

Homework 7.3 Trigonometric Substitution

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1 Integration Problems and Solutions

1.1 Problem 1

Evaluate the integral: $\int \frac{x}{\sqrt{81+x^2}} dx$

Solution

This integral is solved using u-substitution. Let $u = 81 + x^2$. Then $du = 2x dx$, which implies $x dx = \frac{du}{2}$. Substituting these into the integral gives:

$$\int \frac{1}{\sqrt{u}} \frac{du}{2} = \frac{1}{2} \int u^{-1/2} du$$

Using the power rule for integration:

$$\frac{1}{2} \left[\frac{u^{1/2}}{1/2} \right] + C = u^{1/2} + C$$

Substituting back for u: **Answer:** $\sqrt{81 + x^2} + C$

1.2 Problem 2

Evaluate the integral: $\int \frac{x}{\sqrt{x^2-5}} dx$

Solution

This is also solved with a u-substitution. Let $u = x^2 - 5$. Then $du = 2x dx$, so $x dx = \frac{du}{2}$. The integral becomes:

$$\int \frac{1}{\sqrt{u}} \frac{du}{2} = \frac{1}{2} \int u^{-1/2} du$$

Integrating gives:

$$\frac{1}{2} \left[\frac{u^{1/2}}{1/2} \right] + C = u^{1/2} + C$$

Substituting back for u: **Answer:** $\sqrt{x^2 - 5} + C$

1.3 Problem 3

Evaluate the integral: $\int_0^3 \sqrt{x^2 + 9} dx$

Solution

This integral requires trigonometric substitution. Let $x = 3 \tan(\theta)$, so $dx = 3 \sec^2(\theta) d\theta$. The expression $\sqrt{x^2 + 9}$ becomes $\sqrt{9 \tan^2(\theta) + 9} = 3 \sec(\theta)$. Change the limits of integration:

- When $x = 0$, $\tan(\theta) = 0 \implies \theta = 0$.
- When $x = 3$, $\tan(\theta) = 1 \implies \theta = \frac{\pi}{4}$.

The integral transforms to:

$$\int_0^{\pi/4} (3 \sec(\theta))(3 \sec^2(\theta) d\theta) = 9 \int_0^{\pi/4} \sec^3(\theta) d\theta$$

Using the standard integral of $\sec^3(\theta)$:

$$\begin{aligned} & 9 \left[\frac{1}{2} (\sec(\theta) \tan(\theta) + \ln |\sec(\theta) + \tan(\theta)|) \right]_0^{\pi/4} \\ &= \frac{9}{2} [(\sec(\frac{\pi}{4}) \tan(\frac{\pi}{4}) + \ln |\sec(\frac{\pi}{4}) + \tan(\frac{\pi}{4})|) - (\sec(0) \tan(0) + \ln |\sec(0) + \tan(0)|)] \\ &= \frac{9}{2} [(\sqrt{2} \cdot 1 + \ln(\sqrt{2} + 1)) - (1 \cdot 0 + \ln(1))] \end{aligned}$$

Answer: $\frac{9}{2}(\sqrt{2} + \ln(1 + \sqrt{2}))$

1.4 Problem 4

Evaluate $\int \frac{x^3}{\sqrt{16+x^2}} dx$ using $x = 4 \tan(\theta)$.

Solution

Let $x = 4 \tan(\theta)$, so $dx = 4 \sec^2(\theta) d\theta$. Then $x^3 = 64 \tan^3(\theta)$ and $\sqrt{16+x^2} = 4 \sec(\theta)$. Substitute into the integral:

$$\int \frac{64 \tan^3(\theta)}{4 \sec(\theta)} (4 \sec^2(\theta) d\theta) = 64 \int \tan^3(\theta) \sec(\theta) d\theta$$

Rewrite as $64 \int (\sec^2 \theta - 1) \sec \theta \tan \theta d\theta$. Let $u = \sec(\theta)$, so $du = \sec(\theta) \tan(\theta) d\theta$.

$$64 \int (u^2 - 1) du = 64 \left(\frac{u^3}{3} - u \right) + C = \frac{64}{3} \sec^3(\theta) - 64 \sec(\theta) + C$$

From $x = 4 \tan(\theta)$, the triangle gives $\sec(\theta) = \frac{\sqrt{x^2+16}}{4}$. Substituting back:

$$\frac{64}{3} \left(\frac{\sqrt{x^2+16}}{4} \right)^3 - 64 \left(\frac{\sqrt{x^2+16}}{4} \right) + C = \frac{1}{3} (x^2+16)^{3/2} - 16 \sqrt{x^2+16} + C$$

Answer: $\frac{1}{3}(x^2 - 32)\sqrt{x^2 + 16} + C$

1.5 Problem 5

Evaluate $\int \frac{\sqrt{4x^2-25}}{x} dx$ using $x = \frac{5}{2} \sec(\theta)$.

Solution

Let $x = \frac{5}{2} \sec(\theta)$, so $dx = \frac{5}{2} \sec(\theta) \tan(\theta) d\theta$. Then $\sqrt{4x^2 - 25} = \sqrt{25 \sec^2(\theta) - 25} = 5 \tan(\theta)$. Substitute into the integral:

$$\int \frac{5 \tan(\theta)}{\frac{5}{2} \sec(\theta)} \left(\frac{5}{2} \sec(\theta) \tan(\theta) d\theta \right) = \int 5 \tan^2(\theta) d\theta$$

Using the identity $\tan^2(\theta) = \sec^2(\theta) - 1$:

$$5 \int (\sec^2(\theta) - 1) d\theta = 5 \tan(\theta) - 5\theta + C$$

From the substitution, $\tan(\theta) = \frac{\sqrt{4x^2 - 25}}{5}$ and $\theta = \operatorname{arcsec}\left(\frac{2x}{5}\right)$. **Answer:** $\sqrt{4x^2 - 25} - 5 \operatorname{arcsec}\left(\frac{2x}{5}\right) + C$

1.6 Problem 6

Consider $\int \frac{x^4}{\sqrt{1+x^2}} dx$. Transform the integral using a trigonometric substitution.

Solution

The form $\sqrt{1+x^2}$ suggests the substitution $x = \tan(\theta)$, so $dx = \sec^2(\theta) d\theta$.

$$\int \frac{\tan^4(\theta)}{\sqrt{1+\tan^2(\theta)}} \sec^2(\theta) d\theta = \int \frac{\tan^4(\theta)}{\sec(\theta)} \sec^2(\theta) d\theta$$

Answer: $\int \tan^4(\theta) \sec(\theta) d\theta$

1.7 Problem 7

Evaluate the integral: $\int_2^5 \frac{dx}{(x^2-1)^{3/2}}$

Solution

Let $x = \sec(\theta)$, so $dx = \sec(\theta) \tan(\theta) d\theta$. The denominator is $\tan^3(\theta)$. Limits: $x = 2 \implies \theta = \pi/3$ and $x = 5 \implies \theta = \operatorname{arcsec}(5)$.

$$\int_{\pi/3}^{\operatorname{arcsec}(5)} \frac{\sec(\theta) \tan(\theta)}{\tan^3(\theta)} d\theta = \int_{\pi/3}^{\operatorname{arcsec}(5)} \frac{\sec(\theta)}{\tan^2(\theta)} d\theta = \int_{\pi/3}^{\operatorname{arcsec}(5)} \cot(\theta) \csc(\theta) d\theta$$

The integral is $[-\csc(\theta)]_{\pi/3}^{\operatorname{arcsec}(5)} = -\csc(\operatorname{arcsec}(5)) - (-\csc(\pi/3)) = \frac{2}{\sqrt{3}} - \frac{5}{\sqrt{24}}$. **Answer:** $\frac{2\sqrt{3}}{3} - \frac{5\sqrt{6}}{12}$

1.8 Problem 8

Evaluate the integral: $\int_0^4 \frac{dt}{\sqrt{16+t^2}}$

Solution

The antiderivative of $\frac{1}{\sqrt{a^2+t^2}}$ is $\ln|t + \sqrt{a^2+t^2}|$.

$$\begin{aligned} [\ln|t + \sqrt{16+t^2}|]_0^4 &= (\ln|4 + \sqrt{16+16}|) - (\ln|0 + \sqrt{16}|) \\ &= \ln(4 + 4\sqrt{2}) - \ln(4) = \ln\left(\frac{4(1+\sqrt{2})}{4}\right) \end{aligned}$$

Answer: $\ln(1 + \sqrt{2})$

1.9 Problem 9

Evaluate $\int_0^7 \frac{7}{\sqrt{49+t^2}} dt$.

Solution

$$\begin{aligned} 7 \int_0^7 \frac{dt}{\sqrt{49+t^2}} &= 7[\ln |t + \sqrt{49+t^2}|]_0^7 \\ &= 7[(\ln |7 + \sqrt{49+49}|) - (\ln |0 + \sqrt{49}|)] \\ &= 7[\ln(7 + 7\sqrt{2}) - \ln(7)] = 7 \ln \left(\frac{7(1 + \sqrt{2})}{7} \right) \end{aligned}$$

Answer: $7 \ln(1 + \sqrt{2})$

1.10 Problem 10

Evaluate the integral: $\int \frac{\sqrt{x^2-25}}{x^3} dx$

Solution

Let $x = 5 \sec(\theta)$, so $dx = 5 \sec(\theta) \tan(\theta) d\theta$.

$$\int \frac{5 \tan(\theta)}{125 \sec^3(\theta)} 5 \sec(\theta) \tan(\theta) d\theta = \frac{1}{5} \int \frac{\tan^2(\theta)}{\sec^2(\theta)} d\theta = \frac{1}{5} \int \sin^2(\theta) d\theta$$

Using the half-angle identity: $\frac{1}{10} \int (1 - \cos(2\theta)) d\theta = \frac{1}{10}(\theta - \sin(\theta) \cos(\theta)) + C$. From $x = 5 \sec(\theta)$, we have $\theta = \operatorname{arcsec}(x/5)$, $\sin(\theta) = \frac{\sqrt{x^2-25}}{x}$, and $\cos(\theta) = \frac{5}{x}$. **Answer:** $\frac{1}{10} \left[\operatorname{arcsec} \left(\frac{x}{5} \right) - \frac{5\sqrt{x^2-25}}{x^2} \right] + C$

1.11 Problem 11

Evaluate the integral: $\int \frac{\sqrt{4+x^2}}{x} dx$

Solution

Let $x = 2 \tan(\theta)$, so $dx = 2 \sec^2(\theta) d\theta$.

$$\int \frac{2 \sec(\theta)}{2 \tan(\theta)} 2 \sec^2(\theta) d\theta = 2 \int \frac{\sec^3(\theta)}{\tan(\theta)} d\theta = 2 \int (\sec(\theta) \tan(\theta) + \csc(\theta)) d\theta$$

This integrates to $2[\sec(\theta) - \ln |\csc(\theta) + \cot(\theta)|] + C$. From $x = 2 \tan(\theta)$, we have $\sec(\theta) = \frac{\sqrt{x^2+4}}{2}$, $\csc(\theta) = \frac{\sqrt{x^2+4}}{x}$, and $\cot(\theta) = \frac{2}{x}$. **Answer:** $\sqrt{4+x^2} - 2 \ln \left| \frac{\sqrt{4+x^2}+2}{x} \right| + C$

1.12 Problem 12

Evaluate the integral: $\int 3x\sqrt{1-x^4} dx$

Solution

Let $u = x^2$, then $du = 2x dx \implies x dx = \frac{du}{2}$.

$$\int 3\sqrt{1-u^2} \left(\frac{du}{2} \right) = \frac{3}{2} \int \sqrt{1-u^2} du$$

The integral of $\sqrt{1-u^2}$ is a standard form: $\frac{1}{2}(u\sqrt{1-u^2} + \arcsin(u))$.

$$\frac{3}{2} \cdot \frac{1}{2} [u\sqrt{1-u^2} + \arcsin(u)] + C$$

Substituting back $u = x^2$: **Answer:** $\frac{3}{4} [x^2\sqrt{1-x^4} + \arcsin(x^2)] + C$

1.13 Problem 13

Evaluate the integral: $\int x^3 \sqrt{64 + x^2} dx$

Solution

Let $u = 64 + x^2$, so $du = 2x dx$ and $x^2 = u - 64$. Rewrite as $\int x^2 \sqrt{64 + x^2} \cdot (x dx)$. Substitute:

$$\int (u - 64) \sqrt{u} \left(\frac{du}{2} \right) = \frac{1}{2} \int (u^{3/2} - 64u^{1/2}) du$$

Integrate: $\frac{1}{2} \left[\frac{2}{5} u^{5/2} - 64 \cdot \frac{2}{3} u^{3/2} \right] + C = \frac{1}{5} u^{5/2} - \frac{64}{3} u^{3/2} + C$. Factor to simplify: $\frac{1}{15} u^{3/2} (3u - 320) + C$. Substitute back $u = 64 + x^2$:

$$\frac{1}{15} (64 + x^2)^{3/2} (3(64 + x^2) - 320) + C$$

Answer: $\frac{1}{15} (3x^2 - 128)(64 + x^2)^{3/2} + C$

1.14 Problem 14

Evaluate the integral: $\int \frac{x^2}{\sqrt{49 - x^2}} dx$

Solution

Let $x = 7 \sin(\theta)$, so $dx = 7 \cos(\theta) d\theta$.

$$\int \frac{49 \sin^2(\theta)}{7 \cos(\theta)} (7 \cos(\theta) d\theta) = 49 \int \sin^2(\theta) d\theta$$

Use the half-angle identity:

$$\frac{49}{2} \int (1 - \cos(2\theta)) d\theta = \frac{49}{2} \left(\theta - \frac{1}{2} \sin(2\theta) \right) + C = \frac{49}{2} (\theta - \sin(\theta) \cos(\theta)) + C$$

From $x = 7 \sin(\theta)$, we have $\theta = \arcsin(x/7)$ and $\cos(\theta) = \frac{\sqrt{49 - x^2}}{7}$. **Answer:** $\frac{49}{2} \arcsin\left(\frac{x}{7}\right) - \frac{x}{2} \sqrt{49 - x^2} + C$

Summary of Rules, Formulas, and Tricks

This set of problems primarily tests u-substitution and trigonometric substitution.

Key Integration Rules & Formulas

- **Power Rule:**

$$\int u^n du = \frac{u^{n+1}}{n+1} + C, \quad (n \neq -1)$$

- **U-Substitution:** The main strategy is to find a function u in the integrand whose derivative du also appears. This simplifies the integral into a more basic form. Look for an "inside" function and its derivative on the "outside".

- **Trigonometric Identities:**

- **Pythagorean:** $\sin^2 \theta + \cos^2 \theta = 1$ and $1 + \tan^2 \theta = \sec^2 \theta$
- **Half-Angle:** $\sin^2 \theta = \frac{1}{2}(1 - \cos(2\theta))$ and $\cos^2 \theta = \frac{1}{2}(1 + \cos(2\theta))$

Trigonometric Substitution Standard Forms

The trick is to recognize which form the integral takes based on the expression under the square root.

1. **Form** $\sqrt{a^2 - x^2}$

- **Substitution:** $x = a \sin(\theta)$
- **Identity Used:** $a^2 - a^2 \sin^2(\theta) = a^2 \cos^2(\theta)$

2. **Form** $\sqrt{a^2 + x^2}$

- **Substitution:** $x = a \tan(\theta)$
- **Identity Used:** $a^2 + a^2 \tan^2(\theta) = a^2 \sec^2(\theta)$

3. **Form** $\sqrt{x^2 - a^2}$

- **Substitution:** $x = a \sec(\theta)$
- **Identity Used:** $a^2 \sec^2(\theta) - a^2 = a^2 \tan^2(\theta)$

Tricks and Important Concepts Shown

- **Look for U-Sub First:** Before attempting a complex trigonometric substitution, always check if a simple u-substitution will work. It is often much faster.
- **Change Limits of Integration:** For definite integrals, when you substitute variables (e.g., from x to θ), you **must** change the limits of integration to the new variable's values. This avoids the final step of converting back to x .
- **Draw the Triangle:** For indefinite integrals, after you integrate in terms of θ , you must convert back to x . Drawing a right triangle based on your initial substitution (e.g., if $x = a \tan(\theta)$, then $\tan(\theta) = x/a$) is the most reliable way to find expressions for $\sin(\theta)$, $\sec(\theta)$, etc., in terms of x .