10.1: Parametric Equations - Problem Set

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October 2025

Parametric Equations Problem Set

Problems

1. For the parametric equations $x = 3t^2 - 1$, $y = t^3 - t$, find the coordinates of the point for $t = -2$.
2. For the parametric equations $x = e^{2t}$, $y = \ln(t + 1)$, find the coordinates of the point for $t = 0$.
3. Eliminate the parameter to find the Cartesian equation for $x = 2t + 5$, $y = 4t - 1$.
4. Eliminate the parameter to find the Cartesian equation for $x = \sqrt{t-3}$, $y = t+1$. State the domain for the resulting equation.
5. Eliminate the parameter to find the Cartesian equation for $x = e^{-t}$, $y = 3e^{2t}$.
6. Eliminate the parameter to find the Cartesian equation for $x = \frac{1}{t+1}$, $y = \frac{t}{t+1}$.
7. Eliminate the parameter to find the Cartesian equation for $x = 5\cos(t)$, $y = 5\sin(t)$.
8. Eliminate the parameter to find the Cartesian equation for $x = 4\cos(t) + 1$, $y = 3\sin(t) - 2$.
9. Eliminate the parameter to find the Cartesian equation for $x = 3\sec(t)$, $y = 4\tan(t)$.
10. Eliminate the parameter to find the Cartesian equation for $x = \cos(2t)$, $y = \cos(t)$. (Hint: Use a double-angle identity).
11. Sketch the curve for $x = t - 1$, $y = t^2 + 4$ for $-1 \leq t \leq 2$. Indicate the orientation with an arrow.
12. Sketch the curve for $x = t^3 - 3t$, $y = t^2$. Indicate the orientation.
13. Sketch the curve for $x = 2\sin(t)$, $y = \cos^2(t)$. Indicate the orientation.
14. Sketch the curve for $x = \sqrt{t}$, $y = t - 2$. What portion of the Cartesian curve is traced? Indicate the orientation.
15. Sketch the curve for $x = 4\sin(t)$, $y = 4\cos(t)$ for $0 \leq t \leq \pi$. Indicate the orientation.
16. Sketch the curve for $x = 1 + \ln(t)$, $y = t^2$ for $t > 0$. Indicate the orientation.
17. The path of a particle is given by $x = 2 - t^2$, $y = t$. Sketch the curve and indicate the direction of motion as t increases.
18. Sketch the curve defined by $x = e^t$, $y = e^{-t}$. Indicate the orientation.
19. A particle moves according to $x = 6\cos(\pi t)$, $y = 6\sin(\pi t)$. How long does it take to complete one full revolution? Is the motion clockwise or counter-clockwise?
20. A particle moves on an ellipse given by $x = 5\sin(t)$, $y = 2\cos(t)$, for $0 \leq t \leq 4\pi$. Describe the motion.
21. The position of a particle is given by $x = 2t$, $y = \cos(\pi t)$. Describe the particle's horizontal and vertical motion. Is the overall motion periodic?
22. A Lissajous figure is created by $x = \sin(t)$, $y = \sin(2t)$. Sketch the curve for $0 \leq t \leq 2\pi$.

23. Find a set of parametric equations for the line $y = 7x - 3$.

24. Find a set of parametric equations for the parabola $x = y^2 - 4y + 1$.

25. Find a set of parametric equations for the ellipse $\frac{(x-2)^2}{25} + \frac{(y+4)^2}{9} = 1$.

26. Find the parametric equations for the line segment starting at $(1, 6)$ and ending at $(-3, 2)$.

27. A projectile is launched from ground level with an initial speed of 100 m/s at an angle of 30°. Using $g \approx 9.8\ \mathrm{m/s^2}$, the parametric equations are $x(t) = (100\cos(30°))t$ and $y(t) = (100\sin(30°))t - \frac{1}{2}(9.8)t^2$. Find how long the projectile is in the air.

28. The equations for a cycloid (the path traced by a point on a rolling circle of radius r) are $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$. Find the position of the point when the circle has rolled a quarter of a turn ($\theta = \pi/2$) if the radius is 2.

29. Two particles have paths given by $\mathbf{r}_1(t) = \langle t + 3, t^2\rangle$ and $\mathbf{r}_2(s) = \langle s - 1, 2s\rangle$. Find any intersection points of their paths. Do they collide?

30. For the curve given by $x = t^3 - 3t$ and $y = 3t^2 - 9$, find the slope of the tangent line at $t = 2$.

Solutions

Problem 1

Given $x = 3t^2 - 1$, $y = t^3 - t$. For $t = -2$:
$x = 3(-2)^2 - 1 = 3(4) - 1 = 12 - 1 = 11$.
$y = (-2)^3 - (-2) = -8 + 2 = -6$.
The point is **(11, -6)**.

Problem 2

Given $x = e^{2t}$, $y = \ln(t+1)$. For $t = 0$:
$x = e^{2(0)} = e^0 = 1$.
$y = \ln(0+1) = \ln(1) = 0$.
The point is **(1, 0)**.

Problem 3

From $x = 2t + 5$, solve for t: $t = \frac{x-5}{2}$.
Substitute into the y equation: $y = 4\left(\frac{x-5}{2}\right) - 1 = 2(x-5) - 1 = 2x - 10 - 1$.
The Cartesian equation is $\mathbf{y = 2x - 11}$.

Problem 4

From $x = \sqrt{t-3}$, square both sides: $x^2 = t - 3$, so $t = x^2 + 3$.
Substitute into the y equation: $y = (x^2 + 3) + 1$.
The Cartesian equation is $\mathbf{y = x^2 + 4}$.
Since $x = \sqrt{t-3}$, x must be non-negative. The domain is $\mathbf{x \geq 0}$.

Problem 5

From $x = e^{-t}$, we can write $t = -\ln(x)$. Alternatively, notice $x = e^{-t} \implies \frac{1}{x} = e^t$. Also $y = 3e^{2t} = 3(e^t)^2$. Substitute $e^t = \frac{1}{x}$: $y = 3\left(\frac{1}{x}\right)^2$. The Cartesian equation is $\mathbf{y = \frac{3}{x^2}}$.

Problem 6

From $x = \frac{1}{t+1}$, solve for t: $x(t+1) = 1 \implies xt + x = 1 \implies t = \frac{1-x}{x}$.
Substitute into the y equation: $y = \frac{\frac{1-x}{x}}{\frac{1-x}{x}+1} = \frac{\frac{1-x}{x}}{\frac{1-x+x}{x}} = \frac{\frac{1-x}{x}}{\frac{1}{x}} = 1 - x$.
A simpler way: Notice that $x + y = \frac{1}{t+1} + \frac{t}{t+1} = \frac{1+t}{t+1} = 1$. The Cartesian equation is $\mathbf{y = 1 - x}$.

Problem 7

Recognize that this fits the Pythagorean identity. $\cos(t) = x/5$ and $\sin(t) = y/5$.
Since $\cos^2(t) + \sin^2(t) = 1$, we have $(\frac{x}{5})^2 + (\frac{y}{5})^2 = 1$.
The Cartesian equation is $\mathbf{x^2 + y^2 = 25}$, a circle centered at the origin with radius 5.

Problem 8

Isolate the trigonometric terms: $\cos(t) = \frac{x-1}{4}$ and $\sin(t) = \frac{y+2}{3}$.
Using $\cos^2(t) + \sin^2(t) = 1$:
$\left(\frac{x-1}{4}\right)^2 + \left(\frac{y+2}{3}\right)^2 = 1$.
This is the equation of an ellipse centered at $(1, -2)$.

Problem 9

Isolate the trigonometric terms: $\sec(t) = x/3$ and $\tan(t) = y/4$.
Use the identity $\sec^2(t) - \tan^2(t) = 1$.
$\left(\frac{x}{3}\right)^2 - \left(\frac{y}{4}\right)^2 = 1$.
This is the equation of a hyperbola.

Problem 10

Use the double-angle identity for cosine: $\cos(2t) = 2\cos^2(t) - 1$.
From the parametric equations, we have $x = \cos(2t)$ and $y = \cos(t)$.
Substitute these into the identity: $x = 2y^2 - 1$. This is the equation of a parabola opening to the right.
Since $y = \cos(t)$, $-1 \leq y \leq 1$.

Problem 11

Points: $t = -1 \implies (-2, 5)$, $t = 0 \implies (-1, 4)$, $t = 2 \implies (1, 8)$. The curve is a parabola ($y = (x+1)^2 + 4$) opening upwards. The orientation is from left to right.

problem11_sketch.png

Problem 12

This is a self-intersecting curve. At $t = 0$, point is $(0, 0)$. At $t = \pm\sqrt{3}$, $x = 0$, so it crosses the y-axis. The curve starts from the bottom left, moves up and right, loops at the origin, and then moves up and left.

problem12_sketch.png

Problem 13

Eliminate parameter: $x = 2\sin(t) \implies \sin(t) = x/2$. $y = \cos^2(t) = 1 - \sin^2(t) = 1 - (x/2)^2 = 1 - x^2/4$. This is a parabola opening downwards. Since $x = 2\sin(t)$, we have $-2 \leq x \leq 2$. The particle oscillates back and forth along this parabolic arc. At $t = 0$, point is $(0, 1)$. At $t = \pi/2$, point is $(2, 0)$. At $t = \pi$, point is $(0, 1)$. The orientation moves from $(0, 1)$ to $(2, 0)$ and back.

problem13_sketch.png

Problem 14

Eliminate parameter: $x = \sqrt{t} \implies t = x^2$. Substitute: $y = x^2 - 2$. This is a parabola. Restriction: Since $x = \sqrt{t}$, $t \geq 0$ and $x \geq 0$. So, only the right half of the parabola is traced. Orientation: $t = 0 \implies (0, -2)$, $t = 4 \implies (2, 2)$. The curve moves upwards and to the right.

problem14_sketch.png

Problem 15

This is a circle $x^2 + y^2 = 16$. The interval $0 \leq t \leq \pi$ traces a semi-circle. $t = 0 \implies (0, 4)$. $t = \pi/2 \implies (4, 0)$. $t = \pi \implies (0, -4)$. The orientation is **clockwise** along the right semi-circle.

problem15_sketch.png

Problem 16

Eliminate parameter: $x = 1 + \ln(t) \implies \ln(t) = x - 1 \implies t = e^{x-1}$. Substitute into y: $y = (e^{x-1})^2 = e^{2x-2}$. This is an exponential curve. As t increases from near 0 to ∞, $\ln(t)$ goes from $-\infty$ to ∞, so x covers all real numbers. The orientation is from left to right.

problem16_sketch.png

Problem 17

Eliminate parameter: $t = y$. Substitute into x: $x = 2 - y^2$. This is a parabola opening to the left with vertex at $(2, 0)$. Orientation: As t increases, y increases. The particle moves up along the parabola.

problem17_sketch.png

Problem 18

Notice that $y = e^{-t} = 1/e^t = 1/x$. The curve is the hyperbola $y = 1/x$. Restriction: Since $e^t > 0$ for all t, both x and y are positive. The curve is restricted to the first quadrant. Orientation: As t increases from $-\infty$ to ∞, $x = e^t$ increases from 0 to ∞. The orientation is from left to right along the hyperbola branch.

problem18_sketch.png

Problem 19

The equations describe a circle of radius 6. The period T is found when the argument of sine/cosine completes a 2π cycle. $\pi T = 2\pi \implies T = 2$. It takes **2 seconds** to complete one revolution. To find direction, check points: $t = 0 \implies (6, 0)$. $t = 0.5 \implies (0, 6)$. The motion is from the positive x-axis to the positive y-axis, which is **counter-clockwise**.

Problem 20

The curve is an ellipse $\frac{x^2}{25} + \frac{y^2}{4} = 1$. The interval length is 4π, which is two full 2π cycles. Direction: $t = 0 \implies (0, 2)$. $t = \pi/2 \implies (5, 0)$. The motion is from the positive y-axis to the positive x-axis, which is **clockwise**. The particle traverses the entire ellipse **twice in a clockwise direction**.

Problem 21

Horizontal motion: $x = 2t$. The particle moves to the right at a constant speed. Vertical motion: $y = \cos(\pi t)$. The particle oscillates vertically between -1 and 1 with a period of $T = 2\pi/\pi = 2$. The overall motion is not periodic in the sense of returning to a starting point, because the x coordinate always increases. The particle moves along a cosine wave that is stretched horizontally.

Problem 22

This curve traces a "figure-eight" shape. It starts at $(0, 0)$, moves into the first quadrant, crosses the origin at $t = \pi$, moves into the fourth quadrant, and returns to the origin at $t = 2\pi$.

`problem22_sketch.png`

Problem 23

The simplest parameterization is to let $x = t$. Then substitute into the equation to find y. $\mathbf{x = t, y = 7t - 3}$.

Problem 24

Since the equation gives x in terms of y, it's easiest to let $y = t$. $\mathbf{y = t, x = t^2 - 4t + 1}$.

Problem 25

This is an ellipse centered at $(2, -4)$ with semi-major axis $a = 5$ and semi-minor axis $b = 3$. Use the standard parameterization for an ellipse: $\frac{x-h}{a} = \cos(t)$ and $\frac{y-k}{b} = \sin(t)$. $\mathbf{x = 2 + 5\cos(t), y = -4 + 3\sin(t)}$ for $0 \leq t \leq 2\pi$.

Problem 26

Use the formula $x(t) = x_1 + (x_2 - x_1)t$ and $y(t) = y_1 + (y_2 - y_1)t$ for $0 \leq t \leq 1$. $x(t) = 1 + (-3-1)t = 1 - 4t$. $y(t) = 6 + (2-6)t = 6 - 4t$. So, $\mathbf{x = 1 - 4t, y = 6 - 4t}$ for $0 \leq t \leq 1$.

Problem 27

The projectile is in the air until $y(t) = 0$. $y(t) = (100\sin(30°))t - 4.9t^2 = (100 \cdot 0.5)t - 4.9t^2 = 50t - 4.9t^2$. Set $y(t) = 0$: $t(50 - 4.9t) = 0$. The solutions are $t = 0$ (launch) and $t = 50/4.9 \approx 10.2$. The projectile is in the air for approximately **10.2 seconds**.

Problem 28

Given $r = 2$ and $\theta = \pi/2$. $x = 2(\pi/2 - \sin(\pi/2)) = 2(\pi/2 - 1) = \pi - 2$. $y = 2(1 - \cos(\pi/2)) = 2(1 - 0) = 2$. The position is $(\pi - \mathbf{2}, \mathbf{2})$.

Problem 29

Intersection points occur when coordinates are equal, but not necessarily at the same time parameter. Set $x_1(t) = x_2(s)$ and $y_1(t) = y_2(s)$. $t + 3 = s - 1 \implies s = t + 4$. $t^2 = 2s$. Substitute s into the second equation: $t^2 = 2(t + 4) \implies t^2 = 2t + 8 \implies t^2 - 2t - 8 = 0$. $(t - 4)(t + 2) = 0$, so $t = 4$ or $t = -2$. If $t = 4$, the point on path 1 is $(4 + 3, 4^2) = (7, 16)$. If $t = -2$, the point on path 1 is $(-2 + 3, (-2)^2) = (1, 4)$. The intersection points are **(7, 16)** and **(1, 4)**.

Collision: Does $t = s$? Set $x_1(t) = x_2(t)$ and $y_1(t) = y_2(t)$. $t + 3 = t - 1 \implies 3 = -1$, which is impossible. There is **no collision**.

Problem 30

The slope of the tangent line is given by $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$. $x = t^3 - 3t \implies \frac{dx}{dt} = 3t^2 - 3$. $y = 3t^2 - 9 \implies \frac{dy}{dt} = 6t$. So, $\frac{dy}{dx} = \frac{6t}{3t^2-3} = \frac{2t}{t^2-1}$. At $t = 2$, the slope is $\frac{2(2)}{2^2-1} = \frac{4}{4-1} = \frac{4}{3}$. The slope at $t = 2$ is $\mathbf{4/3}$.

Concept Checklist

- **Evaluating Points from Parametric Equations:** Problems 1, 2
- **Eliminating the Parameter (Algebraic Methods):**
  - Linear/Polynomial: Problem 3
  - Radical Expressions: Problem 4
  - Exponential/Logarithmic Expressions: Problems 5, 16
  - Rational Expressions: Problem 6
- **Eliminating the Parameter (Trigonometric Identities):**
  - Circles ($\sin^2 + \cos^2 = 1$): Problem 7
  - Ellipses ($\sin^2 + \cos^2 = 1$): Problem 8
  - Hyperbolas ($\sec^2 - \tan^2 = 1$): Problem 9
  - Double-Angle Identities: Problem 10
- **Sketching Curves and Determining Orientation:**
  - Parabolas: Problems 11, 14, 17
  - Self-Intersecting Curves: Problem 12
  - Oscillating Motion on an Arc: Problem 13
  - Semi-circles/Arcs: Problem 15
  - Hyperbolas: Problem 18
  - Lissajous Figures: Problem 22
- **Analyzing Motion (Period, Direction, Description):** Problems 19, 20, 21
- **Parameterizing a Cartesian Equation:**
  - Line: Problem 23
  - Parabola: Problem 24
  - Ellipse: Problem 25
- **Applications:**
  - Line Segments: Problem 26
  - Projectile Motion: Problem 27
  - Cycloid: Problem 28
- **Advanced Topics:**
  - Intersection vs. Collision: Problem 29
  - Calculus (Derivatives/Tangent Slopes): Problem 30