Homework 7.3 Trigonometric Substitution

Tashfeen Omran

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1 Integration Problems and Solutions

1.1 Problem 1

Evaluate the integral: $\int \frac{x}{\sqrt{81+x^2}}\,dx$

Solution

This integral is solved using u-substitution. Let $u = 81+x^2$. Then $du = 2x\,dx$, which implies $x\,dx = \frac{du}{2}$. Substituting these into the integral gives:

$$\int \frac{1}{\sqrt{u}}\frac{du}{2} = \frac{1}{2}\int u^{-1/2}\,du$$

Using the power rule for integration:

$$\frac{1}{2}\left[\frac{u^{1/2}}{1/2}\right] + C = u^{1/2} + C$$

Substituting back for u: **Answer:** $\sqrt{81+x^2} + C$

1.2 Problem 2

Evaluate the integral: $\int \frac{x}{\sqrt{x^2-5}}\,dx$

Solution

This is also solved with a u-substitution. Let $u = x^2 - 5$. Then $du = 2x\,dx$, so $x\,dx = \frac{du}{2}$. The integral becomes:

$$\int \frac{1}{\sqrt{u}}\frac{du}{2} = \frac{1}{2}\int u^{-1/2}\,du$$

Integrating gives:

$$\frac{1}{2}\left[\frac{u^{1/2}}{1/2}\right] + C = u^{1/2} + C$$

Substituting back for u: **Answer:** $\sqrt{x^2-5} + C$

1.3 Problem 3

Evaluate the integral: $\int_0^3 \sqrt{x^2+9}\,dx$

Solution

This integral requires trigonometric substitution. Let $x = 3\tan(\theta)$, so $dx = 3\sec^2(\theta)\,d\theta$. The expression $\sqrt{x^2+9}$ becomes $\sqrt{9\tan^2(\theta)+9} = 3\sec(\theta)$. Change the limits of integration:

- When $x = 0, \tan(\theta) = 0 \implies \theta = 0$.
- When $x = 3, \tan(\theta) = 1 \implies \theta = \frac{\pi}{4}$.

The integral transforms to:

$$\int_0^{\pi/4} (3\sec(\theta))(3\sec^2(\theta)\,d\theta) = 9\int_0^{\pi/4} \sec^3(\theta)\,d\theta$$

Using the standard integral of $\sec^3(\theta)$:

$$\begin{aligned}
&9\left[\frac{1}{2}(\sec(\theta)\tan(\theta) + \ln|\sec(\theta)+\tan(\theta)|)\right]_0^{\pi/4} \\
=&\frac{9}{2}[(\sec(\tfrac{\pi}{4})\tan(\tfrac{\pi}{4}) + \ln|\sec(\tfrac{\pi}{4})+\tan(\tfrac{\pi}{4})|) - (\sec(0)\tan(0) + \ln|\sec(0)+\tan(0)|)] \\
=&\frac{9}{2}[(\sqrt{2}\cdot 1 + \ln(\sqrt{2}+1)) - (1\cdot 0 + \ln(1))]
\end{aligned}$$

Answer: $\frac{9}{2}(\sqrt{2} + \ln(1+\sqrt{2}))$

1.4 Problem 4

Evaluate $\int \frac{x^3}{\sqrt{16+x^2}}\,dx$ using $x = 4\tan(\theta)$.

Solution

Let $x = 4\tan(\theta)$, so $dx = 4\sec^2(\theta)\,d\theta$. Then $x^3 = 64\tan^3(\theta)$ and $\sqrt{16+x^2} = 4\sec(\theta)$. Substitute into the integral:

$$\int \frac{64\tan^3(\theta)}{4\sec(\theta)}(4\sec^2(\theta)\,d\theta) = 64\int \tan^3(\theta)\sec(\theta)\,d\theta$$

Rewrite as $64\int(\sec^2\theta - 1)\sec\theta\tan\theta\,d\theta$. Let $u = \sec(\theta)$, so $du = \sec(\theta)\tan(\theta)\,d\theta$.

$$64\int (u^2-1)\,du = 64\left(\frac{u^3}{3} - u\right) + C = \frac{64}{3}\sec^3(\theta) - 64\sec(\theta) + C$$

From $x = 4\tan(\theta)$, the triangle gives $\sec(\theta) = \frac{\sqrt{x^2+16}}{4}$. Substituting back:

$$\frac{64}{3}\left(\frac{\sqrt{x^2+16}}{4}\right)^3 - 64\left(\frac{\sqrt{x^2+16}}{4}\right) + C = \frac{1}{3}(x^2+16)^{3/2} - 16\sqrt{x^2+16} + C$$

Answer: $\frac{1}{3}(x^2-32)\sqrt{x^2+16} + C$

1.5 Problem 5

Evaluate $\int \frac{\sqrt{4x^2-25}}{x}\,dx$ using $x = \frac{5}{2}\sec(\theta)$.

Solution

Let $x = \frac{5}{2}\sec(\theta)$, so $dx = \frac{5}{2}\sec(\theta)\tan(\theta)\,d\theta$. Then $\sqrt{4x^2-25} = \sqrt{25\sec^2(\theta)-25} = 5\tan(\theta)$. Substitute into the integral:

$$\int \frac{5\tan(\theta)}{\frac{5}{2}\sec(\theta)}\left(\frac{5}{2}\sec(\theta)\tan(\theta)\,d\theta\right) = \int 5\tan^2(\theta)\,d\theta$$

Using the identity $\tan^2(\theta) = \sec^2(\theta) - 1$:

$$5\int(\sec^2(\theta)-1)\,d\theta = 5\tan(\theta) - 5\theta + C$$

From the substitution, $\tan(\theta) = \frac{\sqrt{4x^2-25}}{5}$ and $\theta = \operatorname{arcsec}(\frac{2x}{5})$. **Answer:** $\sqrt{4x^2-25} - 5\operatorname{arcsec}\left(\frac{2x}{5}\right) + C$

1.6 Problem 6

Consider $\int \frac{x^4}{\sqrt{1+x^2}}\,dx$. Transform the integral using a trigonometric substitution.

Solution

The form $\sqrt{1+x^2}$ suggests the substitution $x = \tan(\theta)$, so $dx = \sec^2(\theta)\,d\theta$.

$$\int \frac{\tan^4(\theta)}{\sqrt{1+\tan^2(\theta)}}\sec^2(\theta)\,d\theta = \int \frac{\tan^4(\theta)}{\sec(\theta)}\sec^2(\theta)\,d\theta$$

Answer: $\int \tan^4(\theta)\sec(\theta)\,d\theta$

1.7 Problem 7

Evaluate the integral: $\int_2^5 \frac{dx}{(x^2-1)^{3/2}}$

Solution

Let $x = \sec(\theta)$, so $dx = \sec(\theta)\tan(\theta)d\theta$. The denominator is $\tan^3(\theta)$. Limits: $x = 2 \implies \theta = \pi/3$ and $x = 5 \implies \theta = \operatorname{arcsec}(5)$.

$$\int_{\pi/3}^{\operatorname{arcsec}(5)} \frac{\sec(\theta)\tan(\theta)}{\tan^3(\theta)}\,d\theta = \int_{\pi/3}^{\operatorname{arcsec}(5)} \frac{\sec(\theta)}{\tan^2(\theta)}\,d\theta = \int_{\pi/3}^{\operatorname{arcsec}(5)} \cot(\theta)\csc(\theta)\,d\theta$$

The integral is $[-\csc(\theta)]_{\pi/3}^{\operatorname{arcsec}(5)} = -\csc(\operatorname{arcsec}(5)) - (-\csc(\frac{\pi}{3})) = \frac{2}{\sqrt{3}} - \frac{5}{\sqrt{24}}$. **Answer:** $\frac{2\sqrt{3}}{3} - \frac{5\sqrt{6}}{12}$

1.8 Problem 8

Evaluate the integral: $\int_0^4 \frac{dt}{\sqrt{16+t^2}}$

Solution

The antiderivative of $\frac{1}{\sqrt{a^2+t^2}}$ is $\ln|t+\sqrt{a^2+t^2}|$.

$$\begin{aligned}[\ln|t+\sqrt{16+t^2}|]_0^4 &= (\ln|4+\sqrt{16+16}|) - (\ln|0+\sqrt{16}|)\\ &= \ln(4+4\sqrt{2}) - \ln(4) = \ln\left(\frac{4(1+\sqrt{2})}{4}\right)\end{aligned}$$

Answer: $\ln(1+\sqrt{2})$

1.9 Problem 9

Evaluate $\int_0^7 \frac{7}{\sqrt{49+t^2}}\,dt$.

Solution

$7\int_0^7 \frac{dt}{\sqrt{49+t^2}} = 7[\ln|t+\sqrt{49+t^2}|]_0^7$

$$= 7[(\ln|7+\sqrt{49+49}|) - (\ln|0+\sqrt{49}|)]$$
$$= 7[\ln(7+7\sqrt{2}) - \ln(7)] = 7\ln\left(\frac{7(1+\sqrt{2})}{7}\right)$$

Answer: $7\ln(1+\sqrt{2})$

1.10 Problem 10

Evaluate the integral: $\int \frac{\sqrt{x^2-25}}{x^3}\,dx$

Solution

Let $x = 5\sec(\theta)$, so $dx = 5\sec(\theta)\tan(\theta)\,d\theta$.

$$\int \frac{5\tan(\theta)}{125\sec^3(\theta)} 5\sec(\theta)\tan(\theta)\,d\theta = \frac{1}{5}\int \frac{\tan^2(\theta)}{\sec^2(\theta)}\,d\theta = \frac{1}{5}\int \sin^2(\theta)\,d\theta$$

Using the half-angle identity: $\frac{1}{10}\int(1-\cos(2\theta))\,d\theta = \frac{1}{10}(\theta - \sin(\theta)\cos(\theta)) + C$. From $x = 5\sec(\theta)$, we have $\theta = \operatorname{arcsec}(x/5)$, $\sin(\theta) = \frac{\sqrt{x^2-25}}{x}$, and $\cos(\theta) = \frac{5}{x}$. **Answer:** $\frac{1}{10}\left[\operatorname{arcsec}\left(\frac{x}{5}\right) - \frac{5\sqrt{x^2-25}}{x^2}\right] + C$

1.11 Problem 11

Evaluate the integral: $\int \frac{\sqrt{4+x^2}}{x}\,dx$

Solution

Let $x = 2\tan(\theta)$, so $dx = 2\sec^2(\theta)\,d\theta$.

$$\int \frac{2\sec(\theta)}{2\tan(\theta)} 2\sec^2(\theta)\,d\theta = 2\int \frac{\sec^3(\theta)}{\tan(\theta)}\,d\theta = 2\int(\sec(\theta)\tan(\theta) + \csc(\theta))\,d\theta$$

This integrates to $2[\sec(\theta) - \ln|\csc(\theta)+\cot(\theta)|] + C$. From $x = 2\tan(\theta)$, we have $\sec(\theta) = \frac{\sqrt{x^2+4}}{2}$, $\csc(\theta) = \frac{\sqrt{x^2+4}}{x}$, and $\cot(\theta) = \frac{2}{x}$. **Answer:** $\sqrt{4+x^2} - 2\ln\left|\frac{\sqrt{4+x^2}+2}{x}\right| + C$

1.12 Problem 12

Evaluate the integral: $\int 3x\sqrt{1-x^4}\,dx$

Solution

Let $u = x^2$, then $du = 2x\,dx \implies x\,dx = \frac{du}{2}$.

$$\int 3\sqrt{1-u^2}\left(\frac{du}{2}\right) = \frac{3}{2}\int \sqrt{1-u^2}\,du$$

The integral of $\sqrt{1-u^2}$ is a standard form: $\frac{1}{2}(u\sqrt{1-u^2} + \arcsin(u))$.

$$\frac{3}{2}\cdot\frac{1}{2}[u\sqrt{1-u^2} + \arcsin(u)] + C$$

Substituting back $u = x^2$: **Answer:** $\frac{3}{4}[x^2\sqrt{1-x^4} + \arcsin(x^2)] + C$

1.13 Problem 13

Evaluate the integral: $\int x^3\sqrt{64+x^2}\,dx$

Solution

Let $u = 64 + x^2$, so $du = 2x\,dx$ and $x^2 = u - 64$. Rewrite as $\int x^2\sqrt{64+x^2}\cdot(x\,dx)$. Substitute:

$$\int (u-64)\sqrt{u}\left(\frac{du}{2}\right) = \frac{1}{2}\int (u^{3/2} - 64u^{1/2})\,du$$

Integrate: $\frac{1}{2}\left[\frac{2}{5}u^{5/2} - 64\cdot\frac{2}{3}u^{3/2}\right] + C = \frac{1}{5}u^{5/2} - \frac{64}{3}u^{3/2} + C$. Factor to simplify: $\frac{1}{15}u^{3/2}(3u - 320) + C$. Substitute back $u = 64 + x^2$:

$$\frac{1}{15}(64+x^2)^{3/2}(3(64+x^2) - 320) + C$$

Answer: $\frac{1}{15}(3x^2 - 128)(64+x^2)^{3/2} + C$

1.14 Problem 14

Evaluate the integral: $\int \frac{x^2}{\sqrt{49-x^2}}\,dx$

Solution

Let $x = 7\sin(\theta)$, so $dx = 7\cos(\theta)\,d\theta$.

$$\int \frac{49\sin^2(\theta)}{7\cos(\theta)}(7\cos(\theta)\,d\theta) = 49\int \sin^2(\theta)\,d\theta$$

Use the half-angle identity:

$$\frac{49}{2}\int (1-\cos(2\theta))\,d\theta = \frac{49}{2}\left(\theta - \frac{1}{2}\sin(2\theta)\right) + C = \frac{49}{2}(\theta - \sin(\theta)\cos(\theta)) + C$$

From $x = 7\sin(\theta)$, we have $\theta = \arcsin(x/7)$ and $\cos(\theta) = \frac{\sqrt{49-x^2}}{7}$. **Answer:** $\frac{49}{2}\arcsin\left(\frac{x}{7}\right) - \frac{x}{2}\sqrt{49-x^2} + C$

Summary of Rules, Formulas, and Tricks

This set of problems primarily tests u-substitution and trigonometric substitution.

Key Integration Rules & Formulas

- **Power Rule:**
$$\int u^n\,du = \frac{u^{n+1}}{n+1} + C, \quad (n \neq -1)$$
- **U-Substitution:** The main strategy is to find a function `u` in the integrand whose derivative `du` also appears. This simplifies the integral into a more basic form. Look for an "inside" function and its derivative on the "outside".
- **Trigonometric Identities:**
  - **Pythagorean:** $\sin^2\theta + \cos^2\theta = 1 \quad$ and $\quad 1 + \tan^2\theta = \sec^2\theta$
  - **Half-Angle:** $\sin^2\theta = \frac{1}{2}(1-\cos(2\theta)) \quad$ and $\quad \cos^2\theta = \frac{1}{2}(1+\cos(2\theta))$

Trigonometric Substitution Standard Forms

The trick is to recognize which form the integral takes based on the expression under the square root.

1. **Form** $\sqrt{a^2 - x^2}$
   - **Substitution:** $x = a\sin(\theta)$
   - **Identity Used:** $a^2 - a^2\sin^2(\theta) = a^2\cos^2(\theta)$
2. **Form** $\sqrt{a^2 + x^2}$
   - **Substitution:** $x = a\tan(\theta)$
   - **Identity Used:** $a^2 + a^2\tan^2(\theta) = a^2\sec^2(\theta)$
3. **Form** $\sqrt{x^2 - a^2}$
   - **Substitution:** $x = a\sec(\theta)$
   - **Identity Used:** $a^2\sec^2(\theta) - a^2 = a^2\tan^2(\theta)$

Tricks and Important Concepts Shown

- **Look for U-Sub First:** Before attempting a complex trigonometric substitution, always check if a simple u-substitution will work. It is often much faster.
- **Change Limits of Integration:** For definite integrals, when you substitute variables (e.g., from x to θ), you **must** change the limits of integration to the new variable's values. This avoids the final step of converting back to x.
- **Draw the Triangle:** For indefinite integrals, after you integrate in terms of θ, you must convert back to x. Drawing a right triangle based on your initial substitution (e.g., if $x = a\tan(\theta)$, then $\tan(\theta) = x/a$) is the most reliable way to find expressions for $\sin(\theta)$, $\sec(\theta)$, etc., in terms of x.