PART- 1

Elementary Transformations, Inverse of a Matrix.

CONCEPT OUTLINE

Elementary Operations :

The following operations on a matrix are called elementary operations :

i. Interchange of two rows or two columns, e.g., $R_1 \leftrightarrow R_2$, $C_1 \leftrightarrow C_2$

ii. Multiplication of a row or column by a non zero number K, e.g., $R_1 \to KR_1$ or $C_1 \to KC_1$

iii. Addition of K times the elements of a row (or column) to the corresponding elements of another row (or column), $K \neq 0$, e.g., $R_1 \to R_1 + KR_3$, $R_2 \to R_2 - 3R_3$, $C_1 \to C_1 + 2C_3$

Inverse of a Matrix : If A and B be two square matrices of same order such that $AB = BA = I$, then B is called inverse of A, *i.e.*, $B = A^{-1}$.

Properties of Inverse :

i. If A is invertible then A^{-1} is also invertible.

ii. $(AB)^{-1} = B^{-1}A^{-1}$

iii. $(A^1)^{-1} = (A^{-1})^1$

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 1.1. **Compute the inverse of the matrix $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$ by employing elementary row transformation.**

Answer

We know that, $A = IA$

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Que 1.3. **Find inverse employing elementary transformation.**

$$A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$$

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Answer

Same as Q. 1.1, Page 1–2C, Unit-1.

Ans : $$A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

PART-2

Rank of Matrix, Solution of System of Linear Equations.

CONCEPT OUTLINE

Rank of a Matrix : A number r is said to be the rank of a matrix A if it possesses the following properties :

i. There is at least one square sub-matrix of A of order r whose determinant is not equal to zero.
ii. If the matrix A contains any square sub-matrix of order $r + 1$, then the determinant of every square sub-matrix of A of order $r + 1$ should be zero.

Therefore the rank of a matrix is the order of any highest order non zero minor of the matrix.

Condition for Consistency : The system of equations $AX = B$ is consistent *i.e.*, possesses a solution if the coefficient matrix A and the augmented matrix $[AB]$ are of same rank.

Case I : If rank $[A]$ < rank $[AB]$, then the system of equations is inconsistent, *i.e.*, they have no solution.

Case II : If rank $[A]$ = rank $[AB] = r$ (say), then the system of equations $AX = B$ is consistent, *i.e.*, they possesses a solution.

1. If $r = n$, then the system of equations has unique solution.
2. If $r < n$, then there will be infinite solution. Only $n - r + 1$ solution will be linearly independent and rest of the solutions will be linear combination of them.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 1.5. **Using elementary transformations, find the rank of the following matrix :**

$$A = \begin{bmatrix} -2 & -1 & 3 & -1 \\ 1 & 2 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

Answer

$$A = \begin{bmatrix} -2 & -1 & 3 & -1 \\ 1 & 2 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & -3 & -1 \\ -2 & -1 & 3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix} \qquad R_1 \leftrightarrow R_2$$

$$= \begin{bmatrix} 1 & 2 & -3 & -1 \\ 0 & 3 & -3 & -3 \\ 0 & -2 & 4 & 2 \\ 0 & 1 & 1 & -1 \end{bmatrix} \qquad R_2 \to R_2 + 2R_1, \; R_3 \to R_3 - R_1$$

$$= \begin{bmatrix} 1 & 2 & -3 & -1 \\ 0 & 3 & -3 & -3 \\ 0 & -2 & 4 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix} \qquad R_4 \to R_4 - R_2 / 3$$

$$= \begin{bmatrix} 1 & 2 & -3 & -1 \\ 0 & 3 & -3 & -3 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \qquad R_3 \to R_3 + 2R_2 / 3$$

$$= \begin{bmatrix} 1 & 2 & -3 & -1 \\ 0 & 3 & -3 & -3 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad R_4 \to R_4 - R_3$$

$$= \begin{bmatrix} 1 & 2 & -3 & -1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad R_2 \to R_2 / 3, \; R_3 \to R_3 / 2$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 16/5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_2 \to R_2 + 5/4R_3 \\ R_4 \to R_4 - 5/16\,R_3 \end{matrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_2 \to 1/5\,R_2 \\ R_3 \to 5/16\,R_3 \end{matrix}$$

$$= \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$

Which is the required normal form.
Since, the non zero rows are 3 hence, the rank of the given matrix is 3.

Que 1.7. Find the rank of the matrix $A = \begin{bmatrix} 1 & 3 & 4 & 2 \\ 2 & -1 & 3 & 2 \\ 3 & -5 & 2 & 2 \\ 6 & -3 & 8 & 6 \end{bmatrix}$ by reducing it to normal form.

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Answer

Same as Q. 1.6, Page 1–7C, Unit-1.
[Answer : Rank of matrix = 3]

Que 1.8. **Investigate for what values of λ and μ, the system of equations $x + y + z = 6$, $x + 2y + 3z = 10$ and $x + 2y + \lambda z = \mu$, has**

i. No solution.
ii. Unique solution.
iii. Infinite number of solutions.

OR

Determine the values 'a' and 'b' for which the following system of equation has.

$$x + y + z = 6$$
$$x + 2y + 3z = 10$$
$$x + 2y + az = b$$

i. No solution.
ii. Unique solution.
iii. Infinite number of solutions.

Answer

$$x + y + z = 6$$

$$\begin{bmatrix}1&1&1\\1&2&3\\1&2&\lambda\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}6\\10\\\mu\end{bmatrix}$$

$$AX = B$$

$$C = [A : B] = \begin{bmatrix}1&1&1&:&6\\1&2&3&:&10\\1&2&\lambda&:&\mu\end{bmatrix} \begin{matrix}R_2 \to R_2 - R_1\\R_3 \to R_3 - R_1\end{matrix}$$

$$= \begin{bmatrix}1&1&1&:&6\\0&1&2&:&4\\0&1&\lambda-1&:&\mu-6\end{bmatrix} R_3 \to R_3 - R_2$$

$$= \begin{bmatrix}1&1&1&:&6\\0&1&2&:&4\\0&0&\lambda-3&:&\mu-10\end{bmatrix}$$

i. For no solution :

$$R(A) \neq R(C)$$

i.e., $\lambda - 3 = 0$ or $\lambda = 3$ and $\mu - 10 \neq 0$ or $\mu \neq 10$.

ii. A unique solution :

$$R(A) = R(C) = 3$$

i.e., $\lambda - 3 \neq 0$ or $\lambda \neq 3$ and μ may have any value.

iii. Infinite solutions :

$$R(A) = R(C) = 2$$

i.e., $\lambda - 3 = 0$ or $\lambda = 3$ and $\mu - 10 = 0$ or $\mu = 10$.

Que 1.9. **Find the value of k for which the system of equations $(3k - 8)x + 3y + 3z = 0$, $3x + (3k - 8)y + 3z = 0$, $3x + 3y + (3k - 8)z = 0$ has a non trivial solution.**

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Answer

For the given system of equations to have a non-trivial solution, the determinant of the coefficient matrix should be zero.

i.e.,
$$\begin{vmatrix}3k-8&3&3\\3&3k-8&3\\3&3&3k-8\end{vmatrix}=0$$

$$\begin{vmatrix}3k-2&3&3\\3k-2&3k-8&3\\3k-2&3&3k-8\end{vmatrix}=0 \qquad [\text{Operating } C_1 + (C_2 + C_3)]$$

$$(3k-2)\begin{vmatrix}1&3&3\\1&3k-8&3\\1&3&3k-8\end{vmatrix}=0$$

$$(3k-2)\begin{vmatrix}1&3&3\\0&3k-11&0\\0&0&3k-11\end{vmatrix}=0 \qquad [\text{Operating } R_2 - R_1; R_3 - R_1]$$

$$(3k-2)(3k-11)^2 = 0$$

$$k = 2/3, 11/3, 11/3.$$

Que 1.10. **For what values of λ and μ the system of linear equations :**

$$x + y + z = 6$$
$$x + 2y + 5z = 10$$
$$2x + 3y + \lambda z = \mu$$

i. A unique solution,
ii. No solution, and
iii. Infinite solution.
Also find the solution for $\lambda = 2$ and $\mu = 8$.

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Answer

Same as Q. 1.8, Page 1–8C, Unit-1.

[Answer :

i. For unique solution : $\lambda \neq 6, \mu \neq 16$
ii. For no solution : $\lambda = 6, \mu \neq 16$
iii. For infinite solution : $\lambda = 6, \mu = 16$]

$$C = [A : B] = \sim\begin{bmatrix}1&1&1&:&6\\0&1&4&:&4\\0&0&\lambda-6&:&\mu-16\end{bmatrix} \qquad ...(1.10.1)$$

Putting $\lambda = 2$ and $\mu = 8$ in eq. (1.10.1), we get

$$\begin{bmatrix}1&1&1&:&6\\0&1&4&:&4\\0&0&-4&:&-8\end{bmatrix} \Rightarrow \begin{bmatrix}1&1&1\\0&1&4\\0&0&-4\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}6\\4\\-8\end{bmatrix}$$

$$x + y + z = 6$$
$$y + 4z = 4$$
$$-4z = -8 \quad \Rightarrow z = 2$$

Putting $z = 2$, we get

$$y + 8 = 4 \quad \Rightarrow y = -4$$

Putting $y = -4$, $z = 2$, we get

$$x - 4 + 2 = 6$$
$$\Rightarrow \quad x = 8$$

Hence, $x = 8, y = -4, z = 2$

Answer

The characteristic equation of the matrix A is $|A - \lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$(2-\lambda)[(1-\lambda)(2-\lambda) - 0] - 1(0-0) + 1(-1+\lambda)$

or $[\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0]$

According to Cayley-Hamilton theorem, we have

$$A^3 - 5A^2 + 7A - 3I = 0 \quad \text{...(1.11.1)}$$

Multiplying eq. (1) by A^{-1}, we get

$$A^2 - 5A + 7I - 3A^{-1} = 0$$

$$A^{-1} = \frac{1}{3}[A^2 - 5A + 7I]$$

But
$$A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4+0+1 & 2+1+1 & 2+0+2 \\ 0+0+0 & 0+1+0 & 0+0+0 \\ 2+0+2 & 1+1+2 & 1+0+4 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$$

$\therefore$
$$A^2 - 5A + 7I = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} - 5\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & 0 \\ -1 & -1 & 2 \end{bmatrix}$$

$$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$$

$$= A^5(A^3 - 5A^2 + 7A - 3I) + A(A^3 - 5A^2 + 7A - 3I) + A^2 + A + I$$

$$= A^2 + A + I \qquad [\because A^3 - 5A^2 + 7A - 3I = 0]$$

$$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

$$\text{Adj}(A-\lambda I) = B_0\lambda^{n-1} + B_1\lambda^{n-2} + \ldots.. + B_{n-2}\lambda + B_{n-1}$$

where $B_0, B_1, \ldots\ldots, B_{n-1}$ are the matrices of type $n \times n$.

Now, $(A-\lambda I)\,\text{adj}(A-\lambda I) = |A-\lambda I|\,I$

$$(A-\lambda I)(B_0\lambda^{n-1} + B_1\lambda^{n-2} + \ldots.. + B_{n-1}) = (-1)^n[\lambda^n + a_1\lambda^{n-1} + \ldots. + a_n]\,I$$

Comparing coefficients of like powers of λ on both sides,

$$\begin{aligned} -IB_0 &= (-1)^n I \\ AB_0 - IB_1 &= (-1)^n a_1 I \\ AB_1 - IB_2 &= (-1)^n a_2 I \\ \vdots\quad & \quad\vdots \\ AB_{n-1} &= (-1)^n a_n I \end{aligned}$$

On multiplying these equations by $A^n, A^{n-1}, \ldots\ldots, I$ respectively and adding, we get

$$0 = (-1)^n[A^n + a_1A^{n-1} + a_2A^{n-2} + \ldots.. + a_nI]$$

Thus, $A^n + a_1A^{n-1} + a_2A^{n-2} + \ldots.. + a_nI = 0$

Que 1.14. **Verify Cayley-Hamilton theorem for $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$.**

Hence find A^{-1}.

Answer

Characteristic equation, $|A-\lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & 4-\lambda & 5 \\ 3 & 5 & 6-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)[(4-\lambda)(6-\lambda)-25] - 2[2(6-\lambda)-15] + 3[10-3(4-\lambda)] = 0$$

$$(1-\lambda)[24-10\lambda+\lambda^2-25] - 2[12-2\lambda-15] + 3[-2+3\lambda] = 0$$

$$(1-\lambda)(\lambda^2-10\lambda-1) + 2(3+2\lambda) + 3(3\lambda-2) = 0$$

$$\lambda^2 - 10\lambda - 1 - \lambda^3 + 10\lambda^2 + \lambda + 6 + 4\lambda + 9\lambda - 6 = 0$$

$$-\lambda^3 + 11\lambda^2 + 4\lambda - 1 = 0$$

$$\lambda^3 - 11\lambda^2 - 4\lambda + 1 = 0$$

For verification of Cayley-Hamilton theorem we need to verify,

$$A^3 - 11A^2 - 4A + I = 0 \qquad \ldots(1.14.1)$$

$$4A = 4\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 12 \\ 8 & 16 & 20 \\ 12 & 20 & 24 \end{bmatrix}$$

$$A^2 = AA = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix}$$

$$\begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)\{(2-\lambda)^2 - 1\} + 1\{-(2-\lambda)+1\} + 1\{1-(2-\lambda)\} = 0$$

$$(2-\lambda)(3-\lambda)(1-\lambda) - 2(1-\lambda) = 0$$

$$(1-\lambda)[6 - 5\lambda + \lambda^2 - 2] = 0$$

$$(1-\lambda)(\lambda^2 - 5\lambda + 4) = 0$$

$$\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$$

To verify Cayley-Hamilton theorem we need to prove,

$$A^3 - 6A^2 + 9A - 4I = 0$$

$$A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

$$A^3 - 6A^2 + 9A - 4I = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - \begin{bmatrix} 36 & -30 & 30 \\ -30 & 36 & -30 \\ 30 & -30 & 36 \end{bmatrix}$$

$$+ \begin{bmatrix} 18 & -9 & 9 \\ -9 & 18 & -9 \\ 9 & -9 & 18 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

Hence Cayley-Hamilton theorem is verified.

Now computing the given equation,

$$= A^6 - 6A^5 + 9A^4 - 2A^3 - 12A^2 + 23A - 9I$$

$$= A^6 - 6A^5 + 9A^4 - 4A^3 + 4A^3 - 2A^3 - 12A^2 + 23A - 9I$$

$$= A^3(A^3 - 6A^2 + 9A - 4I) + 2A^3 - 12A^2 + 23A - 9I$$

$$= A^3(0) + 2(A^3 - 6A^2 + 9A - 4I) + 5A - I$$

$$= A^3(0) + 2(0) + 5A - I$$

$$= 5\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & -5 & 5 \\ -5 & 10 & -5 \\ 5 & -5 & 10 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 9 & -5 & 5 \\ -5 & 9 & -5 \\ 5 & -5 & 9 \end{bmatrix}$$

Que 1.16. **Using Cayley-Hamilton theorem find the inverse of the matrix** $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$.

Also express the polynomial $B = A^8 - 11A^7 - 4A^6 + A^5 + A^4 - 11A^3 - 3A^2 + 2A + I$ as a quadratic polynomial in A and hence find B.

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Answer

A. **Inverse of Matrix :** Refer Q. 1.14, Page 1–14C, Unit-1.

B. $B = A^8 - 11A^7 - 4A^6 + A^5 + A^4 - 11A^3 - 3A^2 + 2A + I$

$$= A^5 (A^3 - 11A^2 - 4A + I) + A (A^3 - 11A^2 - 4A + I) + A^2 + A + I \quad ...(1.16.1)$$

Characteristic equation of matrix

$\lambda^3 - 11\lambda^2 - 4\lambda + I = 0$

By Cayley-Hamilton theorem, $A^3 - 11A^2 - 4A + I = 0$ ...(1.16.2)

On putting value of eq. (1.16.2) in eq. (1.16.1), we get

$B = A^2 + A + I$

$$A^2 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix}$$

$$B = \begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & 27 & 34 \\ 27 & 50 & 61 \\ 34 & 31 & 77 \end{bmatrix}$$

Que 1.17. **Verify Cayley-Hamilton theorem for the matrix** $A = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$ **and hence find A^{-1}.**

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Answer

Same as Q. 1.14, Page 1–14C, Unit-1.

Answer : $A^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & -8 \\ -1 & 0 & -4 \end{bmatrix}$

PART-4

Linear Dependence and Independence of Vectors.

CONCEPT OUTLINE

Linear Dependence and Linear Independence : A set of n-vectors $x_1, x_2, ...x_n$ are said to be linearly dependent if there exists n scalars, $a_1, a_2,.....a_n$ not all zero such that $a_1x_1 + a_2x_2 + ... + a_nx_n = 0$ and if all $a_1, a_2,.....a_n$ are zero, vectors are said to be linearly independent.

Characteristic Matrix : Let A be any square matrix of order n and λ an indeterminate. The matrix $[A - \lambda I]$ is called the characteristic matrix of A where I is an identity matrix of order n.

Also the determinant $|A - \lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & : & a_{1n} \\ a_{21} & a_{22}-\lambda & : & a_{2n} \\ : & : & : & : \\ a_{n1} & a_{n2} & : & a_{nn}-\lambda \end{vmatrix}$

is called the characteristic polynomial of A and the roots of this equation are called the characteristic roots, latent roots or eigen values of A. If λ is a characteristic root of an $n \times n$ matrix A, then a non-zero vector x such that $AX = \lambda X$ is called a characteristic vector or eigen vector of A corresponding to the eigen value λ.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 1.18. **Let $v_1 = (+1, -1, 0)$, $v_2 = (0, 1, -1)$ and $v_3 = (0, 0, 1)$ be element of R^3. Show that the set of vectors $\{v_1, v_2, v_3\}$ is linearly independent.**

Answer

Consider the vector equation

$$\alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n = 0$$

Substituting for v_1, v_2, v_3, we obtain

Hence, there exist scalars not all equal to zero such that equation (1.19.1) is satisfied. Therefore they are linearly dependent.

Que 1.20. **Show that the vectors $x_1 = [1, 2, 3], x_2 = [3, -2, 1], x_3 = [1, -6, -5]$ form a linearly dependent system.**

Answer

$$a_1x_1 + a_2x_2 + a_3x_3 = 0$$

$$a_1[1, 2, 3] + a_2[3, -2, 1] + a_3[1, -6, -5] = 0$$

$$a_1 + 3a_2 + a_3 = 0 \quad ...(1.20.1)$$

$$2a_1 - 2a_2 - 6a_3 = 0 \quad ...(1.20.2)$$

$$3a_1 + a_2 - 5a_3 = 0 \quad ...(1.20.3)$$

Solving the above equations (1.20.2) and (1.20.3), we get

$$\frac{a_1}{2} = \frac{a_2}{-1} = \frac{a_3}{1} = K \text{ (say)}$$

$$a_1 = 2K, a_2 = -K, a_3 = K$$

Putting in equation (1.21.1)

$$2K - 3K + K = 0$$

Thus for any value of K equation is satisfied. Hence the vectors are linearly dependent.

PART-5

Eigen Values and Eigen Vectors.

CONCEPT OUTLINE

Let A be a $n \times n$ matrix. Suppose the linear transformation $Y = AX$ transform X into a scalar multiple of itself *i.e.*,

$$AX = Y = \lambda X$$

i.e., X is an invariant vector.

Then the unknown scalar λ is known as an eigen value of the matrix A and the corresponding non-zero vector X as eigen vector.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{3} = k_3$$

$$\therefore \quad X_3 = k_3 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Que 1.22. **Show that the matrix** $A = \begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$**, has less than three linearly independent eigen vectors. Is it possible to obtain a similarity transformation that will diagonalize this matrix ?**

Answer

Characteristic equation of given matrix is,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 3-\lambda & 10 & 5 \\ -2 & -3-\lambda & -4 \\ 3 & 5 & 7-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)\,[(-3-\lambda)(7-\lambda)+20] - 10\,[-14+2\lambda+12] + 5\,[-10+9+3\lambda] = 0$$

$$(3-\lambda)\,[-21+3\lambda-7\lambda+\lambda^2+20] - 10\,(2\lambda-2) + 5\,(3\lambda-1) = 0$$

$$(3-\lambda)(\lambda^2-4\lambda-1) - 20\lambda + 20 + 15\lambda - 5 = 0$$

$$3\lambda^2 - 12\lambda - 3 - \lambda^3 + 4\lambda^2 + \lambda - 5\lambda + 15 = 0$$

$$-\lambda^3 + 7\lambda^2 - 16\lambda + 12 = 0$$

$$\lambda^3 - 7\lambda^2 + 16\lambda - 12 = 0$$

By solving above equation, we get

$$\lambda = 2, 2, 3$$

Eigen vector for $\lambda = 2$

$$\begin{bmatrix} 1 & 10 & 5 \\ -2 & -5 & -4 \\ 3 & 5 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x + 10y + 5z = 0 \qquad \text{...(1.22.1)}$$

$$-2x - 5y - 4z = 0 \quad \Rightarrow \quad 2x + 5y + 4z = 0 \qquad \text{...(1.22.2)}$$

$$3x + 5y + 5z = 0 \qquad \text{...(1.22.3)}$$

Using cross-Multiplication in eq. (1.22.1) and eq. (1.22.2), we get

$$\frac{x}{40-25} = \frac{y}{10-4} = \frac{z}{5-20}$$

$$\frac{x}{15} = \frac{y}{6} = -\frac{z}{15} \Rightarrow \frac{x}{5} = \frac{y}{2} = \frac{z}{-5} = k$$

$$\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 101 \\ 101 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\begin{bmatrix} 101 \\ 101 \end{bmatrix}$$

$$\begin{bmatrix} 101 \times 4 + 101 \times 2 \\ 101 \times 2 + 101 \times 4 \end{bmatrix} = \begin{bmatrix} 101\lambda_1 \\ 101\lambda_2 \end{bmatrix}$$

$$101\lambda_1 = 101 \times 6 \Rightarrow \lambda_1 = 6$$

$$101\lambda_2 = 101 \times 6 \Rightarrow \lambda_2 = 6$$

Que 1.24. **Find the eigen values and the corresponding eigen vectors of the following matrix,** $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix}$.

AKTU 2020-21, Marks 10

Answer

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

The characteristics equation is given by,

$$\begin{bmatrix} 2-\lambda & 0 & 1 \\ 0 & 3-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{bmatrix} = 0$$

$$(2-\lambda)\,[(3-\lambda)(2-\lambda) - 0] - 0\,(0-0) + 1\,[0-(3-\lambda)] = 0$$

$$(2-\lambda)(3-\lambda)(2-\lambda) - (3-\lambda) = 0$$

$$(3-\lambda)\,[(2-\lambda)^2 - 1] = 0$$

$$(3-\lambda)(\lambda^2 + 4 - 4\lambda - 1) = 0$$

$$(3-\lambda)(\lambda^2 - 4\lambda + 3) = 0$$

$$(3-\lambda)(\lambda^2 - 3\lambda - \lambda + 3) = 0$$

$$(3-\lambda)\,[\lambda(\lambda-3) - 1\,(\lambda-3)] = 0$$

$$(3-\lambda)(\lambda-1)(\lambda-3) = 0$$

$$\lambda = 1, 3, 3$$

The eigen values are 1, 3, 3

Eigen vectors can be calculated as follows :

When $\lambda = 1$, $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$x + z = 0$$

$$2y = 0$$

$$y = 0$$

Let $x = k_1$, $\therefore\ z = -k_1$

so
$$X_1 = k_1\begin{bmatrix}1\\0\\-1\end{bmatrix}$$

When $\lambda = 3$,
$$\begin{bmatrix}-1 & 0 & 1\\0 & 0 & 0\\1 & 0 & -1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$-x + z = 0 \Rightarrow x = z$$

$$y = 0$$

$$x - z = 0$$

$$\therefore\quad X_2 = X_3 = k_2\begin{bmatrix}1\\0\\1\end{bmatrix}$$

PART-6

Complex Matrices, Hermitian, Skew–Hermitian and Unitary Matrices, Applications to Engineering Problems.

CONCEPT OUTLINE

Complex Conjugate of a Matrix : A matrix in which complex elements are replaced by its corresponding conjugate complex numbers is called complex conjugate of that matrix. It is denoted as $\overline{A}$.

For Example : If $A = \begin{bmatrix}1+i & 2+3i\\2 & 3i\end{bmatrix}$

Then $\overline{A} = \begin{bmatrix}1-i & 2-3i\\2 & -3i\end{bmatrix}$

Hermitian Matrix : A square matrix A is said to be Hermitian if $(\overline{A})' = A$ or $A^\theta = A$.

Skew Hermitian Matrix : A square matrix A is said to be skew Hermitian if $(\overline{A})' = -A$ or $A^\theta = -A$.

Unitary Matrix : A square matrix A is said to be unitary if $AA^\theta = A^\theta A = I$.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 1.25. If $A = \begin{bmatrix}3 & 2-3i & 3+5i\\2+3i & 5 & i\\3-5i & -i & 7\end{bmatrix}$ then prove that $\overline{A}$ is Hermitian.

Answer

$$\overline{A}\ \text{(Conjugate of } A) = \begin{bmatrix}3 & 2+3i & 3-5i\\2-3i & 5 & -i\\3+5i & i & 7\end{bmatrix} = B\ \text{(Say)}$$

Then
$$B' = \begin{bmatrix}3 & 2-3i & 3+5i\\2+3i & 5 & i\\3-5i & -i & 7\end{bmatrix}$$

Now
$$(\overline{B})' = \begin{bmatrix}3 & 2+3i & 3-5i\\2-3i & 5 & -i\\3+5i & i & 7\end{bmatrix} = B$$

$\therefore$ B is Hermitian *i.e.*, $\overline{A}$ is Hermitian.

Que 1.26. Show that the matrix $\begin{bmatrix}\alpha+i\gamma & -\beta+i\delta\\\beta+i\delta & \alpha-i\gamma\end{bmatrix}$ is unitary if $\alpha^2+\beta^2+\gamma^2+\delta^2 = 1$.

Answer

$$A = \begin{bmatrix}\alpha+i\gamma & -\beta+i\delta\\\beta+i\delta & \alpha-i\gamma\end{bmatrix}$$

$$\overline{A} = \begin{bmatrix}\alpha-i\gamma & -\beta-i\delta\\\beta-i\delta & \alpha+i\gamma\end{bmatrix}$$

$$A^\theta = (\overline{A})' = \begin{bmatrix}\alpha-i\gamma & \beta-i\delta\\-\beta-i\delta & \alpha+i\gamma\end{bmatrix}$$

For unitary matrix, $AA^\theta = I$

Now, $AA^\theta = \begin{bmatrix} \alpha+i\gamma & -\beta+i\delta \\ \beta+i\delta & \alpha-i\gamma \end{bmatrix}\begin{bmatrix} \alpha-i\gamma & \beta-i\delta \\ -\beta-i\delta & \alpha+i\gamma \end{bmatrix}$

$$= \begin{bmatrix} (\alpha+i\gamma)(\alpha-i\gamma)+(-\beta+i\delta)(-\beta-i\delta) & (\alpha+i\gamma)(\beta-i\delta)+(-\beta+i\delta)(\alpha+i\gamma) \\ (\beta+i\delta)(\alpha-i\gamma)+(\alpha-i\gamma)(-\beta-i\delta) & (\beta+i\delta)(\beta-i\delta)+(\alpha-i\gamma)(\alpha+i\gamma) \end{bmatrix}$$

$$= \begin{bmatrix} \alpha^2+\gamma^2+\beta^2+\delta^2 & 0 \\ 0 & \alpha^2+\beta^2+\gamma^2+\delta^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Hence, A is a unitary matrix.

Que 1.27. **Find non-singular matrices P and Q such that PAQ is normal form, $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix}$.**

AKTU 2020-21, Marks 10

Answer

We know that $A = IAI$

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[R_3 \to R_3 - R_2]{R_1 \to R_1 - R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{C_3 \to C_3 - C_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{C_3 \to C_3 - C_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Now this is required normal form PAQ, therefore on comparing, we get

$$P = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \text{ and } Q = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

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$A(\bar{A})' = I$
$(\bar{A})'$

1.1. Prove that the matrix $\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$ is unitary.

AKTU 2020-21, Marks 02

OR

Prove that the matrix $A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$ is unitary.

Ans.
$$A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

$$A^{\theta} = (\bar{A})' = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

So,
$$AA^{\theta} = \frac{1}{3}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

1.2. If A is a unitary matrix, show that A^{-1} is also unitary.

Ans. $AA^{\theta} = A^{\theta}A = I$, since A is a unitary matrix.

$$(A A^{\theta})^{-1} = (A^{\theta}\ A)^{-1} = (I)^{-1}$$
$$(A^{\theta})^{-1} A^{-1} = A^{-1} (A^{\theta})^{-1} = I$$
$$(A^{-1})^{\theta} A^{-1} = A^{-1} (A^{-1})^{\theta} = I$$

1.3. Show that the matrix $A = \begin{bmatrix} 0 & i & 3 \\ -7 & 0 & 5i \\ 3i & 1 & 0 \end{bmatrix}$ is Hermitian matrix.

Ans.
$$A = \begin{bmatrix} 0 & i & 3 \\ -7 & 0 & 5i \\ 3i & 1 & 0 \end{bmatrix}$$

$A = (\bar{A})'$

$$\bar{A} = \begin{bmatrix} 0 & -i & 3 \\ -7 & 0 & -5i \\ -3i & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix}1 & 2\\0 & -3\end{bmatrix} = \begin{bmatrix}1 & 0\\-5 & 1\end{bmatrix}A \qquad R_2 \to R_2 - 5R_1$$

$$\begin{bmatrix}1 & 0\\0 & -3\end{bmatrix} = \begin{bmatrix}-7/3 & 2/3\\-5 & 1\end{bmatrix}A \qquad R_1 \to R_1 + \frac{2}{3}R_2'$$

$$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}-7/3 & 2/3\\5/3 & -1/3\end{bmatrix}A \qquad R_2 \to \frac{R_2}{-3}$$

$\because$ $I = A^{-1}A$

Thus, $A^{-1} = \begin{bmatrix}-7/3 & 2/3\\5/3 & -1/3\end{bmatrix}$

1.7. Define vector space.

Ans. A vector space is a (non empty) set V of vectors with the same number of components such that with any two vectors $\vec{a}$ and $\vec{b}$ in V all their linear combinations $\alpha\vec{a} + \beta\vec{b}$ (α, β any real numbers) are elements of V.

1.8. Reduce the matrix A to triangular form.

$$A = \begin{bmatrix}1 & 2 & 3\\2 & 5 & 7\\3 & 1 & 2\end{bmatrix}$$

Ans.

$$A = \begin{bmatrix}1 & 2 & 3\\2 & 5 & 7\\3 & 1 & 2\end{bmatrix}$$

$$= \begin{bmatrix}1 & 2 & 3\\0 & 1 & 1\\0 & -5 & -7\end{bmatrix} \qquad \begin{matrix}R_2 \to R_2 - 2R_1\\R_3 \to R_3 - 3R_1\end{matrix}$$

$$= \begin{bmatrix}1 & 2 & 3\\0 & 1 & 1\\0 & 0 & -2\end{bmatrix} \qquad R_3 \to R_3 + 5R_2$$

1.9. Reduce the matrix $\begin{bmatrix}1 & 1 & 1\\3 & 1 & 1\end{bmatrix}$ into normal form and find its rank.

AKTU 2021-22, Marks 02

$$(3-\lambda)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)\,[\{(1-\lambda)^2 - 1\} - 1\,(1-\lambda-1) + 1\,(1-1+\lambda)] = 0$$

$$(3-\lambda)\,[1+\lambda^2 - 2\lambda - 1 + \lambda + \lambda] = 0$$

$$\lambda^2(3-\lambda) = 0$$

$$\therefore \qquad \lambda = 0, 0, 3$$

1.12. Find the rank of the matrix $\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$.

AKTU 2018-19, Marks 02

Ans. $\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - R_1$

$$\begin{bmatrix} 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Rank of matrix = Number of non-zeroes row = 1

1.13. If the eigen values of matrix A are 1, 1, 1, then find the eigen values of $A^2 + 2A + 3I$.

AKTU 2018-19, Marks 02

Ans. Eigen values of A^2 are $1^2, 1^2, 1^2$

Eigen values of $2A$ are 2, 2, 2

Eigen values of $3I$ are 3, 3, 3

$\therefore$ Eigen values of $A^2 + 2A + 3I$ are

$(1 + 2 + 3), (1 + 2 + 3), (1 + 2 + 3) = 6, 6, 6$

1.14. If the matrix $A = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 3 & 5 \\ 0 & 0 & -2 \end{bmatrix}$, **then find the eigen value of $A^3 + 5A + 8I$.**

AKTU 2021-22, Marks 02

1.16. Find the value of 'b' so that rank of $A = \begin{bmatrix} 2 & 4 & 2 \\ 3 & 1 & 2 \\ 1 & 0 & b \end{bmatrix}$ is 2.

AKTU 2019-20, Marks 02

Ans. Since, rank of $A = 2$

So, $|A| = 0$

$$\begin{vmatrix} 2 & 4 & 2 \\ 3 & 1 & 2 \\ 1 & 0 & b \end{vmatrix} = 0$$

$$2(b-0) - 4(3b-2) + 2(0-1) = 0$$

$$2b - 12b + 8 - 2 = 0$$

$$b = 3/5$$

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2 UNIT

Differential Calculus-I (2 Marks Questions)

2.1. Find the 8th derivative of x^2e^x.

OR

If $y = x^2 e^x$, find y_n.

Ans. $y = x^2 e^x$

n^{th} derivative of the given equation is find out by Leibnitz's theorem,

i.e., $y_n = D^n u.v + {}^nC_1 D^{n-1}u.\, Dv + {}^nC_2 D^{n-2}u\, D^2v + ...$

$$= x^2 D^n(e^x) + {}^nC_1 D(x^2) D^{n-1}(e^x) + {}^nC_2 D^2(x^2) D^{n-2}(e^x)$$

$$= e^x x^2 + n.2x\, e^x + \frac{n(n-1)}{2} 2\, e^x = [x^2 + 2nx + n^2 - n]\, e^x$$

$$y_n = [n(n-1) + x^2 + 2nx]\, e^x$$

Hence, $y_8 = x^2e^x + 8(2x)e^x + 8(7)e^x$

$$y_8 = x^2e^x + 16xe^x + 56e^x$$

2.2. Find the n^{th} derivative of $x^{n-1} \log x$.

Ans. Let, $y = x^{n-1} \log x$

$$y_1 = (n-1)\, x^{n-2} \log x + x^{n-1}\left(\frac{1}{x}\right)$$

$$xy_1 = (n-1)\, x^{n-1} \log x + x^{n-1}$$

$$xy_1 = (n-1)\, y + x^{n-1}$$

Differentiating $(n-1)$ times by Leibnitz theorem,

$$xy_n + (n-1)\, y_{n-1} = (n-1)y_{n-1} + (n-1)!$$

$$xy_n = (n-1)!$$

$$y_n = \frac{(n-1)!}{x}$$

2.3 If $y = e^{a \sin^{-1}x}$, find the value of $(1-x^2)\, y_2 - xy_1 - a^2y$.

Ans. $\because$ $y = e^{a \sin^{-1}x}$

On differentiating with respect to x,

$$y_1 = e^{a\sin^{-1}x} \frac{a}{\sqrt{1-x^2}} = \frac{ay}{\sqrt{1-x^2}}$$

or $(1-x^2)\, {y_1}^2 = a^2y^2$

Again differentiating,

$(1-x^2)\, 2y_1 y_2 + (-2x)\, y_1^2 = 2a^2 yy_1$

On dividing by $2y_1$, $(1-x^2)\, y_2 - xy_1 - a^2y = 0$