Digital Assignment-3

MAT2002

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A)

$$f(x) = \begin{cases} x & , \ 0 \le x \le 1 \\ 2-x & , \ 1 \le x \le 2 \end{cases}$$

$b-a=2$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n x}{b-a}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2n\pi x}{b-a}\right)$$

$$a_0 = \frac{2}{b-a} \int_a^b f(x)\,dx$$

$$a_0 = \frac{2}{2} \int_0^1 x\,dx + \int_1^2 (2-x)\,dx$$

$$a_0 = \left|\frac{x^2}{2}\right|_0^1 + \left|2x - \frac{x^2}{2}\right|_1^2$$

$$a_0 = \frac{1}{2} + \left(2(2) - \frac{4}{2}\right) - \left(2(1) - \frac{1}{2}\right)$$

$$a_0 = \frac{1}{2} + \left((4-2) - \left(2 - \frac{1}{2}\right)\right)$$

$$a_0 = \frac{1}{2} + \left(2 - 2 + \frac{1}{2}\right)$$

$$\boxed{a_0 = 1}$$

$$a_n = \frac{2}{b-a} \int_a^b f(x) \cos\left(\frac{2n\pi x}{b-a}\right) dx$$

$$a_n = \frac{2}{2}\left(\int_0^1 x \cos m\pi x \, dx + \int_1^2 (2-x) \cos n\pi x \, dx \right)$$

$$a_n = \left(\frac{x \sin(\pi n x)}{\pi n} - \int \frac{\sin(\pi n x) dx}{\pi n} \right)_0^1 + \left(\frac{-(\pi n x - 2\pi n) \sin(\pi n x) + \cos \pi n x}{\pi^2 n^2} \right)_1^2$$

$$a_n = \frac{\pi n \sin(\pi n) + \cos(\pi n) - 1}{\pi^2 n^2} \quad \frac{-\cos(2\pi n) - \pi n \sin \pi(n) + \cos(\pi n)}{\pi^2 n^2}$$

$$a_n = \frac{(-1)^n - 1}{\pi^2 n^2} \quad \frac{-1 + (-1)^n}{\pi^2 n^2}$$

$$a_n = \frac{2(-1)^n - 2}{\pi^2 n^2}$$

$$\boxed{a_n = \frac{2((-1)^n - 1)}{\pi^2 n^2}}$$

$$b = \frac{2}{b-a}\int_a^b f(x)\sin n\pi x \, dx$$

$$b_n = \frac{2}{2}\int_0^1 x \sin n\pi x \, dx + \int_1^2 (2-x)\sin(n\pi x)\, dx$$

$$b_n = \frac{\operatorname{Sin}(\pi n) - \pi n \operatorname{Cos}(\pi n)}{\pi^2 n^2} - \frac{\operatorname{Sin}(2\pi n) + \operatorname{Sin}(\pi n) + \pi n \operatorname{Cos}(\pi n)}{\pi^2 n^2}$$

$$b_n = \frac{-\pi n(-1)^n}{\pi^2 n^2} + \frac{\pi n(-1)^n}{\pi^2 n^2}$$

$$\boxed{b_n = 0}$$

$$\boxed{f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} 2\left(\frac{(-1)^n - 1}{\pi^2 n^2}\right)\operatorname{Cos} n\pi x}$$

Put $x = 0$, $\therefore \boxed{\operatorname{Cos} n\pi x = 1}$

$$0 = \frac{1}{2} + 2\left(\frac{(-1)^1 - 1}{\pi^2 1^2} + \frac{(-1)^2 - 1}{\pi^2 2^2} + \frac{(-1)^3 - 1}{\pi^3 3^3} + \cdots\cdots\right)$$

$$-\frac{1}{4} = \frac{1}{\pi^2}\left(\frac{-2}{1^2} \quad \frac{-2}{3^2} - \frac{2}{5^3} - \cdots\cdots\right)$$

$$-\frac{1}{4} = -\frac{2}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\cdots\right)$$

$$\boxed{\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\cdots = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}}$$

To find $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4}$

Apply Parsevals Identity,

$a_0 = 1$ $\qquad a_n = \frac{2((-1)^n - 1)}{\pi^2 n^2}$ $\qquad b_n = 0$

Take the function $\qquad x \qquad 0 \le x \le 1 \qquad b - a = 2$

$\qquad\qquad 2 - x \qquad 1 \le x \le 2$

$$\int_0^1 x^2\,dx + \int_1^2 (2-x)^2\,dx = \frac{2}{2}\left[\frac{1}{2} + 4\sum_{n=1}^{\infty}\frac{((-1)^n-1)^2}{\pi^4 n^4}\right]$$

$$\left[\frac{x^3}{3}\right]_0^1 + \left[\frac{x^3}{3} - 2x^2 + 4x\right]_1^2 = \frac{1}{2} + \frac{4}{\pi^4}\sum_{n=1}^{\infty}\frac{((-1)^n-1)^2}{n^4}$$

$$\frac{1}{3} + \frac{1}{3} = \frac{1}{2} + \frac{4}{\pi^4}\left(\frac{(-2)^2}{1^4} + \frac{(-2)^2}{3^4} + \frac{(-2)^2}{5^4} + \cdots\right)$$

$$\frac{2}{3} = \frac{1}{2} + \frac{4}{\pi^4}\times(-2)^2\left(\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right)$$

$$\frac{1}{6} = \frac{+16}{\pi^4}\left(\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right) = \frac{\pi^4}{96}$$

$\frac{2}{3} - \frac{1}{2}$ $\quad 2\times3$

$\frac{4-3}{6}$

$\frac{1}{6}$

$$\boxed{\frac{+\pi^4}{96} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}} \qquad \boxed{\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{8}}$$

B)

θ	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$	$\frac{7\pi}{12}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\frac{11\pi}{12}$
T	0	2.7	5.2	7	8.1	8.3	7.9	6.8	5.5	4.1	2.6	1.2

$$\boxed{N = 12}$$

$$\boxed{b - a = \pi}$$

$$f(\theta) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi\theta}{b-a}$$

$$T(\theta) = \sum_{n=1}^{\infty} b_n \sin n\theta$$

$$T(\theta) = b_1 \sin\theta + b_2 \sin 2\theta + b_3 \sin 3\theta + b_4 \sin 4\theta \ldots$$

$$b_n = 2 \times [\text{Mean value of } f(x) \sin x \text{ in } (0, \pi)]$$

$$b_1 = 2 \times \frac{\sum A \sin\theta}{N}$$

$$b_2 = 2 \times \frac{\sum A \sin 2\theta}{N}$$

θ	$\sin\theta$	$\sin 2\theta$	T	$T\sin\theta$	$T\sin 2\theta$
0	0	0	0	0	0
$\frac{\pi}{12}$	0.25	0.5	2.7	0.67	1.35
$\frac{\pi}{6}$	0.5	0.866	5.2	2.6	4.50
$\frac{\pi}{4}$	0.70	1	7	4.9	7
$\frac{\pi}{3}$	0.866	0.866	8.1	7.01	7.01
$\frac{5\pi}{12}$	0.96	0.5	8.3	8.00	4.15
$\frac{\pi}{2}$	1	0	7.9	7.9	0
$\frac{7\pi}{12}$	0.96	-0.5	6.8	7.96	-4.3
$\frac{2\pi}{3}$	0.866	-0.866	5.5	4.73	-4.73
$\frac{3\pi}{4}$	0.70	-1	4.1	2.87	-4.1
$\frac{5\pi}{6}$	0.5	-0.866	2.6	1.3	-2.25
$\frac{11\pi}{12}$	0.25	-0.5	1.2	0.3	-0.6
				$\Sigma = 48.24$	$\Sigma = 8.03$

$$b_1 = 2 \times \frac{\Sigma T \sin\theta}{N}$$

$$b_1 = 2 \times \frac{48.23}{12}$$

$$\boxed{b_1 = 8.03}$$

$$b_2 = 2 \times \frac{\Sigma T \sin 2\theta}{N}$$

$$b_2 = 2 \times \frac{8.03}{12}$$

$$\boxed{b_2 = 1.33}$$

$$T(\theta) = b_1 \sin\theta + b_2 \sin 2\theta$$ (upto 2 harmonics)

$$\boxed{T(\theta) = 8.03 \sin\theta + 1.33 \sin 2\theta}$$

3)

i) Initial Value Theorem

If $x(n) \xleftrightarrow{ZT} X(z)$

$$\boxed{x(0) = \lim_{z \to \infty} X(z)}$$

Proof :-

$$Z\{x(n)\} = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$X(z) = x(0) + x(1) z^{-1} + x(2) z^{-2} + \dots$$

$$X(z) = x(0) + \frac{x(1)}{z} + \frac{x(2)}{z^2} + \dots$$

Take limit $z \to \infty$ on both sides,

$$\lim_{z \to \infty} X(z) = \lim_{z \to \infty} \left\{ x(0) + \frac{x(1)}{z} + \frac{x(2)}{z^2} + \dots \right\}$$

$$= x(0)$$

$$\therefore \boxed{x(0) = \lim_{z \to \infty} X(z)}$$

Final Value Theorem:-

If $x(n) \xleftrightarrow{ZT} X(z)$ then

$$\boxed{x(\infty) = \underset{z \to 1}{Lt} (z-1) X(z)}$$

Proof:-

$$Z\{x(n+1) - x(n)\} = \sum_{n=0}^{\infty} [x(n+1) - z(n)] z^{-n}$$

$$Z\{x(n+1)\} - Z\{x(n)\} = \sum_{n=0}^{\infty} [x(n+1) - x(n)] z^{-n}$$

$$z X(z) - z X(0) - X(z) = \sum_{n=0}^{\infty} [x(n+1) - X(n)] z^{-n}$$

$$(z-1) X(z) - z X(0) = \sum_{n=0}^{\infty} [x(n+1) - x(n)] z^{-n}$$

Take limit $z \to 1$ on both sides

$$\underset{z \to 1}{Lt} (z-1) X(z) - X(0) = \underset{z \to 1}{Lt} \sum_{n=0}^{\infty} [x(n+1) - x(n)] z^{-n}$$

$$= \sum_{n=0}^{\infty} x(n+1) - x(n)$$

$$= x(1) - x(0) + x(2) - x(1) + \dots (x(\infty+1)) - x(\infty)$$

$$= X(\infty) - X(0)$$

$$\therefore \boxed{X(\infty) = \underset{z \to 1}{Lt} (z-1) X(z)}$$

D)

$$f_1(k) = \left(\frac{1}{4}\right)^k u(k-1)$$

$$f_2(k) = 1 + \left(\frac{1}{2}\right)^k$$

$$Z\{f_1(k)\} = F_1(z)$$

$$Z\{f_2(k)\} = F_2(z)$$

$$\boxed{Z\{f_1(k) * f_2(k)\} = F_1(z) \cdot F_2(z)}$$ (Convolution theorem)

$$Z\left\{\left(\frac{1}{4}\right)^k u(k-1)\right\} = F_1(z)$$

Apply Damping rule and time shifting property,

$$Z(u(k-1)) = z^{-1} Z(u(k))$$ (time shifting)

$$= z^{-1}$$

Apply Damping,

$$Z\left\{\left(\frac{1}{4}\right)^k u(k-1)\right\} = \left(\frac{z}{\frac{1}{4}}\right)^{-1}$$

$$= (4z)^{-1}$$

$$\boxed{Z\left\{\left(\frac{1}{4}\right)^k u(k-1)\right\} = \frac{1}{4z}} = F_1(z)$$

b) $Z\left\{1+\left(\frac{1}{2}\right)^k\right\} = F_2(z)$

Apply Linearity

$$Z(\{1\}) + Z\left(\left\{\frac{1}{2}\right\}^k\right) = F_2(z)$$

$$\frac{z}{z-1} + \frac{z}{z-\frac{1}{2}} = F_2(z)$$

$$\boxed{\frac{z}{z-1} + \frac{2z}{2z-1} = F_2(z)}$$

$$Z\{f_1(k) * f_2(k)\} = F_1(z) * F_2(z) \qquad \text{(Convolution)}$$

$$= \frac{1}{4z}\left(\frac{z}{z-1} + \frac{2z}{2z-1}\right)$$

$$= \frac{1}{4(z-1)} + \frac{\cancel{2}1}{4z-2}$$

$$\boxed{Z\{f_1(k) * f_2(k)\} = \frac{1}{4z-4} + \frac{1}{4z-2}}$$

E) $y(k+2) + 4y(k+1) + 3y(k) = 2^k \cdot k^2$ $\qquad y(0)=0$, $y(1)=0$

RHS

$$z\left(2^k k^2\right)$$

Apply Damping rule,

$$z[\{k^2\}] = \frac{z+z^2}{(z-1)^3}$$

$$z[\{2^k \cdot z^2\}] = \frac{\frac{z}{2} + \left(\frac{z}{2}\right)^2}{\left(\frac{z}{2}-1\right)^3}$$

LHS = RHS

$$[y(z) - y(0) - y(1)z^{-1}] + 4[z(y(z)-y(0))] + 3y(z) = \frac{\frac{z}{2} + \left(\frac{z}{2}\right)^2}{\left(\frac{z}{2}-1\right)^3}$$

$$y(z)[z^2+4z+3] - z = \frac{\frac{z}{2} + \left(\frac{z}{2}\right)^2}{\left(\frac{z}{2}-1\right)^3}$$

$$y(z)((z+1)(z+3)) = \frac{\frac{z}{2} + \left(\frac{z}{2}\right)^2}{\left(\frac{z}{2}-1\right)^3} + z$$

$$y(z)((z+1)(z+3)) = \frac{\frac{z}{2} + \frac{z^2}{4} + z\left(\frac{z}{2}-1\right)^3}{\left(\frac{z}{2}-1\right)^3}$$

$$\frac{y(z)}{z} = \frac{\frac{z}{2} + \frac{z^2}{4} + z\left(\frac{z}{2} - 1\right)^3}{\left(\frac{z}{2} - 1\right)^3 (z+1)(z+3)}$$

$$= \frac{\frac{z}{2} + \frac{z^2}{4} + \frac{z^4}{8} - \frac{3z^3}{4} + \frac{3z^2}{2} - z}{\left(\frac{z}{2} - 1\right)^3 (z+1)(z+3)}$$

$$= \frac{a_0 (z+1)(z+3)\left(\frac{z}{2} - 1\right)^3}{\frac{z}{2} - 1} + \frac{a_1 (z+1)(z+3)\left(\frac{z}{2} - 1\right)^3}{\left(\frac{z}{2} - 1\right)^2} + \frac{a_2 (z+1)(z+3)\left(\frac{z}{2} - 1\right)^3}{\left(\frac{z}{2} - 1\right)^3}$$

$$+ \frac{a_3 (z+1)(z+3)\left(\frac{z}{2} - 1\right)^3}{z+1} + \frac{a_4 (z+1)(z+3)\left(\frac{z}{2} - 1\right)^3}{z+3}$$

$a_0 = \frac{76}{3375} \quad a_1 = \frac{-17}{450} \quad a_2 = \frac{1}{15} \quad a_3 = \frac{25}{54} \quad a_4 = \frac{-127}{250}$

$$\therefore \frac{Y(z)}{z} = \frac{152}{3375(z-2)} - \frac{34}{225(z-2)^2} + \frac{8}{15(z-2)^3} + \frac{25}{54(z+1)} - \frac{127}{250(z+3)}$$

$$\boxed{Y(z) = \frac{152}{3375}(2)^n - \frac{34}{225}/(z-2)^2 + \frac{8}{15(z-2)^3} + \frac{25}{54}(-1)^n - \frac{127}{250}(-3)^n}$$

$$\cancel{\frac{-34(z-2)^2}{240}} \quad \frac{-34z}{225(z-2)^2} + \frac{8z}{15(z-2)^3}$$

$$\frac{-34z(z-2) + 8z}{240(z-2)^3}$$

$$z^{-1}\left(\frac{8z - 34z^2 + 68z}{240(z-2)^3}\right) = \frac{-17}{120} - \frac{1}{4(z-2)} + \frac{1}{15(z-2)^2}$$

$$\boxed{Y(z) = \frac{152}{3375}(2)^n - \frac{17}{120} - \frac{1}{4}(2)^n + \frac{1}{15}n + \frac{25}{54}(-1)^n - \frac{127}{250}(-3)^n}$$