Homework 11.8 Power Series: Practice Problem Set

Generated by Gemini

November 2025

Concept Checklist

This problem set is designed to test your understanding of the following concepts related to Power Series. The goal is to ensure you have mastered the full range of techniques required for this topic.

- **Core Technique: Finding the Radius of Convergence**
  - Applying the Ratio Test to a power series of the form $\sum c_n(x-a)^n$.
  - Applying the Root Test, especially for series where c_n involves an n-th power.
- **Endpoint Analysis: Determining the Interval of Convergence**
  - **p-Series Test:** Recognizing and applying the test for series of the form $\sum \frac{1}{n^p}$ at an endpoint.
  - **Alternating Series Test (AST):** Applying the test for series with a $(-1)^n$ term at an endpoint.
  - **Divergence Test:** Identifying series whose terms do not approach zero at the endpoints.
  - **Limit Comparison Test (LCT):** Using the test for more complex endpoint series by comparing them to known series like $\sum \frac{1}{n}$.
- **Categorization of Power Series**
  - Series centered at $a = 0$.
  - Series centered at $a \neq 0$, including those requiring algebraic manipulation (e.g., rewriting $(kx-c)^n$ as $k^n(x-c/k)^n$).
- **Special Cases for the Radius of Convergence**
  - Series that converge only at their center ($R = 0$).
  - Series that converge for all real numbers ($R = \infty$).
- **Advanced Concepts**
  - Series with more complex terms, such as x^{2n}, x^{n^2}, or factorials in both numerator and denominator.
  - Finding new series through term-by-term differentiation and integration of a known series.

Practice Problems

For each of the following power series, find the radius of convergence, R, and the interval of convergence, I. Show all steps, including the application of the Ratio or Root Test and the full analysis of the endpoints.

Standard Series (Centered at a=0)

1. $\sum_{n=1}^{\infty} \frac{x^n}{n^3}$ 2. $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n \cdot 5^n}$ 3. $\sum_{n=1}^{\infty} \frac{n^2 x^n}{3^n}$ 4. $\sum_{n=1}^{\infty} n^n x^n$ 5. $\sum_{n=2}^{\infty} \frac{x^n}{\ln n}$ 6. $\sum_{n=1}^{\infty} \frac{10^n x^n}{n^3}$ 7. $\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{9^n}$
8. $\sum_{n=1}^{\infty} \frac{x^n}{5n+1}$ 9. $\sum_{n=1}^{\infty} \frac{(n+1)x^n}{n^2+2n}$ 10. $\sum_{n=1}^{\infty} \frac{(-2)^n x^n}{\sqrt{n}}$

Series Centered at a $\neq$ 0

11. $\sum_{n=1}^{\infty} \frac{(x-5)^n}{n^2}$ 12. $\sum_{n=1}^{\infty} \frac{(x+1)^n}{n \cdot 4^n}$ 13. $\sum_{n=0}^{\infty} \frac{(-1)^n (x-2)^n}{3n+1}$ 14. $\sum_{n=1}^{\infty} \frac{n(x+3)^n}{5^n}$ 15. $\sum_{n=1}^{\infty} \frac{(3x-1)^n}{n^3}$
16. $\sum_{n=1}^{\infty} \frac{(2x+5)^n}{n^2 4^n}$

Series with Factorials & Special Radii

17. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ 18. $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2} x^n$ 19. $\sum_{n=1}^{\infty} \frac{n! x^n}{100^n}$ 20. $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ 21. $\sum_{n=1}^{\infty} \frac{n!(x-4)^n}{n^3 2^n}$

Series for the Root Test

22. $\sum_{n=1}^{\infty} \left(\frac{n}{3n+1}\right)^n x^n$ 23. $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2} x^n$ 24. $\sum_{n=2}^{\infty} \frac{(x+2)^n}{(\ln n)^n}$ 25. $\sum_{n=1}^{\infty} \left(\frac{4n+3}{2n+5}\right)^n (x-1)^n$

Term-by-Term Differentiation and Integration

26. The geometric series $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$ converges for $I = (-1, 1)$. Find a power series for $\ln(1+x)$ by integrating this series. What are the new radius and interval of convergence? 27. Using the same geometric series as in problem 26, find a power series for $\frac{1}{(1+x)^2}$ by differentiating the series. What are the radius and interval of convergence?

Mixed and Challenging Problems

28. $\sum_{n=1}^{\infty} \frac{3^n (x-1)^{2n}}{n}$ 29. $\sum_{n=1}^{\infty} \frac{n^2 (x+2)^n}{2^n (n^3+1)}$ 30. $\sum_{n=1}^{\infty} \frac{(n!)^3}{(3n)!} x^n$ 31. $\sum_{n=1}^{\infty} \frac{(-1)^n (x-4)^n}{\sqrt{n^2+9}}$ 32. $\sum_{n=1}^{\infty} \frac{x^{3n}}{n!}$

Solutions

Solutions to Standard Series (Centered at a=0)

1. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^3}$

- **Ratio Test:** $L = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)^3} \cdot \frac{n^3}{x^n} \right| = |x| \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^3 = |x|$.
- For convergence, $|x| < 1$. Thus, $R = 1$.
- **Endpoints:**
  - $x = 1$: $\sum \frac{1}{n^3}$. Converges (p-series, $p = 3 > 1$).
  - $x = -1$: $\sum \frac{(-1)^n}{n^3}$. Converges (absolutely).
- **Answer:** $R = 1, I = [-1, 1]$.

2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n \cdot 5^n}$

- **Ratio Test:** $L = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)5^{n+1}} \cdot \frac{n \cdot 5^n}{x^n} \right| = \frac{|x|}{5} \lim_{n\to\infty} \frac{n}{n+1} = \frac{|x|}{5}$.
- For convergence, $\frac{|x|}{5} < 1 \implies |x| < 5$. Thus, $R = 5$.
- **Endpoints:**
  - $x = 5$: $\sum \frac{(-1)^n 5^n}{n \cdot 5^n} = \sum \frac{(-1)^n}{n}$. Converges (AST).
  - $x = -5$: $\sum \frac{(-1)^n (-5)^n}{n \cdot 5^n} = \sum \frac{5^n}{n \cdot 5^n} = \sum \frac{1}{n}$. Diverges (harmonic).
- **Answer:** $R = 5, I = (-5, 5]$.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 x^n}{3^n}$

- **Ratio Test:** $L = \lim_{n\to\infty} \left| \frac{(n+1)^2 x^{n+1}}{3^{n+1}} \cdot \frac{3^n}{n^2 x^n} \right| = \frac{|x|}{3} \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^2 = \frac{|x|}{3}$.
- For convergence, $|x| < 3$. Thus, $R = 3$.
- **Endpoints:**
  - $x = 3$: $\sum \frac{n^2 3^n}{3^n} = \sum n^2$. Diverges (Divergence Test).
  - $x = -3$: $\sum \frac{n^2 (-3)^n}{3^n} = \sum (-1)^n n^2$. Diverges (Divergence Test).
- **Answer:** $R = 3, I = (-3, 3)$.

4. $\displaystyle\sum_{n=1}^{\infty} n^n x^n$

- **Root Test:** $L = \lim_{n\to\infty} \sqrt[n]{|n^n x^n|} = \lim_{n\to\infty} |nx| = \infty$ for any $x \neq 0$.
- The series diverges for all $x \neq 0$. It only converges at its center, $x = 0$.
- **Answer:** $R = 0, I = \{0\}$.

5. $\sum_{n=2}^{\infty} \frac{x^n}{\ln n}$

- **Ratio Test:** $L = \lim_{n\to\infty} \left| \frac{x^{n+1}}{\ln(n+1)} \cdot \frac{\ln n}{x^n} \right| = |x| \lim_{n\to\infty} \frac{\ln n}{\ln(n+1)} = |x| \cdot 1 = |x|$ (using L'Hopital's Rule).
- For convergence, $|x| < 1$. Thus, $R = 1$.
- **Endpoints:**
  - $x = 1$: $\sum \frac{1}{\ln n}$. Diverges (Direct comparison with $\sum \frac{1}{n}$, since $\ln n < n$ for $n \geq 1$).
  - $x = -1$: $\sum \frac{(-1)^n}{\ln n}$. Converges (AST).
- **Answer:** $R = 1, I = [-1, 1)$.

6. $\sum_{n=1}^{\infty} \frac{10^n x^n}{n^3}$

- **Ratio Test:** $L = \lim_{n\to\infty} \left| \frac{10^{n+1} x^{n+1}}{(n+1)^3} \cdot \frac{n^3}{10^n x^n} \right| = 10|x| \lim_{n\to\infty} (\frac{n}{n+1})^3 = 10|x|$.
- For convergence, $10|x| < 1 \implies |x| < 1/10$. Thus, $R = 1/10$.
- **Endpoints:**
  - $x = 1/10$: $\sum \frac{10^n (1/10)^n}{n^3} = \sum \frac{1}{n^3}$. Converges (p-series, $p = 3 > 1$).
  - $x = -1/10$: $\sum \frac{10^n (-1/10)^n}{n^3} = \sum \frac{(-1)^n}{n^3}$. Converges (absolutely).
- **Answer:** $R = 1/10, I = [-1/10, 1/10]$.

7. $\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{9^n}$

- This is a geometric series in disguise. Let $u = -x^2/9$. The series is $\sum_{n=1}^{\infty} u^n$. It converges if $|u| < 1$.
- $|-x^2/9| < 1 \implies x^2/9 < 1 \implies x^2 < 9 \implies |x| < 3$.
- Thus, $R = 3$.
- **Endpoints:**
  - $x = 3$: $\sum \frac{(-1)^n 3^{2n}}{9^n} = \sum \frac{(-1)^n 9^n}{9^n} = \sum (-1)^n$. Diverges (Divergence Test).
  - $x = -3$: $\sum \frac{(-1)^n (-3)^{2n}}{9^n} = \sum \frac{(-1)^n 9^n}{9^n} = \sum (-1)^n$. Diverges (Divergence Test).
- **Answer:** $R = 3, I = (-3, 3)$.

8. $\sum_{n=1}^{\infty} \frac{x^n}{5n+1}$

- **Ratio Test:** $L = \lim_{n\to\infty} \left| \frac{x^{n+1}}{5(n+1)+1} \cdot \frac{5n+1}{x^n} \right| = |x| \lim_{n\to\infty} \frac{5n+1}{5n+6} = |x|$.
- For convergence, $|x| < 1$. Thus, $R = 1$.
- **Endpoints:**

- $x = 1$: $\sum \frac{1}{5n+1}$. Diverges (Limit Comparison Test with $\sum \frac{1}{n}$).
- $x = -1$: $\sum \frac{(-1)^n}{5n+1}$. Converges (AST).

- **Answer:** $R = 1, I = [-1, 1)$.

9. $\displaystyle\sum_{n=1}^{\infty} \frac{(n+1)x^n}{n^2 + 2n}$

- Simplify term: $\frac{n+1}{n(n+2)}x^n$.
- **Ratio Test:** $L = \lim_{n\to\infty} \left| \frac{(n+2)x^{n+1}}{(n+1)(n+3)} \cdot \frac{n(n+2)}{(n+1)x^n} \right| = |x| \lim_{n\to\infty} \frac{n(n+2)^2}{(n+1)^2(n+3)} = |x| \cdot 1 = |x|$.
- For convergence, $|x| < 1$. Thus, $R = 1$.
- **Endpoints:**
  - $x = 1$: $\sum \frac{n+1}{n(n+2)}$. Diverges (Limit Comparison Test with $\sum \frac{1}{n}$).
  - $x = -1$: $\sum \frac{(-1)^n(n+1)}{n(n+2)}$. Converges (AST).
- **Answer:** $R = 1, I = [-1, 1)$.

10. $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n x^n}{\sqrt{n}}$

- Term is $\frac{(-1)^n 2^n x^n}{\sqrt{n}}$.
- **Ratio Test:** $L = \lim_{n\to\infty} \left| \frac{(-2)^{n+1}x^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{(-2)^n x^n} \right| = |-2x| \lim_{n\to\infty} \sqrt{\frac{n}{n+1}} = 2|x|$.
- For convergence, $2|x| < 1 \implies |x| < 1/2$. Thus, $R = 1/2$.
- **Endpoints:**
  - $x = 1/2$: $\sum \frac{(-2)^n(1/2)^n}{\sqrt{n}} = \sum \frac{(-1)^n}{\sqrt{n}}$. Converges (AST).
  - $x = -1/2$: $\sum \frac{(-2)^n(-1/2)^n}{\sqrt{n}} = \sum \frac{1^n}{\sqrt{n}} = \sum \frac{1}{n^{1/2}}$. Diverges (p-series, $p = 1/2 \le 1$).
- **Answer:** $R = 1/2, I = (-1/2, 1/2]$.

Solutions to Series Centered at $a \neq 0$

11. $\displaystyle\sum_{n=1}^{\infty} \frac{(x-5)^n}{n^2}$

- Center $a = 5$.
- **Ratio Test:** $L = \lim_{n\to\infty} \left| \frac{(x-5)^{n+1}}{(n+1)^2} \cdot \frac{n^2}{(x-5)^n} \right| = |x-5| \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^2 = |x-5|$.
- For convergence, $|x-5| < 1$. Thus, $R = 1$. Interval: $(4, 6)$.
- **Endpoints:**
  - $x = 6$: $\sum \frac{1^n}{n^2} = \sum \frac{1}{n^2}$. Converges (p-series, $p = 2 > 1$).
  - $x = 4$: $\sum \frac{(-1)^n}{n^2}$. Converges (absolutely).
- **Answer:** $R = 1, I = [4, 6]$.

12. $\sum_{n=1}^{\infty} \frac{(x+1)^n}{n \cdot 4^n}$

- Center $a = -1$.
- **Ratio Test:** $L = \lim_{n\to\infty} \left| \frac{(x+1)^{n+1}}{(n+1)4^{n+1}} \cdot \frac{n \cdot 4^n}{(x+1)^n} \right| = \frac{|x+1|}{4} \lim_{n\to\infty} \frac{n}{n+1} = \frac{|x+1|}{4}$.
- For convergence, $\frac{|x+1|}{4} < 1 \implies |x+1| < 4$. Thus, $R = 4$. Interval: $(-5, 3)$.
- **Endpoints:**
  - $x = 3$: $\sum \frac{4^n}{n \cdot 4^n} = \sum \frac{1}{n}$. Diverges (harmonic).
  - $x = -5$: $\sum \frac{(-4)^n}{n \cdot 4^n} = \sum \frac{(-1)^n}{n}$. Converges (AST).
- **Answer:** $R = 4, I = [-5, 3)$.

13. $\sum_{n=0}^{\infty} \frac{(-1)^n (x-2)^n}{3n+1}$

- Center $a = 2$.
- **Ratio Test:** $L = \lim_{n\to\infty} \left| \frac{(x-2)^{n+1}}{3(n+1)+1} \cdot \frac{3n+1}{(x-2)^n} \right| = |x-2| \lim_{n\to\infty} \frac{3n+1}{3n+4} = |x-2|$.
- For convergence, $|x-2| < 1$. Thus, $R = 1$. Interval: $(1, 3)$.
- **Endpoints:**
  - $x = 3$: $\sum \frac{(-1)^n (1)^n}{3n+1} = \sum \frac{(-1)^n}{3n+1}$. Converges (AST).
  - $x = 1$: $\sum \frac{(-1)^n (-1)^n}{3n+1} = \sum \frac{1}{3n+1}$. Diverges (LCT with $\sum \frac{1}{n}$).
- **Answer:** $R = 1, I = (1, 3]$.

14. $\sum_{n=1}^{\infty} \frac{n(x+3)^n}{5^n}$

- Center $a = -3$.
- **Ratio Test:** $L = \lim_{n\to\infty} \left| \frac{(n+1)(x+3)^{n+1}}{5^{n+1}} \cdot \frac{5^n}{n(x+3)^n} \right| = \frac{|x+3|}{5} \lim_{n\to\infty} \frac{n+1}{n} = \frac{|x+3|}{5}$.
- For convergence, $|x+3| < 5$. Thus, $R = 5$. Interval: $(-8, 2)$.
- **Endpoints:**
  - $x = 2$: $\sum \frac{n \cdot 5^n}{5^n} = \sum n$. Diverges (Divergence Test).
  - $x = -8$: $\sum \frac{n(-5)^n}{5^n} = \sum (-1)^n n$. Diverges (Divergence Test).
- **Answer:** $R = 5, I = (-8, 2)$.

15. $\sum_{n=1}^{\infty} \frac{(3x-1)^n}{n^3}$

- Rewrite: $\frac{[3(x-1/3)]^n}{n^3} = \frac{3^n (x-1/3)^n}{n^3}$. Center $a = 1/3$.
- **Ratio Test:** $L = \lim_{n\to\infty} \left| \frac{3^{n+1}(x-1/3)^{n+1}}{(n+1)^3} \cdot \frac{n^3}{3^n (x-1/3)^n} \right| = 3|x - 1/3|$.
- For convergence, $3|x-1/3| < 1 \implies |x - 1/3| < 1/3$. Thus, $R = 1/3$. Interval: $(0, 2/3)$.

- **Endpoints:**
  - $x = 2/3$: $\sum \frac{(2-1)^n}{n^3} = \sum \frac{1}{n^3}$. Converges (p-series, $p = 3 > 1$).
  - $x = 0$: $\sum \frac{(-1)^n}{n^3}$. Converges (absolutely).
- **Answer:** $R = 1/3, I = [0, 2/3]$.

16. $\displaystyle\sum_{n=1}^{\infty} \frac{(2x+5)^n}{n^2 4^n}$

- Rewrite: $\frac{[2(x+5/2)]^n}{n^2 4^n} = \frac{2^n (x+5/2)^n}{n^2 4^n} = \frac{(x+5/2)^n}{n^2 2^n}$. Center $a = -5/2$.
- **Ratio Test:** $L = \lim_{n\to\infty} \left| \frac{(x+5/2)^{n+1}}{(n+1)^2 2^{n+1}} \cdot \frac{n^2 2^n}{(x+5/2)^n} \right| = \frac{|x+5/2|}{2}$.
- For convergence, $\frac{|x+5/2|}{2} < 1 \implies |x + 5/2| < 2$. Thus, $R = 2$. Interval: $(-9/2, -1/2)$.
- **Endpoints:**
  - $x = -1/2$: $\sum \frac{(-1/2+5/2)^n}{n^2 2^n} = \sum \frac{2^n}{n^2 2^n} = \sum \frac{1}{n^2}$. Converges (p-series, $p = 2 > 1$).
  - $x = -9/2$: $\sum \frac{(-9/2+5/2)^n}{n^2 2^n} = \sum \frac{(-2)^n}{n^2 2^n} = \sum \frac{(-1)^n}{n^2}$. Converges (absolutely).
- **Answer:** $R = 2, I = [-9/2, -1/2]$.

Solutions to Series with Factorials & Special Radii

17. $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$

- **Ratio Test:** $L = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n\to\infty} \left| \frac{x}{n+1} \right| = |x| \cdot 0 = 0$.
- Since $L = 0 < 1$ for all x, the series converges for all real numbers.
- **Answer:** $R = \infty, I = (-\infty, \infty)$.

18. $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2} x^n$

- **Ratio Test:** $L = \lim_{n\to\infty} \left| \frac{(2(n+1))! x^{n+1}}{((n+1)!)^2} \cdot \frac{(n!)^2}{(2n)! x^n} \right| = |x| \lim_{n\to\infty} \frac{(2n+2)!}{(2n)!} \frac{(n!)^2}{((n+1)!)^2}$.
- $L = |x| \lim_{n\to\infty} (2n+2)(2n+1) \frac{1}{(n+1)^2} = |x| \lim_{n\to\infty} \frac{4n^2+6n+2}{n^2+2n+1} = 4|x|$.
- For convergence, $4|x| < 1 \implies |x| < 1/4$. Thus, $R = 1/4$. Endpoint analysis is complex and often omitted in standard courses, but both endpoints diverge.
- **Answer:** $R = 1/4, I = (-1/4, 1/4)$.

19. $\displaystyle\sum_{n=1}^{\infty} \frac{n! x^n}{100^n}$

- **Ratio Test:** $L = \lim_{n\to\infty} \left| \frac{(n+1)! x^{n+1}}{100^{n+1}} \cdot \frac{100^n}{n! x^n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)x}{100} \right| = \infty$ for any $x \neq 0$.
- The series only converges at its center, $x = 0$.
- **Answer:** $R = 0, I = \{0\}$.

20. $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

- **Ratio Test:** $L = \lim_{n\to\infty} \left| \frac{x^{2(n+1)+1}}{(2(n+1)+1)!} \cdot \frac{(2n+1)!}{x^{2n+1}} \right| = \lim_{n\to\infty} \left| \frac{x^2}{(2n+3)(2n+2)} \right| = |x^2| \cdot 0 = 0.$
- Since $L = 0 < 1$ for all x, the series converges for all real numbers.
- **Answer:** $R = \infty, I = (-\infty, \infty)$.

21. $\sum_{n=1}^{\infty} \frac{n!(x-4)^n}{n^3 2^n}$

- Center $a = 4$.
- **Ratio Test:** $L = \lim_{n\to\infty} \left| \frac{(n+1)!(x-4)^{n+1}}{(n+1)^3 2^{n+1}} \cdot \frac{n^3 2^n}{n!(x-4)^n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)(x-4)}{2} \left(\frac{n}{n+1}\right)^3 \right| = \infty$ for $x \neq 4$.
- The series only converges at its center, $x = 4$.
- **Answer:** $R = 0, I = \{4\}$.

Solutions to Series for the Root Test

22. $\sum_{n=1}^{\infty} \left(\frac{n}{3n+1}\right)^n x^n$

- **Root Test:** $L = \lim_{n\to\infty} \sqrt[n]{\left| \left(\frac{n}{3n+1}\right)^n x^n \right|} = \lim_{n\to\infty} \left| \frac{nx}{3n+1} \right| = |x| \lim_{n\to\infty} \frac{n}{3n+1} = \frac{|x|}{3}.$
- For convergence, $\frac{|x|}{3} < 1 \implies |x| < 3$. Thus, $R = 3$. Endpoints diverge by Divergence Test.
- **Answer:** $R = 3, I = (-3, 3)$.

23. $\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{n^2} x^n$

- **Root Test:** $L = \lim_{n\to\infty} \sqrt[n]{\left| \left(1 + \frac{1}{n}\right)^{n^2} x^n \right|} = |x| \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = |x| \cdot e = e|x|.$
- For convergence, $e|x| < 1 \implies |x| < 1/e$. Thus, $R = 1/e$. Endpoints diverge.
- **Answer:** $R = 1/e, I = (-1/e, 1/e)$.

24. $\sum_{n=2}^{\infty} \frac{(x+2)^n}{(\ln n)^n}$

- Center $a = -2$.
- **Root Test:** $L = \lim_{n\to\infty} \sqrt[n]{\left| \frac{(x+2)^n}{(\ln n)^n} \right|} = \lim_{n\to\infty} \left| \frac{x+2}{\ln n} \right| = |x+2| \cdot 0 = 0.$
- Since $L = 0 < 1$ for all x, the series converges everywhere.
- **Answer:** $R = \infty, I = (-\infty, \infty)$.

25. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{4n+3}{2n+5}\right)^n (x-1)^n$

- Center $a = 1$.
- **Root Test:** $L = \lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{4n+3}{2n+5}\right)^n (x-1)^n\right|} = |x-1| \lim_{n\to\infty} \frac{4n+3}{2n+5} = 2|x-1|$.
- For convergence, $2|x-1| < 1 \implies |x-1| < 1/2$. Thus, $R = 1/2$. Endpoints diverge by Divergence Test.
- **Answer:** $R = 1/2, I = (1/2, 3/2)$.

Solutions to Term-by-Term Differentiation and Integration

26. Find a series for $\ln(1+x)$.

- $\ln(1+x) = \int \frac{1}{1+x}dx = \int \left(\sum_{n=0}^{\infty}(-1)^n x^n\right) dx = C + \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1}$.
- To find C, let $x = 0$. $\ln(1) = 0$, and the series is also 0. So $C = 0$.
- Series is $\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$
- The radius of convergence is the same, $R = 1$.
- **Endpoints:**
  - $x = 1$: $\sum \frac{(-1)^n}{n+1}$. Converges (AST).
  - $x = -1$: $\sum \frac{(-1)^n(-1)^{n+1}}{n+1} = \sum \frac{-1}{n+1}$. Diverges (harmonic series).
- **Answer:** Series: $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n}$, $R = 1$, $I = (-1, 1]$.

27. Find a series for $\frac{1}{(1+x)^2}$.

- Note that $\frac{d}{dx}\left(\frac{1}{1+x}\right) = -\frac{1}{(1+x)^2}$.
- $\frac{d}{dx}\left(\sum_{n=0}^{\infty}(-1)^n x^n\right) = \sum_{n=1}^{\infty}(-1)^n n x^{n-1}$.
- So, $\frac{1}{(1+x)^2} = -\sum_{n=1}^{\infty}(-1)^n n x^{n-1} = \sum_{n=1}^{\infty}(-1)^{n+1} n x^{n-1} = 1 - 2x + 3x^2 - \dots$
- The radius is the same, $R = 1$.
- **Endpoints:**
  - $x = 1$: $\sum(-1)^{n+1}n$. Diverges (Divergence Test).
  - $x = -1$: $\sum(-1)^{n+1}n(-1)^{n-1} = \sum n$. Diverges (Divergence Test).
- **Answer:** Series: $\displaystyle\sum_{n=0}^{\infty}(-1)^n(n+1)x^n$, $R = 1$, $I = (-1, 1)$.

Solutions to Mixed and Challenging Problems

28. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n(x-1)^{2n}}{n}$

- Let $u = 3(x-1)^2$. The series becomes $\sum \frac{u^n}{n \cdot 3^n}$ without the 3^n factor, or let $u = (x-1)^2$ giving $\sum \frac{3^n u^n}{n}$. Let's use the Ratio Test directly.
- $L = \lim_{n\to\infty} \left| \frac{3^{n+1}(x-1)^{2(n+1)}}{n+1} \cdot \frac{n}{3^n(x-1)^{2n}} \right| = \lim_{n\to\infty} \left| 3(x-1)^2 \frac{n}{n+1} \right| = 3(x-1)^2$.
- For convergence, $3(x-1)^2 < 1 \implies (x-1)^2 < 1/3 \implies |x-1| < 1/\sqrt{3}$.
- $R = 1/\sqrt{3}$. Interval: $(1 - 1/\sqrt{3}, 1 + 1/\sqrt{3})$.
- **Endpoints:**
  - $|x-1| = 1/\sqrt{3} \implies (x-1)^2 = 1/3$. The series becomes $\sum \frac{3^n(1/3)^n}{n} = \sum \frac{1}{n}$. Diverges. At both endpoints, we get divergence. Wait, let me recheck.
  - Correct endpoint test: At $x = 1 \pm 1/\sqrt{3}$, we have $(x-1)^2 = 1/3$. The series becomes $\sum \frac{3^n(1/3)^n}{n} = \sum \frac{1}{n}$. This diverges.
- **Answer:** $R = 1/\sqrt{3}, I = (1 - 1/\sqrt{3}, 1 + 1/\sqrt{3})$.

29. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2(x+2)^n}{2^n(n^3+1)}$

- Center $a = -2$.
- **Ratio Test:** $L = \lim_{n\to\infty} \left| \frac{(n+1)^2(x+2)^{n+1}}{2^{n+1}((n+1)^3+1)} \cdot \frac{2^n(n^3+1)}{n^2(x+2)^n} \right| = \frac{|x+2|}{2} \lim_{n\to\infty} \frac{(n+1)^2(n^3+1)}{n^2((n+1)^3+1)} = \frac{|x+2|}{2}$.
- For convergence, $|x+2| < 2$. Thus, $R = 2$. Interval: $(-4, 0)$.
- **Endpoints:**
  - $x = 0$: $\sum \frac{n^2 2^n}{2^n(n^3+1)} = \sum \frac{n^2}{n^3+1}$. Diverges (LCT with $\sum \frac{1}{n}$).
  - $x = -4$: $\sum \frac{n^2(-2)^n}{2^n(n^3+1)} = \sum \frac{(-1)^n n^2}{n^3+1}$. Converges (AST).
- **Answer:** $R = 2, I = [-4, 0)$.

30. $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^3}{(3n)!} x^n$

- **Ratio Test:** $L = \lim_{n\to\infty} \left| \frac{((n+1)!)^3 x^{n+1}}{(3(n+1))!} \cdot \frac{(3n)!}{(n!)^3 x^n} \right| = |x| \lim_{n\to\infty} \frac{(n+1)^3(n!)^3}{(n!)^3} \frac{(3n)!}{(3n+3)!}$.
- $L = |x| \lim_{n\to\infty} (n+1)^3 \frac{1}{(3n+3)(3n+2)(3n+1)} = |x| \lim_{n\to\infty} \frac{n^3+\dots}{27n^3+\dots} = \frac{|x|}{27}$.
- For convergence, $|x| < 27$. Thus, $R = 27$. Endpoints diverge.
- **Answer:** $R = 27, I = (-27, 27)$.

31. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(x-4)^n}{\sqrt{n^2+9}}$

- Center $a = 4$.

- **Ratio Test:** $L = |x-4| \lim_{n\to\infty} \frac{\sqrt{n^2+9}}{\sqrt{(n+1)^2+9}} = |x-4| \cdot 1 = |x-4|$.
- For convergence, $|x-4| < 1$. Thus, $R = 1$. Interval: $(3, 5)$.
- **Endpoints:**
  - $x = 5$: $\sum \frac{(-1)^n}{\sqrt{n^2+9}}$. Converges (AST).
  - $x = 3$: $\sum \frac{(-1)^n(-1)^n}{\sqrt{n^2+9}} = \sum \frac{1}{\sqrt{n^2+9}}$. Diverges (LCT with $\sum \frac{1}{n}$).
- **Answer:** $R = 1, I = (3, 5]$.

32. $\displaystyle\sum_{n=1}^{\infty} \frac{x^{3n}}{n!}$

- **Ratio Test:** $L = \lim_{n\to\infty} \left| \frac{x^{3(n+1)}}{(n+1)!} \cdot \frac{n!}{x^{3n}} \right| = \lim_{n\to\infty} \left| \frac{x^3}{n+1} \right| = |x^3| \cdot 0 = 0$.
- Since $L = 0 < 1$ for all x, the series converges everywhere.
- **Answer:** $R = \infty, I = (-\infty, \infty)$.

Concept and Problem Mapping

This list maps each concept from the checklist to the problem numbers that specifically test that concept.

- **Core Technique: Finding the Radius of Convergence**
  - **Ratio Test Application:** 1, 2, 3, 5, 6, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 28, 29, 30, 31, 32. (Essentially all non-Root-Test problems).
  - **Root Test Application:** 4, 22, 23, 24, 25.
- **Endpoint Analysis: Determining the Interval of Convergence**
  - **p-Series Test:** 1, 10, 11, 12, 15, 16.
  - **Alternating Series Test (AST):** 2, 5, 8, 9, 10, 12, 13, 26, 29, 31.
  - **Divergence Test:** 3, 4, 7, 14, 19, 21, 22, 23, 25, 27.
  - **Limit Comparison Test (LCT):** 8, 9, 13, 29, 31 (could be used for the divergent endpoint).
- **Categorization of Power Series**
  - **Series centered at $a = 0$:** 1-10, 17, 18, 19, 20, 22, 23, 26, 27, 30, 32.
  - **Series centered at $a \neq 0$:** 11, 12, 13, 14, 21, 24, 25, 28, 29, 31.
  - **Requiring algebraic manipulation to find center:** 15, 16.
- **Special Cases for the Radius of Convergence**
  - **Converges only at center ($R = 0$):** 4, 19, 21.
  - **Converges for all real numbers ($R = \infty$):** 17, 20, 24, 32.
- **Advanced Concepts**
  - **Series with x^{kn} terms:** 7, 20, 28, 32.
  - **Term-by-term Differentiation/Integration:** 26, 27.