COL352: Assignment 2

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1 Question 1

Let L = $\{bin(p)$: p is prime $\}$

Proving by updated version of pumping lemma.
So we need to show:-
$\forall n \geq 0$
$\exists\ p = uwv \in L : |w| \geq n$
$\forall xyz \in \Sigma^* : xyz = w, |xy| \leq n, |y| > 0$
$\exists i \geq 0 : uxy^i zv \notin L$

Let $p = 2^{n+1} + 1$
$v = 1$
w = 2nd bit to nth bit of p => w = 0....0 n times
u = remaining part of p

Now, since w is all zero, any split xyz of w will just differ in their number of zeros.
Let, $|x| = r, |y| = s, |z| = t : r + s \leq n, s > 0$
$=> r + s + t = n$
$p' = uxy^i zv = 2^{r+is+t+1} + 1$
$r + is + t + 1 = r + s + t + (i - 1)s + 1 = n + (i - 1)s + 1$
$=> p' = 2^{n+1+(i-1)s} + 1$
$=> p' = 2^{n+1}2^{(i-1)s} + 1$
$=> p' = (p - 1)2^{(i-1)s} + 1$
$=> p' = p2^{(i-1)s} - (2^{(i-1)s} - 1)$
take i = p
$=> p' = p2^{(p-1)s} - (2^{(p-1)s} - 1)$
We, know that for any prime number $p : a^{p-1} mod p = 1$ (done in course col351, fermat's little theorem)
$=> 2^{p-1} mod p = 1 => 2^{(p-1)s} mod p = 1 => 2^{(p-1)s} - 1 (mod p) = 0$
=> second term is divisible by p then let it be p*k
$=> p' = p(2^{(p-1)s} - k)$
$=> p'$ is divisible by p, hence not prime and hence not in L

2 Question 2

The n-th Fibonacci number is defined as $F_1 = 1, F_2 = 1$, and for all $n \geq 3$, $F_n = F_{n-1} + F_{n-2}$ .Consider the language over $\Sigma = \{a\} L_2 = \{a^m | m = F_n\}$ Is L_2 regular? Justify your answer.

The given language is not regular and we will prove this using the pumping lemma.
To Prove: L_2 is not regular.
Proof: We will use the contrapositive of the pumming lemma here. So let k be the pumping length s.t. $k \geq 1$. Now we pick a fibonacci number $F_n \geq k$ and also $F_{n+1} - F_n > k$. Such a fibonacci number exists clearly because second condition basically comes to $F_{n-1} > k$. So we have to find a fibonacci number which is greater than k and the fibonacci just number before it is also grater than k. That is clearly possible since fibonacci is a fast growing series.
Now,

$$k \geq 1, a^{F_n} \in L_2 \ and \ |a^{F_n}| \geq k$$

Every break up of a^{F_n} can be written as xyz :

$$x = a^r, y = a^s, z = a^t \ where \ r + s + t = F_n$$

$$Also, \ |xy| \leq k \ and \ y \neq \epsilon \implies s \neq 0 \ and \ s \leq k \qquad - (i)$$

Now consider i=2 . Clearly $i \geq 0$. We can pump up y for i=2 to get:

$$xy^2z = a^{r+2*s+t}$$

$$=> a^{F_n+s}$$

$$We \ know \ F_n < F_n + s \leq F_n + k < F_{n+1} \qquad - From \ (i)$$

So we have shown that $F_n + s$ is not a fibonacci number so $xy^2z \notin L_2$. Hence by the contrapositive of the pumping lemma L_2 is not a regular language. Hence Proved.

3 Question 3

If A is any language, let $A_{\frac{1}{2}-}$ denote the set of all first halves of strings in A so that

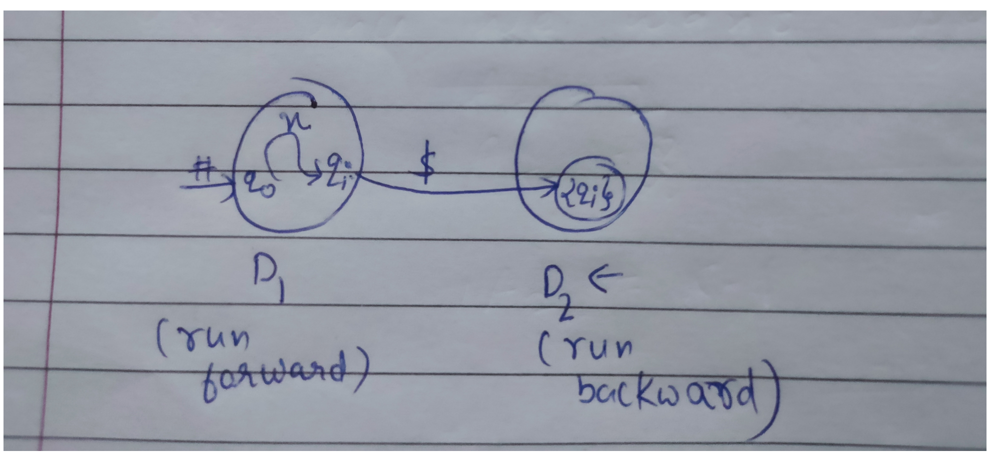
$$A_{\frac{1}{2}-} = \{\mathbf{x} \mid \textbf{for some } \boldsymbol{y}, |\boldsymbol{x}| = |\boldsymbol{y}| \textbf{ and } \mathbf{xy} \in A\}$$

Show that if A is regular, then so is $\boldsymbol{A_{\frac{1}{2}-}}$.

To, show that $A_{\frac{1}{2}-}$ is regular, it is sufficient to show that there exist a 2DFA that accepts $A_{\frac{1}{2}-}$ (as it was proved in class that language accepted by 2DFA is regular).
Now, before constructing the 2DFA, consider the following psuedo code:-

Let DFA D accepts A, and $x \in \Sigma^*$ Start from start state of D
Run the string x on D
Suppos final state is q_i
From, q_i at each step traverse to all states having 1-step transition from q_i
Iteratively, make similar transitions from all states at step i
Perform $|x|$ such steps
If out of all states in last step if any of them is in final state then x is accepted else x is rejected.

So, basically from state q_i we are traversing all states that are $|x|$ transitions away (hence got all possible final states that we can get after running xy such that $|x| = |y|$).
Since $\hat{\delta}(q_0, xy) = \hat{\delta}(\hat{\delta}(q_0, x), y)$



Now, lets folmally define the 2DFA D' that runs above psuedo code:- We will require 2 DFA's D1 and D2 to construct D'.
Let D1 = D = $\{Q, \Sigma, \delta, q_0, F\}$

First run of x would be on D with transition function δ and moving right.
Second run would be on D2 = $\{2^Q, \Sigma, \delta_1, q_0, F'\}$ with transition function δ' and moving left.

So, D' = $\{Q', \Sigma \bigcup \{\#, \$\}, \delta', q_0, q_a, q_r\}$
is defined as:- $Q' = Q \bigcup 2^Q \bigcup q_a \bigcup q_r$
$\delta'(q_i', a)$ defined as:

1. $q_i' \in Q$ (first run)
   $\delta'(q_0, \#) = (q_0, R)$
   $\delta'(q_i, c) = (\delta(q_i, c), R)$ if $c \in \Sigma$
   $\delta'(q_i, \$) = (\{q_r\}, L)$

2. $q_i' \in 2^Q$ (second run)
   $\delta'(q_i', c) = (\bigcup\limits_{q_k \in q_i'} \{\bigcup\limits_{a \in \Sigma} \delta(q_i, a)\}, L)$ if $c \in \Sigma$
   $\delta'(q_i', \#) = (q_a, L)$ if $q_i' \bigcap F \neq \phi$
   $\delta'(q_i', \#) = (q_r, L)$ if $q_i' \bigcap F = \phi$

Claim: Above 2DFA accepts $A_{\frac{1}{2}-}$
Proof: Consider string $x \in \Sigma^*$
if, $x \in A_{\frac{1}{2}-}$
$<=> \exists y$ such that $xy \in A, |x| = |y| =$ n(say)
Consider the run of xy on D.
Let, $\hat{\delta}(q_0, x) = q_i$ $\hat{\delta}(q_i, y) = q_j$
$<=> q_j \in F$
Now, consider the run of x on D'
$(q_0, 0) \xrightarrow{x} (q_i, n)$
$(q_i, n) \xrightarrow{y} (q_i', 2n)$
q_i' is the set of all possible final states rechable from q_i after n transition
Clearly q_j is in q_i' (as $|y| = n$)
$=> q_j \in F <=> q_i' \bigcup F \neq \phi$
$x \in A_{\frac{1}{2}-} <=> x$ accepted by D'
Hence $L(D') = A_{\frac{1}{2}-}$
Hence done.

4 Question 4

If A is any language, let $A_{\frac{1}{3}-\frac{1}{3}}$ denote the set of strings in A with the middle-third removed so that

$$A_{\frac{1}{3}-\frac{1}{3}} = \{ \textbf{ xz } \mid \textbf{ for somey, } |x| = |y| = |z| \textbf{ and } xz \in A \}$$

Show that if A is regular, then $A_{\frac{1}{3}-\frac{1}{3}}$ is not necessarily regular.

To disprove the that $A_{\frac{1}{3}-\frac{1}{3}}$ is not necessarily regular if A is regular we will show a couter-example. That is we need a regular language A such that $A_{\frac{1}{3}-\frac{1}{3}}$ is not regular.
Consider the regular language $A = a^*bc^*$.

Claim: $A_{\frac{1}{3}-\frac{1}{3}}$ is not regular.
Proof:
Proving this via contradiction.
Suppose $A_{\frac{1}{3}-\frac{1}{3}}$ is regular.
Now, consider the following language made by intersection of A and $A_{\frac{1}{3}-\frac{1}{3}}$:
$L = A_{\frac{1}{3}-\frac{1}{3}} \bigcap a^*c^*$

Claim : L is not regular.
Proof:
Any string x in A is of the form $a^nbc^m : n, m \geq 0$.
So, for any string s of $A_{\frac{1}{3}-\frac{1}{3}}$ following cases are possible: -

1. **Case 1:** $s = a^{k_1}c^{k_2}$ where $k_1, k_2 > 0 \& k_1 = k_2$
   This would be the case when b is not in middle third
   Proving $k_1 = k_2$
   Since, b is not in middle third, hence a^{k_1} and c^{k_2} comes from first third and last third respectively.
   And by the defination of $A_{\frac{1}{3}-\frac{1}{3}}$ we have $|a^{k_1}| = |c^{k_2}| => k_1 = k_2$
   $=> s = a^kc^k$

2. **Case 2:** $s = a^{k_1}bc^{k_2}$ where $k_1, k_2 \geq 0$
   This would be the case when b is in middle third

Now, $L = A_{\frac{1}{3}-\frac{1}{3}} \bigcap a^*c^*$
In intersection only first case of s will be considered since b cannot be in second language of intersection.
Hence $L = \{\bigcup s \text{ from case 1 } \} \bigcap a^*c^* = \{a^nc^n \forall n \geq 0\}$
Clearly L is irregular (provved in class).
But L is intersection of 2 regular languages and has to regular by closure of regularity.

That is a contradiction.
Hence our assumption that $A_{\frac{1}{3}-\frac{1}{3}}$ is regular is false.
Hence proved.

5 Question 5

1. **Part a :**

   a) We need to prove A does not accept w if and only if sets $W_i \subseteq Q$ exists such that given condition will hold.
   Proof:
   1:Forward implication
   Lets assume our 2-NFA A does not accept x then there exist a sequence that are subsets of S.
   Let our sets W_i where $0 <= i <= n+1$ be the all states $q \in Q$ such that for each i there is a set of states q when A is run on x i.e there exist a state in A till i when A is run on x.

   (a) **1st condition:** $(S \subseteq W_0)$:
   If $q \in S$ that is set of states q is in start states when A is run on x implies first sequence will be accepted (i=0).So set of all start states will be in first sequence i.e $S \in W_0$

   (b) **2nd condition:** If $u \in W_i$ implies A run on x till ith state seq i.e $q \in W_i$.Given $(v, R) \in \triangle(u, a_i))$ there is A is accepted for a_i on x implies i+1 is also accepting seq and the new state $s_{i+1} \in q$ i.e $v \in W_{i+1}$

   (c) **3rd condition:** If $u \in W_i$ implies A run on x till ith state seq i.e $q \in W_i$.Given $(v, L) \in \triangle(u, a_i))$ there is A is accepted for a_i on x implies i+1 is also accepting seq and the new state $s_{i-1} \in q$ i.e $v \in W_{i-1}$

   (d) **4th condition:** $t \notin W_{n+1})$:Means $q \notin t$
   Proof by contradiction: If $q \in t$ implies on i = n+1 out sequence with states s is still being accepted implies whole string will be accepted i.e A accepts x.Which contradicts our assumption.

   2:Backward implication
   proof by contradiction:Let A accept x implies w runs n+1 times i.e i=n+1 ,$q \in t$ and $W_0, W_1 .....W_n + 1$ satisfies given conditions. Since A accepts x,$S \subseteq W_0$.Every forward state which accepts x will be in part of sequence W_i for each ith state.So 2nd and 3rd condition will also be satisfied. However, the final condition cannot be satisfied since the final state will not in an accept state.

2. **Part b :**
b) A NFA B is defined by the 5 tuple $(Q', \Sigma, \delta, S', F')$ which will accept the complement of the language accepted by A. We define the various elements of the tuple as follows -

1. Q' = $2^Q \cup (2^Q)^2$ which is the union of all the possible sets of states and the pair of all the possible sets of states
2. Σ is the same as before (set of all possible characters of the language)
3. (T,K) $\in \delta(S, a)$ such that (s $\in$ T and (k, R) $\in \Delta(s, a)$) $\implies$ k $\in$ K, (K, U) $\in (\delta(S, M), a)$ such that (k $\in$ K and (u, R) $\in \Delta(k, a)$) $\implies$ u $\in$ U or ((u, L) $\in \Delta(m, a)$) $\implies$ u $\in$ S.
4. S' represents the set of all possible states which contain the start state of A
5. F' is the set of all states which are neither present in the final state of A and neither in the set of possible final pairs

We now take a string x that is accpeted by this NFA and the set of sequences obtained on running this string through the NFA. We define the set of sequences as W_0, (W_0, W_1), (W_1, W_2) .... (W_n, W_{n+1}). Initial state of the NFA is the start state so, S $\subseteq W_0$ holds. For (2) and (3), if there exists a state change from (W_{i-1}, W_i) to (W_i, W_{i+1}) $\implies$ v $\in W_{i+1}$. For (3), if on the transition to L, v $\notin W_{i-1}$, then it cannot be the case that this is a valid run of the NFA, so it is a contradiction. For (4), we can see that by our definition of F', the f $\notin W_{n+1}$ so that means the sequence that we have defined satisfies all the given properties.

We now prove the opposite side of this to show that this is an if and only if statement. We thus take a string x that is not accepted by this NFA. We define the set of sequences as W_0, (W_0, W_1), (W_1, W_2) .... (W_n, W_{n+1}). Base Case: Initial state of the NFA is the start state so, S $\subseteq W_0$ holds. For the induction hypothesis, if there exists a state change from (W_{i-1}, W_i) to (W_i, W_{i+1}) $\implies$ v $\in W_{i+1}$ and v $\notin W_{i-1}$, then it cannot be the case that this is a valid run of the NFA, so it is a contradiction. Now that we have proved the induction holds, we see that by our definition of F', the f $\in W_{n+1}$ which contradicts the 4th property. So, this sequence cannot be the one satisfies all the properties when a string is not accepted by the NFA.

Thus we can show that the set of strings accepted by the NFA is the complement of the set of strings accepted by the 2-NFA. We also know that the language accepted by a NFA is regular and the complement of a regular language is also regular. Thus, the language accepted by the 2NFA is also regular.

6 Question 6

Let M = $(\boldsymbol{Q}, \boldsymbol{\Sigma}, \boldsymbol{q_0}, \boldsymbol{\delta}, \boldsymbol{F})$ be a DFA and let h be a state of M called its "home". A synchronizing sequence for M and h is a string $\boldsymbol{s} \in \boldsymbol{\Sigma^*}$ where $\boldsymbol{\delta(q, s)} =$ h for every $\boldsymbol{q} \in \boldsymbol{Q}$. Say that M is synchronizable if it has a synchronizing sequence for some state h. Prove that if M is a k-state synchronizable DFA, then it has a synchronizing sequence of length at most $\boldsymbol{q^3}$. Can you improve upon this bound?

Given: We have a DFA M=$(Q, \Sigma, q_0, \delta, F)$ which is synchronizable according to the definition given above. Also let us suppose k=$|Q|$ i.e. k is the number of states of the DFA. $h \in Q$ be the home state of M and $s \in \Sigma^*$
be the synchronizing sequence.
To Prove: The upper bound of the synchronizing sequence is k^3.
Proof: If we choose any two states $q_1 \in Q$ and $q_2 \in Q$ such that $q_1 \neq q_2$ then there must exist a sequence of alphabet lets call it $s^{'}$ that takes both q_1 and q_2 to the same state. So $\delta^{'}(q_1, s^{'}) = \delta^{'}(q_2, s^{'})$. This holds because the DFA is synchronizable.
Now the length of the smallest $s^{'}$ such that the above condition is met is at max k*(k-1). This can be proved through the piegon hole principle. Suppose the length of $s^{'}$ is greater than k*(k-1). But we know that size of set of pair of distinct states is k*(k-1). So that would mean that some pair of states is repeated i.e. $\delta^{'}(q_1, s_1s_2...s_i)$=a , $\delta^{'}(q_2, s_1s_2....s_i)$=b and $\delta^{'}(q_1, s_1s_2....s_j)$=a , $\delta^{'}(q_2, s_1s_2....s_j) = b$ where $j > i$ and $s^{'}$=$s_1s_2....s_n$. But since the states are repeated we can omit the alphabet between s_i and s_j and there would be no difference. But that is contradiction because because $s^{'}$ was of the least length. So that proves that length of the string $s^{'}$ is at most $k * (k-1)$.
Now if we run $s^{'}$ we have found on all the states of Q then it would lead us to at most k-1 distinct states. This is true because of the way we constructed $s^{'}$ q_1 and q_2 would lead to same state. Now we will apply this process recursively for smaller number of states till the number of states reach 1. Let the state that was left be $h^{'}$ and the concatination of all the $s^{'}$ obtained at each recursive step be $s^{''}$.
We can say that $s = s^{''}$ and $h = h^{'}$ because the way we have constructed $s^{''}$ and $h^{'}$, every state will reach $h^{'}$ if applied $s^{''}$. So $s^{''}$ is the syncronizi g sequence and $h^{'}$ if the home state.
Now length of all the $s^{'}$ is at most k*(k-1) and we concat k-1 such $s^{'}$ to get $s^{''}$. So the length $s^{''}$ is at most k*(k-1)*(k-1) which is less than k^3. So we have shown that k^3 is an upper bound for the length of synchronizing sequence for a k-state synchronizable DFA. A tighter bound to this is k*(k-1)*(k-1).

7 Question 7

The given function 0.x is basically representation of any number less than 1 in binary. So, set L_θ is set of all numbers x that are $\leq \theta$.

1. **Proving** => Given that θ is regular. Then we need to prove that L_θ is regular.
   We, know that languages accepted by NFA are regular so it is sufficient to show that there is one NFA that accepts L_θ.
   Since, any rational can be either terminating or recurring, lets see both of the possibilities of θ:

   (a) **Case 1:** θ is terminating

   Let $|\theta| = k$ Consider the NFA N with k+2 states as follows:
   N = $\{Q, 0, 1, \delta, q_0, F$
   $Q = q_0, ...q_k, q_f, q_d$
   $F = q_k, q_f$
   δ is defined as:
   $\delta(q_i, 0) = q_f$ if $\theta_i = 1$
   $\delta(q_i, \theta_{i+1}) = q_{i+1} \forall i < k$
   $\delta(q_k, a) = q_d \forall a \in \{0, 1\}$
   $\delta(q_f, a) = q_f \forall a \in \{0, 1\}$

   So, basically this NFA traverses the string $0.x$ in stats q_i untill it is equal to some subset of θ, and once it differs from θ then it either becomes more than theta or less than it. If $x_i = 0$ then $\theta_i = 1$ hence $0.x < 0.\theta$, so in that case x is sent to accepting state q_f that remains on itself untill the string is exhausted. And at the end accepted. If $x_i = 1$ then $\theta_i = 0$ hence $0.x > 0.\theta$, so in that case x is sent to dead state q_d that has no tranistion and hence string gets rejected.
   If $|x| = k$ and $0.x = 0.\theta$ then string ends up in accepting state q_k and hence accepted. Finally, if $|x| > k$ in that case also x is sent to dead state q_d hence string gets rejected.
   Hence NFA accepts all string x such that $0.x \leq 0.\theta$ and only those strings.
   Hence L_θ is accepting language of N

   (b) **Case 2:** θ is recurring

   In this case suppose $\theta = 0.\overline{x_1 x_2 .....x_k}$
   Let x = $x_1 x_2 .....x_k$
   Claim : $L_\theta = {L_x}^*$
   Proof: Suppose, $0.z \in {L_x}^* \Rightarrow \exists n : 0.z \leq 0.xxx...$ (n times)
   $\Rightarrow 0.z \leq 0.\theta \Rightarrow 0.z \in L_\theta$

$=> {L_x}^* \subseteq L_\theta$

Now, let $0.z \in L_\theta$, and $|z| = n$.
$=> 0.z \leq 0.\theta$
But, since $|z| = n$ $0.z \leq 0.xxx....$ ($\log_k n$ times)
$=> 0.z \in {L_x}^*$
$=> {L_x}^* \supseteq L_\theta$

$=> L_\theta = {L_x}^*$

Now, since x is terminating, so we have a NFA N of it from part 1, hence by the closure property of regular languages, there exists a NFA N' for which L_θ is accepted by N'.
Hence, L_θ is regular.

2. **Proving** <= It is sufficient to show that L_θ is irregular if θ is irrational to prove this part.
Proving this via contrapositive of pumping lemma.
So we need to show:-
$\forall n \geq 0$
$\exists$ $0.w \in L : |w| \geq n$
$\forall xyz \in \Sigma^* : xyz = w, |xy| \leq n, |y| > 0$
$\exists i \geq 0 : 0.xy^iz \notin L$

Let, $0.w = \theta$ (since θ is irrational => it has infinite number of alphabets as it is non terminating, So $|w| \geq n \forall n$) hence it is a valid choice for all n
Now, consider $w_1 = xy^0z = xz$ and $w_2 = xyyz$
By pumping lemma both of them should be in l_θ
$=> 0.w1 \leq 0.w$ and $0.w2 \leq 0.w$
$=> 0.xz \leq 0.xyz$ and $0.xyyz \leq 0.xyz$
$=> 0.z \leq 0.yz$ and $0.yyz \leq 0.yz - (1)$
$=> 0.z \leq 0.yz$ and $0.yz \leq 0.z - (2)$
From 1 and 2 we have $0.yz = 0.z => y.z = a.b$ where $a.b = 2^{|y|}0.z$
$=> a = y, b = z$
again, $0.b = 0.yz$ (since $0.b = 0.z = 0.yz$)
So, inductively this will give $0.z = 0.yyyyyy.....$ i.e. w is recurring
But that is not possible since θ is irrational
Hence both w_1, w_2 cannot be in L_θ
Hence L_θ is irregular.