

Homework 7.5 Strategy for integration

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1 Comprehensive Introduction, Context, and Prerequisites

1.1 Core Concepts

The topic "Strategy for Integration" is not about learning a new integration method. Instead, it is the capstone of your integration studies, focusing on how to select the *correct* method from your existing toolbox. Up to this point, you have learned several powerful techniques:

- Basic Antiderivatives (Power Rule, Trig, Exponential, etc.)
- u-Substitution
- Integration by Parts
- Trigonometric Integrals
- Trigonometric Substitution
- Partial Fraction Decomposition

This chapter is about developing a systematic approach to analyze any given integral and determine which of these techniques, or combination of techniques, is the most efficient path to a solution. The core skill is pattern recognition.

1.2 Intuition and Derivation

The "why" behind developing a strategy is efficiency and efficacy. A random approach to integration is like trying random keys on a lock; it's frustrating and rarely works. A strategic approach is like being a locksmith who can identify the type of lock and select the right tool immediately.

The general thought process for tackling an integral should be:

1. **Simplify First:** Can the integrand be simplified using algebra (e.g., expanding a product, combining fractions) or trigonometric identities (e.g., Pythagorean, half-angle, double-angle)? This is often the most overlooked but most powerful step.
2. **Look for Obvious Substitution:** Is there a function inside another function whose derivative is also present (perhaps off by a constant)? If so, a u-substitution is likely the way to go. This should be your second instinct after simplification.
3. **Classify the Integrand Form:** If simplification and obvious substitution don't work, classify the integral's form:
 - **Product of Functions:** Is it a product of two unrelated function types, like a polynomial and a log, or an exponential and a trig function? This is a strong indicator for **Integration by Parts**.
 - **Powers of Trig Functions:** Does it involve powers of sine, cosine, tangent, and secant? This calls for the **Trigonometric Integral** techniques you've learned.
 - **Radicals of Quadratics:** Does it contain expressions like $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, or $\sqrt{x^2 - a^2}$? This is the classic sign for **Trigonometric Substitution**.
 - **Rational Function:** Is the integrand a ratio of two polynomials? If the denominator can be factored, **Partial Fraction Decomposition** is the indicated method.

4. **Try Again:** If none of the above work, don't give up. Sometimes a non-obvious substitution is required, or a combination of techniques is needed. For example, a problem might require a substitution first, which then transforms the integral into one that needs integration by parts.

1.3 Historical Context and Motivation

The development of integral calculus was a monumental achievement of the 17th century, driven by Sir Isaac Newton and Gottfried Wilhelm Leibniz. [2, 6] Before their work, mathematicians like Archimedes had used clever "methods of exhaustion" to find areas and volumes, but these were ad-hoc and lacked a unifying principle. [15] Newton (in England) and Leibniz (in Germany) independently developed the Fundamental Theorem of Calculus, which established the inverse relationship between differentiation and integration. [1, 5, 6]

This breakthrough transformed the problem of finding areas (integration) into the problem of finding antiderivatives. The motivation was immense: physicists needed to calculate work, astronomers needed to compute planetary orbits, and geometers needed to find lengths, areas, and volumes of complex shapes. This created an urgent need to develop a systematic "toolkit" for finding antiderivatives of a wide variety of functions. The techniques you've learned, such as integration by parts and partial fractions, were developed and refined during the 17th and 18th centuries by mathematicians like the Bernoulli brothers and Leonhard Euler to meet this demand, turning calculus into the powerful problem-solving tool it is today. [1]

1.4 Key Formulas

- **Integration by Parts:** $\int u \, dv = uv - \int v \, du$

- **Trigonometric Identities:**

- $\sin^2(x) + \cos^2(x) = 1$
- $\tan^2(x) + 1 = \sec^2(x)$
- $\sin^2(x) = \frac{1 - \cos(2x)}{2}$
- $\cos^2(x) = \frac{1 + \cos(2x)}{2}$
- $\sin(2x) = 2 \sin(x) \cos(x)$

- **Trigonometric Substitutions:**

- For $\sqrt{a^2 - x^2}$, use $x = a \sin(\theta)$
- For $\sqrt{a^2 + x^2}$, use $x = a \tan(\theta)$
- For $\sqrt{x^2 - a^2}$, use $x = a \sec(\theta)$

- **Common Basic Integrals:**

- $\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$
- $\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \arcsin\left(\frac{x}{a}\right) + C$
- $\int \frac{1}{x\sqrt{x^2 - a^2}} \, dx = \frac{1}{a} \operatorname{arcsec}\left(\frac{|x|}{a}\right) + C$

1.5 Prerequisites

Mastery of this topic requires a strong foundation in:

- **Algebra:** Factoring polynomials, polynomial long division, completing the square, and solving systems of linear equations are all critical for partial fractions.
- **Trigonometry:** A deep and automatic recall of trigonometric identities is essential. You must be able to recognize when an identity can simplify an integrand.
- **Differential Calculus:** You must know all basic differentiation rules fluently to compute the 'du' in substitutions and the 'du' and 'v' in integration by parts.
- **Basic Integration:** You must have mastered all the individual integration techniques listed in the "Core Concepts" section. This chapter is about choosing between them, not learning them for the first time.

2 Detailed Homework Solutions

2.1 Problem 1

Part (a)

Evaluate $\int \frac{3x}{1+x^2} dx$.

Solution: This integral is a candidate for u-substitution because the derivative of the denominator, $1+x^2$, is $2x$, which is a constant multiple of the numerator, $3x$. Let $u = 1+x^2$. Then $du = 2x dx$, which means $x dx = \frac{1}{2} du$. Substitute these into the integral:

$$\int \frac{3x}{1+x^2} dx = \int \frac{3}{u} \left(\frac{1}{2} du \right) = \frac{3}{2} \int \frac{1}{u} du$$

Now, integrate with respect to u :

$$\frac{3}{2} \ln |u| + C$$

Finally, substitute back $u = 1+x^2$:

$$\frac{3}{2} \ln |1+x^2| + C$$

Since $1+x^2$ is always positive, the absolute value is not necessary.

$$\boxed{\frac{3}{2} \ln(1+x^2) + C}$$

Part (b)

Evaluate $\int \frac{3}{1+x^2} dx$.

Solution: This integral matches the basic form for the arctangent function.

$$\int \frac{3}{1+x^2} dx = 3 \int \frac{1}{1+x^2} dx$$

The integral of $\frac{1}{1+x^2}$ is $\arctan(x)$.

$$\boxed{3 \arctan(x) + C}$$

Part (c)

Evaluate $\int \frac{3}{1-x^2} dx$.

Solution: This integral involves a rational function whose denominator can be factored. This indicates the use of partial fraction decomposition. First, factor the denominator: $1-x^2 = (1-x)(1+x)$. Set up the decomposition:

$$\frac{3}{(1-x)(1+x)} = \frac{A}{1-x} + \frac{B}{1+x}$$

To find A and B , multiply by the common denominator:

$$3 = A(1+x) + B(1-x)$$

Solve for the constants. If we let $x = 1$: $3 = A(1+1) + B(0) \implies 3 = 2A \implies A = \frac{3}{2}$. If we let $x = -1$: $3 = A(0) + B(1 - (-1)) \implies 3 = 2B \implies B = \frac{3}{2}$. Now substitute these back and integrate:

$$\int \left(\frac{3/2}{1-x} + \frac{3/2}{1+x} \right) dx = \frac{3}{2} \int \frac{1}{1-x} dx + \frac{3}{2} \int \frac{1}{1+x} dx$$

For the first integral, a quick substitution $u = 1-x$, $du = -dx$ is needed.

$$\frac{3}{2}(-\ln |1-x|) + \frac{3}{2} \ln |1+x| + C = \frac{3}{2}(\ln |1+x| - \ln |1-x|) + C$$

Using logarithm properties:

$$\boxed{\frac{3}{2} \ln \left| \frac{1+x}{1-x} \right| + C}$$

2.2 Problem 2

Part (a)

Evaluate $\int 7x\sqrt{x^2-1} \, dx$.

Solution: This is a prime candidate for u-substitution, as the derivative of the expression inside the radical, $x^2 - 1$, is $2x$, a multiple of the term x outside the radical. Let $u = x^2 - 1$. Then $du = 2x \, dx$, so $x \, dx = \frac{1}{2} du$. Substitute:

$$\int 7\sqrt{u} \left(\frac{1}{2} du \right) = \frac{7}{2} \int u^{1/2} du$$

Integrate using the power rule:

$$\frac{7}{2} \frac{u^{3/2}}{3/2} + C = \frac{7}{2} \cdot \frac{2}{3} u^{3/2} + C = \frac{7}{3} u^{3/2} + C$$

Substitute back $u = x^2 - 1$:

$$\boxed{\frac{7}{3}(x^2 - 1)^{3/2} + C}$$

Part (b)

Evaluate $\int \frac{7}{x\sqrt{x^2-1}} \, dx$.

Solution: This integral matches the basic form for the arcsecant function.

$$\int \frac{7}{x\sqrt{x^2-1}} \, dx = 7 \int \frac{1}{x\sqrt{x^2-1}} \, dx$$

The integral of $\frac{1}{x\sqrt{x^2-1}}$ is $\text{arcsec}|x|$.

$$\boxed{7\text{arcsec}|x| + C}$$

Part (c)

Evaluate $\int \frac{7\sqrt{x^2-1}}{x} \, dx$.

Solution: The presence of $\sqrt{x^2-1}$ suggests a trigonometric substitution. Let $x = \sec(\theta)$, so $dx = \sec(\theta) \tan(\theta) \, d\theta$. The radical becomes $\sqrt{\sec^2(\theta) - 1} = \sqrt{\tan^2(\theta)} = \tan(\theta)$. Substitute into the integral:

$$\int \frac{7 \tan(\theta)}{\sec(\theta)} \sec(\theta) \tan(\theta) \, d\theta = \int 7 \tan^2(\theta) \, d\theta$$

Use the identity $\tan^2(\theta) = \sec^2(\theta) - 1$:

$$7 \int (\sec^2(\theta) - 1) \, d\theta = 7(\tan(\theta) - \theta) + C$$

Now, we must convert back to x . From $x = \sec(\theta)$, we have $\theta = \text{arcsec}(x)$. From the substitution, $\tan(\theta) = \sqrt{x^2-1}$. Substitute these back:

$$\boxed{7\sqrt{x^2-1} - 7\text{arcsec}(x) + C}$$

2.3 Problem 3

Part (a)

Evaluate $\int \frac{2 \ln(x)}{x} \, dx$.

Solution: This is a classic u-substitution problem. The derivative of $\ln(x)$ is $1/x$, which is present in the integrand. Let $u = \ln(x)$. Then $du = \frac{1}{x} dx$. Substitute:

$$\int 2u du = 2 \frac{u^2}{2} + C = u^2 + C$$

Substitute back $u = \ln(x)$:

$$\boxed{(\ln(x))^2 + C}$$

Part (b)

Evaluate $\int 2 \ln(2x) dx$.

Solution: This integral requires integration by parts. There's no obvious substitution. Let $u = \ln(2x)$ and $dv = 2 dx$. Then $du = \frac{1}{2x} \cdot 2 dx = \frac{1}{x} dx$ and $v = 2x$. Apply the integration by parts formula $\int u dv = uv - \int v du$:

$$\begin{aligned} \int 2 \ln(2x) dx &= (\ln(2x))(2x) - \int (2x) \left(\frac{1}{x} dx \right) \\ &= 2x \ln(2x) - \int 2 dx \\ &= 2x \ln(2x) - 2x + C \\ &\boxed{2x(\ln(2x) - 1) + C} \end{aligned}$$

Part (c)

Evaluate $\int 2x \ln(x) dx$.

Solution: This is another integration by parts problem, involving a product of a polynomial and a logarithmic function. Let $u = \ln(x)$ and $dv = 2x dx$. Then $du = \frac{1}{x} dx$ and $v = x^2$. Apply the formula:

$$\begin{aligned} \int 2x \ln(x) dx &= (\ln(x))(x^2) - \int x^2 \left(\frac{1}{x} dx \right) \\ &= x^2 \ln(x) - \int x dx \\ &= x^2 \ln(x) - \frac{x^2}{2} + C \\ &\boxed{x^2 \ln(x) - \frac{1}{2}x^2 + C} \end{aligned}$$

2.4 Problem 4

Part (a)

Evaluate $\int 2 \sin^2(x) dx$.

Solution: To integrate even powers of sine or cosine, use the half-angle identity. $\sin^2(x) = \frac{1 - \cos(2x)}{2}$.

$$\int 2 \sin^2(x) dx = \int 2 \left(\frac{1 - \cos(2x)}{2} \right) dx = \int (1 - \cos(2x)) dx$$

Integrate term by term:

$$\begin{aligned} \int 1 dx - \int \cos(2x) dx &= x - \frac{1}{2} \sin(2x) + C \\ &\boxed{x - \frac{1}{2} \sin(2x) + C} \end{aligned}$$

Part (b)

Evaluate $\int 2 \sin^3(x) dx$.

Solution: For odd powers of sine, split one sine factor off and use the Pythagorean identity.

$$\int 2 \sin^3(x) dx = \int 2 \sin^2(x) \sin(x) dx = \int 2(1 - \cos^2(x)) \sin(x) dx$$

Now use u-substitution. Let $u = \cos(x)$, so $du = -\sin(x) dx$.

$$\int 2(1 - u^2)(-du) = -2 \int (1 - u^2) du = -2 \left(u - \frac{u^3}{3} \right) + C$$

Substitute back $u = \cos(x)$:

$$-2 \cos(x) + \frac{2}{3} \cos^3(x) + C$$

$$\boxed{\frac{2}{3} \cos^3(x) - 2 \cos(x) + C}$$

Part (c)

Evaluate $\int 2 \sin(2x) dx$.

Solution: This is a simple u-substitution. Let $u = 2x$, so $du = 2 dx$.

$$\int \sin(u) du = -\cos(u) + C$$

Substitute back $u = 2x$:

$$\boxed{-\cos(2x) + C}$$

2.5 Problem 5

Evaluate $\int \frac{\cos(x)}{3 - \sin(x)} dx$.

Solution: This integral is set up for a u-substitution. The derivative of the denominator is present in the numerator. Let $u = 3 - \sin(x)$. Then $du = -\cos(x) dx$. Substitute:

$$\int \frac{1}{u}(-du) = -\int \frac{1}{u} du = -\ln|u| + C$$

Substitute back $u = 3 - \sin(x)$:

$$\boxed{-\ln|3 - \sin(x)| + C}$$

2.6 Problem 6

Evaluate $\int \frac{x}{x^4 + 16} dx$.

Solution: This looks like an arctangent form, but with x^4 instead of x^2 . We can fix this with a u-substitution. Let $u = x^2$. Then $du = 2x dx$, so $x dx = \frac{1}{2} du$. Substitute:

$$\int \frac{1}{u^2 + 16} \left(\frac{1}{2} du \right) = \frac{1}{2} \int \frac{1}{u^2 + 4^2} du$$

This is now the standard arctan form with $a = 4$.

$$\frac{1}{2} \left[\frac{1}{4} \arctan \left(\frac{u}{4} \right) \right] + C = \frac{1}{8} \arctan \left(\frac{u}{4} \right) + C$$

Substitute back $u = x^2$:

$$\boxed{\frac{1}{8} \arctan \left(\frac{x^2}{4} \right) + C}$$

2.7 Problem 7

Evaluate $\int 5t \sin(t) \cos(t) dt$.

Solution: First, simplify the integrand using the double-angle identity $\sin(2t) = 2\sin(t)\cos(t)$. So, $\sin(t)\cos(t) = \frac{1}{2}\sin(2t)$.

$$\int 5t \left(\frac{1}{2} \sin(2t) \right) dt = \frac{5}{2} \int t \sin(2t) dt$$

This is now a product of a polynomial and a trig function, perfect for integration by parts. Let $u = t$ and $dv = \sin(2t) dt$. Then $du = dt$ and $v = -\frac{1}{2}\cos(2t)$. Apply the formula:

$$\begin{aligned} & \frac{5}{2} \left[(t) \left(-\frac{1}{2} \cos(2t) \right) - \int \left(-\frac{1}{2} \cos(2t) \right) dt \right] \\ &= \frac{5}{2} \left[-\frac{t}{2} \cos(2t) + \frac{1}{2} \int \cos(2t) dt \right] \\ &= \frac{5}{2} \left[-\frac{t}{2} \cos(2t) + \frac{1}{2} \left(\frac{1}{2} \sin(2t) \right) \right] + C \\ &= \frac{5}{2} \left[-\frac{t}{2} \cos(2t) + \frac{1}{4} \sin(2t) \right] + C \\ & \boxed{-\frac{5t}{4} \cos(2t) + \frac{5}{8} \sin(2t) + C} \end{aligned}$$

2.8 Problem 8

Evaluate $\int \frac{2x-3}{x^3+3x} dx$.

Solution: This is a rational function, so we use partial fraction decomposition. First, factor the denominator: $x^3+3x = x(x^2+3)$. The x^2+3 term is an irreducible quadratic. Set up the decomposition:

$$\frac{2x-3}{x(x^2+3)} = \frac{A}{x} + \frac{Bx+C}{x^2+3}$$

Multiply by the common denominator:

$$2x-3 = A(x^2+3) + (Bx+C)x = Ax^2 + 3A + Bx^2 + Cx$$

Group terms by powers of x :

$$2x-3 = (A+B)x^2 + Cx + 3A$$

Equate coefficients:

- x^2 : $A+B=0$
- x^1 : $C=2$
- x^0 : $3A=-3 \implies A=-1$

From $A+B=0$, we get $-1+B=0 \implies B=1$. So, $A=-1, B=1, C=2$. Substitute back into the integral:

$$\int \left(\frac{-1}{x} + \frac{1x+2}{x^2+3} \right) dx = \int \frac{-1}{x} dx + \int \frac{x}{x^2+3} dx + \int \frac{2}{x^2+3} dx$$

Solve each integral:

- $\int \frac{-1}{x} dx = -\ln|x|$
- $\int \frac{x}{x^2+3} dx$: Use substitution $u = x^2+3, du = 2xdx$. This becomes $\frac{1}{2} \int \frac{1}{u} du = \frac{1}{2} \ln|x^2+3|$.
- $\int \frac{2}{x^2+3} dx = 2 \int \frac{1}{x^2+(\sqrt{3})^2} dx = 2 \left(\frac{1}{\sqrt{3}} \arctan\left(\frac{x}{\sqrt{3}}\right) \right)$.

Combine the results:

$$\boxed{-\ln|x| + \frac{1}{2} \ln(x^2+3) + \frac{2}{\sqrt{3}} \arctan\left(\frac{x}{\sqrt{3}}\right) + C}$$

2.9 Problem 9

Evaluate $\int x \sec(x) \tan(x) dx$.

Solution: This is a product of functions, suggesting integration by parts. Let $u = x$ and $dv = \sec(x) \tan(x) dx$. Then $du = dx$ and $v = \sec(x)$ (since the derivative of $\sec(x)$ is $\sec(x) \tan(x)$). Apply the formula:

$$\int x \sec(x) \tan(x) dx = x \sec(x) - \int \sec(x) dx$$

The integral of $\sec(x)$ is a standard result: $\ln |\sec(x) + \tan(x)|$.

$$x \sec(x) - \ln |\sec(x) + \tan(x)| + C$$

2.10 Problem 10

Evaluate $\int 9\theta \tan^2(\theta) d\theta$.

Solution: First, use a trigonometric identity to simplify $\tan^2(\theta)$. Use $\tan^2(\theta) = \sec^2(\theta) - 1$.

$$\begin{aligned} \int 9\theta(\sec^2(\theta) - 1) d\theta &= \int (9\theta \sec^2(\theta) - 9\theta) d\theta \\ &= \int 9\theta \sec^2(\theta) d\theta - \int 9\theta d\theta \end{aligned}$$

The second integral is simple: $\int 9\theta d\theta = \frac{9\theta^2}{2}$. The first integral requires integration by parts. For $\int 9\theta \sec^2(\theta) d\theta$: Let $u = 9\theta$ and $dv = \sec^2(\theta) d\theta$. Then $du = 9 d\theta$ and $v = \tan(\theta)$. Apply the formula:

$$\int 9\theta \sec^2(\theta) d\theta = 9\theta \tan(\theta) - \int 9 \tan(\theta) d\theta$$

The integral of $\tan(\theta)$ is $-\ln |\cos(\theta)|$ or $\ln |\sec(\theta)|$.

$$= 9\theta \tan(\theta) - 9 \ln |\sec(\theta)|$$

Combine all the parts:

$$(9\theta \tan(\theta) - 9 \ln |\sec(\theta)|) - \frac{9\theta^2}{2} + C$$

$$9\theta \tan(\theta) - 9 \ln |\sec(\theta)| - \frac{9}{2}\theta^2 + C$$

3 In-Depth Analysis of Problems and Techniques

3.1 Problem Types and General Approach

The homework problems can be categorized as follows:

- **Direct Recognition Problems (Substitution vs. Basic Form):** Problems 1, 2, and 4 highlight how visually similar integrals require completely different approaches. The key is to look at the numerator. In 1(a), the ‘x’ in the numerator signals u-substitution. In 1(b), the constant numerator signals the basic arctan form. In 1(c), the factorable denominator signals partial fractions.
- **Integration by Parts Problems:** Problems 3(b), 3(c), 7, 9, and 10 all involve products of different function types (polynomial, log, trig). The general approach is to use the LIATE (Logarithmic, Inverse Trig, Algebraic, Trigonometric, Exponential) mnemonic to choose ‘u’. The function type that appears first in LIATE is the best choice for ‘u’.
- **Trigonometric Identity Problems:** Problems 4(a), 4(b), 7, and 10 require an initial simplification using a trig identity. In 4(a), the half-angle identity is needed for an even power. In 4(b), splitting off a factor is needed for an odd power. In 7, the double-angle identity simplifies the integrand before parts is applied. In 10, an identity is used to break the integral into two manageable pieces.

- **Multi-Step Problems:** Problems 6, 7, 8, and 10 require more than one technique.
 - **Problem 6:** U-substitution followed by recognition of the arctan form.
 - **Problem 7:** Trig identity followed by integration by parts.
 - **Problem 8:** Partial fractions, which then required three separate, simple integrations (log, u-sub, arctan).
 - **Problem 10:** Trig identity, which broke the integral into one simple integral and one requiring integration by parts.

3.2 Key Algebraic and Calculus Manipulations

- **Factoring Denominators (Algebra):** Used in Problems 1(c) and 8. This is the crucial first step for any partial fraction decomposition. In Problem 8, recognizing that $x^2 + 3$ is an irreducible quadratic was key.
- **Solving for Coefficients (Algebra):** Used in Problems 1(c) and 8. The method of substituting convenient x-values (Problem 1c) and the method of equating coefficients (Problem 8) were both demonstrated.
- **Applying Trig Identities (Trigonometry):**
 - *Pythagorean Identity* ($\sin^2 x = 1 - \cos^2 x$): Essential in Problem 4(b) to convert the integrand into a form suitable for u-substitution.
 - *Half-Angle Identity* ($\sin^2 x = \frac{1 - \cos(2x)}{2}$): Crucial in Problem 4(a) for integrating an even power of sine.
 - *Double-Angle Identity* ($2 \sin t \cos t = \sin(2t)$): This was the key simplification in Problem 7 that made the subsequent integration by parts possible.
 - *Pythagorean Identity* ($\tan^2 \theta = \sec^2 \theta - 1$): Necessary in Problem 10 to break the difficult integrand into two solvable integrals.
- **LIATE Rule for Integration by Parts (Calculus):** Implicitly used in Problems 3(c), 7, 9, and 10. In each case, choosing ‘u’ as the algebraic part (‘t’, ‘x’, ‘’) and ‘dv’ as the trig/exponential part led to a simpler integral. For 3(b) and 3(c), choosing ‘u’ as the logarithmic part was the correct strategy.
- **Multi-Step Integration Processes (Calculus):** The ability to see the path forward was critical. In Problem 6, one had to recognize that the initial u-substitution ($u = x^2$) was not the final answer, but a step that would transform the problem into a standard form.

4 ”Cheatsheet” and Tips for Success

4.1 Summary of Formulas

- **Integration by Parts:** $\int u \, dv = uv - \int v \, du$
- **Key Identities:** $\sin^2 x + \cos^2 x = 1$, $\tan^2 x + 1 = \sec^2 x$, $\sin^2 x = \frac{1 - \cos(2x)}{2}$, $\cos^2 x = \frac{1 + \cos(2x)}{2}$
- **Inverse Trig Integrals:** $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$, $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \operatorname{arcsec}\left|\frac{x}{a}\right| + C$

4.2 Cheats, Tricks, and Shortcuts

- **The Decision Tree:** Always follow the strategy: 1) Simplify, 2) Obvious Substitution, 3) Classify (Parts, Trig Sub, Partial Fractions), 4) Try again.
- **LIATE Rule:** For Integration by Parts, choose ‘u’ in this order of preference: **L**ogarithmic, **I**nverse Trig, **A**lgebraic, **T**rigonometric, **E**xponential. Whatever is left is ‘dv’.
- **Numerator-Denominator Derivative Check:** Before trying complex methods, always check if the numerator is a constant multiple of the derivative of the denominator (or part of it). If so, it’s a simple u-substitution (like in 1a, 5, and part of 8).

- **Odd Man Out for Trig Integrals:** For powers of \sin/\cos , if one power is odd, save one of those factors and convert the rest using $\sin^2 x + \cos^2 x = 1$. This sets up a u-sub.

4.3 Common Pitfalls and Mistakes

- **Forgetting ‘+ C’:** The constant of integration is required for all indefinite integrals.
- **Sign Errors in Parts:** The formula is $uv - \int vdu$. It’s very easy to drop the minus sign.
- **Algebra Errors:** Mistakes in partial fraction decomposition (solving for A, B, C) are common but preventable. Double-check your algebra.
- **Incorrect ‘du’:** When doing a u-substitution, always solve for ‘dx’ fully. Forgetting a constant (e.g., if $u = 3x$, $du = 3dx$, so $dx = du/3$) will lead to a wrong answer.
- **Mixing up similar forms:** Confusing $\int \frac{1}{1+x^2} dx$ (arctan) with $\int \frac{x}{1+x^2} dx$ (ln via u-sub) is a classic trap.

5 Conceptual Synthesis and The ”Big Picture”

5.1 Thematic Connections

The central theme of this topic is **algorithmic problem-solving and pattern recognition**. This isn’t about rote memorization; it’s about developing a logical framework for dissecting a problem and selecting the appropriate tool. This mirrors problem-solving in many other fields. A computer programmer develops an algorithm to solve a coding problem, a doctor follows a diagnostic tree to identify an illness, and a mechanic follows a troubleshooting guide to fix an engine. In each case, a complex problem is solved by categorizing it and applying a known, systematic procedure. This chapter teaches you to think like a mathematical strategist, not just a calculator.

5.2 Forward and Backward Links

- **Backward Links:** This topic is the culmination of everything you have learned about integration. It fundamentally depends on your mastery of basic derivatives (to find ‘du’), algebraic manipulation (for partial fractions), and trigonometric identities (for trig integrals and substitutions). Without a solid foundation in these prerequisite skills, developing a successful integration strategy is impossible.
- **Forward Links:** Mastery of integration strategies is absolutely critical for what comes next.
 - **Differential Equations:** The entire field of differential equations is about solving equations that involve derivatives. The most common solution method, separation of variables, culminates in an integration problem that often requires these advanced techniques.
 - **Multivariable Calculus:** When you calculate volumes, surface areas, work, or fluid flow, you will set up integrals (often double or triple integrals) that ultimately boil down to an iterated integral. The final step of solving these integrals will rely on the toolbox you’ve built here.
 - **Physics and Engineering:** Calculating concepts like moment of inertia, center of mass, total charge, or the Fourier transform of a signal all rely on evaluating definite integrals that are often non-trivial and require these strategic approaches. [25]

6 Real-World Application and Modeling

6.1 Concrete Scenarios

1. **Pharmacokinetics (Partial Fractions):** A pharmacist models the concentration of a drug in a patient’s bloodstream over time using a rational function, like $C(t) = \frac{100t}{(t+2)(t+5)}$. To find the total exposure to the drug over the first 4 hours (the area under the concentration curve), they must calculate $\int_0^4 C(t)dt$. This requires partial fraction decomposition. [7]

2. **Electrical Engineering (Integration by Parts):** An engineer analyzing an RLC circuit finds that the current I (in Amps) at time t (in seconds) after a power surge is modeled by $I(t) = 5te^{-2t}$. To find the total charge (in Coulombs) that passes through a component in the first 3 seconds, they must compute the integral $Q = \int_0^3 5te^{-2t} dt$. This product of an algebraic and an exponential function requires integration by parts. [26, 28]
3. **Cartography (Trigonometric Substitution):** Creating a Mercator map projection, which is used in nearly all web mapping services like Google Maps, requires preserving angles. The formula to do this involves integrating the secant function, $\int \sec(\theta) d\theta$. One of the classic ways to solve this integral is through a clever substitution that eventually requires trigonometric manipulations, and historically its solution was a crucial step in making accurate navigation possible. [12, 17]

6.2 Model Problem Setup

Let's set up the electrical engineering problem.

- **Scenario:** An engineer needs to determine the total charge that passes through a resistor in an RLC circuit during the first 3 seconds after a voltage is applied.
- **Variables:**
 - Q : Total charge in Coulombs (C).
 - t : Time in seconds (s).
 - $I(t)$: The current as a function of time, in Amperes (A).
- **Function:** The current is modeled by the function $I(t) = 5te^{-2t}$. This is a typical model for a critically damped response in a circuit.
- **Equation to be Solved:** The relationship between charge and current is $Q = \int I(t) dt$. To find the total charge that passes from $t = 0$ to $t = 3$, we must evaluate the definite integral:

$$Q = \int_0^3 5te^{-2t} dt$$

This integral requires integration by parts to solve.

7 Common Variations and Untested Concepts

The provided homework set was excellent but did not cover a few common integration scenarios.

7.1 Completing the Square for Irreducible Quadratics

Sometimes, a denominator contains a quadratic that cannot be factored, and it's not part of a simple u-substitution. The strategy is to complete the square to make it look like the arctan form.

Example: Evaluate $\int \frac{1}{x^2 - 6x + 13} dx$.

Solution: The denominator does not factor. Complete the square: $x^2 - 6x + 13 = (x^2 - 6x + 9) - 9 + 13 = (x - 3)^2 + 4$. The integral becomes:

$$\int \frac{1}{(x - 3)^2 + 2^2} dx$$

Now, let $u = x - 3$, so $du = dx$.

$$\int \frac{1}{u^2 + 2^2} du = \frac{1}{2} \arctan\left(\frac{u}{2}\right) + C$$

Substitute back:

$$\frac{1}{2} \arctan\left(\frac{x - 3}{2}\right) + C$$

7.2 Rationalizing Substitutions

When an integral contains a problematic root, like \sqrt{x} or $\sqrt[3]{x+1}$, a substitution can be made to eliminate the root entirely.

Example: Evaluate $\int \frac{1}{1+\sqrt{x}} dx$.

Solution: Let $u = \sqrt{x}$. Then $u^2 = x$, and $2u du = dx$. Substitute everything in terms of u :

$$\int \frac{1}{1+u} (2u du) = \int \frac{2u}{1+u} du$$

This is a rational function where the degree of the numerator is not less than the denominator. We can use polynomial long division or a simple trick:

$$\frac{2u}{1+u} = \frac{2(u+1)-2}{1+u} = \frac{2(u+1)}{1+u} - \frac{2}{1+u} = 2 - \frac{2}{1+u}$$

Now integrate:

$$\int \left(2 - \frac{2}{1+u} \right) du = 2u - 2 \ln|1+u| + C$$

Substitute back $u = \sqrt{x}$:

$$2\sqrt{x} - 2 \ln(1 + \sqrt{x}) + C$$

7.3 Weierstrass Substitution (Tangent Half-Angle)

This is a powerful, if complex, "last resort" substitution for integrals that are rational functions of $\sin(x)$ and $\cos(x)$. [13, 19] Let $t = \tan(x/2)$. Then:

$$\sin(x) = \frac{2t}{1+t^2}, \quad \cos(x) = \frac{1-t^2}{1+t^2}, \quad dx = \frac{2}{1+t^2} dt$$

Example: Evaluate $\int \frac{1}{1+\cos(x)} dx$.

Solution: Apply the Weierstrass substitution.

$$\begin{aligned} \int \frac{1}{1 + \frac{1-t^2}{1+t^2}} \left(\frac{2}{1+t^2} dt \right) &= \int \frac{1}{\frac{(1+t^2)+(1-t^2)}{1+t^2}} \left(\frac{2}{1+t^2} dt \right) \\ &= \int \frac{1+t^2}{2} \left(\frac{2}{1+t^2} dt \right) = \int 1 dt = t + C \end{aligned}$$

Substitute back $t = \tan(x/2)$:

$$\tan(x/2) + C$$

8 Advanced Diagnostic Testing: "Find the Flaw"

8.1 Problem 1

Problem: Evaluate $\int x e^{2x} dx$.

Flawed Solution: We use integration by parts with $u = x$ and $dv = e^{2x} dx$. Then $du = dx$ and $v = e^{2x}$. Applying the formula $\int u dv = uv - \int v du$:

$$\begin{aligned} \int x e^{2x} dx &= x e^{2x} - \int e^{2x} dx \\ &= x e^{2x} - e^{2x} + C \end{aligned}$$

- Identify the Flaw:

- Explain the Error in One Sentence:
- Provide the Correct Step and Final Answer:

Answer Key: The error is in the calculation of v ; the integral of e^{2x} is $\frac{1}{2}e^{2x}$, not e^{2x} . The correct step is $v = \frac{1}{2}e^{2x}$. The correct final answer is $\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$.

8.2 Problem 2

Problem: Evaluate $\int \frac{5x-5}{x^2-x-6} dx$.

Flawed Solution: First, factor the denominator: $x^2 - x - 6 = (x - 3)(x + 2)$. Set up the partial fraction decomposition:

$$\frac{5x-5}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2}$$

$$5x-5 = A(x+2) + B(x-3)$$

Let $x = 3$: $5(3) - 5 = A(3 + 2) \implies 10 = 5A \implies A = 2$. Let $x = -2$: $5(-2) - 5 = B(-2 - 3) \implies -15 = -5B \implies B = -3$. The integral is:

$$\int \left(\frac{2}{x-3} + \frac{-3}{x+2} \right) dx = 2 \ln |x-3| - 3 \ln |x+2| + C$$

- Identify the Flaw:
- Explain the Error in One Sentence:
- Provide the Correct Step and Final Answer:

Answer Key: The error is in solving for B ; when $x = -2$, $5(-2) - 5 = -15$, and $B(-2 - 3) = -5B$, so $-15 = -5B$ implies $B = 3$, not -3 . The correct setup for the integral is $\int \left(\frac{2}{x-3} + \frac{3}{x+2} \right) dx$. The correct final answer is $2 \ln |x-3| + 3 \ln |x+2| + C$.

8.3 Problem 3

Problem: Evaluate $\int \cos^3(x) \sin^4(x) dx$.

Flawed Solution: Since the power of cosine is odd, we save one cosine factor.

$$\int \cos^2(x) \sin^4(x) \cos(x) dx = \int (1 - \sin^2(x)) \sin^4(x) \cos(x) dx$$

Let $u = \cos(x)$, so $du = -\sin(x)dx$. This substitution does not work. Let's try again. Let $u = \sin(x)$, so $du = \cos(x)dx$.

$$\int (1 - u^2)u^4 du = \int (u^4 - u^6) du = \frac{u^5}{5} - \frac{u^7}{7} + C$$

$$= \frac{\sin^5(x)}{5} - \frac{\sin^7(x)}{7} + C$$

- Identify the Flaw:
- Explain the Error in One Sentence:
- Provide the Correct Step and Final Answer:

Answer Key: The solution correctly identifies the strategy and performs the integration, but the final answer is missing the constant of integration, '+ C'. This is a conceptual error. The steps shown are correct, but the final line should explicitly include '+ C'. The final answer is correct, but was marked to find the flaw of omitting the constant. A more subtle flaw would have been an algebraic one. To make it a calculation flaw: Let's assume an algebra mistake: $\int (1 - u^2)u^4 du = \int (u^4 - u^8) du = \frac{u^5}{5} - \frac{u^9}{9} + C$. The error is in distributing u^4 into $(1 - u^2)$; it should be $u^4 - u^6$, not $u^4 - u^8$. The correct integration is $\frac{u^5}{5} - \frac{u^7}{7} + C$, leading to the final answer $\frac{\sin^5(x)}{5} - \frac{\sin^7(x)}{7} + C$.

8.4 Problem 4

Problem: Evaluate $\int \frac{dx}{\sqrt{9-x^2}}$.

Flawed Solution: This is a trigonometric substitution. Let $x = 3 \tan(\theta)$. Then $dx = 3 \sec^2(\theta) d\theta$. The radical becomes $\sqrt{9 - 9 \tan^2(\theta)} = \sqrt{9(1 - \tan^2(\theta))} = 3\sqrt{1 - \tan^2(\theta)}$. This does not simplify well. This seems like the wrong substitution. The integral looks like the arctan form.

$$\int \frac{dx}{\sqrt{9-x^2}} = \frac{1}{3} \arctan\left(\frac{x}{3}\right) + C$$

- **Identify the Flaw:**
- **Explain the Error in One Sentence:**
- **Provide the Correct Step and Final Answer:**

Answer Key: The error is misidentifying the integral form; the form $\int \frac{dx}{\sqrt{a^2-x^2}}$ corresponds to \arcsin , not \arctan . The correct substitution is $x = 3 \sin(\theta)$ and the correct final answer is $\arcsin(\frac{x}{3}) + C$.

8.5 Problem 5

Problem: Evaluate $\int (2x+1)^5 dx$.

Flawed Solution: This is a simple power rule application.

$$\int (2x+1)^5 dx = \frac{(2x+1)^6}{6} + C$$

- **Identify the Flaw:**
- **Explain the Error in One Sentence:**
- **Provide the Correct Step and Final Answer:**

Answer Key: The error is failing to account for the chain rule when integrating, which requires a u -substitution. Let $u = 2x + 1$, then $du = 2dx$, so $dx = \frac{1}{2} du$. The integral becomes $\int u^5 (\frac{1}{2} du) = \frac{1}{2} \frac{u^6}{6} + C$. The correct final answer is $\frac{(2x+1)^6}{12} + C$.