

Homework 7.8 Improper Integrals

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1 Improper Integral Problems and Solutions

1.1 Problem 1

Determine whether the integral $\int_1^\infty 3x^{-4} dx$ is convergent or divergent. If it is convergent, evaluate it.

Solution

This is a Type 1 improper integral because it has an infinite limit of integration.

$$\begin{aligned}\int_1^\infty 3x^{-4} dx &= \lim_{t \rightarrow \infty} \int_1^t 3x^{-4} dx \\&= \lim_{t \rightarrow \infty} \left[3 \frac{x^{-3}}{-3} \right]_1^t \\&= \lim_{t \rightarrow \infty} [-x^{-3}]_1^t = \lim_{t \rightarrow \infty} \left[-\frac{1}{x^3} \right]_1^t \\&= \lim_{t \rightarrow \infty} \left(-\frac{1}{t^3} - \left(-\frac{1}{1^3} \right) \right) \\&= \lim_{t \rightarrow \infty} \left(-\frac{1}{t^3} + 1 \right) \\&= 0 + 1 = 1\end{aligned}$$

Answer: Convergent. The integral evaluates to **1**.

1.2 Problem 2

Determine whether the integral $\int_{-\infty}^{-1} \frac{1}{\sqrt[3]{x}} dx$ is convergent or divergent. If it is convergent, evaluate it.

Solution

This is a Type 1 improper integral.

$$\begin{aligned}\int_{-\infty}^{-1} x^{-1/3} dx &= \lim_{t \rightarrow -\infty} \int_t^{-1} x^{-1/3} dx \\&= \lim_{t \rightarrow -\infty} \left[\frac{x^{2/3}}{2/3} \right]_t^{-1} = \lim_{t \rightarrow -\infty} \left[\frac{3}{2} x^{2/3} \right]_t^{-1} \\&= \lim_{t \rightarrow -\infty} \left(\frac{3}{2} (-1)^{2/3} - \frac{3}{2} t^{2/3} \right) \\&= \frac{3}{2} (1) - \lim_{t \rightarrow -\infty} \frac{3}{2} t^{2/3} \\&= \frac{3}{2} - \infty = -\infty\end{aligned}$$

As $t \rightarrow -\infty$, $t^{2/3}$ (which is $(\sqrt[3]{t})^2$) approaches $+\infty$. The limit does not exist. **Answer: DIVERGES.**

1.3 Problem 3

Determine whether the integral $\int_{-6}^{\infty} \frac{1}{x+7} dx$ is convergent or divergent.

Solution

This is a Type 1 improper integral.

$$\begin{aligned}\int_{-6}^{\infty} \frac{1}{x+7} dx &= \lim_{t \rightarrow \infty} \int_{-6}^t \frac{1}{x+7} dx \\&= \lim_{t \rightarrow \infty} [\ln |x+7|]_{-6}^t \\&= \lim_{t \rightarrow \infty} (\ln |t+7| - \ln |-6+7|) \\&= \lim_{t \rightarrow \infty} \ln(t+7) - \ln(1) \\&= \infty - 0 = \infty\end{aligned}$$

The limit does not exist. **Answer: DIVERGES.**

1.4 Problem 4

Determine whether the integral $\int_8^{\infty} \frac{1}{(x-7)^{3/2}} dx$ is convergent or divergent.

Solution

This is a Type 1 improper integral.

$$\begin{aligned}\int_8^{\infty} (x-7)^{-3/2} dx &= \lim_{t \rightarrow \infty} \int_8^t (x-7)^{-3/2} dx \\&= \lim_{t \rightarrow \infty} \left[\frac{(x-7)^{-1/2}}{-1/2} \right]_8^t = \lim_{t \rightarrow \infty} \left[\frac{-2}{\sqrt{x-7}} \right]_8^t \\&= \lim_{t \rightarrow \infty} \left(\frac{-2}{\sqrt{t-7}} - \left(\frac{-2}{\sqrt{8-7}} \right) \right) \\&= \lim_{t \rightarrow \infty} \left(\frac{-2}{\sqrt{t-7}} + 2 \right) \\&= 0 + 2 = 2\end{aligned}$$

Answer: Convergent. The integral evaluates to **2**.

1.5 Problem 5

Determine whether the integral $\int_0^{\infty} \frac{1}{\sqrt{1+x}} dx$ is convergent or divergent.

Solution

This is a Type 1 improper integral.

$$\begin{aligned}\int_0^\infty (1+x)^{-1/2} dx &= \lim_{t \rightarrow \infty} \int_0^t (1+x)^{-1/2} dx \\&= \lim_{t \rightarrow \infty} \left[\frac{(1+x)^{1/2}}{1/2} \right]_0^t = \lim_{t \rightarrow \infty} [2\sqrt{1+x}]_0^t \\&= \lim_{t \rightarrow \infty} (2\sqrt{1+t} - 2\sqrt{1+0}) \\&= \infty - 2 = \infty\end{aligned}$$

The limit does not exist. **Answer: DIVERGES.**

1.6 Problem 6

Determine whether the integral $\int_{-\infty}^0 \frac{x}{(x^2+4)^2} dx$ is convergent or divergent.

Solution

This is a Type 1 improper integral. Use u-substitution: $u = x^2 + 4$, $du = 2x dx \implies x dx = du/2$.

$$\begin{aligned}\int_{-\infty}^0 \frac{x}{(x^2+4)^2} dx &= \lim_{t \rightarrow -\infty} \int_t^0 \frac{x}{(x^2+4)^2} dx \\&= \lim_{t \rightarrow -\infty} \int_{x=t}^{x=0} \frac{1}{u^2} \frac{du}{2} \\&= \frac{1}{2} \lim_{t \rightarrow -\infty} \left[-\frac{1}{u} \right]_{x=t}^{x=0} = \frac{1}{2} \lim_{t \rightarrow -\infty} \left[-\frac{1}{x^2+4} \right]_t^0 \\&= \frac{1}{2} \lim_{t \rightarrow -\infty} \left(-\frac{1}{0^2+4} - \left(-\frac{1}{t^2+4} \right) \right) \\&= \frac{1}{2} \lim_{t \rightarrow -\infty} \left(-\frac{1}{4} + \frac{1}{t^2+4} \right) \\&= \frac{1}{2} \left(-\frac{1}{4} + 0 \right) = -\frac{1}{8}\end{aligned}$$

Answer: Convergent. The integral evaluates to **-1/8**.

1.7 Problem 7

Determine whether the integral $\int_1^\infty \frac{x^3+x+1}{x^5} dx$ is convergent or divergent.

Solution

First, simplify the integrand.

$$\frac{x^3+x+1}{x^5} = \frac{x^3}{x^5} + \frac{x}{x^5} + \frac{1}{x^5} = x^{-2} + x^{-4} + x^{-5}$$

Now evaluate the integral:

$$\begin{aligned}
 \int_1^{\infty} (x^{-2} + x^{-4} + x^{-5}) dx &= \lim_{t \rightarrow \infty} \int_1^t (x^{-2} + x^{-4} + x^{-5}) dx \\
 &= \lim_{t \rightarrow \infty} \left[-x^{-1} - \frac{x^{-3}}{3} - \frac{x^{-4}}{4} \right]_1^t \\
 &= \lim_{t \rightarrow \infty} \left[-\frac{1}{x} - \frac{1}{3x^3} - \frac{1}{4x^4} \right]_1^t \\
 &= \lim_{t \rightarrow \infty} \left(\left(-\frac{1}{t} - \frac{1}{3t^3} - \frac{1}{4t^4} \right) - \left(-1 - \frac{1}{3} - \frac{1}{4} \right) \right) \\
 &= (0 - 0 - 0) - \left(-\frac{12}{12} - \frac{4}{12} - \frac{3}{12} \right) = - \left(-\frac{19}{12} \right) = \frac{19}{12}
 \end{aligned}$$

Answer: Convergent. The integral evaluates to **19/12**.

1.8 Problem 8

Determine whether the integral $\int_0^{\infty} \frac{e^x}{(8+e^x)^2} dx$ is convergent or divergent.

Solution

Use u-substitution: $u = 8 + e^x$, $du = e^x dx$.

$$\begin{aligned}
 \int_0^{\infty} \frac{e^x}{(8+e^x)^2} dx &= \lim_{t \rightarrow \infty} \int_0^t \frac{e^x}{(8+e^x)^2} dx \\
 &= \lim_{t \rightarrow \infty} \left[-\frac{1}{8+e^x} \right]_0^t \\
 &= \lim_{t \rightarrow \infty} \left(-\frac{1}{8+e^t} - \left(-\frac{1}{8+e^0} \right) \right) \\
 &= \lim_{t \rightarrow \infty} \left(-\frac{1}{8+e^t} + \frac{1}{9} \right) \\
 &= 0 + \frac{1}{9} = \frac{1}{9}
 \end{aligned}$$

Answer: Convergent. The integral evaluates to **1/9**.

1.9 Problem 9

Determine whether the integral $\int_{-\infty}^{\infty} 9xe^{-x^2} dx$ is convergent or divergent.

Solution

The integral is over $(-\infty, \infty)$, so we split it at an arbitrary point, like $x = 0$.

$$\int_{-\infty}^{\infty} 9xe^{-x^2} dx = \int_{-\infty}^0 9xe^{-x^2} dx + \int_0^{\infty} 9xe^{-x^2} dx$$

Evaluate the second integral first. Use u-substitution: $u = -x^2$, $du = -2x dx \implies 9x dx = -\frac{9}{2} du$.

$$\begin{aligned}
 \int_0^{\infty} 9xe^{-x^2} dx &= \lim_{t \rightarrow \infty} \int_0^t 9xe^{-x^2} dx \\
 &= \lim_{t \rightarrow \infty} \left[-\frac{9}{2} e^{-x^2} \right]_0^t \\
 &= \lim_{t \rightarrow \infty} \left(-\frac{9}{2} e^{-t^2} - \left(-\frac{9}{2} e^0 \right) \right) = 0 + \frac{9}{2} = \frac{9}{2}
 \end{aligned}$$

Now evaluate the first integral:

$$\begin{aligned}\int_{-\infty}^0 9xe^{-x^2} dx &= \lim_{s \rightarrow -\infty} \int_s^0 9xe^{-x^2} dx \\ &= \lim_{s \rightarrow -\infty} \left[-\frac{9}{2}e^{-x^2} \right]_s^0 \\ &= \lim_{s \rightarrow -\infty} \left(-\frac{9}{2}e^0 - \left(-\frac{9}{2}e^{-s^2} \right) \right) = -\frac{9}{2} - 0 = -\frac{9}{2}\end{aligned}$$

Since both integrals converge, the original integral converges. The total value is $\frac{9}{2} + (-\frac{9}{2}) = 0$. **Answer:** Convergent. The integral evaluates to **0**. (Note: This could also be solved by recognizing $f(x) = 9xe^{-x^2}$ is an odd function.)

1.10 Problem 10

Determine whether the integral $\int_{-\infty}^{\infty} \frac{x^5}{x^6+1} dx$ is convergent or divergent.

Solution

Split the integral at $x = 0$. Let's evaluate the part from $[0, \infty)$. Use u-substitution: $u = x^6 + 1, du = 6x^5 dx \implies x^5 dx = du/6$.

$$\begin{aligned}\int_0^{\infty} \frac{x^5}{x^6+1} dx &= \lim_{t \rightarrow \infty} \int_0^t \frac{x^5}{x^6+1} dx \\ &= \lim_{t \rightarrow \infty} \left[\frac{1}{6} \ln |x^6+1| \right]_0^t \\ &= \frac{1}{6} \lim_{t \rightarrow \infty} (\ln(t^6+1) - \ln(1)) = \frac{1}{6}(\infty - 0) = \infty\end{aligned}$$

Since one part of the split integral diverges, the entire integral diverges. **Answer: DIVERGES.**

1.11 Problem 11

Determine whether the integral $\int_0^{\infty} 4 \sin^2(\alpha) d\alpha$ is convergent or divergent.

Solution

Use the half-angle identity: $\sin^2(\alpha) = \frac{1 - \cos(2\alpha)}{2}$.

$$\begin{aligned}\int_0^{\infty} 4 \sin^2(\alpha) d\alpha &= \int_0^{\infty} 4 \left(\frac{1 - \cos(2\alpha)}{2} \right) d\alpha = \int_0^{\infty} (2 - 2 \cos(2\alpha)) d\alpha \\ &= \lim_{t \rightarrow \infty} \int_0^t (2 - 2 \cos(2\alpha)) d\alpha \\ &= \lim_{t \rightarrow \infty} [2\alpha - \sin(2\alpha)]_0^t \\ &= \lim_{t \rightarrow \infty} ((2t - \sin(2t)) - (0 - 0))\end{aligned}$$

As $t \rightarrow \infty$, the $2t$ term goes to infinity. The term $\sin(2t)$ oscillates between -1 and 1, but it cannot stop the $2t$ term from growing infinitely large. The limit does not exist. **Answer: DIVERGES.**

1.12 Problem 12

(a) Evaluate the integral: $\int_0^t 8 \sin^2(\alpha) d\alpha$ (b) Determine whether $\int_0^{\infty} 8 \sin^2(\alpha) d\alpha$ is convergent or divergent.

Solution (a)

This is a standard definite integral. Using the half-angle identity:

$$\begin{aligned}\int_0^t 8 \sin^2(\alpha) d\alpha &= \int_0^t 8 \left(\frac{1 - \cos(2\alpha)}{2} \right) d\alpha = \int_0^t (4 - 4 \cos(2\alpha)) d\alpha \\ &= [4\alpha - 2 \sin(2\alpha)]_0^t \\ &= (4t - 2 \sin(2t)) - (0 - 0)\end{aligned}$$

Answer (a): $4t - 2 \sin(2t)$

Solution (b)

To determine if the improper integral converges, we take the limit of the result from part (a) as $t \rightarrow \infty$.

$$\lim_{t \rightarrow \infty} (4t - 2 \sin(2t)) = \infty$$

The limit does not exist. **Answer (b): DIVERGES.**

1.13 Problem 13

Determine whether the integral $\int_0^\infty \sin(\theta) e^{\cos(\theta)} d\theta$ is convergent or divergent.

Solution

Use u-substitution: $u = \cos(\theta)$, $du = -\sin(\theta) d\theta$.

$$\begin{aligned}\int_0^\infty \sin(\theta) e^{\cos(\theta)} d\theta &= \lim_{t \rightarrow \infty} \int_0^t \sin(\theta) e^{\cos(\theta)} d\theta \\ &= \lim_{t \rightarrow \infty} \left[-e^{\cos(\theta)} \right]_0^t \\ &= \lim_{t \rightarrow \infty} \left(-e^{\cos(t)} - (-e^{\cos(0)}) \right) \\ &= \lim_{t \rightarrow \infty} \left(e - e^{\cos(t)} \right)\end{aligned}$$

As $t \rightarrow \infty$, $\cos(t)$ oscillates between -1 and 1. Therefore, $e^{\cos(t)}$ oscillates between e^{-1} and e^1 . The limit does not settle on a single value. **Answer: DIVERGES.**

1.14 Problem 14

Determine whether the integral $\int_1^\infty \frac{1}{x^2+x} dx$ is convergent or divergent.

Solution

Use partial fraction decomposition: $\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} \implies 1 = A(x+1) + Bx$. If $x = 0$, $A = 1$. If $x = -1$, $B = -1$. So, $\frac{1}{x^2+x} = \frac{1}{x} - \frac{1}{x+1}$.

$$\begin{aligned}\int_1^\infty \left(\frac{1}{x} - \frac{1}{x+1} \right) dx &= \lim_{t \rightarrow \infty} \int_1^t \left(\frac{1}{x} - \frac{1}{x+1} \right) dx \\ &= \lim_{t \rightarrow \infty} [\ln|x| - \ln|x+1|]_1^t \\ &= \lim_{t \rightarrow \infty} \left[\ln \left| \frac{x}{x+1} \right| \right]_1^t \\ &= \lim_{t \rightarrow \infty} \left(\ln \left(\frac{t}{t+1} \right) - \ln \left(\frac{1}{2} \right) \right)\end{aligned}$$

As $t \rightarrow \infty$, $\frac{t}{t+1} \rightarrow 1$, so $\ln(\frac{t}{t+1}) \rightarrow \ln(1) = 0$. The result is $0 - \ln(1/2) = -(-\ln(2)) = \ln(2)$. **Answer:** Convergent. The integral evaluates to **ln(2)**.

1.15 Problem 15

Determine whether the integral $\int_2^\infty \frac{dv}{v^2+2v-3}$ is convergent or divergent.

Solution

Use partial fraction decomposition: $\frac{1}{(v+3)(v-1)} = \frac{A}{v+3} + \frac{B}{v-1} \implies 1 = A(v-1) + B(v+3)$. If $v = 1$, $B = 1/4$. If $v = -3$, $A = -1/4$.

$$\begin{aligned} \int_2^\infty \frac{1/4}{v-1} - \frac{1/4}{v+3} dv &= \frac{1}{4} \lim_{t \rightarrow \infty} \int_2^t \left(\frac{1}{v-1} - \frac{1}{v+3} \right) dv \\ &= \frac{1}{4} \lim_{t \rightarrow \infty} [\ln|v-1| - \ln|v+3|]_2^t \\ &= \frac{1}{4} \lim_{t \rightarrow \infty} \left[\ln \left| \frac{v-1}{v+3} \right| \right]_2^t \\ &= \frac{1}{4} \lim_{t \rightarrow \infty} \left(\ln \left(\frac{t-1}{t+3} \right) - \ln \left(\frac{1}{5} \right) \right) \\ &= \frac{1}{4} (\ln(1) - \ln(1/5)) = \frac{1}{4} (0 - (-\ln(5))) = \frac{\ln(5)}{4} \end{aligned}$$

Answer: Convergent. The integral evaluates to **ln(5)/4**.

1.16 Problem 16

Determine whether the integral $\int_{-\infty}^0 \frac{z}{z^4+81} dz$ is convergent or divergent.

Solution

Use u-substitution: $u = z^2$, $du = 2z dz \implies z dz = du/2$.

$$\begin{aligned} \int_{-\infty}^0 \frac{z}{z^4+81} dz &= \lim_{t \rightarrow -\infty} \int_t^0 \frac{z}{(z^2)^2+81} dz \\ &= \lim_{t \rightarrow -\infty} \frac{1}{2} \int_{z=t}^{z=0} \frac{du}{u^2+81} \\ &= \frac{1}{2} \lim_{t \rightarrow -\infty} \left[\frac{1}{9} \arctan \left(\frac{u}{9} \right) \right]_{z=t}^{z=0} = \frac{1}{18} \lim_{t \rightarrow -\infty} \left[\arctan \left(\frac{z^2}{9} \right) \right]_t^0 \\ &= \frac{1}{18} \lim_{t \rightarrow -\infty} \left(\arctan(0) - \arctan \left(\frac{t^2}{9} \right) \right) \end{aligned}$$

As $t \rightarrow -\infty$, $t^2 \rightarrow \infty$, so $\arctan(t^2/9) \rightarrow \pi/2$. The result is $\frac{1}{18}(0 - \frac{\pi}{2}) = -\frac{\pi}{36}$. **Answer:** Convergent. The integral evaluates to **-π/36**.

1.17 Problem 17

(a) Evaluate the integral: $\int_t^0 \frac{z}{z^4+36} dz$ (b) Determine whether $\int_{-\infty}^0 \frac{z}{z^4+36} dz$ is convergent or divergent.

Solution (a)

Similar to problem 16, use u-substitution $u = z^2, du = 2z dz$.

$$\begin{aligned}\int_t^0 \frac{z}{z^4 + 36} dz &= \frac{1}{2} \int_{z=t}^{z=0} \frac{du}{u^2 + 36} \\ &= \frac{1}{2} \left[\frac{1}{6} \arctan\left(\frac{u}{6}\right) \right]_{z=t}^{z=0} = \frac{1}{12} \left[\arctan\left(\frac{z^2}{6}\right) \right]_t^0 \\ &= \frac{1}{12} \left(\arctan(0) - \arctan\left(\frac{t^2}{6}\right) \right)\end{aligned}$$

Answer (a): $-\frac{1}{12} \arctan\left(\frac{t^2}{6}\right)$

Solution (b)

Take the limit of the result from part (a) as $t \rightarrow -\infty$.

$$\lim_{t \rightarrow -\infty} \left(-\frac{1}{12} \arctan\left(\frac{t^2}{6}\right) \right) = -\frac{1}{12} \left(\frac{\pi}{2} \right) = -\frac{\pi}{24}$$

Answer (b): Convergent. The integral evaluates to $-\pi/24$.

1.18 Problem 18

Determine whether the integral $\int_0^9 \frac{3}{\sqrt[3]{x-1}} dx$ is convergent or divergent.

Solution

This is a Type 2 improper integral because the function has an infinite discontinuity at $x = 1$, which is within the interval $[0, 9]$. We must split the integral at the point of discontinuity.

$$\int_0^9 \frac{3}{(x-1)^{1/3}} dx = \int_0^1 3(x-1)^{-1/3} dx + \int_1^9 3(x-1)^{-1/3} dx$$

Evaluate the first part:

$$\begin{aligned}\lim_{t \rightarrow 1^-} \int_0^t 3(x-1)^{-1/3} dx &= \lim_{t \rightarrow 1^-} \left[3 \frac{(x-1)^{2/3}}{2/3} \right]_0^t = \lim_{t \rightarrow 1^-} \left[\frac{9}{2} (x-1)^{2/3} \right]_0^t \\ &= \lim_{t \rightarrow 1^-} \left(\frac{9}{2} (t-1)^{2/3} - \frac{9}{2} (-1)^{2/3} \right) \\ &= 0 - \frac{9}{2} (1) = -\frac{9}{2}\end{aligned}$$

This part converges. Now evaluate the second part:

$$\begin{aligned}\lim_{s \rightarrow 1^+} \int_s^9 3(x-1)^{-1/3} dx &= \lim_{s \rightarrow 1^+} \left[\frac{9}{2} (x-1)^{2/3} \right]_s^9 \\ &= \lim_{s \rightarrow 1^+} \left(\frac{9}{2} (9-1)^{2/3} - \frac{9}{2} (s-1)^{2/3} \right) \\ &= \frac{9}{2} (8)^{2/3} - 0 = \frac{9}{2} (4) = 18\end{aligned}$$

This part also converges. Since both parts converge, the original integral converges. The total value is $-\frac{9}{2} + 18 = \frac{27}{2}$.

Answer: Convergent. The integral evaluates to **27/2**.

2 Analysis of Problems and Techniques

2.1 Types of Improper Integrals Encountered

1. Type 1: Infinite Intervals

- Integrals over $[a, \infty)$: Problems 1, 3, 4, 5, 7, 8, 11, 12, 13, 14, 15.
- Integrals over $(-\infty, b]$: Problems 2, 6, 16, 17.
- Integrals over $(-\infty, \infty)$: Problems 9, 10. These must be split into two separate improper integrals, e.g., $\int_{-\infty}^c f(x) dx + \int_c^{\infty} f(x) dx$. The integral converges only if BOTH parts converge.

2. Type 2: Infinite Discontinuity

- The integrand has a vertical asymptote at $x = c$ within the interval $[a, b]$. The integral must be split at the discontinuity: $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$. This was seen in Problem 18 at $x = 1$ on the interval $[0, 9]$.

2.2 Convergence and Divergence Rules (p-Integrals)

A key tool for quickly assessing convergence is the **p-integral test**.

- For Type 1 integrals: $\int_a^{\infty} \frac{1}{x^p} dx$ (where $a > 0$) **converges if** $p > 1$ and **diverges if** $p \leq 1$.
 - Problem 1: $p = 4 > 1$, converges.
 - Problem 4: Form is $1/u^{3/2}$, $p = 3/2 > 1$, converges.
 - Problem 2: $p = 1/3 \leq 1$ on an infinite interval, diverges.
 - Problem 5: Form is $1/u^{1/2}$, $p = 1/2 \leq 1$, diverges.
- For Type 2 integrals: $\int_0^a \frac{1}{x^p} dx$ (discontinuity at 0) **converges if** $p < 1$ and **diverges if** $p \geq 1$.
 - Problem 18: Form is $1/u^{1/3}$, $p = 1/3 < 1$, converges.

2.3 Techniques and Algebraic Manipulations Used

- **Limit Definition:** The fundamental technique for all problems was to rewrite the improper integral as a limit of a proper integral.
- **U-Substitution:** Used in Problems 6, 8, 9, 10, 13, 16, 17 to simplify the integrand before integration.
- **Partial Fraction Decomposition:** Necessary for integrating rational functions where the denominator is factorable. Used in Problems 14 and 15.
- **Trigonometric Identities:** The half-angle identity $\sin^2(\alpha) = (1 - \cos(2\alpha))/2$ was crucial for Problems 11 and 12.
- **Splitting Integrals:** Required for Type 1 integrals over $(-\infty, \infty)$ (Problems 9, 10) and for Type 2 integrals with an interior discontinuity (Problem 18).
- **Recognizing Odd/Even Functions:** In Problem 9, the integrand is an odd function integrated over a symmetric interval $(-\infty, \infty)$. If such an integral converges, its value must be 0. In Problem 10, the integrand was also odd, but it was shown to diverge. *Trick: You must still prove convergence of one half of the integral before concluding the value is 0.*
- **Simplifying the Integrand:** In Problem 7, dividing each term in the numerator by the denominator simplified the problem into a sum of p-integrals.

2.4 Essential Limits to Know

For evaluating improper integrals, knowledge of limits at infinity is critical.

- $\lim_{x \rightarrow \infty} \frac{1}{x^p} = 0$ for any $p > 0$.
- $\lim_{x \rightarrow \infty} e^x = \infty$ and $\lim_{x \rightarrow -\infty} e^x = 0$.
- $\lim_{x \rightarrow \infty} \ln(x) = \infty$ and $\lim_{x \rightarrow 0^+} \ln(x) = -\infty$.
- $\lim_{x \rightarrow \infty} \arctan(x) = \frac{\pi}{2}$ and $\lim_{x \rightarrow -\infty} \arctan(x) = -\frac{\pi}{2}$.
- Limits of oscillating functions like $\sin(x)$ or $\cos(x)$ as $x \rightarrow \infty$ do not exist.

2.5 Additional Tricks and Untested Concepts

The provided problems did not cover all aspects of improper integrals. Here are other important concepts:

- **The Comparison Test:** For an integrand that is difficult to integrate directly, you can compare it to a simpler function. If $f(x) \geq g(x) \geq 0$:
 - If $\int_a^\infty f(x) dx$ converges, then $\int_a^\infty g(x) dx$ also converges.
 - If $\int_a^\infty g(x) dx$ diverges, then $\int_a^\infty f(x) dx$ also diverges.
 - **Example:** To determine if $\int_1^\infty \frac{1}{x^2+5} dx$ converges, we can note that $\frac{1}{x^2+5} < \frac{1}{x^2}$. Since $\int_1^\infty \frac{1}{x^2} dx$ converges (p-integral with $p=2$), our integral must also converge.
- **The Limit Comparison Test:** If $f(x)$ and $g(x)$ are positive functions and $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = L$, where L is a finite, positive number, then $\int f(x) dx$ and $\int g(x) dx$ either both converge or both diverge.
 - **Example:** For $\int_1^\infty \frac{x}{x^3-2} dx$, we can compare it to $g(x) = \frac{x}{x^3} = \frac{1}{x^2}$. Since $\lim_{x \rightarrow \infty} \frac{x/(x^3-2)}{1/x^2} = 1$, and we know $\int_1^\infty \frac{1}{x^2} dx$ converges, our integral must also converge.
- **Checking the Domain:** As a trick, always check that the function is defined over the integration interval. A problem like $\int_0^2 \frac{1}{\sqrt{x-3}} dx$ is invalid because the integrand is not real for any value in the interval. The initial (incorrect) reading of Problem 18 as $1/\sqrt{x-1}$ would have made the integral from $[0, 1]$ invalid in the real number system.