Homework 11.1 Sequences: Problem Set

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Concept Checklist

This problem set is designed to test the following concepts related to sequences:

- **Direct Calculation & Basic Properties:**
  - Writing the first few terms of a sequence from an explicit formula.
  - Writing the first few terms of a recursively defined sequence.
  - Understanding the definitions of convergence and divergence.
- **Pattern Recognition:**
  - Finding an explicit formula for an arithmetic sequence.
  - Finding an explicit formula for a geometric sequence.
  - Finding an explicit formula for sequences involving alternating signs, factorials, and powers.
- **Limit Evaluation via Algebraic Manipulation:**
  - Limits of rational functions of n (dividing by the highest power).
  - Limits involving radicals (dividing by highest power, using conjugates).
  - Limits of geometric sequences ($\lim_{n\to\infty} r^n$).
  - Limits involving exponential functions (dividing by the fastest-growing base).
  - Limits involving factorials (simplification).
  - Limits involving logarithmic properties.
- **Limit Evaluation using Calculus Theorems:**
  - **L'Hôpital's Rule:** For indeterminate forms $\frac{\infty}{\infty}$ or $\frac{0}{0}$ involving functions of n (like logarithms, exponentials, and powers).
  - **Squeeze Theorem:** For sequences involving bounded, oscillating terms like $\sin(n)$, $\cos(n)$, or $(-1)^n$.
  - **Continuity:** Evaluating limits by passing the limit inside a continuous function (e.g., $\lim_{n\to\infty} e^{a_n} = e^{\lim a_n}$).
  - **Indeterminate Forms:** Evaluating limits of the form 1^∞, 0^0, or ∞^0 using logarithms and L'Hôpital's Rule.
- **Monotonic Sequence Theorem:**
  - Proving a sequence is monotonic (increasing or decreasing).
  - Proving a sequence is bounded (above and/or below).
  - Using the Monotonic Sequence Theorem to conclude convergence and find the limit of a recursive sequence.

Problem Set

Part 1: Direct Calculation and Pattern Recognition

1. List the first five terms of the sequence $a_n = \frac{n^2-1}{n^2+1}$.
2. List the first five terms of the sequence defined by $a_1 = 2$ and $a_{n+1} = \frac{a_n}{a_n-1}$.
3. Find a formula for the general term a_n of the sequence, assuming the pattern continues:
$$\left\{\frac{3}{4}, -\frac{4}{8}, \frac{5}{16}, -\frac{6}{32}, \dots\right\}$$
4. Find a formula for the general term a_n of the arithmetic sequence $\{11, 8, 5, 2, \dots\}$.
5. Find a formula for the general term a_n of the geometric sequence $\{5, -10/3, 20/9, -40/27, \dots\}$.

Part 2: Determining Convergence or Divergence

For each of the following sequences, determine whether it converges or diverges. If it converges, find the limit.

6. $a_n = \frac{3n^2-5n+2}{8n^2+4n-1}$
7. $a_n = \frac{n}{n^3+1}$
8. $a_n = \frac{n^4-n^2}{n^3+n}$
9. $a_n = \frac{\sqrt{4n^2+1}}{3n-2}$
10. $a_n = \sqrt{n^2+n} - n$
11. $a_n = \frac{5^n+3^n}{5^n-2^n}$
12. $a_n = (-1.01)^n$
13. $a_n = \frac{3^{n+2}}{7^n}$
14. $a_n = \frac{\cos(n)}{n^2}$
15. $a_n = \frac{(-1)^n n!}{n^n}$ (Hint: Write out the terms of $\frac{n!}{n^n}$ and bound it.)
16. $a_n = \frac{5n^2-\sin(2n)}{n^2+n}$
17. $a_n = \frac{\ln(n)}{\sqrt{n}}$
18. $a_n = n^2 e^{-n}$
19. $a_n = \frac{(\ln n)^3}{n}$
20. $a_n = \arctan(2n)$
21. $a_n = \cos\left(\frac{n\pi}{n+1}\right)$
22. $a_n = \frac{(n+1)!-n!}{(n+1)!}$
23. $a_n = \frac{(2n-1)!}{(2n+1)!}$
24. $a_n = \left(1+\frac{3}{n}\right)^n$
25. $a_n = n^{1/n}$
26. $a_n = n\sin\left(\frac{1}{n}\right)$
27. $a_n = \frac{\ln(n^2+1)}{\ln(3n+1)}$
28. $a_n = \frac{2^n}{n!}$
29. $a_n = \frac{n\sin(n\pi)}{2n+1}$

Part 3: Monotonic Sequence Theorem

30. Consider the sequence defined by $a_1 = \sqrt{5}$ and $a_{n+1} = \sqrt{5 + a_n}$.

    a. Show that $\{a_n\}$ is increasing.

    b. Show that $\{a_n\}$ is bounded above by 3.

    c. Explain why the sequence converges and find its limit.

31. Consider the sequence defined by $a_1 = 3$ and $a_{n+1} = \frac{1}{4}(a_n + 6)$.

    a. Show that $\{a_n\}$ is decreasing and bounded below.

    b. Find the limit of the sequence.

Solutions

1. **Solution:** $a_1 = \frac{1-1}{1+1} = 0$, $a_2 = \frac{4-1}{4+1} = \frac{3}{5}$, $a_3 = \frac{9-1}{9+1} = \frac{8}{10} = \frac{4}{5}$, $a_4 = \frac{16-1}{16+1} = \frac{15}{17}$, $a_5 = \frac{25-1}{25+1} = \frac{24}{26} = \frac{12}{13}$. The terms are $\left\{0, \frac{3}{5}, \frac{4}{5}, \frac{15}{17}, \frac{12}{13}, \dots\right\}$.

2. **Solution:** $a_1 = 2$, $a_2 = \frac{2}{2-1} = 2$, $a_3 = \frac{2}{2-1} = 2$, $a_4 = 2$, $a_5 = 2$. The sequence is a constant sequence $\{2, 2, 2, 2, 2, \dots\}$.

3. **Solution:** The signs are alternating, starting negative if we consider $n = 1$ for the first term $-4/8$. Let's re-index to start at $n = 1$ for $3/4$. The sign is $(-1)^{n+1}$. The numerator starts at 3 and increases by 1, so it is $n + 2$. The denominator is a power of 2, starting with $4 = 2^2$, then $8 = 2^3$, etc. The denominator is 2^{n+1}. Thus, $a_n = (-1)^{n+1} \frac{n+2}{2^{n+1}}$.

4. **Solution:** This is an arithmetic sequence with first term $a_1 = 11$ and common difference $d = 8 - 11 = -3$. The formula is $a_n = a_1 + (n-1)d = 11 + (n-1)(-3) = 11 - 3n + 3 = 14 - 3n$.

5. **Solution:** This is a geometric sequence with first term $a_1 = 5$. The common ratio is $r = \frac{-10/3}{5} = -\frac{10}{15} = -\frac{2}{3}$. The formula is $a_n = a_1 r^{n-1} = 5\left(-\frac{2}{3}\right)^{n-1}$.

6. **Solution:** Divide by the highest power of n in the denominator, which is n^2.

$$\lim_{n\to\infty} \frac{3n^2/n^2 - 5n/n^2 + 2/n^2}{8n^2/n^2 + 4n/n^2 - 1/n^2} = \lim_{n\to\infty} \frac{3 - 5/n + 2/n^2}{8 + 4/n - 1/n^2} = \frac{3 - 0 + 0}{8 + 0 - 0} = \frac{3}{8}.$$

   Converges to $\frac{3}{8}$.

7. **Solution:** Divide by n^3.

$$\lim_{n\to\infty} \frac{n/n^3}{n^3/n^3 + 1/n^3} = \lim_{n\to\infty} \frac{1/n^2}{1 + 1/n^3} = \frac{0}{1+0} = 0.$$

   Converges to 0.

8. **Solution:** The degree of the numerator (4) is greater than the degree of the denominator (3).

$$\lim_{n\to\infty} \frac{n^4 - n^2}{n^3 + n} = \lim_{n\to\infty} \frac{n(1 - 1/n^2)}{1 + 1/n^2} = \infty.$$

   The sequence diverges.

9. **Solution:** Divide by n (which is $\sqrt{n^2}$).

$$\lim_{n\to\infty} \frac{\sqrt{4n^2/n^2 + 1/n^2}}{3n/n - 2/n} = \lim_{n\to\infty} \frac{\sqrt{4 + 1/n^2}}{3 - 2/n} = \frac{\sqrt{4+0}}{3-0} = \frac{2}{3}.$$

   Converges to $\frac{2}{3}$.

10. **Solution:** This is an indeterminate form $\infty - \infty$. Multiply by the conjugate.

$$\lim_{n\to\infty} \frac{(\sqrt{n^2+n} - n)(\sqrt{n^2+n} + n)}{\sqrt{n^2+n} + n} = \lim_{n\to\infty} \frac{n^2 + n - n^2}{\sqrt{n^2+n} + n} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2(1+1/n)} + n}$$

$$= \lim_{n\to\infty} \frac{n}{n\sqrt{1+1/n} + n} = \lim_{n\to\infty} \frac{1}{\sqrt{1+1/n} + 1} = \frac{1}{\sqrt{1+0} + 1} = \frac{1}{2}.$$

   Converges to $\frac{1}{2}$.

11. **Solution:** Divide by the fastest-growing term, 5^n.

$$\lim_{n\to\infty} \frac{5^n/5^n + 3^n/5^n}{5^n/5^n - 2^n/5^n} = \lim_{n\to\infty} \frac{1 + (3/5)^n}{1 - (2/5)^n} = \frac{1+0}{1-0} = 1.$$

   Converges to 1.

12. **Solution:** This is a geometric sequence with ratio $r = -1.01$. Since $|r| > 1$, the sequence diverges.

13. **Solution:** Rewrite as $a_n = 9 \cdot \left(\frac{3}{7}\right)^n$. This is a geometric sequence with $|r| = 3/7 < 1$.

$$\lim_{n\to\infty} 9\left(\frac{3}{7}\right)^n = 9 \cdot 0 = 0.$$

Converges to 0.

14. **Solution:** Use the Squeeze Theorem. We know $-1 \le \cos(n) \le 1$.

$$-\frac{1}{n^2} \le \frac{\cos(n)}{n^2} \le \frac{1}{n^2}$$

Since $\lim_{n\to\infty} -\frac{1}{n^2} = 0$ and $\lim_{n\to\infty} \frac{1}{n^2} = 0$, by the Squeeze Theorem, $\lim_{n\to\infty} \frac{\cos(n)}{n^2} = 0$. Converges to 0.

15. **Solution:** Use the Squeeze Theorem. $a_n = (-1)^n \frac{n!}{n^n} = (-1)^n \left(\frac{1}{n} \cdot \frac{2}{n} \cdot \frac{3}{n} \cdots \frac{n}{n}\right)$. We know that $0 \le \frac{n!}{n^n} \le \frac{1}{n}$. So, $-\frac{1}{n} \le (-1)^n \frac{n!}{n^n} \le \frac{1}{n}$. Since $\lim_{n\to\infty} \pm\frac{1}{n} = 0$, the limit of a_n is 0. Converges to 0.

16. **Solution:** Divide by n^2. $a_n = \frac{5-\sin(2n)/n^2}{1+1/n}$. For the term $\frac{\sin(2n)}{n^2}$, we can use the Squeeze Theorem. $-\frac{1}{n^2} \le \frac{\sin(2n)}{n^2} \le \frac{1}{n^2}$, so its limit is 0.

$$\lim_{n\to\infty} a_n = \frac{5-0}{1+0} = 5.$$

Converges to 5.

17. **Solution:** Use L'Hôpital's Rule for the indeterminate form $\frac{\infty}{\infty}$.

$$\lim_{x\to\infty} \frac{\ln(x)}{x^{1/2}} \stackrel{L'H}{=} \lim_{x\to\infty} \frac{1/x}{(1/2)x^{-1/2}} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0.$$

Converges to 0.

18. **Solution:** Use L'Hôpital's Rule for $\frac{\infty}{\infty}$. Let $f(x) = x^2/e^x$.

$$\lim_{x\to\infty} \frac{x^2}{e^x} \stackrel{L'H}{=} \lim_{x\to\infty} \frac{2x}{e^x} \stackrel{L'H}{=} \lim_{x\to\infty} \frac{2}{e^x} = 0.$$

Converges to 0.

19. **Solution:** Use L'Hôpital's Rule repeatedly. Let $f(x) = (\ln x)^3/x$.

$$\lim_{x\to\infty} \frac{(\ln x)^3}{x} \stackrel{L'H}{=} \lim_{x\to\infty} \frac{3(\ln x)^2 \cdot (1/x)}{1} = \lim_{x\to\infty} \frac{3(\ln x)^2}{x} \stackrel{L'H}{=} \lim_{x\to\infty} \frac{6(\ln x) \cdot (1/x)}{1} = \lim_{x\to\infty} \frac{6\ln x}{x} \stackrel{L'H}{=} \lim_{x\to\infty} \frac{6/x}{1} = 0.$$

Converges to 0.

20. **Solution:** As $n \to \infty$, $2n \to \infty$. The range of $\arctan(x)$ is $(-\pi/2, \pi/2)$.

$$\lim_{n\to\infty} \arctan(2n) = \frac{\pi}{2}.$$

Converges to $\pi/2$.

21. **Solution:** The function $\cos(x)$ is continuous.

$$\lim_{n\to\infty} \cos\left(\frac{n\pi}{n+1}\right) = \cos\left(\lim_{n\to\infty} \frac{n\pi}{n+1}\right) = \cos\left(\lim_{n\to\infty} \frac{\pi}{1+1/n}\right) = \cos(\pi) = -1.$$

Converges to -1.

22. **Solution:** Simplify the expression.

$$a_n = \frac{n!(n+1) - n!}{n!(n+1)} = \frac{n!(n+1-1)}{n!(n+1)} = \frac{n}{n+1}.$$

$$\lim_{n\to\infty} \frac{n}{n+1} = 1.$$

Converges to 1.

23. **Solution:** Simplify the expression using $(n+1)! = (n+1)n!$.

$$a_n = \frac{(2n-1)!}{(2n+1)(2n)(2n-1)!} = \frac{1}{(2n+1)(2n)}.$$

$$\lim_{n\to\infty} \frac{1}{4n^2+2n} = 0.$$

Converges to 0.

24. **Solution:** This is the indeterminate form 1^∞. Let $L = \lim_{n\to\infty} \left(1+\frac{3}{n}\right)^n$. Take the natural log: $\ln(L) = \lim_{n\to\infty} n \ln\left(1+\frac{3}{n}\right) = \lim_{n\to\infty} \frac{\ln(1+3/n)}{1/n}$. This is $\frac{0}{0}$, so use L'Hôpital's Rule.

$$\ln(L) = \lim_{n\to\infty} \frac{\frac{1}{1+3/n}\cdot(-3/n^2)}{-1/n^2} = \lim_{n\to\infty} \frac{3}{1+3/n} = 3.$$

So, $\ln(L) = 3$, which means $L = e^3$. Converges to e^3.

25. **Solution:** This is the indeterminate form ∞^0. Let $L = \lim_{n\to\infty} n^{1/n}$. Take the natural log: $\ln(L) = \lim_{n\to\infty} \frac{1}{n}\ln(n)$. This is $\frac{\infty}{\infty}$, so use L'Hôpital's Rule.

$$\ln(L) = \lim_{n\to\infty} \frac{1/n}{1} = 0.$$

So, $\ln(L) = 0$, which means $L = e^0 = 1$. Converges to 1.

26. **Solution:** Indeterminate form $\infty \cdot 0$. Rewrite and use L'Hôpital's Rule. Let $x = 1/n$. As $n \to \infty$, $x \to 0^+$.

$$\lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = \lim_{x\to 0^+} \frac{\sin(x)}{x} \overset{L'H}{=} \lim_{x\to 0^+} \frac{\cos(x)}{1} = 1.$$

Converges to 1.

27. **Solution:** Use L'Hôpital's Rule for $\frac{\infty}{\infty}$.

$$\lim_{x\to\infty} \frac{\ln(x^2+1)}{\ln(3x+1)} \overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{2x}{x^2+1}}{\frac{3}{3x+1}} = \lim_{x\to\infty} \frac{2x(3x+1)}{3(x^2+1)} = \lim_{x\to\infty} \frac{6x^2+2x}{3x^2+3}.$$

This is still $\frac{\infty}{\infty}$. The degrees are equal, so the limit is the ratio of leading coefficients, $6/3 = 2$. Converges to 2.

28. **Solution:** For $n > 2$, $n!$ grows much faster than 2^n. Let's look at the terms: $a_1 = 2, a_2 = 2, a_3 = 8/6, a_4 = 16/24, \ldots$ We can see $0 < a_n = \frac{2\cdot 2\cdots 2}{1\cdot 2\cdots n} = \frac{2}{1}\cdot\frac{2}{2}\cdot\frac{2}{3}\cdots\frac{2}{n} \le 2\cdot 1\cdot\left(\frac{2}{3}\right)^{n-2}$. As $n\to\infty$, this goes to 0. The limit is 0. Converges to 0.

29. **Solution:** $\sin(n\pi) = 0$ for any integer n. So, $a_n = \frac{n\cdot 0}{2n+1} = 0$ for all n. The sequence is $\{0,0,0,\ldots\}$. Converges to 0.

30. **Solution:**

   a. **Increasing:** Use induction. Base Case: $a_1 = \sqrt{5} \approx 2.23$, $a_2 = \sqrt{5+\sqrt{5}} \approx 2.69$. So $a_2 > a_1$. Inductive Step: Assume $a_{k+1} > a_k$ for some $k \ge 1$. Then $5+a_{k+1} > 5+a_k$. Since square root is an increasing function, $\sqrt{5+a_{k+1}} > \sqrt{5+a_k}$, which means $a_{k+2} > a_{k+1}$. By induction, the sequence is increasing.

   b. **Bounded:** Use induction. Base Case: $a_1 = \sqrt{5} < 3$. Inductive Step: Assume $a_k < 3$. Then $a_{k+1} = \sqrt{5+a_k} < \sqrt{5+3} = \sqrt{8} < 3$. By induction, $a_n < 3$ for all n.

   c. **Limit:** Since the sequence is increasing and bounded above, it must converge by the Monotonic Sequence Theorem. Let $L = \lim_{n\to\infty} a_n$. Then $L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{5+a_n} = \sqrt{5+L}$. $L^2 = 5+L \implies L^2 - L - 5 = 0$. Using the quadratic formula, $L = \frac{1\pm\sqrt{1-4(1)(-5)}}{2} = \frac{1\pm\sqrt{21}}{2}$. Since the terms are all positive, the limit must be positive. So, $L = \frac{1+\sqrt{21}}{2}$.

31. **Solution:**

a. **Monotonic and Bounded:** $a_1 = 3$, $a_2 = \frac{1}{4}(3+6) = \frac{9}{4} = 2.25$, $a_3 = \frac{1}{4}(2.25+6) = \frac{8.25}{4} = 2.0625$. The sequence appears to be decreasing. It is bounded below by 2. Proof (Decreasing by induction): Base case $a_2 < a_1$ is true. Assume $a_k < a_{k-1}$. Then $a_k + 6 < a_{k-1} + 6$, so $\frac{1}{4}(a_k+6) < \frac{1}{4}(a_{k-1}+6)$, which is $a_{k+1} < a_k$. Proof (Bounded below by 2): Base case $a_1 > 2$. Assume $a_k > 2$. Then $a_{k+1} = \frac{1}{4}(a_k+6) > \frac{1}{4}(2+6) = 2$.

b. **Limit:** Since the sequence is decreasing and bounded below, it converges. Let the limit be L. $L = \frac{1}{4}(L+6) \implies 4L = L+6 \implies 3L = 6 \implies L = 2$.

Problem Cross-Reference by Concept

- **Direct Calculation & Basic Properties:**
  - Writing first terms (explicit): 1
  - Writing first terms (recursive): 2
  - Definitions are implicitly tested in all limit problems.
- **Pattern Recognition:**
  - Arithmetic: 4
  - Geometric: 5
  - Mixed (alternating, powers, etc.): 3
- **Limit Evaluation via Algebraic Manipulation:**
  - Rational functions of n: 6, 7, 8
  - Radicals: 9, 10
  - Geometric sequences / Exponentials: 11, 12, 13
  - Factorials: 22, 23
  - Logarithmic properties: 27 (can also be solved with L'Hôpital's)
- **Limit Evaluation using Calculus Theorems:**
  - L'Hôpital's Rule: 17, 18, 19, 26, 27
  - Squeeze Theorem: 14, 15, 16, 28 (denominator dominates)
  - Continuity: 20, 21
  - Indeterminate Forms ($1^\infty, \infty^0$): 24, 25
- **Mixed Forms and Special Cases:**
  - Rewriting for L'Hôpital's Rule: 26
  - Recognizing zero terms: 29
- **Monotonic Sequence Theorem:**
  - Proving properties and finding the limit: 30, 31