Lecture (2)

In this Lecture, we consider the function f of two variables, x_1 and x_2, where $x_1 \in \mathbb{R}$ and $x_2 \in \mathbb{R}$.

Consider $\quad \min f(x_1, x_2)$

Critical point:

The critical point is the point that satisfies

$$\frac{\partial f}{\partial x_1} = 0$$

and

$$\frac{\partial f}{\partial x_2} = 0$$

Example

Find the critical point of

$$f(x,y) = x^2 - 2xy - 2x + 2y^2$$

Soln.:

$$\frac{\partial f}{\partial x} = 0 \implies 2x - 2y - 2 = 0 \quad (1)$$

$$\frac{\partial f}{\partial y} = 0 \implies -2x + 4y = 0 \quad (2)$$

$(1) + (2) \implies \quad 2y - 2 = 0 \implies 2y = 2 \implies y = 1$

in $(2) \implies \quad -2x + 4 = 0 \implies -2x = -4 \implies x = 2$

$\therefore$ The critical point is $(2, 1)$.

Let (c,d) be a critical point

① If $f_{xx}(c,d)\, f_{yy}(c,d) - f_{xy}^2(c,d) > 0$

and $f_{xx}(c,d) > 0$

$\Rightarrow$ the critical point (c,d) is a local min

② If $f_{xx}(c,d)\, f_{yy}(c,d) - f_{xy}^2(c,d) > 0$

and $f_{xx}(c,d) < 0$

$\Rightarrow$ the critical point (c,d) is a local max

③ If $f_{xx}(c,d)\, f_{yy}(c,d) - f_{xy}^2(c,d) < 0$

$\Rightarrow$ (c,d) is a saddle point.

Ex Classify the critical point of the last example

Soln From the last example $(2,1)$ is a critical point.

$$f_{xx} = 2$$
$$f_{yy} = 4$$
$$f_{xy} = -2$$

$$D(x,y) = f_{xx} f_{yy} - f_{xy}^2 = 2\times 4 - (-2)^2 = 4 > 0$$

and $f_{xx} = 2 > 0$

$\therefore$ The critical point $(2,1)$ is a min point.

Example

Find all local min, local max, and saddle pts of $f(x,y) = 3x^2 + 6xy + 2y^2 + 4x + 2y + 15$

Solution

$f_x = 0 \Rightarrow 6x + 6y + 4 = 0$ ①

$f_y = 0 \Rightarrow 6x + 4y + 2 = 0$ ②

① − ② $\Rightarrow 2y + 2 = 0 \Rightarrow 2y = -2 \Rightarrow y = -1$

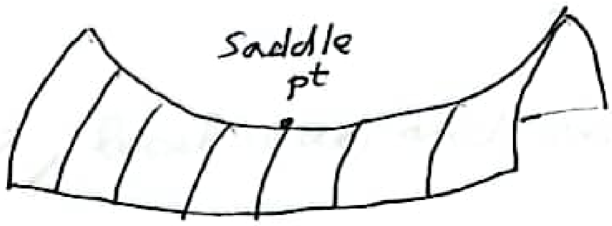
in ① $\Rightarrow 6x - 4 + 2 = 0 \Rightarrow 6x = 2 \Rightarrow x = \frac{1}{3}$

The critical point is $(\frac{1}{3}, -1)$

Now, $f_{xx} = 6$

$f_{yy} = 4$

$f_{xy} = 6$



$D = f_{xx} f_{yy} - f_{xy}^2$

$= 6 * 4 - (6)^2$

$= 24 - 36 = -12 < 0$

$\therefore$ The point is a saddle point

Example

Find all min, max, and saddle pts of $f(x,y) = x^2 + y^2 - 2x - 6y + 14$

Soln

$$f_x = 0 \Rightarrow 2x - 2 = 0 \Rightarrow x = 1$$
$$f_y = 0 \Rightarrow 2y - 6 = 0 \Rightarrow y = 3$$

we have one critical point $(1,3)$

$f_{xx} = 2$

$f_{yy} = 2$

$f_{xy} = 0$

$$D(x,y) = f_{xx} f_{yy} - f_{xy}^2 = 4 - 0 = 4 > 0$$
$$f_{xx} = 2 > 0$$

$\Rightarrow$ the point $(1,3)$ is a local min

Ex Find all local min, local max, and saddle points of

$$f(x,y) = x^4 + y^4 - 4xy + 1$$

Soln

$$f_x = 0 \Rightarrow 4x^3 - 4y = 0 \Rightarrow x^3 = y \quad (1)$$
$$f_y = 0 \Rightarrow 4y^3 - 4x = 0 \Rightarrow y^3 = x \quad (2)$$

from (1) $\quad x^9 = y^3 \quad$ use (2) $\quad x^9 = x$

$$\Rightarrow x^9 - x = 0 \Rightarrow x(x^8 - 1) = 0$$
$$\Rightarrow x(x^4 - 1)(x^4 + 1) = 0$$
$$\Rightarrow x(x^2 - 1)(x^2 + 1)(x^4 + 1) = 0$$
$$\Rightarrow x(x-1)(x+1)(x^2+1)(x^4+1) = 0$$

$\Rightarrow \quad x = 0 \; , \; x = 1 , \; x = -1$

substitute in ②, the critical points are

$(0,0) \; , \; (1,1) \; ,$ and $(-1,-1)$

Now

$$D(x,y) = f_{xx} \, f_{yy} - f_{xy}^2$$

$$= (12x^2)(12y^2) - (-4)^2$$

For the critical point $(1,1)$

$$D(1,1) = f_{xx}(1,1) \, f_{yy}(1,1) - f_{xy}^2(1,1)$$

$$= 12 * 12 - 16 = 144 - 16 > 0$$

$f_{xx}(1,1) = 12 > 0$

$\Rightarrow$ $(1,1)$ is a local min

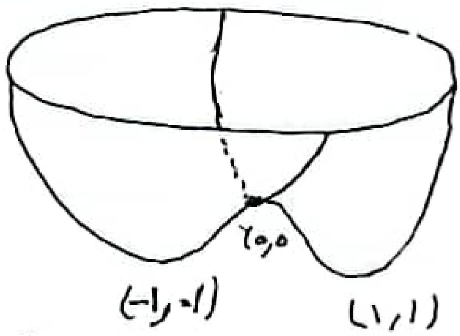
For the critical point $(-1,-1)$

$$D(-1,-1) = f_{xx}(-1,-1) \, f_{yy}(-1,-1) - f_{xy}^2(-1,-1)$$

$$= 12 * 12 - 16 > 0$$

$f_{xx}(-1,-1) = 12 > 0$

$\Rightarrow$ $(-1,-1)$ is a local min



For the critical point $(0,0)$

$$D(0,0) = f_{xx}(0,0) \, f_{yy}(0,0) - f_{xy}^2(0,0)$$

$$= 0 - (-4)^2 = -16 < 0$$

So, $(0,0)$ is a saddle point