Test 1 Revision

Tashfeen Omran

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1 Homework 5.5 The Substitution Rule

Question 1

Evaluate the integral by making the given substitution. $\int \cos(2x)\,dx$, $u = 2x$.

Answer

$\frac{1}{2}\sin(2x) + C$

Question 2

Evaluate the integral by making the given substitution. $\int xe^{-x^2}\,dx$, $u = -x^2$.

Answer

$-\frac{1}{2}e^{-x^2} + C$

Question 3

Evaluate the integral by making the given substitution. $\int x^2\sqrt{x^3+9}\,dx$, $u = x^3 + 9$.

Answer

$\frac{2}{9}(x^3+9)^{3/2} + C$

Question 4

Evaluate the integral by making the given substitution. $\int \sin^4(\theta)\cos(\theta)\,d\theta$, $u = \sin(\theta)$.

Answer

$\frac{1}{5}\sin^5(\theta) + C$

Question 5

Evaluate the integral by making the given substitution. $\int \frac{x^4}{x^5-3}\,dx$, $u = x^5 - 3$.

Answer

$\frac{1}{5}\ln|x^5-3| + C$

Question 6

Evaluate the integral by making the given substitution. $\int \frac{\cos(\sqrt{t})}{\sqrt{t}}\,dt$, $u = \sqrt{t}$.

Answer

$2\sin(\sqrt{t}) + C$

Question 7

Evaluate the indefinite integral. $\int t^4 e^{-t^5}\,dt$.

Answer

$-\frac{1}{5}e^{-t^5} + C$

Question 8

Evaluate the indefinite integral. $\int \sin(t)\sqrt{1+\cos(t)}\,dt$.

Answer

$-\frac{2}{3}(1+\cos(t))^{3/2} + C$

Question 9

Evaluate the indefinite integral. $\int y^2(5-y^3)^{2/3}\,dy$.

Answer

$-\frac{1}{5}(5-y^3)^{5/3} + C$

Question 10

Evaluate the indefinite integral. $\int \frac{\sin(\frac{1}{x^3})}{x^4}\,dx$.

Answer

$\frac{1}{3}\cos(\frac{1}{x^3}) + C$

Question 11

Evaluate the indefinite integral. $\int \frac{(\ln(x))^{28}}{x}\,dx$.

Answer

$\frac{1}{29}(\ln(x))^{29} + C$

Question 12

Evaluate the indefinite integral. $\int \sin(22x)\sin(\cos(22x))\,dx$.

Answer

$\frac{1}{22}\cos(\cos(22x)) + C$

Question 13

Evaluate the indefinite integral. $\int \sec^2(\theta)\tan^9(\theta)\,d\theta$.

Answer

$\frac{1}{10}\tan^{10}(\theta) + C$

Question 14

Evaluate the indefinite integral. $\int x\sqrt{x+5}\,dx$.

Answer

$\frac{2}{5}(x+5)^{5/2} - \frac{10}{3}(x+5)^{3/2} + C$

Question 15

Evaluate the indefinite integral. $\int \sqrt[6]{\cot(x)}\csc^2(x)\,dx$.

Answer

$-\frac{6}{7}(\cot(x))^{7/6} + C$

Question 16

Evaluate the indefinite integral. $\int \frac{\sin(2x)}{45+\cos^2(x)}\,dx$.

Answer

$-\ln(45 + \cos^2(x)) + C$

2 Homework 6.1 Area Between Curves

Question 1

Set up an integral for the area of the shaded region. Evaluate the integral to find the area of the shaded region bounded by $y = 5x - x^2$ and $y = x$.

Answer

$\frac{32}{3}$

Question 2

Set up an integral for the area of the shaded region. Evaluate the integral to find the area of the shaded region bounded by $y = e^x$ and $y = x^6$ from $x = 0$ to $x = 1$.

Answer

$e - \frac{8}{7}$

Question 3

Set up an integral for the area of the shaded region. Evaluate the integral to find the area of the shaded region bounded by $x = y^2 - 5$ and $x = e^y$ from $y = -1$ to $y = 1$.

Answer

$e - \frac{1}{e} + \frac{28}{3}$

Question 4

Set up an integral for the area of the shaded region. Evaluate the integral to find the area of the shaded region bounded by $x = y^2 - 4y$ and $x = 2y - y^2$.

Answer

9

Question 5

Find the area of the shaded region bounded by $y = x^3 - 15x$ and $y = x$.

Answer

128

Question 6

Find the area of the shaded region bounded by $y = x^2$, $y = 8 - 2x$, and $y = \frac{2}{3}x + \frac{16}{3}$.

Answer

$\frac{44}{3}$

Question 7

Set up an integral representing the area A of the region enclosed by the given curves: $x = y^4$, $x = 2 - y^2$.

Answer

$\int_{-1}^{1}(2 - y^2 - y^4)\,dy$

Question 8

Sketch the region enclosed by the given curves and find the area: $y = 3 + x^3$, $y = 5 - x$, $x = -1$, $x = 0$.

Answer

$\frac{11}{4}$

Question 9

Sketch the region enclosed by the given curves and find the area: $y = 4\cos(x)$, $y = 4e^x$, $x = \frac{\pi}{2}$.

Answer

$4e^{\pi/2} - 8$

Question 10

Sketch the region enclosed by the given curves and find the area: $y = x^2 - 4x$, $y = 4x$.

Answer

$\frac{256}{3}$

Question 11

Sketch the region enclosed by the given curves and find the area: $x = 4 - y^2$, $x = y^2 - 4$.

Answer

$\frac{64}{3}$

Question 12

Sketch the region enclosed by the given curves and find the area: $2x + y^2 = 8$, $x = y$.

Answer

18

Question 13

Sketch the region enclosed by the given curves and find the area: $x = 8y^2$, $x = 28 + y^2$.

Answer

$\frac{224}{3}$

Question 14

Sketch the region enclosed by the given curves and find the area: $y = \sqrt{x}$, $y = \frac{1}{5}x$, $0 \leq x \leq 36$.

Answer

$\frac{409}{15}$

Question 15

Sketch the region enclosed by the given curves and find the area: $y = \cos(x)$, $y = 2 - \cos(x)$, $0 \leq x \leq 2\pi$.

Answer

4π

Question 16

Sketch the region enclosed by the given curves and find the area: $y = \cos(x)$, $y = \sin(2x)$, $0 \leq x \leq \frac{\pi}{2}$.

Answer

$\frac{1}{2}$

3 Homework 6.2 Volumes

Question 1

Set up, but do not evaluate, an integral for the volume of the solid obtained by rotating the region bounded by the given curves about the specified line. $y = \ln(x)$, $y = 0$, $x = 2$; about the x-axis.

Answer

$V = \int_1^2 \pi(\ln(x))^2\, dx$

Question 2

Set up, but do not evaluate, an integral for the volume of the solid obtained by rotating the region bounded by the given curves about the specified line. $x = \sqrt{6-y}$, $y = 0$, $x = 0$; about the y-axis.

Answer

$V = \int_0^6 \pi(\sqrt{6-y})^2\, dy$

Question 3

Find the volume V of the solid obtained by rotating the region bounded by the given curves about the specified line. $y = x + 1$, $y = 0$, $x = 0$, $x = 2$; about the x-axis.

Answer

$V = \frac{26\pi}{3}$

Question 4

Find the volume V of the solid obtained by rotating the region bounded by the given curves about the specified line. $y = \sqrt{x-1}$, $y = 0$, $x = 5$; about the x-axis.

Answer

$V = 8\pi$

Question 5

Consider the solid obtained by rotating the region bounded by the given curves about the specified line. $y = \sqrt{x-1}$, $y = 0$, $x = 6$; about the x-axis.

Answer

$V = \frac{25\pi}{2}$

Question 6

Find the volume V of the solid obtained by rotating the region bounded by the given curves about the specified line. $y = e^x$, $y = 0$, $x = -2$, $x = 2$; about the x-axis.

Answer

$V = \frac{\pi}{2}(e^4 - e^{-4})$

Question 7

Find the volume V of the solid obtained by rotating the region bounded by the given curves about the specified line. $x = 6\sqrt[3]{y}$, $x = 0$, $y = 3$; about the y-axis.

Answer

$V = 486\pi$

Question 8

Find the volume V of the solid obtained by rotating the region bounded by the given curves about the specified line. $y = x^2$, $y = 4x$; about the y-axis.

Answer

$V = \frac{128\pi}{3}$

Question 9

Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified line. $y = x^3$, $y = \sqrt{x}$; about the x-axis.

Answer

$V = \frac{5\pi}{14}$

Question 10

Find the volume V of the solid obtained by rotating the region bounded by the given curves about the specified line. $y = x^2$, $x = y^2$; about $y = 1$.

Answer

$V = \frac{11\pi}{30}$

Question 11

Find the volume V of the solid obtained by rotating the region bounded by the given curves about the specified line. $x = y^2$, $x = 1 - y^2$; about $x = 5$.

Answer

$V = 6\sqrt{2}\pi$

Question 12

Find the volume generated by rotating the region about the specified line. Region R_1 bounded by $y = 2x$, $y = 0$, $x = 1$ about the line $y = 2$ (BC).

Answer

$V = \frac{8\pi}{3}$

Question 13

Find the volume generated by rotating the region about the specified line. Region R_2 bounded by $y = 2\sqrt{x}$, $y = 2$, $x = 0$ about the line $x = 1$ (AB).

Answer

$V = \frac{26\pi}{45}$

Question 14

Find the volume V of the solid obtained by rotating the region bounded by the given curves about the specified line. $y = 8x^3$, $y = 0$, $x = 1$; about $x = 2$.

Answer

$V = \frac{24\pi}{5}$

Question 15

Find the volume V of the solid obtained by rotating the region bounded by the given curves about the specified line. $y = 5\sin(x)$, $y = 5\cos(x)$, $0 \leq x \leq \frac{\pi}{4}$; about $y = -1$.

Answer

$(\frac{5}{2} + 10\sqrt{2})\pi$

4 Homework 6.3 Volumes by Cylindrical Shells

Question 1

Use the method of cylindrical shells to find the volume V of S, the solid obtained by rotating the region bounded by $y = 4x(x-1)^2$ about the y-axis.

Answer

$V = \frac{4\pi}{15}$

Question 2

Set up and evaluate an integral using the method of cylindrical shells for the volume of the solid obtained by rotating the region bounded by $y = \sqrt{x}$ and $y = 6 - x$ about the x-axis.

Answer

$V = \frac{32\pi}{3}$

Question 3

Set up, but do not evaluate, an integral for the volume of the solid obtained by rotating the region bounded by $y = \ln(x)$, $y = 0$, $x = 9$ about the y-axis.

Answer

$V = \int_1^9 2\pi x \ln(x)\, dx$

Question 4

Set up, but do not evaluate, an integral for the volume of the solid obtained by rotating the region bounded by $y = x^3$, $y = 27$, $x = 0$ about the x-axis.

Answer

$V = \int_0^{27} 2\pi y \sqrt[3]{y}\, dy$

Question 5

Use the method of cylindrical shells to find the volume generated by rotating the region bounded by $y = \sqrt{x}$, $y = 0$, $x = 9$ about the y-axis.

Answer

$V = \frac{972\pi}{5}$

Question 6

Use the method of cylindrical shells to find the volume generated by rotating the region bounded by $y = x^3$, $y = 0$, $x = 1$, $x = 3$ about the y-axis.

Answer

$V = \frac{484\pi}{5}$

Question 7

Use the method of cylindrical shells to find the volume generated by rotating the region bounded by $y = 8x - x^2$ and $y = x$ about the y-axis.

Answer

$V = \frac{2401\pi}{6}$

Question 8

Use the method of cylindrical shells to find the volume of the solid obtained by rotating the region bounded by $xy = 2$, $x = 0$, $y = 2$, $y = 4$ about the x-axis.

Answer

$V = 8\pi$

Question 9

Use the method of cylindrical shells to find the volume of the solid obtained by rotating the region bounded by $y = \sqrt{x}$, $x = 0$, $y = 5$ about the x-axis.

Answer

$V = \frac{625\pi}{2}$

Question 10

Use the method of cylindrical shells to find the volume of the solid obtained by rotating the region bounded by $y = x^{3/2}$, $y = 8$, $x = 0$ about the x-axis.

Answer

$V = 192\pi$

Question 11

Use the method of cylindrical shells to find the volume of the solid obtained by rotating the region bounded by $x = 2 + (y - 4)^2$ and $x = 3$ about the x-axis.

Answer

$V = \frac{32\pi}{3}$

Question 12

Use the method of cylindrical shells to find the volume generated by rotating the region bounded by $y = x^3$, $y = 8$, $x = 0$ about the line $x = 7$.

Answer

$V = \frac{744\pi}{5}$

Question 13

Use the method of cylindrical shells to find the volume generated by rotating the region bounded by $y = 5x - x^2$ and $y = 4$ about the line $x = 1$.

Answer

$V = \frac{27\pi}{2}$

Question 14

Use the method of cylindrical shells to find the volume generated by rotating the region bounded by $x = 7y^2$, $y \geq 0$, $x = 7$ about the line $y = 2$.

Answer

$V = \frac{91\pi}{6}$

Question 15

Use the method of cylindrical shells to find the volume generated by rotating the region bounded by $x = 2y^2$ and $x = y^2 + 16$ about the line $y = -17$.

Answer

$V = \frac{8704\pi}{3}$

Question 16

Find the volume of the solid obtained by rotating the shaded region bounded by $y = 8 - 6x^2$ and $y = 2x^2$ about the x-axis.

Answer

$V = \frac{192\pi}{5}$

Question 17

A solid is obtained by rotating the shaded region bounded by $x = 6y - y^2$ and $x = 5$ about the line $y = 6$.

Answer

$V = 64\pi$

Question 18

The region bounded by $x^2 + (y - 2)^2 = 4$ is rotated about the y-axis. Find the volume of the resulting solid by any method.

Answer

$V = \frac{32\pi}{3}$

5 Homework 7.1 Integration by Parts

Question 1

Evaluate the integral using integration by parts with the indicated choices of u and dv. $\int xe^{6x}\,dx$; $u = x$, $dv = e^{6x}\,dx$.

Answer

$\frac{1}{6}xe^{6x} - \frac{1}{36}e^{6x} + C$

Question 2

Evaluate the integral using integration by parts with the indicated choices of u and dv. $\int x\cos(9x)\,dx$; $u = x$, $dv = \cos(9x)\,dx$.

Answer

$\frac{1}{9}x\sin(9x) + \frac{1}{81}\cos(9x) + C$

Question 3

Evaluate the integral. $\int w\ln(w)\,dw$.

Answer

$\frac{1}{2}w^2\ln(w) - \frac{1}{4}w^2 + C$

Question 4

Evaluate the integral. $\int \frac{\ln(x)}{x^2}\,dx$.

Answer

$-\frac{\ln(x)}{x} - \frac{1}{x} + C$

Question 5

Evaluate the integral. $\int \ln(\sqrt{x})\,dx$.

Answer

$x\ln(\sqrt{x}) - \frac{x}{2} + C$

Question 6

Evaluate the integral. $\int e^{8\theta}\sin(9\theta)\,d\theta$.

Answer

$\frac{1}{145}e^{8\theta}(8\sin(9\theta) - 9\cos(9\theta)) + C$

Question 7

Evaluate the integral. $\int e^{3\theta}\sin(4\theta)\,d\theta$.

Answer

$\frac{1}{25}e^{3\theta}(3\sin(4\theta) - 4\cos(4\theta)) + C$

Question 8

Evaluate the integral. $\int_0^{2\pi} x\sin(x)\cos(x)\,dx$.

Answer

$-\frac{\pi}{2}$

Question 9

Evaluate the integral. $\int_0^t 7e^s \sin(t-s)\,ds$.

Answer

$\frac{7}{2}(e^t - \sin(t) - \cos(t))$

Question 10

First make a substitution and then use integration by parts to evaluate the integral. $\int_0^{\pi} e^{\cos(t)}\sin(2t)\,dt$.

Answer

$\frac{4}{e}$

Question 11

Use integration by parts to prove the reduction formula. $\int (\ln(x))^n\,dx = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx$.

Proof

Let $u = (\ln x)^n$ and $dv = dx$. Then $du = n(\ln x)^{n-1}\frac{1}{x}dx$ and $v = x$. $\int (\ln(x))^n\,dx = x(\ln x)^n - \int x \cdot n(\ln x)^{n-1}\frac{1}{x}\,dx = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx$.

Question 12

Use integration by parts to determine which of the reduction formulas is correct.

Answer

$\int 9x^n e^x\,dx = 9x^n e^x - 9n\int x^{n-1}e^x\,dx$

Question 13

Use integration by parts to determine which of the reduction formulas is correct.

Answer

$\int 2\tan^n(x)\,dx = 2\frac{\tan^{n-1}(x)}{n-1} - \int 2\tan^{n-2}(x)\,dx, (n \neq 1)$

Question 14

Evaluate the integral using integration by parts with the indicated choices of u and dv. $\int xe^{9x}\,dx$; $u = x$, $dv = e^{9x}\,dx$.

Answer

$\frac{1}{9}xe^{9x} - \frac{1}{81}e^{9x} + C$

Question 15

Evaluate the integral. $\int te^{8t}\,dt$.

Answer

$\frac{1}{8}te^{8t} - \frac{1}{64}e^{8t} + C$

Question 16

Evaluate the integral. $\int(\pi - x)\cos(\pi x)\,dx$.

Answer

$\frac{1}{\pi}(\pi - x)\sin(\pi x) - \frac{1}{\pi^2}\cos(\pi x) + C$

6 Homework 7.2 Trigonometric Integrals

Question 1

Evaluate the integral. $\int 2\sin^2(x)\cos^3(x)\,dx$.

Answer

$\frac{2}{3}\sin^3(x) - \frac{2}{5}\sin^5(x) + C$

Question 2

Evaluate the integral. $\int \sin^3(y)\cos^4(y)\,dy$.

Answer

$-\frac{1}{5}\cos^5(y) + \frac{1}{7}\cos^7(y) + C$

Question 3

Evaluate the integral. $\int_0^{\pi/2} \cos^{13}(x)\sin^5(x)\,dx$.

Answer

$\frac{1}{504}$

Question 4

Evaluate the integral. $\int_0^{\pi/2} 9\sin^2(x)\cos^2(x)\,dx$.

Answer

$\frac{9\pi}{16}$

Question 5

Evaluate the integral. $\int_0^{\pi/2} 5\cos^2(\theta)\,d\theta$.

Answer

$\frac{5\pi}{4}$

Question 6

Evaluate the integral. $\int \sqrt{\cos(\theta)}\sin^3(\theta)\,d\theta$.

Answer

$\frac{2}{7}(\cos\theta)^{7/2} - \frac{2}{3}(\cos\theta)^{3/2} + C$

Question 7

Evaluate the integral. $\int \sin(3x)\sec^5(3x)\,dx$.

Answer

$\frac{1}{12}\sec^4(3x) + C$

Question 8

Evaluate the integral. $\int 4\tan(x)\sec^3(x)\,dx$.

Answer

$\frac{4}{3}\sec^3(x) + C$

Question 9

Evaluate the integral. $\int 5\tan^2(x)\,dx$.

Answer

$5\tan(x) - 5x + C$

Question 10

Evaluate the integral. $\int 11\tan^4(x)\sec^6(x)\,dx$.

Answer

$\frac{11}{9}\tan^9(x) + \frac{22}{7}\tan^7(x) + \frac{11}{5}\tan^5(x) + C$

Question 11

Evaluate the integral. $\int \tan^3(x)\sec(x)\,dx$.

Answer

$\frac{1}{3}\sec^3(x) - \sec(x) + C$

Question 12

Evaluate the integral. $\int \tan^5(x)\sec^6(x)\,dx$.

Answer

$\frac{1}{8}\tan^8(x) + \frac{1}{3}\tan^6(x) + \frac{1}{4}\tan^4(x) + C$

Question 13

Evaluate the integral. $\int_0^{\pi/6} \tan^4(t)\,dt$.

Answer

$\frac{\pi}{6} - \frac{8}{9\sqrt{3}}$

Question 14

Evaluate the integral. $\int \tan^5(x)\,dx$.

Answer

$\frac{1}{4}\sec^4(x) - \tan^2(x) + \ln|\sec(x)| + C$ (Note: Answer in image is equivalent: $\frac{1}{4}\tan^4(x) - \frac{1}{2}\tan^2(x) + \ln|\sec x| + C$ is also common)

Question 15

Evaluate the integral. $\int \frac{\tan(x)\sec^2(x)}{\cos(x)}\,dx$.

Answer

$\frac{1}{3}\sec^3(x) + C$

Question 16

Evaluate the integral. $\int_{\pi/6}^{\pi/2} 5\cot^2(x)\,dx$.

Answer

$5\sqrt{3} - \frac{5\pi}{3}$

Question 17

Evaluate the integral. $\int \sin(8x)\cos(5x)\,dx$.

Answer

$-\frac{1}{26}\cos(13x) - \frac{1}{6}\cos(3x) + C$

Question 18

Evaluate the integral. $\int 5\tan^2(x)\sec(x)\,dx$.

Answer

$\frac{5}{2}(\sec(x)\tan(x) - \ln|\sec(x) + \tan(x)|) + C$

7 Homework 7.3 Trigonometric Substitution

Question 1

Evaluate the integral. $\int \frac{x}{\sqrt{81+x^2}}\,dx$.

Answer

$\sqrt{81+x^2}+C$

Question 2

Evaluate the integral. $\int \frac{x}{\sqrt{x^2-5}}\,dx$.

Answer

$\sqrt{x^2-5}+C$

Question 3

Evaluate the integral. $\int_0^3 \sqrt{x^2+9}\,dx$.

Answer

$\frac{9}{2}(\sqrt{2}+\ln(1+\sqrt{2}))$

Question 4

Evaluate the integral using the indicated trigonometric substitution. $\int \frac{x^3}{\sqrt{16+x^2}}\,dx$, $x=4\tan(\theta)$.

Answer

$\frac{1}{3}(x^2+16)^{3/2}-16\sqrt{x^2+16}+C$

Question 5

Evaluate the integral using the indicated trigonometric substitution. $\int \frac{\sqrt{4x^2-25}}{x}\,dx$, $x=\frac{5}{2}\sec(\theta)$.

Answer

$\sqrt{4x^2-25}-5\sec^{-1}(\frac{2x}{5})+C$

Question 6

Determine an appropriate trigonometric substitution for $\int \frac{x^4}{\sqrt{1+x^2}}\,dx$ and transform the integral.

Answer

Substitution: $x=\tan(\theta)$. Transformed Integral: $\int \tan^4(\theta)\sec(\theta)\,d\theta$.

Question 7

Evaluate the integral. $\int_2^5 \frac{dx}{(x^2-1)^{3/2}}$.

Answer

$\frac{5\sqrt{6}}{12} + \frac{2\sqrt{3}}{3}$ (Note: The image answer seems simplified incorrectly. It should be $\frac{5}{\sqrt{24}} - \frac{2}{\sqrt{3}} = \frac{5\sqrt{6}}{12} - \frac{2\sqrt{3}}{3}$)

Question 8

Evaluate the integral. $\int_0^4 \frac{dt}{\sqrt{16+t^2}}$.

Answer

$\ln(1+\sqrt{2})$

Question 9

Evaluate the integral. $\int_0^7 \frac{dt}{\sqrt{49+t^2}}$.

Answer

$\ln(1+\sqrt{2})$

Question 10

Evaluate the integral. $\int \frac{\sqrt{x^2-25}}{x^3}\, dx$.

Answer

$\frac{1}{10}\sec^{-1}(\frac{x}{5}) - \frac{\sqrt{x^2-25}}{2x^2} + C$

Question 11

Evaluate the integral. $\int \frac{\sqrt{4+x^2}}{x}\, dx$.

Answer

$\sqrt{4+x^2} + 2\ln|\frac{\sqrt{x^2+4}-2}{x}| + C$

Question 12

Evaluate the integral. $\int 3x\sqrt{1-x^4}\, dx$.

Answer

$\frac{3}{4}\sin^{-1}(x^2) + \frac{3}{4}x^2\sqrt{1-x^4} + C$

Question 13

Evaluate the integral. $\int x^3\sqrt{64+x^2}\, dx$.

Answer

$\frac{1}{5}(x^2+64)^{5/2} - \frac{64}{3}(x^2+64)^{3/2} + C$

Question 14

Evaluate the integral. $\int \frac{x^2}{\sqrt{49-x^2}}\, dx$.

Answer

$\frac{49}{2}\sin^{-1}(\frac{x}{7}) - \frac{1}{2}x\sqrt{49-x^2} + C$

8 Homework 7.4 integration by fraction decomposition

Question 1

Write out the form of the partial fraction decomposition. (a) $\frac{x-42}{x^2+x-42}$ (b) $\frac{1}{x^2+x^4}$

Answer

(a) $\frac{A}{x+7} + \frac{B}{x-6}$ (b) $\frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}$

Question 2

Write out the form of the partial fraction decomposition. (a) $\frac{x^5+36}{(x^2-x)(x^4+12x^2+36)}$ (b) $\frac{x^2}{x^2+x-20}$

Answer

(a) $\frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+6} + \frac{Ex+F}{(x^2+6)^2}$ (b) $1 + \frac{A}{x+5} + \frac{B}{x-4}$

Question 3

Write out the form of the partial fraction decomposition. (a) $\frac{x^6}{x^2-4}$ (b) $\frac{x^4}{(x^2-x+1)(x^2+4)^2}$

Answer

(a) $x^4 + 4x^2 + 16 + \frac{A}{x-2} + \frac{B}{x+2}$ (b) $\frac{Ax+B}{x^2-x+1} + \frac{Cx+D}{x^2+4} + \frac{Ex+F}{(x^2+4)^2}$

Question 4

Evaluate the integral. $\int \frac{3}{(x-1)(x+2)}\, dx$.

Answer

$\ln|x-1| - \ln|x+2| + C$

Question 5

Evaluate the integral. $\int \frac{82x+8}{(9x+1)(x-1)}\, dx$.

Answer

$\frac{1}{9}\ln|9x+1| + 9\ln|x-1| + C$

Question 6

Evaluate the integral. $\int \frac{5t-2}{t+1}\, dt$.

Answer

$5t - 7\ln|t+1| + C$

Question 7

Evaluate the integral. $\int \frac{2}{(t^2-4)^2}\, dt$.

Answer

$\frac{1}{16}\ln|\frac{t-2}{t+2}| - \frac{t}{4(t^2-4)} + C$

Question 8

Evaluate the integral. $\int \frac{17}{(x-1)(x^2+16)}\,dx$.

Answer

$\ln|x-1| - \frac{1}{2}\ln(x^2+16) - \frac{1}{4}\arctan(\frac{x}{4}) + C$

Question 9

Evaluate the integral. $\int \frac{x^2-x+12}{x^3+2x}\,dx$.

Answer

$6\ln|x| - \frac{5}{2}\ln(x^2+2) - \frac{\sqrt{2}}{2}\arctan(\frac{x}{\sqrt{2}}) + C$

Question 10

Evaluate the integral. $\int \frac{5x^2+x+5}{(x^2+1)^2}\,dx$.

Answer

$5\arctan(x) - \frac{1}{2(x^2+1)} + C$

Question 11

Evaluate the integral. $\int \frac{x+12}{x^2+14x+53}\,dx$.

Answer

$\frac{1}{2}\ln(x^2+14x+53) + \frac{5}{2}\arctan(\frac{x+7}{2}) + C$

Question 12

Evaluate the integral. $\int_0^1 \frac{x^3+3x}{x^4+6x^2+3}\,dx$.

Answer

$\frac{1}{4}\ln(\frac{10}{3})$

Question 13

Evaluate the integral. $\int_0^1 \frac{2}{2x^2+3x+1}\,dx$.

Answer

$2\ln(\frac{4}{3})$ or $\ln(\frac{16}{9})$

Question 14

Evaluate the integral. $\int \frac{x^2}{x-5}\,dx$.

Answer

$\frac{1}{2}x^2 + 5x + 25\ln|x-5| + C$

Comprehensive Guide to Integration Problem Types and Tricks

9 Section 5.5: The Substitution Rule (U-Substitution)

The core idea of u-substitution is to simplify an integral by replacing a part of the integrand with a single variable, u, effectively reversing the chain rule. The key is to choose u such that its derivative, du, also appears in the integral (perhaps off by a constant).

9.1 Problem Types and Tricks in Your Homework

9.1.1 Problem Type 1: Direct Substitution with Constant Adjustment

This is the most common type, where the derivative of your chosen u is present in the integral, but might be off by a constant multiplier.

Example (Q2): $\int xe^{-x^2}\,dx$

Solution Strategy:

- Identify the "inner function." Here, it's the exponent, $-x^2$.
- Let $u = -x^2$.
- Find the differential: $du = -2x\,dx$.
- Notice the integral has $x\,dx$, not $-2x\,dx$. Algebraically solve for what you have: $x\,dx = -\frac{1}{2}du$.
- Substitute both u and the expression for $x\,dx$ into the integral: $\int e^u(-\frac{1}{2}du) = -\frac{1}{2}\int e^u\,du$.
- Integrate with respect to u: $-\frac{1}{2}e^u + C$.
- Substitute back for x: $-\frac{1}{2}e^{-x^2} + C$.

Tricks Used:

- **Identifying the Inner Function:** Recognizing that the derivative of the exponent (x^2) is related to the other factor (x).
- **Constant Adjustment:** Manipulating the differential ($du = -2x\,dx$) to solve for the parts available in the integrand ($x\,dx$).

9.1.2 Problem Type 2: "Change of Variables" or Back-Substitution

This occurs when, after substituting for u and du, you still have a variable x left in the integrand. You must then use your original substitution equation to solve for x in terms of u.

Example (Q14): $\int x\sqrt{x+5}\,dx$

Solution Strategy:

- The complicated part is the radical, so let $u = x + 5$.
- The differential is simple: $du = dx$.
- Substituting gives: $\int x\sqrt{u}\,du$. We still have an x.
- Go back to the substitution equation, $u = x + 5$, and solve for x: $x = u - 5$.
- Substitute this back into the integral: $\int(u-5)\sqrt{u}\,du = \int(u-5)u^{1/2}\,du$.
- Distribute and integrate using the power rule: $\int(u^{3/2} - 5u^{1/2})\,du = \frac{2}{5}u^{5/2} - 5(\frac{2}{3})u^{3/2} + C$.
- Substitute back for x: $\frac{2}{5}(x+5)^{5/2} - \frac{10}{3}(x+5)^{3/2} + C$.

Tricks Used:

- **Solving for x:** Recognizing that the leftover variable must be eliminated by using the original substitution equation.
- **Algebraic Simplification:** Distributing the radical term to create a simple polynomial in u that can be integrated with the power rule.

9.2 Other Common Problem Types and Tricks

9.2.1 Problem Type 3: Splitting the Integrand

Sometimes, an integrand can be split into two parts: one that can be solved with u-substitution and another that has a different form (often leading to an arctangent).

Example: $\int \frac{2x+5}{x^2+9}\,dx$

Solution Strategy:

- Split the fraction: $\int \frac{2x}{x^2+9}\,dx + \int \frac{5}{x^2+9}\,dx$.
- For the first integral, use u-sub: $u = x^2 + 9$, $du = 2x\,dx$. This becomes $\int \frac{1}{u}\,du = \ln|u| = \ln(x^2 + 9)$.
- For the second integral, recognize the arctan form: $5\int \frac{1}{x^2+3^2}\,dx = 5(\frac{1}{3}\arctan(\frac{x}{3}))$.
- Combine the results: $\ln(x^2 + 9) + \frac{5}{3}\arctan(\frac{x}{3}) + C$.

Tricks Used:

- **Fraction Splitting:** A key algebraic step to separate the integral into manageable parts.
- **Recognizing Standard Forms:** Identifying both a ln pattern (du/u) and an arctan pattern ($1/(x^2 + a^2)$) in the same problem.

9.2.2 Problem Type 4: U-Substitution with Completing the Square (leading to arctan)

When the denominator is an irreducible quadratic that isn't a simple sum of squares, you must complete the square first.

Example: $\int \frac{1}{x^2+6x+13}\,dx$

Solution Strategy:

- Complete the square in the denominator: $x^2 + 6x + 9 - 9 + 13 = (x + 3)^2 + 4$.
- The integral becomes: $\int \frac{1}{(x+3)^2+2^2}\,dx$.
- Now use u-substitution: Let $u = x + 3$, so $du = dx$.
- The integral is now in the standard arctan form: $\int \frac{1}{u^2+2^2}\,du = \frac{1}{2}\arctan(\frac{u}{2}) + C$.
- Substitute back: $\frac{1}{2}\arctan(\frac{x+3}{2}) + C$.

Tricks Used:

- **Completing the Square:** A fundamental algebraic technique to transform a quadratic into a more useful form.
- **Shifting the Variable:** The substitution $u = x + 3$ handles the horizontal shift.

9.2.3 Problem Type 5: U-Substitution with Definite Integrals

When evaluating a definite integral with u-substitution, you have two options: (1) change the limits of integration to be in terms of u, or (2) integrate, substitute back to x, and then use the original x limits. Changing the limits is usually faster and less error-prone.

Example: $\int_0^2 x(x^2+1)^3\,dx$

Solution Strategy (Changing Limits):

- Let $u = x^2 + 1$, so $du = 2x\,dx \Rightarrow x\,dx = \frac{1}{2}du$.
- Change the limits: When $x = 0$, $u = 0^2 + 1 = 1$. When $x = 2$, $u = 2^2 + 1 = 5$.
- The new integral is: $\int_1^5 u^3(\frac{1}{2}du) = \frac{1}{2}\left[\frac{u^4}{4}\right]_1^5$.
- Evaluate with the new limits: $\frac{1}{8}(5^4 - 1^4) = \frac{1}{8}(625 - 1) = \frac{624}{8} = 78$.

Tricks Used:

- **Changing Integration Limits:** This is the most important trick for definite integrals with substitution. It avoids the need to substitute back to x.

10 Section 7.1: Integration by Parts (IBP)

The formula is $\int u\,dv = uv - \int v\,du$. The goal is to choose u and dv such that the new integral, $\int v\,du$, is simpler than the original. The acronym LIATE (Logarithmic, Inverse Trig, Algebraic, Trigonometric, Exponential) is a useful guideline for choosing u.

10.1 Problem Types and Tricks in Your Homework

10.1.1 Problem Type 1: Standard Application (Polynomial x Transcendental)

This is the classic IBP problem where you differentiate the polynomial part down to a constant.

Example (Q1): $\int xe^{6x}\,dx$

Solution Strategy:

- Using LIATE, choose the algebraic part for u: $u = x$.
- The rest is dv: $dv = e^{6x}dx$.
- Differentiate u and integrate dv: $du = dx$ and $v = \frac{1}{6}e^{6x}$.
- Apply the formula: $uv - \int v\,du = x(\frac{1}{6}e^{6x}) - \int \frac{1}{6}e^{6x}\,dx$.
- The new integral is simple: $\frac{1}{6}xe^{6x} - \frac{1}{36}e^{6x} + C$.

Tricks Used:

- **LIATE Rule:** A heuristic for choosing u to ensure the new integral is simpler.

10.1.2 Problem Type 2: Integrating Logarithmic or Inverse Trig Functions

When the integrand is just a log or inverse trig function, there appears to be only one part. The trick is to choose dv = dx.

Example (Q3): $\int w \ln(w)\, dw$

Solution Strategy:

- Using LIATE, choose u as the log: $u = \ln(w)$.
- The rest is dv: $dv = w\, dw$.
- Differentiate and integrate: $du = \frac{1}{w} dw$ and $v = \frac{1}{2}w^2$.
- Apply the formula: $uv - \int v\, du = (\ln w)(\frac{1}{2}w^2) - \int \frac{1}{2}w^2(\frac{1}{w}dw)$.
- Simplify and solve the new integral: $\frac{1}{2}w^2 \ln(w) - \frac{1}{2}\int w\, dw = \frac{1}{2}w^2 \ln(w) - \frac{1}{4}w^2 + C$.

Tricks Used:

- **Strategic dv choice:** Even though the log function is the "L" in LIATE, here it's paired with an algebraic term, making the choice straightforward. For just $\int \ln(x)dx$, the trick is to choose $dv = dx$.

10.1.3 Problem Type 3: Looping Integration by Parts

This advanced technique is used for integrals of the form $\int e^{ax}\sin(bx)\, dx$ or $\int e^{ax}\cos(bx)\, dx$. Applying IBP twice will result in the original integral appearing on the right side of the equation, allowing you to solve for it algebraically.

Example (Q6): $\int e^{8\theta}\sin(9\theta)\, d\theta$

Solution Strategy:

- Let $u = \sin(9\theta)$ and $dv = e^{8\theta}d\theta$. Then $du = 9\cos(9\theta)d\theta$ and $v = \frac{1}{8}e^{8\theta}$.
- First IBP: $I = \frac{1}{8}e^{8\theta}\sin(9\theta) - \frac{9}{8}\int e^{8\theta}\cos(9\theta)\, d\theta$.
- Apply IBP to the new integral: Let $U = \cos(9\theta)$ and $dV = e^{8\theta}d\theta$. Then $dU = -9\sin(9\theta)d\theta$ and $V = \frac{1}{8}e^{8\theta}$.
- Substitute this back in: $I = \frac{1}{8}e^{8\theta}\sin(9\theta) - \frac{9}{8}\left[\frac{1}{8}e^{8\theta}\cos(9\theta) - \int \frac{1}{8}e^{8\theta}(-9\sin(9\theta)d\theta)\right]$.
- Simplify: $I = \frac{1}{8}e^{8\theta}\sin(9\theta) - \frac{9}{64}e^{8\theta}\cos(9\theta) - \frac{81}{64}\int e^{8\theta}\sin(9\theta)\, d\theta$.
- Notice the original integral I has reappeared. Substitute I: $I = \cdots - \frac{81}{64}I$.
- Solve for I algebraically: $I + \frac{81}{64}I = \cdots \Rightarrow \frac{145}{64}I = \cdots \Rightarrow I = \frac{64}{145}[\dots]$.

Tricks Used:

- **Repeating IBP:** Recognizing that a single application isn't enough.
- **The "Boomerang" or "Loop":** Identifying that the original integral has returned.
- **Algebraic Solution:** Treating the integral I as a variable and solving the equation for it.

10.2 Other Common Problem Types and Tricks

10.2.1 Problem Type 4: The Tabular Method

This is a fantastic shortcut for repeated IBP when one function is a polynomial (differentiates to zero) and the other can be repeatedly integrated.

Example: $\int x^3 e^{2x}\, dx$

Solution Strategy:

- Create two columns: D (for derivatives) and I (for integrals).
- Place the polynomial (x^3) in the D column and the other function (e^{2x}) in the I column.
- Differentiate down the D column until you reach zero. Integrate down the I column the same number of times.
- Add alternating signs to the D column (+, -, +, -).
- The answer is the sum of the products of the diagonal terms.

Signs	D	I
$+$	x^3	e^{2x}
-	$3x^2$	$\frac{1}{2}e^{2x}$
$+$	$6x$	$\frac{1}{4}e^{2x}$
-	6	$\frac{1}{8}e^{2x}$
$+$	0	$\frac{1}{16}e^{2x}$

- **Answer:** $x^3(\frac{1}{2}e^{2x}) - 3x^2(\frac{1}{4}e^{2x}) + 6x(\frac{1}{8}e^{2x}) - 6(\frac{1}{16}e^{2x}) + C$.

Tricks Used:

- **Organizational Shortcut:** The tabular method organizes the repeated applications of IBP, reducing the chance of algebraic errors.

11 Section 7.2: Trigonometric Integrals

This section focuses on integrals containing powers of trigonometric functions. The strategy depends entirely on the powers (even or odd) and which functions are present. It relies heavily on trigonometric identities.

11.1 Problem Types and Tricks in Your Homework

11.1.1 Problem Type 1: Powers of Sine and Cosine (At least one is ODD)

If the power of cosine is odd, save one cosine factor and convert the rest to sines. If the power of sine is odd, save one sine factor and convert the rest to cosines.

Example (Q1): $\int 2\sin^2(x)\cos^3(x)\,dx$

Solution Strategy:

- The power of cosine (3) is odd.
- Split off one cosine factor: $\int 2\sin^2(x)\cos^2(x)\cos(x)\,dx$.
- Use the Pythagorean Identity $\cos^2(x) = 1 - \sin^2(x)$ to convert the remaining even-powered cosines: $\int 2\sin^2(x)(1 - \sin^2(x))\cos(x)\,dx$.
- Now use u-substitution. Let $u = \sin(x)$, so $du = \cos(x)dx$.
- The integral becomes a simple polynomial: $\int 2u^2(1-u^2)\,du = \int(2u^2 - 2u^4)\,du$.
- Integrate and substitute back: $\frac{2}{3}u^3 - \frac{2}{5}u^5 + C = \frac{2}{3}\sin^3(x) - \frac{2}{5}\sin^5(x) + C$.

Tricks Used:

- **Identity Conversion:** Using $\sin^2\theta + \cos^2\theta = 1$ is the core trick.
- **"Save One" Strategy:** Saving a single factor to serve as the du in the subsequent u-substitution.

11.1.2 Problem Type 2: Powers of Sine and Cosine (Both are EVEN)

If both powers are even, you must use the half-angle (or power-reducing) identities repeatedly.

Example (Q4): $\int_0^{\pi/2} 9\sin^2(x)\cos^2(x)\,dx$

Solution Strategy:

- Both powers (2) are even.
- Use the identities: $\sin^2(x) = \frac{1-\cos(2x)}{2}$ and $\cos^2(x) = \frac{1+\cos(2x)}{2}$.
- Substitute: $9\int_0^{\pi/2}(\frac{1-\cos(2x)}{2})(\frac{1+\cos(2x)}{2})\,dx = \frac{9}{4}\int_0^{\pi/2}(1-\cos^2(2x))\,dx$.
- Notice we have another even power, $\cos^2(2x)$. Apply the half-angle identity again: $\cos^2(2x) = \frac{1+\cos(4x)}{2}$.
- Substitute and simplify: $\frac{9}{4}\int_0^{\pi/2}(1-\frac{1+\cos(4x)}{2})\,dx = \frac{9}{8}\int_0^{\pi/2}(1-\cos(4x))\,dx$.
- Integrate and evaluate.

Tricks Used:

- **Half-Angle/Power-Reducing Identities:** This is the essential tool for this case.
- **Repeated Application:** Recognizing that you may need to apply the identities more than once.
- An alternative trick for this specific example is to use $\sin(x)\cos(x) = \frac{1}{2}\sin(2x)$, which simplifies the integrand to $\frac{9}{4}\int \sin^2(2x)\,dx$ before using the half-angle identity once.

11.1.3 Problem Type 3: Powers of Tangent and Secant

There are two main sub-cases here.

Case A: Power of Secant is EVEN (and ≥ 2)

Example (Q10): $\int 11\tan^4(x)\sec^6(x)\,dx$

Strategy: Save a $\sec^2(x)$ factor. Convert the remaining secants to tangents using $\sec^2(x) = 1+\tan^2(x)$. Use $u = \tan(x)$.

$$\int 11\tan^4(x)\sec^4(x)\sec^2(x)\,dx = \int 11\tan^4(x)(1+\tan^2 x)^2\sec^2(x)\,dx. \text{ Let } u = \tan x.$$

Case B: Power of Tangent is ODD (and secant exists)

Example (Q8): $\int 4\tan(x)\sec^3(x)\,dx$

Strategy: Save a $\sec(x)\tan(x)$ factor. Convert the remaining tangents to secants using $\tan^2(x) = \sec^2(x) - 1$. Use $u = \sec(x)$.

$$\int 4\sec^2(x)(\sec(x)\tan(x))\,dx. \text{ Let } u = \sec x.$$

Tricks Used:

- **Secant/Tangent Pythagorean Identity:** The core tool is $\tan^2(x) + 1 = \sec^2(x)$.
- **Strategic Factoring:** Saving the correct factor ($\sec^2 x$ or $\sec x\tan x$) to serve as du.

11.1.4 Problem Type 4: Product-to-Sum Integrals

For integrals of $\sin(mx)\cos(nx)$, $\sin(mx)\sin(nx)$, or $\cos(mx)\cos(nx)$.

Example (Q17): $\int \sin(8x)\cos(5x)\,dx$

Strategy: Use the identity $\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$.

$$\int \frac{1}{2}[\sin(3x) + \sin(13x)]\,dx. \text{ This is now a simple integral.}$$

Tricks Used:

- **Product-to-Sum Identities:** These must be known or looked up. They convert a difficult product into a simple sum.

11.2 Other Common Problem Types and Tricks

11.2.1 Problem Type 5: Powers of Tangent and Secant (ODD secant, EVEN tangent)

This is the hardest case and often requires integration by parts.

Example (Q18): $\int 5\tan^2(x)\sec(x)\,dx$

Solution Strategy:

- Convert $\tan^2(x)$ to $\sec^2(x) - 1$.
- $\int 5(\sec^2(x) - 1)\sec(x)\,dx = 5\int(\sec^3(x) - \sec(x))\,dx$.
- The integral of $\sec(x)$ is a standard result: $\ln|\sec(x) + \tan(x)|$.
- The integral of $\sec^3(x)$ is a famous and tricky one that requires integration by parts (let $u = \sec x, dv = \sec^2 x dx$). This will lead to a looping result similar to the $e^x \sin x$ type.

Tricks Used:

- **Identity Conversion:** Starting with the Pythagorean identity.
- **Decomposition into Known Integrals:** Breaking the problem down.
- **IBP for Trig Functions:** Applying integration by parts to solve a purely trigonometric integral.

12 Section 7.3: Trigonometric Substitution

This technique is used to evaluate integrals containing radical expressions like $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, and $\sqrt{x^2 - a^2}$. The substitution turns the radical into a simple trig function using Pythagorean identities.

12.1 Problem Types and Tricks in Your Homework

The problem types are defined by the form of the radical.

12.1.1 Problem Type 1: Form $\sqrt{a^2 + x^2}$ (or $a^2 + x^2$)

Example (Q4): $\int \frac{x^3}{\sqrt{16+x^2}}\,dx$

Solution Strategy:

- Identify $a^2 = 16 \Rightarrow a = 4$. This is the form $a^2 + x^2$.
- **Substitution:** Let $x = 4\tan(\theta)$. Then $dx = 4\sec^2(\theta)d\theta$.
- Simplify the radical: $\sqrt{16 + 16\tan^2\theta} = \sqrt{16\sec^2\theta} = 4\sec\theta$.
- Substitute everything into the integral: $\int \frac{(4\tan\theta)^3}{4\sec\theta}(4\sec^2\theta)\,d\theta$.
- Simplify and solve the resulting trig integral (Section 7.2 skills).
- **Draw the Reference Triangle:** From $x = 4\tan\theta \Rightarrow \tan\theta = x/4$. This triangle has opposite side 'x', adjacent side '4', and hypotenuse $\sqrt{x^2 + 16}$.
- Use the triangle to substitute back from θ to x.

Tricks Used:

- **Choosing the Correct Substitution:** 'tan' for sum of squares.
- **Reference Triangle:** This is the crucial, non-negotiable step for converting the answer back to the original variable 'x'.

12.1.2 Problem Type 2: Form $\sqrt{x^2 - a^2}$ (or $x^2 - a^2$)

Example (Q5): $\int \frac{\sqrt{4x^2-25}}{x}\,dx$

Solution Strategy:

- Rewrite as $\sqrt{(2x)^2 - 5^2}$. Let $u = 2x$ or substitute directly. Let's use $2x$.
- Identify $a^2 = 25 \Rightarrow a = 5$.
- **Substitution:** Let $2x = 5\sec(\theta)$. Then $2dx = 5\sec\theta\tan\theta d\theta$. Also $x = \frac{5}{2}\sec\theta$.
- Simplify the radical: $\sqrt{25\sec^2\theta - 25} = \sqrt{25\tan^2\theta} = 5\tan\theta$.
- Substitute, solve the trig integral, draw the triangle (from $\sec\theta = 2x/5$), and convert back.

Tricks Used:

- **Choosing the Correct Substitution:** 'sec' for (variable squared) - (constant squared).
- **Algebraic pre-processing:** Recognizing $4x^2$ as $(2x)^2$.

12.1.3 Problem Type 3: Form $\sqrt{a^2 - x^2}$ (or $a^2 - x^2$)

Example (from Q14): $\int \frac{x^2}{\sqrt{49-x^2}}\,dx$

Solution Strategy:

- Identify $a^2 = 49 \Rightarrow a = 7$.
- **Substitution:** Let $x = 7\sin(\theta)$. Then $dx = 7\cos(\theta)d\theta$.
- Simplify the radical: $\sqrt{49 - 49\sin^2\theta} = \sqrt{49\cos^2\theta} = 7\cos\theta$.
- Substitute, solve the trig integral, draw the triangle (from $\sin\theta = x/7$), and convert back.

Tricks Used:

- **Choosing the Correct Substitution:** 'sin' for (constant squared) - (variable squared).

12.2 Other Common Problem Types and Tricks

12.2.1 Problem Type 4: Trig Substitution after Completing the Square

This combines the trick from U-Sub with the methods of this section. It's used for integrals with a full quadratic under the radical.

Example: $\int \frac{1}{\sqrt{x^2-4x+13}}\,dx$

Solution Strategy:

- Complete the square: $x^2 - 4x + 4 - 4 + 13 = (x-2)^2 + 9$.
- The integral becomes: $\int \frac{1}{\sqrt{(x-2)^2+3^2}}\,dx$.
- First, do a simple u-sub: let $u = x - 2, du = dx$. Integral becomes $\int \frac{1}{\sqrt{u^2+3^2}} du$.
- Now, perform a trig substitution on 'u': let $u = 3\tan\theta$.
- Solve the resulting simple trig integral, convert back to 'u' using a triangle, then convert back to 'x'.

Tricks Used:

- **Completing the Square:** The essential first step to reveal the trig sub form.
- **Multi-step Substitution:** A u-substitution followed by a trig substitution.

13 Section 7.4: Partial Fraction Decomposition (PFD)

This is an algebraic technique to break down complex rational functions (polynomials in the numerator and denominator) into simpler fractions that are easy to integrate.

13.1 Problem Types and Tricks in Your Homework

13.1.1 Problem Type 1: Improper Fractions (requiring Long Division)

If the degree of the numerator is greater than or equal to the degree of the denominator, you MUST perform polynomial long division first.

Example (Q14): $\int \frac{x^2}{x-5}\,dx$

Solution Strategy:

- Degree of top (2) $\geq$ Degree of bottom (1). Perform long division.
- $x^2 \div (x-5)$ gives a quotient of $x+5$ and a remainder of 25.
- So, $\frac{x^2}{x-5} = x + 5 + \frac{25}{x-5}$.
- Integrate the new expression: $\int (x + 5 + \frac{25}{x-5})\,dx = \frac{1}{2}x^2 + 5x + 25\ln|x-5| + C$.

Tricks Used:

- **Polynomial Long Division:** The critical first step for any improper rational function.

13.1.2 Problem Type 2: Distinct Linear Factors in Denominator

This is the simplest PFD case. For each factor $(x-a)$, the decomposition gets a term $\frac{A}{x-a}$.

Example (Q4): $\int \frac{3}{(x-1)(x+2)}\,dx$

Solution Strategy:

- Decomposition: $\frac{3}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$.
- Multiply by the common denominator: $3 = A(x+2) + B(x-1)$.
- Solve for A and B. A quick way is the **Heaviside Cover-up Method**:
  - To find A, let $x = 1$: $3 = A(1+2) + B(0) \Rightarrow 3 = 3A \Rightarrow A = 1$.
  - To find B, let $x = -2$: $3 = A(0) + B(-2-1) \Rightarrow 3 = -3B \Rightarrow B = -1$.
- Integrate the simple fractions: $\int (\frac{1}{x-1} - \frac{1}{x+2})\,dx = \ln|x-1| - \ln|x+2| + C$.

Tricks Used:

- **Heaviside Cover-up Method:** A very fast way to find coefficients for distinct linear factors.

13.1.3 Problem Type 3: Irreducible Quadratic Factors in Denominator

If the denominator has a factor like x^2+c^2 that cannot be factored further, its term in the decomposition is $\frac{Ax+B}{x^2+c^2}$.

Example (Q8): $\int \frac{17}{(x-1)(x^2+16)}\,dx$

Solution Strategy:

- Decomposition: $\frac{17}{(x-1)(x^2+16)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+16}$.
- Multiply by the common denominator and group terms: $17 = A(x^2+16) + (Bx+C)(x-1)$.
- You can't use the cover-up method for B and C. You must expand and equate coefficients of powers of ‘x‘:
  - x^2 terms: $0 = A + B$.
  - x terms: $0 = -B + C$.
  - Constant terms: $17 = 16A - C$.
- Solve this system of 3 equations.
- Integrate the resulting terms. The term with the irreducible quadratic will often split into a ‘ln‘ (from a u-sub) and an ‘arctan‘ (see PFD Trick 1 below).

Tricks Used:

- **Equating Coefficients:** The general method for finding unknown constants when the cover-up method isn't fully applicable.

13.1.4 Problem Type 4: Repeated Irreducible Quadratic Factors

If you have a factor like $(x^2+c^2)^n$, you need a term for each power up to ‘n‘.

Example (Q10): $\int \frac{5x^2+x+5}{(x^2+1)^2}\,dx$

Solution Strategy:

- Decomposition: $\frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}$.
- Solve the system of equations for A, B, C, D.
- Integrate each term. The integral of the $(x^2+1)^2$ term may require trig substitution.

Tricks Used:

- **Systematic Decomposition:** Remembering to include a term for each power of the repeated factor.

13.2 Other Common Problem Types and Tricks

13.2.1 PFD Trick 1: Splitting the Irreducible Quadratic Integral

When you integrate a term like $\int \frac{Bx+C}{x^2+a^2}\,dx$, you almost always split it.

Strategy: $\int \frac{Bx}{x^2+a^2}\,dx + \int \frac{C}{x^2+a^2}\,dx$.

- The first part is solved with a u-sub, $u = x^2 + a^2$, and becomes a natural log.
- The second part is a direct arctangent integral.

13.2.2 PFD Trick 2: Repeated Linear Factors

Similar to repeated quadratics, a factor of $(x-a)^n$ requires terms for each power.

Example: $\frac{x+5}{(x-2)^3}$

Decomposition: $\frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{(x-2)^3}$.

14 Sections 6.1, 6.2, 6.3: Area and Volume Applications

These sections are about setting up the correct integral based on geometry. The integration itself often uses simpler techniques, but the setup is the key challenge.

14.1 Section 6.1: Area Between Curves

14.1.1 Problem Type 1: Integrating with Respect to x (Top - Bottom)

Used when the region is clearly bounded by a function on the top and a function on the bottom.

Formula: $A = \int_a^b [y_{\text{top}}(x) - y_{\text{bottom}}(x)]\,dx$

Example (Q1): Bounded by $y = 5x - x^2$ and $y = x$.

Strategy:

- Find intersection points: $5x - x^2 = x \Rightarrow 4x - x^2 = 0 \Rightarrow x(4-x) = 0 \Rightarrow x = 0, 4$. These are your limits 'a' and 'b'.
- Determine which function is on top in the interval. Pick a test point like $x = 1$: $y = 5(1) - 1^2 = 4$ and $y = 1$. The parabola is on top.
- Set up the integral: $A = \int_0^4 [(5x - x^2) - (x)]\,dx = \int_0^4 (4x - x^2)\,dx$.

Tricks Used:

- **Finding Intersections:** Setting the functions equal to each other to find the limits of integration.
- **Test Points:** Plugging in a value within the interval to determine the top vs. bottom function.

14.1.2 Problem Type 2: Integrating with Respect to y (Right - Left)

Used when integrating with respect to ‘x‘ would require multiple integrals, but the region has a consistent right and left boundary.

Formula: $A = \int_c^d [x_{\text{right}}(y) - x_{\text{left}}(y)]\, dy$

Example (Q7): Bounded by $x = y^4$ and $x = 2 - y^2$.

Strategy:

- This is much harder to do with respect to ‘x‘. It’s set up for ‘y‘.
- Find intersections: $y^4 = 2 - y^2 \Rightarrow y^4 + y^2 - 2 = 0 \Rightarrow (y^2 + 2)(y^2 - 1) = 0 \Rightarrow y = \pm 1$. These are your limits ‘c‘ and ‘d‘.
- Determine right vs. left. At $y = 0$, $x = 0$ and $x = 2$. So $x = 2 - y^2$ is the right function.
- Set up integral: $A = \int_{-1}^{1} [(2 - y^2) - (y^4)]\, dy$.

Tricks Used:

- **Changing Perspective:** Deciding to integrate with respect to ‘y‘ when it’s simpler. This requires solving equations for ‘x‘ in terms of ‘y‘.

14.2 Sections 6.2 & 6.3: Volumes of Revolution

The most important decision is choosing the method: Disk/Washer or Cylindrical Shells.

- **Disk/Washer:** Use when your representative rectangle is **PERPENDICULAR** to the axis of rotation.
- **Shells:** Use when your representative rectangle is **PARALLEL** to the axis of rotation.

Key Trick: Finding the Radius R(x) or r(x)

This is the most common point of confusion.

- **If rotating around the x-axis ($y = 0$):** The radius is simply the function value, $R(x) = f(x)$.
- **If rotating around a horizontal line $y = k$:**
  - If the region is *above* the line, the radius is ‘function - k‘.
  - If the region is *below* the line, the radius is ‘k - function‘.
  - **General Rule:** Radius is always the ‘(Farther Boundary) - (Closer Boundary)‘. For the outer radius $R(x)$, it’s $|y_{\text{top}} - k|$. For the inner radius $r(x)$, it’s $|y_{\text{bottom}} - k|$.
- **If rotating around the y-axis ($x = 0$):** The radius is $R(y) = f(y)$.
- **If rotating around a vertical line $x = k$:**
  - The same logic applies. The radius is $|x_{\text{right}} - k|$ or $|x_{\text{left}} - k|$.

14.2.1 Problem Type 1: Disk Method (Rotation about x- or y-axis)

This is a washer with an inner radius of 0.

Example (from Q3): Region under $y = x + 1$ from $x = 0$ to $x = 2$, rotated about the x-axis.

Strategy (Washer/Disk):

- Axis is horizontal (x-axis), so integrate w.r.t ‘x‘. Rectangle is vertical (perpendicular).
- Radius $R(x) = (x + 1) - 0 = x + 1$.
- Volume: $V = \int_0^2 \pi[R(x)]^2\, dx = \int_0^2 \pi(x + 1)^2\, dx$.

14.2.2 Problem Type 2: Washer Method (Rotation about off-axis line)

Example (Q10): Region between $y = x^2$ and $x = y^2$ (or $y = \sqrt{x}$), rotated about $y = 1$.

Strategy (Washer/Disk):

- Axis is horizontal ($y = 1$), so integrate w.r.t ‘x‘. Rectangle is vertical.
- Outer Radius $R(x)$: The axis is $y = 1$. The bottom curve is $y = x^2$. This is farther from the axis. Radius is ‘(axis) - (curve)‘ $= 1 - x^2$.
- Inner Radius $r(x)$: The top curve is $y = \sqrt{x}$. This is closer to the axis. Radius is ‘(axis) - (curve)‘ $= 1 - \sqrt{x}$.
- Volume: $V = \int_0^1 \pi[(1 - x^2)^2 - (1 - \sqrt{x})^2]\, dx$.

14.2.3 Problem Type 3: Cylindrical Shells Method

Sometimes this method avoids solving for y in terms of x (or vice-versa).

Example (Q6): Region under $y = x^3$ from $x = 1$ to $x = 3$, rotated about the y-axis.

Strategy (Shells):

- Axis is vertical (y-axis). To use shells, we need a rectangle parallel to the axis, so a vertical rectangle, meaning we integrate w.r.t ‘x‘.
- Shell radius: The distance from the y-axis to the rectangle is simply ‘x‘.
- Shell height: The height of the rectangle is the function value, ‘h(x) = x^3‘.$Volume$:V $= \int_1^3 2\pi(\text{radius})(\text{height})\, dx = \int_1^3 2\pi(x)(x^3)\, dx = \int_1^3 2\pi x^4\, dx$.

14.2.4 Problem Type 4: Shells with Off-Axis Rotation

Example (Q12): Region bounded by $y = x^3, y = 8, x = 0$, rotated about $x = 7$.

Strategy (Shells):

- Axis is vertical ($x = 7$). For shells, we need a parallel (vertical) rectangle. Integrate w.r.t ‘x‘.
- Shell radius: The distance from the axis of rotation ($x = 7$) to the rectangle at x is $7 - x$.
- Shell height: The height of the rectangle is ‘top curve - bottom curve‘ $= 8 - x^3$.
- Volume: $V = \int_0^2 2\pi(7 - x)(8 - x^3)\, dx$.

Tricks Used:

- **Choosing the right method:** Ask yourself, "Is it easier to express functions in terms of x or y?" and "Does one method avoid multiple integrals?"
- **Correctly defining the radius:** This is the most critical trick. Always think of it as a distance: '—curve - axis of rotation—'.
- **Correctly defining the height/width:** For washers, it's 'dx' or 'dy'. For shells, the height 'h(x)' is 'top-bottom' and the width is 'dx'.