REGRESSION MODELS FOR COUNT DATA WITH R

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LINEAR REGRESSION: REVIEW

- *Linear Regression* model is

$$y = \beta_0 + \beta_1 x_1 + \cdots + \beta_k x_k + \varepsilon$$

where ε is a $N(0, \sigma^2)$ *random error*.

- Equivalently, y is a normally distribution random variable with mean

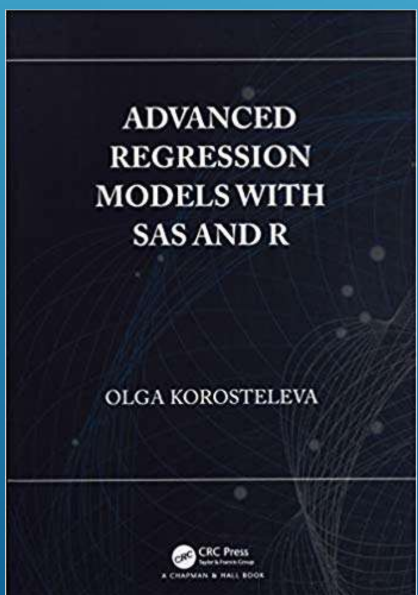
$Ey = \beta_0 + \beta_1 x_1 + \cdots + \beta_k x_k$ and variance σ^2.

- Parameters are $\beta_0, \beta_1, \ldots \beta_k$, and σ^2.
- Fitted model is $\hat{E}y = \hat{\beta}_0 + \hat{\beta}_1 x_1 + \cdots + \hat{\beta}_k x_k$.

WE USE LINEAR REGRESSION FOR:

- Prediction $y^0 = \hat{\beta}_0 + \hat{\beta}_1 x_1^0 + \cdots + \hat{\beta}_k x_k^0$.
- Interpretation of fitted coefficients:
  - If x_1 is continuous, since $\hat{\beta}_1 = \hat{E}y|_{x_1+1} - \hat{E}y|_{x_1}$, as x_1 increases by one unit, the estimated mean of y changes by $\hat{\beta}_1$.
  - If x_1 is 0 -1 variable, since $\hat{\beta}_1 = \hat{E}y|_{x_1=1} - \hat{E}y|_{x_1=0}$, the difference of the estimated means of y for $x_1 = 1$ and $x_1 = 0$ is $\hat{\beta}_1$.

OTHER REGRESSION MODELS



Idea:

- Model y as having certain distribution defined by the setting.

- Model mean Ey as a certain function of linear regression $\beta_0 + \beta_1 x_1 + \cdots + \beta_k x_k$: $g(Ey) = \beta_0 + \beta_1 x_1 + \cdots + \beta_k x_k$ where $g(.)$ is called a *link function.*

- *Predict as* $y^0 = g^{-1}(\hat{\beta}^0 + \hat{\beta}_1 x_1^0 + \cdots + \hat{\beta}_k x_k^0)$.

- *Interpret as*
  - If x_1 is continuous, $\hat{\beta}_1 = g^{-1}(\hat{E}y)|_{x_1+1} - g^{-1}(\hat{E}y)|_{x_1}$.
  - If x_1 is 0 -1 variable, $\hat{\beta}_1 = g^{-1}(\hat{E}y)|_{x_1=1} - g^{-1}(\hat{E}y)|_{x_1=0}$.

- *My recently published book "Advanced Regression Models with SAS and R" discusses 60 different regressions.*

QUICK EXAMPLE: BINARY LOGISTIC REGRESSION

- Suppose $y = 1$ with probability $\pi = P(y = 1)$, and 0, otherwise. Then y has a *Bernoulli* (or *binary*) distribution with mean $Ey = 1 \cdot \pi + 0 \cdot (1 - \pi) = \pi = P(y = 1)$.

- This mean lies between 0 and 1, so we can relate it to the linear regression via the *logistic* function $\frac{\exp(x)}{1+\exp(x)}$:

$$\pi = P(y = 1) = \frac{Exp(\beta_0+\beta_1 x_1+\cdots+\beta_k x_k)}{1+Exp(\beta_0+\beta_1 x_1+\cdots+\beta_k x_k)}.$$

- *Binary logistic regression* models the mean of y through the *logit* link function $g(x) = ln\frac{x}{1-x}$:

$$ln\frac{\pi}{1-\pi} = \beta_0 + \beta_1 x_1 + \cdots + \beta_k x_k.$$

BINARY LOGISTIC REGRESSION EXAMPLE CONTINUES

- Fitted model is $\hat{\pi} = \hat{P}(y=1) = \frac{Exp(\hat{\beta}_0+\hat{\beta}_1 x_1+\cdots+\hat{\beta}_k x_k)}{1+Exp(\hat{\beta}_0+\hat{\beta}_1 x_1+\cdots+\hat{\beta}_k x_k)}$. Equivalently, the fitted *odds in favor of* $y = 1$ can be written as

$$\frac{\hat{\pi}}{1-\hat{\pi}} = Exp(\hat{\beta}_0 + \hat{\beta}_1 x_1 + \cdots + \hat{\beta}_k x_k).$$

- Interpretation:
  - If x_1 is continuous, as x_1 increases by one unit, the estimated odds change by $\frac{\widehat{odds}_{x_1+1}-\widehat{odds}_{x_1}}{\widehat{odds}_{x_1}} \cdot 100\% = \left(Exp(\hat{\beta}_1)-1\right) \cdot 100\%.$
  - If x_1 is 0 -1 variable, the ratio of estimated odds for $x_1 = 1$ and $x_1 = 0$ is $\frac{\widehat{odds}_{x_1=1}}{\widehat{odds}_{x_1=0}} \cdot 100\% = Exp(\hat{\beta}_1) \cdot 100\%.$

POISSON MODEL FOR COUNT DATA

- *Count data* means that y assumes values 0, 1, 2, etc.
- Suppose 0 is quite a common value and so is 1; 2 is more rare; 3, 4, 5 are even less frequent; 6, 7, 8 are very infrequent. Overall, we can model y as having a Poisson distribution with mean λ and probability mass function $P(Y = y) = \frac{\lambda^y}{y!} e^{-\lambda}, y = 0, 1, 2, \dots.$
- We know that λ must be positive, thus we can model

$$\lambda = Ey = Exp(\beta_0 + \beta_1 x_1 + \cdots + \beta_k x_k).$$

- *Poisson regression* models y as having Poisson distribution, and the mean relating to the linear regression through the *log* link function

$$\ln(\lambda) = \beta_0 + \beta_1 x_1 + \cdots + \beta_k x_k.$$

POISSON MODEL CONTINUES

- Fitted model is $\hat{\lambda} = \hat{E}y = Exp(\hat{\beta}_0 + \hat{\beta}_1 x_1 + \cdots + \hat{\beta}_k x_k)$.
- Prediction: $y^0 = Exp\ (\hat{\beta}^0 + \hat{\beta}_1 x_1^0 + \cdots + \hat{\beta}_k x_k^0)$.
- Interpretation of fitted coefficients:
  - If x_1 is continuous, as x_1 increases by one unit, the estimated mean changes by $\frac{\hat{\lambda}_{x_1+1} - \hat{\lambda}_{x_1}}{\hat{\lambda}_{x_1}} \cdot 100\% = \left(Exp(\hat{\beta}_1) - 1\right) \cdot 100\%$.
  - If x_1 is 0 -1 variable, the ratio of estimated means for $x_1 = 1$ and x_1 =0 is $\frac{\hat{\lambda}_{x_1=1}}{\hat{\lambda}_{x_1=0}} \cdot 100\% = Exp(\hat{\beta}_1) \cdot 100\%$.

EXAMPLE: POISSON REGRESSION

- Number of days of hospital stay was recorded for 45 patients along with their gender, age, and history of chronical cardiac illness.

1	F	31	yes	0	F	28	no	0	M	52	yes
1	M	72	yes	0	F	29	no	0	F	30	no
1	M	74	no	2	M	30	yes	2	F	72	no
1	M	58	no	2	F	28	no	2	F	65	no
2	M	65	no	1	M	52	no	4	M	51	no
2	F	63	no	0	F	31	no	1	F	47	yes
1	M	49	no	2	M	71	yes	2	M	48	no
2	F	47	no	0	F	31	no	3	M	44	yes
3	M	44	no	3	M	54	yes	4	F	72	yes
4	M	56	yes	3	F	73	yes	1	F	46	no
3	M	58	no	4	M	70	yes	2	M	36	no
1	M	50	no	1	M	59	no	0	M	52	no
6	M	68	yes	2	F	41	no	1	M	31	yes
1	M	69	no	3	M	73	no	3	F	77	yes
2	F	54	no	4	M	69	yes	5	M	68	yes

POISSON REGRESSION EXAMPLE CONTINUES

- ❑ We fit Poisson regression model using R:

```
hospitalstay.data<-read.csv(file= “./data.csv” , header=TRUE, sep= “,” )
summary(fitted.model<- glm(days ~ gender + age + illness,
data=hospitalstay.data, family=poisson(link=log)))
```

- ❑ The fitted model is
  $\hat{\lambda} = Exp(-0.8263+0.2264*\text{male}+0.0205*\text{age}+0.4477*\text{illness})$.
- ❑ Prediction: The predicted length of stay for a 55-year old male with no chronic cardiac illness is computed as
  $$y^0 = Exp(-0.8263+0.2264+0.0205*55) = 1.6949.$$
- ❑ Interpretation of estimated regression coefficients:

- (gender) Estimated average length of hospital stay for males is exp{0.2264} · 100% = 125.41% of that for females.
- (age) For a one-year increase in patient's age, the estimated average number of days of hospital stay increases by (exp{0.0205}−1) ·100% = 2.07%.
- (illness) The estimated average number of days of hospital stay for patients with a chronic cardiac illness is exp{0.4477}·100% = 156.47% of that for patients without it.

ZERO-TRUNCATED POISSON MODEL FOR COUNT DATA

- Suppose y follows a Poisson distribution but no zeros are observed.
- Then y can be modeled via a *zero-truncated Poisson regression* where

  the distribution of y is $P(Y = y) = \frac{\lambda^y}{y!} \cdot \frac{e^{-\lambda}}{1-e^{-\lambda}}, \; y = 1, 2, \ldots,$

  with $\lambda = Exp(\beta_0 + \beta_1 x_1 + \cdots + \beta_k x_k)$.
- Fitted *model is* $\hat{\lambda} = Exp\left(\hat{\beta}_0 + \hat{\beta}_1 x_1 + \cdots + \hat{\beta}_k x_k\right)$.
- Prediction $y^0 = \hat{E}y = \frac{\widehat{\lambda}}{1-E \quad \left(-\widehat{\lambda}\right)} = \frac{Exp\,\left(\widehat{\beta}^0+\widehat{\beta}_1 x_1^0+\cdots+\widehat{\beta}_k x_k^0\right)}{1-Exp\left(-Exp\,\left(\widehat{\beta}^0+\widehat{\beta}_1 x_1^0+\cdots+\widehat{\beta}_k x_k^0\right)\right)}.$
- Interpretation of estimated regression coefficients is the same as in Poisson model.

EXAMPLE: ZERO-TRUNCATED POISSON REGRESSION

- Suppose in the previous example, the data were reduced to the 38 patients who spent at least one day in the hospital. We run a zero-truncated Poisson model using R:

```
hospitalstay.data<-read.csv(file= “./data.csv” , header=TRUE, sep= “,” )
#eliminating zeros from the original data set
hospitaldays.data<-hospitalstay.data[which(hospitalstay.data$days!=0),]
install.packages("VGAM")
library(VGAM)
summary(fitted.model<- vglm(days ~ gender + age + illness,
data=hospitaldays.data, family=pospoisson()))
```

- The fitted model is
  $\hat{\lambda} = Exp(-0.7041+0.2146*male+0.01604*age+0.5903*illness)$.

- Prediction:

To predict the number of days of hospital stay for a 55-year old male without a chronic cardiac illness, we calculate

$$y^0 = \frac{\exp\{-0.7041+0.2146+0.01604*55\}}{1-\exp\left\{-\exp\{-0.7041+0.2146+0.01604*55\}\right\}} = 1.9169.$$

ZERO-TRUNCATED POISSON REGRESSION EXAMPLE CONTINUES

- Interpretation of estimated regression coefficients:

  - (gender) Estimated average length of hospital stay for males is $\exp\{0.2146\} \cdot 100\% = 123.94\%$ of that for females.
  - (age) For a one-year increase in patient's age, the estimated average number of days of hospital stay increases by $(\exp\{0.01604\}-1) \cdot 100\% = 1.62\%$.
  - (illness) The estimated average number of days of hospital stay for patients with a chronic cardiac illness is $\exp\{0.5903\} \cdot 100\% = 180.45\%$ of that for patients without it.

ZERO-INFLATED POISSON MODEL FOR COUNT DATA

- Suppose y follows a Poisson distribution but too many zeros are observed. For example, suppose that one of the variables recorded during a health survey is the number of cigarettes the respondent smoked yesterday. Some respondents may have reported zero number of cigarettes smoked because they either do not smoke at all (*structural zero*), or they happened not to smoke a single cigarette that day (*chance zero*).
- Then y can be modeled via a *zero-inflated Poisson (ZIP) regression* where the distribution of y is

$$\mathbb{P}(Y=y)=\begin{cases}\pi+(1-\pi)\exp\{-\lambda\}, & \text{if } y=0,\\ (1-\pi)\dfrac{\lambda^y\exp\{-\lambda\}}{y!}, & \text{if } y=1,2,\ldots,\end{cases}$$

where

$$\pi=\frac{\exp\{\beta_0+\beta_1 x_1+\cdots+\beta_m x_m\}}{1+\exp\{\beta_0+\beta_1 x_1+\cdots+\beta_m x_m\}},$$

and

$$\lambda=\exp\left\{\gamma_0+\gamma_1 x_{m+1}+\cdots+\gamma_{k-m} x_k\right\}.$$

ZIP MODEL (CONTINUED)

- The fitted model is

$$\widehat{\pi} = \frac{\exp\{\widehat{\beta}_0 + \widehat{\beta}_1 x_1 + \cdots + \widehat{\beta}_m x_m\}}{1 + \exp\{\widehat{\beta}_0 + \widehat{\beta}_1 x_1 + \cdots + \widehat{\beta}_m x_m\}},$$

$$\widehat{\lambda} = \exp\left\{\widehat{\gamma}_0 + \widehat{\gamma}_1 x_{m+1} + \cdots + \widehat{\gamma}_{k-m} x_k\right\}.$$

- The fitted mean is

$$\hat{E}y = (1 - \hat{\pi}) \cdot \hat{\lambda} = \frac{\exp(\widehat{\gamma}_0 + \widehat{\gamma}_1 x_{m+1} + \cdots + \widehat{\gamma}_{k-m} x_k)}{1 + \exp(\widehat{\beta}_0 + \widehat{\beta}_1 x_1 + \cdots + \widehat{\beta}_m x_m)}.$$

- Prediction:

$$y^0 = \frac{\exp(\widehat{\gamma}_0 + \widehat{\gamma}_1 x_{m+1}^0 + \widehat{\gamma}_{k-m} x_k^0)}{1 + \exp(\widehat{\beta}_0 + \widehat{\beta}_1 x_1^0 + \cdots + \widehat{\beta}_m x_m^0)}.$$

ZIP MODEL (CONTINUED)

- Interpretation of estimated regression coefficients:
- Probability of structural zero π is modeled as in the binary logistic regression, thus, estimated beta coefficients are interpreted in terms of estimated odds.
- The mean of y is $Ey = (1 - \pi) \cdot \lambda$, and since we assume x variables are non-overlapping in π and λ, interpretation of gamma coefficients in λ is the same as in Poisson regression model.
- Note that it is possible to use the same x variables in the regression parts of π and λ, but the estimates of the regression coefficients won't be easily interpretable. Can be useful for prediction.

EXAMPLE: ZERO-INFLATED POISSON REGRESSION

- A health survey was been administered to a random sample of 40 people aged between 25 and 50. Their gender, self-reported health condition (excellent or good), age, and the number of cigarettes they smoked yesterday were recorded. The data are:

M	good	34	3	F	exclnt	48	1	M	exclnt	26	0	M	good	39	0
F	good	27	1	M	good	28	5	F	good	44	1	M	exclnt	30	0
F	exclnt	26	0	F	good	38	2	F	good	40	1	F	exclnt	31	0
M	good	27	3	F	exclnt	34	1	F	good	36	2	F	exclnt	34	2
F	exclnt	39	0	F	good	42	1	F	good	48	4	M	good	32	5
M	good	47	2	M	good	29	3	M	exclnt	38	0	F	good	50	4
M	good	30	3	M	good	38	2	M	good	31	6	F	exclnt	33	0
F	good	28	0	F	good	42	3	M	exclnt	28	0	M	good	31	2
F	exclnt	31	0	F	exclnt	42	0	F	good	44	4	F	good	39	1

ZERO-INFLATED POISSON REGRESSION EXAMPLE CONTINUES

- We fit a ZIP model with health condition modeling structural zeros and gender and age predicting the Poisson part:

```
smoking.data<-read.csv(file="./data.csv", header=TRUE, sep=",")
install.packages("pscl")
library(pscl)
#specifying reference category
health.rel<- relevel(smoking.data$health, ref="good")
#fitting zero-inflated Poisson model
summary(fitted.model<- zeroinfl(cigarettes ~ gender +
age|health.rel, data=smoking.data))
```

- The fitted model is

$$\hat{\pi} = \frac{\exp\{-3.7950 + 4.9195 * excellent\_health\}}{1 + \exp\{-3.7950 + 4.9195 * excellent\_health\}},$$

$$\hat{\lambda} = \exp\{-0.1381 + 0.0186 * age + 0.7268 * male\}.$$

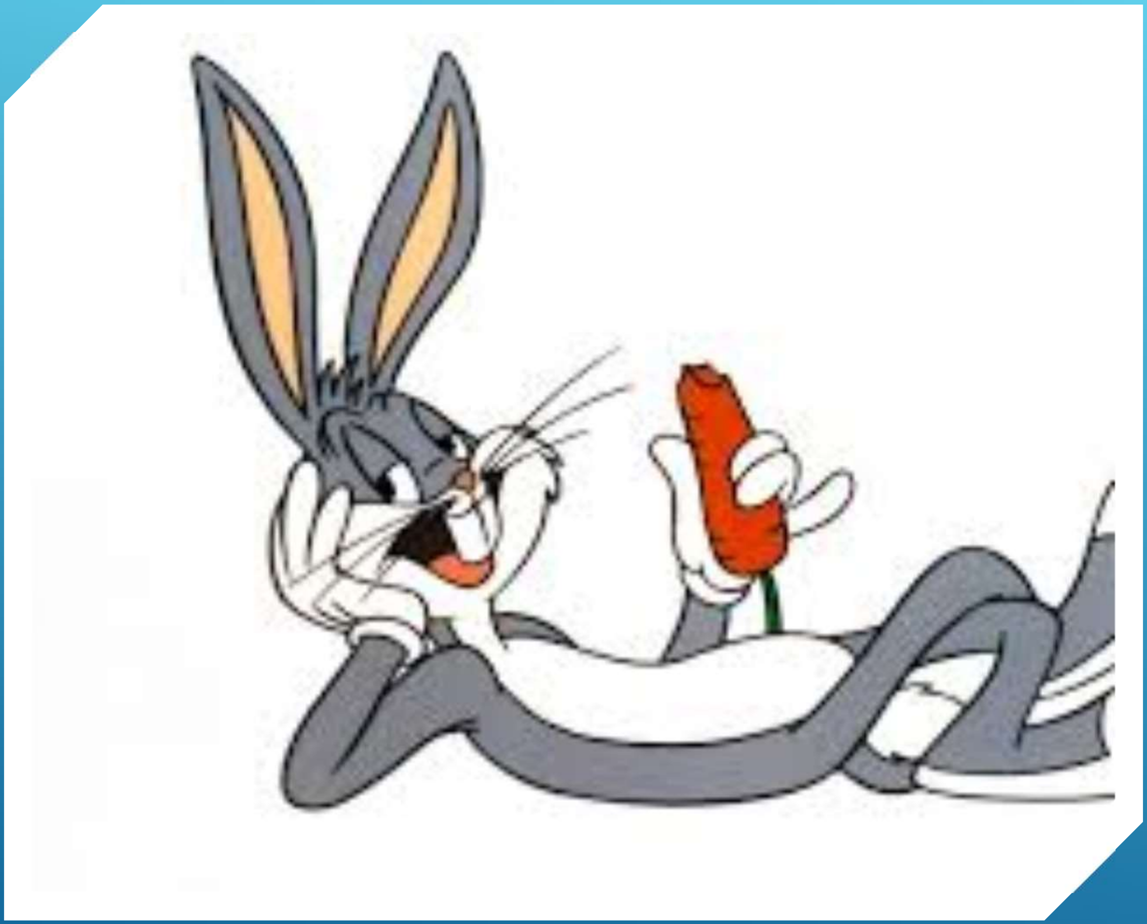
ZIP REGRESSION EXAMPLE CONTINUES

- Prediction: The predicted number of cigarettes smoked per day by a 50-year old male who is in good health is found as

$$y^0 = \frac{\exp(-0.1381 + 0.0186 * 50 + 0.7268)}{1 + \exp(-3.7950)} = 4.4659.$$

- Interpretation of estimated regression coefficients:

- (health condition) The estimated odds of not smoking for people in excellent health is $\exp\{4.9195\} \cdot 100\% = 13{,}694.26\%$ of those for people in good health.
- (age) As age increases by one year, the estimated average number of cigarettes smoked in a day increases by $(\exp\{0.0186\}-1) \cdot 100\% = 1.88\%$.
- (gender) The estimated average number of cigarettes smoked in a day by men is $\exp\{0.7268\} \cdot 100\% = 206.85\%$ of that by women.



THANK YOU

PLEASE ATTEND
MY PRESENTATION
ON OCTOBER 5