7.8: Improper Integrals - Problem Set

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October 2025

Part I: Problems

Problem 1

Determine whether the integral is convergent or divergent. If it is convergent, evaluate it.

$$\int_2^{\infty} \frac{5}{x^3}\,dx$$

Problem 2

Determine whether the integral is convergent or divergent. If it is convergent, evaluate it.

$$\int_1^{\infty} \frac{1}{\sqrt[4]{x}}\,dx$$

Problem 3

Determine whether the integral is convergent or divergent. If it is convergent, evaluate it.

$$\int_0^{\infty} e^{-2x}\,dx$$

Problem 4

Determine whether the integral is convergent or divergent. If it is convergent, evaluate it.

$$\int_e^{\infty} \frac{1}{x(\ln x)^2}\,dx$$

Problem 5

Determine whether the integral is convergent or divergent. If it is convergent, evaluate it.

$$\int_1^{\infty} \frac{x^2+2}{x^3}\,dx$$

Problem 6

Determine whether the integral is convergent or divergent. If it is convergent, evaluate it.

$$\int_{-\infty}^{0} \frac{1}{(1-x)^{3/2}}\,dx$$

Problem 7

Determine whether the integral is convergent or divergent. If it is convergent, evaluate it.

$$\int_{-\infty}^{-1} \frac{1}{x^5}\,dx$$

Problem 8

Determine whether the integral is convergent or divergent. If it is convergent, evaluate it.

$$\int_{-\infty}^{0} \frac{x}{(x^2+1)^2}\,dx$$

Problem 9

Determine whether the integral is convergent or divergent. If it is convergent, evaluate it.

$$\int_{-\infty}^{\infty} \frac{x}{1+x^2}\,dx$$

Problem 10

Determine whether the integral is convergent or divergent. If it is convergent, evaluate it.

$$\int_{-\infty}^{\infty} \frac{1}{x^2+4}\,dx$$

Problem 11

Determine whether the integral is convergent or divergent. If it is convergent, evaluate it.

$$\int_{-\infty}^{\infty} x^2 e^{-x^3}\,dx$$

Problem 12

Determine whether the integral is convergent or divergent. If it is convergent, evaluate it.

$$\int_{0}^{1} \frac{1}{\sqrt[3]{x}}\,dx$$

Problem 13

Determine whether the integral is convergent or divergent. If it is convergent, evaluate it.

$$\int_{0}^{2} \frac{1}{(x-2)^2}\,dx$$

Problem 14

Determine whether the integral is convergent or divergent. If it is convergent, evaluate it.

$$\int_{0}^{3} \frac{1}{\sqrt{3-x}}\,dx$$

Problem 15

Determine whether the integral is convergent or divergent. If it is convergent, evaluate it.

$$\int_{-1}^{8} \frac{1}{\sqrt[3]{x}}\,dx$$

Problem 16

Determine whether the integral is convergent or divergent. If it is convergent, evaluate it.

$$\int_{1}^{\infty} \frac{1}{x^2+x}\,dx$$

Problem 17

Determine whether the integral is convergent or divergent. If it is convergent, evaluate it.

$$\int_{2}^{\infty} \frac{4}{x^2 - 1}\, dx$$

Problem 18

Determine whether the integral is convergent or divergent. If it is convergent, evaluate it.

$$\int_{0}^{\infty} xe^{-x}\, dx$$

Problem 19

Determine whether the integral is convergent or divergent. If it is convergent, evaluate it.

$$\int_{1}^{\infty} \frac{\ln x}{x^2}\, dx$$

Problem 20

Determine whether the integral is convergent or divergent. If it is convergent, evaluate it.

$$\int_{0}^{\infty} \cos(x)\, dx$$

Problem 21

Determine whether the integral is convergent or divergent. If it is convergent, evaluate it.

$$\int_{0}^{\infty} 2\cos^2(x)\, dx$$

Problem 22

Determine whether the integral is convergent or divergent. If it is convergent, evaluate it.

$$\int_{1}^{\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}}\, dx$$

Problem 23

Determine whether the integral is convergent or divergent. If it is convergent, evaluate it.

$$\int_{-\infty}^{0} xe^{x}\, dx$$

Problem 24

Determine whether the integral is convergent or divergent. If it is convergent, evaluate it.

$$\int_{0}^{1} \frac{1}{4y - 1}\, dy$$

Problem 25

Determine whether the integral is convergent or divergent. If it is convergent, evaluate it.

$$\int_{1}^{\infty} \frac{\arctan(x)}{x^2 + 1}\, dx$$

Problem 26

Determine whether the integral is convergent or divergent. If it is convergent, evaluate it.

$$\int_0^{\pi/2} \tan(x)\,dx$$

Problem 27

Determine whether the integral is convergent or divergent. If it is convergent, evaluate it.

$$\int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}}\,dx$$

Problem 28

Determine whether the integral is convergent or divergent. If it is convergent, evaluate it.

$$\int_0^1 \ln(x)\,dx$$

Problem 29

Determine whether the integral is convergent or divergent. If it is convergent, evaluate it.

$$\int_1^{\infty} \frac{1}{x\sqrt{x^2-1}}\,dx$$

Problem 30

Determine whether the integral is convergent or divergent. If it is convergent, evaluate it.

$$\int_{-\infty}^{1} \frac{1}{x^2-4x+5}\,dx$$

Part II: Detailed Solutions

Solution 1

This is a Type 1 improper integral, which is a p-integral with $p = 3 > 1$, so it converges.

$$\begin{aligned}
\int_2^\infty 5x^{-3}\,dx &= \lim_{t\to\infty}\int_2^t 5x^{-3}\,dx \\
&= \lim_{t\to\infty}\left[\frac{5x^{-2}}{-2}\right]_2^t = \lim_{t\to\infty}\left[-\frac{5}{2x^2}\right]_2^t \\
&= \lim_{t\to\infty}\left(-\frac{5}{2t^2} - \left(-\frac{5}{2(2)^2}\right)\right) \\
&= 0 + \frac{5}{8} = \frac{5}{8}
\end{aligned}$$

Answer: Convergent, value is 5/8.

Solution 2

This is a Type 1 improper integral, which is a p-integral with $p = 1/4 \leq 1$, so it diverges.

$$\begin{aligned}
\int_1^\infty x^{-1/4}\,dx &= \lim_{t\to\infty}\int_1^t x^{-1/4}\,dx \\
&= \lim_{t\to\infty}\left[\frac{x^{3/4}}{3/4}\right]_1^t = \lim_{t\to\infty}\left[\frac{4}{3}x^{3/4}\right]_1^t \\
&= \lim_{t\to\infty}\left(\frac{4}{3}t^{3/4} - \frac{4}{3}(1)^{3/4}\right) \\
&= \infty - \frac{4}{3} = \infty
\end{aligned}$$

Answer: Diverges.

Solution 3

This is a Type 1 improper integral.

$$\begin{aligned}
\int_0^\infty e^{-2x}\,dx &= \lim_{t\to\infty}\int_0^t e^{-2x}\,dx \\
&= \lim_{t\to\infty}\left[-\frac{1}{2}e^{-2x}\right]_0^t \\
&= \lim_{t\to\infty}\left(-\frac{1}{2}e^{-2t} - \left(-\frac{1}{2}e^{0}\right)\right) \\
&= 0 + \frac{1}{2} = \frac{1}{2}
\end{aligned}$$

Answer: Convergent, value is 1/2.

Solution 4

This is a Type 1 improper integral. Use u-substitution with $u = \ln x$, so $du = \frac{1}{x}dx$. When $x = e, u = 1$. When $x \to \infty, u \to \infty$.

$$\begin{aligned}
\int_e^\infty \frac{1}{x(\ln x)^2}\,dx &= \int_1^\infty \frac{1}{u^2}\,du \\
&= \lim_{t\to\infty}\int_1^t u^{-2}\,du = \lim_{t\to\infty}\left[-u^{-1}\right]_1^t \\
&= \lim_{t\to\infty}\left(-\frac{1}{t} - (-1)\right) = 0 + 1 = 1
\end{aligned}$$

Answer: Convergent, value is 1.

Solution 5

This is a Type 1 improper integral. First, simplify the integrand.

$$\begin{aligned}
\int_1^\infty \left(\frac{x^2}{x^3} + \frac{2}{x^3}\right) dx &= \int_1^\infty \left(\frac{1}{x} + 2x^{-3}\right) dx \\
&= \lim_{t\to\infty} \int_1^t \left(\frac{1}{x} + 2x^{-3}\right) dx \\
&= \lim_{t\to\infty} \left[\ln|x| - x^{-2}\right]_1^t \\
&= \lim_{t\to\infty} \left((\ln t - \frac{1}{t^2}) - (\ln 1 - 1)\right) \\
&= (\infty - 0) - (0 - 1) = \infty
\end{aligned}$$

The integral diverges because the $\int \frac{1}{x} dx$ part diverges ($p = 1$). **Answer:** Diverges.

Solution 6

This is a Type 1 improper integral.

$$\begin{aligned}
\int_{-\infty}^0 (1-x)^{-3/2}\, dx &= \lim_{t\to-\infty} \int_t^0 (1-x)^{-3/2}\, dx \\
&= \lim_{t\to-\infty} \left[2(1-x)^{-1/2}\right]_t^0 \\
&= \lim_{t\to-\infty} \left(2(1)^{-1/2} - 2(1-t)^{-1/2}\right) \\
&= \lim_{t\to-\infty} \left(2 - \frac{2}{\sqrt{1-t}}\right) \\
&= 2 - 0 = 2
\end{aligned}$$

Answer: Convergent, value is 2.

Solution 7

This is a Type 1 improper integral. The p-integral with $p = 5 > 1$ converges on $[1, \infty)$, and similarly converges on $(-\infty, -1]$.

$$\begin{aligned}
\int_{-\infty}^{-1} x^{-5}\, dx &= \lim_{t\to-\infty} \int_t^{-1} x^{-5}\, dx \\
&= \lim_{t\to-\infty} \left[\frac{x^{-4}}{-4}\right]_t^{-1} \\
&= \lim_{t\to-\infty} \left(\frac{(-1)^{-4}}{-4} - \frac{t^{-4}}{-4}\right) \\
&= \lim_{t\to-\infty} \left(-\frac{1}{4} + \frac{1}{4t^4}\right) = -\frac{1}{4} + 0 = -\frac{1}{4}
\end{aligned}$$

Answer: Convergent, value is $-1/4$.

Solution 8

This is a Type 1 improper integral. Use u-substitution with $u = x^2 + 1$, $du = 2x\,dx$. When $x = 0, u = 1$. When $x \to -\infty, u \to \infty$.

$$\begin{aligned}
\int_{-\infty}^{0} \frac{x}{(x^2+1)^2}\,dx &= \lim_{t\to-\infty} \int_t^0 \frac{x}{(x^2+1)^2}\,dx \\
&= \int_\infty^1 \frac{1}{u^2}\frac{du}{2} = -\frac{1}{2}\int_1^\infty u^{-2}\,du \\
&= -\frac{1}{2}\lim_{t\to\infty}\left[-u^{-1}\right]_1^t \\
&= -\frac{1}{2}\lim_{t\to\infty}\left(-\frac{1}{t} - (-1)\right) = -\frac{1}{2}(0+1) = -\frac{1}{2}
\end{aligned}$$

Answer: Convergent, value is $-1/2$.

Solution 9

This is a Type 1 integral over $(-\infty, \infty)$. We split it at $x = 0$.

$$\int_{-\infty}^{\infty} \frac{x}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{x}{1+x^2}\,dx + \int_0^\infty \frac{x}{1+x^2}\,dx$$

Let's evaluate the second part. Use $u = 1 + x^2, du = 2x\,dx$.

$$\begin{aligned}
\int_0^\infty \frac{x}{1+x^2}\,dx &= \lim_{t\to\infty}\int_0^t \frac{x}{1+x^2}\,dx \\
&= \lim_{t\to\infty}\left[\frac{1}{2}\ln(1+x^2)\right]_0^t \\
&= \frac{1}{2}\lim_{t\to\infty}(\ln(1+t^2) - \ln(1)) = \infty
\end{aligned}$$

Since one part diverges, the whole integral diverges. Note: The integrand is an odd function, but for the integral to be 0, it must first converge. **Answer:** Diverges.

Solution 10

This is a Type 1 integral over $(-\infty, \infty)$. Split at $x = 0$. The integrand is even.

$$\int_{-\infty}^{\infty} \frac{1}{x^2+4}\,dx = 2\int_0^\infty \frac{1}{x^2+4}\,dx$$

$$\begin{aligned}
2\lim_{t\to\infty}\int_0^t \frac{1}{x^2+2^2}\,dx &= 2\lim_{t\to\infty}\left[\frac{1}{2}\arctan\left(\frac{x}{2}\right)\right]_0^t \\
&= \lim_{t\to\infty}\left(\arctan\left(\frac{t}{2}\right) - \arctan(0)\right) \\
&= \frac{\pi}{2} - 0 = \frac{\pi}{2}
\end{aligned}$$

The original integral is $2 \times (\pi/2) = \pi$. **Answer:** Convergent, value is π.

Solution 11

This is a Type 1 integral over $(-\infty, \infty)$. Split at $x = 0$.

$$\int_{-\infty}^{0} x^2 e^{-x^3}\,dx + \int_0^\infty x^2 e^{-x^3}\,dx$$

Let's evaluate the second part. Use $u = -x^3, du = -3x^2\,dx$.

$$
\begin{aligned}
\int_0^\infty x^2 e^{-x^3}\,dx &= \lim_{t\to\infty} \int_0^t x^2 e^{-x^3}\,dx \\
&= \lim_{t\to\infty} \left[-\frac{1}{3}e^{-x^3}\right]_0^t \\
&= -\frac{1}{3}\lim_{t\to\infty}(e^{-t^3} - e^0) = -\frac{1}{3}(0-1) = \frac{1}{3}
\end{aligned}
$$

The first part diverges:

$$
\begin{aligned}
\int_{-\infty}^0 x^2 e^{-x^3}\,dx &= \lim_{t\to-\infty} \int_t^0 x^2 e^{-x^3}\,dx \\
&= \lim_{t\to-\infty} \left[-\frac{1}{3}e^{-x^3}\right]_t^0 \\
&= -\frac{1}{3}\lim_{t\to-\infty}(e^0 - e^{-t^3}) = -\frac{1}{3}(1-\infty) = \infty
\end{aligned}
$$

Since one part diverges, the whole integral diverges. **Answer:** Diverges.

Solution 12

This is a Type 2 improper integral with a discontinuity at $x = 0$. It's a p-integral with $p = 1/3 < 1$, so it converges.

$$
\begin{aligned}
\int_0^1 x^{-1/3}\,dx &= \lim_{t\to0^+} \int_t^1 x^{-1/3}\,dx \\
&= \lim_{t\to0^+} \left[\frac{3}{2}x^{2/3}\right]_t^1 \\
&= \lim_{t\to0^+} \left(\frac{3}{2}(1)^{2/3} - \frac{3}{2}t^{2/3}\right) = \frac{3}{2} - 0 = \frac{3}{2}
\end{aligned}
$$

Answer: Convergent, value is $3/2$.

Solution 13

This is a Type 2 improper integral with a discontinuity at $x = 2$. It's a p-integral with $p = 2 > 1$, so it diverges.

$$
\begin{aligned}
\int_0^2 (x-2)^{-2}\,dx &= \lim_{t\to2^-} \int_0^t (x-2)^{-2}\,dx \\
&= \lim_{t\to2^-} \left[-(x-2)^{-1}\right]_0^t \\
&= \lim_{t\to2^-} \left(-\frac{1}{t-2} - \left(-\frac{1}{-2}\right)\right) \\
&= -(-\infty) - \frac{1}{2} = \infty
\end{aligned}
$$

Answer: Diverges.

Solution 14

This is a Type 2 improper integral with a discontinuity at $x = 3$.

$$
\begin{aligned}
\int_0^3 (3-x)^{-1/2}\,dx &= \lim_{t\to3^-} \int_0^t (3-x)^{-1/2}\,dx \\
&= \lim_{t\to3^-} \left[-2(3-x)^{1/2}\right]_0^t \\
&= \lim_{t\to3^-} \left(-2\sqrt{3-t} - (-2\sqrt{3})\right) \\
&= 0 + 2\sqrt{3} = 2\sqrt{3}
\end{aligned}
$$

Answer: Convergent, value is $2\sqrt{3}$.

Solution 15

This is a Type 2 improper integral with a discontinuity at $x = 0$ inside the interval. We must split it.

$$\int_{-1}^{8} x^{-1/3}\,dx = \int_{-1}^{0} x^{-1/3}\,dx + \int_{0}^{8} x^{-1/3}\,dx$$

First part:

$$\begin{aligned}\lim_{t\to 0^-}\int_{-1}^{t} x^{-1/3}\,dx &= \lim_{t\to 0^-}\left[\frac{3}{2}x^{2/3}\right]_{-1}^{t} \\ &= \lim_{t\to 0^-}\left(\frac{3}{2}t^{2/3} - \frac{3}{2}(-1)^{2/3}\right) = 0 - \frac{3}{2} = -\frac{3}{2}\end{aligned}$$

Second part:

$$\begin{aligned}\lim_{t\to 0^+}\int_{t}^{8} x^{-1/3}\,dx &= \lim_{t\to 0^+}\left[\frac{3}{2}x^{2/3}\right]_{t}^{8} \\ &= \lim_{t\to 0^+}\left(\frac{3}{2}(8)^{2/3} - \frac{3}{2}t^{2/3}\right) = \frac{3}{2}(4) - 0 = 6\end{aligned}$$

Both parts converge, so the total is $-\frac{3}{2} + 6 = \frac{9}{2}$. **Answer:** Convergent, value is $9/2$.

Solution 16

This is a Type 1 integral. Use partial fractions: $\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$.

$$\begin{aligned}\int_{1}^{\infty}\left(\frac{1}{x} - \frac{1}{x+1}\right)dx &= \lim_{t\to\infty}\int_{1}^{t}\left(\frac{1}{x} - \frac{1}{x+1}\right)dx \\ &= \lim_{t\to\infty}\left[\ln|x| - \ln|x+1|\right]_{1}^{t} \\ &= \lim_{t\to\infty}\left[\ln\left|\frac{x}{x+1}\right|\right]_{1}^{t} \\ &= \lim_{t\to\infty}\left(\ln\left(\frac{t}{t+1}\right) - \ln\left(\frac{1}{2}\right)\right) \\ &= \ln(1) - \ln(1/2) = 0 - (-\ln 2) = \ln 2\end{aligned}$$

Answer: Convergent, value is $\ln 2$.

Solution 17

This is a Type 1 integral. Use partial fractions: $\frac{4}{x^2-1} = \frac{2}{x-1} - \frac{2}{x+1}$.

$$\begin{aligned}\int_{2}^{\infty}\left(\frac{2}{x-1} - \frac{2}{x+1}\right)dx &= \lim_{t\to\infty}\left[2\ln|x-1| - 2\ln|x+1|\right]_{2}^{t} \\ &= 2\lim_{t\to\infty}\left[\ln\left|\frac{x-1}{x+1}\right|\right]_{2}^{t} \\ &= 2\lim_{t\to\infty}\left(\ln\left(\frac{t-1}{t+1}\right) - \ln\left(\frac{1}{3}\right)\right) \\ &= 2(\ln(1) - \ln(1/3)) = 2(0 - (-\ln 3)) = 2\ln 3\end{aligned}$$

Answer: Convergent, value is $2\ln 3$.

Solution 18

This is a Type 1 integral. Use integration by parts with $u = x, dv = e^{-x}dx$. Then $du = dx, v = -e^{-x}$.

$$\begin{aligned}
\int_0^\infty xe^{-x}\,dx &= \lim_{t\to\infty}\int_0^t xe^{-x}\,dx \\
&= \lim_{t\to\infty}\left(\left[-xe^{-x}\right]_0^t - \int_0^t -e^{-x}dx\right) \\
&= \lim_{t\to\infty}\left([-xe^{-x} - e^{-x}]_0^t\right) \\
&= \lim_{t\to\infty}\left((-\frac{t}{e^t} - \frac{1}{e^t}) - (0 - e^0)\right) \\
&= (0-0) - (-1) = 1
\end{aligned}$$

(Used L'Hôpital's Rule for $\lim_{t\to\infty} t/e^t = 0$). **Answer:** Convergent, value is 1.

Solution 19

This is a Type 1 integral. Use integration by parts with $u = \ln x, dv = x^{-2}dx$. Then $du = 1/xdx, v = -x^{-1}$.

$$\begin{aligned}
\int_1^\infty \frac{\ln x}{x^2}\,dx &= \lim_{t\to\infty}\int_1^t (\ln x)(x^{-2})\,dx \\
&= \lim_{t\to\infty}\left(\left[-\frac{\ln x}{x}\right]_1^t - \int_1^t -\frac{1}{x^2}dx\right) \\
&= \lim_{t\to\infty}\left([-\frac{\ln x}{x} - \frac{1}{x}]_1^t\right) \\
&= \lim_{t\to\infty}\left((-\frac{\ln t}{t} - \frac{1}{t}) - (-\frac{\ln 1}{1} - \frac{1}{1})\right) \\
&= (0-0) - (0-1) = 1
\end{aligned}$$

(Used L'Hôpital's Rule for $\lim_{t\to\infty} \ln t/t = 0$). **Answer:** Convergent, value is 1.

Solution 20

This is a Type 1 integral with an oscillating function.

$$\begin{aligned}
\int_0^\infty \cos(x)\,dx &= \lim_{t\to\infty}\int_0^t \cos(x)\,dx \\
&= \lim_{t\to\infty}[\sin(x)]_0^t \\
&= \lim_{t\to\infty}(\sin(t) - \sin(0)) = \lim_{t\to\infty}\sin(t)
\end{aligned}$$

The limit does not exist as $\sin(t)$ oscillates between -1 and 1. **Answer:** Diverges.

Solution 21

This is a Type 1 integral. Use the power-reducing identity $\cos^2(x) = \frac{1+\cos(2x)}{2}$.

$$\begin{aligned}
\int_0^\infty 2\left(\frac{1+\cos(2x)}{2}\right)\,dx &= \int_0^\infty (1+\cos(2x))\,dx \\
&= \lim_{t\to\infty}\int_0^t (1+\cos(2x))\,dx \\
&= \lim_{t\to\infty}\left[x + \frac{1}{2}\sin(2x)\right]_0^t \\
&= \lim_{t\to\infty}\left((t + \frac{1}{2}\sin(2t)) - 0\right) = \infty
\end{aligned}$$

The limit is infinite. **Answer:** Diverges.

Solution 22

This is a Type 1 integral. Use u-substitution with $u = -\sqrt{x}, du = -\frac{1}{2\sqrt{x}}dx$.

$$\begin{aligned}
\int_1^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx &= \lim_{t\to\infty}\int_1^t \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx \\
&= \lim_{t\to\infty}\left[-2e^{-\sqrt{x}}\right]_1^t \\
&= \lim_{t\to\infty}\left(-2e^{-\sqrt{t}} - (-2e^{-1})\right) \\
&= 0 + \frac{2}{e} = \frac{2}{e}
\end{aligned}$$

Answer: Convergent, value is $2/e$.

Solution 23

This is a Type 1 integral. It is the same integral as problem 18, but over a different interval. Use integration by parts with $u = x, dv = e^x dx$.

$$\begin{aligned}
\int_{-\infty}^0 xe^x\,dx &= \lim_{t\to-\infty}\int_t^0 xe^x\,dx \\
&= \lim_{t\to-\infty}\left[xe^x - e^x\right]_t^0 \\
&= \lim_{t\to-\infty}\left((0-e^0) - (te^t - e^t)\right) \\
&= -1 - (0-0) = -1
\end{aligned}$$

(Used L'Hôpital's Rule for $\lim_{t\to-\infty} te^t = \lim_{t\to-\infty} t/e^{-t} = 0$). **Answer:** Convergent, value is -1.

Solution 24

This is a Type 2 integral with a discontinuity at $y = 1/4$, which is inside $[0,1]$. Must split.

$$\int_0^{1/4}\frac{1}{4y-1}\,dy + \int_{1/4}^1 \frac{1}{4y-1}\,dy$$

Let's evaluate the first part.

$$\begin{aligned}
\lim_{t\to 1/4^-}\int_0^t \frac{1}{4y-1}\,dy &= \lim_{t\to 1/4^-}\left[\frac{1}{4}\ln|4y-1|\right]_0^t \\
&= \frac{1}{4}\lim_{t\to 1/4^-}\left(\ln|4t-1| - \ln|-1|\right) \\
&= \frac{1}{4}(-\infty - 0) = -\infty
\end{aligned}$$

Since one part diverges, the whole integral diverges. **Answer:** Diverges.

Solution 25

This is a Type 1 integral. Use u-substitution with $u = \arctan(x), du = \frac{1}{1+x^2}dx$. When $x = 1, u = \pi/4$. When $x \to \infty, u \to \pi/2$.

$$\begin{aligned}
\int_1^\infty \frac{\arctan(x)}{x^2+1}\,dx &= \int_{\pi/4}^{\pi/2} u\,du \\
&= \left[\frac{u^2}{2}\right]_{\pi/4}^{\pi/2} \\
&= \frac{1}{2}\left(\left(\frac{\pi}{2}\right)^2 - \left(\frac{\pi}{4}\right)^2\right) \\
&= \frac{1}{2}\left(\frac{\pi^2}{4} - \frac{\pi^2}{16}\right) = \frac{1}{2}\left(\frac{3\pi^2}{16}\right) = \frac{3\pi^2}{32}
\end{aligned}$$

Answer: Convergent, value is $3\pi^2/32$.

Solution 26

This is a Type 2 integral since $\tan(x)$ has a vertical asymptote at $x = \pi/2$.

$$\begin{aligned}
\int_0^{\pi/2} \tan(x)\,dx &= \lim_{t\to\pi/2^-} \int_0^t \tan(x)\,dx \\
&= \lim_{t\to\pi/2^-} [-\ln|\cos(x)|]_0^t \\
&= \lim_{t\to\pi/2^-} (-\ln|\cos(t)| - (-\ln|\cos(0)|)) \\
&= -(-\infty) + \ln(1) = \infty
\end{aligned}$$

Answer: Diverges.

Solution 27

This is a Type 1 integral over $(-\infty, \infty)$. Let $u = e^x, du = e^x dx$. When $x \to -\infty, u \to 0$. When $x \to \infty, u \to \infty$.

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{e^x}{1+(e^x)^2}\,dx &= \int_0^{\infty} \frac{1}{1+u^2}\,du \\
&= \lim_{t\to\infty} \int_0^t \frac{1}{1+u^2}\,du \\
&= \lim_{t\to\infty} [\arctan(u)]_0^t \\
&= \lim_{t\to\infty} (\arctan(t) - \arctan(0)) = \frac{\pi}{2} - 0 = \frac{\pi}{2}
\end{aligned}$$

Answer: Convergent, value is $\pi/2$.

Solution 28

This is a Type 2 integral with a discontinuity at $x = 0$. Use integration by parts with $u = \ln x, dv = dx$. Then $du = 1/x dx, v = x$.

$$\begin{aligned}
\int_0^1 \ln(x)\,dx &= \lim_{t\to 0^+} \int_t^1 \ln(x)\,dx \\
&= \lim_{t\to 0^+} \left([x\ln x]_t^1 - \int_t^1 1\,dx \right) \\
&= \lim_{t\to 0^+} [x\ln x - x]_t^1 \\
&= \lim_{t\to 0^+} ((1\ln 1 - 1) - (t\ln t - t)) \\
&= (0-1) - (0-0) = -1
\end{aligned}$$

(Used L'Hôpital's Rule for $\lim_{t\to 0^+} t\ln t = \lim_{t\to 0^+} \frac{\ln t}{1/t} = 0$). **Answer:** Convergent, value is -1.

Solution 29

This is a Type 2 integral with a discontinuity at $x = 1$. The antiderivative of the integrand is $\operatorname{arcsec}(x)$.

$$\begin{aligned}
\int_1^\infty \frac{1}{x\sqrt{x^2-1}}\,dx &= \text{We must split this integral, for example at } x = 2. \\
&= \int_1^2 \frac{1}{x\sqrt{x^2-1}}\,dx + \int_2^\infty \frac{1}{x\sqrt{x^2-1}}\,dx \\
\text{First part: } \lim_{t\to 1^+} \int_t^2 \frac{1}{x\sqrt{x^2-1}}\,dx &= \lim_{t\to 1^+} [\operatorname{arcsec}(x)]_t^2 \\
&= \operatorname{arcsec}(2) - \lim_{t\to 1^+} \operatorname{arcsec}(t) = \frac{\pi}{3} - 0 = \frac{\pi}{3} \\
\text{Second part: } \lim_{t\to\infty} \int_2^t \frac{1}{x\sqrt{x^2-1}}\,dx &= \lim_{t\to\infty} [\operatorname{arcsec}(x)]_2^t \\
&= \lim_{t\to\infty} \operatorname{arcsec}(t) - \operatorname{arcsec}(2) = \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}
\end{aligned}$$

Both parts converge. Total value is $\frac{\pi}{3} + \frac{\pi}{6} = \frac{\pi}{2}$. **Answer:** Convergent, value is $\pi/2$.

Solution 30

This is a Type 1 integral. Complete the square for the denominator: $x^2 - 4x + 5 = (x^2 - 4x + 4) + 1 = (x-2)^2 + 1$.

$$\begin{aligned}
\int_{-\infty}^1 \frac{1}{(x-2)^2+1}\,dx &= \lim_{t\to-\infty} \int_t^1 \frac{1}{(x-2)^2+1}\,dx \\
&= \lim_{t\to-\infty} [\arctan(x-2)]_t^1 \\
&= \lim_{t\to-\infty} (\arctan(1-2) - \arctan(t-2)) \\
&= \arctan(-1) - (-\pi/2) = -\frac{\pi}{4} + \frac{\pi}{2} = \frac{\pi}{4}
\end{aligned}$$

Answer: Convergent, value is $\pi/4$.

Concept and Problem Number Index

Here is a list of the concepts tested and the corresponding problem numbers.

- **Type 1 Integrals, upper limit** ∞: 1, 2, 3, 4, 5, 16, 17, 18, 19, 21, 22, 25
- **Type 1 Integrals, lower limit** $-\infty$: 6, 7, 8, 23, 30
- **Type 1 Integrals, on** $(-\infty, \infty)$: 9, 10, 11, 27
- **Type 2 Integrals, discontinuity at endpoint**: 12, 13, 14, 26, 28
- **Type 2 Integrals, discontinuity inside interval**: 15, 24
- **Mixed Type 1 and Type 2**: 29
- **p-Test (Direct or after substitution)**:
  - Convergent ($p > 1$): 1, 4, 7, 8
  - Divergent ($p \leq 1$): 2, 5, 9
- **u-Substitution**: 4, 6, 8, 11, 22, 25, 27, 30
- **Integration by Parts**: 18, 19, 23, 28
- **Partial Fraction Decomposition**: 16, 17
- **Trigonometric Functions/Identities**: 21 (Power-reducing), 26 (tan(x))
- **Oscillating Functions (leading to divergence)**: 20, 21
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- **Algebraic Simplification**: 5
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