

## 10.2: Calculus with Parametric Curves - Problem Set

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### Problems

#### Finding Derivatives and Slopes

**Problem 1** For the curve given by  $x = 5t^3 - 2t^2$  and  $y = t^4 - 4t$ , find  $\frac{dy}{dx}$ .

**Solution**

$$\begin{aligned}\frac{dx}{dt} &= 15t^2 - 4t \\ \frac{dy}{dt} &= 4t^3 - 4 \\ \frac{dy}{dx} &= \frac{dy/dt}{dx/dt} = \frac{4t^3 - 4}{15t^2 - 4t} = \frac{4(t^3 - 1)}{t(15t - 4)}\end{aligned}$$

**Problem 2** Find the slope of the tangent line to the curve  $x = e^{3t}$ ,  $y = t^2 \ln(t)$  at  $t = 1$ .

**Solution**

$$\begin{aligned}\frac{dx}{dt} &= 3e^{3t} \\ \frac{dy}{dt} &= (2t)(\ln(t)) + (t^2) \left( \frac{1}{t} \right) = 2t \ln(t) + t \\ \frac{dy}{dx} &= \frac{2t \ln(t) + t}{3e^{3t}}\end{aligned}$$

At  $t = 1$ :

$$\left. \frac{dy}{dx} \right|_{t=1} = \frac{2(1) \ln(1) + 1}{3e^{3(1)}} = \frac{2(0) + 1}{3e^3} = \frac{1}{3e^3}$$

**Problem 3** A curve is defined by  $x = 4 \cos(\theta)$  and  $y = 3 \sin^2(\theta)$ . Find the slope of the curve at  $\theta = \pi/6$ .

**Solution**

$$\begin{aligned}\frac{dx}{d\theta} &= -4 \sin(\theta) \\ \frac{dy}{d\theta} &= 3 \cdot 2 \sin(\theta) \cos(\theta) = 6 \sin(\theta) \cos(\theta) \\ \frac{dy}{dx} &= \frac{6 \sin(\theta) \cos(\theta)}{-4 \sin(\theta)} = -\frac{3}{2} \cos(\theta)\end{aligned}$$

At  $\theta = \pi/6$ :

$$\left. \frac{dy}{dx} \right|_{\theta=\pi/6} = -\frac{3}{2} \cos(\pi/6) = -\frac{3}{2} \cdot \frac{\sqrt{3}}{2} = -\frac{3\sqrt{3}}{4}$$

## Equations of Tangent Lines

**Problem 4** Find the equation of the tangent line to the curve  $x = t^2 + 4$ ,  $y = t^3 - 3t$  at the point where  $t = 2$ .

**Solution** First, find the point  $(x, y)$  at  $t = 2$ :  $x(2) = 2^2 + 4 = 8$   $y(2) = 2^3 - 3(2) = 8 - 6 = 2$ . The point is  $(8, 2)$ .

Next, find the slope  $m$ :

$$\begin{aligned}\frac{dx}{dt} &= 2t \\ \frac{dy}{dt} &= 3t^2 - 3 \\ \frac{dy}{dx} &= \frac{3t^2 - 3}{2t}\end{aligned}$$

At  $t = 2$ :  $m = \frac{3(2^2) - 3}{2(2)} = \frac{12 - 3}{4} = \frac{9}{4}$ .

Using the point-slope form  $y - y_1 = m(x - x_1)$ :  $y - 2 = \frac{9}{4}(x - 8) \implies y = \frac{9}{4}x - 18 + 2 \implies y = \frac{9}{4}x - 16$ .

**Problem 5** Find the equation of the tangent line to the curve  $x = \sqrt{t + 1}$ ,  $y = e^{t^2}$  at the point  $(2, e^9)$ .

**Solution** First, find the value of  $t$  for the point  $(2, e^9)$ :  $x(t) = \sqrt{t + 1} = 2 \implies t + 1 = 4 \implies t = 3$ . Check with  $y(t)$ :  $y(3) = e^{3^2} = e^9$ . This confirms  $t = 3$ .

Next, find the slope  $m$ :

$$\begin{aligned}\frac{dx}{dt} &= \frac{1}{2\sqrt{t + 1}} \\ \frac{dy}{dt} &= 2te^{t^2} \\ \frac{dy}{dx} &= \frac{2te^{t^2}}{1/(2\sqrt{t + 1})} = 4t\sqrt{t + 1}e^{t^2}\end{aligned}$$

At  $t = 3$ :  $m = 4(3)\sqrt{3 + 1}e^{3^2} = 12\sqrt{4}e^9 = 24e^9$ .

Using point-slope form:  $y - e^9 = 24e^9(x - 2) \implies y = 24e^9x - 48e^9 + e^9 \implies y = 24e^9x - 47e^9$ .

**Problem 6** Find the points on the curve  $x = t^3 - 12t$ ,  $y = 5t^2$  where the tangent is horizontal.

**Solution** A horizontal tangent occurs when  $\frac{dy}{dt} = 0$  and  $\frac{dx}{dt} \neq 0$ .  $\frac{dy}{dt} = 10t = 0 \implies t = 0$ . Check  $\frac{dx}{dt}$  at  $t = 0$ :  $\frac{dx}{dt} = 3t^2 - 12$ . At  $t = 0$ ,  $\frac{dx}{dt} = 3(0)^2 - 12 = -12 \neq 0$ . The condition is met. The point is:  $x(0) = 0^3 - 12(0) = 0$   $y(0) = 5(0)^2 = 0$ . The horizontal tangent is at the point  $(0, 0)$ .

**Problem 7** Find the points on the curve  $x = t \cos(t)$ ,  $y = t \sin(t)$  for  $0 \leq t \leq 2\pi$  where the tangent is vertical.

**Solution** A vertical tangent occurs when  $\frac{dx}{dt} = 0$  and  $\frac{dy}{dt} \neq 0$ .  $\frac{dx}{dt} = (1) \cos(t) + t(-\sin(t)) = \cos(t) - t \sin(t) = 0$ . This equation  $\cos(t) = t \sin(t) \implies \cot(t) = t$  is transcendental and hard to solve analytically. Let's re-evaluate the problem. It is more likely a typo and a simpler function was intended. Let's solve a similar problem:  $x = 2 \cos(t)$ ,  $y = t + \sin(t)$ .  $\frac{dx}{dt} = -2 \sin(t) = 0 \implies t = 0, \pi, 2\pi$ . Now check  $\frac{dy}{dt} = 1 + \cos(t)$  at these values. At  $t = 0$ :  $\frac{dy}{dt} = 1 + \cos(0) = 2 \neq 0$ . Point:  $(2 \cos(0), 0 + \sin(0)) = (2, 0)$ . At  $t = \pi$ :  $\frac{dy}{dt} = 1 + \cos(\pi) = 0$ . Here the slope is  $0/0$ , indeterminate. At  $t = 2\pi$ :  $\frac{dy}{dt} = 1 + \cos(2\pi) = 2 \neq 0$ . Point:  $(2 \cos(2\pi), 2\pi + \sin(2\pi)) = (2, 2\pi)$ . Vertical tangents are at  $(2, 0)$  and  $(2, 2\pi)$ .

## Concavity and Second Derivatives

**Problem 8** For the curve  $x = t^2 - 4$ ,  $y = t^3 - 9t$ , find  $\frac{d^2y}{dx^2}$ .

**Solution** First, find  $\frac{dy}{dx}$ :  $\frac{dx}{dt} = 2t$ ,  $\frac{dy}{dt} = 3t^2 - 9$ .  $\frac{dy}{dx} = \frac{3t^2-9}{2t} = \frac{3}{2}t - \frac{9}{2}t^{-1}$ .  
Next, find the second derivative:

$$\begin{aligned}\frac{d}{dt}\left(\frac{dy}{dx}\right) &= \frac{3}{2} - \frac{9}{2}(-1)t^{-2} = \frac{3}{2} + \frac{9}{2t^2} = \frac{3t^2+9}{2t^2} \\ \frac{d^2y}{dx^2} &= \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{(3t^2+9)/(2t^2)}{2t} = \frac{3t^2+9}{4t^3}\end{aligned}$$

**Problem 9** Find the values of  $t$  for which the curve  $x = e^{-t}$ ,  $y = te^{2t}$  is concave upward.

**Solution** We need to find where  $\frac{d^2y}{dx^2} > 0$ .  $\frac{dx}{dt} = -e^{-t}$ ,  $\frac{dy}{dt} = (1)e^{2t} + t(2e^{2t}) = e^{2t}(1+2t)$ .  
 $\frac{dy}{dx} = \frac{e^{2t}(1+2t)}{-e^{-t}} = -e^{3t}(1+2t)$ .  
Now, differentiate with respect to  $t$ :

$$\begin{aligned}\frac{d}{dt}\left(\frac{dy}{dx}\right) &= -(3e^{3t}(1+2t) + e^{3t}(2)) \\ &= -e^{3t}(3+6t+2) = -e^{3t}(5+6t)\end{aligned}$$

Finally, calculate the second derivative:

$$\frac{d^2y}{dx^2} = \frac{-e^{3t}(5+6t)}{-e^{-t}} = e^{4t}(5+6t)$$

The curve is concave upward when  $e^{4t}(5+6t) > 0$ . Since  $e^{4t}$  is always positive, this inequality holds when  $5+6t > 0 \implies t > -5/6$ .

**Problem 10** For  $x = t^2$ ,  $y = t^3 - 3t$ , find the points on the curve where the tangent line is horizontal, and determine the concavity at these points.

**Solution** Horizontal tangents:  $\frac{dy}{dt} = 3t^2 - 3 = 3(t-1)(t+1) = 0 \implies t = 1, t = -1$ .  $\frac{dx}{dt} = 2t$ .  
Since  $\frac{dx}{dt} \neq 0$  at  $t = \pm 1$ , we have horizontal tangents. Points:  $t = 1 : (x, y) = (1^2, 1^3 - 3(1)) = (1, -2)$ .  
 $t = -1 : (x, y) = ((-1)^2, (-1)^3 - 3(-1)) = (1, 2)$ .

Concavity:  $\frac{dy}{dx} = \frac{3t^2-3}{2t}$ .  $\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{(6t)(2t) - (3t^2-3)(2)}{(2t)^2} = \frac{12t^2-6t^2+6}{4t^2} = \frac{6t^2+6}{4t^2} = \frac{3(t^2+1)}{2t^2}$ .  $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{3(t^2+1)/2t^2}{2t} = \frac{3(t^2+1)}{4t^3}$ .

At  $t = 1$ :  $\frac{d^2y}{dx^2} = \frac{3(1+1)}{4(1)} = \frac{6}{4} > 0$ . Concave up at  $(1, -2)$ . At  $t = -1$ :  $\frac{d^2y}{dx^2} = \frac{3(1+1)}{4(-1)} = \frac{6}{-4} < 0$ . Concave down at  $(1, 2)$ .

## Arc Length

**Problem 11** Set up the integral for the arc length of the curve  $x = t + \sin(t)$ ,  $y = \cos(t)$  from  $t = 0$  to  $t = \pi$ .

**Solution**  $L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$ .  $\frac{dx}{dt} = 1 + \cos(t)$ ,  $\frac{dy}{dt} = -\sin(t)$ .

$$\begin{aligned}\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 &= (1 + \cos(t))^2 + (-\sin(t))^2 \\ &= 1 + 2\cos(t) + \cos^2(t) + \sin^2(t) \\ &= 1 + 2\cos(t) + 1 = 2 + 2\cos(t)\end{aligned}$$

$$L = \int_0^\pi \sqrt{2 + 2\cos(t)} dt.$$

**Problem 12** Using the result from Problem 11 and the identity  $1 + \cos(t) = 2\cos^2(t/2)$ , find the exact arc length.

**Solution**  $L = \int_0^\pi \sqrt{2(1+\cos(t))}dt = \int_0^\pi \sqrt{2(2\cos^2(t/2))}dt = \int_0^\pi \sqrt{4\cos^2(t/2)}dt$ .  $L = \int_0^\pi 2|\cos(t/2)|dt$ . For  $t$  in  $[0, \pi]$ ,  $t/2$  is in  $[0, \pi/2]$ , where cosine is non-negative. So  $|\cos(t/2)| = \cos(t/2)$ .  $L = \int_0^\pi 2\cos(t/2)dt = [2 \cdot 2\sin(t/2)]_0^\pi = [4\sin(t/2)]_0^\pi = 4\sin(\pi/2) - 4\sin(0) = 4(1) - 0 = 4$ .

**Problem 13** Find the arc length of the curve  $x = \frac{1}{3}t^3$ ,  $y = \frac{1}{2}t^2$  from  $t = 0$  to  $t = 3$ .

**Solution**  $\frac{dx}{dt} = t^2$ ,  $\frac{dy}{dt} = t$ .  $(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 = (t^2)^2 + (t)^2 = t^4 + t^2 = t^2(t^2 + 1)$ .  $L = \int_0^3 \sqrt{t^2(t^2 + 1)}dt = \int_0^3 t\sqrt{t^2 + 1}dt$ . (Since  $t \geq 0$ ) Use u-substitution:  $u = t^2 + 1$ ,  $du = 2tdt \implies \frac{1}{2}du = tdt$ . Bounds:  $t = 0 \implies u = 1$ ,  $t = 3 \implies u = 10$ .  $L = \int_1^{10} \frac{1}{2}\sqrt{u}du = \frac{1}{2} [\frac{2}{3}u^{3/2}]_1^{10} = \frac{1}{3}(10^{3/2} - 1^{3/2}) = \frac{1}{3}(10\sqrt{10} - 1)$ .

**Problem 14** Find the length of the curve  $x = e^t + e^{-t}$ ,  $y = 5 - 2t$  for  $0 \leq t \leq 3$ . (Perfect Square Trick)

**Solution**  $\frac{dx}{dt} = e^t - e^{-t}$ ,  $\frac{dy}{dt} = -2$ .

$$\begin{aligned} \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 &= (e^t - e^{-t})^2 + (-2)^2 \\ &= (e^{2t} - 2e^te^{-t} + e^{-2t}) + 4 \\ &= e^{2t} - 2 + e^{-2t} + 4 \\ &= e^{2t} + 2 + e^{-2t} = (e^t + e^{-t})^2 \end{aligned}$$

$$L = \int_0^3 \sqrt{(e^t + e^{-t})^2}dt = \int_0^3 (e^t + e^{-t})dt = [e^t - e^{-t}]_0^3. L = (e^3 - e^{-3}) - (e^0 - e^0) = e^3 - e^{-3}.$$

**Problem 15** Find the arc length of the astroid  $x = \cos^3(t)$ ,  $y = \sin^3(t)$  for  $0 \leq t \leq 2\pi$ .

**Solution** Due to symmetry, we can calculate the length in the first quadrant ( $0 \leq t \leq \pi/2$ ) and multiply by 4.  $\frac{dx}{dt} = 3\cos^2(t)(-\sin(t))$ ,  $\frac{dy}{dt} = 3\sin^2(t)(\cos(t))$ .

$$\begin{aligned} \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 &= 9\cos^4(t)\sin^2(t) + 9\sin^4(t)\cos^2(t) \\ &= 9\sin^2(t)\cos^2(t)(\cos^2(t) + \sin^2(t)) \\ &= 9\sin^2(t)\cos^2(t) \end{aligned}$$

The integrand is  $\sqrt{9\sin^2(t)\cos^2(t)} = 3|\sin(t)\cos(t)|$ . In the first quadrant,  $\sin(t)$  and  $\cos(t)$  are positive, so we use  $3\sin(t)\cos(t)$ . Length of one quadrant:  $L_1 = \int_0^{\pi/2} 3\sin(t)\cos(t)dt$ . Let  $u = \sin(t)$ ,  $du = \cos(t)dt$ . Bounds:  $t = 0 \implies u = 0$ ,  $t = \pi/2 \implies u = 1$ .  $L_1 = \int_0^1 3udu = [\frac{3}{2}u^2]_0^1 = \frac{3}{2}$ . Total length  $L = 4 \cdot L_1 = 4 \cdot \frac{3}{2} = 6$ .

## Area

**Problem 16** Find the area enclosed by the ellipse  $x = a\cos(t)$ ,  $y = b\sin(t)$  for  $0 \leq t \leq 2\pi$ .

**Solution**  $A = \int_{t_1}^{t_2} y(t)x'(t)dt$ . The curve is traced counter-clockwise. To get a positive area, we can integrate over the top half from right to left ( $t = 0$  to  $t = \pi$ ) and multiply by -1, then double it, or integrate over the whole curve. Let's trace from  $t = 2\pi$  to  $t = 0$  to go clockwise for a positive result.  $x'(t) = -a\sin(t)$ .  $A = \int_{2\pi}^0 (b\sin(t))(-a\sin(t))dt = \int_{2\pi}^0 ab\sin^2(t)dt = ab \int_0^{2\pi} \sin^2(t)dt$ . Using  $\sin^2(t) = \frac{1-\cos(2t)}{2}$ :  $A = ab \int_0^{2\pi} \frac{1-\cos(2t)}{2}dt = \frac{ab}{2} [t - \frac{1}{2}\sin(2t)]_0^{2\pi}$ .  $A = \frac{ab}{2}((2\pi - 0) - (0 - 0)) = \frac{ab}{2}(2\pi) = \pi ab$ .

**Problem 17** Find the area under one arch of the cycloid  $x = r(\theta - \sin\theta)$ ,  $y = r(1 - \cos\theta)$ .

**Solution** One arch is traced from  $\theta = 0$  to  $\theta = 2\pi$ .  $x'(\theta) = r(1 - \cos\theta)$ .  $A = \int_0^{2\pi} y(\theta)x'(\theta)d\theta = \int_0^{2\pi} r(1 - \cos\theta) \cdot r(1 - \cos\theta)d\theta$ .  $A = r^2 \int_0^{2\pi} (1 - \cos\theta)^2d\theta = r^2 \int_0^{2\pi} (1 - 2\cos\theta + \cos^2\theta)d\theta$ . Using  $\cos^2\theta = \frac{1+\cos(2\theta)}{2}$ :  $A = r^2 \int_0^{2\pi} (1 - 2\cos\theta + \frac{1}{2} + \frac{1}{2}\cos(2\theta))d\theta$ .  $A = r^2 \int_0^{2\pi} (\frac{3}{2} - 2\cos\theta + \frac{1}{2}\cos(2\theta))d\theta$ .  $A = r^2 [\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin(2\theta)]_0^{2\pi}$ .  $A = r^2((\frac{3}{2}(2\pi) - 0 + 0) - (0 - 0 + 0)) = r^2(3\pi) = 3\pi r^2$ .

**Problem 18** Find the area of the region enclosed by the curve  $x = t^2 - 2t$ ,  $y = \sqrt{t}$  and the y-axis.

**Solution** The curve intersects the y-axis when  $x = 0$ .  $t^2 - 2t = t(t - 2) = 0 \implies t = 0, t = 2$ . The portion of the curve is traced for  $t$  from 0 to 2.  $x'(t) = 2t - 2$ .  $A = \int_0^2 y(t)x'(t)dt = \int_0^2 \sqrt{t}(2t - 2)dt = \int_0^2 (2t^{3/2} - 2t^{1/2})dt$ . Note: at  $t = 1$ ,  $x(1) = -1$ ,  $x(0) = 0$ ,  $x(2) = 0$ . The curve traces from right-to-left for  $t \in [0, 1]$  and left-to-right for  $t \in [1, 2]$ . The area integral will be negative. We should take the absolute value.  $A = \left| \left[ 2\frac{t^{5/2}}{5/2} - 2\frac{t^{3/2}}{3/2} \right]_0^2 \right| = \left| \left[ \frac{4}{5}t^{5/2} - \frac{4}{3}t^{3/2} \right]_0^2 \right|$ .  $A = \left| \left( \frac{4}{5}2^{5/2} - \frac{4}{3}2^{3/2} \right) - 0 \right| = \left| \frac{4}{5}(4\sqrt{2}) - \frac{4}{3}(2\sqrt{2}) \right|$ .  $A = \left| \frac{16\sqrt{2}}{5} - \frac{8\sqrt{2}}{3} \right| = \left| \frac{48\sqrt{2} - 40\sqrt{2}}{15} \right| = \frac{8\sqrt{2}}{15}$ .

## Mixed and Challenging Problems

**Problem 19** For the curve  $x = t^3 + 1$ ,  $y = t^2 - t$ , find the equation of the tangent line at the point  $(9, -2)$ .

**Solution** Find  $t$ :  $x(t) = t^3 + 1 = 9 \implies t^3 = 8 \implies t = 2$ . Let's check with  $y$ :  $y(-2) = (-2)^2 - (-2) = 4 + 2 = 6 \neq -2$ . Wait, there is a typo in the question point. Let's assume the question meant  $y = t - t^2$ .  $y(2) = 2 - 2^2 = -2$ . This works. Let's proceed with  $y = t - t^2$ . Slope:  $\frac{dx}{dt} = 3t^2$ ,  $\frac{dy}{dt} = 1 - 2t$ .  $m = \frac{1-2t}{3t^2}|_{t=2} = \frac{1-4}{3(4)} = \frac{-3}{12} = -\frac{1}{4}$ . Equation:  $y - (-2) = -\frac{1}{4}(x - 9) \implies y + 2 = -\frac{1}{4}x + \frac{9}{4} \implies y = -\frac{1}{4}x + \frac{1}{4}$ .

**Problem 20** A particle's position is given by  $x(t) = 2\sin(t)$ ,  $y(t) = \cos(2t)$ . Find all points where the particle is momentarily stopped.

**Solution** The particle is stopped when its speed is zero, which means both  $\frac{dx}{dt}$  and  $\frac{dy}{dt}$  are zero simultaneously.  $\frac{dx}{dt} = 2\cos(t) = 0 \implies t = \frac{\pi}{2}, \frac{3\pi}{2}, \dots$ .  $\frac{dy}{dt} = -2\sin(2t) = -2(2\sin(t)\cos(t)) = -4\sin(t)\cos(t) = 0$ . This is zero when  $\sin(t) = 0$  or  $\cos(t) = 0$ . The values of  $t$  for which both derivatives are zero are when  $\cos(t) = 0$ , i.e.,  $t = \frac{\pi}{2} + n\pi$  for any integer  $n$ . At these times, the particle stops. Let's find the points: If  $t = \pi/2$ ,  $(x, y) = (2\sin(\pi/2), \cos(\pi)) = (2, -1)$ . If  $t = 3\pi/2$ ,  $(x, y) = (2\sin(3\pi/2), \cos(3\pi)) = (-2, -1)$ . The particle stops at  $(2, -1)$  and  $(-2, -1)$ .

**Problem 21** Find  $\frac{d^2y}{dx^2}$  for the curve  $x = a\cos(t)$ ,  $y = b\sin(t)$  and interpret the result for concavity.

**Solution**  $\frac{dx}{dt} = -a\sin(t)$ ,  $\frac{dy}{dt} = b\cos(t)$ .  $\frac{dy}{dx} = \frac{b\cos(t)}{-a\sin(t)} = -\frac{b}{a}\cot(t)$ .  $\frac{d}{dt}\left(\frac{dy}{dx}\right) = -\frac{b}{a}(-\csc^2(t)) = \frac{b}{a}\csc^2(t)$ .  $\frac{d^2y}{dx^2} = \frac{\frac{b}{a}\csc^2(t)}{-a\sin(t)} = -\frac{b}{a^2\sin^3(t)}$ . Concavity: If  $0 < t < \pi$ ,  $\sin(t) > 0$ , so  $\frac{d^2y}{dx^2} < 0$ . The top half of the ellipse is concave down. If  $\pi < t < 2\pi$ ,  $\sin(t) < 0$ , so  $\frac{d^2y}{dx^2} > 0$ . The bottom half of the ellipse is concave up. This matches our geometric intuition.

**Problem 22** Set up, but do not evaluate, an integral for the surface area generated by rotating the curve  $x = t^3$ ,  $y = t^2$ ,  $0 \leq t \leq 1$  about the x-axis.

**Solution**  $S = \int_a^b 2\pi y(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$ .  $\frac{dx}{dt} = 3t^2$ ,  $\frac{dy}{dt} = 2t$ . The radical term is  $\sqrt{(3t^2)^2 + (2t)^2} = \sqrt{9t^4 + 4t^2} = \sqrt{t^2(9t^2 + 4)} = t\sqrt{9t^2 + 4}$  (for  $t \geq 0$ ).  $S = \int_0^1 2\pi(t^2)(t\sqrt{9t^2 + 4})dt = \int_0^1 2\pi t^3 \sqrt{9t^2 + 4} dt$ .

**Problem 23** Find the total distance traveled by a particle whose position is given by  $x = 3\cos^2(t)$ ,  $y = 3\sin^2(t)$  for  $0 \leq t \leq \pi$ .

**Solution** This is an arc length problem.  $\frac{dx}{dt} = 3 \cdot 2\cos(t)(-\sin(t)) = -6\cos(t)\sin(t)$ .  $\frac{dy}{dt} = 3 \cdot 2\sin(t)(\cos(t)) = 6\cos(t)\sin(t)$ .  $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 36\cos^2(t)\sin^2(t) + 36\cos^2(t)\sin^2(t) = 72\cos^2(t)\sin^2(t)$ .  $L = \int_0^\pi \sqrt{72\cos^2(t)\sin^2(t)} dt = \int_0^\pi \sqrt{72} |\cos(t)\sin(t)| dt$ .  $\sqrt{72} = 6\sqrt{2}$ .  $L = 6\sqrt{2} \int_0^\pi |\cos(t)\sin(t)| dt$ . Since  $\sin(t) \geq 0$  on  $[0, \pi]$ , we only care about the sign of  $\cos(t)$ .  $L = 6\sqrt{2} \left( \int_0^{\pi/2} \cos(t)\sin(t) dt + \int_{\pi/2}^\pi -\cos(t)\sin(t) dt \right)$ .

Let  $u = \sin(t)$ ,  $du = \cos(t)dt$ .  $\int \cos(t) \sin(t)dt = \int udu = \frac{1}{2}u^2 = \frac{1}{2}\sin^2(t)$ .  $L = 6\sqrt{2} \left( \left[ \frac{1}{2}\sin^2(t) \right]_0^{\pi/2} - \left[ \frac{1}{2}\sin^2(t) \right]_{\pi/2}^{\pi} \right)$ .  
 $L = 6\sqrt{2} \left( \left( \frac{1}{2}(1)^2 - 0 \right) - \left( \frac{1}{2}(0)^2 - \frac{1}{2}(1)^2 \right) \right) = 6\sqrt{2} \left( \frac{1}{2} + \frac{1}{2} \right) = 6\sqrt{2}$ .

**Problem 24** Find the area of the region bounded by the x-axis and the curve  $x = t^3 + t$ ,  $y = 1 - t^2$ .

**Solution** The curve intersects the x-axis when  $y = 0$ .  $1 - t^2 = 0 \implies t = \pm 1$ .  $x(-1) = -2$ ,  $x(1) = 2$ . The curve is traced from left to right as  $t$  goes from  $-1$  to  $1$ .  $x'(t) = 3t^2 + 1$ .  $A = \int_{-1}^1 (1 - t^2)(3t^2 + 1)dt = \int_{-1}^1 (3t^2 + 1 - 3t^4 - t^2)dt$ .  $A = \int_{-1}^1 (-3t^4 + 2t^2 + 1)dt$ . Since the integrand is an even function:  
 $A = 2 \int_0^1 (-3t^4 + 2t^2 + 1)dt = 2 \left[ -\frac{3}{5}t^5 + \frac{2}{3}t^3 + t \right]_0^1$ .  $A = 2 \left( -\frac{3}{5} + \frac{2}{3} + 1 \right) = 2 \left( \frac{-9+10+15}{15} \right) = 2 \left( \frac{16}{15} \right) = \frac{32}{15}$ .

**Problem 25** The velocity components of a particle are  $\frac{dx}{dt} = t^2$  and  $\frac{dy}{dt} = \sqrt{t}$ . What is the acceleration vector  $\vec{a}(t)$  and the slope of the curve at  $t = 4$ ?

**Solution** The velocity vector is  $\vec{v}(t) = \langle t^2, \sqrt{t} \rangle$ . The acceleration vector is the derivative of the velocity vector:  $\vec{a}(t) = \langle \frac{d}{dt}(t^2), \frac{d}{dt}(\sqrt{t}) \rangle = \langle 2t, \frac{1}{2\sqrt{t}} \rangle$ .

The slope of the curve is  $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\sqrt{t}}{t^2} = t^{-3/2}$ . At  $t = 4$ , the slope is  $4^{-3/2} = (4^{1/2})^{-3} = 2^{-3} = \frac{1}{8}$ .

**Problem 26** Find the arc length of  $x = t^2$ ,  $y = 2t$  from  $t = 0$  to  $t = \sqrt{3}$ .

**Solution**  $\frac{dx}{dt} = 2t$ ,  $\frac{dy}{dt} = 2$ .  $(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 = (2t)^2 + 2^2 = 4t^2 + 4 = 4(t^2 + 1)$ .  $L = \int_0^{\sqrt{3}} \sqrt{4(t^2 + 1)}dt = \int_0^{\sqrt{3}} 2\sqrt{t^2 + 1}dt$ . This requires a trig substitution. Let  $t = \tan \theta$ ,  $dt = \sec^2 \theta d\theta$ .  $L = \int_0^{\pi/3} 2\sqrt{\tan^2 \theta + 1} \sec^2 \theta d\theta = \int_0^{\pi/3} 2\sec^3 \theta d\theta$ . Using the reduction formula  $\int \sec^n(x)dx = \frac{\sec^{n-2}(x)\tan(x)}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2}(x)dx$ :  $L = 2 \left[ \frac{\sec \theta \tan \theta}{2} + \frac{1}{2} \int \sec \theta d\theta \right]_0^{\pi/3} = [\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|]_0^{\pi/3}$ .  $L = (\sec(\pi/3)\tan(\pi/3) + \ln |\sec(\pi/3) + \tan(\pi/3)|) - (\sec(0)\tan(0) + \ln |\sec(0) + \tan(0)|)$ .  $L = (2\sqrt{3} + \ln |2 + \sqrt{3}|) - (0 + \ln |1 + 0|) = 2\sqrt{3} + \ln(2 + \sqrt{3})$ .

**Problem 27** Consider the curve  $x = t^2$ ,  $y = kt^3 - t^2$ . Find the value of  $k$  such that the curve has a vertical tangent at  $t = 0$ . Explain your reasoning.

**Solution** A vertical tangent requires  $\frac{dx}{dt} = 0$  and  $\frac{dy}{dt} \neq 0$ .  $\frac{dx}{dt} = 2t$ . This is zero at  $t = 0$ .  $\frac{dy}{dt} = 3kt^2 - 2t$ . At  $t = 0$ ,  $\frac{dy}{dt} = 3k(0)^2 - 2(0) = 0$ . Since both derivatives are zero at  $t = 0$ , the slope is of the indeterminate form  $0/0$ . There is no value of  $k$  for which the tangent is strictly vertical at  $t = 0$  based on the standard definition. Using L'Hopital's rule on the slope:  $\lim_{t \rightarrow 0} \frac{dy/dt}{dx/dt} = \lim_{t \rightarrow 0} \frac{3kt^2 - 2t}{2t} = \lim_{t \rightarrow 0} \frac{6kt - 2}{2} = -1$ . The slope approaches  $-1$ , so the curve has a defined tangent at the origin, but it is not vertical.

**Problem 28** A curve is given by  $x = \sin(t)$ ,  $y = \sin(2t)$ . Find the area of the loop enclosed by the curve.

**Solution** The curve creates a loop. We need to find the  $t$ -values where it self-intersects.  $\sin(t_1) = \sin(t_2)$  and  $\sin(2t_1) = \sin(2t_2)$  for  $t_1 \neq t_2$ . This occurs for example when  $t_1 = 0$  and  $t_2 = \pi$ .  $x(0) = 0$ ,  $y(0) = 0$ .  $x(\pi) = 0$ ,  $y(\pi) = 0$ . The loop is traced between  $t = 0$  and  $t = \pi$ .  $x'(t) = \cos(t)$ .  $A = \int_0^\pi y(t)x'(t)dt = \int_0^\pi \sin(2t)\cos(t)dt$ .  $A = \int_0^\pi (2\sin(t)\cos(t))\cos(t)dt = \int_0^\pi 2\sin(t)\cos^2(t)dt$ . Let  $u = \cos(t)$ ,  $du = -\sin(t)dt$ . Bounds:  $t = 0 \implies u = 1$ ,  $t = \pi \implies u = -1$ .  $A = \int_1^{-1} 2u^2(-du) = \int_{-1}^1 2u^2du = 2 \left[ \frac{u^3}{3} \right]_{-1}^1 = \frac{2}{3}(1^3 - (-1)^3) = \frac{2}{3}(2) = \frac{4}{3}$ .

**Problem 29** The curve  $x = \sec(t)$ ,  $y = \tan(t)$  for  $-\pi/2 < t < \pi/2$  is a hyperbola. Find its Cartesian equation and use it to find  $\frac{dy}{dx}$ . Verify your answer using parametric differentiation.

**Solution** We know the identity  $1 + \tan^2(t) = \sec^2(t)$ . Substituting  $x$  and  $y$ :  $1 + y^2 = x^2 \implies x^2 - y^2 = 1$ . Differentiating with respect to  $x$ :  $2x - 2y \frac{dy}{dx} = 0 \implies \frac{dy}{dx} = \frac{2x}{2y} = \frac{x}{y}$ .

Using parametric differentiation:  $\frac{dx}{dt} = \sec(t) \tan(t)$ ,  $\frac{dy}{dt} = \sec^2(t)$ .  $\frac{dy}{dx} = \frac{\sec^2(t)}{\sec(t) \tan(t)} = \frac{\sec(t)}{\tan(t)} = \frac{1/\cos(t)}{\sin(t)/\cos(t)} = \frac{1}{\sin(t)} = \csc(t)$ . To verify they are the same:  $\frac{x}{y} = \frac{\sec(t)}{\tan(t)} = \csc(t)$ . The results match.

**Problem 30** Explain the "second derivative trap". For the curve  $x = t^3, y = t^2$ , show that using the trap formula  $\frac{y''(t)}{x''(t)}$  gives the wrong answer for  $\frac{d^2y}{dx^2}$ .

**Solution** The "second derivative trap" is the common mistake of thinking that  $\frac{d^2y}{dx^2}$  is equal to the ratio of the second derivatives with respect to the parameter  $t$ , i.e.,  $\frac{d^2y/dt^2}{d^2x/dt^2}$ . This is incorrect because the chain rule must be applied to the first derivative,  $\frac{dy}{dx}$ , which is itself a function of  $t$ .

For  $x = t^3, y = t^2$ :  $x'(t) = 3t^2, y'(t) = 2t$ .  $x''(t) = 6t, y''(t) = 2$ . The incorrect trap formula gives:  $\frac{y''(t)}{x''(t)} = \frac{2}{6t} = \frac{1}{3t}$ .

The correct method: First, find  $\frac{dy}{dx} = \frac{2t}{3t^2} = \frac{2}{3t}$ . Next, differentiate this with respect to  $t$ :  $\frac{d}{dt} \left( \frac{2}{3t} \right) = -\frac{2}{3t^2}$ . Finally, divide by  $\frac{dx}{dt}$ :  $\frac{d^2y}{dx^2} = \frac{-2/(3t^2)}{3t^2} = -\frac{2}{9t^4}$ . Clearly,  $-\frac{2}{9t^4} \neq \frac{1}{3t}$ , demonstrating that the trap formula is wrong.

# Concept Checklist and Problem Index

This index maps the core concepts of Calculus with Parametric Curves to the problem numbers in this document that test them.

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