

Homework 8.2: Area of a Surface of Revolution

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Introduction, Context, and Prerequisites

1 Core Concepts

The "Area of a Surface of Revolution" is a concept in integral calculus used to calculate the total surface area of a three-dimensional shape formed by rotating a two-dimensional curve around an axis.

Imagine taking a simple curve, like a parabola or a line segment, on a graph. Now, spin that curve around the x -axis or y -axis. The path it sweeps out creates a 3D surface. For example, rotating a semicircle around its diameter creates a sphere. Rotating a line segment that doesn't touch the axis creates a "frustum"—a cone with its tip cut off.

The fundamental idea is to approximate this curved surface with an infinite number of these tiny frustums. We know the formula for the surface area of a single frustum. By treating a tiny segment of our curve (ds) as the slant height of an infinitesimally thin frustum and integrating (summing up) the surface areas of all these frustums along the curve, we can find the exact total surface area.

2 Intuition and Derivation

The formula for the surface area of a frustum is $A = 2\pi rL$, where L is the slant height and r is the average radius of the frustum.

In our calculus model, we consider an infinitesimally small segment of the curve, called the arc length element, ds . When this tiny segment is rotated around an axis, it forms an infinitesimally thin band or ribbon.

- The "slant height" of this band is the arc length element, $L = ds$.
- The "average radius" is the distance from the axis of rotation to the curve segment, r .

If we rotate around the **x -axis**, the radius at any point x is its corresponding y -value. So, $r = y$. If we rotate around the **y -axis**, the radius at any point y is its corresponding x -value. So, $r = x$.

The surface area of one infinitesimal band, dS , is therefore $dS = 2\pi r \cdot ds$. To get the total surface area S , we integrate these infinitesimal areas:

$$S = \int dS = \int 2\pi r \cdot ds$$

The final piece is the arc length element, ds , which we know from the study of arc length is derived from the Pythagorean theorem: $ds^2 = dx^2 + dy^2$. We can express it in two ways:

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \quad \text{or} \quad ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$

Substituting these into our integral gives us the master formulas.

3 Historical Context and Motivation

The problem of finding the surface areas of solids has ancient roots, with Greek mathematicians like Archimedes famously calculating the surface area of a sphere and cylinder over 2000 years ago. However,

his methods were geometric and specific to certain shapes. The development of calculus by Isaac Newton and Gottfried Wilhelm Leibniz in the 17th century provided a universal tool to solve these problems for any continuous curve.

The motivation was both theoretical and practical. In physics, understanding the surface area of objects is crucial for problems involving fluid dynamics (drag), thermodynamics (heat transfer), and optics (designing lenses and mirrors). Engineers and architects need to calculate surface areas to determine material costs for domes, towers, and machine parts. The formulas derived from calculus allowed for precise calculations for complex, custom shapes that were previously impossible to analyze.

4 Key Formulas

Let a smooth curve be defined by $y = f(x)$ on $[a, b]$ or $x = g(y)$ on $[c, d]$.

1. **Rotation about the x-axis:** The radius is $r = y$.

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \quad \text{or} \quad S = \int_c^d 2\pi y \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$

2. **Rotation about the y-axis:** The radius is $r = x$.

$$S = \int_a^b 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \quad \text{or} \quad S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$

5 Prerequisites

- **Algebra:** Strong skills in simplifying complex fractions and expressions involving square roots. Recognizing patterns for factoring, especially perfect squares.
- **Calculus I (Derivatives):** Proficiency with all differentiation rules (power, product, quotient, chain).
- **Calculus I (Integrals):** A solid understanding of the concept of the definite integral as an accumulation.
- **Calculus II (Integration Techniques):** Mastery of u-substitution is essential. Some problems may require more advanced techniques like trigonometric substitution or integration by parts.
- **Arc Length:** The concept and formula for the arc length element, ds , is a direct prerequisite.

Detailed Homework Solutions

6 Problem 1

The curve $y = \sqrt[3]{x}$ for $1 \leq x \leq 8$ is rotated about the x-axis.

6.1 Part (a): Integrate with respect to x

1. **Formula:** For rotation about the x-axis, $S = \int_a^b 2\pi y \sqrt{1 + (y')^2} dx$.
2. **Derivative:** $y = x^{1/3}$, so $\frac{dy}{dx} = \frac{1}{3}x^{-2/3}$.
3. **Setup:** The integral is:

$$S = \int_1^8 2\pi x^{1/3} \sqrt{1 + \left(\frac{1}{3}x^{-2/3}\right)^2} dx$$

$$S = \int_1^8 2\pi x^{1/3} \sqrt{1 + \frac{1}{9}x^{-4/3}} dx$$

Answer for (a): The integrand is $2\pi x^{1/3} \sqrt{1 + \frac{1}{9}x^{-4/3}}$.

6.2 Part (b): Integrate with respect to y

1. **Formula:** $S = \int_c^d 2\pi y \sqrt{1 + (x')^2} dy$.
2. **Rewrite function and bounds:** If $y = x^{1/3}$, then $x = y^3$. The bounds are: if $x = 1, y = 1^{1/3} = 1$; if $x = 8, y = 8^{1/3} = 2$. So $1 \leq y \leq 2$.
3. **Derivative:** $x = y^3$, so $\frac{dx}{dy} = 3y^2$.
4. **Setup:** The integral is:

$$S = \int_1^2 2\pi y \sqrt{1 + (3y^2)^2} dy$$

$$S = \int_1^2 2\pi y \sqrt{1 + 9y^4} dy$$

Answer for (b): The integral is $\int_1^2 2\pi y \sqrt{1 + 9y^4} dy$.

7 Problem 2

The curve $x = y + y^3$ for $0 \leq y \leq 3$.

7.1 Part (a): Set up integrals

1. **Derivative:** We are given x in terms of y , so let's use dy .

$$\frac{dx}{dy} = 1 + 3y^2$$

2. **Arc Length Element:**

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy = \sqrt{1 + (1 + 3y^2)^2} dy = \sqrt{1 + (1 + 6y^2 + 9y^4)} dy = \sqrt{2 + 6y^2 + 9y^4} dy$$

3. (i) **Rotation about the x-axis (Radius $r = y$):**

$$S_x = \int_0^3 2\pi y \sqrt{2 + 6y^2 + 9y^4} dy$$

4. (ii) **Rotation about the y-axis (Radius $r = x$):**

$$S_y = \int_0^3 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy = \int_0^3 2\pi (y + y^3) \sqrt{2 + 6y^2 + 9y^4} dy$$

7.2 Part (b): Evaluate using a calculator

Using a numerical integrator:

1. (i) **x-axis:** $S_x = \int_0^3 2\pi y \sqrt{2 + 6y^2 + 9y^4} dy \approx 814.7388$
2. (ii) **y-axis:** $S_y = \int_0^3 2\pi (y + y^3) \sqrt{2 + 6y^2 + 9y^4} dy \approx 8168.6416$

8 Problem 4

Find the exact area of the surface obtained by rotating $y = \sqrt{1 + e^x}$ for $0 \leq x \leq 1$ about the x-axis.

1. **Formula:** Rotate about x-axis, so $S = \int 2\pi y \sqrt{1 + (y')^2} dx$.

2. **Derivative:** $y = (1 + e^x)^{1/2}$

$$\frac{dy}{dx} = \frac{1}{2}(1 + e^x)^{-1/2} \cdot e^x = \frac{e^x}{2\sqrt{1 + e^x}}$$

3. **Simplify $1 + (y')^2$:**

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{e^{2x}}{4(1 + e^x)} = \frac{4(1 + e^x) + e^{2x}}{4(1 + e^x)} = \frac{4 + 4e^x + e^{2x}}{4(1 + e^x)}$$

The numerator is a perfect square: $e^{2x} + 4e^x + 4 = (e^x + 2)^2$.

$$1 + \left(\frac{dy}{dx}\right)^2 = \frac{(e^x + 2)^2}{4(1 + e^x)}$$

4. **Simplify the square root term:**

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{\frac{(e^x + 2)^2}{4(1 + e^x)}} = \frac{e^x + 2}{2\sqrt{1 + e^x}}$$

5. **Set up the integral:**

$$S = \int_0^1 2\pi y \left(\frac{e^x + 2}{2\sqrt{1 + e^x}}\right) dx = \int_0^1 2\pi(\sqrt{1 + e^x}) \left(\frac{e^x + 2}{2\sqrt{1 + e^x}}\right) dx$$

The $\sqrt{1 + e^x}$ terms cancel, as does the 2.

$$S = \int_0^1 \pi(e^x + 2) dx$$

6. **Evaluate:**

$$S = \pi [e^x + 2x]_0^1 = \pi ((e^1 + 2(1)) - (e^0 + 2(0))) = \pi(e + 2 - 1) = \pi(e + 1)$$

Final Answer: $\pi(e + 1)$.

9 Problem 6

Set up the integral for $y = x^4$, $0 \leq x \leq 1$.

1. **Derivative and Arc Length Element:**

$$\frac{dy}{dx} = 4x^3 \implies ds = \sqrt{1 + (4x^3)^2} dx = \sqrt{1 + 16x^6} dx$$

2. (a) **about the x-axis (r=y):**

$$S_x = \int_0^1 2\pi y ds = \int_0^1 2\pi x^4 \sqrt{1 + 16x^6} dx$$

3. (b) **about the y-axis (r=x):**

$$S_y = \int_0^1 2\pi x ds = \int_0^1 2\pi x \sqrt{1 + 16x^6} dx$$

10 Problem 9

Find the exact area of the surface obtained by rotating $y = \sqrt{8-x}$ for $2 \leq x \leq 8$ about the x-axis.

1. **Formula:** Rotate about x-axis, so $S = \int 2\pi y \sqrt{1 + (y')^2} dx$.

2. **Derivative:** $y = (8-x)^{1/2}$

$$\frac{dy}{dx} = \frac{1}{2}(8-x)^{-1/2} \cdot (-1) = \frac{-1}{2\sqrt{8-x}}$$

3. **Simplify** $1 + (y')^2$:

$$1 + \left(\frac{-1}{2\sqrt{8-x}} \right)^2 = 1 + \frac{1}{4(8-x)} = \frac{4(8-x) + 1}{4(8-x)} = \frac{32 - 4x + 1}{4(8-x)} = \frac{33 - 4x}{4(8-x)}$$

4. **Set up the integral:**

$$S = \int_2^8 2\pi y \sqrt{\frac{33-4x}{4(8-x)}} dx = \int_2^8 2\pi \sqrt{8-x} \frac{\sqrt{33-4x}}{2\sqrt{8-x}} dx$$

The $\sqrt{8-x}$ and 2 terms cancel.

$$S = \int_2^8 \pi \sqrt{33-4x} dx$$

5. **Evaluate (u-substitution):** Let $u = 33 - 4x$, so $du = -4dx$, or $dx = -du/4$. Change bounds:
 $x = 2 \implies u = 33 - 8 = 25$. $x = 8 \implies u = 33 - 32 = 1$.

$$S = \int_{25}^1 \pi \sqrt{u} \left(-\frac{du}{4} \right) = -\frac{\pi}{4} \int_{25}^1 u^{1/2} du = \frac{\pi}{4} \int_1^{25} u^{1/2} du$$

$$S = \frac{\pi}{4} \left[\frac{2}{3} u^{3/2} \right]_1^{25} = \frac{\pi}{6} [25^{3/2} - 1^{3/2}] = \frac{\pi}{6} [125 - 1] = \frac{124\pi}{6} = \frac{62\pi}{3}$$

Final Answer: $\frac{62\pi}{3}$.

11 Problem 10

Find the exact area of the surface for $y = \frac{1}{4}x^2 - \frac{1}{2}\ln(x)$, $4 \leq x \leq 5$, rotated about the y-axis.

1. **Formula:** Rotate about y-axis, so $S = \int 2\pi x \sqrt{1 + (y')^2} dx$.

2. **Derivative:**

$$\frac{dy}{dx} = \frac{1}{4}(2x) - \frac{1}{2} \left(\frac{1}{x} \right) = \frac{x}{2} - \frac{1}{2x}$$

3. **Simplify** $1 + (y')^2$ (**Perfect Square Trick**):

$$\begin{aligned} 1 + \left(\frac{x}{2} - \frac{1}{2x} \right)^2 &= 1 + \left(\frac{x^2}{4} - 2 \left(\frac{x}{2} \right) \left(\frac{1}{2x} \right) + \frac{1}{4x^2} \right) \\ &= 1 + \left(\frac{x^2}{4} - \frac{1}{2} + \frac{1}{4x^2} \right) = \frac{x^2}{4} + \frac{1}{2} + \frac{1}{4x^2} \end{aligned}$$

This is the expansion of $\left(\frac{x}{2} + \frac{1}{2x} \right)^2$.

$$1 + \left(\frac{dy}{dx} \right)^2 = \left(\frac{x}{2} + \frac{1}{2x} \right)^2$$

4. **Set up the integral:**

$$S = \int_4^5 2\pi x \sqrt{\left(\frac{x}{2} + \frac{1}{2x}\right)^2} dx = \int_4^5 2\pi x \left(\frac{x}{2} + \frac{1}{2x}\right) dx$$

Distribute the $2\pi x$:

$$S = \int_4^5 \left(2\pi x \cdot \frac{x}{2} + 2\pi x \cdot \frac{1}{2x}\right) dx = \int_4^5 (\pi x^2 + \pi) dx = \pi \int_4^5 (x^2 + 1) dx$$

5. **Evaluate:**

$$\begin{aligned} S &= \pi \left[\frac{x^3}{3} + x \right]_4^5 = \pi \left[\left(\frac{5^3}{3} + 5 \right) - \left(\frac{4^3}{3} + 4 \right) \right] \\ &= \pi \left[\left(\frac{125}{3} + \frac{15}{3} \right) - \left(\frac{64}{3} + \frac{12}{3} \right) \right] = \pi \left[\frac{140}{3} - \frac{76}{3} \right] = \frac{64\pi}{3} \end{aligned}$$

Final Answer: $\frac{64\pi}{3}$.

12 Problem 11 (Gabriel's Horn)

Determine the surface area for $y = 1/x$, $x \geq 1$, rotated about the x-axis.

1. **Formula and Derivative:** $S = \int_1^\infty 2\pi y \sqrt{1 + (y')^2} dx$. $y = x^{-1}$, so $\frac{dy}{dx} = -x^{-2} = -\frac{1}{x^2}$.

2. **Set up the integral:**

$$\begin{aligned} S &= \int_1^\infty 2\pi \left(\frac{1}{x}\right) \sqrt{1 + \left(-\frac{1}{x^2}\right)^2} dx = \int_1^\infty \frac{2\pi}{x} \sqrt{1 + \frac{1}{x^4}} dx \\ S &= \int_1^\infty \frac{2\pi}{x} \sqrt{\frac{x^4 + 1}{x^4}} dx = \int_1^\infty \frac{2\pi}{x} \frac{\sqrt{x^4 + 1}}{x^2} dx = 2\pi \int_1^\infty \frac{\sqrt{x^4 + 1}}{x^3} dx \end{aligned}$$

3. **Test for Convergence:** We use the Limit Comparison Test. Let our integrand be $f(x) = \frac{\sqrt{x^4 + 1}}{x^3}$. Let's compare it to a simpler function $g(x)$. For large x , $x^4 + 1 \approx x^4$, so $\sqrt{x^4 + 1} \approx \sqrt{x^4} = x^2$. Therefore, $f(x) \approx \frac{x^2}{x^3} = \frac{1}{x}$. Let's use $g(x) = \frac{1}{x}$.

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} &= \lim_{x \rightarrow \infty} \frac{\sqrt{x^4 + 1}/x^3}{1/x} = \lim_{x \rightarrow \infty} \frac{x\sqrt{x^4 + 1}}{x^3} = \lim_{x \rightarrow \infty} \frac{\sqrt{x^4 + 1}}{x^2} \\ &= \lim_{x \rightarrow \infty} \sqrt{\frac{x^4 + 1}{x^4}} = \lim_{x \rightarrow \infty} \sqrt{1 + \frac{1}{x^4}} = \sqrt{1 + 0} = 1 \end{aligned}$$

Since the limit is a finite positive number (1), our integral $\int f(x) dx$ behaves the same as $\int g(x) dx$. We know that $\int_1^\infty \frac{1}{x} dx$ (the p-integral with $p = 1$) diverges. Therefore, the integral for the surface area also diverges.

Final Answer: The surface area is infinite.

13 Problem 14

For $y = x^3$, $0 \leq x \leq 3$, rotated about the x-axis. Set up and find the exact area.

13.1 Part (a): Set up integral

1. **Derivative:** $\frac{dy}{dx} = 3x^2$.

2. **Arc Length Element:** $ds = \sqrt{1 + (3x^2)^2} dx = \sqrt{1 + 9x^4} dx$.

3. **Setup:** Rotating about the x-axis means $r = y$.

$$S = \int_0^3 2\pi y ds = \int_0^3 2\pi x^3 \sqrt{1 + 9x^4} dx$$

13.2 Part (b): Find exact area

1. **Integration (u-substitution):** Let $u = 1 + 9x^4$. Then $du = 36x^3 dx$, which means $x^3 dx = \frac{du}{36}$.
2. **Change bounds:** If $x = 0$, $u = 1 + 9(0)^4 = 1$. If $x = 3$, $u = 1 + 9(3)^4 = 1 + 9(81) = 1 + 729 = 730$.
3. **Evaluate:**

$$\begin{aligned} S &= \int_1^{730} 2\pi\sqrt{u} \left(\frac{du}{36} \right) = \frac{2\pi}{36} \int_1^{730} u^{1/2} du = \frac{\pi}{18} \int_1^{730} u^{1/2} du \\ &= \frac{\pi}{18} \left[\frac{2}{3} u^{3/2} \right]_1^{730} = \frac{\pi}{27} \left[u^{3/2} \right]_1^{730} = \frac{\pi}{27} (730^{3/2} - 1^{3/2}) \end{aligned}$$

$$730^{3/2} = (\sqrt{730})^3 = 730\sqrt{730}.$$

Final Answer: $\frac{\pi}{27}(730\sqrt{730} - 1)$.

14 Problem 15

This is the same as problem 14, but with bounds $0 \leq x \leq 2$. Using the same setup and u-substitution:

1. **Integral:** $S = \int_0^2 2\pi x^3 \sqrt{1 + 9x^4} dx$.
2. **Change bounds:** If $x = 0$, $u = 1$. If $x = 2$, $u = 1 + 9(2)^4 = 1 + 9(16) = 1 + 144 = 145$.
3. **Evaluate:** Using the antiderivative from problem 14:

$$S = \frac{\pi}{27} \left[u^{3/2} \right]_1^{145} = \frac{\pi}{27} (145^{3/2} - 1^{3/2}) = \frac{\pi}{27} (145\sqrt{145} - 1)$$

Final Answer: $\frac{\pi}{27}(145\sqrt{145} - 1)$.

In-Depth Analysis of Problems and Techniques

15 Problem Types and General Approach

- **Setup Only Problems (1, 2, 6, 7, 8, 12, 13):** These problems test the fundamental understanding of the formulas. The approach is to: 1. Identify the axis of rotation to determine the radius ($r = y$ for x-axis, $r = x$ for y-axis). 2. Choose the variable of integration (dx or dy). 3. Compute the correct derivative (dy/dx or dx/dy). 4. Substitute all components into the correct integral formula.
- **Direct Integration via U-Substitution (9, 14, 15):** These problems result in an integral that is solvable with a standard u-substitution. The general approach is to complete the setup, then identify a composition of functions (e.g., a square root of a polynomial) where the derivative of the "inside" function is present as a factor outside.
- **Radical Cancellation Problems (4, 9):** A special type of solvable problem where the function itself (y) is a square root. After simplifying the ds term, a $\sqrt{\dots}$ in the denominator of ds cancels with the $y = \sqrt{\dots}$ term representing the radius, leaving a much simpler integral.
- **The "Perfect Square Trick" Problems (10):** The hallmark of this type is a function of the form $y = ax^n \pm b/x^n$. The algebraic expression $1 + (dy/dx)^2$ miraculously simplifies into a perfect square, which eliminates the square root from the integral entirely. The key is to recognize the algebraic pattern $1 + (A - B)^2 = (A + B)^2$.
- **Improper Integral Problems (11):** Characterized by an infinite bound of integration. The approach is to set up the definite integral, then analyze its long-term behavior using convergence tests (like the Limit Comparison Test) to determine if the area is finite or infinite.

16 Key Algebraic and Calculus Manipulations

- **Finding a Common Denominator (Problems 4, 9):** This was the critical first step inside the square root for simplifying $1 + (y')^2$. For example, in Problem 9, $1 + \frac{1}{4(8-x)}$ becomes $\frac{4(8-x)+1}{4(8-x)}$.
- **Recognizing Perfect Squares (Problem 4):** After finding a common denominator in Problem 4, the numerator became $4 + 4e^x + e^{2x}$, which had to be recognized as $(2 + e^x)^2$ to simplify the square root.
- **The Perfect Square Trick (Problem 10):** This is the most important algebraic trick in this section. The expression for $1 + (dy/dx)^2$ was $1 + (\frac{x^2}{4} - \frac{1}{2} + \frac{1}{4x^2})$. The crucial step was to combine the constants $1 - \frac{1}{2} = +\frac{1}{2}$, changing the expression to $\frac{x^2}{4} + \frac{1}{2} + \frac{1}{4x^2}$, which is the expansion of $(\frac{x}{2} + \frac{1}{2x})^2$. This trick eliminated the square root, making an impossible integral trivial.
- **Strategic U-Substitution (Problems 14, 15):** The integral was $\int 2\pi x^3 \sqrt{1 + 9x^4} dx$. The key was to choose $u = 1 + 9x^4$, because its derivative, $du = 36x^3 dx$, contains the x^3 factor needed to substitute away all x terms.
- **Limit Comparison Test (Problem 11):** To handle the improper integral, we compared the complex integrand $\frac{\sqrt{x^4+1}}{x^3}$ to the simpler, dominant function $\frac{1}{x}$ to prove divergence. This avoids trying to solve an unsolvable integral.

”Cheatsheet” and Tips for Success

17 Summary of Formulas

- Rotation about **x-axis** (radius $r = y$): $S = \int 2\pi y \, ds$
- Rotation about **y-axis** (radius $r = x$): $S = \int 2\pi x \, ds$
- Arc Length Element ds : $ds = \sqrt{1 + (y')^2} dx$ or $ds = \sqrt{1 + (x')^2} dy$

18 Tricks and Shortcuts

- **Perfect Square Alert:** If you see a function like $y = \frac{x^n}{A} + \frac{1}{Bx^{n-2}}$, immediately expect the ”perfect square trick” to eliminate the square root in ds .
- **Radical Cancellation:** If y itself is a square root, there’s a good chance it will cancel with a term in the denominator of ds after simplification.
- **Variable Choice:** If $y = f(x)$ is complex to solve for x , stick with dx . If $x = g(y)$ is complex to solve for y , stick with dy . Choose the path of least algebraic resistance.

19 Common Pitfalls and Mistakes

- **Wrong Radius:** The most common mistake. Remember: rotation about **x-axis** \implies **radius is y**. Rotation about **y-axis** \implies **radius is x**.
- **Algebra Errors:** Squaring the derivative incorrectly or making a mistake when finding a common denominator under the radical. Be meticulous.
- **Forgetting to Change Bounds:** When performing a u-substitution, you **MUST** calculate the new bounds of integration in terms of u .
- **Dropping the 2π :** Forgetting to include the constant 2π in the setup and final answer.

Conceptual Synthesis and The "Big Picture"

20 Thematic Connections

The central theme of this topic, like many in calculus, is the **accumulation of infinitesimal pieces**. We are applying the fundamental principle of the integral to a new geometric context. This exact same theme is the foundation for:

- **Area Under a Curve:** Summing the areas of infinite, infinitesimally thin rectangles.
- **Volumes of Revolution:** Summing the volumes of infinite, infinitesimally thin disks, washers, or cylindrical shells.
- **Arc Length:** Summing the lengths of infinite, infinitesimally small straight line segments (ds).

Calculating the area of a surface of revolution is just one more application of this powerful idea: we chop a complex problem into an infinite number of simple pieces (frustums) and use the integral to sum them up.

21 Forward and Backward Links

- **Backward Links:** This topic is a direct and logical extension of **Arc Length**. The ds component in the surface area formula is precisely the arc length element. Without a firm grasp of how to derive and calculate ds , surface area problems are impossible. It also relies heavily on all prior differentiation and integration techniques.
- **Forward Links:** This topic is a crucial stepping stone to **Surface Integrals** in Multivariable Calculus (Calc III). In this chapter, we find the area *of* a surface. In Calc III, we will learn to integrate a function *over* a surface. This allows us to calculate concepts like the flux of a vector field (e.g., the rate of fluid flowing through a membrane) or the center of mass of a thin shell. Understanding how to set up an integral for the area of a surface of revolution provides the foundational intuition for these more advanced topics.

Real-World Application and Modeling

22 Concrete Scenarios

1. **Aerospace Engineering:** An engineer designing the nose cone of a supersonic aircraft needs to calculate its surface area. The nose cone is formed by rotating a curve (like a parabola or a custom spline) around the x-axis. This calculation is vital for determining the effects of aerodynamic heating and for estimating the amount of heat-shielding material required.
2. **Manufacturing:** A company manufactures custom glass lenses by rotating a curve $y = f(x)$ around the x-axis. To apply a special anti-reflective coating, they need to know the exact surface area of the lens to calculate the precise amount of coating material needed per lens, which directly impacts production cost and quality control.
3. **Civil Engineering:** The iconic shape of a nuclear power plant cooling tower is a hyperboloid, formed by rotating a hyperbola around its central axis (the y-axis). Civil engineers must calculate the surface area to estimate the amount of reinforced concrete needed for construction and to analyze wind loads on the structure.

23 Model Problem Setup

Let's model the engineering problem of the cooling tower.

- **Problem:** An engineer needs to find the surface area of a cooling tower whose shape is modeled by rotating the hyperbola $x = 40\sqrt{1 + \frac{y^2}{75^2}}$ from $y = -50$ m to $y = 100$ m about the y-axis.
- **Variables:** x is the radius of the tower at height y .
- **Function:** $x(y) = 40\sqrt{1 + \frac{y^2}{5625}}$ on the interval $[-50, 100]$.
- **Integral Setup:** The surface area S is given by the integral for rotation about the y-axis:

$$S = \int_{-50}^{100} 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$

First, we would need to calculate the derivative $\frac{dx}{dy}$:

$$\frac{dx}{dy} = 40 \cdot \frac{1}{2\sqrt{1 + y^2/5625}} \cdot \frac{2y}{5625} = \frac{40y}{5625\sqrt{1 + y^2/5625}}$$

Substituting everything into the integral gives the formula that must be solved (likely numerically) to find the total surface area of concrete required:

$$S = \int_{-50}^{100} 2\pi \left(40\sqrt{1 + \frac{y^2}{5625}} \right) \sqrt{1 + \left(\frac{40y}{5625\sqrt{1 + y^2/5625}} \right)^2} dy$$

Common Variations and Untested Concepts

The provided homework was thorough but did not include problems involving curves defined parametrically or rotations about arbitrary lines.

24 Surfaces from Parametric Curves

If a curve is defined by $x = f(t)$ and $y = g(t)$ for $\alpha \leq t \leq \beta$, the arc length element is $ds = \sqrt{(dx/dt)^2 + (dy/dt)^2} dt$. The formulas become:

- **About x-axis:** $S = \int_{\alpha}^{\beta} 2\pi y(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$
- **About y-axis:** $S = \int_{\alpha}^{\beta} 2\pi x(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$

24.0.1 Worked Example: Surface Area of a Sphere

Find the surface area generated by rotating the semicircle defined by $x = \cos(t)$, $y = \sin(t)$ for $0 \leq t \leq \pi$ about the x-axis.

1. **Derivatives:** $\frac{dx}{dt} = -\sin(t)$, $\frac{dy}{dt} = \cos(t)$.

2. **Arc Length Element:**

$$ds = \sqrt{(-\sin t)^2 + (\cos t)^2} dt = \sqrt{\sin^2 t + \cos^2 t} dt = \sqrt{1} dt = dt$$

3. **Integral Setup (about x-axis, $r = y(t)$):**

$$S = \int_0^{\pi} 2\pi y(t) ds = \int_0^{\pi} 2\pi \sin(t) dt$$

4. **Evaluate:**

$$S = 2\pi[-\cos(t)]_0^{\pi} = -2\pi[\cos(\pi) - \cos(0)] = -2\pi[-1 - 1] = 4\pi$$

This correctly gives the surface area of a unit sphere.

25 Rotation About Arbitrary Lines

If we rotate a curve around a horizontal line $y = k$ or a vertical line $x = h$, the ds element remains the same, but the radius r changes. The radius is the perpendicular distance from the curve to the line of rotation.

- For rotation about $y = k$, the radius is $r = |y - k|$.
- For rotation about $x = h$, the radius is $r = |x - h|$.

25.0.1 Worked Example: Rotation About $y = -2$

Set up the integral for the surface area generated by rotating $y = x^3$ for $0 \leq x \leq 1$ about the line $y = -2$.

1. **Radius:** The curve is above the line $y = -2$. The distance is $r = y - (-2) = y + 2 = x^3 + 2$.
2. **Arc Length Element:** From Problem 14, we know $ds = \sqrt{1 + 9x^4} dx$.
3. **Integral Setup:**

$$S = \int_0^1 2\pi r \, ds = \int_0^1 2\pi(x^3 + 2)\sqrt{1 + 9x^4} dx$$

Advanced Diagnostic Testing: "Find the Flaw"

For each problem below, a complete solution is provided. However, each solution contains one subtle but critical error. Your task is to locate the flaw, explain it in one sentence, and then provide the correct step and final answer.

26 Problem 1

Find the area of the surface generated by rotating $y = x^2$ for $0 \leq x \leq 2$ about the **y-axis**.

26.1 Flawed Solution

1. The formula for rotation about the y-axis is $S = \int 2\pi y \, ds$.
2. Derivative: $\frac{dy}{dx} = 2x$.
3. Arc length: $ds = \sqrt{1 + (2x)^2} dx = \sqrt{1 + 4x^2} dx$.
4. Setup: $S = \int_0^2 2\pi y \, ds = \int_0^2 2\pi x^2 \sqrt{1 + 4x^2} dx$.
5. Let $u = 1 + 4x^2$, then $du = 8x dx$. This integral is complex.

Find the Flaw

- **The Flaw Is:** The radius for rotation about the y-axis is x , but the formula in step 1 incorrectly uses y .
- **Correction:** The formula should be $S = \int_0^2 2\pi x \, ds = \int_0^2 2\pi x \sqrt{1 + 4x^2} dx$. Let $u = 1 + 4x^2$, $du = 8x dx \implies x dx = du/8$. Bounds become $u(0) = 1$, $u(2) = 17$.

$$S = \int_1^{17} 2\pi \sqrt{u} \frac{du}{8} = \frac{\pi}{4} \int_1^{17} u^{1/2} du = \frac{\pi}{4} \left[\frac{2}{3} u^{3/2} \right]_1^{17} = \frac{\pi}{6} (17\sqrt{17} - 1)$$

27 Problem 2

Find the area of the surface generated by rotating $y = \frac{x^3}{6} + \frac{1}{2x}$ for $1 \leq x \leq 2$ about the x-axis.

27.1 Flawed Solution

1. Derivative: $\frac{dy}{dx} = \frac{3x^2}{6} - \frac{1}{2x^2} = \frac{x^2}{2} - \frac{1}{2x^2}$.
2. Simplify $1 + (y')^2$: $1 + \left(\frac{x^2}{2} - \frac{1}{2x^2}\right)^2 = 1 + \left(\frac{x^4}{4} - \frac{1}{2} + \frac{1}{4x^4}\right) = \frac{x^4}{4} - \frac{1}{2} + \frac{1}{4x^4}$.
3. This doesn't simplify nicely, so the integral will be $\int_1^2 2\pi\left(\frac{x^3}{6} + \frac{1}{2x}\right)\sqrt{\frac{x^4}{4} - \frac{1}{2} + \frac{1}{4x^4}} dx$.

Find the Flaw

- **The Flaw Is:** In step 2, the leading '1' was incorrectly combined with the '-1/2' inside the parenthesis instead of after the expansion, and should have been $1 - 1/2 = +1/2$.
- **Correction:** The correct simplification is $1 + \left(\frac{x^4}{4} - \frac{1}{2} + \frac{1}{4x^4}\right) = \frac{x^4}{4} + \frac{1}{2} + \frac{1}{4x^4} = \left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2$. The square root is now eliminated.

$$S = \int_1^2 2\pi \left(\frac{x^3}{6} + \frac{1}{2x} \right) \left(\frac{x^2}{2} + \frac{1}{2x^2} \right) dx$$

This is now a polynomial integral which can be solved by expanding the terms.

28 Problem 3

Find the area of the surface generated by rotating $y = e^x$ for $0 \leq x \leq 1$ about the x-axis. The integral is $S = \int_0^1 2\pi e^x \sqrt{1 + e^{2x}} dx$. Use u-substitution $u = e^x$.

28.1 Flawed Solution

1. Let $u = e^x$, then $du = e^x dx$.
2. Substitute into the integral: $S = \int_0^1 2\pi \sqrt{1 + u^2} du$.
3. The antiderivative of $\sqrt{1 + u^2}$ is $\frac{u}{2}\sqrt{1 + u^2} + \frac{1}{2} \ln |u + \sqrt{1 + u^2}|$.
4. Evaluate from 0 to 1: $S = 2\pi \left[\frac{u}{2}\sqrt{1 + u^2} + \frac{1}{2} \ln |u + \sqrt{1 + u^2}| \right]_0^1$.
5. $S = 2\pi \left[\left(\frac{1}{2}\sqrt{2} + \frac{1}{2} \ln(1 + \sqrt{2}) \right) - \left(0 + \frac{1}{2} \ln(1) \right) \right] = \pi(\sqrt{2} + \ln(1 + \sqrt{2}))$.

Find the Flaw

- **The Flaw Is:** The bounds of integration were not changed after performing the u-substitution.
- **Correction:** The bounds should change to $u(0) = e^0 = 1$ and $u(1) = e^1 = e$. The integral must be evaluated from 1 to e.

$$S = 2\pi \left[\frac{u}{2}\sqrt{1 + u^2} + \frac{1}{2} \ln |u + \sqrt{1 + u^2}| \right]_1^e$$

$$S = \pi \left[(e\sqrt{1 + e^2} + \ln(e + \sqrt{1 + e^2})) - (\sqrt{2} + \ln(1 + \sqrt{2})) \right]$$

29 Problem 4

Set up an integral for the area of the surface generated by rotating $x = y^2$ for $0 \leq y \leq 2$ about the y-axis.

29.1 Flawed Solution

1. Rotation about y-axis, integrate with respect to y . $r = x$. $S = \int_0^2 2\pi x \, ds$.
2. We need ds in terms of y . We have $y = \sqrt{x}$ so $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$.
3. $ds = \sqrt{1 + (\frac{dy}{dx})^2} dy = \sqrt{1 + \frac{1}{4x}} dy = \sqrt{1 + \frac{1}{4y^2}} dy$.
4. Setup: $S = \int_0^2 2\pi y^2 \sqrt{1 + \frac{1}{4y^2}} dy$.

Find the Flaw

- **The Flaw Is:** The formula for ds in step 3 incorrectly mixed differentials, using dy with a derivative taken with respect to x .
- **Correction:** Since we are integrating with respect to y , we must use $ds = \sqrt{1 + (dx/dy)^2} dy$. From $x = y^2$, we have $dx/dy = 2y$.

$$ds = \sqrt{1 + (2y)^2} dy = \sqrt{1 + 4y^2} dy$$
$$S = \int_0^2 2\pi x \, ds = \int_0^2 2\pi y^2 \sqrt{1 + 4y^2} dy$$

30 Problem 5

Find the area of the surface generated by rotating $y = \cos(x)$ for $0 \leq x \leq \pi/2$ about the x-axis.

30.1 Flawed Solution

1. Rotation about x-axis, $r = y$. $S = \int_0^{\pi/2} 2\pi y \, ds$.
2. Derivative: $\frac{dy}{dx} = \cos(x)$.
3. Arc Length: $ds = \sqrt{1 + \cos^2(x)} dx$.
4. Setup: $S = \int_0^{\pi/2} 2\pi \cos(x) \sqrt{1 + \cos^2(x)} dx$.
5. Let $u = \cos(x)$, then $du = -\sin(x) dx$. This doesn't seem to work.

Find the Flaw

- **The Flaw Is:** The derivative of $\cos(x)$ in step 2 is $-\sin(x)$, not $\cos(x)$.
- **Correction:** The derivative should be $\frac{dy}{dx} = -\sin(x)$.

$$ds = \sqrt{1 + (-\sin x)^2} dx = \sqrt{1 + \sin^2 x} dx$$

The correct integral setup is:

$$S = \int_0^{\pi/2} 2\pi \cos(x) \sqrt{1 + \sin^2 x} dx$$

This can now be solved with u-substitution where $u = \sin x$, $du = \cos x dx$. Bounds become $u(0) = 0$, $u(\pi/2) = 1$.

$$S = \int_0^1 2\pi \sqrt{1 + u^2} du$$