Ill-conditioned Matrices

A condition number for a matrix A is given by cond(A)=$\| A \| \| A^{-1} \|$

If cond(A) > 1, then the matrix A is ill-condition, otherwise, it is well-condition

Consider systems

$$\begin{aligned} x+y&=2 \\ x+1.001y&=2 \end{aligned} \quad \text{and} \quad \begin{aligned} x+y&=2 \\ x+1.001y&=2.001 \end{aligned}$$

The system on the left has solution x = 2, y = 0 while the one on the right has solution x = 1, y = 1. The coefficient matrix is called ill-conditioned because a small change in the constant coefficients results in a large change in the solution

In the presence of rounding errors, ill-conditioned systems are inherently difficult to handle. When solving systems where round-off errors occur, one must avoid ill-conditioned systems whenever possible; this means that the usual row reduction algorithm must be modified. Consider the system

$$\begin{aligned} 0.001x + y &= 1 \\ x + y &= 2 \end{aligned}$$

We see that the solution is x = 1000/999 ≈ 1, y = 998/999 ≈ 1 which does not change much if the coefficients are altered slightly (condition number 4).

The usual row reduction algorithm, however, gives an ill-conditioned system. Adding a multiple of the first to the second row gives the system on the left below, then dividing by −999 and rounding to 3 places on 998/999 = .99899≈ 1.00 gives the system on the right:

$$\begin{aligned} 0.001x + y &= 1 \qquad \Rightarrow \qquad 0.001x + y = 1 \\ -999y &= -998 \qquad \Rightarrow \qquad y = 1 \end{aligned}$$

The solution for the last system is x = 0, y = 1 which is wildly inaccurate (and the condition number is 2002).

Gauss Elimination Method (GEM)

In matrix algebra, the solution vector x of the equation $Ax = b$ can be obtained by multiplying both sides by A^{-1} :

- a square system of linear equations in MATLAB one may use

$$x = \boldsymbol{inv}(A) * b$$

- The method of solving a system of linear equations by using ***matrix inverse*** is limited to small square systems, where the coefficients matrix must be non-singular.

- A more practical method called the LU factorization based on Gauss Elimination Method (restricted only for square systems) will be considered.

We can apply the elementary operations simply on the rows of the augmented matrix of the system $Ax=b$:

$$[A|b]=\begin{bmatrix} a_{11} & a_{12} & a_{13} & | b_1 \\ a_{21} & a_{22} & a_{23} & | b_2 \\ a_{31} & a_{32} & a_{33} & | b_3 \end{bmatrix} \xrightarrow[\frac{-a_{31}}{a_{11}}R_1+R_3]{\frac{-a_{21}}{a_{11}}R_1+R_2} \begin{bmatrix} a_{11} & a_{12} & a_{13} & | b_1' \\ 0 & a_{22}' & a_{23}' & | b_2' \\ 0 & a_{32}' & a_{33}' & | b_3' \end{bmatrix} \xrightarrow{\frac{-a_{32}'}{a_{22}'}R_2+R_3} \begin{bmatrix} a_{11} & a_{12} & a_{13} & | b_1'' \\ 0 & a_{22}' & a_{23}' & | b_2'' \\ 0 & 0 & a_{33}'' & | b_3'' \end{bmatrix}$$

Then we can get the solution of the original system by solving the equivalent triangular system by back substitution.

The LU Factorization Method

- The LU factorization for a square matrix A is viewed as a matrix representation of the Gauss elimination method.

- The LU factorization is based on converting a given square matrix Ainto the product of a lower triangular matrix $L$and an upper triangular matrix U, where $A = LU$.

- For an n$\times n$ coefficients matrix, GEM requires $O(n^3)$calculations for the row operations.

- In addition, it needs $O(n^2)$calculations for solving the final linear system by back substitution

LU-factorization

This method is a variant of the Gauss elimination method that decomposed a matrix as a product of a lower triangular matrix and an upper triangular matrix. This method is the most widely used method on computers for solving a linear system. The following linear system

$$
\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + \ldots\ldots a_{1n}x_n = b_1 \\
&a_{21}x_1 + a_{22}x_2 + \ldots\ldots a_{2n}x_n = b_2 \\
&. \\
&. \\
&a_{n1}x_1 + a_{n2}x_2 + \ldots\ldots a_{nn}x_n = b_n
\end{aligned}
$$

This system can be written as AX=b where

$$
A = \begin{bmatrix} a_{11} & a_{12} & .. & a_{1n} \\ a_{21} & a_{22} & .. & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & .. & a_{nn} \end{bmatrix} \qquad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}
$$

If we can decompose A to upper triangular matrix U and lower triangular matrix L .i.e $A=LU$

Then we can write the linear system as follows

Since $Ax=b$, them $LUx=b$. Let $Ux=z$, then we can obtain the vector Z by solving the system

$Lz=b$. Hence we can obtain the vector X by solving the system $Ux=z$

Now to decompose the matrix

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \xrightarrow[\frac{-a_{31}}{a_{11}}R_1+R_3]{\frac{-a_{21}}{a_{11}}R_1+R_2} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & a'_{32} & a'_{33} \end{bmatrix} \xrightarrow{\frac{-a'_{32}}{a'_{22}}R_2+R_3} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & 0 & a''_{33} \end{bmatrix}=U$$

The matrix L has the form

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{a_{21}}{a_{11}} & 1 & 0 \\ \frac{a_{31}}{a_{11}} & \frac{a'_{32}}{a'_{22}} & 1 \end{bmatrix}$$

Example: Using LU decomposition to find the solution of the following linear system

$$\begin{aligned} 6x_1 - 2x_2 - 4x_3 + 4x_4 &= 2 \\ 3x_1 - 3x_2 - 6x_3 + x_4 &= -4 \\ -12x_1 + 8x_2 + 21x_3 - 8x_4 &= 8 \\ -6x_1 \qquad\qquad - 10x_3 + 7x_4 &= -43 \end{aligned}$$

Solution: We can write A and obtained U as follows

$$A=\begin{bmatrix} 6 & -2 & -4 & 4 \\ 3 & -3 & -6 & 1 \\ -12 & 8 & 21 & -8 \\ -6 & 0 & -10 & 7 \end{bmatrix} \xrightarrow[\substack{\frac{12}{6}R_1+R_3 \\ \frac{6}{6}R_1+R_4}]{\frac{-3}{6}R_1+R_2} \begin{bmatrix} 6 & -2 & -4 & 4 \\ 0 & -2 & -4 & -1 \\ 0 & 4 & 13 & 0 \\ 0 & -2 & -14 & 11 \end{bmatrix} \xrightarrow[\frac{2}{-2}R_2+R_4]{\frac{-4}{-2}R_2+R_3} \begin{bmatrix} 6 & -2 & -4 & 4 \\ 0 & -2 & -4 & -1 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & -10 & 12 \end{bmatrix} \xrightarrow{\frac{10}{5}R_3+R_4} \begin{bmatrix} 6 & -2 & -4 & 4 \\ 0 & -2 & -4 & -1 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & 0 & 8 \end{bmatrix}=U$$

Then

$$L=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 3/6 & 1 & 0 & 0 \\ -12/6 & -4/2 & 1 & 0 \\ -6/6 & 2/2 & -10/5 & 1 \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ -2 & -2 & 1 & 0 \\ -1 & 1 & -2 & 1 \end{bmatrix}$$

Since L z=b, then

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ -2 & -2 & 1 & 0 \\ -1 & 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 8 \\ -43 \end{bmatrix}$$

$$z_1 = 2, \quad z_2 = -5, \quad z_3 = 2, \quad z_4 = -32$$

Next, we solve U x= z and then we has the value of x as follows

$$\begin{bmatrix} 6 & -2 & -4 & 4 \\ 0 & -2 & -4 & -1 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & 0 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ 2 \\ -32 \end{bmatrix}$$

Then $x_1 = 4.5, \quad x_2 = 6.9, \quad x_3 = -1.2, \quad x_4 = -4$

Do all square matrices have an LU factorization?

No, sometimes it is not possible to write a square matrix in the form of the product of lower and upper triangular matrices.

Reason: An invertible matrix A has an LU factorization if all its leading submatrices have non-zero determinants.

Iterative Methods

- GEM and the LU factorization are direct methods for solving systems of linear equations.
- Direct methods, in theory, give the exact solution within a finite number of steps.
- Direct methods stand in contrast to iterative methods(like root-finding methods for solving nonlinear equations).
- The iterative method starts with an initial guess to the solution and refine it at each iteration.
- This approximate solution under certain conditions on the coefficients matrix converges to the solution of the system.
- Iterative methods are suitable for large square linear systems.

Iterative method for solving linear system

A linear system Ax=b can be solved by using selected iteration scheme . This method using in large scale problem

Jacobi's method: in this method we start by initial value $(x_1^0, x_2^0, \ldots, x_n^0)$ to solve The linear system

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots\ldots a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots\ldots a_{2n}x_n &= b_2 \\
&\vdots \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \ldots\ldots a_{nn}x_n &= b_n
\end{aligned}
$$

By substituting from $(x_1^0, x_2^0, \ldots, x_n^0)$ in the right hand side of the following system

$$x_1^1 = \frac{1}{a_{11}}[b_1 - a_{12}x_2^0 - \ldots\ldots - a_{1n}x_n^0]$$

$$x_2^1 = \frac{1}{a_{22}}[b_2 - a_{21}x_1^0 \ldots\ldots - a_{2n}x_n^0]$$

$$\vdots$$

$$x_n^1 = \frac{1}{a_{nn}}[b_n - a_{n1}x_1^0 \ldots\ldots - a_{n,n-1}x_{n-1}^0]$$

We repeat this until we obtain approximate solution for the system Ax=b

Example: Using Jacobi's method to find approximate solution of the following linear system with starting point (0.5,1,1)

$$\begin{aligned} 10x_1 + 2x_2 - 3x_3 &= 5 \\ 2x_1 + 8x_2 + x_3 &= 6 \\ 3x_1 - x_2 + 15x_3 &= 12 \end{aligned} \quad \longrightarrow \quad \begin{aligned} x_1^1 &= \frac{1}{10}[5 - 2x_2^0 + 3x_3^0] \\ x_2^1 &= \frac{1}{8}[6 - 2x_1^0 - x_3^0] \\ x_3^1 &= \frac{1}{15}[12 - 3x_1^0 + x_2^0] \end{aligned}$$

$$x_1^1=\frac{1}{10}[5-2(1)+3(1)]=0.6$$

$$x_2^1=\frac{1}{8}[6-2(0.5)-(1)]=0.5$$

$$x_3^1=\frac{1}{15}[12-3(0.5)+1]=0.76667$$

Second iteration: begin by (0.6, 0.5,0.76667) we have

$$x_1^2=\frac{1}{10}[5-2(0.5)+3(0.76667)]=0.63$$

$$x_2^2=\frac{1}{8}[6-2(0.6)-0.76667]=0.504167$$

$$x_3^2=\frac{1}{15}[12-3(0.6)+0.5]=0.713333$$

Continue this iteration until iteration 8 where the solution approximate to (0.611831,0.508104,0.711507). The exact solution by Gauss elimination is to (0.611832,0.508104,0.711507).

Gauss-Seidel iterative method

Convergence of the Jacobi iterations is often slow. It seems reasonable to try speed up the iteration by making use of the improved estimate of the x_j as they are calculated

$$x_1^1 = \frac{1}{a_{11}}[b_1 - a_{12}x_2^0 - ..... - a_{1n}x_n^0]$$

$$x_2^1 = \frac{1}{a_{22}}[b_2 - a_{21}x_1^1 ..... - a_{2n}x_n^0]$$

$$\vdots$$

$$x_n^1 = \frac{1}{a_{nn}}[b_n - a_{n1}x_1^1 ..... - a_{n,n-1}x_{n-1}^1]$$

Example: Using the Gauss-Seidel method to find an approximate solution of the following linear system with starting point (1,1,1)

solution:

$$\begin{aligned} 4x_1 + 2x_2 + x_3 &= 11 \\ -x_1 + 2x_2 + 0x_3 &= 3 \\ 2x_1 + x_2 + 4x_3 &= 16 \end{aligned} \quad \rightarrow \quad \begin{aligned} x_1^1 &= \frac{1}{4}[11 - 2x_2^0 - x_3^0] \\ x_2^1 &= \frac{1}{2}[3 + x_1^1] \\ x_3^1 &= \frac{1}{4}[16 - 2x_1^1 - x_2^1] \end{aligned}$$

Then

$$x_1^1 = \frac{1}{4}[11 - 2x_2^0 - x_3^0] = \frac{1}{4}[11 - 2(1) - 1] = 2$$

$$x_2^1 = \frac{1}{2}[3 + x_1^1] = \frac{1}{2}[3 + 2] = 2.5$$

$$x_3^1 = \frac{1}{4}[16 - 2x_1^1 - x_2^1] = \frac{1}{4}[16 - 2(2) - 2.5] = \frac{19}{8}$$

Second iteration: begin by (2, 2.5,19/8) we have

$$x_1^2 = \frac{1}{4}[11 - 2x_2^1 - x_3^1] = \frac{1}{4}[11 - 2(2.5) - \tfrac{19}{8}] = \tfrac{29}{32}$$

$$x_2^2 = \frac{1}{2}[3 + x_1^2] = \frac{1}{2}[3 + \tfrac{29}{32}] = \tfrac{125}{64}$$

$$x_3^2 = \frac{1}{4}[16 - 2x_1^2 - x_2^2] = \frac{1}{4}[16 - 2(\tfrac{29}{32}) - \tfrac{125}{64}] = \frac{783}{256}$$

Continue this iteration until $\left| x^{k+1} - x^k \right| < \varepsilon$

Exercises: Solve the following system of equations

1] $$\begin{aligned} 5x_1 - 2x_2 - 4x_3 &= 17 \\ 2x_1 + x_2 + x_3 &= 0 \\ x_1 - 5x_2 - 3x_3 &= 3 \end{aligned}$$

2] $$\begin{aligned} 3x_1 + x_2 + x_3 &= 1 \\ 10x_1 - 3x_2 - x_3 &= 6 \\ x_1 - 2x_2 + 3x_3 &= 6 \end{aligned}$$

3] $$\begin{aligned} 7x_1 + x_2 - 2x_3 &= 9 \\ -35x_1 - 8x_2 + 11x_3 &= -35 \\ 21x_1 + 15x_2 - 8x_3 &= -11 \end{aligned}$$

4] $$\begin{aligned} 2x_1 - 3x_2 - 4x_3 &= 1 \\ 4x_1 + 6x_2 + x_3 &= 3 \\ 8x_1 + 3x_2 + 2x_3 &= 7 \end{aligned}$$

5] $$\begin{aligned} 4x_1 - x_2 - 3x_3 + 2x_4 &= 1 \\ 12x_1 - x_2 - 14x_3 + 4x_4 &= -1 \\ 4x_1 + 3x_2 - 12x_3 + x_4 &= -11 \\ -8x_1 + 12x_2 - 22x_3 - 25x_4 &= -6 \end{aligned}$$

6] $$\begin{aligned} 2x_1 - 3x_2 + 7x_3 + 8x_4 &= -3 \\ -10x_1 + 5x_2 - 34x_3 - 36x_4 &= 15 \\ -4x_1 + 6x_2 - 13x_3 - 14x_4 &= 8 \\ 8x_1 - 7x_2 + 31x_3 + 37x_4 &= -5 \end{aligned}$$