CORSO DI LAUREA IN INFORMATICA APPLICATA

ANALISI MATEMATICA 1

ESERCIZI SU

PRINCIPIO DI INDUZIONE

Esercizio 1.
Dimostrare per induzione la validità delle seguenti affermazioni:

1. $\forall\, n \in \mathbb{N}\,, 3^n \geq n2^{n-1}$
2. $\forall\, n \geq 2\,, 2^n + 4^n \leq 5^n$
3. $\forall\, n \geq 6\,, n^n \geq 2^n n!$
4. $\forall\, n \in \mathbb{N}\,, n! \geq 2^{n-1}$
5. $\displaystyle\sum_{k=0}^{n} 3^k = \frac{3^{n+1}-1}{2}$
6. $\forall\, n \in \mathbb{N}\,, 2^n \geq 2n$
7. $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$
8. $\forall\, n \geq 0\,, 3^n \geq 1 + 2n$
9. $\displaystyle\sum_{k=0}^{n} 3^k = \frac{3^{n+1}-1}{2}$
10. $\forall\, n \in \mathbb{N}\,, n(n+1)$ è pari

1 $\forall n \in \mathbb{N} \quad 3^n \geq n\, 2^{n-1}$

BASE: $n = 1$

$3 \geq 1 \cdot 2^0 \Rightarrow 3 \geq 1$ VERO

INDUTTIVO:

$$3^{n+1} \geq (n+1)\, 2^{n+1-1}$$

$3^n + 3^n + 3^n$

$(n+1)\, 2^{n-1} \cdot 2 =$

$= n \cdot 2^{n-1} \cdot 2 + 2^{n-1} \cdot 2 =$

$= n\, 2^{n-1} + n\, 2^{n-1} + 2^{n-1} \cdot 2$

$$\underbrace{3^n}+\underbrace{3^n}+\underbrace{3^n} \geq \underbrace{n\, 2^{n-1}} + \underbrace{n\, 2^{n-1}} + \underbrace{2^{n-1} \cdot 2}$$

$3^n \geq n\, 2^{n-1}$ (PER CASO BASE)

$3^n \geq 2^n$ VERO $\forall n \in \mathbb{N}$

$\ln(3) n - \ln(2) n > 0$

$n(\ln(3) - \ln(2)) > 0$

$n > 0$

2 $\forall m \geq 2 \qquad 2^m + 4^m \leq 5^m$

CASO BASE: $m = 2$

$4 + 16 \leq 25$ VERO

PRINCIPIO INDUTTIVO:

$$2^{m+1} + 4^{m+1} \leq 5^{m+1}$$

$$\cancel{2^m} + \cancel{2^m} + 4^m + 4^m + \cancel{4^m} + \cancel{4^m} \leq \cancel{5^m} + \cancel{5^m} + 5^m + 5^m + 5^m$$

$$\cancel{4^m} + \cancel{4^m} \leq \cancel{5^m} + \cancel{5^m} + 5^m$$

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3 RISOLTO DA PROF

$\forall m \geq 6 \qquad m^m \geq 2^m m!$

BASE: $m = 6 \qquad 6^6 \geq 2^6 \cdot 6!$ VERO

INDUTTIVO: $(m+1)^{m+1} \geq 2^{m+1} \cdot (m+1)!$

$$(m+1)^{m+1} = \left(m\left(1+\frac{1}{m}\right)\right)^{m+1}$$

$$= m^{m+1}\left(1+\frac{1}{m}\right)^{m+1}$$

$$= m^m \cdot m \underline{\left(1 \cdot \frac{1}{m}\right)^{m+1}} \geq 2^m \cdot m! \left(1 \cdot \frac{1}{m}\right)^{m+1}$$

$$n\left(1+\frac{1}{n}\right)^{n+1} \geq 2(n+1)$$

$$\left(1+\frac{1}{n}\right)^{n+1} \geq 2\,\frac{n+1}{n} = 2\left(1+\frac{1}{n}\right)$$

$$\left(1+\frac{1}{n}\right)^{n} \geq 2 \quad \text{VERO}$$

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4 $\forall n \in \mathbb{N} \quad n! \geq 2^{n-1}$

BASE: $n=0 \qquad 0! \geq 2^0 \qquad 1 \geq 1$ VERO

INDUTTIVO: $(n+1)! \geq 2^n$

$$= (n+1)!$$

$$= (n+1)\,n! \geq 2^{n-1}(n+1)$$

$$n \geq 1$$

$$\geq 2^{n-1}(1+1) \geq 2^n$$

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5 $\forall n \in \mathbb{N} \quad \sum_{k=0}^{n} 3^k = \frac{3^{n+1}-1}{2}$

BASE: $n=0 \qquad \sum_{k=0}^{0} 3^0 = \frac{3-1}{2} \qquad 1=1$ VERO

INDUTTIVO:

$$\sum_{k=0}^{n+1} 3^k = \frac{3^{n+2}-1}{2}$$

$$\sum_{k=0}^{n} 3^k + 3^{n+1} = \frac{3^{n+1}-1}{2} + 3^{n+1} = \frac{3^{n+1}-1+2\cdot 3^{n+1}}{2} =$$

$$= \frac{3^{n+1}\cdot 3 - 1}{2} = \frac{3^{n+2}-1}{2}$$

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6 $\forall n \in \mathbb{N} \quad 2^n \geq 2n$

BASE: $n = 1 \qquad 2 \geq 2$ VERO

INDUTTIVO:

$$2^{n+1} \geq 2(n+1)$$

$$2^n + 2^n \geq 2n + 2$$

$2^n \geq 2n$ CASO BASE

$2^n \geq 2$ VERO

---

7 $\forall n \in \mathbb{N} \quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

BASE: $n = 1 \qquad 1 = \frac{2\cdot 3}{6}$ VERO

INDUTTIVO:

$$\sum_{k=1}^{n+1} k^2 = \frac{(n+1)(n+2)(2(n+1)+1)}{6}$$

$$\sum_{k=1}^{n} k^2 + (n+1)^2 = \frac{(n+1)(n+2)(2n+2+1)}{6}$$

$$\frac{n(n+1)(2n+1)}{6} + (n+1)^2 = \frac{(n+1)(n+2)(2n+2+1)}{6}$$

$$n(n+1)(2n+1) + 6(n+1)^2 = (n+1)(n+2)(2n+3)$$

$$n(2n^2+n+2n+1) + 6(n^2+2n+1) = (n^2+3n+2)(2n+3)$$

$$\cancel{2n^3} + \cancel{3n^2} + \cancel{n} + \cancel{6n^2} + \cancel{12}n + \cancel{6} = \cancel{2n^3} + \cancel{6n^2} + \cancel{4n} + \cancel{3n^2} + 9n + \cancel{6}$$

$0 = 0$ VERO

8 $\forall n \geq 0 \quad 3^n \geq 1 + 2n$

BASE: $n = 0 \qquad 1 \geq 1$ VERO

INDUTTIVO,

$$3^{n+1} \geq 1 + 2(n+1)$$

$$3^n \cdot 3 \geq 1 + 2n + 2$$

$$3^n + 3^n + 3^n \geq 2n + 1 + 2$$

$3^n \geq 2n+1$ CASO BASE

$3^n + 3^n \geq 2$ VERO

10

$\forall n \in \mathbb{N} \quad n(n+1)$ è PARI

BASE: $n=1 \qquad 1(1+1) = 2$ VERO

INDUTTIVO:

$$(n+1)(n+2) = 2k$$
$$n^2+2n+n+2 = 2k$$
$$n(n+1)+2(n+1) = 2k$$
$$n(n+1) = 2\underbrace{(-n-1+k)}_{=\bar{k}}$$
$$n(n+1) = 2\bar{k} \qquad \text{PARI}$$

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2 RISOLTO DA PROF

$$2^{n+1}+4^{n+1} = 2\cdot 2^n + 4\cdot 4^n$$
$$\le 4\cdot 2^n + 4\cdot 2^n$$
$$\le 4(2^n+4^n)$$
$$\le 4\cdot 5^n$$
$$\le 5\cdot 5^n = 5^{n+1}$$

CORSO DI LAUREA IN INFORMATICA APPLICATA

ANALISI MATEMATICA 1

ESERCIZI SU

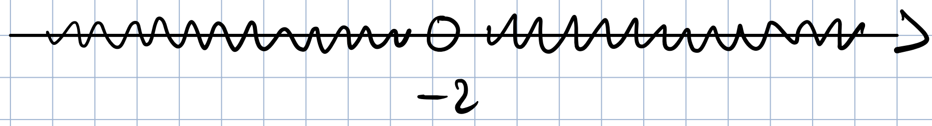
FUNZIONI DI UNA VARIABILE

Esercizio 1.
Determinare e rappresentare graficamente il dominio delle seguenti funzioni:

1. $f(x) = \frac{x-1}{x+2}$
2. $f(x) = \sqrt{x-2}$
3. $f(x) = x\sqrt{x+1}$
4. $f(x) = \frac{\sqrt[3]{x^2-1}}{x}$
5. $f(x) = \frac{2x+1}{\sqrt{x-2}}$
6. $f(x) = \sqrt{\frac{x}{x^2-1}}$
7. $f(x) = e^{\sin x^3}$
8. $f(x) = \sqrt{|x-2|}$
9. $f(x) = \log \frac{x}{x+3}$
10. $f(x) = \cos \sqrt{x+2}$
11. $f(x) = \log(\sqrt{x^2-6x+5})$
12. $f(x) = \sin(x - \sqrt{1-2x})$
13. $f(x) = e^{\frac{x+3}{x+4}}$
14. $f(x) = \sqrt{|x|-3}$
15. $f(x) = \log \frac{x+5}{x^3}$
16. $f(x) = \sqrt[4]{x-x^2}$
17. $f(x) = \frac{1}{\sqrt[3]{x^2-4}}$

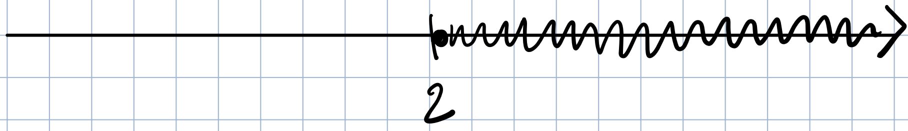
1 $f(x) = \dfrac{x-1}{x+2}$

$D := \quad x+2 \neq 0 \quad \Rightarrow \quad x \neq -2$



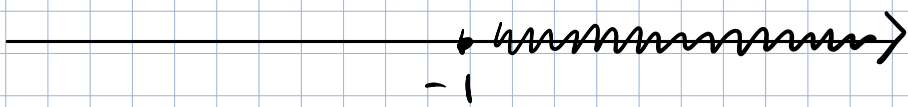
2 $f(x) = \sqrt{x-2}$

$D := \quad x-2 \geq 0 \qquad x \geq 2$



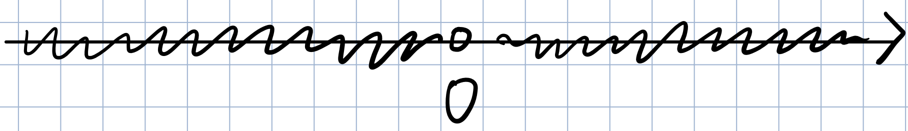
3 $f(x) = x\sqrt{x+1}$

$D := \quad x+1 \geq 0 \qquad x \geq -1$



4 $f(x) = \dfrac{\sqrt[3]{x^2-1}}{x}$

$D := \quad x \neq 0$

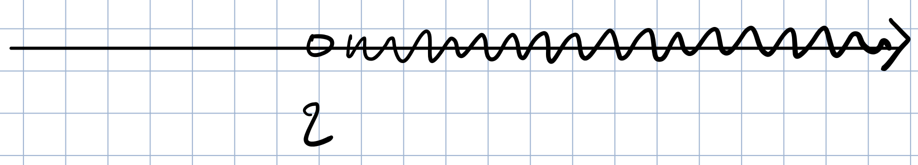


5

$$f(x) = \frac{2x+1}{\sqrt{x-2}}$$

$$\sqrt{x-2} \neq 0 \qquad x \neq 2$$

$$x - 2 \geq 0 \qquad x > 2$$



6

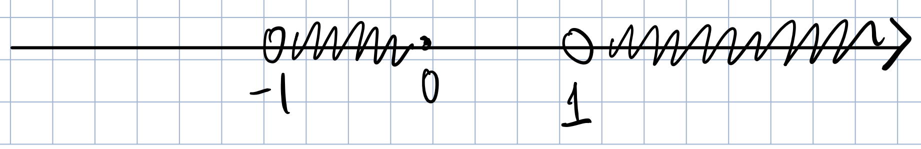
$$f(x) = \sqrt{\frac{x}{x^2-1}}$$

$$x^2 - 1 \neq 0 \qquad x \neq \pm 1$$

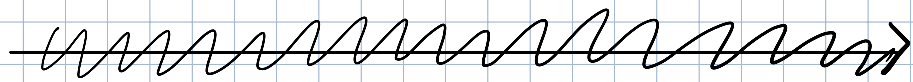
$$\frac{x}{x^2-1} \geq 0 \qquad \begin{cases} x \geq 0 \\ x^2 - 1 \geq 0 \quad]-\infty, -1] \cup [1, +\infty[\end{cases}$$

		-1		0		1	
x	$-$		$-$		$+$		$+$
x^2-1	$+$		$-$		$-$		$+$
	$-$		$+$		$-$		$+$

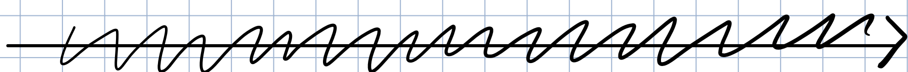
$$]-1, 0] \cup]1, +\infty[$$



7 $f(x) = e^{\sin x^3}$ $\quad]-\infty, +\infty[$



8 $f(x) = \sqrt{|x-2|}$ $\quad]-\infty, +\infty[$

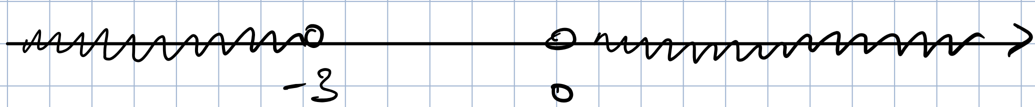


9 $f(x) = \log \dfrac{x}{x+3}$

$$\frac{x}{x+3} > 0$$

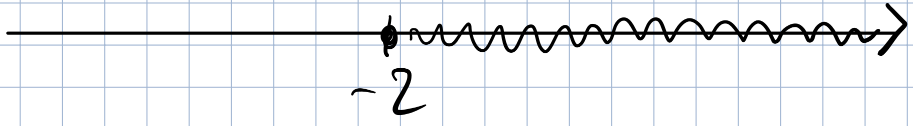
		-3		0	
$x > 0$	$-$		$-$		$+$
$x > -3$	$-$		$+$		$+$
	$+$		$-$		$+$

$]-\infty, -3[\cup]0, +\infty[$



10 $f(x) = \cos\sqrt{x+2}$

$x + 2 \geq 0 \qquad x \geq -2$



11 $f(x) = \log\left(\sqrt{x^2 - 6x + 5}\right)$

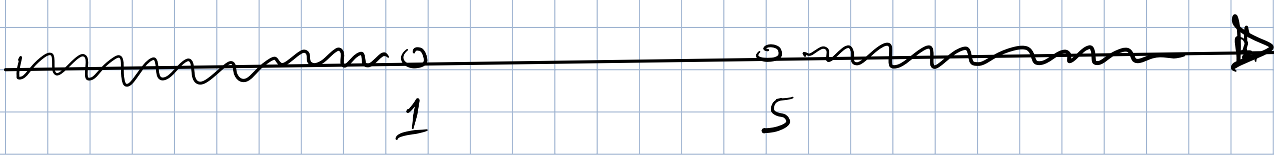
$x^2 - 6x + 5 = (x-5)(x-1)$

$x - 5 \geq 0 \qquad x \geq 5$

$x - 1 \geq 0 \qquad x \geq 1$

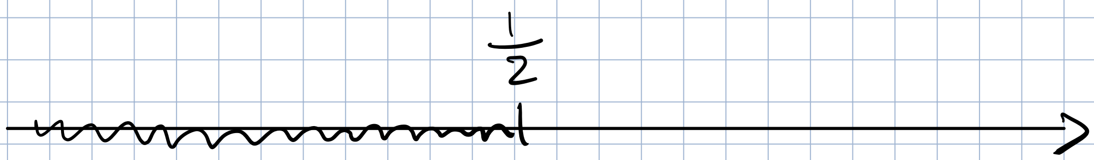
		1		5	
$x-5$	$-$		$-$		$+$
$x-1$	$-$		$+$		$+$
	$+$		$-$		$+$

$]-\infty, 1[\cup]5, +\infty[$



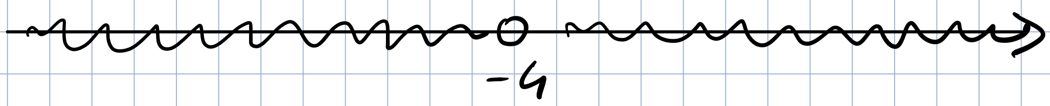
12 $f(x) = \sin\left(x - \sqrt{1-2x}\right)$

$1 - 2x \geq 0$ $\quad$ $x \leq \frac{1}{2}$



13 $f(x) = e^{\frac{x+3}{x+4}}$

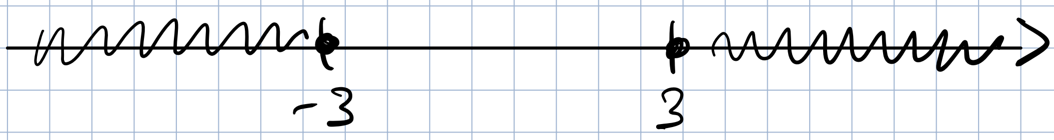
$x + 4 \neq 0$ $\quad$ $x \neq -4$



14 $f(x) = \sqrt{|x| - 3}$

$|x| - 3 \geq 0$ $\quad$ $|x| \geq 3$

$]-\infty, -3] \cup [3, +\infty[$



15

$f(x) = \log \frac{x+5}{x^3}$

$\frac{x+5}{x^3} \neq 1$

$x+5 \neq x^3$

$x^3 \neq 0 \qquad x \neq 0$

$\frac{x+5}{x^3} > 0 \Rightarrow \begin{matrix} x+5 > 0 & x > -5 \\ x^3 > 0 & x > 0 \end{matrix}$

	-5		0	
$-$		$+$		$+$
$-$		$-$		$+$
$+$		$-$		$+$

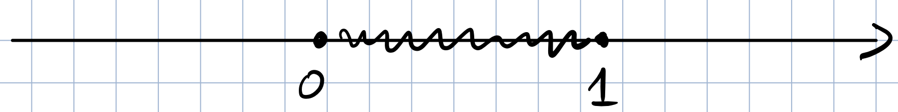
$\boxed{]-\infty, -5[\cup]0, +\infty[}$

16

$f(x) = \sqrt[4]{x - x^2}$

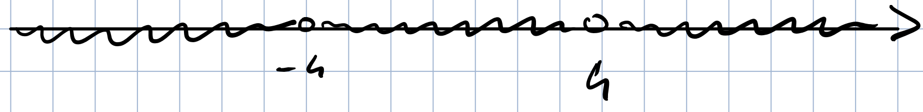
$x - x^2 \geq 0 \qquad x \geq x^2$

$\boxed{0 \geq x \geq 1}$



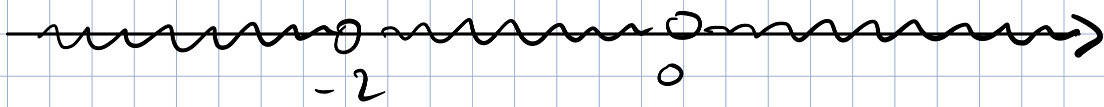
17 $f(x) = \dfrac{1}{\sqrt[3]{x^2-4}}$

$\sqrt[3]{x^2-4} \neq 0$ $\quad x \neq \pm 4$



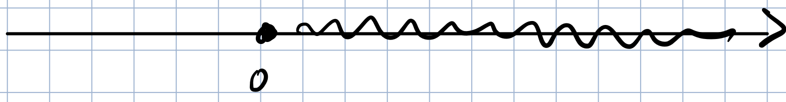
18 $f(x) = \dfrac{x+5}{x\sqrt{x+2}}$

$x\sqrt{x+2} \neq 0$ $\quad \begin{cases} x \neq 0 \\ x \neq -2 \end{cases}$



19 $f(x) = \dfrac{x\ln\sqrt{x}}{e^{x+1}}$

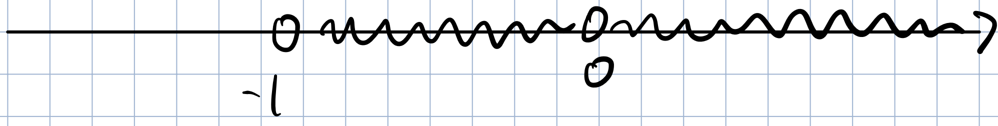
$x \geq 0$



20 $f(x) = \dfrac{2x}{\log(x+1)}$

$x+1 > 0$ $\quad x > -1$

$x+1 \neq 1$ $\quad x \neq 0$



18. $f(x) = \frac{x+5}{x\sqrt{x+2}}$

19. $f(x) = \frac{\sin\sqrt{x}}{e^{x+1}}$

20. $f(x) = \frac{2x}{\log x+1}$

Esercizio 2.
Date le funzioni f e g, scrivere, se è possibile farlo, le funzioni $g \circ f$ e $f \circ g$, indicando anche il rispettivo dominio:

1. $f(x) = x - 2,\ g(x) = 4 - 3x$
2. $f(x) = \sqrt{x^2 - 2x + 3} - 1,\ g(x) = \log x$
3. $f(x) = \sin x + \cos x,\ g(x) = \sqrt{2x - 2}$
4. $f(x) = 3|x|,\ g(x) = e^{x+2}$
5. $f(x) = \frac{1}{x+1},\ g(x) = \log x$
6. $f(x) = e^{\sqrt{x+5}},\ g(x) = \frac{1}{x}$
7. $f(x) = \frac{x^2-4}{x+3},\ g(x) = \sqrt{2x}$
8. $f(x) = \sqrt[3]{x^2 - 2},\ g(x) = \frac{x+2}{x}$
9. $f(x) = e^{x^2-4},\ g(x) = x + 2$
10. $f(x) = \sqrt{x},\ g(x) = x^4 + x^2$

1

$f(x) = x-2$
$g(x) = 4-3x$

$f \circ g = (4-3x)-2 = 4-5x$

$g \circ f = 4-3(x-2) = 4-3x+6 = -3x+10$

$D := \mathbb{R}$

2

$f(x) = \sqrt{x^2-2x+3} - 1$ $\quad D := \mathbb{R}$ PERCHÉ $\Delta < 0$

$g(x) = \log x$ $\quad D := x > 0, x \neq 1$

$f \circ g = \sqrt{(\log x)^2 - 2(\log x) + 3} - 1$

$g \circ f = \log\left(\sqrt{x^2-2x+3} - 1\right)$

$D := x > 0, x \neq 1$

3

$f(x) = \sin x + \cos x$ $\quad D := \mathbb{R}$

$g(x) = \sqrt{2x-2}$ $\quad D := x \geq 1$

$f \circ g = \sin\left(\sqrt{2x-2}\right) + \cos\left(\sqrt{2x-2}\right)$

$g \circ f = \sqrt{2(\sin x + \cos x) - 2}$

4

$f(x) = 3|x|$ $\quad D := \mathbb{R}$

$g(x) = e^{x+2}$ $\quad D := \mathbb{R}$

$f \circ g = 3|e^{x+2}|$

$g \circ f = e^{3|x|+4}$

$D := \mathbb{R}$

5

$f(x) = \frac{1}{x+1}$ $\quad D := x \neq -1$

$g(x) = \log x$ $\quad D := x > 0, x \neq 1$

$f \circ g = \frac{1}{\log(x)+1}$

$g \circ f = \log\left(\frac{1}{x+1}\right)$

$D := x > 0, x \neq 1$

6

$f(x) = e^{\sqrt{x+5}}$ $\quad D := x \geq -5$

$g(x) = \frac{1}{x}$ $\quad D := x \neq 0$

$f \circ g = e^{\sqrt{\frac{1}{x}+5}}$

$g \circ f = \frac{1}{e^{\sqrt{x+5}}}$

$D := x \geq -5, x \neq 0$

7

$f(x) = \frac{x^2-4}{x+3}$ $\quad D := x \neq -3$

$g(x) = \sqrt{2x} \quad D := x \geq 0$

$f \circ g = \dfrac{2x-4}{\sqrt{2x}+3}$

$g \circ f = \sqrt{2\dfrac{x^2-4}{x+3}}$

$D := x \geq 0$

8 $f(x) = \sqrt[3]{x^2-2} \quad D := \mathbb{R}$

$g(x) = \dfrac{x+2}{x} \quad D := x \neq 0$

$f \circ g = \sqrt[3]{\left(\dfrac{x+2}{x}\right)^2-2}$

$g \circ f = \dfrac{\sqrt[3]{x^2-2}+2}{\sqrt[3]{x^2-2}}$

$D := x \neq 0$

9 $f(x) = e^{x^2-4} \quad D := \mathbb{R}$

$g(x) = x+2 \quad D := \mathbb{R}$

$f \circ g = e^{(x+2)^2-4} = e^{x^2+2x}$ $\quad D :=$

$g \circ f = e^{x^2-4}+2$

10 $f(x) = \sqrt{x} \quad D := x \geq 0$

$g(x) = x^4 + x^2 \quad D := \mathbb{R}$

$f \circ g = \sqrt{x^2(x^2+1)}$

$g \circ f = \left(\sqrt[2]{x}\right)^4 + \left(\sqrt{x}\right)^2$

Limiti Notevoli: Tabella

funzioni goniometriche	
$\lim\limits_{x\to 0}\dfrac{sen\ x}{x}=1$	$\lim\limits_{x\to 0}\dfrac{tg\ x}{x}=1$
$\lim\limits_{x\to 0}\dfrac{1-\cos x}{x}=0$	$\lim\limits_{x\to 0}\dfrac{arcsen\ x}{x}=1$
$\lim\limits_{x\to 0}\dfrac{1-\cos x}{x^2}=\dfrac{1}{2}$	$\lim\limits_{x\to 0}\dfrac{arctg\ x}{x}=1$
$\lim\limits_{x\to 0}\dfrac{sen\ mx}{sen\ nx}=\dfrac{m}{n}$	$\lim\limits_{x\to 1}\dfrac{(\arccos x)^2}{1-x}=2$
funzioni esponenziali e logaritmiche	
$\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^x=e$	$\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}}=e$
$\lim\limits_{x\to 0}\dfrac{\log_a(1+x)}{x}=\log_a e$	$\lim\limits_{x\to 0}\dfrac{\ln(1+x)}{x}=1$
$\lim\limits_{x\to 0}\dfrac{a^x-1}{x}=\ln a$	$\lim\limits_{x\to 0}\dfrac{e^x-1}{x}=1$
$\lim\limits_{x\to 0}\dfrac{(1+x)^k-1}{x}=k$	$\lim\limits_{x\to\infty}\dfrac{x^n}{a^x}=0 \qquad (a>1)$
$\lim\limits_{x\to\infty}\dfrac{a^n}{n!}=0$	$\lim\limits_{x\to\infty}\dfrac{1}{1+a^x}=\begin{cases}1 & se\ a<1\\ \dfrac{1}{2} & se\ a=1\\ 0 & se\ a>1\end{cases}$

CORSO DI LAUREA IN INFORMATICA APPLICATA

ANALISI MATEMATICA 1

ESERCIZI SU

LIMITI DI FUNZIONI

Esercizio 1.
Calcolare, se esistono, i seguenti limiti:

1. $\lim_{x\to+\infty} \frac{x^2-2x}{2x^2+1}$
2. $\lim_{x\to0} \frac{x^2-2x}{2x^3+x^4}$
3. $\lim_{x\to0} e^x(\cos x - 2\sin x)$
4. $\lim_{x\to-\infty} (1+e^x)\sin x$
5. $\lim_{x\to+\infty} (\sqrt{x+1}-\sqrt{x})$
6. $\lim_{x\to0} \frac{1-\cos x}{\sin^2 x}$
7. $\lim_{x\to0} \frac{e^x-1}{\log(1+x)}$
8. $\lim_{x\to-\infty} \frac{x^2+\sin e^x}{2x}$
9. $\lim_{x\to0} \frac{\sin x}{x+\sqrt{3x}}$
10. $\lim_{x\to0} \frac{x+3\sqrt{x}}{2x-5\sqrt{x}}$
11. $\lim_{x\to+\infty} \frac{x+3\sqrt{x}}{2x-5\sqrt{x}}$
12. $\lim_{x\to0} \sqrt{\log\cos x}$
13. $\lim_{x\to1} \frac{2\sin(x-1)}{x^2-1}\cdot\frac{e^{x^2-1}-1}{x-1}$
14. $\lim_{x\to-1} \frac{1-\cos(x+1)}{x+1}\cdot\frac{3(x-1)}{e^{x^2-1}-1}$
15. $\lim_{x\to0} \frac{x(2^x-3^x)}{1-\cos 3x}$

1 $$\lim_{x\to+\infty} \frac{x^2-2x}{2x^2+1} = \frac{1}{2}$$

POLINOMI DI STESSO GRADO

2 $$\lim_{x\to 0} \frac{x^2-2x}{2x^3+x^4} = \frac{x(x-2)}{x^3(2+x)} = \underbrace{\frac{1}{x^2}}_{+\infty} \underbrace{\frac{(x-2)}{(x+2)}}_{-1} = -\infty$$

3 $$\lim_{x\to 0} e^x(\cos x - 2\sin x) = e^0(\cos 0 - 2\sin 0) = 1$$

4 $$\lim_{x\to-\infty} (1+e^x)\sin x = \sin x + \sin x e^x \quad \nexists$$

$$\sin x = \sin x + \sin x e^x - \sin x e^x$$

5 $$\lim_{x\to+\infty} \left(\sqrt{x+1}-\sqrt{x}\right) = \frac{\left(\sqrt{x+1}-\sqrt{x}\right)\left(\sqrt{x+1}+\sqrt{x}\right)}{\left(\sqrt{x+1}+\sqrt{x}\right)} =$$

$$= \frac{1}{\underbrace{\sqrt{x+1}+\sqrt{x}}_{+\infty}} = 0$$

6 $$\lim_{x\to 0} \frac{1-\cos x}{\sin^2 x} = \frac{1-\cos x}{\sin^2 x} \cdot \frac{1+\cos x}{1+\cos x} = \frac{\cancel{\sin^2 x}}{\cancel{\sin^2 x}(1+\cos x)} =$$

$$= \frac{1}{(1+\cos x)} = \frac{1}{2}$$

7 $$\lim_{x\to 0} \frac{e^x-1}{\log(1+x)} = e^x-1 \cdot \frac{1}{\log(1+x)} \cdot \frac{x}{x} =$$

$$= \underbrace{\frac{e^x-1}{x}}_{1} \cdot \frac{x}{\log(1+x)}$$

$x = e^y - 1$ PERCHÉ PER $x\to 0$ $y\to 0$

$$\frac{e^y-1}{y} \to 1 = 1$$

8 $$\lim_{x \to -\infty} \frac{x^2 + \sin e^x}{2x} = \underbrace{\frac{x}{2}}_{-\infty} + 2\underbrace{\frac{\sin \overbrace{e^x}^{0}}{x}}_{-\infty} = -\infty$$

9 $$\lim_{x \to 0} \frac{\sin x}{x + \sqrt{3x}}$$

$$\lim_{x \to 0^+} \frac{\sin x}{x + \sqrt{3x}} = \frac{\overbrace{\frac{\sin x}{x}}^{1}}{1 + \underbrace{\sqrt{\frac{3}{x}}}_{\infty}} = 0$$

$$\lim_{x \to 0^-} \frac{\sin x}{x + \sqrt{3x}} = \nexists$$

10 $$\lim_{x \to 0} \frac{x + 3\sqrt{x}}{2x - 5\sqrt{x}} = \nexists$$

$$\lim_{x \to 0^+} \frac{x + 3\sqrt{x}}{2x - 5\sqrt{x}} = \frac{\sqrt{x}\,(3 + \sqrt{x})}{\sqrt{x}\,(-5 + 2\sqrt{x})} = \frac{3 + \overbrace{\sqrt{x}}^{0}}{-5 + 2\underbrace{\sqrt{x}}_{0}} = -\frac{3}{5}$$

$$\lim_{x \to 0^-} \frac{x + 3\sqrt{x}}{2x - 5\sqrt{x}} = \nexists$$

11 $$\lim_{x \to \infty} \frac{x + 3\sqrt{x}}{2x - 5\sqrt{x}} = \frac{1}{2}$$

12 $$\lim_{x\to 0} \sqrt{\log(\underbrace{\cos x}_{1})} = 0$$

(the log term tends to 0)

13 $$\lim_{x\to 1} \frac{2\sin(x-1)}{x^2-1} \cdot \frac{e^{x^2-1}-1}{x-1} =$$

$$\frac{2\sin(x-1)}{(x-1)(x+1)} \cdot \frac{e^{x^2-1}-1}{(x-1)} = \underbrace{2\cdot}_{2} \underbrace{\frac{\sin(x-1)}{(x-1)}}_{1} \cdot \underbrace{\frac{e^{x^2-1}-1}{x^2-1}}_{1} = 2$$

14 $$\lim_{x\to -1} \frac{1-\cos(x+1)}{x+1} \cdot \frac{3(x-1)}{e^{x^2-1}-1}$$

$x = y-1 \qquad y = x+1$

$$\lim_{y\to 0} \frac{1-\cos(y)}{y} \cdot \frac{3(y-2)}{e^{y(y-2)}-1} =$$

$$\frac{y}{y} \cdot \frac{1-\cos(y)}{y} \cdot \frac{3(y-2)}{e^{y(y-2)}-1}$$

$\frac{1}{2}$

15 $$\lim_{x\to 0} \frac{x(2^x-3^x)}{1-\cos 3x} =$$

$$\lim_{x\to 0} \underbrace{\frac{(3x)^2}{1-\cos(3x)}}_{1} \cdot \underbrace{\frac{x(2^x-3^x)}{9x^2}}_{-2^x\frac{\left(\left(\frac{3}{2}\right)^x-1\right)}{x}} = -\frac{2}{9}\ln\frac{3}{2}$$

16

$$\lim_{x\to 0^+} \frac{7^x-2^x}{\sin^4\sqrt{x}} = \frac{2^x\left(\left(\frac{7}{2}\right)^x-1\right)}{\sin^4\sqrt{x}}\,\frac{(\sqrt{x})^4}{(\sqrt{x})^2(\sqrt{x})^2} =$$

$$= \underbrace{\frac{2^x\left(\left(\frac{7}{2}\right)^x-1\right)}{x}}_{\frac{7}{2}} \cdot \underbrace{\frac{(\sqrt{x})^4}{\sin^4\sqrt{x}}}_{1} \cdot \underbrace{\frac{1}{x}}_{+\infty}$$

17

$$\lim_{x\to 1^+} \tan\left(\frac{\pi}{4}x\right)\frac{\sin(2^x-2)}{x^2-1} =$$

$$\tan\left(\frac{\pi}{4}x\right)\frac{\sin(2^x-\ )}{x^2-1}\cdot\frac{2^x-2}{2^x-2}$$

$$\frac{2^x-2}{x^2-1} = \frac{2^x-2}{(x+1)(x-1)} = \frac{2(2^{x-1}-1)}{(x+1)(x-1)}$$

$$\underbrace{\tan\left(\frac{\pi}{4}x\right)}_{1}\ \underbrace{\frac{\sin(2^x-2)}{2^x-2}}_{1}\cdot\frac{2(2^{x-1}-1)}{(x+1)(x-1)} = \log 2$$

$\log 2$; 1

18

$$\lim_{x\to 1^-} \frac{1}{x-1}+\frac{x}{1-x^2} = \frac{1}{x-1}-\frac{x}{x^2-1} = \frac{\overbrace{x^2-x-1}^{-1}}{\underbrace{x^2-1}_{0}} = -\infty$$

19

$$\lim \frac{\sin x-1}{\ldots} = \frac{\sin x-1}{\ldots}\cdot\frac{\sin x+1}{\ldots} =$$

$$x \to \frac{\pi}{2} \quad \left(x - \frac{\pi}{2}\right)^2 \qquad x^2 - \pi x + \left(\frac{\pi}{2}\right)^2 \qquad \sin x + 1$$

$$y = x - \frac{\pi}{2} \qquad x = y + \frac{\pi}{2}$$

$$\lim_{y \to 0} \frac{\sin\left(y + \frac{\pi}{2}\right) - 1}{y^2} = \frac{\cos(y) - 1}{y^2}$$

19. $\lim_{x \to \frac{\pi}{2}} \frac{\sin x - 1}{\left(x - \frac{\pi}{2}\right)^2}$

$$= -\frac{1 - \cos(y)}{y^2} = -\frac{1}{2}$$

20 $\lim_{x \to 0} x^2 \log \cos x$

$$\lim_{x \to 0} \underbrace{x^2}_{0} \cdot \overbrace{\log(\underbrace{\cos x}_{1})}^{0} = 0 \cdot 0 = 0$$

21 $\lim_{x \to +\infty} \frac{x^3 - e^x}{x^2 + 3 \log x}$

$$\lim_{x \to +\infty} \frac{x^3 - e^x}{x^2 + 3 \log x} = \frac{x^3\left(1 - \frac{e^x}{x^3}\right)}{x^2\left(1 + \frac{3 \log x}{x^2}\right)} = \frac{x\overbrace{\left(1 - \frac{e^x}{x^3}\right)}^{-\infty}}{\underbrace{1}_{1} + \underbrace{\frac{3 \log x}{x^2}}_{0}} = -\infty$$

22 $\lim_{x \to \pi} \frac{\tan x}{x - \pi}$

$$-\frac{\tan(\pi - x)}{\pi - x} = 1$$

23 $\lim_{x \to 0} \sin x \cos \frac{1}{x}$

$$\lim_{x \to 0} \sin x \cdot \lim_{x \to 0} \cos \frac{1}{x} \quad ?$$

24 $\lim_{x \to 0} \frac{\log(1 + 3x^2)}{x(e^{2x} - 1)}$

$$\frac{\log(1 + 3x^2)}{x(e^{2x} - 1)} \cdot \frac{3x^2}{3x^2}$$

$$\frac{\log(1+3x^2)}{\underbrace{3x^2}_{1}} \cdot \frac{3x^2}{x(e^{2x}-1)} =$$

$$1 \cdot \frac{3x}{e^{2x}-1} = \frac{3}{\underbrace{e^x+1}_{\frac{3}{2}}} \cdot \frac{x}{\underbrace{e^x-1}_{1}} = \frac{3}{2}$$

25 $\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4}$

$$\frac{\sqrt{x}-2}{x-4} \cdot \frac{\sqrt{x}+2}{\sqrt{x}+2} =$$

$$= \frac{\cancel{x-4}}{(\sqrt{x}+2)(\cancel{x-4})} = \frac{1}{\underbrace{\sqrt{x}+2}_{\frac{1}{4}}} = \frac{1}{4}$$

26 $\lim_{x\to 1} \frac{\log x}{x-1}$

$$y = x-1 \qquad x = y+1$$

$$\lim_{y\to 0} \frac{\log(y+1)}{y} = 1$$

27 $\lim_{x\to 0} (x^2+x)\log x$

$$\lim_{x\to 0^+} x^2\log x + x\log x = x\log x(1+x)$$

28 $\lim_{x\to 0} \frac{\tan x + x^2}{\sin^3 x + 2x}$

$$\frac{\tan x}{\sin^3 x + 2x} + \frac{x^2}{\sin^3 x + 2x}$$

16. $\lim_{x\to 0^+} \dfrac{7^x - 2^x}{\sin^4 \sqrt{x}}$
17. $\lim_{x\to 1^+} \tan\left(\dfrac{\pi}{4}x\right) \dfrac{\sin(2^x - 2)}{x^2 - 1}$
18. $\lim_{x\to 1^-} \left(\dfrac{1}{x-1} + \dfrac{x}{1-x^2}\right)$
19. $\lim_{x\to \frac{\pi}{2}} \dfrac{\sin x - 1}{(x - \frac{\pi}{2})^2}$
20. $\lim_{x\to 0} x^2 \log\cos x$
21. $\lim_{x\to +\infty} \dfrac{x^3 - e^x}{x^2 + 3\log x}$
22. $\lim_{x\to \pi} \dfrac{\tan x}{x - \pi}$
23. $\lim_{x\to 0} \sin x \cos \dfrac{1}{x}$
24. $\lim_{x\to 0} \dfrac{\log(1+3x^2)}{x(e^{2x} - 1)}$
25. $\lim_{x\to 4} \dfrac{\sqrt{x} - 2}{x - 4}$
26. $\lim_{x\to 1} \dfrac{\log x}{x - 1}$
27. $\lim_{x\to 0} (x^2 + x)\log x$
28. $\lim_{x\to 0} \dfrac{\tan x + x^2}{\sin^3 x + 2x}$
29. $\lim_{x\to -\infty} e^x \sqrt[3]{x+1}$
30. $\lim_{x\to +\infty} x\left(e^{1/x} - 1\right)$

CORSO DI LAUREA IN INFORMATICA APPLICATA

ANALISI MATEMATICA 1

ESERCIZI SU

LIMITI DI SUCCESSIONI

Esercizio 1.
Calcolare, se esistono, i seguenti limiti:

1. $\lim_{n\to+\infty}\left(\frac{n+2}{3-n}-\frac{1}{n^2}+\frac{1}{n^3(n+1)}\right)$
2. $\lim_{n\to+\infty}\frac{n^3-2n^2}{n+3}$
3. $\lim_{n\to+\infty}\frac{(\sqrt{n})^3-2(\sqrt{n})^5}{1+3(\sqrt[3]{n})^7}$
4. $\lim_{n\to+\infty}\left(\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{1+2n}}\right)$
5. $\lim_{n\to+\infty}\sqrt[n]{\frac{n!}{2^n+1}}$
6. $\lim_{n\to+\infty}\left(\frac{2^n+4^n}{3^n}\right)$
7. $\lim_{n\to+\infty}(5+\cos n)^n$
8. $\lim_{n\to+\infty}\left(\sqrt{n+1}-\sqrt{n-1}\right)$
9. $\lim_{n\to+\infty}\frac{n!}{(2n)^{2n}}$
10. $\lim_{n\to+\infty}\left(\frac{n}{\sqrt{n+1}}-\frac{n}{\sqrt{n-1}}\right)$
11. $\lim_{n\to+\infty}\left((n^2+\sin n)\sin\frac{2}{n}\right)$
12. $\lim_{n\to+\infty}\left(1-\frac{1}{2n^2}\right)^{n^2}$
13. $\lim_{n\to+\infty}\left(n\cos\frac{\pi}{n}\sin\frac{2\pi}{n}\right)$
14. $\lim_{n\to+\infty}\left(\frac{1}{3^n}+\frac{(-1)^n}{7^n}\right)$

15. $\lim_{n\to+\infty}\left(\frac{\sin\frac{1}{n}}{1-\cos\frac{1}{\sqrt{n}}}\right)$
16. $\lim_{n\to+\infty}\sqrt[n]{2^n+1}$
17. $\lim_{n\to+\infty}e^{n\sin\frac{1}{n}}$
18. $\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^{\frac{1}{\sin\frac{1}{n}}}$
19. $\lim_{n\to+\infty}\left(\frac{(3+\sin n)^n+n^4}{(n-3)!-5^n}\right)$
20. $\lim_{n\to+\infty}\sqrt{n}\left(\sqrt{n+1}-\sqrt{n-1}\right)$

CORSO DI LAUREA IN INFORMATICA APPLICATA

ANALISI MATEMATICA 1

ESERCIZI SU

FUNZIONI CONTINUE

Esercizio 1.
Stabilire se le seguenti funzioni sono continue nel loro dominio:

1. $f(x) = \begin{cases} x^2 \log|x| & \text{se } x \neq 0 \\ 0 & \text{se } x = 0 \end{cases}$

2. $f(x) = \begin{cases} \dfrac{4(1-\sqrt{x})}{x^2-1} & \text{se } x \in [0,1) \cup (1,+\infty) \\ -1 & \text{se } x = 1 \end{cases}$

3. $f(x) = \begin{cases} \dfrac{e^x}{x} & \text{se } x \neq 0 \\ 1 & \text{se } x = 0 \end{cases}$

4. $f(x) = \begin{cases} \dfrac{\sin 2x}{e^{-x}-1} & \text{se } x \neq 0 \\ -2 & \text{se } x = 0 \end{cases}$

5. $f(x) = \begin{cases} \dfrac{\cos x}{2x-\pi} & \text{se } x \neq \frac{\pi}{2} \\ 1 & \text{se } x = \frac{\pi}{2} \end{cases}$

Esercizio 2.
Stabilire se le seguenti funzioni sono prolungabili con continuità in $x = 0$:

1. $f(x) = \frac{1}{\log|x|}$

2. $f(x) = e^{-1/x^2}$

Esercizio 3.
Stabilire per quali valori del parametro $\alpha \in \mathbb{R}$ le seguenti funzioni sono continue nel loro dominio:

1. $f(x) = \begin{cases} 5 + x & \text{se } x \neq 0 \\ \alpha & \text{se } x = 0 \end{cases}$

2. $f(x) = \begin{cases} 1 - \dfrac{x}{|x|} & \text{se } x \neq 0 \\ \alpha & \text{se } x = 0 \end{cases}$

3. $f(x) = \begin{cases} 1 - x + x^2 & \text{se } x \geq 0 \\ \alpha(1 + \cos x) & \text{se } x < 0 \end{cases}$

4. $f(x) = \begin{cases} \dfrac{x^3 - 8}{x^2 - 4} & \text{se } x > 2 \\ \alpha(x + 1) & \text{se } x \leq 2 \end{cases}$

5. $f(x) = \begin{cases} x \cos \dfrac{x}{x^2 + 1} & \text{se } x \geq 0 \\ \alpha \dfrac{(x + 1)^2 - 1}{\log(x + 1)} & \text{se } x \in (-1, 0) \end{cases}$

Derivate

derivate delle funzioni elementari	
$D\ k = 0$ dove k è una costante	$D\ senx = cosx$
$D\ x^n = n\ x^{n-1}$	$D\ cosx = -\ senx$
$D\ \frac{1}{x^n} = Dx^{-n} = -\ n\ x^{-n-1} = -\frac{n}{x^{n+1}}$	$D\ tgx = \frac{1}{cos^2x} = 1 + tg^2x$
$D\sqrt[n]{x} = \frac{1}{n\sqrt[n]{x^{n-1}}}$	$D\ cotgx = -\frac{1}{sen^2x} = -1 - cotg^2x$
$D\ \sqrt{x} = \frac{1}{2\sqrt{x}}$	$D\ arcsenx = \frac{1}{\sqrt{1-x^2}}$
$D\ log_a x = \frac{1}{x} log_a e = \frac{1}{x} \cdot \frac{1}{lna}$	$D\ arccosx = -\frac{1}{\sqrt{1-x^2}}$
$D \ln x = \frac{1}{x}$	$D\ arctgx = \frac{1}{1+x^2}$
$D\ a^x = a^x\ ln\, a = a^x \cdot \frac{1}{log_a e}$	$D\ arccotgx = -\frac{1}{1+x^2}$
$D\ e^x = e^x$	$D\ \lvert x\rvert = \frac{x}{\lvert x\rvert} = \frac{\lvert x\rvert}{x}$

regole di derivazione	
$D\ k \cdot f(x) = k \cdot f'(x)$	prodotto di una **costante** k per una funzione
$D\ f(x) \pm g(x) \pm h(x) = f'(x) \pm g'(x) \pm h'(x)$	**somma** di due o più funzioni
$D\ f(x) \cdot g(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$	**prodotto** di due funzioni
$D\ f(x) \cdot g(x) \cdot h(x) = f'(x) \cdot g(x) \cdot h(x) + f(x) \cdot g'(x) \cdot h(x) + f(x) \cdot g(x) \cdot h'(x)$	**prodotto** di tre funzioni
$D\ \frac{f(x)}{g(x)} = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{[g(x)]^2}$	**rapporto** di due funzioni
$Df[g(x)] = f'[g(x)] \cdot g'(x)$	funzione **composta**
$Df(x)^{g(x)} = f(x)^{g(x)} \cdot \left[g'(x) \cdot ln[f(x)] + g(x) \cdot \frac{f'(x)}{f(x)}\right]$	funzione **elevata** ad una funzione

CORSO DI LAUREA IN INFORMATICA APPLICATA

ANALISI MATEMATICA 1

ESERCIZI SU

FUNZIONI DERIVABILI

Esercizio 1.
Stabilire, motivando la risposta, se le seguenti funzioni sono derivabili nel loro dominio e, in caso affermativo, calcolarne la derivata:

1. $f(x) = 2x + 4x^3$ $\quad 2 + 12x^2$
2. $f(x) = \sqrt{x}e^x$ $\quad \frac{1}{2\sqrt{x}} \cdot e^x + \sqrt{x}\,e^x = e^x\left(\frac{1}{2\sqrt{x}} + \sqrt{x}\right)$
3. $f(x) = \frac{\log x}{x}$ $\quad \frac{1 - \log x}{x^2}$
4. $f(x) = (\log x)^4$ $\quad \frac{4(\log x)^3}{x}$
5. $f(x) = x \cos|x|$ $\quad \cos|x| + x\left(-\sin|x| \cdot \frac{|x|}{x}\right)$
6. $f(x) = (x + x^2)\log x$ $\quad (1+2x)\log x + \frac{(x+x^2)}{x}$
7. $f(x) = \dfrac{x}{1 + x^2}$ $\quad \frac{3x^2+1}{(1+x^2)^2}$
8. $f(x) = e^{-x}(\sin x + \cos x)$ $\quad -e^{-x}(\sin x + \cos x) + e^{-x}(\cos x - \sin x)$
9. $f(x) = \log(\frac{1-x}{1+x})$ $\quad -\frac{2}{(1-x)^2}$
10. $f(x) = |x|e^{x^2}$
11. $f(x) = \arctan \frac{1}{x}$
12. $f(x) = x^3 e^{2x}$
13. $f(x) = x^2 \sin \frac{x}{x+1}$
14. $f(x) = \arctan(x^2 - x \log x)$
15. $f(x) = e^{x^2 \cos x} \sin x^3$

Esercizio 2.
Stabilire, motivando la risposta, se le seguenti funzioni sono derivabili nel loro dominio e, in caso affermativo, calcolarne la derivata:

1. $f(x) = \begin{cases} 1 & \text{se } x \leq 0 \\ 1 - x^2 & \text{se } 0 < x < 1 \\ \log x & \text{se } x \geq 1 \end{cases}$

2. $f(x) = \begin{cases} x^2 \sin \dfrac{1}{x^2} & \text{se } x \neq 0 \\ 0 & \text{se } x = 0 \end{cases}$

3. $f(x) = \begin{cases} \dfrac{\sin |x|}{x} & \text{se } x \neq 0 \\ 0 & \text{se } x = 0 \end{cases}$

4. $f(x) = \begin{cases} \dfrac{\sin 2x}{e^{-x} - 1} & \text{se } x \neq 0 \\ 1 & \text{se } x = 0 \end{cases}$

5. $f(x) = \begin{cases} \dfrac{\tan x}{2x} & \text{se } x \neq 0 \\ 1 & \text{se } x = 0 \end{cases}$

Esercizio 3.
Stabilire, motivando la risposta, per quali valori dei parametri le seguenti funzioni sono derivabili nel loro dominio:

1. $f(x) = \begin{cases} -x^2 + \alpha x - \beta & \text{se } x < 0 \\ e^x & \text{se } x \geq 0 \end{cases} \qquad \alpha, \beta \in \mathbb{R}$

2. $f(x) = x^\alpha\,, \ \ \alpha > 0$

3. $f(x) = \begin{cases} x^\alpha \sin \dfrac{1}{x} & \text{se } x > 0 \\ 0 & \text{se } x \leq 0 \end{cases} \qquad \alpha \in \mathbb{R}$

4. $f(x) = \begin{cases} e^x + \alpha x & \text{se } x \geq 1 \\ x^2 + \beta & \text{se } x < 1 \end{cases} \qquad \alpha, \beta \in \mathbb{R}$

Esercizio 4.
Classificare, motivando la risposta, gli eventuali punti di non derivabilità delle seguenti funzioni:

1. $f(x) = 3x + |x-1|$
2. $f(x) = \sqrt[3]{1-x}$
3. $f(x) = \sqrt[3]{(x-2)^2}$
4. $f(x) = \sqrt[3]{x}\sin x$
5. $f(x) = \begin{cases} x^2 \cos \dfrac{1}{x} & \text{se } x \neq 0 \\ 0 & \text{se } x = 0 \end{cases}$

Esercizio 5.
Data la funzione
$$f(x) = e^{2x} + x^3\,,$$
stabilire se f è invertibile nel suo dominio e, in caso affermativo, calcolare $(f^{-1})'(1)$.

Esercizio 6.
Data la funzione
$$f(x) = \log x + 2x + 1\,,$$
stabilire se f è invertibile nel suo dominio e, in caso affermativo, calcolare $(f^{-1})'(3)$.

Esercizio 7.
Data la funzione
$$f(x) = \log(2 + \sin x)\,,$$
scrivere, se esiste, l'equazione della retta tangente al grafico di f nel punto $(\pi, f(\pi))$.

Esercizio 8.
Data la funzione
$$f(x) = 1 - \frac{1}{x^2}\,,$$
scrivere, se esiste, l'equazione della retta tangente al grafico di f nel punto $(1, f(1))$.

CORSO DI LAUREA IN INFORMATICA APPLICATA

ANALISI MATEMATICA 1

ESERCIZI SU

MASSIMI E MINIMI PER FUNZIONI DI UNA VARIABILE

Esercizio 1.
Stabilire, motivando la risposta, se le seguenti funzioni ammettono massimi o minimi locali nel loro dominio e, in caso affermativo, determinarli:

1. $f(x) = (x^2 - 8)e^x$
2. $f(x) = x + \sqrt{1 - x^2}$
3. $f(x) = xe^{-x^2}$
4. $f(x) = \dfrac{x}{(x+1)^2}$
5. $f(x) = x + \dfrac{1}{x}$
6. $f(x) = \dfrac{1}{2}\sin 2x + \cos x$
7. $f(x) = x^3 \log x$
8. $f(x) = e^{-\sqrt[3]{x^2}}$
9. $f(x) = \arctan(x(1-x))$
10. $f(x) = \dfrac{\sin x + \cos x}{\cos 2x}$
11. $f(x) = e^{-|x|}$
12. $f(x) = x\sqrt{1 - x^2}$
13. $f(x) = \dfrac{1 - \cos x}{\sin x + \cos x}$
14. $f(x) = e^{-\frac{1}{x^2+1}}$
15. $f(x) = (3 - x)e^{x^2+1}$

16. $f(x) = \arctan\left(-\dfrac{1}{x^2+1}\right)$
17. $f(x) = (x^2-2)e^{x+3}$
18. $f(x) = \dfrac{\sin x}{1+2\sin^2 x}$
19. $f(x) = |x|e^x$
20. $f(x) = \dfrac{1-8x^2}{4x^2+8x+4}$
21. $f(x) = \sqrt{x(10-x)}$
22. $f(x) = x^2e^{-x}$
23. $f(x) = \sin^2 x - \sin x$
24. $f(x) = \begin{cases} x^2+1 & \text{se } x<0 \\ \cos x & \text{se } 0 \le x < \frac{3}{2}\pi \\ 3\pi - 2x & \text{se } x \ge \frac{3}{2}\pi \end{cases}$
25. $f(x) = \begin{cases} e^{-x^2} & \text{se } x \ge 0 \\ 1+e^{1/x} & \text{se } x<0 \end{cases}$

Esercizio 2.
Stabilire, motivando la risposta, se le seguenti funzioni ammettono massimo o minimo assolti nell'insieme indicato e, in caso affermativo, determinarli:

1. $f(x) = \dfrac{x^2+2x-4}{x^2+x-2}\,, \qquad x \in [-1, 1/2]$
2. $f(x) = x\sqrt{1-x^2}\,, \qquad x \in [0,1]$
3. $f(x) = e^x \cos x\,, \qquad x \in [0, \pi/2]$
4. $f(x) = \log \dfrac{x(4x-3)}{x+1}\,, \qquad x \in [1,2]$
5. $f(x) = x(x^2-1)\,, \qquad x \in [-3,2]$
6. $f(x) = \sqrt{x^2-4}\,, \qquad x \in [-3,-2]$
7. $f(x) = \cos x + x\sin x\,, \qquad x \in [0,\pi]$
8. $f(x) = |x^3-3x^2+3x|\,, \qquad x \in [-2,3]$
9. $f(x) = \dfrac{1+x}{\sqrt{x}}\,, \qquad x \in [1/2, 3]$
10. $f(x) = \dfrac{1-\sin x}{1+\cos x}\,, \qquad x \in [0, 2/3\pi]$

11. $f(x) = (2+x^2)e^{-x^2}\,, \qquad x \in [-1,2]$

12. $f(x) = \arctan|x|\,, \qquad x \in [-1,\sqrt{3}]$

13. $f(x) = |x^2-4x+3|\,, \qquad x \in [0,5]$

14. $f(x) = \sqrt{2x+3}\,, \qquad x \in [0,1]$

15. $f(x) = \sin x - \cos x + 1\,, \qquad x \in [-\pi/2, \pi/2]$

16. $f(x) = \log\sqrt{4-x^2}\,, \qquad x \in [-1, 3/2]$

17. $f(x) = (2-x)e^x\,, \qquad x \in [-1,2]$

18. $f(x) = \big|\cos|x+1|\big|\,, \qquad x \in [-2,2]$

19. $f(x) = 2x^3+3x^2-12x+1\,, \qquad x \in [-1,5]$

20. $f(x) = x + \log x\,, \qquad x \in [1,e]$

21. $f(x) = \arctan\dfrac{1+x}{1-x}\,, \qquad x \in [-1,0]$

22. $f(x) = |x|e^x\,, \qquad x \in [-5,1]$

23. $f(x) = \sqrt{1+|x|}\,, \qquad x \in [-1,3]$

24. $f(x) = \dfrac{\sin x}{1+2\sin^2 x}\,, \qquad x \in [0,\pi]$

25. $f(x) = \begin{cases} (x+1)\log^2(x+1) & \text{se } x \in (-1,0] \\ 0 & \text{se } x = -1 \end{cases}$

Esercizio 3.
Dimostrare le seguenti disuguaglianze:

1. $\dfrac{x-1}{x} \le \log x \le x-1 \qquad \text{per ogni} \quad x > 0$
2. $x \ge \dfrac{\tan x}{1+\tan^2 x} \qquad \text{per ogni} \quad x \ge 0$
3. $xe^{1/\sqrt{x}} > 1 \qquad \text{per ogni} \quad x > 1$
4. $x^3\sqrt{1-x^2} > -\dfrac{1}{2} \qquad \text{per ogni} \quad x \in [-1,1]$

Esercizio 4.
Verificare che la funzione

$$f(x) = x\sin x - \cos 2x$$

ha un punto stazionario in $x = 0$ e determinarne la natura.

Esercizio 5.
Verificare che i punti di ascissa 1 e -1 sono rispettivamente di minimo e di massimo locali per la funzione
$$f(x) = \frac{x^2+x+1}{x}\,.$$

Esercizio 6.
Verificare che in $x = \pi/4$ la funzione
$$f(x) = \frac{\cos x}{1+2\cos^2 x}$$
ha un massimo locale e assoluto.

Esercizio 7.
Verificare che in $x = 0$ la funzione
$$f(x) = \begin{cases} |x|\cos^2 \dfrac{1}{x} & \text{se } x \neq 0 \\ 0 & \text{se } x = 0 \end{cases}$$
ha un minimo locale.

Esercizio 8.
Verificare che in $x = -1$ la funzione
$$f(x) = \frac{(x+1)^3}{x}$$
ha un flesso a tangente verticale.

Esercizio 9.
Stabilire se $x = 0$ è un punto di massimo assoluto per la funzione
$$f(x) = \begin{cases} \dfrac{-3x^2-x^3}{2x^2} & \text{se } x > 0 \\ (x+2)^3+5x^3 & \text{se } x \leq 0 \end{cases}$$
nell'intervallo $[-1, 10]$.

Esercizio 10.
Stabilire se $x = 0$ è un punto di minimo assoluto per la funzione
$$f(x) = \begin{cases} \dfrac{x^2-2x^3}{3x^2} & \text{se } x < 0 \\ -4(x-2)^2+x^2 & \text{se } x \geq 0 \end{cases}$$
nell'intervallo $[-5, 2]$.

Esercizio 11.
Stabilire se $x = 2$ è un punto di massimo assoluto per la funzione
$$f(x) = e^{2x}\left(\frac{5}{2} - x\right)$$

nel suo dominio.

Esercizio 12.
Data la funzione
$$f(x) = |x-1| + |x|\,,$$
disegnarne il grafico e individuarne i punti di massimo o minimo locali e globali.

Esercizio 13.
Tra tutti i triangoli rettangoli di ipotenusa 2ℓ, trovare quello di area massima.

Esercizio 14.
Sia dato il cono circolare retto di altezza h e raggio r. Trovare il cilindro, inscritto nel cono, di superficie laterale massima.

Esercizio 15.
Data un cilindro di volume fissato V, determinare quello di superficie totale massima.

Esercizio 16.
Data la curva di equazione
$$y + x^2 = 0\,,$$
determinare il punto della curva che ha distanza minima da $(2, -1/2)$.

Esercizio 17.
Tra tutti i trapezi isosceli inscritti in una semicirconferenza di raggio r, determinare quello di area massima e quello di perimetro massimo.

Esercizio 18.
Trovare due numeri, la cui somma è $2a$ con $a > 0$, tali che la somma delle loro radici quadrate sia massima.

Esercizio 19.
Decomporre un numero in due fattori in modo che sia minima la somma dei loro quadrati.

Esercizio 20.
Verificare che la somma dei cubi di due numeri reali di prodotto assegnato $p > 0$ è minima quando i due numeri sono uguali.

CORSO DI LAUREA IN INFORMATICA APPLICATA

ANALISI MATEMATICA 1

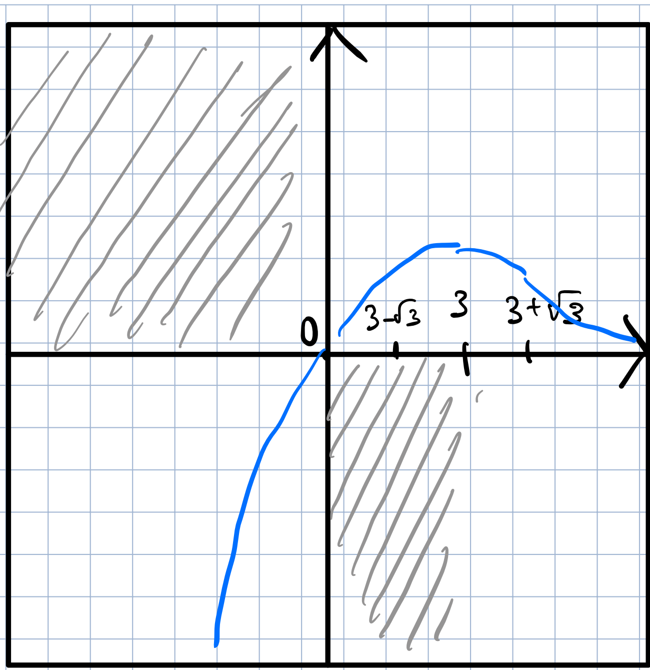
ESERCIZI SU

STUDIO DI FUNZIONI DI UNA VARIABILE

Esercizio 1.
Studiare le seguenti funzioni e disegnarne il grafico:

1. $f(x) = x^3 e^{-x}$
2. $f(x) = \dfrac{x}{\log x}$
3. $f(x) = \frac{(x-1)^2}{x}$
4. $f(x) = \dfrac{1+\cos x}{\cos x - \sin x}$
5. $f(x) = |\sin x| + \sin 2x$
6. $f(x) = \sqrt{x} + \dfrac{1}{\sqrt{x}} - 4$
7. $f(x) = \log(x^2 + 5x - 6)$
8. $f(x) = \sqrt{x} - \arcsin x$
9. $f(x) = \dfrac{|x^2 - x - 2|}{x^2}$
10. $f(x) = x\sqrt[3]{(\log |x|)^2}$
11. $f(x) = (x-2)e^{-1/x}$
12. $f(x) = \arcsin \frac{x^2-1}{x^2+1}$
13. $f(x) = x^3 \log |x|$
14. $f(x) = \sqrt{x^2 - 4x} - x$
15. $f(x) = \dfrac{\cos x}{1 - \cos x}$
16. $f(x) = \dfrac{x^2 - x - 2}{x^2 + x - 2}$

1. $f(x) = x^3 e^{-x}$



DOMINIO: $\mathbb{R}$

SIMMETRIE:

$$x^3 e^{-x} = -x^3 e^{x} \quad \text{NON PARI}$$

$$-x^3 e^{x} = -\left(x^3 e^{-x}\right) \quad \text{NON DISPARI}$$

INTERSEZIONE ASSI:

$$\begin{cases} y = f(x) \\ y = 0 \end{cases} \Rightarrow (0,0) \qquad \begin{cases} y = f(x) \\ x = 0 \end{cases} \Rightarrow (0,0)$$

SEGNO:

$$x^3 e^{-x} \geq 0$$

$$x \geq 0$$

LIMITI

$$\lim_{x \to +\infty} x^3 e^{-x} = \frac{x^3}{e^x} = 0$$

$$\lim_{x \to -\infty} x^3 e^{-x} = \frac{x^3}{e^x} = -\infty$$

DERIVABILITÀ

COMPOSIZIONE DI FUNZIONI DERIVABILI

$$f'(x) = 3x^2 \cdot e^{-x} - e^{-x} x^3$$

$$= e^{-x}(3x^2 - x^3)$$

$$\frac{x^2(-x+3)}{e^x} = 0$$

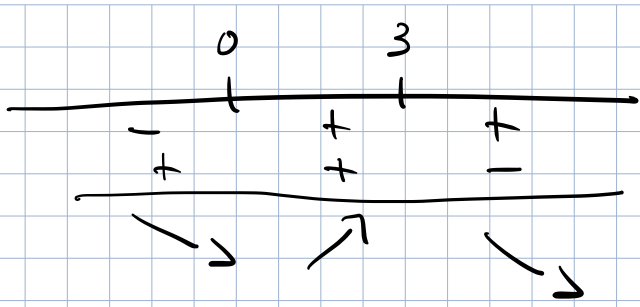
$$x = 0$$

$$x = 3$$

MASSIMO E MINIMO

$$x^2(-x+3) \geq 0$$

$x \leq 3 \qquad x \geq 0$



DERIVATA SECONDA

$$f''(x) = x^3 \cdot e^{-x} - 6x^2 \cdot e^{-x} + 6x \cdot e^{-x}$$

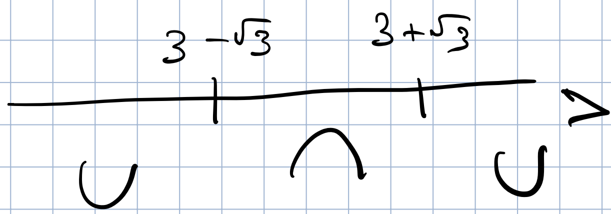
$$= e^{-x}(x^3 - 6x^2 + 6x)$$

$x(x^2 - 6x + 6) = 0 \qquad x = 0$

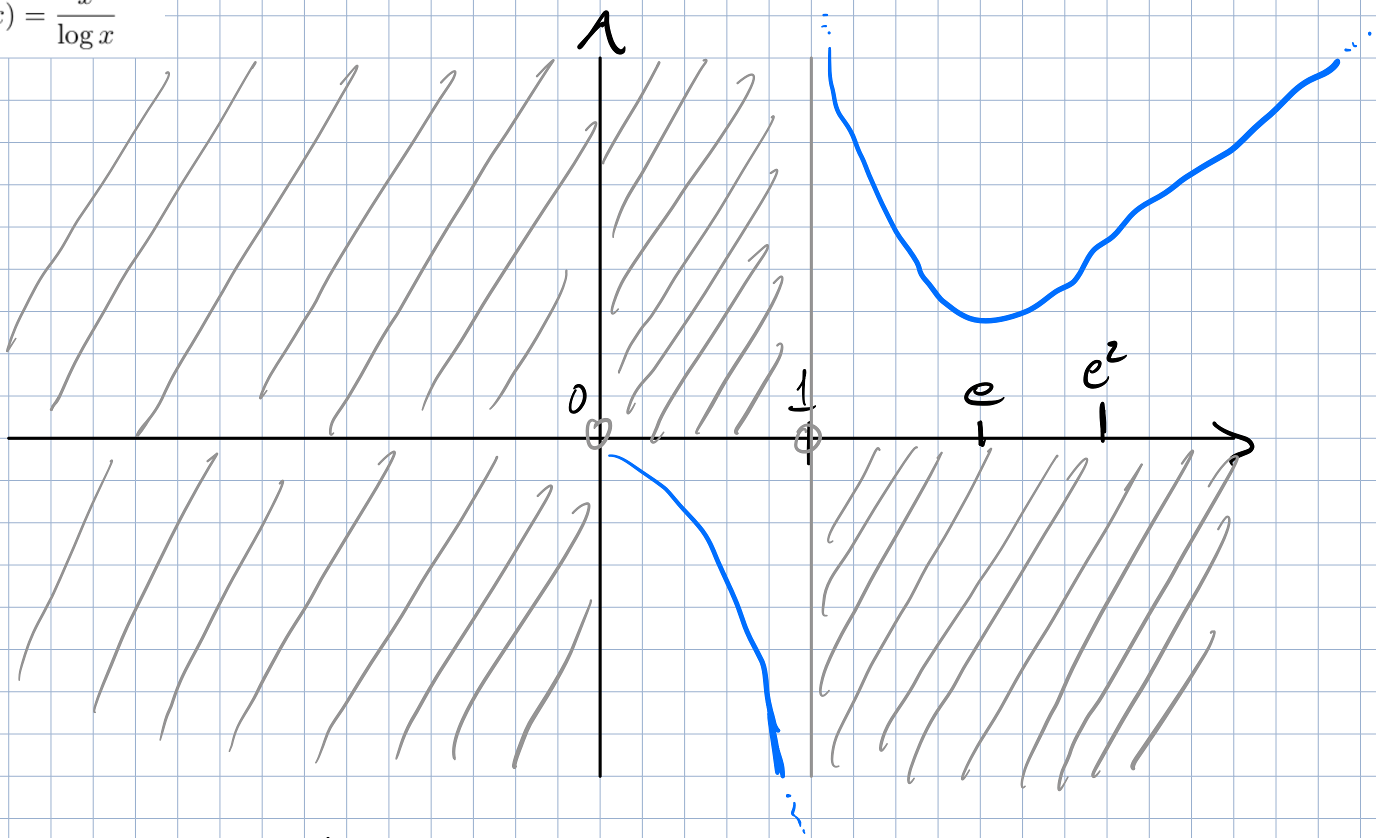
$$\frac{6 \pm \sqrt{36-24}}{2} = \frac{6 \pm \sqrt{12}}{2} = \frac{\cancel{6} \pm \cancel{2}\sqrt{3}}{\cancel{2}} = 3 \pm \sqrt{3}$$

PUNTI DI FLESSO

$$x(x^2 - 6x + 6) \geq 0$$



2. $f(x) = \dfrac{x}{\log x}$



DOMINIO: $\mathbb{R}^+ - \{1\}$

SIMMETRIE:

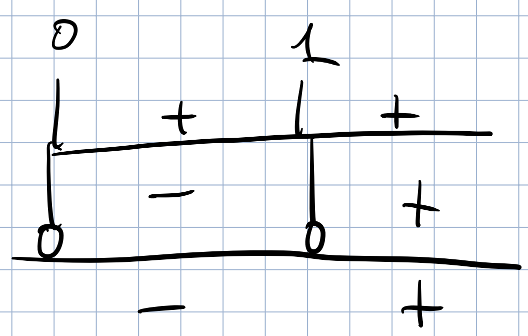
NON CI SONO ESSENDO DEFINITO SOLO IN $\mathbb{R}^+$

SEGNO

$$\frac{x}{\log x} \geq 0$$

$x \geq 0$

$\log x > 0$



INTERSEZIONI

$$\begin{cases} y = f(x) \\ y = 0 \end{cases}$$

NON INTERSECA GLI ASSI

LIMITI E ASINTOTI

$$\lim_{x \to +\infty} \frac{x}{\log x} = \frac{\infty}{\infty} = +\infty$$ NO ORIZZONTALE

NO OB.

$$\lim_{x \to +\infty} \frac{\frac{x}{\log x}}{x} = \frac{x}{x \cdot \log x} = \frac{1}{\log x} = 0$$

$$\lim_{x \to 0^+} \frac{x}{\log x} = \frac{0}{-\infty} = 0$$ NO VERT.

$$\lim_{x \to 1^-} \frac{x}{\log x} = -\infty$$ ASINTOTO VERT.

$$\lim_{x \to 1^+} \frac{x}{\log x} = +\infty$$ ASINTOTO VERT.

DERIVABILITA'

f È DERIVABILE POICHÉ RAPPORTO DI FUNZIONI DERIVABILI

MINIMI E MASSIMI

$$f'(x) = \frac{\log x - x \cdot \frac{1}{x}}{\log^2(x)} = \frac{\log x - 1}{\log^2(x)}$$

$\log(x) - 1 = 0$ SOLO SE $x = e$

MONOTONIA

$$f'(x) \geq 0 \qquad \frac{\log x - 1}{\log^2(x)} \geq 0$$

$$\log x - 1 \geq 0 \qquad \log x \geq 1$$

$$x \geq e$$

e MINIMO LOCALE

FLESSI

$$f''(x) = \frac{\frac{1}{x}\cdot \log^2(x) - (\log x - 1)\frac{2\log x}{x}}{\log^4(x)}$$

$$\frac{\frac{\log^{\cancel{2}} x}{x} - (\log x - 1)\, 2\,\frac{\cancel{\log x}}{x}}{\log^4(x)} = \frac{\log x - 2(\log x - 1)}{\log^3(x)}$$

$$= \frac{\log x - 2\log x + 2}{\log^3 x} = \frac{-\log(x) + 2}{\log^3(x)}$$

$$\frac{-\log(x) + 2}{\log^3(x)} = 0 \qquad \text{QUANDO} \qquad \log(x) = 2$$

$$x = e^2$$

CONCAVITÀ

$$\frac{-\log(x)+2}{\log^3(x)} \geq 0 \quad \Rightarrow \quad \log x \leq 2$$

$$x \leq e^2$$

	e^2	
$\cap$		$\cup$

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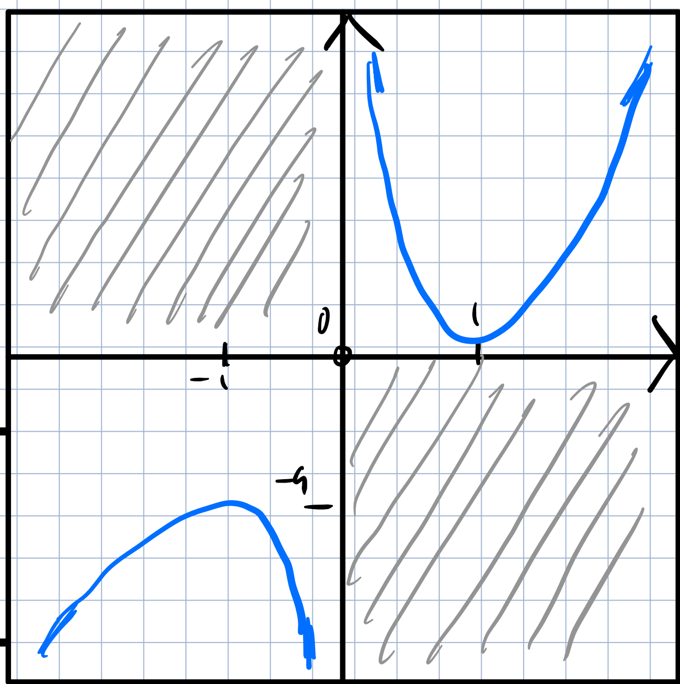
3. $f(x) = \frac{(x-1)^2}{x}$



DOMINIO =

$\mathbb{R} - \{0\}$

SIMMETRIE =

NO POICHÉ POLINOMIO

SEGNO =

$$\frac{(x-1)^2}{x} \geq 0$$

$(x-1)^2 \geq 0$ SEMPRE

$x \geq 0$

	-1	0	1	
	$+$	$+$	$+$	$+$
	$-$	$-$	$+$	$+$
	$-$	$-$	$+$	$+$

INTERSEZIONI

$$\begin{cases} y = f(x) \\ y = 0 \end{cases} \Downarrow (1, 0)$$

$$\begin{cases} y = f(x) \\ x = 0 \end{cases} \Downarrow \nexists$$

LIMITI

$$\lim_{x \to +\infty} \frac{(x-1)^2}{x} = +\infty \qquad \lim_{x \to -\infty} \frac{(x-1)^2}{x} = -\infty$$

$$\lim_{x \to 0^+} \frac{x^2-2x+1}{x} = \frac{0-0+1}{0} = +\infty$$

$$\lim_{x \to 0^-} \frac{x^2-2x+1}{x} = \frac{0+0+1}{0} = -\infty$$

DERIVABILITÀ

POLINOMI SONO DERIVABILI

$$f'(x) = \frac{(2x-2)\cdot x - (x^2-2x+1)}{x^2}$$

$$= \frac{2x^2-\cancel{2x}-x^2+\cancel{2x}-1}{x^2} = \frac{x^2-1}{x^2}$$

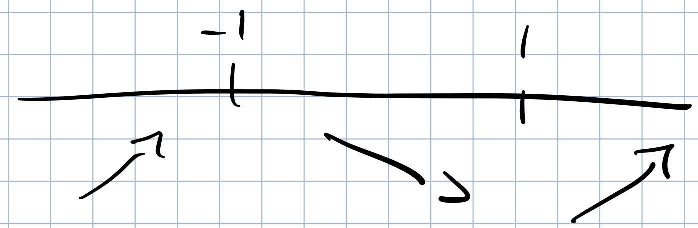
$$= \frac{\cancel{x^2}}{\cancel{x^2}} - \frac{1}{x^2} = 1 - \frac{1}{x^2}$$

MASSIMI E MINIMI

$$1 - \frac{1}{x^2} = 0 \qquad x = \pm 1$$

$$1 - \frac{1}{x^2} \geq 0$$

$$x^2 - 1 \geq 0$$



$\frac{(x-1)^2}{x}$ $\quad f(1)=0 \quad f(-1)=\frac{4}{-1}=-4$

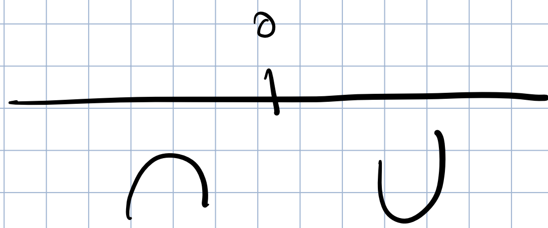
$(1,0) \qquad (-1,-4)$

CONCAVITÀ E PUNTI DI FLESSO

$$f''(x) = 1-\frac{1}{x^2} = \frac{2}{x^3}$$

$\frac{2}{x^3}=0$ MAI

$\frac{2}{x^3}\geq 0 \qquad x\geq 0$

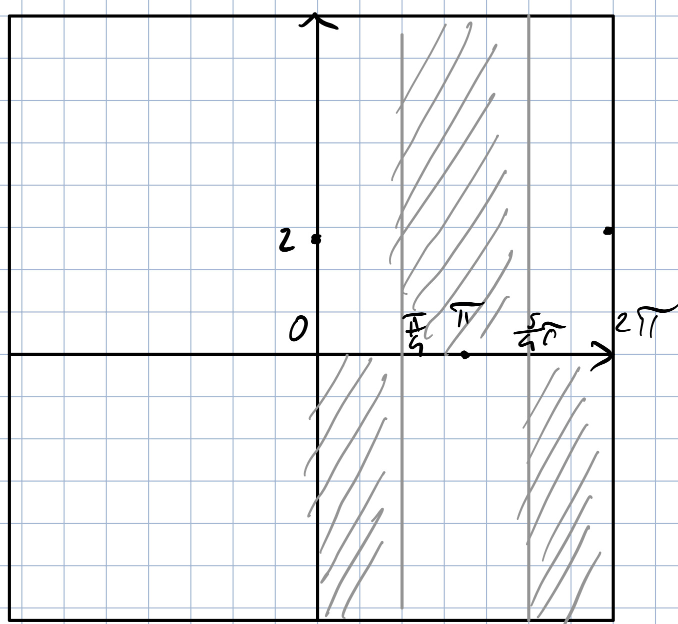


4. $f(x)=\dfrac{1+\cos x}{\cos x-\sin x}$

DOMINIO

$\cos x - \sin x \neq 0$

$x \neq \frac{\pi}{4}+k\pi$



PERIODICITÀ

$$f(x+2\pi) = \frac{1+\cos(x+2\pi)}{\cos(x+2\pi)-\sin(x+2\pi)}$$

ESSENDO FUNZIONI DI PERIODICITÀ 2π

$$= \frac{1+\cos x}{\cos x-\sin x}$$

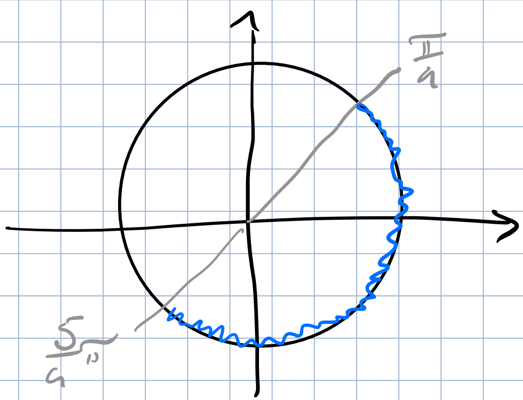
PERIODO 2π

SEGNO

$$\frac{1+\cos x}{\cos x - \sin x} \geq 0$$

$1+\cos x \geq 0$ $\quad$ $\cos x \geq -1$

$\cos x \geq \sin x$



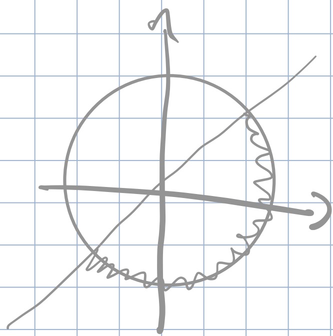
$0 < x < \frac{\pi}{4}$

$\frac{5}{4} < x < 2\pi$

$\sin x < \cos x$

$$\frac{\sin x}{\cos x} < \frac{\cos x}{\cos x}$$

$\begin{cases} \tan x < 1 \\ \cos x > 0 \end{cases}$ $\quad$ $\begin{cases} \tan x > 1 \\ \cos x < 0 \end{cases}$



INTERSEZIONI

$(0,2)$, $(\pi, 0)$

DERIVABILITÀ

COMPOSIZIONI ...

$$f'(x) = \frac{1+\cos x + \sin x}{(\cos x - \sin x)^2}$$

$\sin x + \cos x + 1 = 0$

APPUNTO SU PERIODICITÀ:

$g(x) = \sin 3x$

$g(x+t) = \sin(3(x+t))$

$= \sin(3x+3t) = \sin 3x$

$3t = 2\pi$

$T = \frac{2}{3}\pi$

DA FINIRE

6. $f(x) = \sqrt{x} + \frac{1}{\sqrt{x}} - 4$

DOMINIO: $x > 0$

SIMMETRIE: NON CI SONO ESSENDO DEFINITO SOLO IN $\mathbb{R}^+$

PERIODICITA': NO

$$\begin{cases} y = 0 \\ y = f(x) \end{cases} \Rightarrow (7 \pm 4\sqrt{3}, 0) \qquad \begin{cases} x = 0 \\ y = f(x) \end{cases} \Rightarrow (0, -4)$$

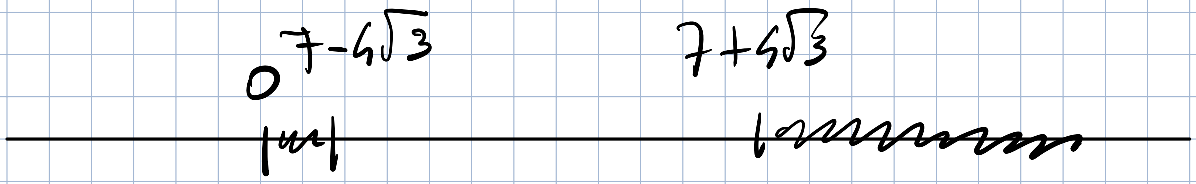
SEGNO:

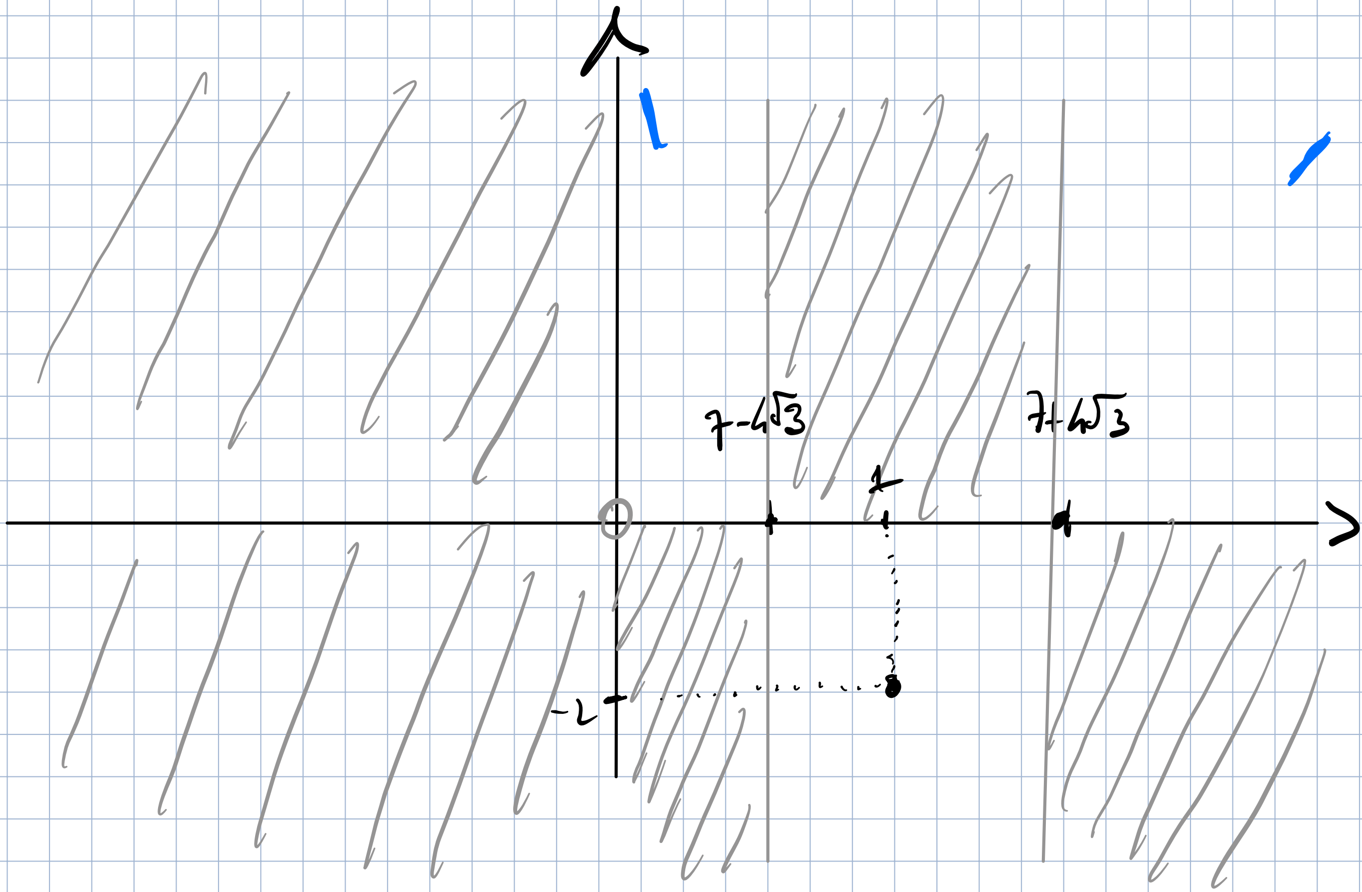
$$\sqrt{x} + \frac{1}{\sqrt{x}} - 4 \geq 0 \qquad \frac{x + 1 - 4\sqrt{x}}{\sqrt{x}} \geq 0$$

$$\frac{x+1}{\sqrt{x}} = 4 \qquad \frac{x^2 + 2x + 1}{x} = 16$$

$$x^2 - 14x + 1 = 0$$

$$\frac{14 \pm 8\sqrt{3}}{2} = \begin{cases} 7 + 4\sqrt{3} \\ 7 - 4\sqrt{3} \end{cases}$$





LIMITI

$$\lim_{x \to +\infty} \sqrt{x} + \frac{1}{\sqrt{x}} - 4 = +\infty$$

$$\lim_{x \to 0} \sqrt{x} + \frac{1}{\sqrt{x}} - 4 = +\infty$$

DERIVABILITA'

LA FUNZIONE È DERIVABILE TRANNE CHE IN 0.

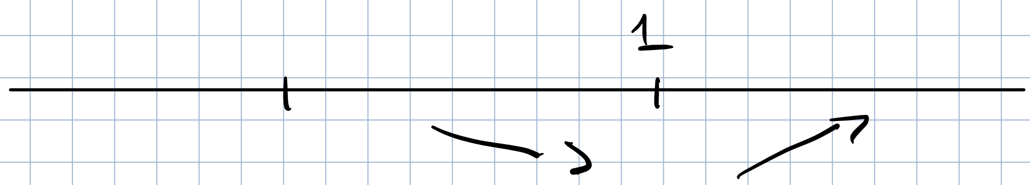
$$f'(x) = \left(\sqrt{x} + \frac{1}{\sqrt{x}} - 4\right)'$$

$$\frac{1}{2\sqrt{x}} + \frac{1}{2}\cdot\frac{1}{\sqrt{x^3}} = \frac{1}{2\sqrt{x}} - \frac{1}{2\sqrt{x^3}}$$

$$f'(x) = 0 \qquad \frac{1}{2\sqrt{x}} - \frac{1}{2\sqrt{x^3}} = 0$$

$$\frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x^3}}$$

$$\frac{1}{2\sqrt{x}} - \frac{1}{2\sqrt{x^3}} \geq 0 \qquad 0 < x < 1$$



$$x = 1 \qquad f(1) = 1 + 1 - 4 = 0 \qquad y = -2$$

CONCAVITA'

$$\left(\frac{1}{2\sqrt{x}} - \frac{1}{2\sqrt{x^3}}\right)' = \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}} =$$

$$= \frac{1}{2}\left(-\frac{1}{2\sqrt{x^3}} + \frac{3}{2\sqrt{x^5}}\right) = \frac{3}{4}\sqrt{x^5} - \frac{1}{4}\sqrt{x^3}$$

$$\frac{3}{4}\sqrt{x^5} - \frac{1}{4}\sqrt{x^3} \geq 0$$

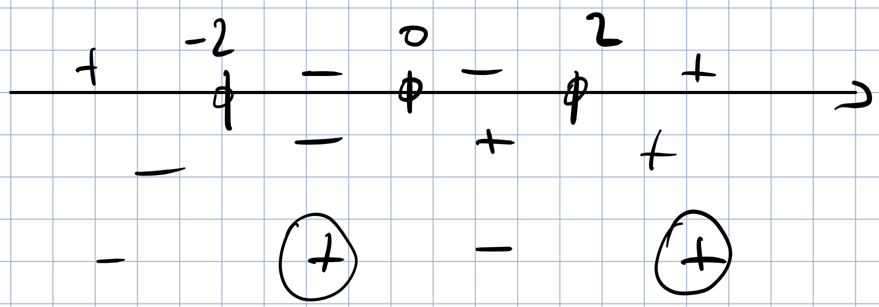
$$\frac{9}{16}x^5 - 2\left(\frac{3}{4}\sqrt{}\right.$$

$$f(x) = \log\left(\frac{x}{x^2-4}\right)$$

$$\begin{cases} \dfrac{x}{x^2-4} > 0 \\ x^2 - 4 \neq 0 \end{cases}$$

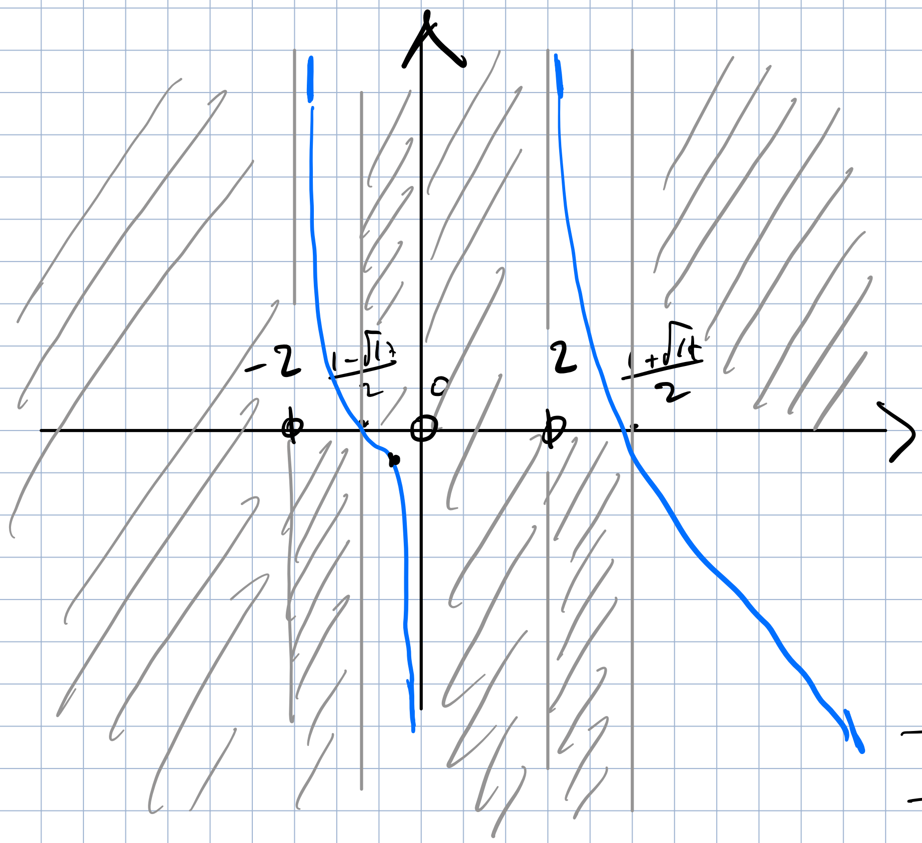
STUDIO SEGNO $\dfrac{x}{x^2-4}$

$$(-\infty, -2] \cup [2, +\infty)$$



INTERSEZIONI CON GLI ASSI:

$$\begin{cases} y = 0 \\ f(x) \end{cases} \qquad \frac{x}{x^2-4} = 1 \qquad \frac{1 \pm \sqrt{17}}{2}$$



SEGNO:

$$\frac{x}{x^2-4} \geq 1$$

$$x < -2, \quad x > 2$$

$$x > \frac{1-\sqrt{17}}{2}, \quad x < \frac{1+\sqrt{17}}{2}$$

	-2		$\frac{1-\sqrt{17}}{2}$		2		$\frac{1+\sqrt{17}}{2}$	
+		−		−		+		+
−		−		+		+		−
		+				+		

LIMITI

$$\lim_{x \to +\infty} \log \frac{x}{x^2-4} = -\infty$$

$$\lim_{x \to -2^+} \log \frac{x}{x^2-4} = +\infty$$

$$\lim_{x \to 0^-} \log \frac{x}{x^2-4} = -\infty$$

$$\lim_{x \to 2^+} \log \frac{x}{x^2-4} = +\infty$$

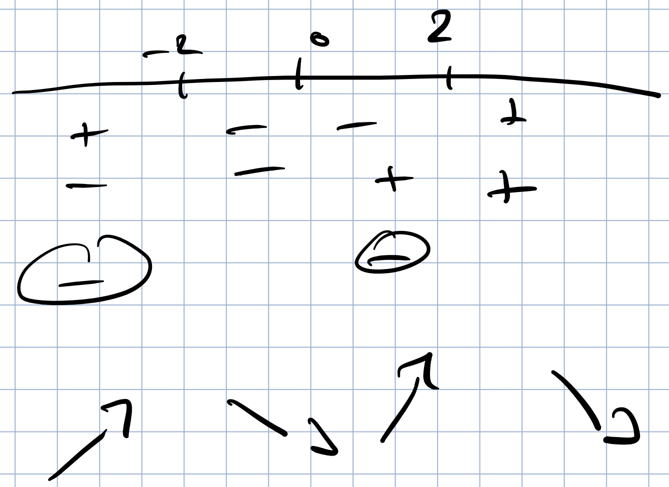
MONOTONIA

$$f'(x) = \frac{x^2-4}{x} \cdot \frac{x^2-4-x\cdot 2x}{(x^2-4)^2} =$$

$$= -\frac{x^2+4}{x(x^2-4)}$$

$$\frac{-(x^2+4)}{x(x^2-4)} \geq 0$$

$$x(x^2-4) < 0$$



$$f'(x) = 0 \text{ MAI} \Rightarrow \text{NO MAX/MIN}$$

CONCAVITÀ

$$f''(x) = \frac{2x \cdot x(x^2-4) - (x^2+4)(3x^2-4)}{x^2(x^2-4)^2}$$

$$= \frac{x^4+16x^2-16}{x^2(x^2-4)^2}$$

$$x^4 + 16x^2 - 16 = 0$$

$$x^2_{1,2} = -8 \pm \sqrt{64+16} = -8 \pm \sqrt{80} = -8 \pm 4\sqrt{5}$$

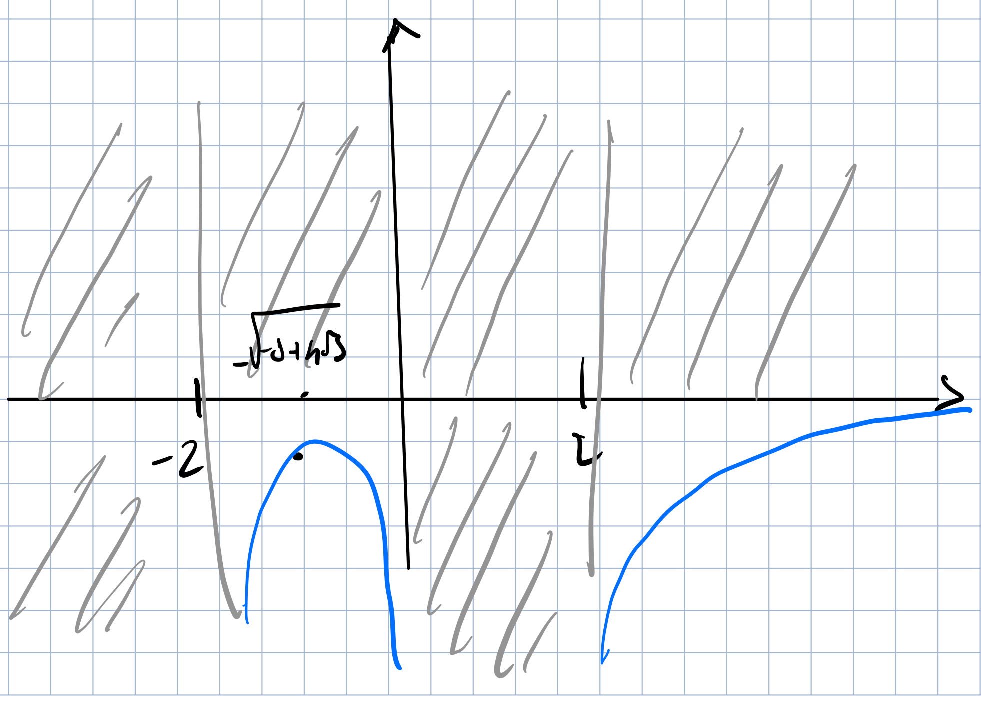
$$x^2 = -8 + 4\sqrt{5}$$

$$x_{1,2} = \pm\sqrt{-8+4\sqrt{5}}$$

$$f''(x) > 0 \iff x < -\sqrt{-8+4\sqrt{5}} \,,\; x > \sqrt{-8+4\sqrt{5}}$$

FLESSO

DATO $f(x) = \log_2 \frac{x}{x^2-4}$ DISEGNARE $f'(x)$:

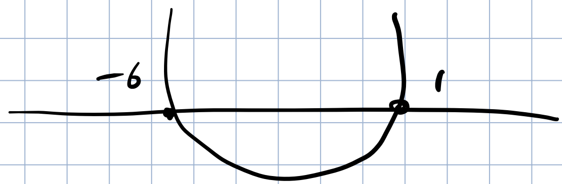


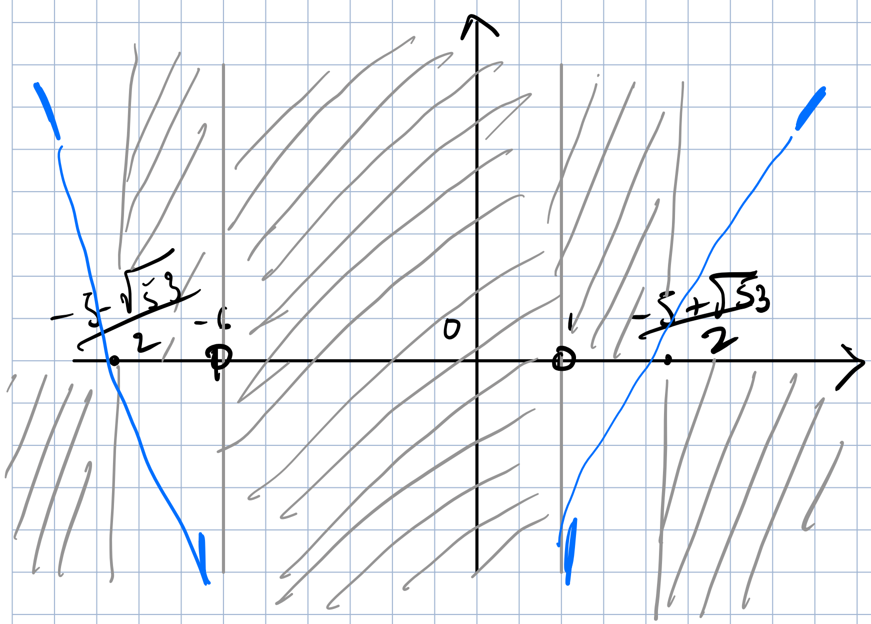
$f(x) = \log(x^2 + 5x - 6)$

DOMINIO: $x^2+5x-6>0$

$$\frac{-5\pm\sqrt{25+24}}{2} = \begin{cases} 1 \\ -6 \end{cases}$$

$x>1, \quad x<-6$





INTERSEZIONI:

$$\begin{cases} y=0 \\ y=f(x) \end{cases} \qquad x=\frac{-5\pm\sqrt{53}}{2}$$

SEGNO:

$x^2+5x-6>1$

$x^2+5x-7>0$

$$\frac{-5\pm\sqrt{25+28}}{2} = \frac{-5\pm\sqrt{53}}{2}$$

$$x<\frac{-5-\sqrt{53}}{2}, \quad x>\frac{-5+\sqrt{53}}{2}$$

LIMITI

$$\lim_{x \to +\infty} \log(x^2+5x-6) = +\infty$$

$$\lim_{x \to -\infty} \log(x^2+5x-6) = \log\left(x^2\left(1+\frac{5}{x}-\frac{6}{x^2}\right)\right) = +\infty$$

$$\lim_{x \to -\infty} \frac{\log(x^2+5x-6)}{x} = 0 \qquad \nexists \text{ ASINTOTO OB.}$$

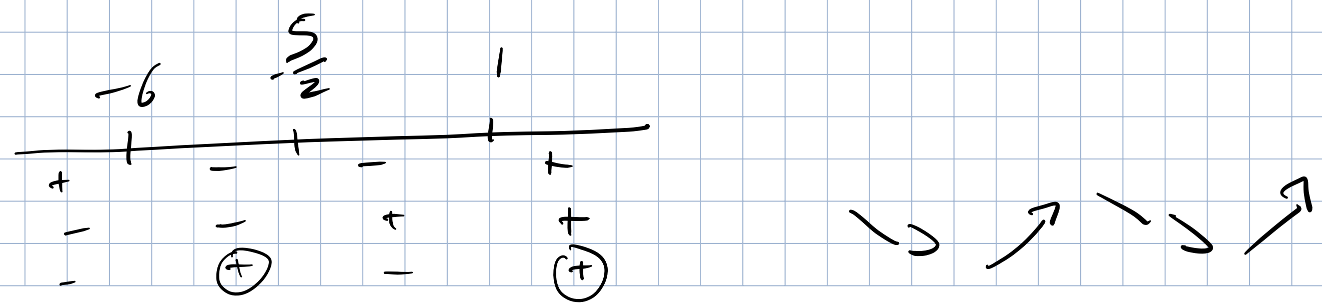
$$\lim_{x \to -6^-} \log(x^2+5x-6) = \log(36-30-6) = \log(0^+) = -\infty$$

$$\lim_{x \to 1^+} \log(x^2+5x-6) = \log(1+5-6) = \log(0^+) = -\infty$$

MONOTONIA

$$f'(x) = \frac{1}{x^2+5x-6}(2x+5)$$

$$\frac{2x+5}{x^2+5x-6} \geq 0 \qquad \begin{cases} x > -\frac{5}{2} \\ x < -6 \quad , x > 1 \end{cases}$$

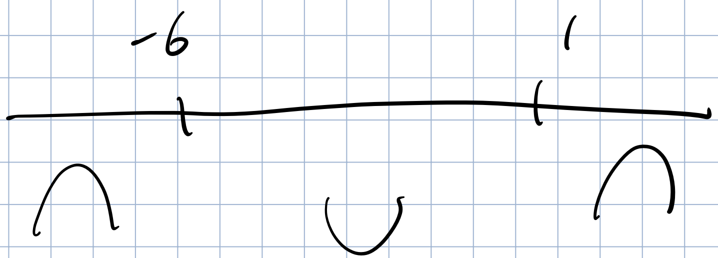


NON PRESENTA MAX POICHÉ $-\frac{5}{2} \notin D$

CONCAVITÀ

$$f''(x) = \frac{2\cdot\cancel{(x^2+5x-6)}\cdot(2x+5)\cdot(2x+5)}{(x^2+5x-6)^{\cancel{2}}}$$

$$= \frac{2(2x+5)^2}{x^2+5x-6} \geq 0 \qquad x = -\frac{5}{2}$$



$f(x) = \sqrt{x} - \arcsin x$

DOMINIO:

$0 \geq x \geq 1$

SIMMETRIE / PERODICITÀ:

NO SIMMETRIE / NO PERIODICITÀ

INTERSEZIONI

$$\begin{cases} y = 0 \\ y = f(x) \end{cases}$$

$\sqrt{x} - \arcsin x = 0$

$\sqrt{x} = \arcsin x$ $\qquad x = 0$ $\qquad (0,0)$

SEGNO

$\sqrt{x} - \operatorname{arcsin} x \geq 0$

$\sqrt{x} - \frac{1}{\sin x} \geq 0 \qquad \sin x \sqrt{x} - 1 \geq 0$

?

9. $f(x) = \frac{|x^2 - x - 2|}{x^2}$

DOMINIO $\mathbb{R} - \{0\}$

SIMMETRIA: $f(x) = f(-x)$
$f(-x) = -f(x)$

$\frac{|x^2 - x - 2|}{x^2} = \frac{|x^2 + x - 2|}{x^2}$ NO PARI

$\frac{|x^2 + x - 2|}{x^2} = -\frac{|x^2 - x - 2|}{x^2}$ NO DISPARI

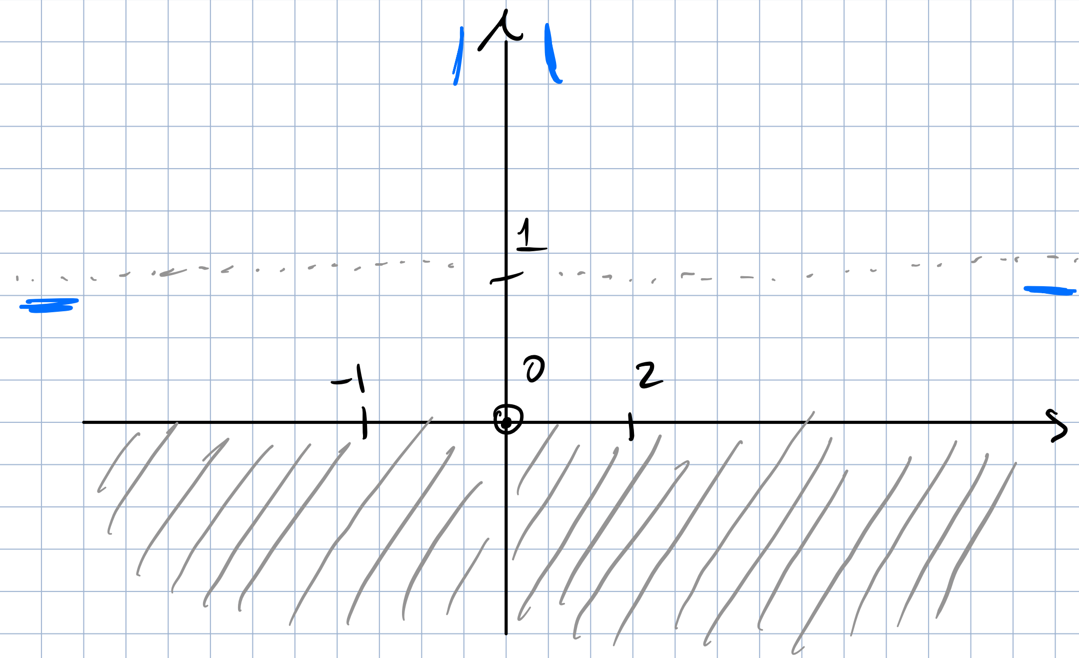
INTERSEZIONI

$\begin{cases} y = 0 \\ y = f(x) \end{cases}$ $x^2 - x - 2 = 0$

$\frac{1 \pm \sqrt{1+8}}{2} \Big\langle \frac{1 \pm 3}{2} \Big\langle \begin{matrix} 2 \\ -1 \end{matrix}$

SEGNO

$$\frac{|x^2-x-2| > 0}{x^2 \quad > 0}$$



LIMITI

$$\lim_{x\to+\infty}\frac{|x^2-x-2|}{x^2} = \begin{cases} \dfrac{-x^2+x+2}{x^2} = -1 \\ \dfrac{x^2-x-2}{x^2} = 1 \end{cases}$$

$$\lim_{x\to-\infty}\frac{|x^2-x-2|}{x^2} = \begin{cases} \dfrac{-x^2+x+2}{x^2} = -1 \\ \dfrac{x^2-x-2}{x} = 1 \end{cases}$$

$$\lim_{x\to0^+}\frac{|x^2-x-2|}{x^2} = +\infty$$

$$\lim_{x\to0^-}\frac{|x^2-x-2|}{x^2} = +\infty$$

MONOTONIA

$$f'(x) = \frac{\frac{|x^2-x-2|}{x^2-x-2}\cdot(2x-1)\cdot 3x^3 - 2x\,|x^2-x-2|}{x^4}$$

$\frac{|x|}{x}$

DA FINIRE

$f(x) = x\sqrt[3]{(\log|x|)^2}$

DOMINIO: $x \neq 0$

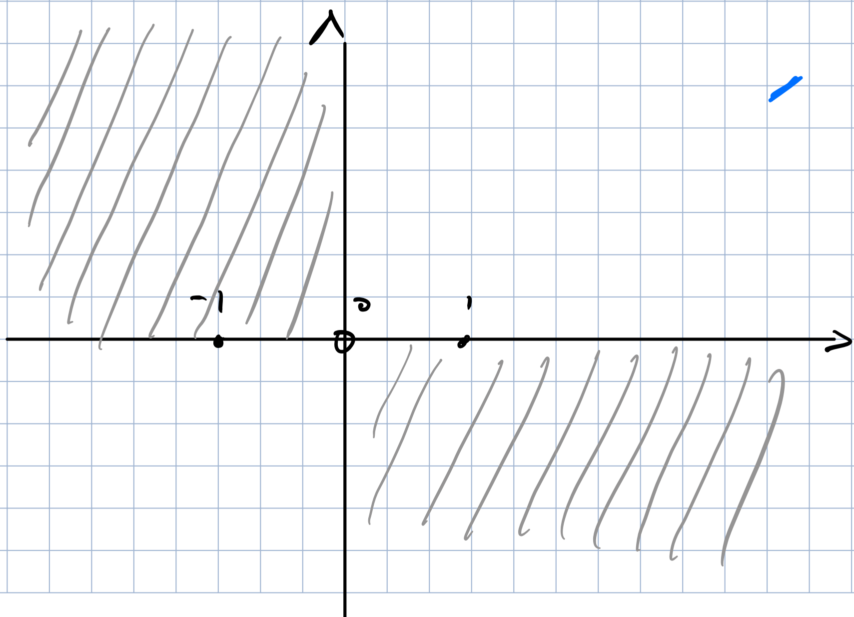
SIMMETRIE: NO

INTERSEZIONI:

$$\begin{cases} y = 0 \\ y = f(x) \end{cases} \qquad x \cdot \sqrt[3]{(\log|x|)^2} = 0 \qquad \begin{matrix} x = 0 \\ x = \pm 1 \end{matrix}$$

SEGNO:

$$x \sqrt[3]{(\log|x|)^2} \geq 0 \qquad x > 0$$



LIMITI

$$\lim_{x \to +\infty} x \sqrt[3]{(\log|x|)^2} = +\infty$$

NO ASINTOTI OR.

$$\lim_{x \to -\infty} x \sqrt[3]{(\log|x|)^2} =$$

$$\lim_{x \to +\infty} \frac{\cancel{x} \sqrt[3]{(\log|x|)^2}}{\cancel{x}} = +\infty$$

NO ASINTOTI OB.

$$\lim_{x \to 0^+} x \cdot \sqrt[3]{(\log|x|)^2} = 0^+$$

$$\lim_{x \to 0^-} x \cdot \sqrt[3]{(\log|x|)^2} = 0^-$$

DERIVABILITA'

$$\lim_{x \to 1} (\log|x|)^2 = 0$$

DERIVABILE

$$\lim_{x \to -1} (\log|x|)^2 = 0$$

MONOTONIA

$$f'(x) = 1 \cdot \left(\frac{1}{3} \cdot (\log|x|)^{-\frac{2}{3}} \cdot \frac{1}{|\cancel{x}|} \cdot \frac{|\cancel{x}|}{x} \cdot 2(\log|x|) \right) + x \cdot \sqrt[3]{[\log|x|]^2}$$

$$=$$

$$f(x) = \frac{x^2-4}{x+1}$$

D: $x+1 \neq 0 \qquad x \neq -1$

SIMMETRIE / PERIODICITA':

NO POICHÉ POLINOMI DI GRADO DIVERSO

INTERSEZIONI:

$y = 0$

$x^2 - 4 = 0$

$x = \pm 2$

$(-4,0) \quad (4,0)$

$x = 0$

$y = -4$

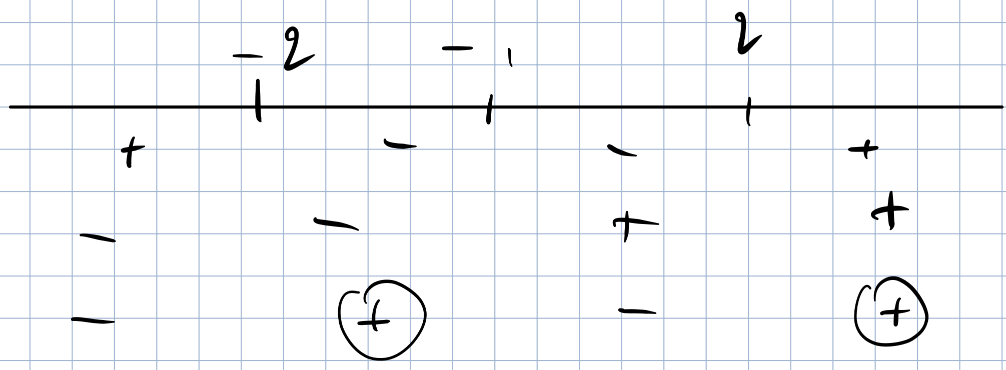
$(0;4)$

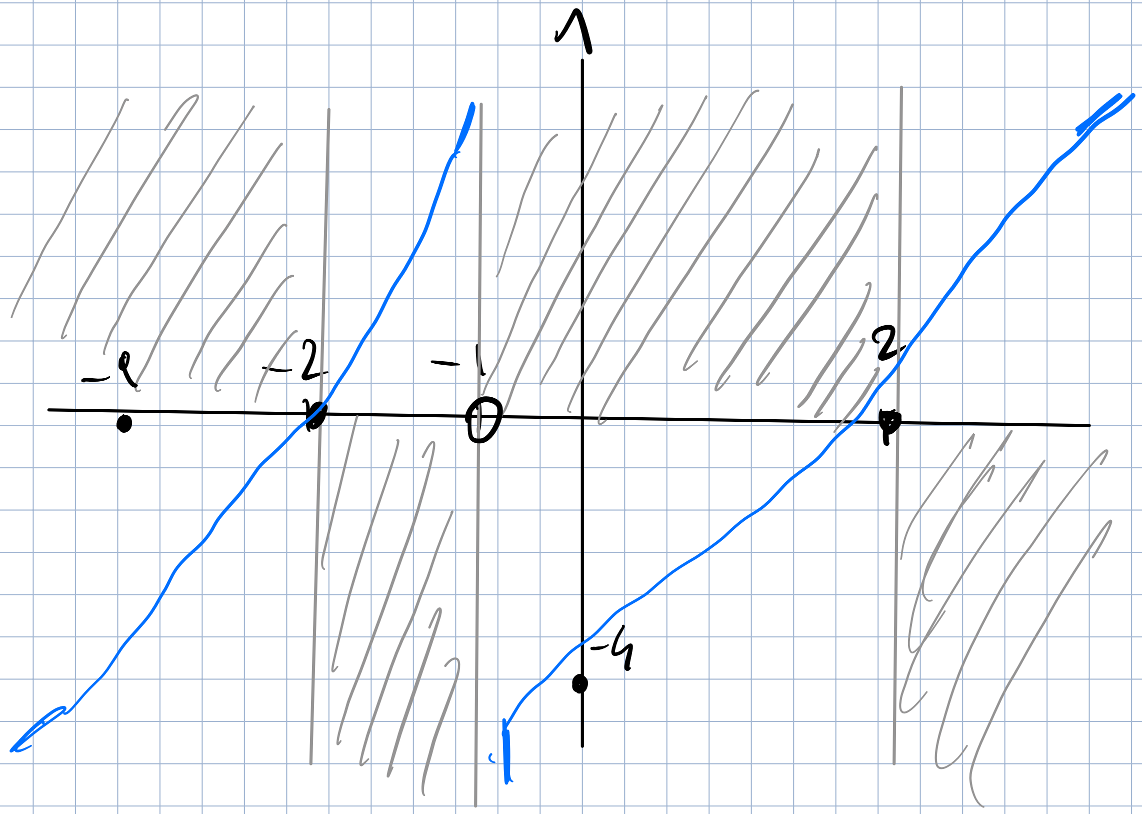
SEGNO:

$$\frac{x^2-4}{x+1} \geq 0$$

$x^2 - 4 \geq 0 \qquad -2 > x > 2$

$x + 1 \geq 0 \qquad x \geq 1$





LIMITI

$$\lim_{x \to -\infty} \frac{x^2-4}{x+1} = -\infty$$

$$\lim_{x \to +\infty} \frac{x^2-4}{x+1} = +\infty$$

$$\lim_{x \to -1^+} \frac{x^2-4}{x+1} = -\infty$$

$$\lim_{x \to -1^-} \frac{x^2-4}{x+1} = +\infty$$

MONOTONIA:

DERIVABILE IN TUTTO $\mathbb{R}$ POICHÉ POLINOMI

$$f'(x) = \frac{2x \cdot (x+1) - 1 \cdot (x^2-4)}{(x+1)^2}$$

$$= \frac{x^2+2x+4}{(x+1)^2}$$

NON CI SONO MASSIMI ; $\Delta < 0$

CONCAVITA'

$$f''(x) = \frac{(2x+2) \cdot (x+1) - (x^2+2x+4) \cdot 2}{(x+1)^4}$$

$$= \frac{-6}{(x+1)^3} \neq 0 \quad \forall x \in D$$

NO FLESSI.

14. $f(x) = \sqrt{x^2 - 4x} - x$

D $= \quad x^2 - 4x \geq 0$

$$x_{1,2} = \frac{4 \pm \sqrt{16-4}}{2} = \frac{4 \pm 2\sqrt{3}}{2} \begin{cases} 2+\sqrt{3} \\ 2-\sqrt{3} \end{cases}$$

SEGNO E INTERSEZIONI

$$\begin{cases} y=0 \\ y=f(x) \end{cases} \qquad x^2 - 4x - x = 0$$

$$x_{1,2} = \frac{4 \pm \sqrt{16+4}}{2} \begin{cases} 2+\sqrt{5} \\ 2-\sqrt{5} \end{cases}$$

$$\begin{cases} x=0 \\ y=0 \end{cases}$$

$(2+\sqrt{5}, 0)$

$(2-\sqrt{5}, 0)$

$(0, 0)$

$f(x) = (1+x)\log^2(1+x)$

D: $x > -1$

SEGNO E INTERSEZIONI

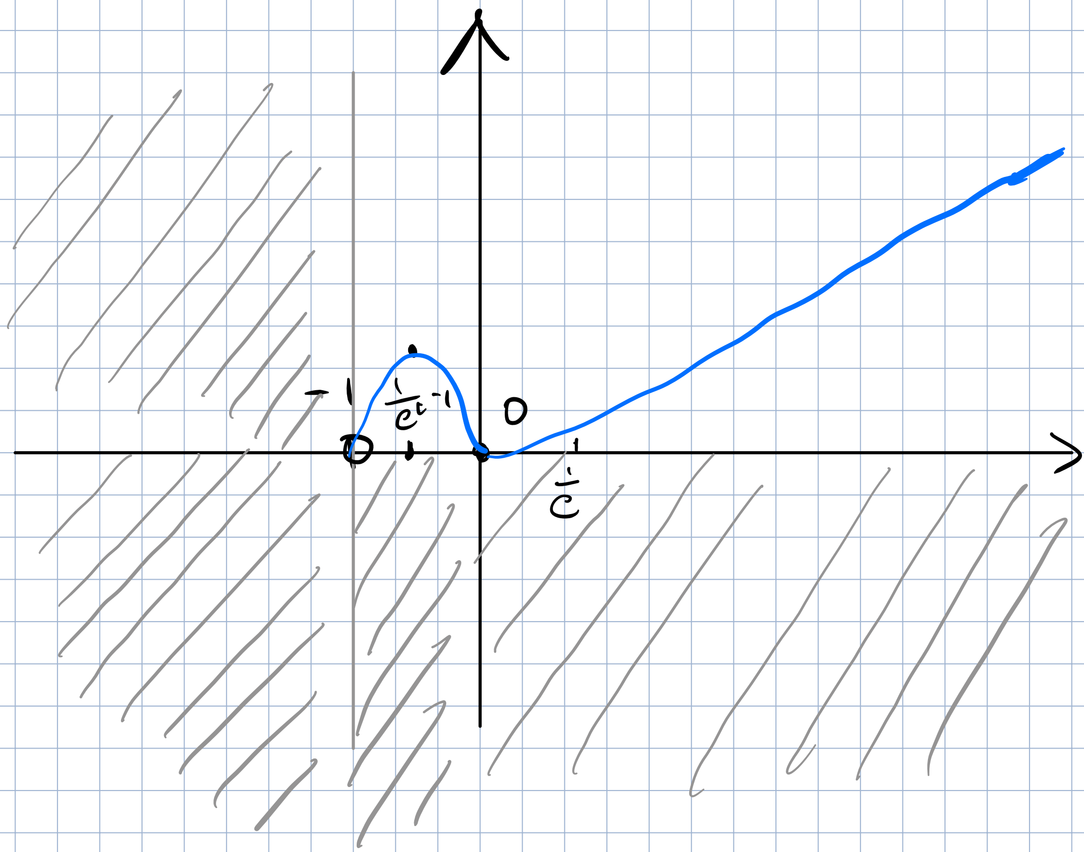
$$\begin{cases} y=0 \\ f(x)=0 \end{cases} \qquad (1+x)\log^2(1+x) = 0 \qquad (0, 0)$$

$x = 0$

$$(1+x)\log^2(1+x) \geq 0$$

$$1+x \geq 0$$

$x \geq -1 \Rightarrow$ SEMPRE POSITIVA IN D



MONOTONIA, MAX - MIN

$$\lim_{x \to 0^+} (1+x)\log^2(1+x) = 0^+$$

$$\lim_{x \to 0^-} (1+x)\log^2(1+x) = 0^+$$

$$f'(x) = 2(\log(1+x)) \cdot \frac{1}{\cancel{1+x}} \cdot \cancel{(1+x)} + \log^2(1+x)$$

$$= 2\log(1+x) + \log^2(1+x)$$

$$t = \log(1+x)$$

$$t^2 + 2t = 0 \qquad \frac{-2 \pm \sqrt{4}}{2} = \begin{cases} 0 \\ -2 \end{cases}$$

$t = 0$

$\log(1+x) = 0$

$x = 0$

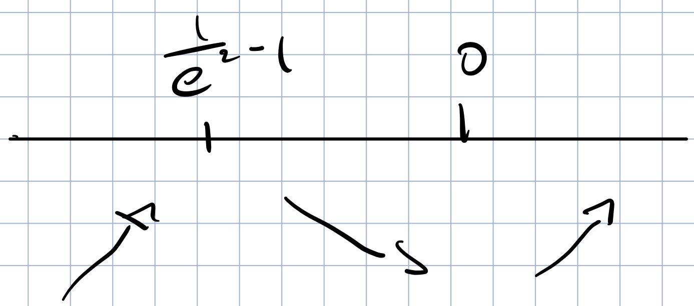
$t = -2$

$\log(1+x) = -2$

$1 + x = e^{-2}$

$x = \frac{1}{e^2} - 1$

$$t^2 + 2t \geq 0$$



LIMITI

$$\lim_{x \to +\infty} (1+x)\log^2(1+x) = +\infty$$

$$\lim_{x \to -1^+} (1+x)\log^2(1+x) \overset{H}{=} 0$$

CONCAVITÀ

$$f''(x) \quad \frac{2}{1+x} + \frac{2}{1+x} = \frac{2+2\log(1+x)}{1+x}$$

$$\frac{4}{1+x} > 0 \quad \forall x \in D$$

	-1		$\frac{1}{e}$	
$-$		$+$		$+$
$-$		$-$		$+$
$+$		$-$		$+$
$\cap$		$\cup$		$\cap$

$$f(x) = e^{\frac{1+x}{1+x^2}}$$

$D: \mathbb{R}$

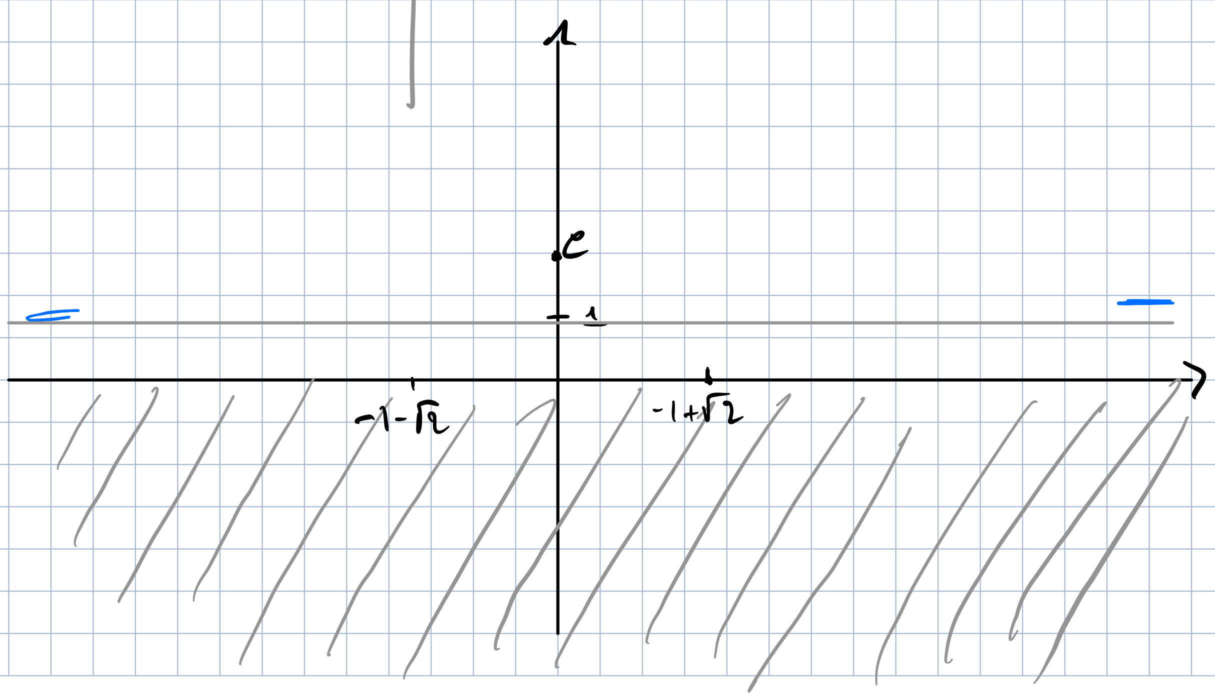
INTERSEZIONI

$$\begin{cases} y = 0 \\ f(x) = 0 \end{cases}$$

$$\begin{cases} x = 0 \\ y = e \end{cases}$$

SEGNO

$$e^{\frac{1+x}{1+x^2}} \geq 0 \quad \text{SEMPRE}$$



LIMITI

$$\lim_{x\to+\infty} e^{\frac{1+x}{1+x^2}} = 1$$

$$\lim_{x\to-\infty} e^{\frac{1+x}{1+x^2}} = 1$$

MONOTONIA

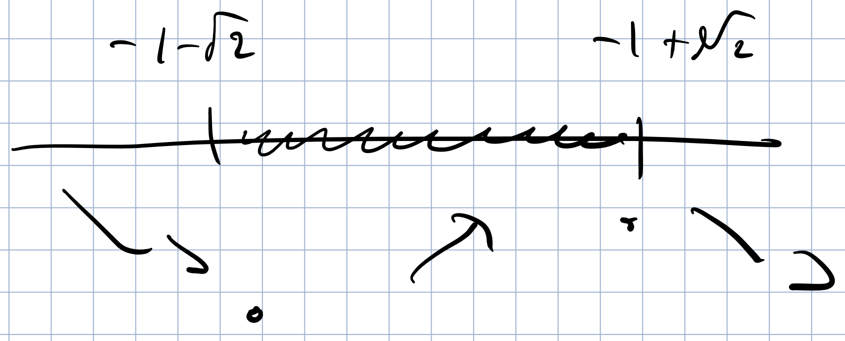
$$f'(x) = e^{\frac{1+x}{1+x^2}} \cdot \left(\frac{(1+x^2)-(1+x)\cdot 2x}{(1+x^2)^2}\right)$$

$$= e^{\frac{1+x}{1+x^2}} \cdot \left(\frac{1+x^2-2x-2x^2}{(1+x^2)^2}\right) = e^{\frac{1+x}{1+x^2}} \cdot \left(\frac{-x^2-2x+1}{(1+x^2)^2}\right)$$

$$f'(x) = 0$$

$$-x^2-2x+1 = 0$$
$$x^2+2x-1 = 0$$

$$\frac{-2\pm\sqrt{8}}{2} = \begin{cases} -1+\sqrt{2} \\ -1-\sqrt{2} \end{cases}$$



MONOTONIA

$$|x-2|\, e^{-(x-2)^2}$$

$D: \mathbb{R}$

SEGNO E INT

$y=0$

$x=2$

$$2 \cdot e^{-4} = \frac{2}{e^4}$$

$x=0$

$\left(0, \frac{2}{e^4}\right)$

LIMITI

$$\lim_{x \to +\infty} |x-2|\, e^{-(x-2)^2} = \frac{|x-2|}{e^{(x-2)^2}} = 0$$

$$\lim_{x \to -\infty} \frac{|x-2|}{e^{(x-2)^2}} = 0$$

DERIV. MONOT.

$$f(x) = \begin{cases} (x-2)\cdot e^{-(x-2)^2} & x \geq 2 \\ -(x-2)\cdot e^{-(x-2)^2} & x < 2 \end{cases}$$

PER $x=2$ É DERIV. POICHÉ PRODOTTO DI f DERIV.:

$$f' = \frac{e^{-(x-2)^2} - (x-2)\cdot(-2(x-2))\cdot e^{-(x-2)^2}}{\left(e^{-(x-2)^2}\right)^2}$$

$$= \frac{\cancel{e^{-(x-2)^2}}\left(1-(x-2)\cdot(-2x+2)\right)}{\left(e^{-(x-2)^2}\right)^{\cancel{2}}}$$

$$= e^{-(x-2)^2}\cdot\left[1-2(x-2)^2\right]$$

$$\lim_{x\to 2^+} e^{-(x-2)^2}\cdot\left[1-2(x-2)^2\right] = 1$$

$$\lim_{x\to 2^-} e^{-(x-2)^2}\cdot\left[1-2(x-2)^2\right] = -1$$

$$f'(x) = 0$$

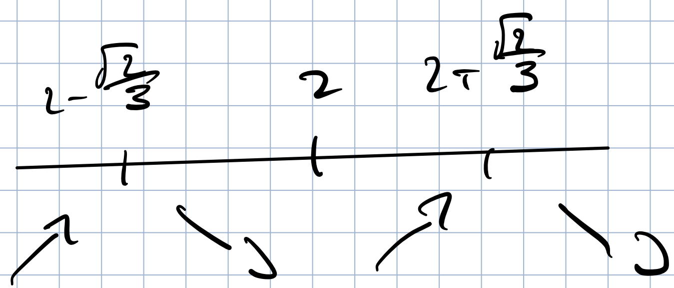
$$(x-2)^2 = \frac{1}{2}$$

$$x - 2 = \pm \frac{\sqrt{2}}{2}$$

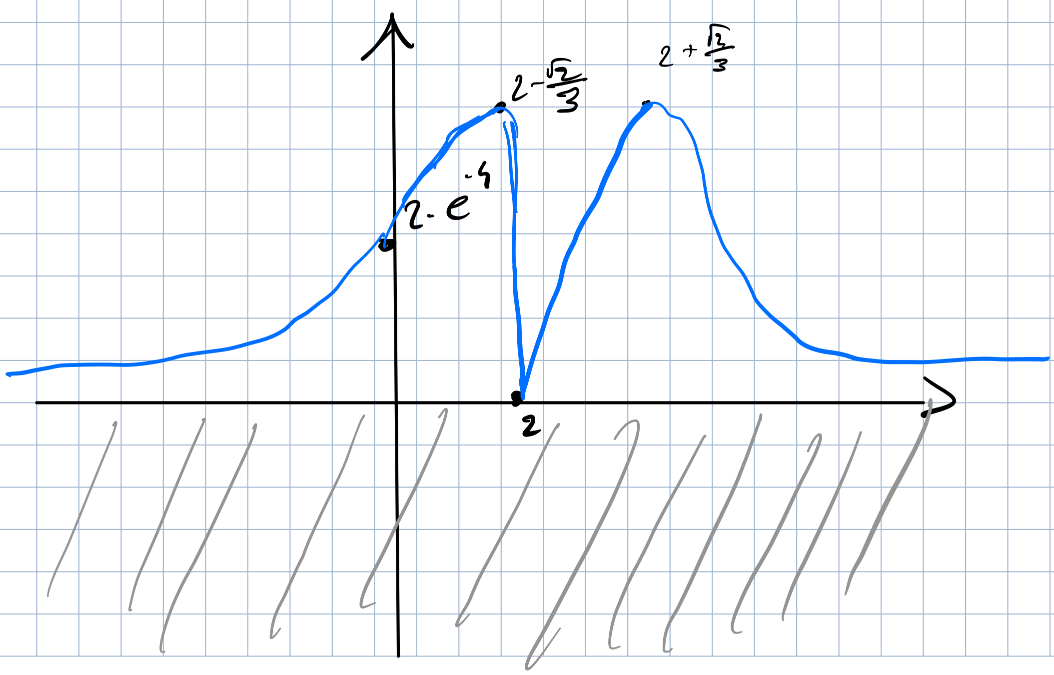
$$x = 2 \pm \frac{\sqrt{2}}{2}$$

$$f'(x) \geq 0$$

$$\begin{cases} 1-2(x-2)^2 \geq 0 & x > 2 \\ 1-2(x-2)^2 \leq 0 & x < 2 \end{cases}$$



$$2 - \frac{\sqrt{2}}{3} < x < 2 + \frac{\sqrt{2}}{3}$$



17. $f(x) = \dfrac{xe^x}{e^x - 1}$
18. $f(x) = \sin x + \dfrac{1}{4\sin x}$
19. $f(x) = x^2 \sqrt[3]{x-1}$
20. $f(x) = \arctan \dfrac{-1}{\sqrt{x^2+4}}$
21. $f(x) = (1+x)\log^2(1+x)$
22. $f(x) = e^{-|x^2-1|}$
23. $f(x) = 2\sin^3 - 3\sin x$
24. $f(x) = x^2(\log x - 1)^2$
25. $f(x) = \dfrac{\sin x}{\cos x + \sqrt{2}}$
26. $f(x) = \sqrt{|2x-5| - 3}$
27. $f(x) = \dfrac{6x^2 + 2x + 3}{2(2x^2+1)}$
28. $f(x) = e^{\frac{1+x}{1+x^2}}$
29. $f(x) = \dfrac{\sqrt{1+x^2}}{2+x^2}$
30. $f(x) = \dfrac{x\log x}{1+\log x}$
31. $f(x) = \arcsin x + \arccos x + |2x-1|$
32. $f(x) = \sqrt[3]{1-\cos x}$
33. $f(x) = \log(e^x - e^{-x})$
34. $f(x) = \arctan \frac{1}{1-x}$
35. $f(x) = e^{-|x|}\sqrt{x^2-3x+2}$
36. $f(x) = \dfrac{1+\sin x}{1-|\cos x|}$
37. $f(x) = \arcsin(2x\sqrt{1-x^2})$
38. $f(x) = \dfrac{\sqrt{x^4-5x^2+6}}{x}$
39. $f(x) = |x|\sqrt{9-x^2}$
40. $f(x) = \sqrt{x+1} - \arcsin x$

Esercizio 2.
Al variare del parametro $\alpha \in \mathbb{R}$ disegnare i grafici delle seguenti funzioni:

1. $f(x) = x^3 - \alpha x$
2. $f(x) = e^{\alpha x^2}$
3. $f(x) = x^\alpha \log x$

Esercizio 3.
Studiare la seguente funzione e disegnarne il grafico:

$$f(x) = \begin{cases} \arctan\left(-\dfrac{2x}{\sqrt{4x^2+2}}\right) & \text{se } x \geq 0 \\ x\sqrt{|x-5|-3} & \text{se } x < 0\,. \end{cases}$$

Esercizio 4.
Dopo aver disegnato il grafico delle seguenti funzioni, dedurre da questo le proprietà del grafico di f' e disegnarlo:

1. $f(x) = \frac{x^2}{x+1} + \log(x+1)$
2. $f(x) = (x-1)\, e^{-\sqrt[3]{x-1}}$
3. $f(x) = \sqrt[3]{\dfrac{(x-3)^2}{x-2}}$
4. $f(x) = \dfrac{e^{x-3}-1}{e^{3(x-3)}}$
5. $f(x) = \log_e^2(x-3) - \log_e(x-3)$
6. $f(x) = \dfrac{\cos^2 x}{1+2\sin x}$
7. $f(x) = \dfrac{x^2-4}{x+1}$
8. $f(x) = \sqrt{\dfrac{x-1}{x+1}}$
9. $f(x) = \dfrac{x+2}{\sqrt{x^2-x}}$
10. $f(x) = x^2 e^{-2x}$
11. $f(x) = \dfrac{e^x-2}{x}$
12. $f(x) = \log|x^2-5x+4|$
13. $f(x) = \dfrac{1-2\log x}{x^2}$
14. $f(x) = \dfrac{x+1}{x+5} e^x$

15. $f(x) = \dfrac{\sin x}{\cos^2 x - 2}$
16. $f(x) = \log \dfrac{x^2}{x-1}$
17. $f(x) = \sqrt{x} \log x$
18. $f(x) = \sqrt[3]{8 - x^3}$
19. $f(x) = \sin x \cos x + \cos^2 x$
20. $f(x) = x(x-1)^2$

Formulario: tavola degli integrali indefiniti

Indice

Definizione

$$\int f(x)dx = F(x) + c \Leftrightarrow F'(x) = f(x)$$

Proprietà dell'integrale indefinito

$$\int k \cdot f(x)dx = k \cdot \int f(x)dx$$

$$\int [f_1(x) + f_2(x) + ... + f_n(x)]dx = \int f_1(x)dx + \int f_2(x)dx + ... + \int f_n(x)dx$$

Integrali indefiniti fondamentali

$$\int f'(x)dx = f(x) + c$$

$$\int a\, dx = ax + c$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c \text{, con } n \neq -1$$

$$\int \frac{1}{x} dx = \log|x| + c$$

$$\int \sin x dx = -\cos x + c$$

$$\int \cos x dx = \sin x + c$$

$$\int (1 + \tan^2 x)dx = \int \frac{1}{\cos^2 x} dx = \tan x + c$$

$$\int (1 + \cot^2 x)dx = \int \frac{1}{\sin^2 x} dx = -\cot x + c$$

$$\int \sinh\ x dx = \cosh x + c$$

$$\int \cosh x dx = \sinh\ x + c$$

$$\int e^x dx = e^x + c$$

$$\int e^{kx} dx = \frac{e^{kx}}{k} + c$$

$$\int a^x dx = \frac{a^x}{\log_e a} + c$$

Integrali notevoli

$$\int \frac{1}{\sin x} dx = \log\left|\frac{\tan x}{2}\right| + c$$

$$\int \frac{1}{\cos x} dx = \log\left|\frac{\tan x}{2} + \frac{\pi}{4}\right| + c$$

$$\int \frac{1}{\sqrt{1-x^2}} dx = \begin{cases} arc\sin x + c \\ -arc\cos x + c \end{cases}$$

$$\int \frac{-1}{\sqrt{1-x^2}} dx = \begin{cases} arc\cos x + c \\ -arc\sin x + c \end{cases}$$

$$\int \frac{1}{1+x^2} dx = arc\tan x + c$$

$$\int \frac{1}{1-x^2} dx = \frac{1}{2}\log\left|\frac{1+x}{1-x}\right| + c$$

$$\int \frac{1}{\sqrt{x^2-1}} dx = \log\left|x + \sqrt{x^2-1}\right| + c$$

$$\int \frac{1}{\sqrt{1+x^2}} dx = \begin{cases} arc\sinh\ x + c \\ \log\left(x + \sqrt{1+x^2}\right) + c \end{cases}$$

$$\int \frac{1}{\sqrt{x^2 \pm a^2}} dx = \log\left|x + \sqrt{x^2 \pm a^2}\right| + c$$

$$\int \sqrt{(x^2 \pm a^2)}dx = \frac{x}{2}\sqrt{x^2 \pm a^2} \pm \frac{a^2}{2}\log\left(x + \sqrt{x^2 \pm a^2}\right) + c$$

$$\int \sqrt{(a^2 - x^2)}dx = \frac{1}{2}\left(a^2 arc\sin\frac{x}{a} + x\sqrt{a^2 - x^2}\right) + c$$

$$\int \sin^2 x dx = \frac{1}{2}(x - \sin\ x\cos x) + c$$

$$\int \cos^2 x dx = \frac{1}{2}(x + \sin\ x\cos x) + c$$

$$\int \frac{1}{\cosh^2 x} dx = \int (1 - \tanh^2 x)dx + c = \tanh x + c$$

CORSO DI LAUREA IN INFORMATICA APPLICATA

ANALISI MATEMATICA 1

ESERCIZI SU

INTEGRALI DI FUNZIONI DI UNA VARIABILE

Esercizio 1.
Determinare una primitiva delle seguenti funzioni:

1. $f(x) = x^2$;
2. $f(x) = e^{3x}$;
3. $f(x) = \sin(2x)$;
4. $f(x) = x + 1$;
5. $f(x) = \dfrac{3}{x}$.

Esercizio 2.
Calcolare i seguenti integrali riconducibili a integrali di funzioni elementari:

1. $\displaystyle\int_1^2 x^2\ dx$;
2. $\displaystyle\int_{-1}^2 x^3\ dx$;
3. $\displaystyle\int_2^3 \frac{1}{x}\ dx$;
4. $\displaystyle\int_1^2 e^{3x}\ dx$;
5. $\displaystyle\int_\pi^{2\pi} \cos x\ dx$;
6. $\displaystyle\int_0^\pi \sin 2x\ dx$;
7. $\displaystyle\int (2x + \cos x)\, dx$;
8. $\displaystyle\int \left(2\sqrt{x} + 3\sqrt[3]{x}\right) dx$;
9. $\displaystyle\int \frac{6\,(x-1)}{\sqrt[3]{x^4}}\ dx$;

ESERCIZIO 1

1. $f(x) = x^2$;

$$\int x^2 dx = \frac{x^3}{3} + c$$

2. $f(x) = e^{3x}$;

$$\int e^{3x} dx = e^{3x} + c$$

3. $f(x) = \sin(2x)$;

$$2x = t$$

$$x = \frac{t}{2}$$

$$dx = \frac{1}{2} dt$$

$$\int \sin(t) \frac{1}{2} dt$$

$$= \frac{1}{2} \int \sin(t) dt =$$

$$= -\frac{1}{2} \cos t = -\frac{1}{2} \cos(2x) + c$$

4. $f(x) = x + 1$;

$$\int x+1 \, dx = \int x dx + \int 1 dx = \frac{x^2}{2} + x + c$$

5. $f(x) = \frac{3}{x}$.

$$\int \frac{3}{x} dx = 3 \int \frac{1}{x} dx = 3 \log|x| + c$$

ESERCIZIO 2

1. $\int_1^2 x^2\, dx;$

$$\int_1^2 x^2\, dx = \frac{2^3}{3} - \frac{1^3}{3} = \frac{7}{3}$$

2. $\int_{-1}^2 x^3\, dx;$

$$\int_{-1}^2 x^3\, dx = \frac{x^4}{4}\Big|_{1}^{b} = \frac{2^4}{4} - \frac{1^4}{4} = 16-1 = \frac{15}{4}$$

3. $\int_2^3 \frac{1}{x}\, dx;$

$$\int_2^3 \frac{1}{x}\, dx = \log|x|\Big|_2^3 = \log|3| - \log|2| = \log\frac{3}{2} + c$$

4. $\int_1^2 e^{3x}\, dx;$

$$\int_1^2 e^{3x}\, dx = e^{3x}\Big|_1^2 = e^6 - e^3 = e^3(e^3-1)$$

5. $\int_\pi^{2\pi} \cos x\, dx;$

$$\int_\pi^{2\pi} \cos x\, dx = \sin x\Big|_\pi^{2\pi} = 0$$

6. $\int_0^{\pi} \sin 2x \ dx;$

$$\int_0^{\pi} \sin 2x \, dx = -\frac{1}{2}\cos 2x \Big|_0^{\pi} = -\frac{1}{2}\cos 2\pi + \frac{1}{2}\cos(0) = -\frac{1}{2} + \frac{1}{2} = 0$$

7. $\int (2x + \cos x)\, dx;$

$$\int (2x + \cos x)\, dx = \int 2x\, dx + \int \cos x \, dx = 2\cdot\frac{x^2}{2} + \sin x$$

$$= \sin x + x + C$$

8. $\int (2\sqrt{x} + 3\sqrt[3]{x})\, dx;$

$$\int 2\sqrt{x} + 3\sqrt[3]{x} \, dx = \int 2\sqrt{x}\,dx + \int 3\sqrt[3]{x}\, dx = 2\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + 3\cdot\frac{x^{\frac{1}{3}+1}}{\frac{1}{3}+1} =$$

$$= 2\cdot\frac{2\sqrt{x^3}}{3} + 3\frac{3\sqrt[3]{x^4}}{4} = \frac{4\sqrt{x^3}}{3} + \frac{9\sqrt[3]{x^4}}{4} + C$$

9. $\int \frac{6\,(x-1)}{\sqrt[3]{x^4}}\, dx;$

$$\int \frac{6(x-1)}{\sqrt[3]{x^4}}\, dx = 6\left(\int \frac{x}{x^{\frac{4}{3}}}\, dx - \int \frac{1}{\sqrt[3]{x^4}}\, dx\right) =$$

$$6\left(\int \frac{1}{\sqrt[3]{x}}\, dx - \int \frac{1}{\sqrt[3]{x^4}}\right) = 6\left(\frac{x^{-\frac{1}{3}+1}}{-\frac{1}{3}+1} - \frac{x^{-\frac{4}{3}+1}}{-\frac{4}{3}+1}\right) =$$

$$= 6\left(\frac{\sqrt[3]{x^2}}{\frac{2}{3}} - \frac{\frac{1}{\sqrt[3]{x}}}{-\frac{1}{3}}\right) = 6\left(\frac{3\sqrt[3]{x^2}}{2} + \frac{3}{\sqrt[3]{x}}\right) + C$$

10. $\displaystyle\int \frac{1+\cos^3 x}{\cos^2 x}\ dx;$

11. $\displaystyle\int \frac{e^{2x}-2e^x}{e^x}\ dx;$

12. $\displaystyle\int \frac{x^2}{1+x^2}\ dx;$

13. $\displaystyle\int_{-2}^{2} |x|\, dx;$

14. $\displaystyle\int_{0}^{2} |(x-1)(x+3)|\, dx;$

15. $\displaystyle\int_{-1}^{1} e^{-|x|}\ dx.$

Esercizio 3.
Calcolare i seguenti integrali per decomposizione in somma di funzioni razionali elementari:

1. $\displaystyle\int \frac{x^3}{x^2+1}\ dx;$
2. $\displaystyle\int \frac{2x-1}{x^2+2x+4}\ dx$ (completamento del quadrato);
3. $\displaystyle\int \frac{x+2}{x^2+4x+3}\ dx$ (completamento del quadrato);
4. $\displaystyle\int \frac{x+5}{x^2(x+1)}\ dx;$
5. $\displaystyle\int \frac{1-2x}{(x+1)^2(x^2+1)}\ dx;$
6. $\displaystyle\int_0^1 \frac{1}{x^3+1}\ dx.$

Esercizio 4.
Calcolare i seguenti integrali, utilizzando il metodo di integrazione per sostituzione:

1. $\displaystyle\int \tan x\ dx;$
2. $\displaystyle\int \frac{1}{e^x+e^{-x}}\ dx;$
3. $\displaystyle\int \frac{1}{x^2+4x+5}\ dx;$
4. $\displaystyle\int \frac{1}{\sqrt{e^{2x}-1}}\ dx;$
5. $\displaystyle\int \frac{\sqrt{x}}{2+\sqrt{x}}\ dx;$
6. $\displaystyle\int \frac{1}{x\sqrt{x+4}}\ dx;$

10. $\displaystyle\int \frac{1+\cos^3 x}{\cos^2 x}\,dx;$

$$\int \frac{1+\cos^3 x}{\cos^2 x}\,dx = \int \frac{1}{\cos^2 x} + \cos x \,dx =$$

$$\int \frac{1}{\cos^2 x}\,dx + \int \cos x\,dx = \log\left|\frac{\tan x}{2} + \frac{\pi}{4}\right| + \sin x + c$$

11. $\displaystyle\int \frac{e^{2x}-2e^x}{e^x}\,dx;$

$$\int \frac{e^{2x}-2e^x}{e^x}\,dx = \int e^x\,dx - 2\int 1\,dx = e^x - 2x + c$$

12. $\displaystyle\int \frac{x^2}{1+x^2}\,dx;$

$t = x^2+1 \qquad x = \sqrt{t-1} \qquad dt = 2x\,dx$

$dx = \frac{1}{2x}\,dt$

$$\int \frac{x^2}{1+x^2}\,dx = \int \frac{x^2+1}{x^2+1} - \frac{1}{x^2+1}\,dx = \int 1 - \frac{1}{x^2+1}\,dx$$

$$= x - \arctan x + c$$

13. $\displaystyle\int_{-2}^{2} |x|\,dx;$

$$\int_{-2}^{2} |x|\,dx = \int_{-2}^{0} -x\,dx + \int_{0}^{2} x\,dx = 2+2 = 4$$

14. $\displaystyle\int_{0}^{2} |(x-1)(x+3)|\,dx;$

$$\int_0^2 |(x-1)(x+3)|\,dx = \qquad -3 > x > 1 \text{ POSITIVA}$$

$$\int_1^2 (x-1)(x+3)\,dx + \int_0^1 -x^2-2x+3\,dx$$

$$\int_0^1 x^2+2x-3\,dx + \int_1^2 -x^2-2x+3\,dx =$$

$$= \frac{x^3}{3}+2\frac{x^2}{2}-3x\,\Big|_1^2 + \left(-\frac{x^3}{3}-2\frac{x^2}{2}+3x\right)\Big|_0^1 =$$

$$= \left(\frac{8}{3}+4-6-\left(\frac{1}{3}+1-3\right)\right)+\left(-\frac{1}{3}-1+3\right)$$

$$= \frac{7}{3}+\frac{5}{3} = \frac{12}{3}$$

15. $\int_{-1}^{1} e^{-|x|}\,dx.$

$$\int_{-1}^{1} e^{-|x|}\,dx = \int_{-1}^{0} e^{x}\,dx + \int_0^1 e^{-x}\,dx$$

$$= e^x\Big|_{-1}^{0} - e^{-x}\Big|_0^1 = 1-\frac{1}{e^x}+\left(-\frac{1}{e^x}+1\right) = -\frac{2}{e^x}+2$$

1. $\int \frac{x^3}{x^2+1}\,dx;$

$$x^2+1 = t$$

$$x = \sqrt{t-1}$$

$$dt = 2x\,dx \qquad dx = \frac{1}{2x}\,dt$$

$$\int \frac{x^3}{x^2+1}\,dx =$$

$$\int \frac{x^3}{t}\cdot\frac{1}{2\cancel{x}}\,dx = \frac{1}{2}\int\frac{x^2}{t}\,dt$$

$$= \frac{1}{2}\int\frac{t-1}{t}\,dt = \frac{1}{2}\left(\int 1\,dt - \int\frac{1}{t}\,dt\right) = \frac{t-\log|t|}{2}$$

$$= \frac{x^2+1-\log|x^2+1|}{2}+c$$

$$\int \frac{2x-1}{x^2+2x+4}\,dx$$

$$\int \frac{2x-1+2-2}{x^2+2x+4}\,dx$$

$$\int \frac{2x+2}{x^2+2x+4}\,dx - 3\int \frac{1}{x^2+2x+4}\,dx$$

3. $\int \frac{x+2}{x^2+4x+3}\,dx$

$$\int \frac{x+2}{x^2+4x+3} = \int \frac{\cancel{x+2}}{x^2+3x+1+\cancel{x+2}}\,dx =$$

$$\int \frac{\cancel{x}+2}{t} \cdot \frac{1}{2\cancel{x+4}}\,dt \qquad t = x^2+4x+3$$

$$\int \frac{1}{2t}\,dt = \frac{1}{2}\int \frac{1}{t}\,dt = \qquad \frac{dt}{dx} = 2x+4$$

$$= \frac{1}{2}\log\left|x^2+4x+3\right| \qquad dx = \frac{1}{2x+4}\,dt$$

4. $\int \frac{x+5}{x^2(x+1)}\,dx;$

$$\int \frac{\cancel{x}}{\cancel{x}^2(x+1)}\,dx + \int \frac{5}{x^2(x+1)}\,dx =$$

$$\frac{5}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$

$$\frac{Ax(x+1) + B(x+1) + Cx^2}{x^2(x+1)}$$

$$Ax^2 + Ax + Bx + B + Cx^2 = 5$$

$$x^2(A+C) + x(A+B) + B = 5$$

$$\begin{cases} A + C = 0 \\ A + B = 0 \\ B = 5 \end{cases} \qquad \begin{aligned} C &= 5 \\ A &= -5 \end{aligned}$$

$$\int \frac{-5}{x} + \frac{5}{x^2} \qquad \frac{5}{x+1} \, dx$$

$$\int \frac{5}{x^2(x+1)} \qquad -5 \log|x| + \frac{5}{x} - 5\log|x+1| + C$$

$$\int \frac{1}{x(x+1)} = \int \frac{-1}{x} + \frac{1}{x^2} + \frac{1}{x+1} \, dx$$

$$B = 1 \qquad = -\log|x| + \frac{1}{x} + \log|x+1| + C$$

$$A = -1$$

$$C = 1$$

$$= -6\log|x| + 6\frac{1}{x} - 4\log|x+1| + c$$

5 $\int \frac{1-2x}{(x+1)^2(x^2+1)}\,dx;$

$$\int \frac{1}{(x+1)^2(x^2+1)} - \frac{2x}{(x+1)^2(x^2+1)}\,dx$$

$$\int \frac{1}{(x+1)^2(x^2+1)}\,dx = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{Cx+D}{(x^2+1)} =$$

$$= A(x+1)(x^2+1) + B(x^2+1) + (Cx+D)(x+1)^2 = 1$$

$$A(x^3+x+x^2+1) + Bx^2 + B + Cx^3 + 2Cx^2 + Cx + Dx^2 + 2Dx + D = 1$$

$$\cancel{Ax^3} + \cancel{Ax^2} + \cancel{Ax} + A + \cancel{Bx^2} + B + \cancel{Cx^3} + 2\cancel{Cx^2} + \cancel{Cx} + \cancel{Dx^2} + \cancel{2Dx} + D = 1$$

$$x^3(A+C) + x^2(A+B+2C+D) + x(A+C+2D) + A+B+D = 1$$

$$\begin{cases} A+B+D=1 \\ A+C+2D=0 \\ A+B+2C+D=0 \\ A+C=0 \end{cases}$$

$B-C=1 \quad B = 1+C$

$D=0$

$A=-C$

$$-C+1+C+2C=0$$

$C = -\frac{1}{2} \quad A = \frac{1}{2} \quad B = \frac{1}{2} \quad D=0$

$$\int \frac{1}{2(x+1)} + \frac{1}{2(x+1)^2} - \frac{x}{2(x^2+1)}\, dx =$$

$$= \frac{1}{2}\log|x+1| - \frac{1}{2x+1} - \frac{1}{4}\log|x^2+1|$$

$$2\int \frac{x}{(x+1)^2(x^2+1)}\, dx = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{Cx+D}{(x^2+1)} =$$

$$\begin{cases} A+B+D=0 \\ A+C+2D=1 \\ A+B+2C+D=0 \\ A+C=0 \end{cases}$$

$A=-C \quad A=0$

$D=\frac{1}{2}$

$-C+B+\frac{1}{2}=0$

$B=C-\frac{1}{2} \quad B=-\frac{1}{2}$

$-C+C-\cancel{\frac{1}{2}}+2C+\cancel{\frac{1}{2}}=0$

$C=0$

$$\int \frac{-1}{2(x+1)^2} + \frac{1}{2(x^2+1)}\, dx$$

$$= -\frac{1}{2x+1} + \frac{1}{2}\arctan(x)$$

$$== \frac{1}{2}\log|x+1| - \frac{1}{2x+1} - \frac{1}{4}\log|x^2+1| - \frac{1}{2x+1} + \frac{1}{2}\arctan(x)$$

6. $\int_0^1 \frac{1}{x^3+1}\, dx.$

$$x^3 + 1^3 = (x+1)(x^2 - x + 1)$$

$$\int \frac{1}{(x+1)(x^2-x+1)}\, dx = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$$

$$Ax^2 - Ax + A + Bx^2 + Bx + Cx + C = 1$$

$$\begin{cases} A + B = 0 \\ -A + B + C = 0 \\ A + C = 1 \end{cases}$$

$$A = 1 - C$$

$$1 - C + B = 0 \qquad B = C - 1$$

$$C - 1 + C - 1 + C = 0$$

$$3C - 2 = 0$$

$$C = \frac{2}{3} \qquad B = -\frac{1}{3} \qquad A = \frac{1}{3}$$

$$\int \frac{1}{3(x+1)} + \frac{-x+2}{3(x^2-x+1)}\, dx$$

$$= \frac{1}{3} \log|x+1| + \frac{1}{3}\int \frac{-x+2}{x^2-x+1}\, dx$$

DA FINIRE

1. $\int \tan x \, dx;$

$$\int f(g(x)) \cdot g'(x)\, dx = \int f(u)\, du$$

$$u = g(x)$$

$$\int \frac{\sin x}{\cos x} dx$$

$$u = \cos x$$

$$du = -\sin x$$

$$dx = \left(-\frac{1}{\sin x}\right) du$$

$$\int \frac{\sin x}{u} \cdot -\frac{1}{\sin x} du = -\frac{1}{u} du$$

$$= -\log|\cos x| + c$$

2. $\int \frac{1}{e^x + e^{-x}} dx;$

3. $\int \frac{1}{x^2 + 4x + 5} dx;$

4. $\int \frac{1}{\sqrt{e^{2x} - 1}} dx;$

5. $\int \frac{\sqrt{x}}{2 + \sqrt{x}} dx;$

6. $\int \frac{1}{x\sqrt{x + 4}} dx;$

2. $\int \frac{1}{e^x + e^{-x}} dx;$

$$t = e^x$$

$$dt = e^x$$

$$\log t = x$$

$$dx = \frac{1}{t} dt$$

$$\frac{1}{t + \frac{1}{t}}$$

$$\int \frac{1}{t + \frac{1}{t}} \cdot \frac{1}{t} dt = \int \frac{1}{t^2+1} dt = \arctan(e^x) + c$$

3. $\int \frac{1}{x^2+4x+5}\,dx;$

$$x^2+4x+5 \quad +2^2-2^2$$

$$(x+2)^2+5-2^2$$

$$\int \frac{1}{(x+2)^2+1}\,dx \qquad t=(x+2) \qquad dt=dx$$

$$\int \frac{1}{t^2+1} = \arctan(x+2)+c$$

4. $\int \frac{1}{\sqrt{e^{2x}-1}}\,dx;$

$$t=\sqrt{e^{2x}-1}$$

$$dt=\frac{2\cdot e^{2x}}{\sqrt{e^{2x}-1}}\,dx$$

$$\int \frac{1}{\cancel{t}}\cdot\frac{\cancel{t}}{2\cdot e^{2x}}\,dt \qquad dx=\frac{\sqrt{e^{2x}-1}}{2\cdot e^{2x}}$$

$$\frac{1}{2}\int \frac{1}{e^{2x}}\,dt$$

$$\frac{1}{2}\int \frac{1}{t^2+1}\,dt = 2\arctan\left(\sqrt{e^{2x}-1}\right)+c$$

5. $\int \frac{\sqrt{x}}{2+\sqrt{x}}\, dx;$

$$t = 2+\sqrt{x}$$

$$dt = \frac{1}{2\sqrt{x}}\, dx$$

$$2\sqrt{x}\, dt = dx$$

$$2(t-2)$$

$$(2t-2)\, dt = dx$$

$$\int \frac{t-2}{t} \cdot (2t-2)\, dt$$

$$\int \frac{(t-2)(2t-2)}{t}\, dt = \int \frac{2t^2-2t-4t+2}{t}\, dt$$

$$= \int \frac{2t^2-6t+2}{t}\, dt = \int 2t\, dt + \int -6\, dt + 2\int \frac{1}{t}\, dt$$

$$= 2\frac{t^2}{2} - 6t + 2\log|t|$$

$$= 4 + 4\sqrt{x} + x - 6(2+\sqrt{x}) + 2\log|2+\sqrt{x}|$$

$$= 4 + 4\sqrt{x} + x - 12 - 6\sqrt{x} + 2\log(2+\sqrt{x})$$

$$= x - 2\sqrt{x} - 8 + 2\log|2+\sqrt{x}|$$

6. $\int \frac{1}{x\sqrt{x+4}}\, dx;$

$$t = \sqrt{x+4}$$

$$dt = \frac{1}{2}\cdot\frac{1}{\sqrt{x+4}} = \frac{1}{2t}\, dx$$

$$2t\, dt = dx$$

$$\int \frac{1}{(t^2-4)\cancel{t}} \cdot 2\cancel{t}\, dt$$

$$2\int \frac{1}{t^2-4}\, dt = 2\int \frac{1}{-4\left(-\left(\frac{t}{2}\right)^2+1\right)}\, dt = 2\cdot -\frac{1}{4}\int \frac{1}{1-\left(\frac{t}{2}\right)^2}\, dt$$

$$= -\frac{1}{4} \log\left|\frac{1+\sqrt{x+4}}{1-\sqrt{x+4}}\right| + c$$

$$\int f(x)\cdot g'(x)\,dx = f(x)\cdot g(x) - \int f'(x)\cdot g(x)\,dx$$

1. $\int (x-2)e^x \, dx;$

$$(x-2)\cdot e^x - \int e^x\,dx = (x-2)\cdot e^x - e^x + c$$
$$= e^x(x-3) + c$$

2 $\int x \cos x \, dx;$

$$x\cdot \sin x - \int \sin x\,dx = x\cdot\sin x + \cos x\,dx$$

3. $\int x^3 \log x \, dx;$

$$\log x \cdot \frac{x^4}{4} - \int \frac{1}{\cancel{x}}\cdot\frac{x^{\cancel{4}}}{4}\,dx =$$

$$= \frac{1}{4}\cdot\log x\cdot x^4 - \frac{1}{4}\,\frac{x^4}{4} = -\frac{1}{4}x^4\left(\frac{1}{4} - \log x\right) + c$$

4. $\int e^{2x} \sin x \, dx;$

$$e^{2x}\cdot -\cos x + 2\int e^{2x}\cdot -\cos x\,dx$$

$$+2\left(e^2\sin x - 2\int e^{2x}\cdot\sin x\,dx\right)$$

$$-e^{2x}\cdot\cos x + 2e^2\sin x - 4\int e^{2x}\cdot\sin x\,dx =$$

$$= \frac{-e^{2x} \cdot \cos x}{5} + \frac{2e^{2x} \sin x}{5} + C$$

5. $\int x^5 e^x \, dx;$

$$x^5 \cdot e^x - \int 5x^4 \cdot e^x \, dx$$

$$= x^5 \cdot e^x - 5\int x^4 \cdot e^x \, dx =$$

$$= x^5 \cdot e^x - 5\left(x^4 \cdot e^x - 4\int x^3 \cdot e^x \, dx\right) =$$

$$= x^5 \cdot e^x - 5\left(x^4 \cdot e^x - 4\left(x^3 \cdot e^x - 3\int x^2 \cdot e^x \, dx\right)\right) =$$

$$= x^5 \cdot e^x - 5\left(x^4 \cdot e^x - 4\left(x^3 \cdot e^x - 3\left(x^2 \cdot e^x - 2\int x \cdot e^x \, dx\right)\right)\right) =$$

$$= x^5 \cdot e^x - 5\left(x^4 \cdot e^x - 4\left(x^3 \cdot e^x - 3\left(x^2 \cdot e^x - 2\left(x \cdot e^x - e^x\right)\right)\right)\right)$$

6. $\int \frac{\log(\log x)}{x} \, dx;$

$$\int \log(\log(x)) \cdot \frac{1}{x} \, dx =$$

$$= \log(\log(x)) \cdot \log|x| - \int \frac{1}{\cancel{\log x}} \cdot \frac{1}{x} \cdot \cancel{\log|x|} \, dx$$

$$= \log(\log(x)) \cdot \log|x| - \log|x| =$$

$$= \log|x| \left(\log(\log(x)) - 1\right) + C$$

7. $\int \arccos x \; dx;$

$$\arccos x \cdot x - \int \frac{x}{\sqrt{1-x^2}} \, dx$$

$t = 1 - x^2$

$dt = 2x \, dx$

$\frac{1}{2x} dt = dx$

$$\int \frac{\cancel{x}}{\sqrt{t}} \cdot \frac{1}{2\cancel{x}} dt$$

$$\frac{1}{2} \int \frac{1}{\sqrt{t}} dt = \frac{1}{\cancel{2}} \sqrt{t} \cdot \cancel{2}$$

$$\arccos x \cdot x - \sqrt{1-x^2} + C$$

8. $\int x \sin^2 x \; dx.$

$$x \cdot \int \sin^2 x \, dx - \int \sin^2 x \, dx$$

$$\int \frac{1-\cos(2x)}{2} dx = \frac{1}{2}\left(x - \underline{\int \cos(2x)\, dx}\right)$$

$$\frac{1}{2} \int \cos(t)\, dt = \frac{1}{2} \cdot \sin(2x)$$

$t = 2x$

$dt = 2\, dx$

$\frac{1}{2} dt = dx$

$$= \left(2x - \sin(2x)\right) x =$$

$$= (2x - \sin 2x) \cdot (x - 1) + C$$

7. $\displaystyle\int x\sqrt{\frac{1-x}{1+x}}\ dx;$
8. $\displaystyle\int_0^1 \frac{1+\sqrt{x}}{1+x+\sqrt{x}}\ dx;$
9. $\displaystyle\int \frac{\sqrt[3]{x}}{\sqrt{x}+1}\ dx;$
10. $\displaystyle\int \sqrt{e^x-1}\ dx;$
11. $\displaystyle\int \frac{1}{e^{2x}-3e^x+2}\ dx;$
12. $\displaystyle\int \frac{e^x}{3e^{2x}-e^x+2}\ dx.$

Esercizio 5.
Calcolare i seguenti integrali, utilizzando il metodo di integrazione per parti:

1. $\displaystyle\int (x-2)e^x\ dx;$
2. $\displaystyle\int x\cos x\ dx;$
3. $\displaystyle\int x^3\log x\ dx;$
4. $\displaystyle\int e^{2x}\sin x\ dx;$
5. $\displaystyle\int x^5 e^x\ dx;$
6. $\displaystyle\int \frac{\log(\log x)}{x}\ dx;$
7. $\displaystyle\int \arccos x\ dx;$
8. $\displaystyle\int x\sin^2 x\ dx.$

Esercizio 6.
Calcolare i seguenti integrali di funzioni trigonometriche:

1. $\displaystyle\int \frac{\sin x}{1+\cos^2 x}\ dx;$
2. $\displaystyle\int \cos(\log x)\ dx;$
3. $\displaystyle\int \cos^2 x\ dx;$
4. $\displaystyle\int \sin^2(3x)\ dx;$
5. $\displaystyle\int \cos x\sin^2 x\ dx;$
6. $\displaystyle\int \cos^8 x\sin^3 x\ dx.$

Esercizio 7.
Calcolare l'area della regione di piano delimitata dal grafico della funzione $y = \log(3x + 1)$ e dalle rette $y = 0$ $x = 0$ e $x = 1$.

Esercizio 8.
Calcolare l'area della regione di piano delimitata dal grafico della funzione $y = e^x - 1$ e dalle rette $y = 0$ $x = -2$ e $x = 0$.

Esercizio 9.
Calcolare l'area della regione di piano delimitata dal grafico della funzione $y = x - 2$ e dalle rette $y = 0$ $x = 0$ e $x = 3$.

Esercizio 10.
Calcolare l'area della regione di piano delimitata dai grafici delle funzioni $y = x$, $y = x^2$ e dalle rette $x = 0$ e $x = 1$.

Esercizio 11.
Calcolare l'area della regione di piano delimitata dai grafici delle funzioni $y = x^2$, $y = x$ e dalle rette $x = 0$ e $x = 2$.

Esercizio 12.
Studiare la convergenza e la divergenza dei seguenti integrali impropri, utilizzando i criteri del confronto e del confronto asintotico:

1. $\displaystyle\int_1^{\infty} \frac{x+1}{x^6 + 5x^4 - 1}\, dx;$
2. $\displaystyle\int_0^{1} \frac{x-1}{(x+2)\sqrt{x}}\, dx;$
3. $\displaystyle\int_1^{+\infty} \frac{1}{(x^2 + 2x + 3)\, e^x}\, dx;$
4. $\displaystyle\int_0^{+\infty} \frac{x^2 - x}{e^x}\, dx;$
5. $\displaystyle\int_0^{3} \frac{\log x}{x\sqrt{x}}\, dx;$
6. $\displaystyle\int_3^{+\infty} \frac{1 + \sin 2x}{x^2 + 1}\, dx;$
7. $\displaystyle\int_3^{+\infty} \frac{\cos^2(2x-3)}{e^{3x}(x+1)}\, dx;$
8. $\displaystyle\int_1^{\infty} \frac{1}{(x+3)^2}\, dx;$
9. $\displaystyle\int_5^{\infty} e^{-5x}\, dx;$

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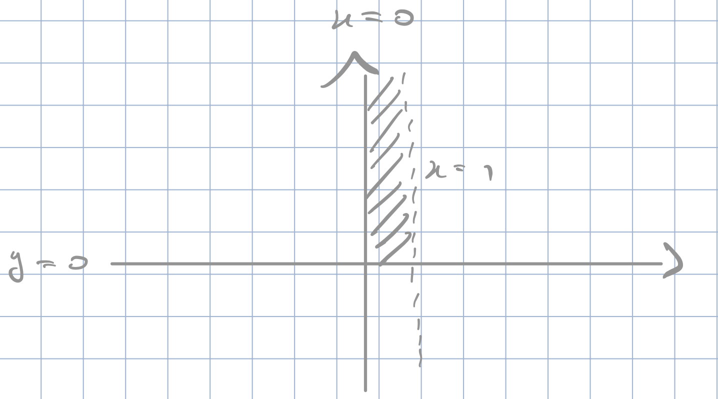
Esercizio 7.
Calcolare l'area della regione di piano delimitata dal grafico della funzione $y = \log(3x+1)$ e dalle rette $y = 0$ $x = 0$ e $x = 1$.

$$f(x) = \log(3x+1)$$

$$\int \log(3x+1)\,dx$$



$$\frac{1}{3}\int \log(t)\,dt$$

$$t = 3x+1$$
$$dt = 3\,dx$$
$$\frac{1}{3}\,dt = dx$$

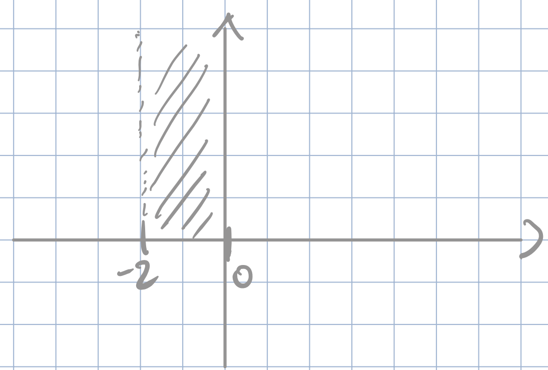
$$\frac{1}{3}\left(\log(t)\cdot t - \int \frac{1}{\cancel{t}}\cdot \cancel{t}\,dt\right) =$$

$$= \frac{1}{3}\left(\log(t)\cdot t - t\right) = \frac{1}{3}(3x+1)\left(\log(3x+1) - 1\right)$$

$$= \frac{1}{3}(4)\left(\log 4 - 1\right) + \frac{1}{3} = \frac{4\log(4) - 3}{3}$$

Esercizio 8.
Calcolare l'area della regione di piano delimitata dal grafico della funzione $y = e^x - 1$ e dalle rette $y = 0$ $x = -2$ e $x = 0$.

$$\int e^x - 1\,dx = e^x - x$$



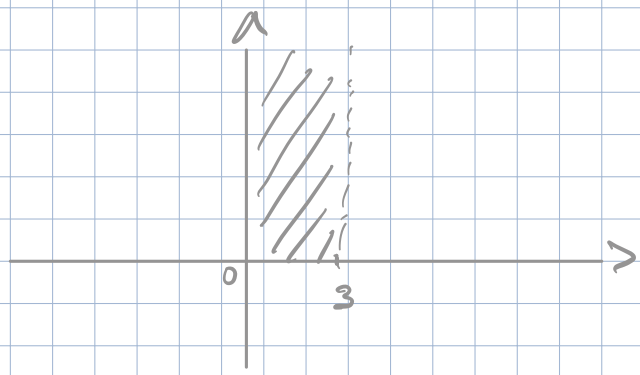
$$1 - \frac{1}{e^2} + 2 = 3 - \frac{1}{e^2}$$

Esercizio 9.
Calcolare l'area della regione di piano delimitata dal grafico della funzione $y = x - 2$ e dalle rette $y = 0$ $x = 0$ e $x = 3$.

$$\int x-2\,dx = \frac{x^2}{2} - 2x$$

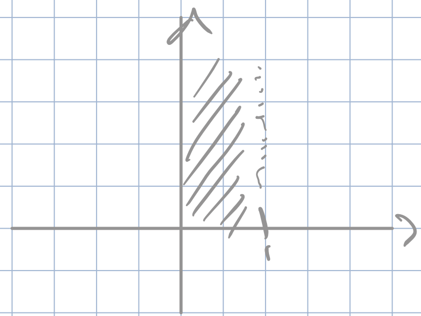
$$\frac{9}{2} - 6$$

NEGATIVA?



Esercizio 10.
Calcolare l'area della regione di piano delimitata dai grafici delle funzioni $y = x$, $y = x^2$ e dalle rette $x = 0$ e $x = 1$.

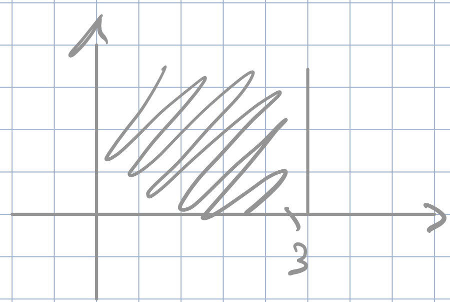
$$\int x\,dx = \frac{x^2}{2} \qquad \int x^2\,dx = \frac{x^3}{3}$$



$$\frac{1}{2} - \frac{1}{3} = \frac{1}{5}$$

Esercizio 11.
Calcolare l'area della regione di piano delimitata dai grafici delle funzioni $y = x^2$, $y = x$ e dalle rette $x = 0$ e $x = 2$.

$$\int x\,dx = \frac{x^2}{2} \qquad \int x^2\,dx = \frac{x^3}{3}$$



$$\frac{8}{3} - 2 = \frac{2}{3}$$

1. $\int_1^{\infty} \frac{x+1}{x^6+5x^4-1}\, dx;$ $\simeq \frac{1}{x^5} \rightarrow$ CONVERGE

$$\lim_{x\to+\infty} \frac{x^6+x^5}{x^6+5x^4-1} = 1$$

$$\int_0^1 \frac{1}{x^\alpha}\, dx = \begin{cases} \alpha > 1 & \text{DIVERGE} \\ \alpha \leq 1 & \text{CONVERGE} \end{cases}$$

$$\int_1^{+\infty} \frac{1}{x^\alpha}\, dx = \begin{cases} \alpha > 1 & \text{CONVERGE} \\ \alpha \leq 1 & \text{DIVERGE} \end{cases}$$

2. $\int_0^1 \frac{x-1}{(x+2)\sqrt{x}}\, dx;$

$$\int_0^1 \frac{1-x}{(x+2)\sqrt{x}} \qquad \frac{1-x}{x\sqrt{x}+2\sqrt{x}} \simeq \frac{1}{\sqrt{x}}$$

$\Rightarrow$ CONVERGE

3. $\int_1^{+\infty} \frac{1}{(x^2+2x+3)\,e^x}\, dx;$ $\leq \frac{1}{e^x}$

$$\int_1^{+\infty} e^x \Rightarrow \text{CONVERGE A } \frac{1}{e}$$

$\Rightarrow$ CONVERGE

4. $\displaystyle\int_0^{+\infty} \frac{x^2 - x}{e^x}\, dx;$

$$\int_0^{+\infty} \frac{x^2 - x}{e^x}$$

10. $\displaystyle\int\limits_0^3 \frac{1}{9-x^2}\ dx;$

11. $\displaystyle\int\limits_0^2 \frac{1}{x\sqrt{2-x}}\ dx;$

12. $\displaystyle\int\limits_0^\infty xe^{-x}\ dx;$

13. $\displaystyle\int\limits_{-\infty}^{+\infty} \frac{|x|}{x^2+1}\ dx;$

14. $\displaystyle\int\limits_{-\infty}^{+\infty} \frac{x}{x^4+1}\ dx;$

15. $\displaystyle\int_1^{+\infty} e^{-x}\sin x\ dx.$

CORSO DI LAUREA IN INFORMATICA APPLICATA

ANALISI MATEMATICA 1

ESERCIZI SU

NUMERI COMPLESSI

Esempio 1.
Esprimere $(1+i)^5$ in forma algebrica.
Svolgimento: innanzitutto si determinino il modulo e l'argomento di $1+i$. Risulta $\rho = |1+i| = \sqrt{2}$ e $\theta = \arg(1+i) = \frac{\pi}{4}$, essendo $\sin\theta = \cos\theta = \frac{\sqrt{2}}{2}$. Allora, applicando la formula di De Moivre

$$z^n = \rho^n (\cos(n\theta) + i\sin(n\theta)), \text{dove } z = \rho(\cos\theta + i\sin\theta) \in \mathbb{C},$$

si ottiene $(1+i)^5 = 4\sqrt{2}\left(\cos\left(\frac{5}{4}\pi\right) + i\sin\left(\frac{5}{4}\pi\right) = ...\right)$.

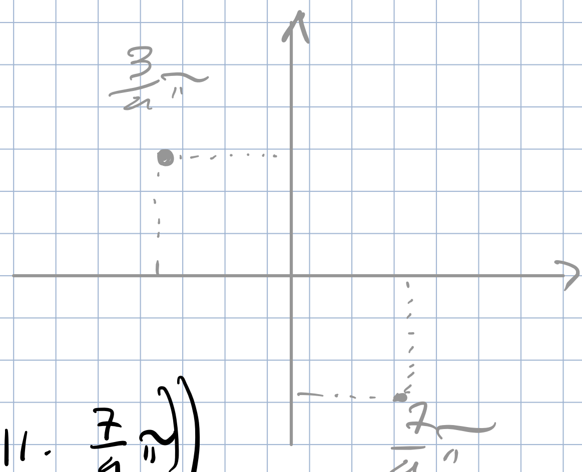
Esempio 2.
Semplificare l'espressione $\dfrac{2+3i}{4-i}$.
Svolgimento: risulta $\dfrac{2+3i}{4-i} = \dfrac{(2+3i)(4+i)}{(4-i)(4+i)} = \dfrac{5}{17} + \dfrac{14}{17}i$.
In alternativa si possono utilizzare le seguenti formule di De Moivre

1. $|z_1 \cdot z_2| = |z_1| \cdot |z_2| \quad$ e $\quad \arg(z_1 \cdot z_2) = \arg(z_1) + \arg(z_2), \quad z_1, z_2 \in \mathbb{C}$;
2. $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|} \quad$ e $\quad \arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2), \quad z_1, z_2 \in \mathbb{C}, z_2 \neq 0$.

Esercizio 1.
Scrivere in forma algebrica i seguenti numeri complessi:

1. $(1-i)^{11}$;
2. $(2+i)^3$;
3. $\left(1+\sqrt{3}i\right)^4$;
4. $\dfrac{2}{i}$;
5. $\dfrac{i}{1+\sqrt{3}i}$;
6. $\dfrac{1+i3}{2-i}$;

1 $(1-i)^{11}$ $\rho = \sqrt{2}$



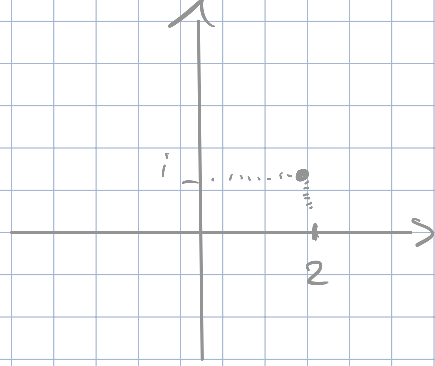
$\cos x = -\sin x$

$$z^{11} = \left(\sqrt{2}\right)^{11} \cdot \left(\cos\left(11 \cdot \frac{7}{4}\pi\right) - i\sin\left(11 \cdot \frac{7}{4}\pi\right)\right)$$

$$\frac{77}{4}\pi = \frac{76+1}{4}\pi = 19\pi + \frac{\pi}{4} = \pi + \frac{\pi}{4}$$

$$\left(\sqrt{2}\right)^{11}\left(\cos\left(\frac{\pi}{4}+\pi\right) - i\sin\left(\frac{\pi}{4}+\pi\right)\right) = \sqrt{2^{11}}\left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)$$

2 $(2+i)^3 =$



$= 8 + 3\cdot 4\cdot i - 3\cdot 2 - i =$

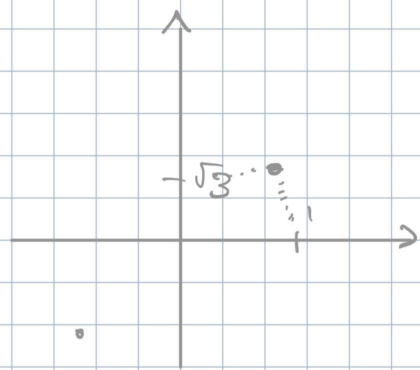
$= 8 + 12i - 6 - i =$

$= 2 + 11i$

3 $(1+\sqrt{3}i)^4$

$\rho = 2$

$\theta = \frac{\pi}{3}$



$= 2^4\left(\cos\left(\frac{4}{3}\pi\right) + i\sin\left(\frac{4}{3}\pi\right)\right)$

$= 2^4\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$

$\sqrt{3} = m$

$\frac{4}{3}\pi = \frac{\pi}{3} + \pi$

4 $\quad \dfrac{2}{i} = \dfrac{2}{i} \cdot \dfrac{i}{i} = -2i$

5 $\quad \dfrac{i}{1+\sqrt{3}i} = \dfrac{i}{1+\sqrt{3}i} \cdot \dfrac{1-\sqrt{3}i}{1-\sqrt{3}i} = \dfrac{i+\sqrt{3}}{4} = \dfrac{\sqrt{3}}{4} + \dfrac{1}{4}i$

6 $\quad \dfrac{1+3i}{2-i} = \dfrac{1+3i}{2-i} \cdot \dfrac{2+i}{2+i} = \dfrac{2+i+6i-3}{5} = -\dfrac{1}{5} + \dfrac{7}{5}i$

7 $\quad \dfrac{1+i}{i(2+3i)} = \dfrac{1+i}{i(2+3i)} \cdot \dfrac{i(2-3i)}{i(2-3i)}$

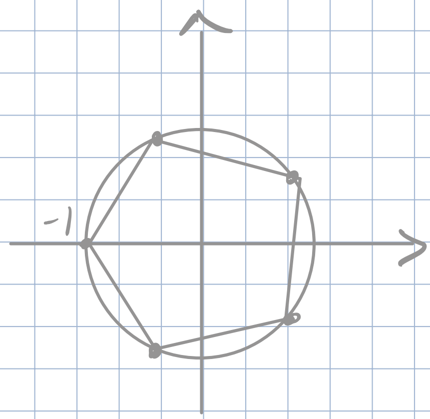
7. $\dfrac{1+i}{i\,(2+3i)}$;

8. $\dfrac{(1+2i)\,(2-3i)}{(2-i)\,(3+2i)}$.

$= \dfrac{(1+i)(2i+3)}{13} = \dfrac{2i+3-2+3i}{13} = \dfrac{1}{13} + \dfrac{5}{13}i$

8 $\quad \dfrac{(1+2i)(2-3i)}{(2-i)(3+2i)} = \dfrac{\cancel{2}-\cancel{3i}+\cancel{4i}+\cancel{6}}{\cancel{6}+\cancel{4i}-\cancel{3i}+\cancel{2}} = 1$

1 $\sqrt[5]{-1}$ $\qquad \rho = 1 \qquad \theta = \pi$



$$\cos\left(\frac{\pi}{5} + \frac{2k}{5}\pi\right) + i\sin\left(\frac{\pi}{5} + \frac{2k}{5}\pi\right)$$

$k = 0 \qquad \cos\left(\frac{\pi}{5}\right) + i\sin\left(\frac{\pi}{5}\right)$

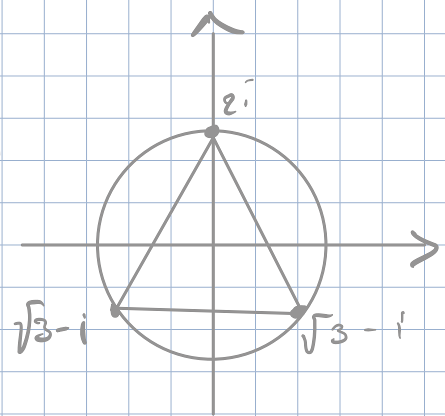
$k = 1 \qquad \cos\left(\frac{3}{5}\pi\right) + i\sin\left(\frac{3}{5}\pi\right)$

$k = 2 \qquad \cos\left(\pi\right) + i\sin\left(\pi\right) = -1$

$k = 3 \qquad \cos\left(\frac{7}{5}\pi\right) + i\sin\left(\frac{7}{5}\pi\right)$

$k = 4 \qquad \cos\left(\frac{9}{5}\pi\right) + i\sin\left(\frac{9}{5}\pi\right)$

2 $\sqrt[3]{-8i}$ $\qquad \rho = 8 \qquad \theta = -\frac{\pi}{2}$



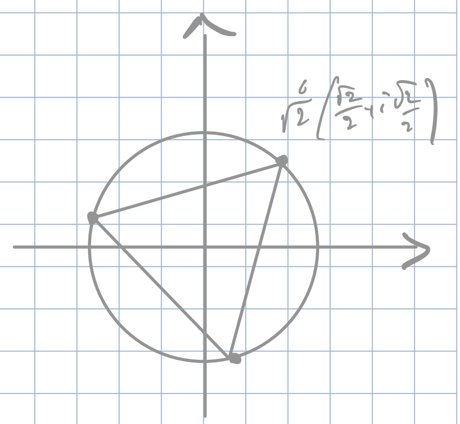
$$2\left(\cos\left(-\frac{\pi}{6} + \frac{4k}{6}\pi\right) + i\sin\left(-\frac{\pi}{6} + \frac{4k}{6}\pi\right)\right)$$

$k = 0 \qquad 2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right) = \sqrt{3} - i$

$k = 1 \qquad 2\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right) = 2i$

$k = 2 \qquad 2\left(\cos\left(\frac{7}{6}\pi\right) + i\sin\left(\frac{7}{6}\pi\right)\right) = -\sqrt{3} - i$

3 $\sqrt[3]{-1+i}$ $\qquad \rho = \sqrt{2} \qquad \theta = \frac{3}{4}\pi$



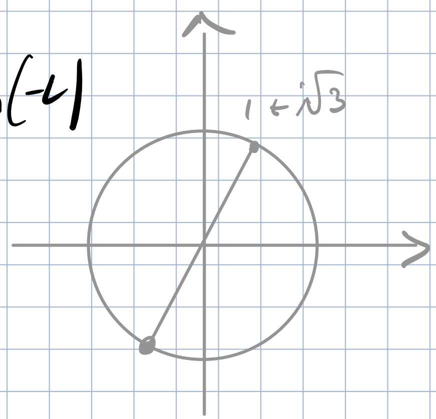
$$\sqrt[6]{2}\left(\cos\left(\frac{3}{12}\pi + \frac{8k}{12}\pi\right) + i\sin\left(\frac{3}{12}\pi + \frac{8k}{12}\pi\right)\right)$$

$k=0 \quad \sqrt[6]{2}\left(\cos\left(\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{4}\right)\right) = \sqrt[6]{2}\left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)$

$k=1 \quad \sqrt[6]{2}\left(\cos\left(\frac{11}{12}\pi\right)+i\sin\left(\frac{11}{12}\pi\right)\right)$

$k=2 \quad \sqrt[6]{2}\left(\cos\left(\frac{19}{12}\pi\right)+i\sin\left(\frac{19}{12}\pi\right)\right)$

4 $\sqrt{-1+2i}$ $\qquad \rho = \sqrt{5} \qquad \theta = \arctan(-2)$

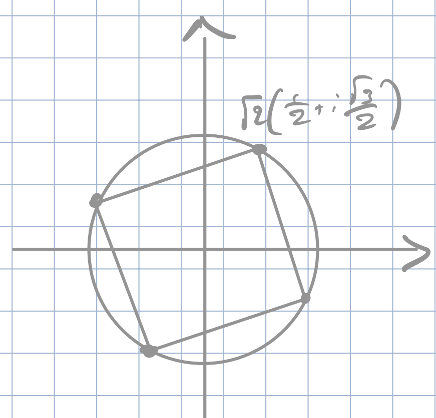


$$\sqrt[10]{5}\left(\cos\left(\frac{2}{6}\pi + k\pi\right) + i\sin\left(\frac{2}{6}\pi + k\pi\right)\right)$$

$k=0 \quad \sqrt[10]{5}\left(\cos\left(\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{3}\right)\right) = \left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\sqrt[10]{5}$

$k=1 \quad \sqrt[10]{5}\left(\cos\left(\frac{\pi}{3}+\pi\right)+i\sin\left(\frac{\pi}{3}+\pi\right)\right) = \left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)\sqrt[10]{5}$

5 $\sqrt[4]{-2-2\sqrt{3}\,i}$ $\qquad \rho = 4 \qquad \theta = \frac{4}{3}\pi$



$$\sqrt{2}\left(\cos\left(\frac{4}{12}\pi + \frac{6k}{12}\pi\right) + i\sin\left(\frac{4}{12}\pi + \frac{6k}{12}\pi\right)\right)$$

$k=0 \quad \sqrt{2}\left(\cos\left(\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{3}\right)\right) = \left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\sqrt{2}$

$k=1 \quad \sqrt{2}\left(\cos\left(\frac{5}{6}\pi\right)+i\sin\left(\frac{5}{6}\pi\right)\right) = \left(\frac{\sqrt{3}}{2}+i\frac{1}{2}\right)\sqrt{2}$

$k=2 \quad \sqrt{2}\left(\cos\left(\frac{\pi}{3}+\pi\right)+i\sin\left(\frac{\pi}{3}+\pi\right)\right) = \left(-\frac{1}{2}-\frac{\sqrt{3}}{2}\right)\sqrt{2}$

$k=3 \quad \sqrt{2}\left(\cos\left(\frac{11}{6}\pi\right)+i\sin\left(\frac{11}{6}\pi\right)\right) = \frac{\sqrt{3}}{2}-\frac{1}{2}i\,)\sqrt{2}$

7. $\dfrac{1+i}{i\,(2+3i)}$;

8. $\dfrac{(1+2i)\,(2-3i)}{(2-i)\,(3+2i)}$.

Esempio 3.
Calcolare le quattro radici quarte di -4.
Svolgimento: Se $z = \rho\,(\cos\theta + i\sin\theta) \in \mathbb{C}$ allora esistono esattamente n radici n-esime complesse $z_k = \rho_k\,(\cos\theta_k + i\sin\theta_k)$ $(k = 0, 1, 2, ..., n-1)$ di z, dove

$$\rho_k = \sqrt[n]{\rho} \quad e \quad \theta_k = \frac{\theta + 2k\pi}{n} \quad k = 0, 1, 2, ..., n-1\,.$$

La radice $z_0 = \sqrt[n]{\rho}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right)$ si dice *radice principale.*

Allora nel caso delle radici quarte di -4 si ha che la radice principale è $1+i$. Le altre radici sono $-1+i$, $-1-i$ e $1-i$.

Esercizio 2.
Calcolare le seguenti radici e rappresentarle nel piano complesso:

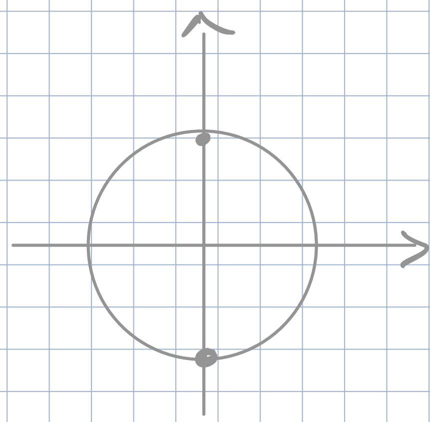
1. $\sqrt[5]{-1}$;
2. $\sqrt[3]{-8i}$;
3. $\sqrt[3]{-1+i}$;
4. $\sqrt{-1+2i}$;
5. $\sqrt[4]{-2-2\sqrt{3}i}\,$.

Esercizio 3.
Determinare le soluzioni complesse delle seguenti equazioni:

1. $x^2 + 1 = 0$;
2. $2x^2 + 4 = 0$;
3. $x^2 + 2x + 3 = 0$;
4. $x^2 + 2ix - \sqrt{3}i = 0$;
5. $x^3 - 1 = 0$;
6. $x^4 + 2x^2 - 3 = 0\,$.

1. $x^2 + 1 = 0;$

$\rho = 1 \quad \theta = \pi$



$z = \sqrt{-1}$

$\cos\left(\frac{\pi}{2} + k\pi\right) + i\sin\left(\frac{\pi}{2} + k\pi\right)$

$k = 0 \quad \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) = i$

$k = 1 \quad \cos\left(\frac{\pi}{2} + \pi\right) + i\sin\left(\frac{\pi}{2} + \pi\right) = -i$

2. $2x^2 + 4 = 0;$

$z = \sqrt{-2} \qquad \rho = 2 \quad \theta = \pi$

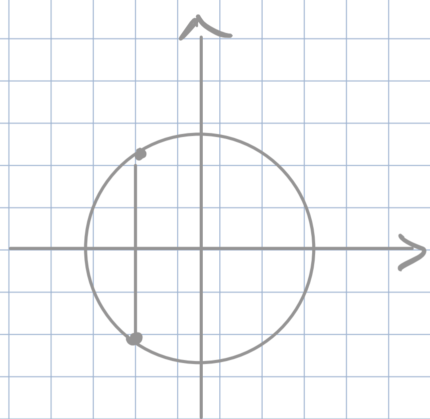
$\sqrt{2}\left(\cos\left(\frac{\pi}{2} + k\pi\right) + i\sin\left(\frac{\pi}{2} + k\pi\right)\right)$

$k = 0 \quad \sqrt{2}\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right) = \sqrt{2}\, i$

$k = 1 \quad \sqrt{2}\left(\cos\left(\frac{\pi}{2} + \pi\right) + i\sin\left(\frac{\pi}{2} + \pi\right)\right) = -\sqrt{2}\, i$

3. $x^2 + 2x + 3 = 0;$

$$x_{1,2} = \frac{-2 \pm 2\sqrt{-2}}{2} \begin{cases} -1 + \sqrt{-2} = -1 + \sqrt{2}i \\ -1 - \sqrt{-2} = -1 - \sqrt{2}i \end{cases}$$

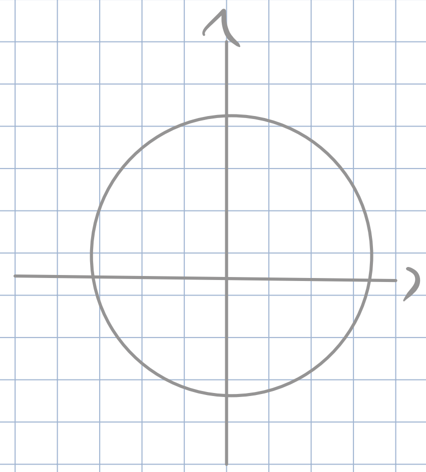


$x^2 + 2ix - \sqrt{3}i = 0;$

$$x_{1,2} = \frac{-2i \pm \sqrt{-4 + 4\cdot\sqrt{3}i}}{2}$$

$$= \frac{-2i \pm 2\sqrt{\sqrt{3}i - 1}}{2}$$

$$= -i \pm \sqrt{\sqrt{3}i - 1}$$



$$\rho = \sqrt{(\sqrt{3})^2 + (-1)^2} = 2 \qquad \theta = \frac{2}{3}\pi$$

$$\sqrt{2}\left(\cos\left(\frac{\pi}{3} + k\pi\right) + i\sin\left(\frac{\pi}{3} + k\pi\right)\right)$$

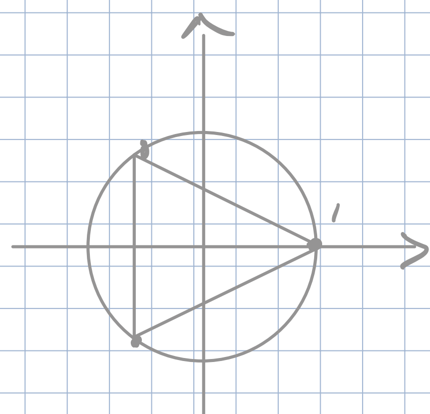
$k=0 \quad \sqrt{2}\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{2}}{2} + i\sqrt{2}\cdot\frac{\sqrt{3}}{2}$

$k=1 \quad \sqrt{2}\left(\cos\left(\frac{\pi}{3} + \pi\right) + i\sin\left(\frac{\pi}{3} + \pi\right)\right) = \sqrt{2}\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = -\frac{\sqrt{2}}{2} - i\sqrt{2}\cdot\frac{\sqrt{3}}{2}$

$$x_1 = -\frac{\sqrt{2}}{2} - i\left(1 + \frac{\sqrt{6}}{2}\right), \quad x_2 = \frac{\sqrt{2}}{2} - i\left(1 - \frac{\sqrt{6}}{2}\right)$$

5. $x^3 - 1 = 0;$ $\qquad \rho = 1 \qquad \theta = 0$

$$x = \sqrt[3]{1}$$



$$\cos\left(\frac{2k}{3}\pi\right) + i\sin\left(\frac{2k}{3}\pi\right)$$

$k=0 \quad \cos(0) + i\sin(0) = 1$

$k=1 \quad \cos\left(\frac{2}{3}\pi\right) + i\sin\left(\frac{2}{3}\pi\right) = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

$k=2 \quad \cos\left(\frac{4}{3}\pi\right) + i\sin\left(\frac{4}{3}\pi\right) = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

6. $x^4 + 2x^2 - 3 = 0$.

$t = x^2$ $\qquad$ $t^2 + 2t - 3 = 0$

$$\frac{-2 \pm \sqrt{4+12}}{2} = \begin{cases} \frac{-2+4}{2} = 1 & x^2 = 1 \\ \frac{-2-4}{2} = -3 & x^2 = -3 \end{cases}$$

$x = \sqrt{1}$ $\quad$ $x = \pm 1$

$x = \sqrt{-3}$ $\quad$ $x = \pm i\sqrt{3}$

CORSO DI LAUREA IN INFORMATICA APPLICATA

ANALISI MATEMATICA 1

ESERCIZI SU

SERIE NUMERICHE

Esercizio 1.
Studiare le seguenti serie geometriche e, nel caso in cui risultino convergenti, calcolarne la somma:

1. $\displaystyle\sum_{n=1}^{\infty} \frac{2^{n+1}}{5^{n-1}}$;

2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^n}{4^{2n}}$;

3. $\displaystyle\sum_{n=1}^{\infty} 3^{-n} 5^{n+2}$;

4. $\displaystyle\sum_{n=3}^{\infty} \frac{1}{4^n}$;

5. $\displaystyle\sum_{n=2}^{\infty} \frac{(-7)^{n+1}}{3 \cdot 2^n}$.

Esercizio 2.
Studiare le seguenti serie numeriche utilizzando la condizione necessaria di convergenza:

1. $\displaystyle\sum_{n=1}^{\infty} (4n^2+1)2^n$;

2. $\displaystyle\sum_{n=1}^{\infty} 3$;

3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{1+\frac{3}{n^3}}}$;

1

$$\sum_{n=1}^{\infty} \frac{2^{n+1}}{5^{n-1}} = \frac{2}{\left(\frac{1}{5}\right)} \cdot \left(\sum_{n=0}^{\infty} \left(\frac{2}{5}\right)^n - 4 \right) = 10 \left(\frac{5}{3} - 4 \right)$$

$$\lim_{n \to +\infty} \sqrt[n]{\frac{2^n}{5^n} \cdot \frac{2}{5^{-1}}} = \frac{2}{5} \cdot \sqrt[n]{\frac{2}{5^{-1}}} = \frac{2}{5} \cdot \left(\frac{2}{5^{-1}}\right)^{\frac{1}{n}} = \frac{2}{5} < 1$$

CONVERGE

2

$$\sum_{n=1}^{\infty} \frac{(-3)^n}{4^{2n}} =$$

$$\lim_{n \to +\infty} \frac{\frac{(-3)^{n+1}}{4^{2(n+1)}}}{\frac{(-3)^n}{4^{2(n)}}} = \frac{\cancel{(-3)}^{n+1}}{\cancel{4}^{2(n+1)}} \cdot \frac{\cancel{4^{2n}}}{\cancel{(-3)^n}} = \frac{3}{16} < 1 \quad \text{CONVERGE}$$

3

$$\sum_{n=1}^{\infty} 3^{-n} \cdot 5^{n+2} = \frac{1}{3^n} \cdot 5^n \cdot 5^2$$

$$\lim_{n \to +\infty} \frac{\cancel{\frac{1}{3^n}} \cdot \frac{1}{3} \cdot \cancel{5^n} \cdot 5 \cdot \cancel{5^2}}{\cancel{\frac{1}{3^n}} \cdot \cancel{5^n} \cdot \cancel{5^2}} = \frac{5}{3} > 1 \quad \text{DIVERGE}$$

4 $$\sum_{n=3}^{\infty} \frac{1}{4^n} = \sum_{n=0}^{\infty} \frac{1}{4^n} - 1 - \frac{1}{4} - \frac{1}{16} = \frac{1}{1-\frac{1}{4}} - 1 - \frac{1}{4} - \frac{1}{16} = \frac{1}{48}$$

5 $$\sum_{n=2}^{\infty} \frac{(-7)^{n+1}}{3 \cdot 2^n}$$

$$\lim_{n \to +\infty} \frac{\frac{(-7)^{n+2}}{3 \cdot 2^{n+1}}}{\frac{(-7)^{n+1}}{3 \cdot 2^n}} = \frac{\cancel{(-7)}^{n+2}}{\cancel{3 \cdot 2}^{n+1}} \cdot \frac{\cancel{3 \cdot 2^n}}{\cancel{(-7)^{n+1}}} = -\frac{7}{2} = \frac{7}{2}$$

DIVERGE

1 $$\sum_{n=1}^{\infty} (4n^2+1)\, 2^n$$

REGOLARE

$$\lim_{n \to +\infty} (4n^2+1)\, 2^n = (+\infty) \cdot +\infty = +\infty$$

DIVERGE

2 $$\sum_{n=1}^{\infty} 3$$ DIVERGE POICHÉ COSTANTE

3 $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{1+\frac{3}{n^3}}}$ REGOLARE

$$\lim_{n\to+\infty} \frac{1}{\sqrt{1+\frac{3}{n^3}}} = \frac{1}{\sqrt{1+0}} = 1 \Rightarrow \text{DIVERGE}$$

4 $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{5}{2}\right)^n$ REGOLARE

$$\lim_{n\to+\infty} \frac{\left(\frac{5}{2}\right)^n}{n} = \frac{\infty}{\infty} = +\infty \quad \text{DIVERGE}$$

1 $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

$$\lim_{n\to+\infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} =$$

$$= \frac{(n+1)\,n^n}{(n+1)^{n+1}} = \frac{n^n}{(n+1)^n} = \left(\frac{n}{n+1}\right)^n$$

$$= \frac{1}{e}$$

$$\lim_{x\to+\infty} \left(\frac{x}{x+k}\right)^x = e^{-k}$$

DIVERGE

2 $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n - n} \cong \frac{1}{2^n}$

$$\lim_{n \to +\infty} \frac{2^n}{2^n - n} = 1 \implies \text{CONV.}$$

3 $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n^n}$ REGOLARE

$$\lim_{n \to +\infty} \sqrt[n]{\frac{3^n}{n^n}} = \frac{3}{n} = 0 \implies$$

4 $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 + \log_2 n + 1}{n^4 + n - 1} \cong \frac{1}{n^2} \implies \text{CONV.}$

$$\lim_{n \to +\infty} \frac{n^4 + n^2 \log_2 n + n^2}{n^4 + n - 1} = 1$$

5 $\displaystyle\sum_{n=1}^{\infty} \frac{2n-1}{2^n} =$

$$\lim_{n \to +\infty} \frac{2n+1}{\cancel{2^n} \cdot 2} \cdot \frac{\cancel{2^n}}{2n-1} = \frac{1}{2}$$

CONVERGE

6 $\sum_{n=1}^{\infty} \log\left(1+\frac{1}{\sqrt{n}}\right) \overset{\sim}{=} \frac{1}{\sqrt{n}} \Rightarrow$ DIVERGE

7 $\sum_{n=1}^{\infty} \left(\frac{4n+1}{2n-1}\right)^n$ REGOLARE

$$\lim_{n\to+\infty} \left(\frac{4n+1}{2n-1}\right)^n = +\infty \to \text{DIVERGE}$$

8 $\sum_{n=0}^{\infty} \frac{(\log(2))^n}{2n+3}$

$$\lim_{n\to+\infty} \frac{\cancel{(\log(2))^n}\,(\log(2))}{2n+5} \cdot \frac{2n+3}{\cancel{(\log(2))^n}} = \log(2)$$

$\Rightarrow$ CONVERGE

9 $\sum_{n=1}^{\infty} \frac{\log(n)}{n} \leq \frac{n}{n} = 1$

$\Rightarrow$ DIVERGE

10 $\sum_{n=1}^{\infty} \left(n^4 - n^2 + \log(n)\right) \arctan\left(\frac{2}{n^5}\right)$ $\arctan\left(\frac{2}{n^5}\right) \simeq \frac{2}{n^5}$

$$\overset{\sim}{=} \left(n^4 - n^2 + \log(n)\right) \cdot \frac{2}{n^5} = \frac{2}{n} - \frac{2}{n^3} + \log(n)$$

$$\leq \frac{1}{n} \Rightarrow \text{DIVERGE}$$

11 $$\sum_{n=1}^{\infty} \frac{\sin(n)+2}{n} \geq \frac{-1+2}{n} \Rightarrow \text{DIVERGE}$$

$$-1 \geq \sin(n) \geq 1$$

12 $$\sum_{n=1}^{\infty} \left(\frac{n}{3n-1}\right)^{2n-1}$$

$$\lim_{n \to +\infty} \frac{\left(\frac{n}{3n-1}\right)^{2n}}{\left(\frac{n}{3n-1}\right)} = 1 \Rightarrow \text{DIVERGE}$$

13

$$\sum^{\infty} \frac{2^n \cdot \frac{1}{2}}{n^n} = \frac{1}{2} \sum^{\infty} \frac{2^n}{n^n}$$

$n=1$ $\quad$ n $\quad$ $n=1$

$$\lim_{n \to +\infty} \sqrt[n]{\frac{2^n}{n^n}} = \frac{2}{n} = 0 < 1 \quad \text{CONV}$$

14 $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{e^n}$

$$\lim_{n \to +\infty} \frac{(n+1)^3}{\cancel{e^n} \cdot e} \cdot \frac{\cancel{e^n}}{n^3} = \frac{1}{e} \cdot \frac{(n+1)^3}{n^3} = \frac{1}{e} < 1$$

$\Rightarrow$ CONV.

15 $\displaystyle\sum_{n=6}^{\infty} \frac{n-5}{n^2+7} \log\left(\frac{1}{(n-5)^3} + 1\right)$

$$\leq \frac{\cancel{n-5}}{n^2+7} \cdot \frac{1}{(n-5)^{\cancel{3}}} + 1$$

$$= \frac{1}{(n^2+7)(n-5)^2} + 1 \geq \frac{1}{n^4} \Rightarrow \text{CONV}$$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{5}{2}\right)^{n}$.

Esercizio 3.
Studiare la convergenza delle seguenti serie numeriche a termini non negativi, utilizzando i criteri di convergenza del confronto, del confronto asintotico, del rapporto e della radice:

1. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$;
2. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n - n}$;
3. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n^n}$;
4. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 + \log n + 1}{n^4 + n - 1}$;
5. $\displaystyle\sum_{n=1}^{\infty} \frac{2n - 1}{2^n}$;
6. $\displaystyle\sum_{n=1}^{\infty} \log\left(1 + \frac{1}{\sqrt{n}}\right)$;
7. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{4n+1}{2n-1}\right)^{n}$;
8. $\displaystyle\sum_{n=0}^{\infty} \frac{(\log 2)^n}{2n+3}$;
9. $\displaystyle\sum_{n=1}^{\infty} \frac{\log n}{n}$;
10. $\displaystyle\sum_{n=1}^{\infty} (n^4 - n^2 + \log n) \arctan\left(\frac{2}{n^5}\right)$;
11. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n + 2}{n}$;
12. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{3n-1}\right)^{2n-1}$;

13. $\displaystyle\sum_{n=1}^{\infty} \frac{2^{n-1}}{n^n}$;

14. $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{e^n}$;

15. $\displaystyle\sum_{n=6}^{\infty} \frac{n-5}{n^2+7} \log\left(\frac{1}{(n-5)^3}+1\right)$.

Esercizio 4.
Studiare la convergenza assoluta delle seguenti serie numeriche:

1. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n}{n(1+n^2)}$;

2. $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{n^2}{n^3+3}\right)$;

3. $\displaystyle\sum_{n=1}^{\infty} \frac{n \tan n^2}{\sqrt{n!}\,(|\tan n^2|+n)}$.

Esercizio 5.
Studiare la convergenza, semplice e assoluta, delle seguenti serie numeriche a segni alternati:

1. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}+\log n}$;

2. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \left(\frac{2n+1}{3n+1}\right)^n$;

3. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n^3}{e^n - n^2}$;

4. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{3n+1}$;

5. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n^2+2(-1)^n n}$;

6. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (\log \frac{1}{2})^n}{n(\log 2)^{n-1}}$ (osservare che si tratta di una serie a termini positivi) ;

1 $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n}{n(1+n^2)}$

$$\sum_{n=1}^{\infty} \left| \frac{\cos n}{n(1+n^2)} \right| \leq \left| \frac{1}{n^3+n} \right| \simeq \left(\frac{1}{n^3} \right) \Rightarrow \text{conv}$$

2 $\displaystyle\sum_{n=1}^{\infty} \sin \left(\frac{n^2}{n^3+3} \right)$

$$= \left| \sin \left(\frac{n^2}{n^3+3} \right) \right| \leq \left| \frac{n^2}{n^3+3} \right|$$

3 $\displaystyle\sum_{n=1}^{\infty} \frac{n \cdot \tan(n^2)}{\sqrt{n!}\,(|\tan(n^2)| + n)}$

1. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n}{n(1+n^2)}$;

2. $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{n^2}{n^3+3}\right)$;

3. $\displaystyle\sum_{n=1}^{\infty} \frac{n \tan n^2}{\sqrt{n!}\,(|\tan n^2| + n)}$.

7. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}-1}\,;$

8. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n-1}{n^2}\,.$

Esercizio 6.
Studiare la convergenza delle seguenti serie numeriche dipendenti da un parametro:

1. $\displaystyle\sum_{n=0}^{\infty} \left(\frac{t}{t-1}\right)^n, \quad t \in \mathbb{R} \setminus \{1\}\,;$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{|t-12|^{n+1}}{2n^2 e^{-n}}\,, \quad t \in \mathbb{R}\,;$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{6^{(t-1)n}}{n^2 3^n}\,, \quad t \in \mathbb{R}\,;$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{e^{n(t^2+t+1)}}{2n}\,, \quad t \in \mathbb{R}\,;$

5. $\displaystyle\sum_{n=2}^{\infty} \frac{[2(\log n)^{t-7}]^n}{n^2}\,, \quad t \in \mathbb{R}\,;$

6. $\displaystyle\sum_{n=0}^{\infty} \left[1-\cos\left(\frac{1}{n+2}\right)^t\right] n^4\,, \quad t \in \mathbb{R}^+\,;$

7. $\displaystyle\sum_{n=1}^{\infty} \left(\sqrt{1+\frac{1}{n^t}}-1\right) n^{2-t}\,, \quad t \in \mathbb{R}^+\,;$

8. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}\,|2\log t|^n}{n^4+n^2+1}\,, \quad t \in \mathbb{R}^+\,;$

9. $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^{3t}}\right) n^{\frac{3}{2}-2t}\,, \quad t \in \mathbb{R}^+\,;$

10. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1-t^2}}\,, \quad t \in \mathbb{R}\,.$

Alcuni sviluppi di McLaurin notevoli (si sottintende ovunque che i resti sono trascurabili per $x \to 0$)

e^x	$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + o(x^n)$	$= \sum_{k=0}^{n} \frac{x^k}{k!} + o(x^n)$
$\sinh x$	$= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^{2n+1}}{(2n+1)!} + o(x^{2n+2})$	$= \sum_{k=0}^{n} \frac{x^{2k+1}}{(2k+1)!} + o(x^{2n+2})$
$\cosh x$	$= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2n}}{(2n)!} + o(x^{2n+1})$	$= \sum_{k=0}^{n} \frac{x^{2k}}{(2k)!} + o(x^{2n+2})$
$\tanh x$	$= x - \frac{1}{3}x^3 + \frac{2}{15}x^5 + o(x^6)$	
$\ln(1+x)$	$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n-1}\frac{x^n}{n} + o(x^n)$	$= \sum_{k=1}^{n} (-1)^{k-1}\frac{x^k}{k} + o(x^n)$
$\sin x$	$= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + o(x^{2n+2})$	$= \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!} + o(x^{2n+2})$
$\cos x$	$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + o(x^{2n+1})$	$= \sum_{k=0}^{n} (-1)^k \frac{x^{2k}}{(2k)!} + o(x^{2n+1})$
$\tan x$	$= x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + o(x^6)$	
$\arcsin x$ $\arccos x$	$= x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \cdots + \left\lvert\binom{-1/2}{n}\right\rvert \frac{x^{2n+1}}{2n+1} + o(x^{2n+2})$ $= \frac{\pi}{2} - \arcsin x$	$= \sum_{k=0}^{n} \left\lvert\binom{-1/2}{k}\right\rvert \frac{x^{2k+1}}{2k+1} + o(x^{2n+2})$
$\arctan x$	$= x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + o(x^{2n+2})$	$= \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{2k+1} + o(x^{2n+2})$
$(1+x)^\alpha$	$= 1 + \alpha x + \binom{\alpha}{2}x^2 + \binom{\alpha}{3}x^3 + \cdots + \binom{\alpha}{n}x^n + o(x^n)$	$= \sum_{k=0}^{n} \binom{\alpha}{k}x^k + o(x^n)$
$\frac{1}{1+x}$	$= 1 - x + x^2 - x^3 + x^4 + \cdots + (-1)^n x^n + o(x^n)$	$= \sum_{k=0}^{n} (-1)^k x^k + o(x^n)$
$\frac{1}{1-x}$	$= 1 + x + x^2 + x^3 + x^4 + \cdots + x^n + o(x^n)$	$= \sum_{k=0}^{n} x^k + o(x^n)$
$\sqrt{1+x}$	$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \cdots + \binom{1/2}{n}x^n + o(x^n)$	$= \sum_{k=0}^{n} \binom{1/2}{k}x^k + o(x^n)$
$\frac{1}{\sqrt{1+x}}$	$= 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \cdots + \binom{-1/2}{n}x^n + o(x^n)$	$= \sum_{k=0}^{n} \binom{-1/2}{k}x^k + o(x^n)$
$\sqrt[3]{1+x}$	$= 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3 + \cdots + \binom{1/3}{n}x^n + o(x^n)$	$= \sum_{k=0}^{n} \binom{1/3}{k}x^k + o(x^n)$
$\frac{1}{\sqrt[3]{1+x}}$	$= 1 - \frac{1}{3}x + \frac{2}{9}x^2 - \frac{7}{81}x^3 + \cdots + \binom{-1/3}{n}x^n + o(x^n)$	$= \sum_{k=0}^{n} \binom{-1/3}{k}x^k + o(x^n)$

Si ricordi che $\forall \alpha \in \mathbb{R}$ si pone $\binom{\alpha}{0} = 1$ e $\binom{\alpha}{n} = \frac{\overbrace{\alpha(\alpha-1)\cdots(\alpha-n+1)}^{n \text{ fattori}}}{n!}$ se $n \geq 1$.

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + o\left((x-x_0)^n\right)$$

CORSO DI LAUREA IN INFORMATICA APPLICATA

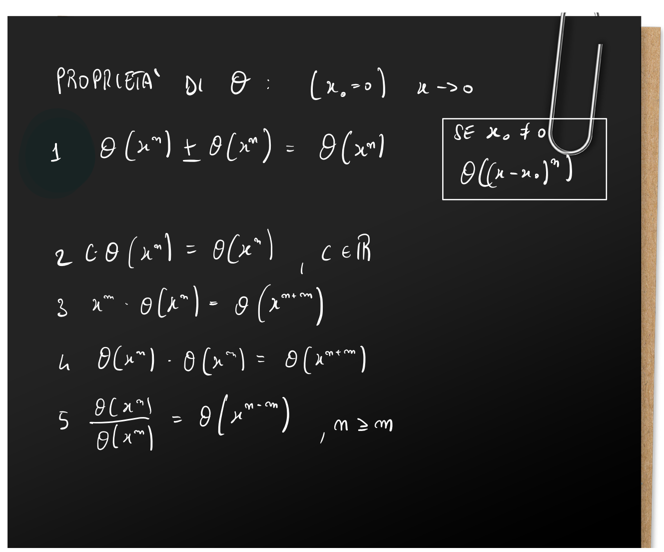
ANALISI MATEMATICA 1

ESERCIZI SU

FORMULA DI TAYLOR E SERIE DI TAYLOR

Esercizio 1.
Scrivere i seguenti polinomi di Taylor (in tutto il seguito indicheremo con $T_n^{x_0}(x)$ il polinomio di Taylor di grado n centrato in x_0):

1. $T_5^0(x)$ di $f(x) = \sin^3(x)$;
2. $T_5^0(x)$ di $f(x) = (e^x - 1)^2$;
3. $T_5^0(x)$ di $f(x) = \dfrac{1}{1+x+x^2}$;
4. $T_7^0(x)$ di $f(x) = \dfrac{x - \sin x}{x^2}$;
5. $T_4^1(x)$ di $f(x) = \dfrac{1}{1+x^2}$;
6. $T_3^2(x)$ di $f(x) = \sqrt{1+x}$;
7. $T_6^0(x)$ di $f(x) = \log(1+x^2)$;
8. $T_8^0(x)$ di $f(x) = x^3 e^{-x^2}$;
9. $T_5^0(x)$ di $f(x) = x \sin^2 x$;
10. $T_3^0(x)$ di $f(x) = \tan(3x)$.



Esercizio 2.
Utilizzando gli sviluppi di Taylor, provare che

$$\lim_{x \to 0} \frac{\log(1+\sin x)}{x} = 1\,.$$

Suggerimento: basta sviluppare sia $\log(1+x)$ *che* $\sin x$ *al primo ordine.*

Esercizio 3.
Utilizzando gli sviluppi di Taylor, provare che

$$\lim_{x \to 0} \frac{(x - \log(1+x))\sin^2(x)}{\arctan(x^4)} = \frac{1}{2}\,.$$

Suggerimento: sviluppare $\log(1+x)$ *al secondo ordine,* $\sin x$ *al primo ordine,* $\arctan x$ *al primo ordine e poi comporlo poi con la funzione* x^4.

$$\lim_{x\to 0}\frac{\log(1+\sin x)}{x} = 1\,.$$

$$\sin x = \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!} + o\left(x^{2n+2}\right) = x + o(x)$$

$$\log(1+x) = \sum_{k=1}^{n} (-1)^{k-1} \cdot \frac{x^k}{k} + o(x^n) = x + o(x)$$

$$\lim_{x\to 0} \frac{x + o(x)}{x} = 1$$

$$\lim_{x\to 0}\frac{(x-\log(1+x))\sin^2(x)}{\arctan(x^4)} = \frac{1}{2}\,.$$

$$\log(1+x) = \sum_{k=1}^{n} (-1)^{k-1} \cdot \frac{x^k}{k} + o(x^n) = x - \frac{x^2}{2} + o(x^2)$$

$$\sin(x) = \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!} + o\left(x^{2n+2}\right) = x + o(x)$$

$$\arctan(x^4) = \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{2k+1} + o\left(x^{2n+2}\right) = x^4 + o(x)$$

$$\frac{\left(x - x + \frac{x^2}{2} + o(x^2)\right)\left(x^2 + o(x)\right)}{x^4 + o(x)} = \underline{\frac{x^4}{2} + o(x)}$$

1. $\lim\limits_{x \to 0} \dfrac{x^2 - \sin^2 x}{x^3(e^x - \cos x)}$;

$$\sum_{k=0}^{\infty} (-1)^k \cdot \frac{x^{2k+1}}{(2k+1)!}$$

Esercizio 4.
Calcolare i seguenti limiti, utilizzando gli sviluppi di Taylor:

1. $\displaystyle\lim_{x\to 0} \frac{x^2 - \sin^2 x}{x^3(e^x - \cos x)}$;
2. $\displaystyle\lim_{x\to +\infty} \log(1+x) \sin\left(\frac{1}{\log(2x-1)}\right)$;
3. $\displaystyle\lim_{x\to 0} \frac{\tan x - x}{(1-\cos x)\sin x}$;
4. $\displaystyle\lim_{x\to 1} \frac{(x+1)^3 \log x}{\sqrt[3]{x+2}\ \sin^2(x-1)}$;
5. $\displaystyle\lim_{x\to 0} \frac{x\sin^2 x - x^3}{5x\log(1+x^4)}$;
6. $\displaystyle\lim_{x\to 0} \frac{e^{-x} - \log(x+1) - (x-1)^2}{x^3}$;
7. $\displaystyle\lim_{x\to 1} \frac{\sin^2(x-1)}{1-\cos(x-1)}$;
8. $\displaystyle\lim_{x\to 0^+} \frac{\sin^2(3x) + \tan x + \log(1+\sqrt{x})}{\arctan x + 1 - \cos x + \sqrt{x}}$;
9. $\displaystyle\lim_{x\to +\infty} x^2\left(\cos\frac{1}{x} - e^{1/x}\right)$;
10. $\displaystyle\lim_{x\to 2^+} \frac{(e^{x-2}-1)(x-2)}{\log(x-1)}$.

Esempio 1.
Data $f(x) = \sin x$, stimare l'errore che si commette valutando nel punto $x = 1$ anziché la funzione f il suo polinomio di Taylor di grado 3.
Svolgimento: l'errore commesso è dato da

$$R(x) = f(x) - T_3^0(x)\,.$$

Usando il resto nella forma di Lagrange si ha

$$R(x) = f(x) - T_3^0(x) = \frac{\cos(c)}{4!}x^4\,,$$

dove $c \in (0,x)$. Allora una stima dell'errore commesso è data da

$$|R(1)| \le \frac{1}{4!}\,.$$

Esempio 2.
Sia $f(x) = \sqrt{x}$. Calcolare $\sqrt{26}$ usando $T_2^{25}(x)$ e stimare l'errore commesso.
Svolgimento: calcolando le derivate di f si ha

$$T_2^{25}(x) = 5 + \frac{1}{10}(x-25) - \frac{1}{1000}(x-25)^2\,.$$

Quindi $\sqrt{26}$ si può stimare con $T_2^{25}(26) = 5,099$.
Scrivendo il resto in forma di Lagrange si ha:

$$R(x) = f(x) - T_2^{25}(x) = \frac{f^{(3)}(c)}{6}(x-25)^3\,,$$

SVILUPPI DI TAYLOR — RESTO DI PEANO

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} + o(x^n)$$

$$\sin x = \sum_{k=0}^{\infty} (-1)^k \cdot \frac{x^{2k+1}}{(2k+1)!} + o(x^{2n+2})$$

$$\log(1+x) = \sum_{k=0}^{\infty} (-1)^{k-1} \frac{x^k}{k} + o(x^n)$$

$$\cos x = \sum_{k=0}^{\infty} (-1)^k \cdot \frac{x^{2k}}{(2k)!} + o(x^{2n+1})$$

SVILUPPI DI TAYLOR — RESTO DI LAGRANGE

$$f(x) = \underbrace{\sum_{k=0}^{n-1} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k}_{\text{valore approssimativo di } f(x)} + \underbrace{\frac{f^{(n)}(\bar{x})}{n!}(x-x_0)^n}_{\text{resto di Lagrange}}$$

SVILUPPI DI TAYLOR — RESTO INTEGRALE

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \underbrace{\int_{x_0}^{x} \frac{(x-t)^n}{n!} f^{(n+1)}(t)\, dt}_{\text{resto integrale}}$$

dove $f^{(3)}(c) = \frac{3}{8}c^{-\frac{5}{2}}$ e $c \in (25,26)$. Allora una stima dell'errore commesso è data da

$$|R(26)| \leq \frac{3}{8 \cdot 25^{\frac{5}{2}} \cdot 6} = 0,00002\,.$$

Esercizio 5.
Valutare l'errore commesso nella formula $e \approx 2 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!}$.

Esercizio 6.
Calcolare $\sqrt[3]{e}$ con un errore inferiore a 10^{-4} .

Esercizio 7.
Calcolare $\frac{\pi}{6}$ con un errore inferiore a $0,001$ (scrivere $\frac{\pi}{6} = \arcsin\left(\frac{1}{2}\right)$) .

Esercizio 8.
Approssimare $\sqrt{2}$ con il polinomio $T_5^1(x)$ della funzione $f(x) = \sqrt{x}$ e stimare l'errore commesso.

Esercizio 9.
Calcolare $\sin 1$ con un errore inferiore a 10^{-5} .

Esempio 3.
Scrivere lo sviluppo in serie di Taylor $S(x)$ centrata in $x_0 = 0$ della funzione $f(x) = \frac{1}{1+2x}$.
Svolgimento: si ha

$$f(x) = \frac{1}{1+2x} = \frac{1}{1-(-2x)}\,.$$

Allora, usando lo sviluppo in serie di Taylor di punto iniziale $x_0 = 0$

$$\frac{1}{1-x} = \sum_{n=0}^{+\infty} x^n\,, \quad x \in (-1,1)$$

e sostituendo in tale sviluppo $-2x$ al posto di x, si ottiene lo sviluppo di f.
Risulta

$$f(x) = \sum_{n=0}^{+\infty} (-1)^n 2^n x^n\,, \quad x \in \left(-\frac{1}{2}, \frac{1}{2}\right)\,.$$

Esempio 4.
Scrivere la serie di Taylor centrata in $x_0 = 2$ della funzione $f(x) = \log x$.
Svolgimento: dopo aver calcolato le prime tre derivate, osservare che per la derivata di ordine n vale

$$f^{(n)}(x) = (-1)^{n-1}\frac{(n-1)!}{x^n}$$

e quindi

$$f^{(n)}(2) = (-1)^{n-1}\frac{(n-1)!}{2^n}\,.$$

Allora si ottiene

$$S(x) = \log 2 + \sum_{n=1}^{+\infty} (-1)^{n-1}\frac{(x-2)^n}{2^n \cdot n}\,.$$

Esempio 5.
Scrivere lo sviluppo in serie di Taylor centrata in $x_0 = 0$ della funzione $f(x) = \log(2+4x)$.
Svolgimento: si ha

$$\log(2+4x) = \log(2 \cdot (1+2x)) = \log 2 + \log(1+2x)\,.$$

Allora, usando lo sviluppo in serie di Taylor di punto iniziale $x_0 = 0$

$$\log(1+x) = \sum_{n=1}^{+\infty} (-1)^{n+1} \frac{x^n}{n}, \quad x \in (-1,1],$$

si ha

$$\log(2+4x) = \log 2 + \sum_{n=1}^{+\infty} (-1)^{n+1} \frac{2^n}{n} x^n, \quad x \in \left(-\frac{1}{2}, \frac{1}{2}\right].$$

Esercizio 10.
Scrivere lo sviluppo in serie di Taylor centrata in x_0 delle seguenti funzioni:

1. $f(x) = \dfrac{1}{\sqrt{1+x}}$ in $x_0 = 0$;
2. $f(x) = \cos(x-1)^2$ in $x_0 = 1$;
3. $f(x) = \dfrac{1}{1+x^2}$ in $x_0 = 0$;
4. $f(x) = \sin(3x)$ in $x_0 = 0$;
5. $f(x) = e^{x^2}$ in $x_0 = 0$;
6. $f(x) = x^2 \sin x$ in $x_0 = 0$;
7. $f(x) = \log\left(\dfrac{1+x}{1-x}\right)$ in $x_0 = 0$;
8. $f(x) = \dfrac{1}{x}$ in $x_0 = 1$;
9. $f(x) = x \cos \dfrac{x}{2}$ in $x_0 = 0$;
10. $f(x) = x^3 e^{-2x}$ in $x_0 = 0$ e altri.

Esempio 6.
Dimostrare che

$$\sum_{n=0}^{\infty} \frac{3^n}{n!} = e^3.$$

Svolgimento: sia $f(x) = e^x$. La Formula di Mac Laurin (Taylor di centro $x_0 = 0$) di f di ordine n è

$$e^x = T_n(x) + R_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!} + \frac{e^c x^{n+1}}{(n+1)!},$$

dove $c \in (0,x)$ (oppure $c \in (x,0)$) opportuno. Supponendo che $x \in (-a,a)$ risulta $f^{(n)}(x) = e^x < e^a$, e dunque per x fissato si ha che

$$\lim_{n\to\infty} R_n(x) = 0.$$

Allora

$$e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!}, \quad x \in (-a,a).$$

Data l'arbitrarietà di a si ha che

$$e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!}\,, \quad x \in \mathbb{R}\,.$$

In particolare, scegliendo $x = 3$ si ottiene

$$\sum_{n=0}^{+\infty} \frac{3^n}{n!} = e^3\,.$$

Esercizio 11.
Dimostrare le seguenti uguaglianze:

1. $\displaystyle\sum_{n=1}^{+\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}$ (usare lo sviluppo in serie di Taylor di $\arctan x$);
2. $\displaystyle\sum_{n=0}^{+\infty} \frac{(-1)^{n+1}}{(2n)!}\pi^{2n} = 1$ (usare lo sviluppo in serie di Taylor di $\cos x$).