MC 123 PROJECT

CATALAN NUMBERS — LATTICE PATHS
SCHRÖDER NUMBERS

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ABSTRACT

This self exploratory project for the MC 123 course covers the topics of Catalan Numbers, Lattice Paths and Schröder Numbers.

The Catalan Numbers are introduced with the help of prior knowledge of the counting sequences. Then it is observed how Catalan Numbers help us to perceive some specific type of Lattice Path problems. Further, some theorems are explored related to the Lattice Paths and HVD Lattice Paths from which the concept of Schröder Numbers is developed.

Contents

1 INTRODUCTION

We have been using a number of **counting sequences** to count the number of objects or ways in which objects are arranged.

For Example:
1) The sequence of Natural Numbers. (1, 2, 3, 4,...)
2) The sequence of Factorials. (1!, 2!, 3!, 4!,...)
3) The sequence of Combinations. $(\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \binom{n}{3}, \ldots)$
4) The sequence of Fibonacci Numbers. (1, 1, 2, 3, 5,...)

There are other **special counting sequences** like :

- **Catalan Numbers sequence**
- **Stirling Numbers sequence** (first and second kind)
- **Schröder Numbers sequence** (small and large)
- **Sequence of Partitions**

which help us to count various objects and their arrangements under different contexts.

2 CATALAN NUMBERS (C_n)

The `Catalan Sequence` is the sequence :

$$C_0, C_1, C_2, C_3, C_4, \ldots\ C_n, \ldots$$

where,

$$C_n = \frac{1}{n+1}\binom{2n}{n}, \qquad (n = 0, 1, 2, 3, \ldots)$$

is the n^{th} `Catalan number`.[1]

The first few `Catalan numbers` are:

$C_0 = 1$	$C_5 = 42$
$C_1 = 1$	$C_6 = 132$
$C_2 = 2$	$C_7 = 429$
$C_3 = 5$	$C_8 = 1430$
$C_4 = 14$	$C_9 = 4862$

2.1 Properties

1) The alternative expressions for C_n are :

$$C_n = \binom{2n}{n} - \binom{2n}{n+1} = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!n!} \tag{1}$$

This shows that C_n is an integer.[3]

2) The `Catalan numbers` satisfy the recurrence relation :

$$C_0 = 1 \qquad and \qquad C_{n+1} = \frac{2(2n+1)}{n+2}C_n \text{[3]} \tag{2}$$

3) Asymptotically the Catalan numbers grow as :

$$C_n = \mathcal{O}\left(\frac{4^n}{n^{\frac{3}{2}}\sqrt{\pi}}\right) \tag{3}$$

The quotient of the n^{th} `Catalan number` and the R.H.S. of expression (3) tends toward 1 as $n \to \infty$. This can be proved by using Stirling's approximation for n!.[3]

4) The `Catalan numbers` C_n which are odd are those for which $n = 2^k - 1$. All others are even.[3]

5) The only prime `Catalan numbers` are $C_2 = 2$ and $C_3 = 5$.[3]

2.2 Applications

1) C_n is the number of Dyck words of length 2n. A Dyck word is a string consisting of n X's and n Y's such that no initial segment of the string has more Y's than X's. For example the following are the Dyck words of length 6 :[3]

XXXYYY XYXXYY XYXYXY XXYYXY XXYXYY

2) Reinterpreting the symbol X as an open parenthesis and Y as a close parenthes, C_n counts the number of expressions containing n pairs of parenthesis which are correctly matched :[3]

((())) ()(()) ()()() (())() (()())

3) C_n is a number of different ways in plus one factors can be completely parenthesized or the number of ways of associating n applications of binary operator as in the matrix chain multiplication problem. For n = 3, for example we have the following five different parenthesization of four factors:[3]

((ab)c)d (a(bc))d (ab)(cd) a((bc)d) a(b(cd))

4) Successive applications of a binary operator can be represented in terms of a full binary tree. (A rooted binary tree is full if every vertex has either two children or no children.) It follows that C_n is the number of full binary trees with n + 1 leaves:[3]

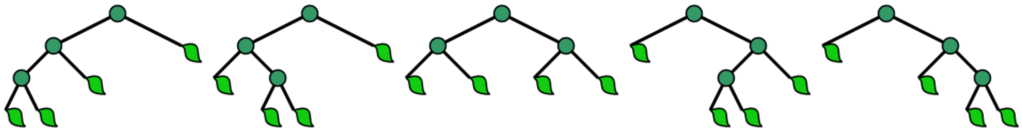

Figure 1: Number of full binary trees with 4 leaves.[3]

5) C_n is the number of monotonic lattice paths along the edges of a grid with $n \times n$ square cells, which do not pass above the diagonal. A monotonic path is one which starts in the lower left corner, finishes in

the upper right corner, and consists entirely of edges pointing rightwards or upwards. Counting such paths is equivalent to counting Dyck words: X stands for "move right" and Y stands for "move up".[3]
The following diagrams show the case n = 4 :

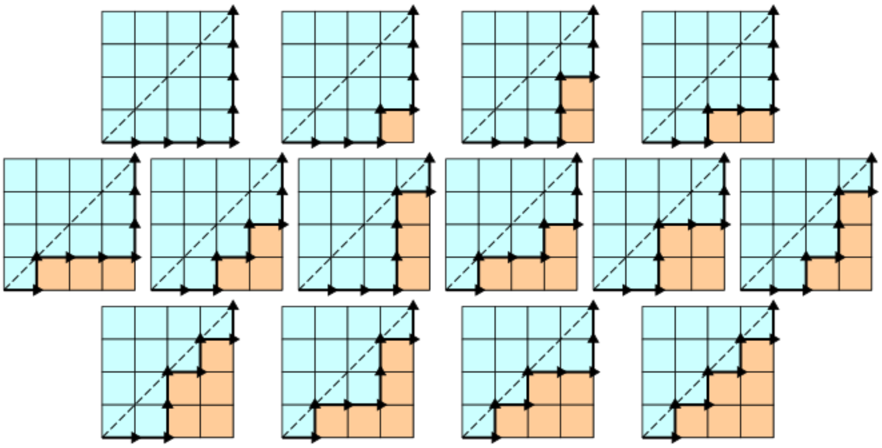

Figure 2: Number of monotonic lattice paths in 4x4 grid which do not cross the diagonal. [3]

6) C_n is the number of permutations of $\{1, ..., n\}$ that avoid the permutation pattern 123 (or, alternatively, any of the other patterns of length 3); that is, the number of permutations with no three-term increasing sub-sequence.[3]
For n = 3, these permutations are 132, 213, 231, 312 and 321.
For n = 4, they are 1432, 2143, 2413, 2431, 3142, 3214, 3241, 3412, 3421, 4132, 4213, 4231, 4312 and 4321.

7) A convex polygon with n + 2 sides can be cut into triangles by connecting vertices with non-crossing line segments (a form of polygon triangulation). The number of triangles formed is n and the number of different ways that this can be achieved is C_n.[3]
The following hexagons illustrate the case n = 4:

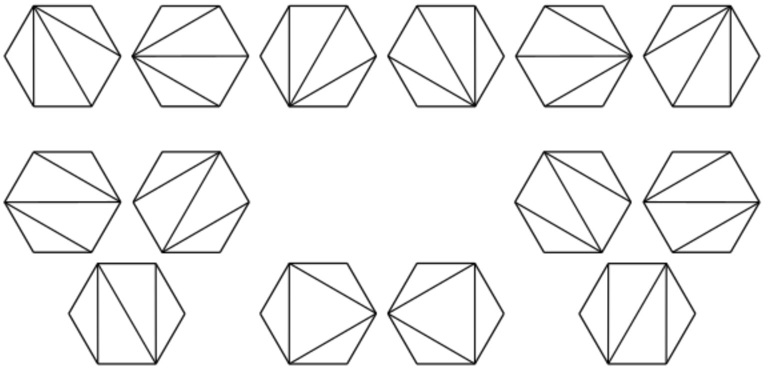

Figure 3: Number of triangles into which a hexagon can be cut with non-crossing line segments. [3]

8) C_n is the number of ways to tile a stair-step shape of height n with n rectangles. Cutting across the anti-diagonal and looking at only the edges gives full binary trees.[3]
The following figure illustrates the case n = 4:

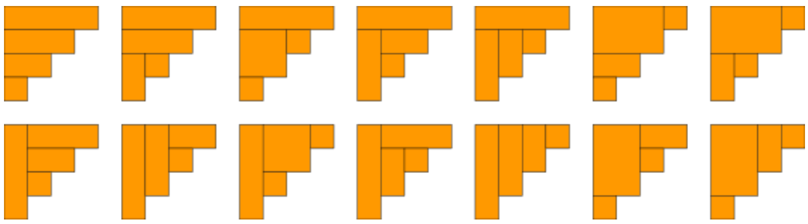

Figure 4: Number of ways of tiling a stair-step of height 4 with 4 rectangles. [3]

9) In chemical engineering C_{n-1} is the number of possible separation sequences which can separate a mixture of n components.[3]

2.3 Proof of Formula

There are several ways of explaining why the formula solves all the combinatorial problems listed above. Here we will see two proofs which use bijection to count a collection of some kind of objects to arrive at the formula.

2.3.1 Theorem

The number of sequences of 2n terms,

$$a_1, a_2, a_3, ..., a_{2n} \tag{1}$$

that can be formed by using exactly n *+1s* and exactly n *-1s* whose partial sums are always positive:

$$a_1 + a_2 + a_3 + ... + a_k \geq 0, \qquad (k = 1, 2, ..., 2n) \tag{2}$$

equals the n^{th} `Catalan number`

$$C_n = \frac{1}{n+1}\binom{2n}{n}.[1]$$

Proof:

A sequence like (1) containing n *+1s* and n *-1s* is acceptable only if it is satisfying (2).

Let A_n denote the number of `acceptable` sequences and U_n denote the number of `unacceptable` sequences.
The total number of sequences of n *+1s* and n *-1s* is hence,

$$\frac{(2n)!}{n!n!} = \binom{2n}{n},$$

since it is the number of permutations of 2n objects out of which n objects are of one kind and the other n objects are of the second kind.

Thus,

$$A_n + U_n = \binom{2n}{n}$$

Now, consider an unacceptable sequence of n *+1s* and n *-1s*. Since the sequence is unacceptable there exists a k such that the partial sum

$$a_1 + a_2 + ... + a_k$$

is negative. Because k is first, there are equal number of $+1$s and -1s preceding a_k.

Hence we have

$$a_1 + a_2 + ... + a_{k-1} = 0$$

and $a_k = -1$.
In particular, k is an odd integer. We now reverse the signs of each of the first k terms, i.e. we replace a_i with $-a_i$ for each i = 1, 2, 3, ... ,k and keep the remaining terms unchanged.

The resulting sequence

$$a'_1, a'_2, ..., a'_{2n}$$

is a sequence of (n+1) +1s and (n-1) -1s.

This process is reversible :
Given a sequence of (n+1) +1s and (n-1) -1s, there exists a first instance when the number of +1s exceed the number of -1s(since there are more +1s than -1s). Reversing the signs of +1s and -1s up to that point results in an unacceptable sequence of n *+1s* and n *-1s.*

Thus, there are as many number of unacceptable sequences as there are sequences of (n+1) +1s and (n-1) -1s. This number represented by U_n is :

$$\frac{(2n)!}{(n+1)!(n-1)!}$$

since it is the number of permutations of 2n objects among which (n+1) objects are of one kind while the other (n-1) objects are of the other kind.

Now since,

$$A_n + U_n = \binom{2n}{n}$$

.

$$
\begin{aligned}
A_n &= \frac{(2n)!}{n!n!} - U_n \\
&= \frac{(2n)!}{n!(n-1)!}\left(\frac{1}{n} - \frac{1}{n+1}\right) \\
&= \frac{(2n)!}{n!(n-1)!}\left(\frac{1}{n(n+1)}\right) \\
A_n &= \frac{1}{n+1}\binom{2n}{n}
\end{aligned}
$$
[1]

Another Proof[3]

Suppose we are given a monotonic path, which may happen to cross the diagonal. Here the right displacement can be considered to be a **+1** and a vertical displacement can be considered to be a **-1**. When the path touches the diagonal the partial sum of +1s and -1s is equal. But if the path goes beyond the diagonal then the partial sum becomes negative since there will be more -1s than +1s till the point where the path crosses the diagonal. The exceedance of the path is defined to be the number of vertical edges which lie above the diagonal. Hence we need the exceedance to be zero. For example in figure 5 the edges lying above the diagonal are marked in red, so the exceedance of the path is 5.

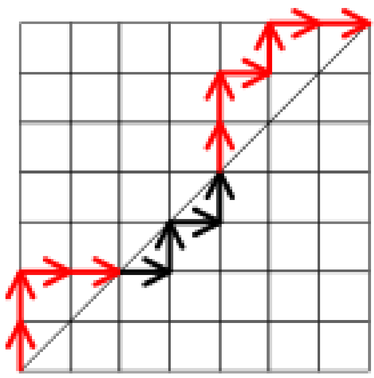

Figure 5: A path with exceedance 5.[3]

If we are given a monotonic path whose exceedance is not zero, then

we can apply the following algorithm to construct a new path whose exceedance is one less than the one we started with.
1) Starting from the bottom left, follow the path until it first travels above the diagonal.
2) Continue to follow the path until it touches the diagonal again. Denote by X the first such edge that is reached.
3) Swap the portion of the path occurring before X with the portion occurring after X.
In Figure 6, the black dot indicates the point where the path first crosses the diagonal. The black edge is X, and we place the last lattice point of the red portion in the top-right corner, and the first lattice point of the green portion in the bottom-left corner, and place X accordingly, to make a new path, shown in the second diagram.

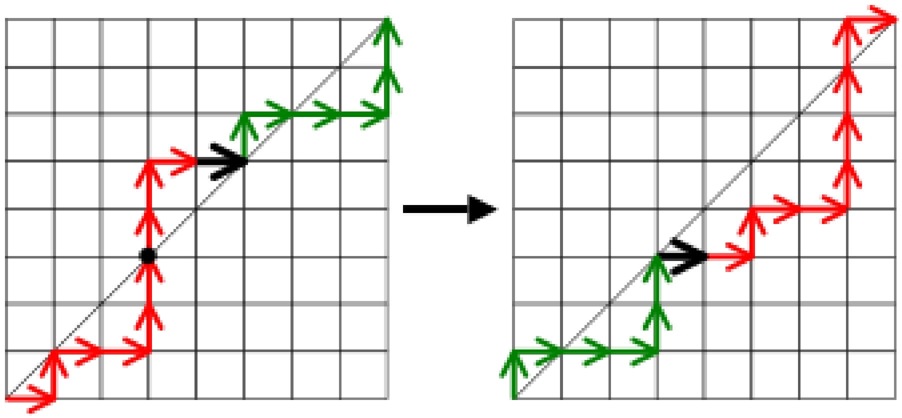

Figure 6: The red and green portions are exchanged.[3]

The exceedance has drops from three to two. The algorithm will cause the exceedance to decrease by one, for any path that we feed it, because the first vertical step starting on the diagonal (at the point marked with a black dot) is the unique vertical edge that under the operation passes from above the diagonal to below it; all other vertical edges stay on the same side of the diagonal.

Also this process is reversible: given any path P whose exceedance is less than n, there is exactly one path which yields P when the algorithm is applied to it. Indeed, the (black) edge X, which originally was the first horizontal step ending on the diagonal, has become the last horizontal step starting on the diagonal. Alternatively, reverse the original algorithm to look for the first edge that passes below the diagonal.

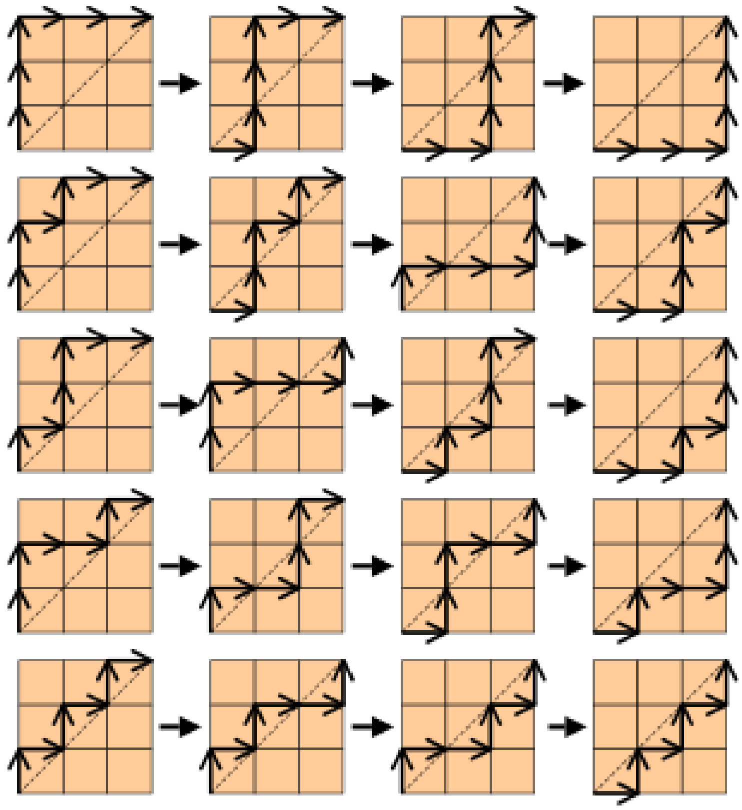

Figure 7: All monotonic paths in a 3×3 grid, illustrating the exceedance-decreasing algorithm.[3]

This implies that the number of paths of exceedance n is equal to the number of paths of exceedance n - 1, which is equal to the number of paths of exceedance n - 2, and so on, down to zero. In other words, we have split up the set of all monotonic paths into n + 1 equally sized classes, corresponding to the possible exceedances between 0 and n. Since there are $\binom{2n}{n}$ paths in total, the number of monotonic paths of exceedance zero will be

$$\boxed{C_n = \frac{1}{n+1}\binom{2n}{n}} \quad [3]$$

2.4 Eugene Charles Catalan (1814-1894)

Figure 8: Eugene Charles Catalan [5]

Eugène Charles Catalan (30 May 1814 —14 February 1894) was a French and Belgian mathematician who worked on continued fractions, descriptive geometry, number theory and combinatorics.
His notable contributions included discovering a periodic minimal surface in the space $\mathcal{R}^3$; stating the famous Catalan's conjecture, which was eventually proved in 2002; and, introducing the Catalan number to solve a combinatorial problem.[5]

3 LATTICE PATHS

A **Lattice path** (**L**) in $\mathcal{Z}^d$ of length k with steps in S is a sequence $v_0, v_1, v_2, ..., v_k \in \mathcal{Z}^d$ such that each consecutive difference $v_i - v_{i-1}$ lies in S.[2]

A **Lattice path** may lie in any lattice $\mathcal{R}^d$. However, integer lattice $\mathcal{Z}^d$ is most commonly used.
For example:

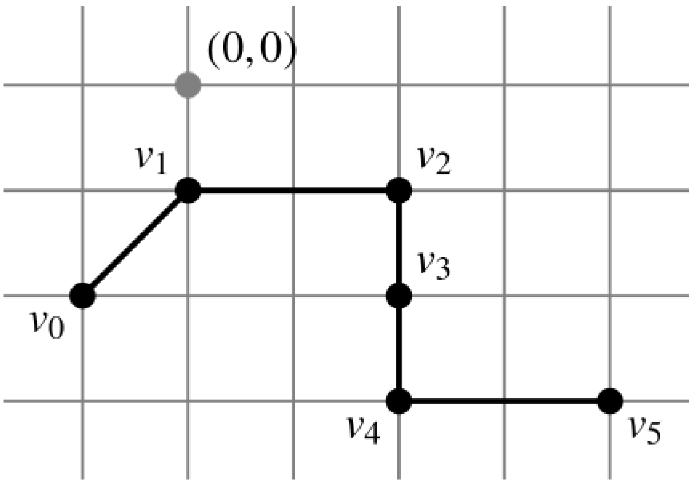

Figure 9: Lattice path[2] of length 5 in $\mathcal{Z}^2$ with S $= \{(2,0),(1,1),(0,-1)\}$

3.1 North-East Lattice Paths or HV Lattice Paths

A North-East (NE)[2] lattice path is a lattice path in $\mathcal{Z}^2$ with steps in S $= \{(0,1),(1,0)\}$. The (0,1) steps are called North steps and denoted by N's; the (1,0) steps are called East steps and denoted by E's.

NE lattice paths most commonly begin at the origin. This convention allows us to encode all the information about a NE lattice path **L** in a single permutation word. The length of the word gives us the

number of steps of the lattice path, k. The order of the N's and E's communicates the sequence of **L**. Furthermore, the number of N's and the number of E's in the word determines the end point of **L**.
If the permutation word for a NE lattice path contains n N-steps and e E-steps, and if the path begins at the origin, then the path necessarily ends at (e,n). This follows because you have "walked" exactly n steps North and e steps East from (0,0).

We consider an integral lattice of points in the coordinate plane with integer coordinates. Given two points (p,q) and (r,s), with $p \geq r$ and $q \geq s$, a rectangular lattice path from (r,s) to (p,q) is a path from (r,s) to (p,q) that is made up of horizontal steps $\mathbf{H} = (1,0)$ and vertical steps $\mathbf{V} = (0,1)$.[1]

Thus a rectangular lattice path from (r,s) to (p,q) starts at (r,s) and gets to (p,q) using unit horizontal and vertical segments.

For example:
Following figure shows a path from (0,0) to (5,5) consisting of five horizontal steps **(H)** and five vertical steps **(V)**.

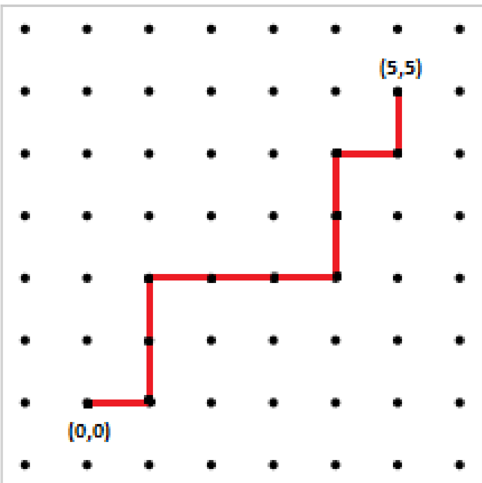

Figure 10: An HV lattice path from (0,0) to (5,5)

Given that the path starts at (0,0) it is uniquely determined by the sequence

$$H, V, V, H, H, H, V, V, H, V.$$

of five **H**s and five **V**s.

3.2 HVD Lattice Paths

These are the **Lattice Paths** where, in addition to horizontal steps $\mathbf{H} = (1, 0)$ and vertical steps $\mathbf{V} = (0,1)$, we allow ***diagonal steps*** $\mathbf{D} = (1, 1)$.

For example:
The following figure shows a path from (0,0) to (5,5) consisting of 3 horizontal steps **H**, 3 vertical steps **V** and 2 diagonal steps **D**.

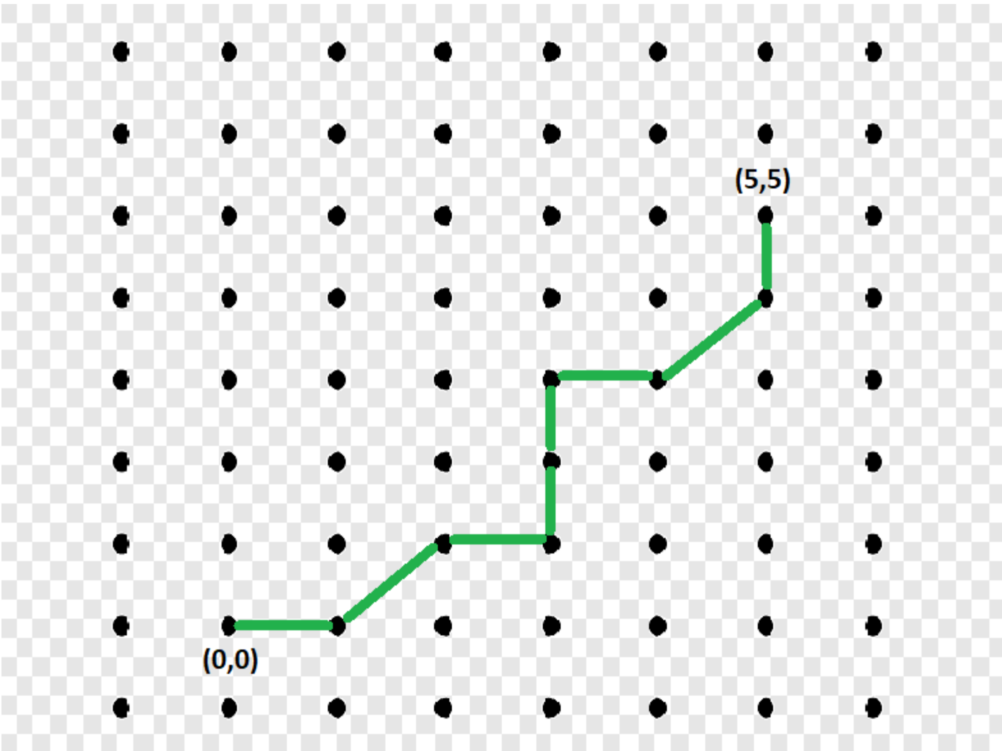

Figure 11: An HVD lattice path from (0,0) to (5,5)

Given that the path starts at (0,0) it is uniquely determined by the sequence

$$H, D, H, V, V, H, D, V.$$

3.3 Monotonic Lattice Paths

The lattice path is **monotonic** if it is either increasing or decreasing. If it is increasing then, the path contains only right(East) and up(North) steps and vice versa.

3.4 Sub-Diagonal Lattice Paths

Rectangular lattice paths from (0,0) to (p,q) which are restricted to lie on or below the line $y = x$ in the coordinate plane are called ***sub-diagonal lattice paths***.

For example:

A monotonic sub-diagonal lattice path from (0,0) to (9,9) is shown below.

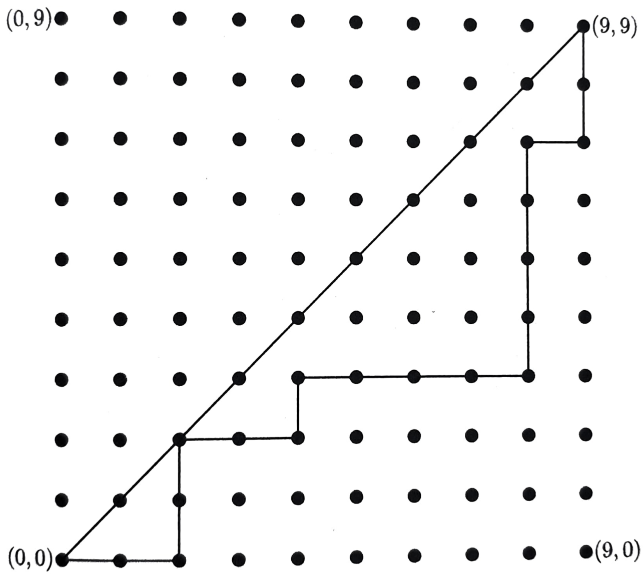

Figure 12: A monotonic sub-diagonal lattice path from (0,0) to (9,9).[1]

3.5 Theorems

3.5.1 The number of rectangular lattice paths from (r,s) to (p,q) equals the binomial coefficient [1]

$$\binom{p-r+q-s}{p-r} = \binom{p-r+q-s}{q-s}.$$

Proof:
A rectangular lattice path from (r,s) to (p,q) is uniquely determined by its sequence of $p-r$ horizontal steps **H** and $q-s$ vertical steps **V**, and every such sequence determines a rectangular lattice path from (r,s) to (p,q). Hence the number of paths equal to the number of permutations of $p-r+q-s$ objects of which $p-r$ are **H**s and $q-s$ are **V**s. This number is the binomial coefficient

$$\binom{p-r+q-s}{p-r} = \binom{p-r+q-s}{q-s}.$$

Consider a path from (r,s) to (p,q) where $p \geq r$ and $q \geq s$. Such a path has exactly $(p-r)+(q-s)$ steps.Since we can translate (r,s) to (0,0) and (p,q) to (p - r, q - s), We obtain a one-to-one correspondence between lattice paths from (r,s) to (p,q) and those from (0,0) to (p - r, q - s).
By theorem 3.5.1 if $p \geq 0$ and $q \geq 0$ then number of lattice paths from (0,0) to (p,q) equals

$$\binom{p+q}{p} = \binom{p+q}{q}.$$

3.5.2 Let n be a non-negative integer. Then the number of sub-diagonal rectangular lattice paths from (0,0) to (n,n) equals the n^{th} Catalan Number

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

Proof:
This has been already proved in section 2.3.

3.5.3 Let p and q be positive integers with $p \geq q$. Then the number of sub diagonal rectangular lattice paths from (0,0) to (p,q) equals [1]

$$\frac{p-q+1}{p+1}\binom{p+q}{q}$$

Proof:
This theorem is the generalisation of the theorem 2.3.1.
We determine the number of unacceptable paths γ from (0,0) to (p,q) that cross the diagonal and then subtract them from the total number of possible rectangular lattice paths.

The number $n(\gamma)$ is the same as number of paths γ' from (0,-1) to (p,q-1) that touch(or possibly cross) the diagonal line $y = x$. This is done by shifting the paths down one unit, thereby shifting a path γ into a path γ', establishing a one-to-one correspondence between the paths.
Consider a path (0,-1) to (p,q-1) that touches the diagonal line $y = x$. Let γ_1' be the sub-path of γ' from (0,-1) to the first diagonal point (d,d) touched γ'. Let γ_2' be the path from (d,d) to (p,q-1).

We reflect the path γ_1' about the line $y = x$ to obtain a path γ_1^* from (-1,0) to (d,d). Paths γ_1^* and γ_2 combined give us a path γ^* from (-1,0) to (p,q-1).

Now every lattice paths from (-1,0) to (p,q-1) must cross the diagonal line $y = x$, since (-1,0) is above the diagonal line and (p,q-1) is below. If we reflect the part of path from (-1,0) to the first crossing point, we get a path from (0,-1) to (p,q-1) that touches the line $y = x$. this shows that there is one-to-one correspondence between the γ' and γ^* and hence $n(\gamma)$ equals the number of rectangular lattice paths from

(-1,0) to (p,q-1).

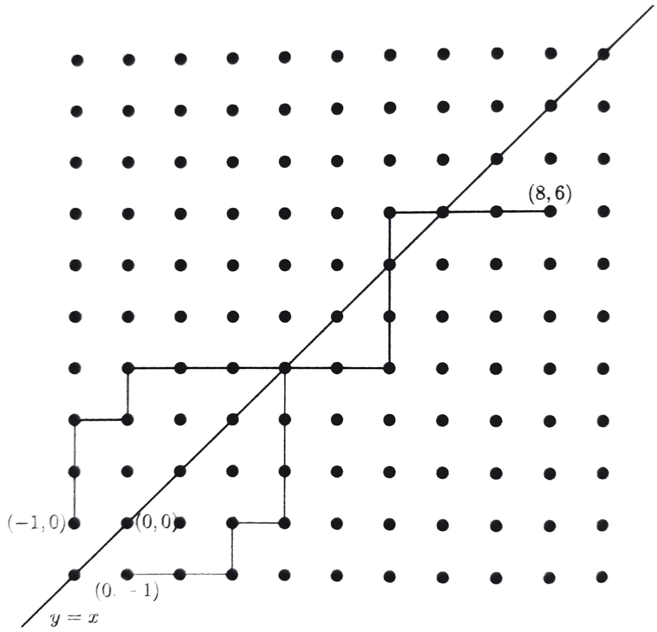

Figure 13: Reflecting the lattice path about line $y = x$

By theorem 3.5.1

$$n(\gamma) = \binom{p+1+q-1}{q-1} = \binom{p+q}{q-1}$$

.

Therefore the number of sub-diagonal lattice paths from (0,0) to (p,q) is :

$$\begin{aligned}\binom{p+q}{q} - n(\gamma) &= \binom{p+q}{q} - \binom{p+q}{q-1} \\ &= \frac{(p+q)!}{p!q!} - \frac{(p+q)!}{(q-1)!(p+1)!} \\ &= \frac{p-q+1}{p+1}\binom{p+q}{q}\end{aligned}$$

3.5.4 Let p and q be non-negative integers and let $K(p,q)$ be the number of HVD lattice paths from (0,0) to (p,q) and $K(p,q:r\boldsymbol{D})$ be the number of such paths from that use exactly r diagonal steps $\boldsymbol{D}$. Then [1]

$$K(p,q:r\boldsymbol{D}) = \frac{(p+q-r)!}{(p-r)!(q-r)!r!}$$

and

$$K(p,q) = \sum_{r=0}^{min\{p,q\}} \frac{(p+q-r)!}{(p-r)!(q-r)!r!}$$

Proof:
An HVD lattice path from (0,0) to (p,q) that uses r diagonal steps D must use $p-r$ horizontal steps **H** and $q-r$ vertical steps **V** and is uniquely determined by its sequence of $p-r$ **H**s, $q-r$ **V**s and r **D**s. Thus the number of such paths is the permutation of $(p-r)+(q-r)+r = p+q-r$ objects among which $p-r$ are of one type, $q-r$ are of second type and r objects are of the third type.
Hence

$$K(p,q:rD) = \frac{(p+q-r)!}{(p-r)!(q-r)!r!}$$

If we do not specify the number r of diagonal steps then by summing $K(p,q:rD)$ from $r=0$ to $r=min\{p,q\}$ we obtain $K(p,q)$ as given in theorem.

Note:
$K(p,q:rD) = 0$ if $r > min\{p,q\}$.
If $r=0$ then

$$K(p,q:rD) = \frac{(p+q)!}{p!q!} = \binom{p+q}{p}$$

which is in agreement with theorem 3.5.1.

3.6 Problem

A big city lawyer works n blocks north and n blocks east of her place of residence. Every day she walks 2n blocks to work (See the map below for n = 4). How many routes are possible if she never crosses (but may touch) the diagonal line from home to office?

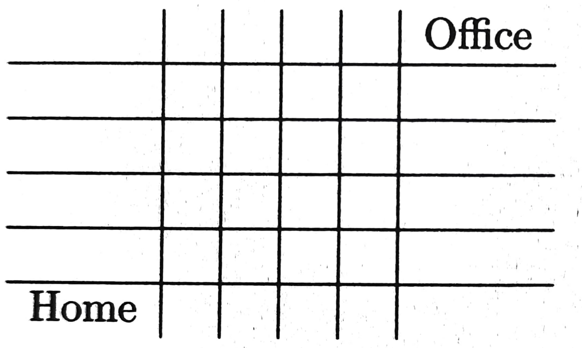

Figure 14: Map for $n = 4$

Each acceptable route either stays above the diagonal or stays below the diagonal. We find the number of acceptable routes below the diagonal and multiply by 2.

As we have seen the number of sub-diagonal paths is equal to the n^{th} Catalan Number and hence the total number of acceptable routes is :

$$2C_n = \frac{2}{n+1}\binom{2n}{n}$$

4 SCHRÖDER NUMBERS

The sub-diagonal HVD lattice paths from (0,0) to (n,n) are called ***Schröder Paths***. The number of Schröder Paths in an $n \times n$ lattice are represented by the **Schröder Numbers**

There are two types of Schröder Numbers :

- `Large Schröder Numbers` ($\mathcal{S}_n$)
- `Small Schröder Numbers` (s_n)

4.1 Large Schröder Numbers ($\mathcal{S}_n$)

The ***Large Schröder Numbers*** ($\mathcal{S}_n$) describe the number of Schröder Paths from (0,0) to (n,n).

Mathematical Expression :

$$\mathcal{S}_n = \sum_{r=0}^{n} \left(\frac{1}{n-r+1} \; \frac{(2n-r)!}{r!((n-r)!)^2} \right)$$

The sequence of Large Schröder Numbers $\mathcal{S}_0$, $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$, ... ,$\mathcal{S}_n$, ... begins as :

$$1, 2, 6, 22, 90, 394, 1806, ...$$

4.1.1 Theorem [1]

Let p and q be positive integers with $p \geq q$ and let r be a non-negative integer with $r \leq q$. Let $R(p, q)$ equal the number of sub-diagonal HVD lattice paths from (0,0) to (p,q). Also let $\mathrm{R}(p, q : rD)$ be the number of sub-diagonal HVD lattice paths from (0,0) to (p,q) that use exactly r diagonal steps D.

Then,

$$R(p,q:rD) = \frac{p-q+1}{p-r+1}\,\frac{(p+q-r)!}{r!(p-r)!(q-r)!}$$

and

$$R(p,q) = \sum_{r=0}^{q}\left(\frac{p-q+1}{p-r+1}\,\frac{(p+q-r)!}{r!(p-r)!(q-r)!}\right)$$

Proof :

A sub-diagonal **HVD** lattice path γ from (0,0) to (p,q) with r diagonal steps D becomes a sub-diagonal rectangular lattice path π from (0,0) to (p-r,q-r) after removing the r diagonal steps D. Conversely a sub-diagonal rectangular lattice path π from (0,0) to (p-r,q-r) becomes a sub-diagonal HVD lattice path with r diagonal steps, from (0,0) to (p,q) by inserting r diagonal steps in any of the p + q - 2 r + 1 places before, between, and after the horizontal and vertical steps.
The number of ways to insert the diagonal steps in π equals to be

$$\binom{p+q-2r+1+r-1}{r} = \binom{p+q-r}{r}$$

Thus, to each sub-diagonal rectangular lattice path from (0,0) to (p-r,q-r), there correspond a number of sub-diagonal HVD lattice paths from (0,0) to (p,q) with r diagonal steps, and this number is given by

$$R(p,q:rD) = \binom{p+q-r}{r} R(p-r,q-r:0D)$$

Therefore using theorem 3.5.3 we get

$$R(p,q:rD) = \binom{p+q-r}{r}\frac{p-q+1}{p-r+1}\binom{p+q-2r}{q-r}$$

which simplifies to

$$R(p,q:rD) = \frac{p-q+1}{p-r+1}\,\frac{(p+q-r)!}{r!(p-r)!(q-r)!}$$

Summing $R(p,q:rD)$ from $r=0$ to q we get the formula for $R(p,q)$ given in the theorem.
Notice that by taking $r=0$ we get theorem 3.5.3.

We now suppose that $p=q=n$ which gives us the **Large Schröder Number :**

$$\mathcal{S}_n = R(n,n) = \sum_{r=0}^{n} \left(\frac{1}{n-r+1} \frac{(2n-r)!}{r!((n-r)!)^2} \right)$$

4.2 Small Schröder Numbers (s_n)

The ***Small Schröder Numbers*** (s_n) are defined in terms of constructs called *bracketings.*[1]

Let $n \geq 1$ and let $a_1, a_2, ..., a_n$ be a sequence of n symbols. Each symbol a_i is itself a bracketing; and any consecutive sequence of two or more bracketings enclosed by a set of parenthesis is a bracketing. We remove parenthesis from single a_i for clarity of notation.

For example :
When $n=3$ we have the following 3 bracketings:

$$a_1a_2a_3, (a_1a_2)a_3, a_1(a_2a_3)$$

When $n=4$ we have the following 11 bracketings:

$$a_1a_2a_3a_4, (a_1a_2)a_3a_4, a_1(a_2a_3a_4), (a_1a_2a_3)a_4, a_1(a_2a_3)a_4, a_1a_2(a_3a_4)$$
$$((a_1a_2)a_3)a_4, (a_1(a_2a_3)a_4), a_1((a_2a_3)a_4), a_1(a_29a_3a_4)), (a_1a_2)(a_3a_4)$$

Bracketings built up of binary operations only are called binary bracketings. Last five of above bracketings are binary.[6]

Algorithm to Construct Bracketings[1]

Start with a sequence $a_1, a_2, ..., a_n$.

1. Let γ equal $a_1 a_2 ... a_n$
2. While γ has at least 3 symbols do the following :
   (a) Put a set of parenthesis around any number $k \geq 2$ of consecutive symbols, say, $a_i, a_{i+1}, ..., a_{i+k-1}$, to form a new symbol $(a_i a_{i+1} ... a_{i+k-1})$.
   (b) Replace γ with the expression in which $(a_i a_{i+1} ... a_{i+k-1})$ is one symbol.
3. Output the current expression.

For $n \geq 1$ the ***small Schröder Numbers*** (s_n) are defined to be the bracketings of a sequence $a_1, a_2, ..., a_n$ of n symbols.

The sequence ($\mathrm{s}_n : n = 1, 2, 3, ...$) begins as :

$$1, 1, 3, 11, 45, 197, 903, ...$$

Comparing with the initial part of the sequence of **Large Schröder Numbers** we observe that

$$\mathcal{S}_n = 2\mathrm{s}_{n+1}$$

with $\mathcal{S}_0 = 1$.

4.3 Ernst Schröder (1841-1902)

Friedrich Wilhelm Karl Ernst Schröder (25 November 1841 in Mannheim, Baden, Germany – 16 June 1902 in Karlsruhe, Germany) was a German mathematician mainly known for his work on algebraic logic. He is a major figure in the history of mathematical logic, by virtue of summarizing and extending the work of George Boole, Augustus De Morgan,

Figure 15: Friedrich Wilhelm Karl Ernst Schröder [4]

Hugh MacColl, and especially Charles Peirce. He is best known for his monumental Vorlesungen über die Algebra der Logik (Lectures on the Algebra of Logic, 1890–1905), in three volumes, which prepared the way for the emergence of mathematical logic as a separate discipline in the twentieth century by systematizing the various systems of formal logic of the day.The Schröder Numbers were named after him.[4]

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