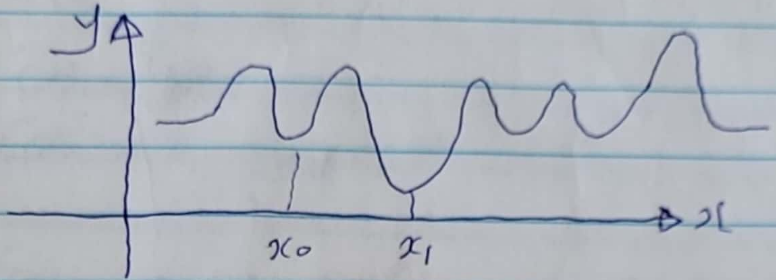
Lecture # 1

Optimization problem

Function of one variable

$$y = f(x) \qquad x \in R$$



Definition:

local min: x_0 is a local min

$$f(x_0) < f(x) \qquad \forall x \in (x_0 - \varepsilon, x_0 + \varepsilon)$$

Global min: x_1 is the global min

$$f(x_1) < f(x) \qquad \forall x \in R$$

Theorem:

If f is continuous on a closed interval $[a, b]$ then f attains a global min and a global max in $[a, b]$

Theorem: If f has a local max or a local min at c and if $f'(c)$ exists then $f'(c) = 0$

Definition: critical point

If $c \in D$, D is the domain of f, then c is a critical point if $f'(c) = 0$

Ex find critical points of $f(x) = 3x - x^2$

Solution

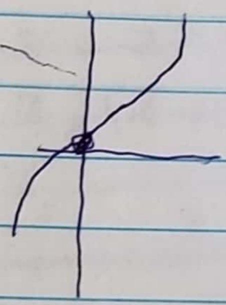
$f(x) = 3x - x^2$

$f'(x) = 3 - 2x$

$f'(x) = 0 \Rightarrow x = 1.5$

Note $f'(c) = 0$ is not enough to say that c is a min point or a max point

Ex $f(x) = x^3$



$f'(x) = 3x^2$

$f'(x) = 0 \Rightarrow x = 0$

but $x = 0$ is not a max point or a min point.

Theorem

If f has a local max or min at $x = x_0$ then x_0 is a critical point of f.

the converse is not true

What does f' say about f?

(1) If $f'(x) > 0$ on an interval, then f is increasing in this interval

(2) If $f'(x) < 0$ on an interval, then f is decreasing in this interval

Ex Find the intervals of increase and the intervals of decrease

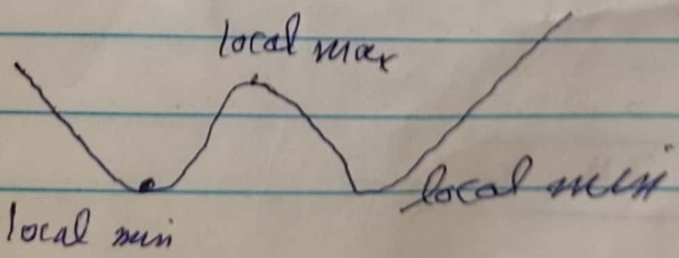
$$f(x) = 3x^4 - 4x^3 - 12x^2 + 5$$

Solution

$$f'(x) = 12x^3 - 12x^2 - 24x$$
$$= 12x(x^2 - x - 2)$$
$$= 12x(x-2)(x+1)$$

$-2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4$

	$x<-1$	$-1<x<0$	$0<x<2$	$x>2$
$12x$	−	−	+	+
$(x-2)$	−	−	−	+
$(x+1)$	−	+	+	+
sign of $f'(x)$	−	+	−	+
shape of f	↘	↗	↘	↗



The first derivative test

Suppose that c is a critical point of a continuous function f

a) If f' changes from positive to negative at c then f has a local max at c

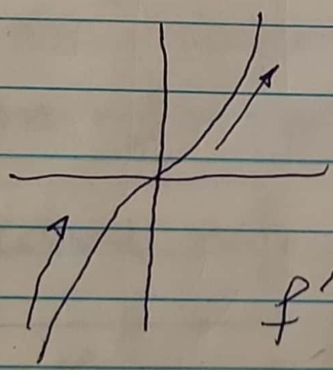
b) If f' changes from negative to positive at c then f has a local min at c

c) If f' does not change sign at c (ie f' is positive on both sides of c or negative on both sides of c) then f has no max no min at c

Ex Consider $f(x) = x^3$

$$f'(x) = 3x^2$$

$f'(x) = 0 \Rightarrow x = 0$ is a critical pt



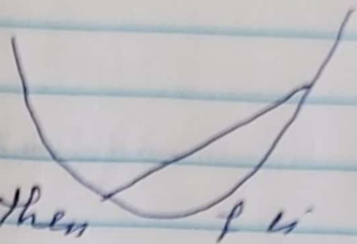
$$f'(x) = 3x^2$$
$$f'(-0.1) > 0$$
$$f'(0.1) > 0$$

f' is positive on both sides of $x=0$

$\therefore$ $x=0$ is no max no min

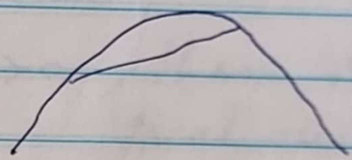
Convex function

Let f be a continuous fn defined on an interval then f is



Convex: If every line sigment joining two points on the graph is never below the graph

Concave: if every line sigment joining two points on the graph never lies above the graph



What does f'' say about f?

Let f be a function of one variable defined on an interval I

f is convex if $f''(x) \geq 0 \quad \forall x \in I$

f is concave if $f'' \leq 0 \quad \forall x \in I$

EX discuss the convexity of $f(x) = x^3 - x^2$

soln

$f(x) = x^3 - x^2$

$f'(x) = 3x^2 - 2x$

$f''(x) = 6x - 2 \quad = 6\left(x - \frac{1}{3}\right)$

So f is convex on $[\frac{1}{3}, \infty)$ and concave $(-\infty, \frac{1}{3}]$

The second derivative test

Suppose f'' is continuous at c

1) If $f'(c) = 0$ and $f''(c) > 0$, then c is a local mini of f

2) If $f'(c) = 0$ and $f''(c) < 0$, then c is a local max of f

3) If $f'(c) = 0$ and $f''(c) = 0$ no conclusion can be made about c

Ex discuss the convexity of
$y = x^4 - 4x^3$

Solution

$f(x) = x^4 - 4x^3$

$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$

$f''(x) = 12x^2 - 24x = 12x(x-2)$

Critical points $x = 0$, $x = 3$

then $f(0) = 0$