left[1 - \frac{t^2}{1!} + \frac{t^4}{2!} - \frac{t^6}{3!} + \cdots\right] dt = \frac{2}{\sqrt{\pi}}\left[x - \frac{x^3}{3(1!)} + \frac{x^5}{5(2!)} - \frac{x^7}{7(3!)} + \cdots\right]$$

64. Integrate by parts and use the result of problem 61.

65. $$\int_0^{\infty} e^{-t^2 - 2\alpha t} dt = \int_0^{\infty} e^{-(t+\alpha)^2 + \alpha^2} dt = e^{\alpha^2}\int_0^{\infty} e^{-(t+\alpha)^2} dt$$

$$= e^{\alpha^2}\int_{\alpha}^{\infty} e^{-u^2} du = \frac{\sqrt{\pi}}{2} e^{\alpha^2} erfc(\alpha) = \frac{\sqrt{\pi}}{2} e^{\alpha^2}[1 - erf(\alpha)].$$

66. Let $x = \alpha u$.

$$I = \int_0^\infty e^{-\alpha^2 u^2} \cos(2b\alpha\, u)\,(\alpha\, du) = \frac{\sqrt{\pi}}{2}\, e^{-b^2}$$

or

$$\int_0^\infty e^{-\alpha^2 u^2} \cos(2b\alpha\, u)\, du = \frac{\sqrt{\pi}}{2\alpha}\, e^{-b^2}.$$

Integrating with respect to p, from $p = 0$ to $p = s$, we get

$$\int_0^\infty e^{-\alpha^2 u^2} \left[\frac{\sin(pu)}{u}\right]_0^s du = \frac{\sqrt{\pi}}{2\alpha} \int_0^s e^{-p^2/(4\alpha^2)}\, dp$$

or

$$\int_0^\infty e^{-\alpha^2 u^2} \left(\frac{\sin(su)}{u}\right) du = \frac{\sqrt{\pi}}{2\alpha} \int_0^{s/(2\alpha)} e^{-u^2}\,(2\alpha\, du) = (\sqrt{\pi})\left(\frac{\sqrt{\pi}}{2}\right) erf\left(\frac{s}{2\alpha}\right) = \frac{\pi}{2}\, erf\left(\frac{s}{2\alpha}\right)$$

Replace the dummy variable u by x.

Exercise 1.4

1. Point of inflection: (1, 9); concave upward in the interval $(1, \infty)$; concave downward in the interval $(-\infty, 1)$.
2. Points of inflection: (0, 0) and (1/2, – 1/6); concave upward in the intervals $(-\infty, 0)$ and $(1/2, \infty)$; concave downward in the interval (0, 1/2).
3. Concave upward in $(-\infty, \infty)$.
4. Concave downward in $(0, \infty)$.
5. Concave upward in the interval $(3, \infty)$; concave downward in the interval $(-\infty, 3)$.
6. Points of inflection: $(-2\sqrt{3}, -3\sqrt{3}/2)$, (0, 0) and $(2\sqrt{3}, 3\sqrt{3}/2)$; concave upward in the intervals $(-\infty, -2\sqrt{3})$ and $(0, 2\sqrt{3})$; concave downward in the intervals $(-2\sqrt{3}, 0)$ and $(2\sqrt{3}, \infty)$.
7. Point of inflection: (0, 0); concave upward in the interval $(0, \pi)$; concave downward in the interval $(-\pi, 0)$.
8. Points of inflection: $(1, 2e^{-1})$, $(3, 10e^{-3})$; concave upward in the intervals $(-\infty, 1)$ and $(3, \infty)$; concave downward in the interval (1, 3).
9. Point of inflection: $(e^{-5/6}, -(5/6)\, e^{-5/2})$; concave upward in the interval $(e^{-5/6}, \infty)$; concave downward in the interval $(0, e^{-5/6})$.
10. Point of inflection: (0, 0); concave upward in the interval $(0, \infty)$; concave downward in the interval $(-\infty, 0)$.
11. 2, 1/2, (3, 3/2).
12. $6/(13\sqrt{13}\,a)$, $(13\sqrt{13}\,a/6)$, $(-11a/2, 16a/3)$.
13. $4\sqrt{2}/a$, $a/4\sqrt{2}$, $(7a/8, 7a/8)$.
14. $1/a$, a, $(0, 2a)$.
15. $\sqrt{2}/(8a)$, $8a/\sqrt{2}$, $(-2a, 5a)$.
16. $1/\sqrt{2}$, $\sqrt{2}$, $(-1, 1)$.
17. 2, 1/2, (0, 3/2).
18. 1, 1, $(\pi/2, 0)$.
19. $4/(a\pi)$, $a\pi/4$, $(a/\sqrt{2}, a/\sqrt{2})$.
20. $1/a$, a, $(0, 2a)$.