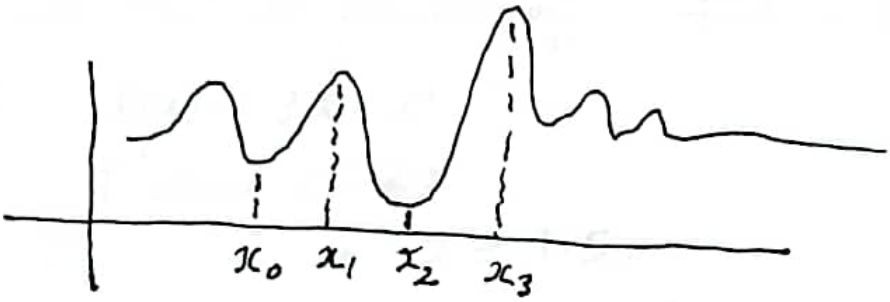
Lecture # 1

Function of one variable

Let $f(x)$ be a continious function of a variable x.



The points x_0, x_2 are local minimum points
The point x_0 is a local min.
The point x_2 is a global min.

Definition

local min: x_0 is a local min if $f(x_0) \leq f(x)$
$\forall\, x \in (x_0 - \varepsilon, x_0 + \varepsilon)$ for some $\varepsilon > 0$

Global min: x_2 is a global min if $f(x_2) \leq f(x)$
$\forall\, x \in$ Domain of f.

Theorem If f is continious on a closed interval $[a,b]$ then f attain a global min and a global max.

Theorem If f has a local max or a local min at c and if $f'(c)$ exists, then $f'(c) = 0$.
Fermat's theorem

This is known as Fermat's theorem. It's important to note that this condition is necessary, but not sufficient. There are points where the derivative of a function is zero that are not local maxima or minima, these points are typically referred to as saddle points or inflection points.

<u>Definition</u> Critical point

Let $c \in$ Domain of f. The point c is a critical point of f, if $f'(c) = 0$

<u>Example</u>

Find all critical points of $f(x) = 3x - x^2$

<u>Solution</u> $f(x) = 3x - x^3$

$$f'(x) = 3 - 2x$$

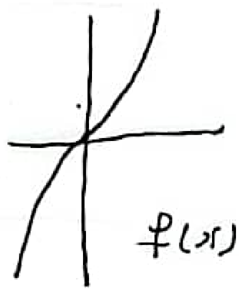
set $f'(x) = 0 \Rightarrow x = 1.5.$

$\therefore$ The function has one critical point $x = 1.5$

<u>Note</u>: $f'(c) = 0$ is not enough to say that c is a min point or a max point.

<u>Example</u>

The function $f(x) = x^3$ has a critical point at $x = 0$ because $f'(x) = 3x^2$ and $f'(x) = 0 \Rightarrow x = 0.$



$f(x) = x^3$

But the point $x = 0$ is not a max point nor a min point.

<u>Theorem</u> If f has a local max or min at $x = x_0$ then x_0 is a critical point

The converse is not true.

$$\text{min or max pt} \Longrightarrow \text{critical pt}$$

$$\text{min or max pt} \;\not\Longleftarrow\; \text{critical pt}$$

What does f' say about f?

① If $f'(x) > 0$ on an interval I, then f is increasing in I.

② If $f'(x) < 0$ on an interval I, then f is decreasing in I.

Example

Find the intervals of increase and the intervals of decrease of $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$

Solution

$$f'(x) = 12x^3 - 12x^2 - 24x$$
$$= 12x(x^2 - x - 2)$$
$$= 12x(x-2)(x+1).$$

$-1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4$

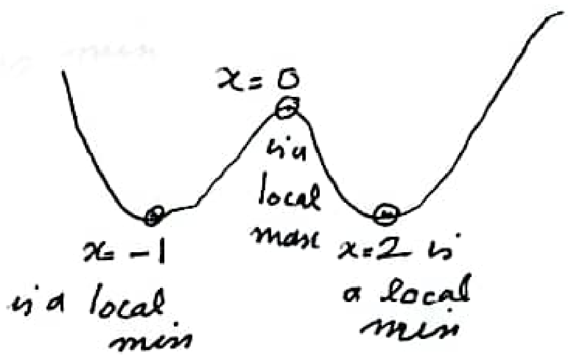
$(12x)$	$-$	$-$	$+$	$+$
$(x-2)$	$-$	$-$	$-$	$+$
$(x+1)$	$-$	$+$	$+$	$+$
Sign of f'	$-$	$+$	$-$	$+$
Shape of f	$\searrow$	$\nearrow$	$\searrow$	$\nearrow$

Intervals of increase of f

$(-1, 0)$ and $(2, \infty)$

Intervals of decrease of f

$(-\infty, -1)$ and $(0, 2)$



The first derivative test

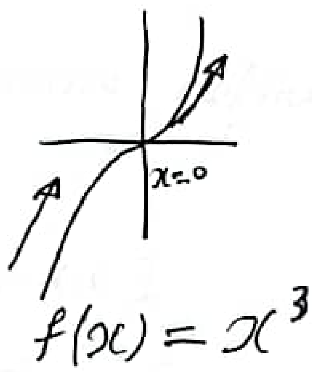
Suppose that c is a critical point of a continuous function f, then

a) If $f'(x)$ changes from positive to negative at $x=c$ then f has a local max at c

b) If $f'(x)$ changes from negative to positive at $x=c$ then f has a local min at c.

c) If $f'(x)$ does not change sign at $x=c$ [i.e, $f'(x)$ is positive on both sides of c or $f'(x)$ is negative on both sides of c] then f has no max no min at c.

<u>Example</u> Consider $f(x) = x^3$



$f'(x) = 3x^2$

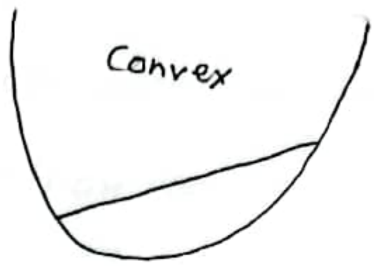
$f'(-0.1) > 0$

$f'(0.1) > 0$

$f'(x)$ is positive on both sides of $x=0$

$\therefore$ $x=0$ is no max no min

Convex function



Let f be a continious fn defined on an interval I, then

a) f is convex if every line sigment joining two points on the graph is never below the graph

b) f is concave if every line sigment joining two points on the graph is never above the graph.

Concave

What does f'' say about f?

Let f be a function of one variable defined on an interval I

f is convex if $f''(x) \geq 0 \quad \forall x \in I$ local min

f is concave if $f''(x) \leq 0 \quad \forall x \in I$ local max

Example

Discuss the convexity of $f(x) = x^3 - x^2$

Solution $\quad f(x) = x^3 - x^2$

$f'(x) = 3x^2 - 2x$

$$f''(x) = 6x - 2 = 6(x - \tfrac{1}{3})$$

So, f is convex on $(\frac{1}{3}, \infty)$ and concave on $(-\infty, \frac{1}{3})$

The point $x = \frac{1}{3}$ is called inflection point.

The second derivative test

1) If $f'(c) = 0$ and $f''(c) > 0$, then $x = c$ is a local minimum point of f.

2) If $f'(c) = 0$ and $f''(c) < 0$, then $x = c$ is a local max point of f.

3) If $f'(c) = 0$ and $f''(c) = 0$, then no conclusion can be made about c.

Example

Discuss the convexity of $y = x^4 - 4x^3$ and classify all critical points

Solution $f(x) = x^4 - 4x^3$

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3)$$

$$f''(x) = 12x^2 - 24x = 12x(x - 2)$$

Critical points at $x = 0$, $x = 3$

Inflection points at $x = 0$, $x = 2$

$f''(x) = 12x(x-2)$

	-1	0	1	2	3
$12x$	$-$		$+$		$+$
$x-2$	$-$		$-$		$+$
sign of f''	$+$		$-$		$+$
Shape of f	$\cup$		$\cap$		$\cup$

The function is convex on $(-\infty, 0) \cup (2, \infty)$
The function is concave on $(0, 2)$

Since $f'(3) = 0$ and $f''(3) > 0$, the point $x = 3$ is a local min.

Now, $f'(0) = 0$ and $f''(0) = 0$ no conclusion can be made by the second derivative test.

By the first derivative test

$f'(0) = 0$ and $f'(-0.1) < 0$ $f'(0.1) < 0$

the critical point $x = 0$ is no max no min