

# Solving parabolic PDEs in parallel via time discretization using Laplace transforms

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December 3, 2012

# The Problem

$$u_t + Au = f(t), \quad \text{for } t > 0, \quad \text{with } u(0) = u_0$$

where  $A$  is the second-order differential operator ( $\nabla^2$ )  
and  $u_0$  and  $f(t)$  are given.

i.e. The heat equation...

# The Transformed Problem

$$(zI + A)w(z) = u_0 + \hat{f}(z)$$

where  $z$  is the transform variable,  $I$  is the identity matrix,  $\hat{f}(z) = \mathcal{L}\{f\}$ , and  $w(z) = \mathcal{L}\{u\}$ .

Note: all functions above have implied spatial terms based on the dimensionality of the problem

## The Solution: space

We find an approximation for  $w(z)$  using the finite difference method with  $m - 1$  interior points,  $m$  a chosen parameter. In one spacial dimension,  $A$  is the  $m - 1 \times m - 1$  discrete Laplacian given by:

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 & \cdots & \cdots & \cdots \\ -1 & 2 & -1 & 0 & 0 & \cdots & \cdots \\ 0 & -1 & 2 & -1 & 0 & 0 & \cdots \\ \vdots & & & \ddots & & & \vdots \\ \vdots & & & & \ddots & & \vdots \\ 0 & 0 & \cdots & \cdots & -1 & 2 & -1 \\ 0 & 0 & \cdots & \cdots & \cdots & -1 & 2 \end{bmatrix}$$

# The Solution: space (continued)

The tridiagonal system we solve becomes

$$\frac{1}{h^2}(zI + A)w(z) = g(t)$$

where

$$g(t) = \vec{u}_0 + \vec{\hat{f}}(z) + \frac{1}{h^2} \begin{bmatrix} \alpha \\ 0 \\ 0 \\ \vdots \\ \beta \end{bmatrix}$$

with

$$h = \frac{b-a}{m}, \text{ } a \text{ and } b \text{ the endpoints of our domain, } \alpha = u(a), \beta = u(b),$$

and  $\vec{u}_0 + \vec{\hat{f}}(z)$  vectors of size  $m - 1$  containing  $u_0$  and  $\hat{f}(z)$  evaluated at  $x_i = a + ih, \quad i = 1, 2, \dots, m - 1$ .

# The Solution: time

Using a trapezoidal rule:

$$u(t) \approx U_{N,\tau}(t) = 2\operatorname{Re} \left( \frac{1}{N\tau} \sum_{j=0}^{N-1} {}' \tilde{\mu}_j e^{z_j t} w(z_j) \right)$$

where  $z_j$  are the quadrature points on the transformed contour,  
 $\tilde{\mu}_j$  are the weights associated with each  $z_j$ ,  
 $\tau$  is a time scaling parameter,  
and the ' denotes halving the first term in the sum.

The first term is halved and the sum from 0 to  $N - 1$  is doubled because of the even symmetry of the contour.

# Parallel Approach

We parallelized the sum from the previous slide using MPI.

Using this approach, each core, except the first and last cores, computes a local sum corresponding to  $\frac{N-1}{P}$  quadrature points.

The first core computes the first term in the full sum, and adds to that a local sum corresponding to  $\frac{N-1}{P}$  quadrature points.

The last core computes a local sum corresponding to  $\frac{N-1}{P} + \text{Mod}(N - 1, P)$  quadrature points.

We then use MPI\_Reduce with the master core to reduce the local sums into a full sum.

# Test PDE

$$u_t - u_{xx} = f(x, t), \quad \text{for } 0 < x < \pi, \quad t > 0,$$

$$u(x, t) = 0 \quad \text{for } x = 0 \text{ and } \pi, \quad t > 0,$$

with

$$u(x, 0) = u_0(x), \quad \text{for } 0 < x < \pi$$

and

$$f(x, t) = e^{-t} \sin(x) + e^{-2t} (2\cos(t) - \sin(t)) \sin(2x),$$

$$u_0(x) = \sin(x) + \sin(2x),$$

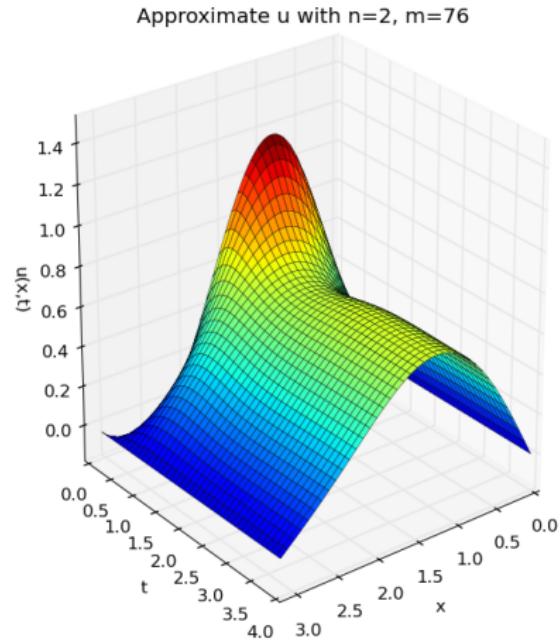
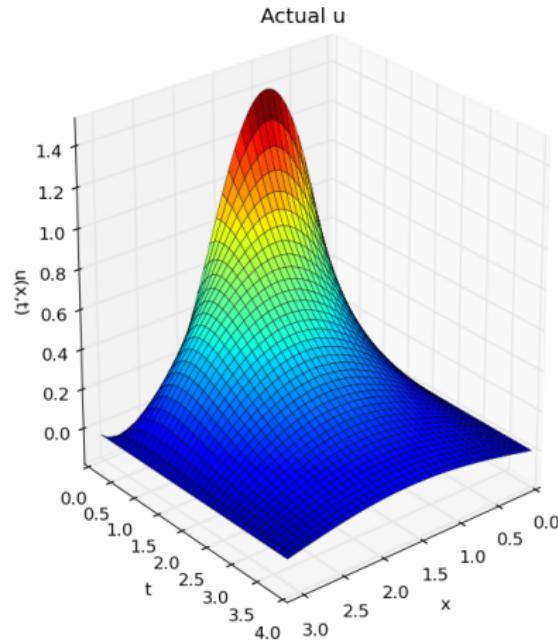
$$\hat{f}(x, z) = \frac{1}{1+z} \sin(x) + \frac{2z+3}{(z+2)^2+1} \sin(2x)$$

# Actual Solution

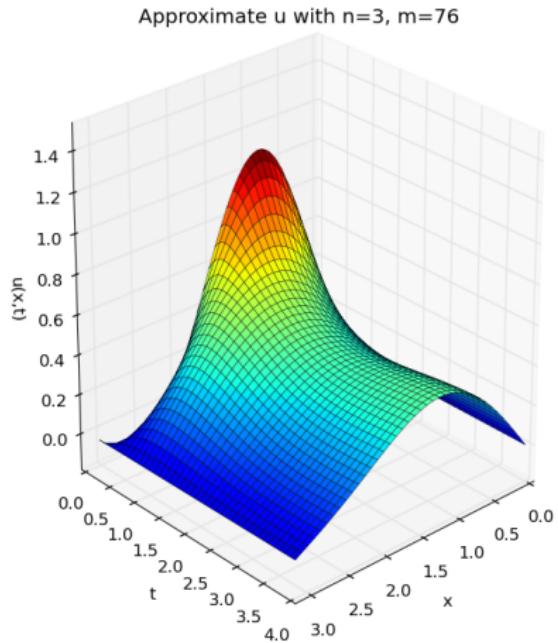
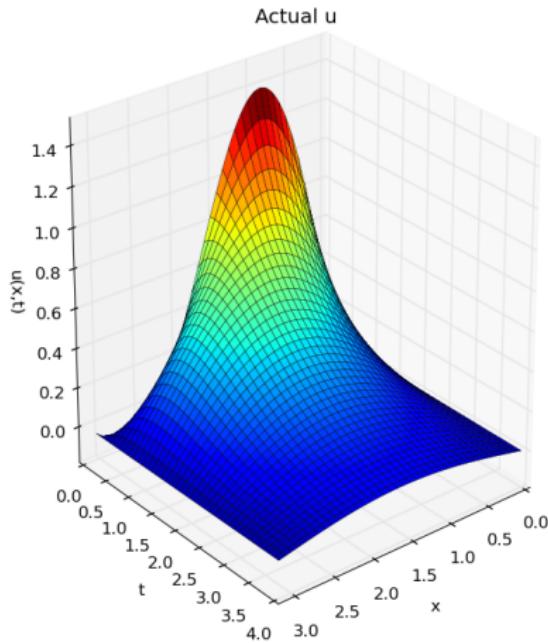
$$u(x, t) = (1 + t)e^{-t} \sin(x) + \cos(t)e^{-2t} \sin(2x)$$

animation.mpg

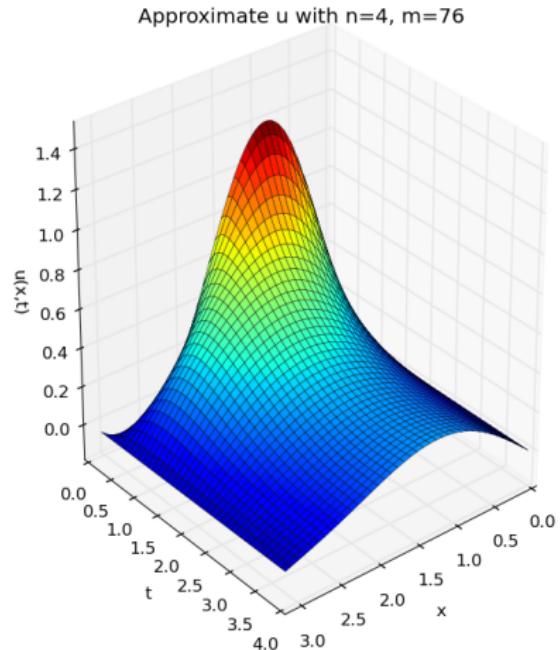
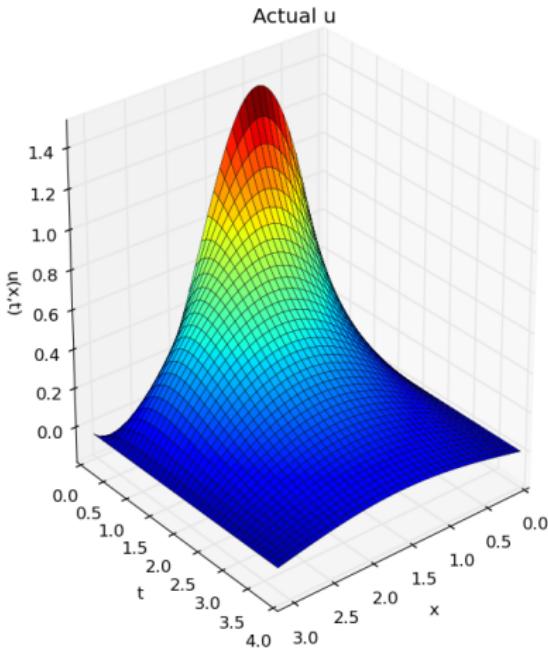
# Results



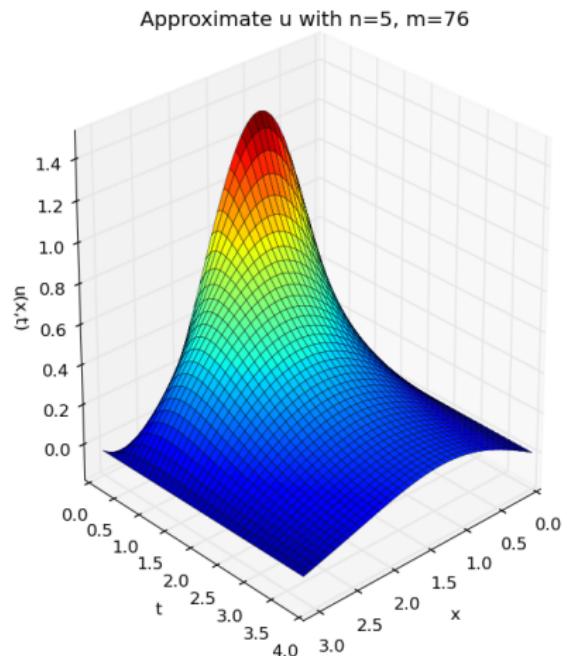
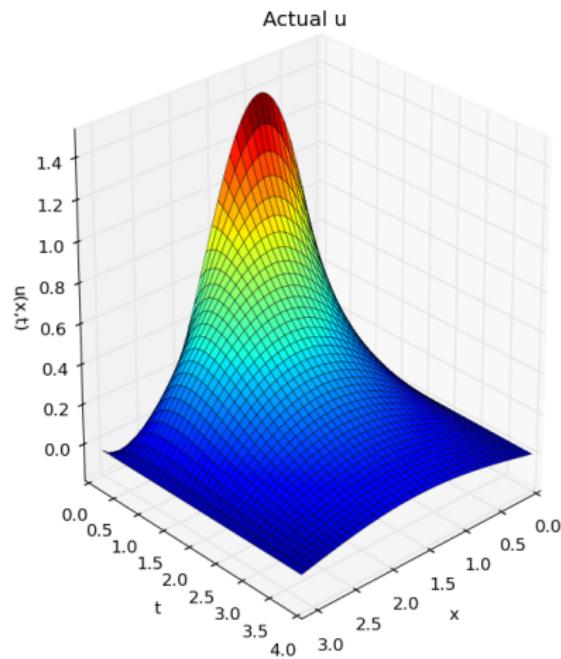
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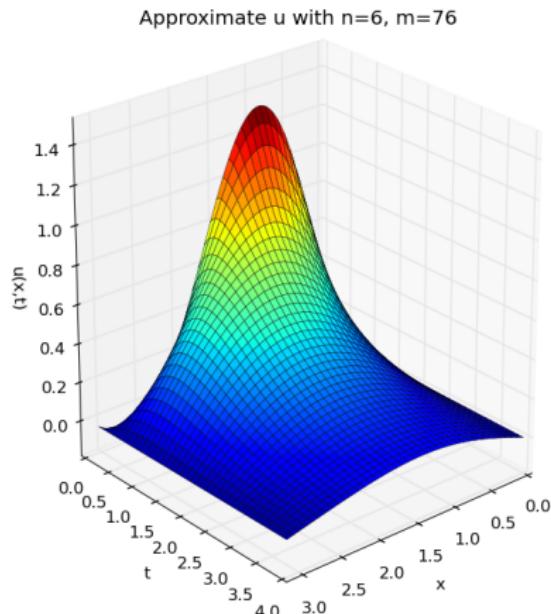
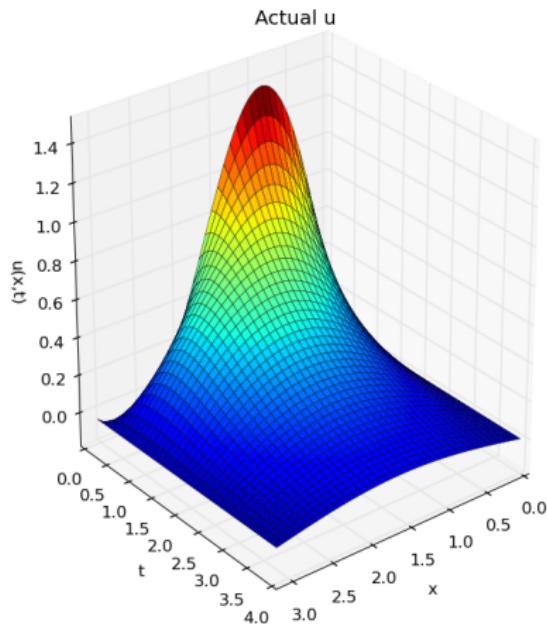
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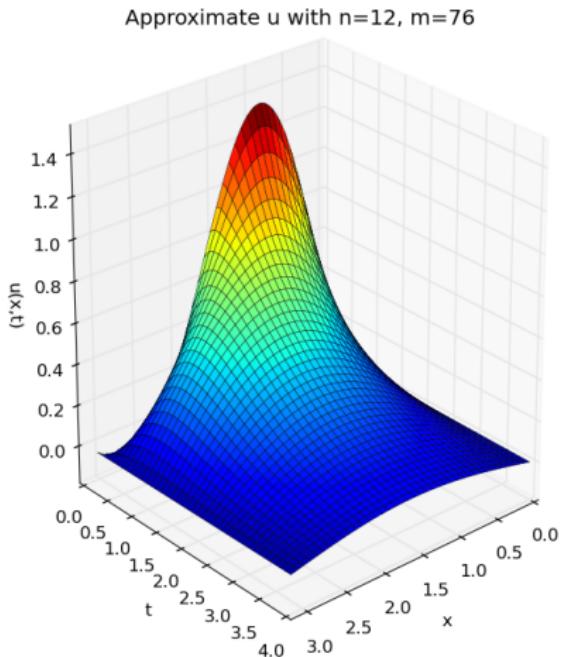
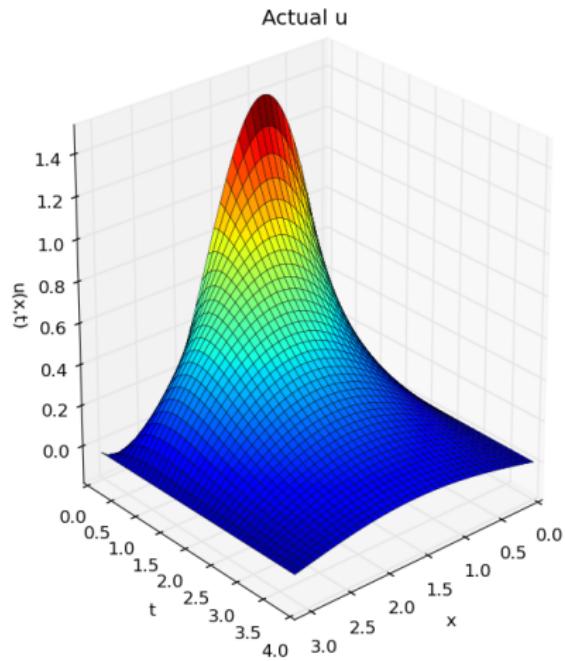
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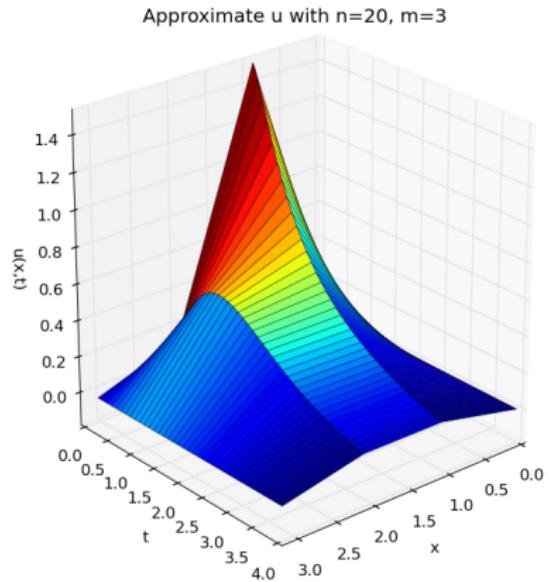
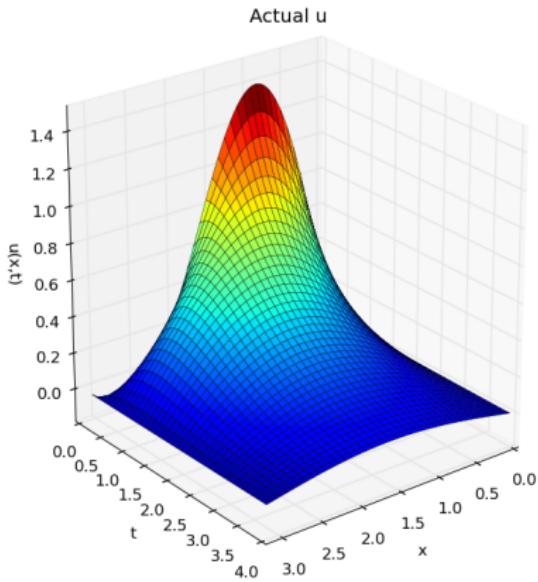
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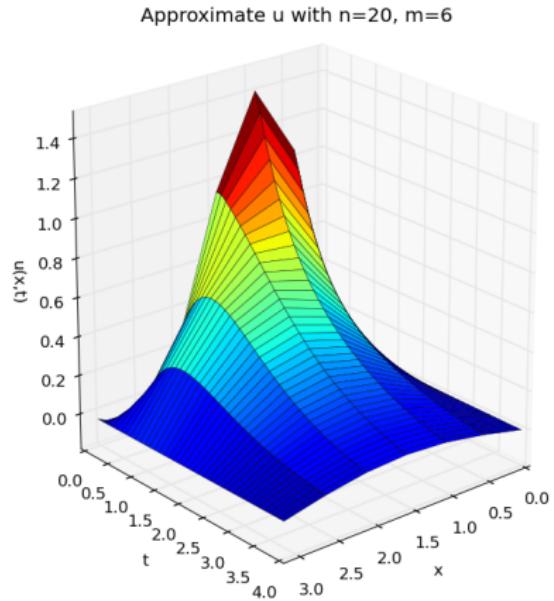
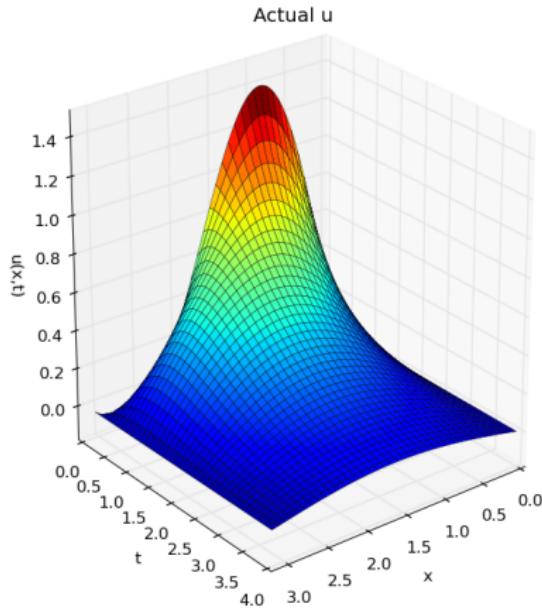
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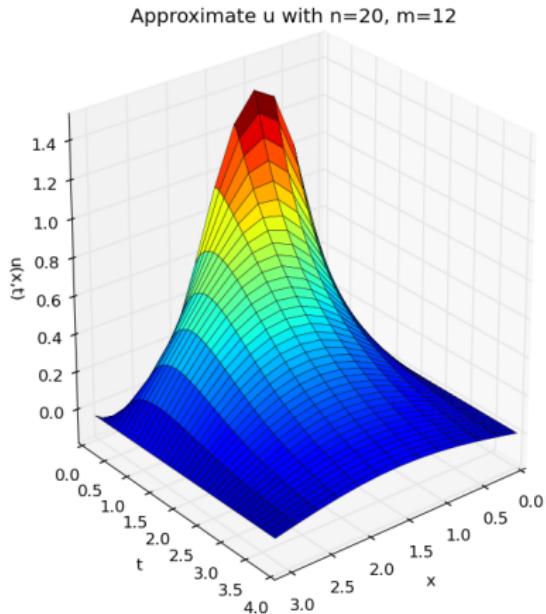
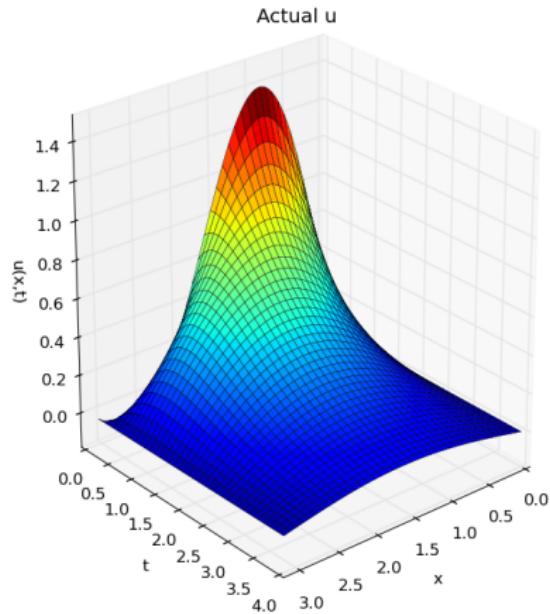
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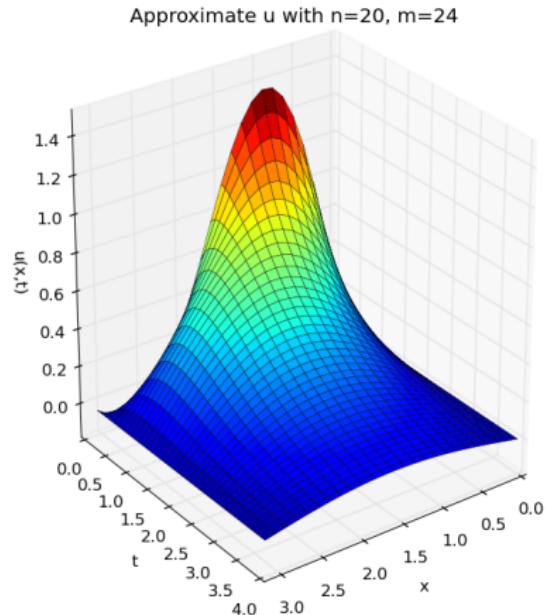
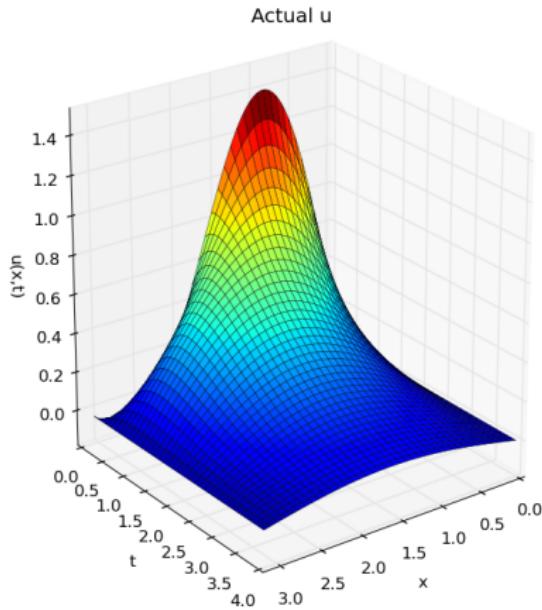
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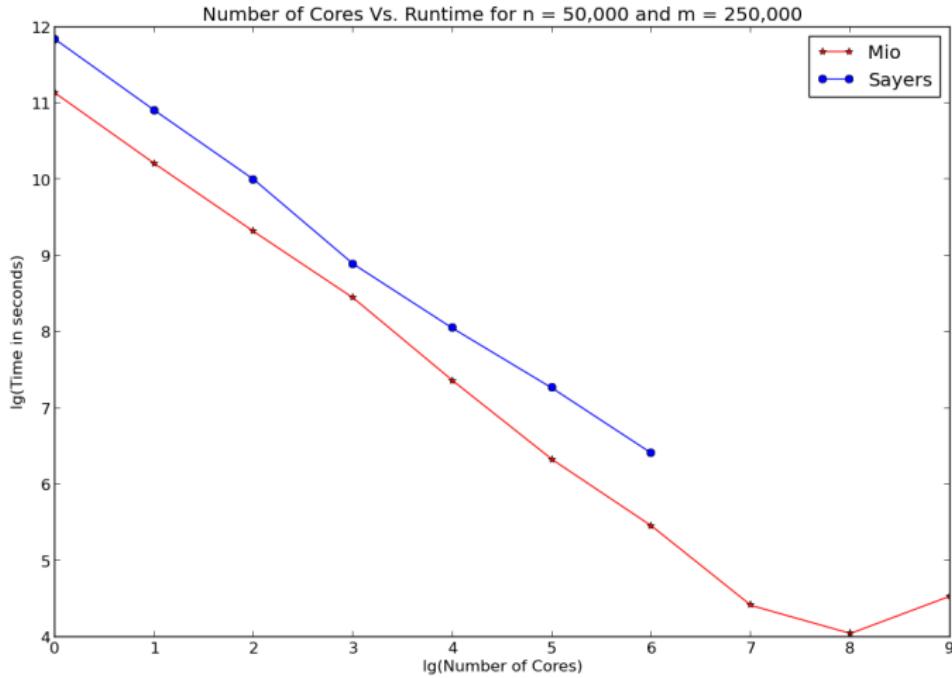
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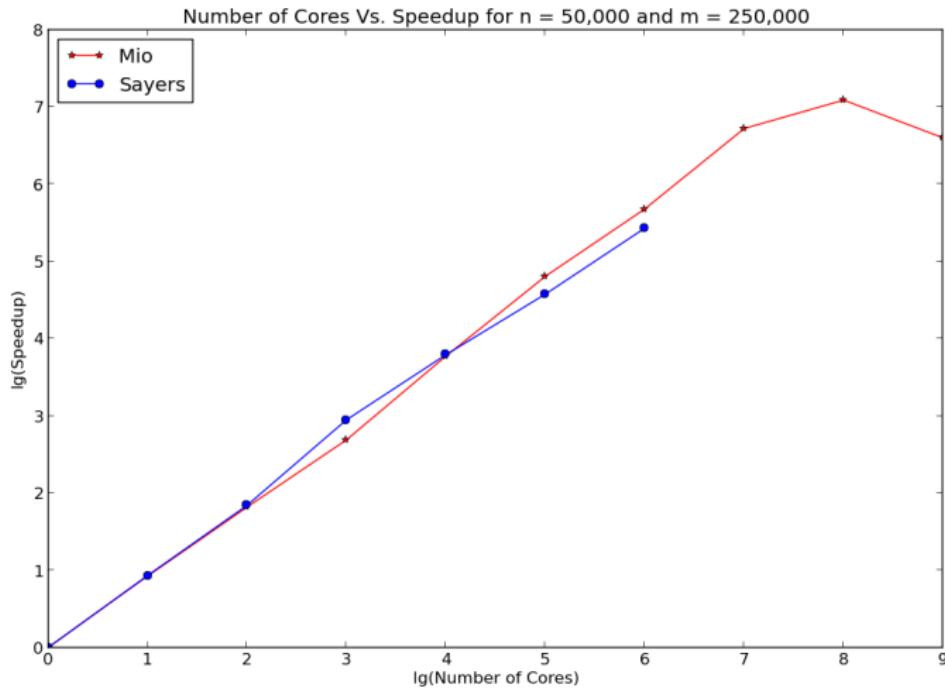
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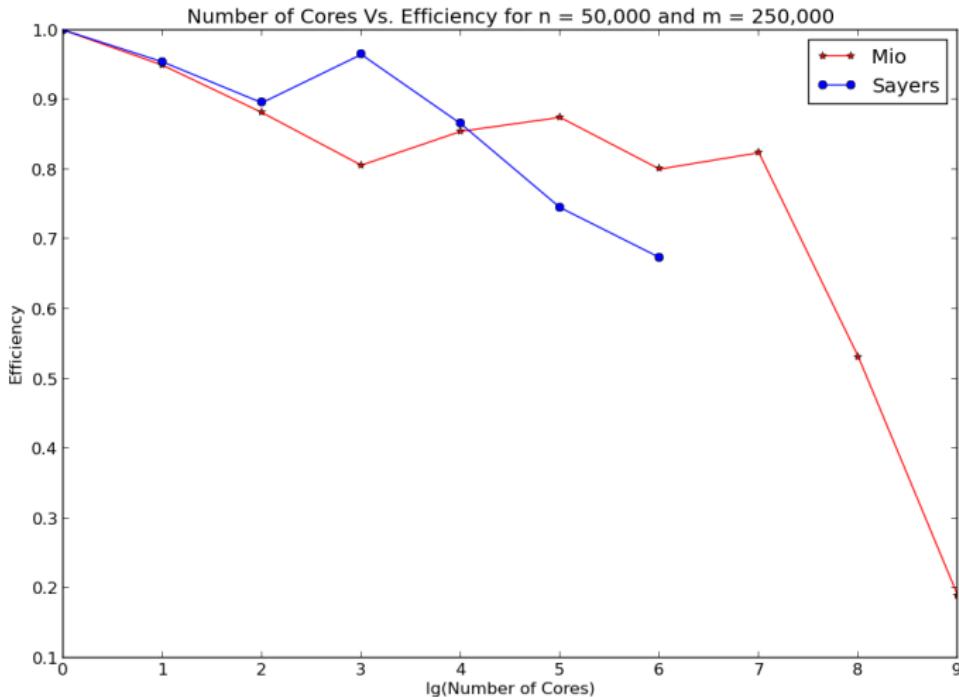
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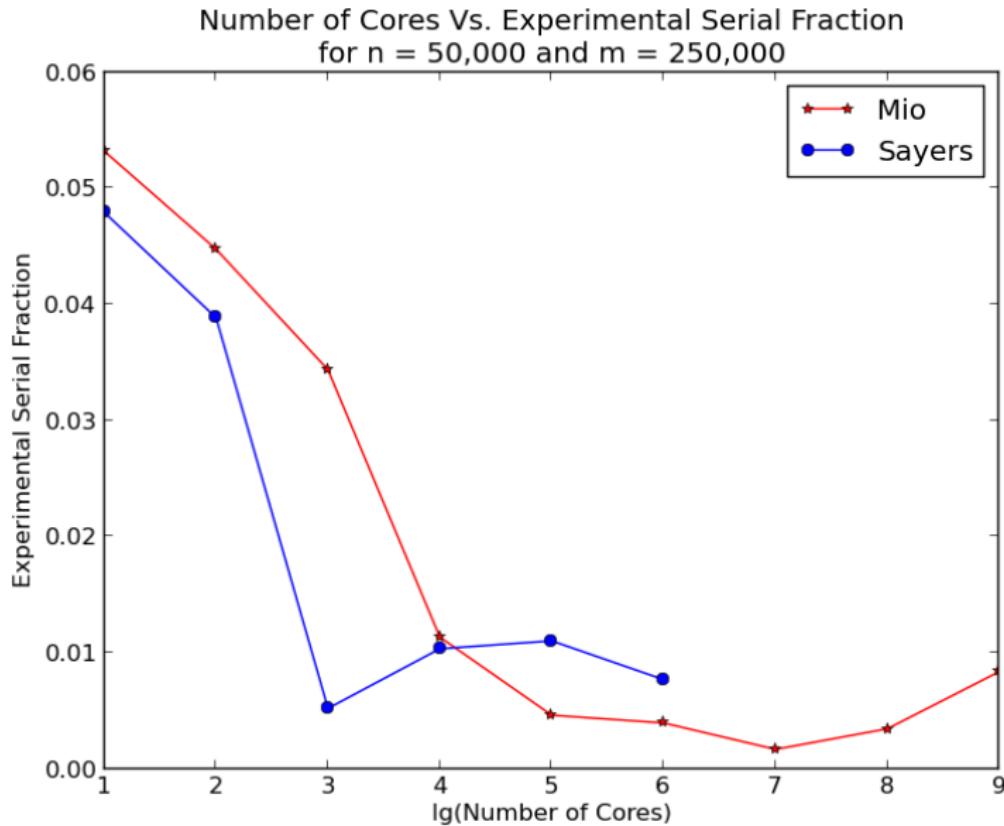
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## References

Dongwoo Sheen, Ian H. Sloan, and Vidar Thomée, *A parallel method for time discretization of parabolic equations based on Laplace transformation and quadrature*. IMA Journal of Numerical Analysis (2003) 23, 269-299

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