2. Solving nonlinear equations

Goal: Find a root (a solution) of $f(x) = 0$

Examples: $x^2 + 4x^2 - 10 = 0$ nonlinear

$$e^x - x = 0$$
$$\cos x = x + 1$$

Simple case: $ax^2 + bx + c = 0$

$$x_1, x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Methods (Iterative methods)

Closed methods
(Bisection method)
(False position method)

Open methods
Newton-Raphson
Secant method

Theorem (Intermediate value theorem)

Let f be continuous on $[a,b]$ ($a < b$)

If $f(a) f(b) < 0$ then there exists $x = c$

such that $f(c) = 0$

Bisection method

Input $[a,b]$ such that $f(a) f(b) < 0$, $\varepsilon = 10^{-6}$, x^*

1. Compute $c = \frac{a+b}{2} = f(c)$
2. If $|f(c)| < \varepsilon$
3. If $f(a) f(c) < 0$, $b = c$ else $a = c$
4. Repeat

$[a,b]$

EX. Carry out two iterations by using the bisection method to solve $x^3 + 4x^2 - 10 = 0$, $1 \le x \le 2$

Solution: $[a,b] = [1, 2]$

$f(x) = x^3 + 4x^2 - 10$

$f(1) = 1 + 4 - 10 = -5$

$f(2) = 8 + 16 - 10 = 14$

$f(a) f(b) = f(1) f(2) < 0$ so $x \in (1, 2)$ such that $f(x) = 0$

$c = \frac{a+b}{2} = \frac{1+2}{2} = 1.5$

$f(c) = f(1.5) = 2.375$ $f(a) = -0.875$

$f(a) f(c) = f(1) f(1.5) = (-5)(2.375) < 0$, $c = 1.5$, $b = 1.5$

$[a,b] = [1, 1.5]$ End of 1st iter.

$a = c = 1.5$, $b = 1 + 1.5 = 1.25$

$f(c) = f(1.25) = 1.125^3 + 4(1.25)^2 - 10 = -1.7968$

$f(a) f(c) = (-5)(-1.7968) > 0$, $a = 1.25$, $b = 1.5$

$a = c = 1.25$, $b = 1.5$

$[1.25, 1.5]$ End of 2nd iter. $1.25 < c < 1.5$

2. Method of false position

$(a, f(a)), (b, f(b)) =$

The eqn of the line joining the two pts

$$y - f(a) = \frac{f(b) - f(a)}{b - a}(x - a)$$

$$0 - f(a) = \frac{f(b) - f(a)}{b - a}(c - a)$$

$$c - a = -f(a) \cdot \frac{(b-a)}{f(b) - f(a)}$$

$$c = a - \frac{f(a)(b-a)}{f(b) - f(a)}$$

$$c(f(b) - f(a)) = a(f(b) - f(a))$$

$$\boxed{\frac{a f(b) - b f(a)}{f(b) - f(a)}}$$

EX. Carry out two iterations by the false position method to approximate the root of $x^3 + 4x^2 - 10 = 0$

Solution: $f(x) = x^3 + 4x^2 - 10$ $1 \le x \le 2$

$a = 1, b = 2$

$f(a) = f(1) = -5$, $f(b) = f(2) = 14$

$f(a) f(b) = (-5)(14) < 0$

Iter 1: $c = \frac{a f(b) - b f(a)}{f(b) - f(a)}$

$$= \frac{(1)(14) - (2)(-5)}{14 - (-5)} = 1.263158$$

$f(c) = f(1.263158) = -1.602274$

$f(a) f(c) = (-5)(-1.602274) > 0$ $a = 1.263158$ $b = 2$

$a = c = 1.263158 < b = 2$

$[a, b] = [1.263158, 2]$ 1st iter.

Iter 2: $c = \frac{a f(b) - b f(a)}{f(b) - f(a)}$

$$= \frac{(1.263158)(14) - 2(-1.602274)}{14 - (-1.602274)}$$

$$\boxed{c = 1.338828}$$

$f(c) = f(1.338828) = -0.430465$

$f(a) f(c) = (-1.602274)(-0.430465) > 0$

$a = c = 1.3388$

$[a, b] = [1.338828, 2]$.

End of 2nd iter. $x_0 = 1.3662$