Solving parabolic PDEs in parallel via time discretization using Laplace transforms

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$$U_{N,\tau}(t) = 2\mathrm{Re}\left(\frac{1}{N\tau}\sum_{j=0}^{N-1}{}'\tilde{\mu}_j \mathrm{e}^{z_j t} w(z_j)\right)$$

$$(zI + A)w(z) = u_0 + \hat{f}(z)$$

$$u_t + Au = f(t), \quad \text{for } t > 0, \quad \text{with } u(0) = u_0$$

$$u_t - u_{xx} = f(x,t), \quad \text{for } 0 < x < \pi,\ t > 0,$$

$$u(x,t) = 0 \quad \text{for } x = 0 \text{ and } \pi,\ t > 0, \quad \text{with } u(x,0) = u_0(x), \quad \text{for } 0 < x < \pi$$

$$u(x,t) = (1+t)\mathrm{e}^{-t}\sin(x) + \cos(t)\mathrm{e}^{-2t}\sin(2x)$$

$$u_0(x) = \sin(x) + \sin(2x)$$

$$\hat{f}(x,z) = \frac{1}{1+z}\sin(x) + \frac{2z+3}{(z+2)^2+1}\sin(2x)$$

Summary

In a normal space-time setting, partial differential equations can be impossible to solve using parallel methods. For each unit of time, the unknown function relies on the values at previous times. Given a parabolic PDE, one can transform the problem into many discrete, space-frequency problems, independent of one another. Following this transformation, parallel methods can be applied to solving the problem with high speed and accuracy. The methods used in this project can be applied to many fields involving differential equations in time, reducing the time needed to solve problems which previously could only be solved with a serial mentality.

Project Description

Aims and Background

The aim of this project is to produce a working program for numerically approximating solutions to space-time differntial equations. We will use a Laplace transform to transform the problem from temporal space into a frequency space. Aiding us in our research and development of this project is an article[1] written in the IMA Journal of Numerical Analysis on the subject matter. We will attempt to replicate the results gained by the authors in this article.

Approach and Methodology

We will begin our studies with producing a working serial version of the algorithm for ordinary differential equations with known solutions, with which we can then compare computed results. Following successful completion of said serial algorithm, we will make the necessary adjustments as to allow for partial differential equation computations, still performing all computations in serial. Once a fully working serial version of the code is completed, we will proceed to parallelize the algorithm to produce our intended final result. By approaching the problem with this methodology, we can work in steps in which we feel comfortable. Between each step, should we need outside guidance, we can approach Dr. Ganesh with any problems we are having, while still showing that we are making progress toward a final production. Throughout the time spent working through these milestones, we can further our knowledge on the subject matter by continuing to digest more of the material in the provided article[1] as well as learn new material in the field of numerical solutions to PDEs through Dr. Ganesh's private lectures.

Expected Outcome

Since we are replicating the process in the article by Sheen, Sloan, and Thomée, we expect to get the same results the authors show in section 6 (Numerical examples) in their paper.[1]

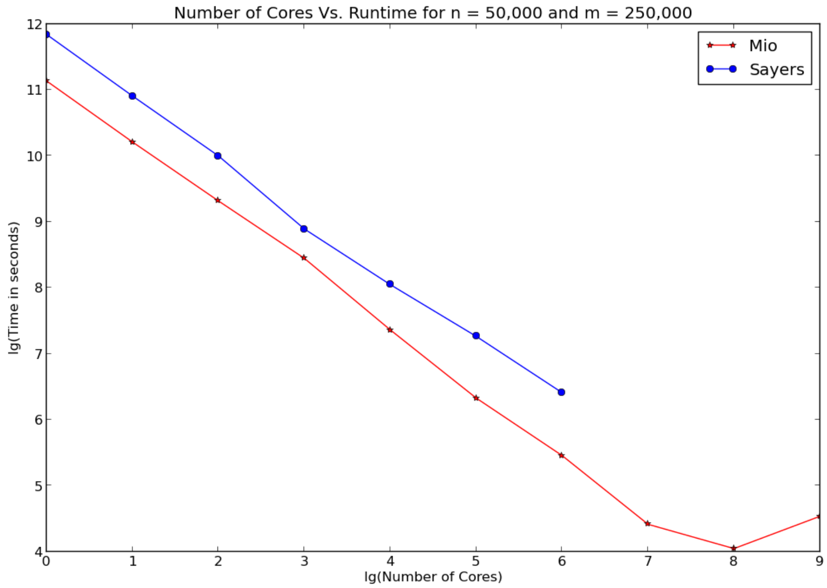
Tables and Plots

t	e_10	e_20	p_20	e_40	p_40	e_80	p_80
0.2	3.715E-04	2.018E-04	0.88	3.574E-04	-0.82	3.451E-04	0.05
0.4	4.808E-04	1.654E-04	1.54	3.972E-05	2.06	1.710E-06	4.54
0.6	1.704E-05	2.586E-05	-0.60	9.553E-06	1.44	1.705E-06	2.49
0.8	1.441E-05	1.837E-07	6.29	1.175E-06	-2.68	3.131E-07	1.91
1.0	9.086E-06	1.158E-06	2.97	4.299E-08	4.75	4.452E-08	-0.05
1.2	1.480E-06	2.628E-07	2.49	2.781E-08	3.24	5.071E-09	2.46
1.4	5.214E-06	2.419E-08	7.75	9.365E-09	1.37	2.872E-10	5.03
1.6	9.582E-06	2.534E-08	8.56	1.248E-09	4.34	6.188E-11	4.33
1.8	1.545E-05	7.777E-09	10.96	1.674E-10	5.54	2.650E-11	2.66
2.0	2.305E-05	7.890E-09	11.51	1.221E-10	6.01	5.444E-12	4.49
3.0	9.117E-05	2.193E-08	12.02	1.247E-13	17.42	6.523E-16	7.58
4.0	2.640E-04	6.540E-08	11.98	5.787E-15	23.43	6.939E-18	9.70

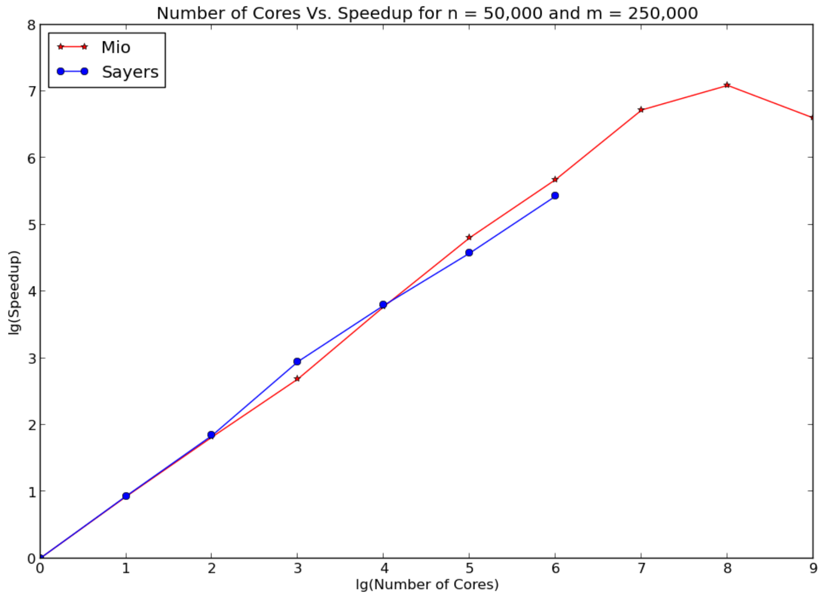
	Tau = 0.25			Tau = 1.0		
t	e_40	e_80	p_80	e_40	e_80	p_80
0.2	1.954E-05	6.513E-07	4.91	4.940E-03	2.792E-03	0.82
0.4	4.151E-07	1.258E-07	1.72	1.344E-03	1.060E-03	0.34
0.6	9.456E-09	1.612E-09	2.55	7.211E-04	3.211E-04	1.17
0.8	7.712E-10	2.510E-11	4.94	6.373E-05	2.477E-05	1.36
1.0	4.394E-10	1.504E-12	8.19	6.359E-05	3.183E-05	1.00
1.2	4.342E-10	2.195E-14	14.27	2.678E-05	3.323E-06	3.01
1.4	1.454E-09	1.832E-15	19.60	8.530E-07	2.766E-06	-1.70
1.6	2.700E-09	4.163E-16	22.63	4.399E-06	8.962E-07	2.30
1.8	4.220E-09	4.441E-16	23.18	1.281E-06	1.798E-07	2.83
2.0	6.075E-09	5.829E-16	23.31	3.676E-07	1.596E-07	1.20
3.0	2.355E-08	2.318E-15	23.28	3.466E-09	1.814E-09	0.93
4.0	7.010E-08	6.994E-15	23.26	6.142E-10	3.219E-11	4.25

t	e_10	e_20	p_20	e_40	p_40	e_80	p_80
0.2	1.269E-02	2.102E-02	-0.73	2.381E-02	-0.18	2.450E-02	-0.04
0.4	2.140E-02	8.455E-03	1.34	5.226E-03	0.69	4.416E-03	0.24
0.6	1.522E-02	3.802E-03	2.00	9.256E-04	2.04	2.070E-04	2.16
0.8	1.212E-02	2.912E-03	2.06	6.515E-04	2.16	8.384E-05	2.96
1.0	9.506E-03	2.351E-03	2.02	6.009E-04	1.97	1.661E-04	1.86
1.2	7.370E-03	1.811E-03	2.02	4.520E-04	2.00	1.141E-04	1.99
1.4	5.859E-03	1.470E-03	2.00	3.650E-04	2.01	8.993E-05	2.02
1.6	5.148E-03	1.276E-03	2.01	3.186E-04	2.00	7.977E-05	2.00
1.8	4.663E-03	1.176E-03	1.99	2.937E-04	2.00	7.348E-05	2.00
2.0	4.468E-03	1.114E-03	2.00	2.783E-04	2.00	6.958E-05	2.00
3.0	3.091E-03	7.691E-04	2.01	1.927E-04	2.00	4.823E-05	2.00
4.0	1.825E-03	4.533E-04	2.01	1.133E-04	2.00	2.854E-05	1.99

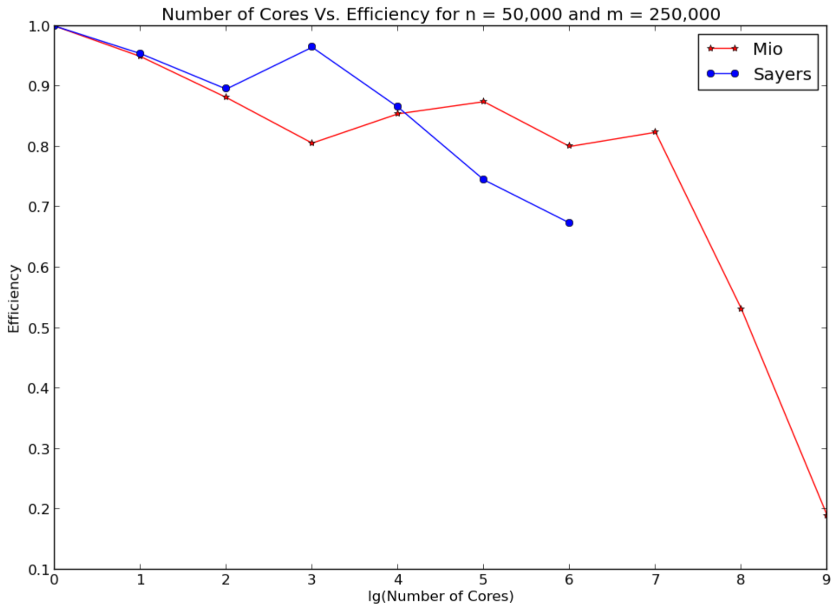
Cores	Sayers Lab	Mio
1.0	3661.0877999999998	2245.8103000000001
2.0	1918.2696000000001	1182.6335999999999
4.0	1021.9571	636.83158000000003
8.0	474.32260000000002	348.26038
16.0	264.32758000000001	164.24718999999999
32.0	153.50402	80.234003999999999
64.0	84.893333999999996	43.834428000000003
128.0	N/A	21.293009999999999
256.0	N/A	16.503080000000001
512.0	N/A	23.166782000000001



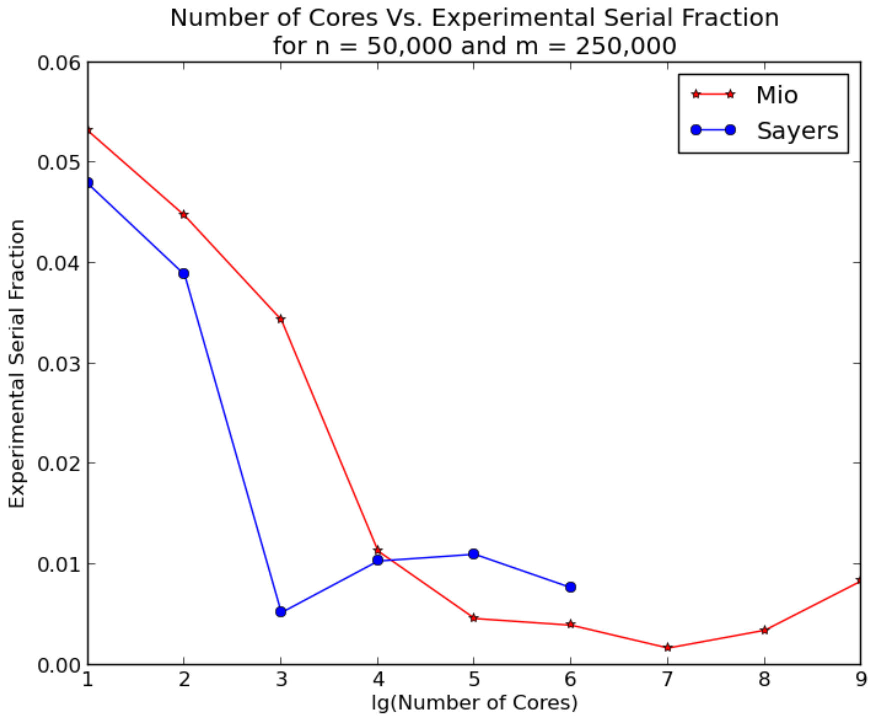
Cores	Sayers Lab	Mio
1.0	1.0	1.0
2.0	1.9085366311388137	1.898990777870678
4.0	3.5824280686537624	3.5265372675142777
8.0	7.718560743257858	6.4486528728878092
16.0	13.850570568534692	13.673355994705298
32.0	23.850110244669814	27.990754393860239
64.0	43.125739413179367	51.233936484810521
128.0	N/A	105.47171583538449
256.0	N/A	136.08431274646915
512.0	N/A	96.940969185966352



Cores	Sayers Lab	Mio
1.0	1.0	1.0
2.0	0.95426831556940683	0.94949538893533902
4.0	0.8956070171634406	0.88163431687856941
8.0	0.96482009290723225	0.80608160911097615
16.0	0.86566066053341828	0.85458474966908116
32.0	0.74531594514593169	0.87471107480813248
64.0	0.67383967833092762	0.80053025757516438
128.0	N/A	0.8239977799639413
256.0	N/A	0.53157934666589512
512.0	N/A	0.18933783044134053



Cores	Sayers Lab	Mio
2.0	0.047923297551072164	0.053191001929236759
4.0	0.03885371628253953	0.044752372896321647
8.0	0.0052089514409975812	0.034367025567564609
16.0	0.010345804508339492	0.011343930518085162
32.0	0.01102299598743091	0.0046204722317963586
64.0	0.0076830559693734967	0.0039551114498611187
128.0	N/A	0.0016818543519778373
256.0	N/A	0.0034556341402023306
512.0	N/A	0.0083787958735268026



References

[1] Dongwoo Sheen, Ian H. Sloan, and Vidar Thomée, *A parallel method for time discretization of parabolic equations based on Laplace transformation and quadrature*. IMA Journal of Numerical Analysis (2003) 23, 269-299