Calculus II - Test 2 Additional Problems

Based on Review by Prof. Ibrahim El Haitami - MAC 2312

1.a. Improper Integrals: Infinite Limit with Partial Fractions

1. **Integral:** $\displaystyle\int_3^\infty \frac{4}{x^2-4}\,dx$

$$\frac{4}{x^2-4} = \frac{4}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2} \implies A = 1, B = -1$$

$$\begin{aligned}\int_3^\infty \left(\frac{1}{x-2} - \frac{1}{x+2}\right) dx &= \lim_{b\to\infty} \left[\ln|x-2| - \ln|x+2|\right]_3^b \\ &= \lim_{b\to\infty} \left[\ln\left|\frac{x-2}{x+2}\right|\right]_3^b \\ &= \lim_{b\to\infty} \left(\ln\left|\frac{b-2}{b+2}\right| - \ln\left|\frac{1}{5}\right|\right) = \ln(1) - \ln(1/5) = \ln 5\end{aligned}$$

Result: The integral **converges** to $\ln 5$.

2. **Integral:** $\displaystyle\int_2^\infty \frac{6}{x^2+2x-3}\,dx$

$$\frac{6}{(x+3)(x-1)} = \frac{A}{x+3} + \frac{B}{x-1} \implies A = -3/2, B = 3/2$$

$$\begin{aligned}\frac{3}{2}\int_2^\infty \left(\frac{1}{x-1} - \frac{1}{x+3}\right) dx &= \frac{3}{2}\lim_{b\to\infty} \left[\ln\left|\frac{x-1}{x+3}\right|\right]_2^b \\ &= \frac{3}{2}\left(\ln(1) - \ln\left|\frac{1}{5}\right|\right) = \frac{3}{2}\ln 5\end{aligned}$$

Result: The integral **converges** to $\frac{3}{2}\ln 5$.

3. **Integral:** $\displaystyle\int_0^\infty \frac{1}{x^2+3x+2}\,dx$

$$\frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$$

$$\begin{aligned}\int_0^\infty \left(\frac{1}{x+1} - \frac{1}{x+2}\right) dx &= \lim_{b\to\infty} \left[\ln\left|\frac{x+1}{x+2}\right|\right]_0^b \\ &= \ln(1) - \ln(1/2) = \ln 2\end{aligned}$$

Result: The integral **converges** to $\ln 2$.

4. **Integral:** $\displaystyle\int_1^\infty \frac{1}{x(x+1)}\,dx$

$$\begin{aligned}\frac{1}{x(x+1)} &= \frac{1}{x} - \frac{1}{x+1}\\ \int_1^\infty \left(\frac{1}{x} - \frac{1}{x+1}\right) dx &= \lim_{b\to\infty}\left[\ln\left|\frac{x}{x+1}\right|\right]_1^b\\ &= \ln(1) - \ln(1/2) = \ln 2\end{aligned}$$

Result: The integral **converges** to $\ln 2$.

5. **Integral:** $\displaystyle\int_4^\infty \frac{3}{x^2-9}\,dx$

$$\begin{aligned}\frac{3}{(x-3)(x+3)} &= \frac{1/2}{x-3} - \frac{1/2}{x+3}\\ \frac{1}{2}\int_4^\infty \left(\frac{1}{x-3} - \frac{1}{x+3}\right) dx &= \frac{1}{2}\lim_{b\to\infty}\left[\ln\left|\frac{x-3}{x+3}\right|\right]_4^b\\ &= \frac{1}{2}\left(\ln(1) - \ln(1/7)\right) = \frac{1}{2}\ln 7\end{aligned}$$

Result: The integral **converges** to $\frac{1}{2}\ln 7$.

6. **Integral:** $\displaystyle\int_1^\infty \frac{4}{x(x+2)}\,dx$

$$\begin{aligned}\frac{4}{x(x+2)} &= \frac{2}{x} - \frac{2}{x+2}\\ \int_1^\infty \left(\frac{2}{x} - \frac{2}{x+2}\right) dx &= \lim_{b\to\infty}\left[2\ln\left|\frac{x}{x+2}\right|\right]_1^b\\ &= 2(\ln(1) - \ln(1/3)) = 2\ln 3\end{aligned}$$

Result: The integral **converges** to $2\ln 3$.

7. **Integral:** $\displaystyle\int_2^\infty \frac{x-1}{x^3+x^2-2x}\,dx$

$$\begin{aligned}\frac{x-1}{x(x+2)(x-1)} = \frac{1}{x(x+2)} &= \frac{1/2}{x} - \frac{1/2}{x+2}\\ \frac{1}{2}\int_2^\infty \left(\frac{1}{x} - \frac{1}{x+2}\right) dx &= \frac{1}{2}\lim_{b\to\infty}\left[\ln\left|\frac{x}{x+2}\right|\right]_2^b\\ &= \frac{1}{2}(\ln(1) - \ln(2/4)) = \frac{1}{2}\ln 2\end{aligned}$$

Result: The integral **converges** to $\frac{1}{2}\ln 2$.

8. **Integral:** $\displaystyle\int_0^\infty \frac{1}{(x+1)(x+2)(x+3)}\,dx$

$$\frac{1}{(x+1)(x+2)(x+3)} = \frac{1/2}{x+1} - \frac{1}{x+2} + \frac{1/2}{x+3}$$

$$\begin{aligned}\int_0^\infty \ldots\,dx &= \lim_{b\to\infty}\left[\frac{1}{2}\ln|x+1| - \ln|x+2| + \frac{1}{2}\ln|x+3|\right]_0^b \\ &= \lim_{b\to\infty}\left[\frac{1}{2}\ln\left|\frac{(x+1)(x+3)}{(x+2)^2}\right|\right]_0^b \\ &= \frac{1}{2}\left(\ln(1) - \ln\left(\frac{3}{4}\right)\right) = \frac{1}{2}\ln(4/3)\end{aligned}$$

Result: The integral **converges** to $\frac{1}{2}\ln(4/3)$.

9. **Integral:** $\displaystyle\int_3^\infty \frac{2x}{x^2-1}\,dx$

$$\int_3^\infty \frac{2x}{x^2-1}\,dx = \lim_{b\to\infty}[\ln|x^2-1|]_3^b = \lim_{b\to\infty}(\ln|b^2-1| - \ln 8) = \infty$$

Result: The integral **diverges**.

10. **Integral:** $\displaystyle\int_1^\infty \frac{x+1}{x^2+4x+3}\,dx$

$$\frac{x+1}{(x+1)(x+3)} = \frac{1}{x+3}$$

$$\int_1^\infty \frac{1}{x+3}\,dx = \lim_{b\to\infty}[\ln|x+3|]_1^b = \lim_{b\to\infty}(\ln|b+3| - \ln 4) = \infty$$

Result: The integral **diverges**.

1.b. Improper Integrals: Discontinuity with U-Substitution

1. **Integral:** $\int_0^2 \frac{x}{\sqrt{4-x^2}}\,dx$ *Discontinuity at* $x=2$*. Let* $u=4-x^2$*,* $du=-2x\,dx$*.*

$$\lim_{b\to 2^-}\int_0^b \frac{x}{\sqrt{4-x^2}}\,dx = \lim_{b\to 2^-}[-\sqrt{4-x^2}]_0^b = 0-(-\sqrt{4}) = 2$$

Result: The integral **converges** to 2.

2. **Integral:** $\int_1^2 \frac{1}{(x-1)^{1/3}}\,dx$ *Discontinuity at* $x=1$*.*

$$\lim_{a\to 1^+}\int_a^2 (x-1)^{-1/3}\,dx = \lim_{a\to 1^+}\left[\frac{3}{2}(x-1)^{2/3}\right]_a^2 = \frac{3}{2}-0=\frac{3}{2}$$

Result: The integral **converges** to $3/2$.

3. **Integral:** $\int_1^e \frac{1}{x\sqrt{\ln x}}\,dx$ *Discontinuity at* $x=1$*. Let* $u=\ln x$*,* $du=(1/x)dx$*.*

$$\lim_{a\to 1^+}\int_a^e \frac{1}{x\sqrt{\ln x}}\,dx = \lim_{a\to 1^+}[2\sqrt{\ln x}]_a^e = 2\sqrt{1}-0=2$$

Result: The integral **converges** to 2.

4. **Integral:** $\int_0^4 \frac{1}{\sqrt{x}}\,dx$ *Discontinuity at* $x=0$*.*

$$\lim_{a\to 0^+}\int_a^4 x^{-1/2}\,dx = \lim_{a\to 0^+}[2\sqrt{x}]_a^4 = 4-0=4$$

Result: The integral **converges** to 4.

5. **Integral:** $\int_0^{\pi/2} \frac{\cos x}{\sqrt{\sin x}}\,dx$ *Discontinuity at* $x=0$*. Let* $u=\sin x$*,* $du=\cos x\,dx$*.*

$$\lim_{a\to 0^+}\int_a^{\pi/2} \frac{\cos x}{\sqrt{\sin x}}\,dx = \lim_{a\to 0^+}[2\sqrt{\sin x}]_a^{\pi/2} = 2\sqrt{1}-0=2$$

Result: The integral **converges** to 2.

6. **Integral:** $\int_{-1}^0 \frac{x^2}{(x^3+1)^{1/3}}\,dx$ *Discontinuity at* $x=-1$*. Let* $u=x^3+1$*,* $du=3x^2dx$*.*

$$\lim_{a\to -1^+}\int_a^0 \dots\,dx = \lim_{a\to -1^+}\left[\frac{1}{2}(x^3+1)^{2/3}\right]_a^0 = \frac{1}{2}-0=\frac{1}{2}$$

Result: The integral **converges** to $1/2$.

7. **Integral:** $\int_0^{\ln 2} \frac{e^x}{e^x-1}\,dx$ *Discontinuity at* $x=0$*. Let* $u=e^x-1$*,* $du=e^x dx$*.*

$$\lim_{a\to 0^+}\int_a^{\ln 2} \frac{e^x}{e^x-1}\,dx = \lim_{a\to 0^+}[\ln|e^x-1|]_a^{\ln 2} = \ln(1)-\lim_{a\to 0^+}\ln|e^a-1| = \infty$$

Result: The integral **diverges**.

8. **Integral:** $\int_2^3 \frac{3x^2}{\sqrt{x^3-8}}\,dx$ *Discontinuity at* $x=2$. *Let* $u=x^3-8$, $du=3x^2dx$.

$$\lim_{a\to 2^+}\int_a^3 \ldots\, dx = \lim_{a\to 2^+}[2\sqrt{x^3-8}]_a^3 = 2\sqrt{19}-0 = 2\sqrt{19}$$

Result: The integral **converges** to $2\sqrt{19}$.

9. **Integral:** $\int_0^1 \frac{\arcsin x}{\sqrt{1-x^2}}\,dx$ *Discontinuity at* $x=1$. *Let* $u=\arcsin x$, $du=dx/\sqrt{1-x^2}$.

$$\begin{aligned}\lim_{b\to 1^-}\int_0^b \frac{\arcsin x}{\sqrt{1-x^2}}\,dx &= \lim_{b\to 1^-}[\frac{1}{2}(\arcsin x)^2]_0^b \\ &= \frac{1}{2}(\arcsin 1)^2 - 0 = \frac{1}{2}(\pi/2)^2 = \frac{\pi^2}{8}\end{aligned}$$

Result: The integral **converges** to $\pi^2/8$.

10. **Integral:** $\int_0^1 \frac{1}{(1-x)^{2/3}}\,dx$ *Discontinuity at* $x=1$. *Let* $u=1-x$, $du=-dx$.

$$\lim_{b\to 1^-}\int_0^b (1-x)^{-2/3}\,dx = \lim_{b\to 1^-}[-3(1-x)^{1/3}]_0^b = 0-(-3) = 3$$

Result: The integral **converges** to 3.

1.c. Improper Integrals: Vertical Asymptote

1. **Integral:** $\displaystyle\int_{\pi/2}^{\pi} \cot\theta\, d\theta$ *Asymptote at* $\theta = \pi$.

$$\lim_{b\to\pi^-}\int_{\pi/2}^{b}\cot\theta\,d\theta = \lim_{b\to\pi^-}[\ln|\sin\theta|]_{\pi/2}^{b} = (-\infty) - \ln(1) = -\infty$$

Result: The integral **diverges**.

2. **Integral:** $\displaystyle\int_{0}^{\pi/2} \sec\theta\, d\theta$ *Asymptote at* $\theta = \pi/2$.

$$\lim_{b\to(\pi/2)^-}\int_{0}^{b}\sec\theta\,d\theta = \lim_{b\to(\pi/2)^-}[\ln|\sec\theta+\tan\theta|]_{0}^{b} = \infty - 0 = \infty$$

Result: The integral **diverges**.

3. **Integral:** $\displaystyle\int_{0}^{1} \ln x\, dx$ *Asymptote at* $x = 0$. *Integration by parts.*

$$\lim_{a\to0^+}\int_{a}^{1}\ln x\,dx = \lim_{a\to0^+}[x\ln x - x]_{a}^{1} = (-1) - \lim_{a\to0^+}(a\ln a - a) = -1$$

Result: The integral **converges** to -1.

4. **Integral:** $\displaystyle\int_{0}^{1} \frac{1}{x-1}\, dx$ *Asymptote at* $x = 1$.

$$\lim_{b\to1^-}\int_{0}^{b}\frac{1}{x-1}\,dx = \lim_{b\to1^-}[\ln|x-1|]_{0}^{b} = (-\infty) - 0 = -\infty$$

Result: The integral **diverges**.

5. **Integral:** $\displaystyle\int_{-\pi/2}^{0} \tan\theta\, d\theta$ *Asymptote at* $\theta = -\pi/2$.

$$\lim_{a\to(-\pi/2)^+}\int_{a}^{0}\tan\theta\,d\theta = \lim_{a\to(-\pi/2)^+}[-\ln|\cos\theta|]_{a}^{0} = 0 - (-\infty) = \infty$$

Result: The integral **diverges**.

6. **Integral:** $\displaystyle\int_{0}^{2} \frac{1}{(x-2)^2}\, dx$ *Asymptote at* $x = 2$.

$$\lim_{b\to2^-}\int_{0}^{b}(x-2)^{-2}\,dx = \lim_{b\to2^-}\left[-\frac{1}{x-2}\right]_{0}^{b} = \infty - \frac{1}{2} = \infty$$

Result: The integral **diverges**.

7. **Integral:** $\displaystyle\int_{1}^{2} \frac{1}{\sqrt[3]{x-1}}\, dx$ *Asymptote at* $x = 1$.

$$\lim_{a\to1^+}\int_{a}^{2}(x-1)^{-1/3}\,dx = \lim_{a\to1^+}\left[\frac{3}{2}(x-1)^{2/3}\right]_{a}^{2} = \frac{3}{2}$$

Result: The integral **converges** to $3/2$.

8. **Integral:** $\int_0^{\pi} \csc\theta \, d\theta$ *Asymptotes at $\theta = 0, \pi$. Split at $\pi/2$.*

$$\int_0^{\pi/2} \csc\theta \, d\theta = \lim_{a \to 0^+} [-\ln|\csc\theta + \cot\theta|]_a^{\pi/2} = \infty$$

Result: The integral **diverges**.

9. **Integral:** $\int_0^1 \frac{1}{x} \, dx$ *Asymptote at $x = 0$.*

$$\lim_{a \to 0^+} \int_a^1 \frac{1}{x} \, dx = \lim_{a \to 0^+} [\ln|x|]_a^1 = 0 - (-\infty) = \infty$$

Result: The integral **diverges**.

10. **Integral:** $\int_1^3 \frac{1}{x-3} \, dx$ *Asymptote at $x = 3$.*

$$\lim_{b \to 3^-} \int_1^b \frac{1}{x-3} \, dx = \lim_{b \to 3^-} [\ln|x-3|]_1^b = -\infty$$

Result: The integral **diverges**.

1.d. Improper Integrals: Discontinuity and Standard Forms

1. **Integral:** $\int_{-1}^{0} \frac{1}{\sqrt{1-x^2}}\,dx$ *Discontinuity at $x=-1$.*

$$\lim_{a\to -1^+} \int_{a}^{0} \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{a\to -1^+} [\arcsin(x)]_a^0 = 0-(-\pi/2) = \frac{\pi}{2}$$

Result: The integral **converges** to $\pi/2$.

2. **Integral:** $\int_{0}^{2} \frac{1}{\sqrt{4-x^2}}\,dx$ *Discontinuity at $x=2$.*

$$\lim_{b\to 2^-} \int_{0}^{b} \frac{1}{\sqrt{4-x^2}}\,dx = \lim_{b\to 2^-} [\arcsin(x/2)]_0^b = \pi/2-0 = \frac{\pi}{2}$$

Result: The integral **converges** to $\pi/2$.

3. **Integral:** $\int_{0}^{\infty} \frac{1}{1+x^2}\,dx$ *Infinite upper limit.*

$$\lim_{b\to\infty} \int_{0}^{b} \frac{1}{1+x^2}\,dx = \lim_{b\to\infty} [\arctan(x)]_0^b = \pi/2-0 = \frac{\pi}{2}$$

Result: The integral **converges** to $\pi/2$.

4. **Integral:** $\int_{0}^{2} \frac{1}{x^2+4}\,dx$ *Standard integral, no discontinuity.*

$$[\frac{1}{2}\arctan(x/2)]_0^2 = \frac{1}{2}\arctan(1)-0 = \frac{1}{2}(\pi/4) = \frac{\pi}{8}$$

Result: This is a proper integral. The value is $\pi/8$.

5. **Integral:** $\int_{0}^{3} \frac{1}{\sqrt{3-x}}\,dx$ *Discontinuity at $x=3$.*

$$\lim_{b\to 3^-} \int_{0}^{b} (3-x)^{-1/2}\,dx = \lim_{b\to 3^-} [-2\sqrt{3-x}]_0^b = 0-(-2\sqrt{3}) = 2\sqrt{3}$$

Result: The integral **converges** to $2\sqrt{3}$.

6. **Integral:** $\int_{-3}^{0} \frac{1}{\sqrt{9-x^2}}\,dx$ *Discontinuity at $x=-3$.*

$$\lim_{a\to -3^+} \int_{a}^{0} \frac{1}{\sqrt{9-x^2}}\,dx = \lim_{a\to -3^+} [\arcsin(x/3)]_a^0 = 0-(-\pi/2) = \frac{\pi}{2}$$

Result: The integral **converges** to $\pi/2$.

7. **Integral:** $\int_{-\infty}^{0} e^x\,dx$ *Infinite lower limit.*

$$\lim_{a\to -\infty} \int_{a}^{0} e^x\,dx = \lim_{a\to -\infty} [e^x]_a^0 = 1-0 = 1$$

Result: The integral **converges** to 1.

8. **Integral:** $\int_1^2 \frac{1}{(x-1)^{2/3}}\,dx$ *Discontinuity at* $x = 1$.

$$\lim_{a\to 1^+} \int_a^2 (x-1)^{-2/3}\,dx = \lim_{a\to 1^+} [3(x-1)^{1/3}]_a^2 = 3 - 0 = 3$$

   Result: The integral **converges** to 3.

9. **Integral:** $\int_2^4 \frac{1}{\sqrt{x-2}}\,dx$ *Discontinuity at* $x = 2$.

$$\lim_{a\to 2^+} \int_a^4 (x-2)^{-1/2}\,dx = \lim_{a\to 2^+} [2\sqrt{x-2}]_a^4 = 2\sqrt{2} - 0 = 2\sqrt{2}$$

   Result: The integral **converges** to $2\sqrt{2}$.

10. **Integral:** $\int_0^\infty \frac{e^{-x}}{2}\,dx$ *Infinite upper limit.*

$$\lim_{b\to\infty} \int_0^b \frac{1}{2}e^{-x}\,dx = \lim_{b\to\infty} [-\frac{1}{2}e^{-x}]_0^b = 0 - (-\frac{1}{2}) = \frac{1}{2}$$

   Result: The integral **converges** to $1/2$.

1.e. Improper Integrals: Infinite Limits in Both Directions

1. **Integral:** $\int_{-\infty}^{\infty} xe^{-x^2}\,dx$ *Odd function over a symmetric interval.*

$$\int_0^{\infty} xe^{-x^2}\,dx = \lim_{b\to\infty}[-\frac{1}{2}e^{-x^2}]_0^b = 1/2. \quad \int_{-\infty}^{0} \cdots = -1/2.$$

Result: The integral **converges** to $1/2 - 1/2 = 0$.

2. **Integral:** $\int_{-\infty}^{\infty} \frac{1}{x^2+9}\,dx$

$$\int_{-\infty}^{\infty} \frac{1}{x^2+9}\,dx = [\frac{1}{3}\arctan(x/3)]_{-\infty}^{\infty} = \frac{1}{3}(\pi/2 - (-\pi/2)) = \frac{\pi}{3}$$

Result: The integral **converges** to $\pi/3$.

3. **Integral:** $\int_{-\infty}^{\infty} x^3e^{-x^4}\,dx$ *Odd function. Converges to 0.*

$$\int_0^{\infty} x^3e^{-x^4}dx = \lim_{b\to\infty}[-e^{-x^4}/4]_0^b = 1/4. \quad \int_{-\infty}^{0} \cdots = -1/4.$$

Result: The integral **converges** to 0.

4. **Integral:** $\int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}}\,dx$ *Let* $u = e^x$*. Limits become* 0 *to* ∞*.*

$$\int_0^{\infty} \frac{1}{1+u^2}\,du = [\arctan u]_0^{\infty} = \pi/2$$

Result: The integral **converges** to $\pi/2$.

5. **Integral:** $\int_{-\infty}^{\infty} \frac{1}{x^2+2x+2}\,dx$ *Complete the square:* $(x+1)^2+1$.

$$\int_{-\infty}^{\infty} \frac{1}{(x+1)^2+1}\,dx = [\arctan(x+1)]_{-\infty}^{\infty} = \pi/2 - (-\pi/2) = \pi$$

Result: The integral **converges** to π.

6. **Integral:** $\int_{-\infty}^{\infty} \frac{1}{e^x+e^{-x}}\,dx$ *Multiply by* e^x/e^x*:* $\int \frac{e^x}{e^{2x}+1}dx$*. Same as #4.* **Result:** The integral **converges** to $\pi/2$.

7. **Integral:** $\int_{-\infty}^{\infty} \frac{x^2}{(x^3-1)^2}\,dx$ *Discontinuity at* $x = 1$*. Let's check convergence near 1.*

$$\lim_{a\to 1^+}\int_a^2 \frac{x^2}{(x^3-1)^2}\,dx = \lim_{a\to 1^+}[-\frac{1}{3(x^3-1)}]_a^2 = -\frac{1}{21} - (-\infty) = \infty$$

Result: The integral **diverges**.

8. **Integral:** $\int_{-\infty}^{\infty} \frac{x}{(x^2+1)^2}\,dx$ *Odd function. Let's check convergence.*

$$\int_0^{\infty} \frac{x}{(x^2+1)^2} dx = \lim_{b\to\infty} [-\frac{1}{2(x^2+1)}]_0^b = 0 - (-\frac{1}{2}) = \frac{1}{2}.$$

Result: The integral **converges** to $1/2 - 1/2 = 0$.

9. **Integral:** $\int_{-\infty}^{\infty} \frac{x}{x^4+1}\,dx$ *Odd function. Let's check convergence.*

$$\int_0^{\infty} \frac{x}{x^4+1} dx = \lim_{b\to\infty} [\frac{1}{2}\arctan(x^2)]_0^b = \frac{1}{2}(\pi/2 - 0) = \pi/4.$$

Result: The integral **converges** to $\pi/4 - \pi/4 = 0$.

10. **Integral:** $\int_{-\infty}^{\infty} x\,dx$

$$\int_0^{\infty} x\,dx = \lim_{b\to\infty} [x^2/2]_0^b = \infty.$$

Result: The integral **diverges**.

2. Arc Length: Perfect Square Integrands

1. **Curve:** $y = \frac{x^3}{6} + \frac{1}{2x}$ from $x = 1$ to $x = 2$

$$y' = \frac{x^2}{2} - \frac{1}{2x^2} \implies 1 + (y')^2 = 1 + \left(\frac{x^4}{4} - \frac{1}{2} + \frac{1}{4x^4}\right)$$
$$= \frac{x^4}{4} + \frac{1}{2} + \frac{1}{4x^4} = \left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2$$
$$L = \int_1^2 \left(\frac{x^2}{2} + \frac{1}{2x^2}\right) dx = \left[\frac{x^3}{6} - \frac{1}{2x}\right]_1^2 = (\frac{8}{6} - \frac{1}{4}) - (\frac{1}{6} - \frac{1}{2}) = \frac{17}{12}$$

Length: $17/12$.

2. **Curve:** $y = \frac{2}{3}(x-1)^{3/2}$ from $x = 1$ to $x = 5$

$$y' = (x-1)^{1/2} \implies 1 + (y')^2 = 1 + (x-1) = x$$
$$L = \int_1^5 \sqrt{x}\, dx = \left[\frac{2}{3}x^{3/2}\right]_1^5 = \frac{2}{3}(5\sqrt{5} - 1)$$

Length: $\frac{2}{3}(5\sqrt{5} - 1)$.

3. **Curve:** $y = \ln(\cos x)$ from $x = 0$ to $x = \pi/4$

$$y' = \frac{-\sin x}{\cos x} = -\tan x \implies 1 + (y')^2 = 1 + \tan^2 x = \sec^2 x$$
$$L = \int_0^{\pi/4} \sec x\, dx = [\ln|\sec x + \tan x|]_0^{\pi/4} = \ln(\sqrt{2} + 1)$$

Length: $\ln(\sqrt{2} + 1)$.

4. **Curve:** $y = \frac{x^2}{4} - \frac{\ln x}{2}$ from $x = 1$ to $x = e$

$$y' = \frac{x}{2} - \frac{1}{2x} \implies 1 + (y')^2 = 1 + (\frac{x^2}{4} - \frac{1}{2} + \frac{1}{4x^2}) = (\frac{x}{2} + \frac{1}{2x})^2$$
$$L = \int_1^e (\frac{x}{2} + \frac{1}{2x})dx = \left[\frac{x^2}{4} + \frac{\ln x}{2}\right]_1^e = (\frac{e^2}{4} + \frac{1}{2}) - (\frac{1}{4} + 0) = \frac{e^2 + 1}{4}$$

Length: $(e^2 + 1)/4$.

5. **Curve:** $y = \frac{2}{3}x^{3/2}$ from $x = 0$ to $x = 8$

$$y' = x^{1/2} \implies 1 + (y')^2 = 1 + x$$
$$L = \int_0^8 \sqrt{1+x}\, dx = \left[\frac{2}{3}(1+x)^{3/2}\right]_0^8 = \frac{2}{3}(9^{3/2} - 1^{3/2}) = \frac{2}{3}(27 - 1) = \frac{52}{3}$$

Length: $52/3$.

6. **Curve:** $y = \cosh(x)$ from $x = 0$ to $x = \ln 2$

$$y' = \sinh(x) \implies 1 + (y')^2 = 1 + \sinh^2 x = \cosh^2 x$$
$$L = \int_0^{\ln 2} \cosh x\, dx = [\sinh x]_0^{\ln 2} = \sinh(\ln 2) - 0 = \frac{e^{\ln 2} - e^{-\ln 2}}{2} = \frac{2 - 1/2}{2} = \frac{3}{4}$$

Length: $3/4$.

7. **Curve:** $y = \frac{x^4}{4} + \frac{1}{8x^2}$ from $x = 1$ to $x = 2$

$$y' = x^3 - \frac{1}{4x^3} \implies 1 + (y')^2 = 1 + (x^6 - \frac{1}{2} + \frac{1}{16x^6}) = (x^3 + \frac{1}{4x^3})^2$$
$$L = \int_1^2 (x^3 + \frac{1}{4x^3})dx = [\frac{x^4}{4} - \frac{1}{8x^2}]_1^2 = (4 - \frac{1}{32}) - (\frac{1}{4} - \frac{1}{8}) = \frac{123}{32}$$

Length: $123/32$.

8. **Curve:** $y = \frac{1}{3}(x^2 - 2)^{3/2}$ from $x = \sqrt{2}$ to $x = 3$

$$y' = x\sqrt{x^2 - 2} \implies 1 + (y')^2 = 1 + x^2(x^2 - 2) = x^4 - 2x^2 + 1 = (x^2 - 1)^2$$
$$L = \int_{\sqrt{2}}^3 (x^2 - 1)dx = [\frac{x^3}{3} - x]_{\sqrt{2}}^3 = (9 - 3) - (\frac{2\sqrt{2}}{3} - \sqrt{2}) = 6 + \frac{\sqrt{2}}{3}$$

Length: $6 + \sqrt{2}/3$.

9. **Curve:** $y = \ln(\csc x - \cot x)$ from $x = \pi/6$ to $x = \pi/2$

$$y' = \frac{-\csc x \cot x + \csc^2 x}{\csc x - \cot x} = \frac{\csc x(\csc x - \cot x)}{\csc x - \cot x} = \csc x$$
$$1 + (y')^2 = 1 + \csc^2 x \quad \text{This does not simplify well. Typo in problem design.}$$

Corrected Curve: $y = \ln(\sin x)$ from $x = \pi/4$ to $x = \pi/2$.

$$y' = \frac{\cos x}{\sin x} = \cot x \implies 1 + (y')^2 = 1 + \cot^2 x = \csc^2 x$$
$$L = \int_{\pi/4}^{\pi/2} \csc x dx = [-\ln|\csc x + \cot x|]_{\pi/4}^{\pi/2}$$
$$= (-\ln|1 + 0|) - (-\ln|\sqrt{2} + 1|) = \ln(\sqrt{2} + 1)$$

Length: $\ln(\sqrt{2} + 1)$.

10. **Curve:** $y = \frac{x^5}{10} + \frac{1}{6x^3}$ from $x = 1$ to $x = 2$

$$y' = \frac{x^4}{2} - \frac{1}{2x^4} \implies 1 + (y')^2 = 1 + (\frac{x^8}{4} - \frac{1}{2} + \frac{1}{4x^8}) = (\frac{x^4}{2} + \frac{1}{2x^4})^2$$
$$L = \int_1^2 (\frac{x^4}{2} + \frac{1}{2x^4})dx = [\frac{x^5}{10} - \frac{1}{6x^3}]_1^2 = (\frac{32}{10} - \frac{1}{48}) - (\frac{1}{10} - \frac{1}{6}) = \frac{779}{240}$$

Length: $779/240$.

3. Surface Area of Revolution

1. **Curve:** $y = \sqrt{9 - x^2}$, $0 \leq x \leq 3$; about the x-axis.

$$y' = \frac{-x}{\sqrt{9 - x^2}} \implies 1 + (y')^2 = 1 + \frac{x^2}{9 - x^2} = \frac{9}{9 - x^2}$$

$$S = \int_0^3 2\pi\sqrt{9 - x^2}\sqrt{\frac{9}{9 - x^2}}dx = \int_0^3 2\pi(3)dx = [6\pi x]_0^3 = 18\pi$$

Surface Area: 18π. (Surface of a hemisphere).

2. **Curve:** $y = x^3$ from $x = 0$ to $x = 1$; about the x-axis.

$$y' = 3x^2 \implies 1 + (y')^2 = 1 + 9x^4$$

$$S = \int_0^1 2\pi x^3\sqrt{1 + 9x^4}dx \quad (u = 1 + 9x^4, du = 36x^3dx)$$

$$= \frac{2\pi}{36}\int_1^{10} \sqrt{u}du = \frac{\pi}{18}[\frac{2}{3}u^{3/2}]_1^{10} = \frac{\pi}{27}(10\sqrt{10} - 1)$$

Surface Area: $\frac{\pi}{27}(10\sqrt{10} - 1)$.

3. **Curve:** $y = \sqrt{x}$, $1 \leq x \leq 4$; about the x-axis.

$$y' = \frac{1}{2\sqrt{x}} \implies 1 + (y')^2 = 1 + \frac{1}{4x} = \frac{4x + 1}{4x}$$

$$S = \int_1^4 2\pi\sqrt{x}\sqrt{\frac{4x + 1}{4x}}dx = \int_1^4 \pi\sqrt{4x + 1}dx \quad (u = 4x + 1)$$

$$= \frac{\pi}{4}\int_5^{17} \sqrt{u}du = \frac{\pi}{4}[\frac{2}{3}u^{3/2}]_5^{17} = \frac{\pi}{6}(17\sqrt{17} - 5\sqrt{5})$$

Surface Area: $\frac{\pi}{6}(17\sqrt{17} - 5\sqrt{5})$.

4. **Curve:** $y = \cosh(x)$, $0 \leq x \leq 1$; about the x-axis.

$y' = \sinh(x) \implies 1 + (y')^2 = \cosh^2(x)$

$$S = \int_0^1 2\pi\cosh(x)\sqrt{\cosh^2(x)}dx = 2\pi\int_0^1 \cosh^2(x)dx$$

$$= \pi\int_0^1 (1 + \cosh(2x))dx = \pi[x + \frac{1}{2}\sinh(2x)]_0^1 = \pi(1 + \frac{1}{2}\sinh$$

Surface Area: $\pi(1 + \frac{1}{2}\sinh 2)$.

5. **Curve:** $y = e^{-x}$, $0 \leq x \leq \infty$; about the x-axis.

$$y' = -e^{-x} \implies 1 + (y')^2 = 1 + e^{-2x}$$

$$S = \int_0^\infty 2\pi e^{-x}\sqrt{1 + e^{-2x}}dx \quad (u = e^{-x}, du = -e^{-x}dx)$$

$$= 2\pi\int_1^0 \sqrt{1 + u^2}(-du) = 2\pi\int_0^1 \sqrt{1 + u^2}du \quad (\text{Trig sub } u = \tan\theta)$$

$$= 2\pi[\frac{u}{2}\sqrt{1 + u^2} + \frac{1}{2}\ln|u + \sqrt{1 + u^2}|]_0^1 = \pi(\sqrt{2} + \ln(1 + \sqrt{2}))$$

Surface Area: $\pi(\sqrt{2} + \ln(1 + \sqrt{2}))$.

6. **Curve:** $y = \frac{x^3}{6} + \frac{1}{2x}$, $1 \le x \le 2$; about the x-axis.

$$y' = \frac{x^2}{2} - \frac{1}{2x^2} \implies 1 + (y')^2 = (\frac{x^2}{2} + \frac{1}{2x^2})^2$$

$$\begin{aligned} S &= \int_1^2 2\pi(\frac{x^3}{6} + \frac{1}{2x})(\frac{x^2}{2} + \frac{1}{2x^2})dx \\ &= 2\pi \int_1^2 (\frac{x^5}{12} + \frac{x}{12} + \frac{x}{4} + \frac{1}{4x^3})dx = 2\pi[\frac{x^6}{72} + \frac{x^2}{6} - \frac{1}{8x^2}]_1^2 \end{aligned}$$

Surface Area: $47\pi/16$.

7. **Curve:** $x = \frac{1}{3}(y^2+2)^{3/2}$, $1 \le y \le 2$; about the y-axis.

$$\begin{aligned} \frac{dx}{dy} &= y\sqrt{y^2+2} \implies 1 + (dx/dy)^2 = (y^2+1)^2 \\ S &= \int_1^2 2\pi y\sqrt{(y^2+1)^2}dy = \int_1^2 2\pi y(y^2+1)dy \\ &= 2\pi[\frac{y^4}{4} + \frac{y^2}{2}]_1^2 = 2\pi[(4+2) - (\frac{1}{4} + \frac{1}{2})] = \frac{21\pi}{2} \end{aligned}$$

Surface Area: $21\pi/2$.

8. **Curve:** $y = 1 - x^2$, $0 \le x \le 1$; about the y-axis.

$$\begin{aligned} S = \int_0^1 2\pi x\sqrt{1+(-2x)^2}dx &= 2\pi \int_0^1 x\sqrt{1+4x^2}dx \quad (u = 1+4x^2) \\ &= \frac{2\pi}{8}\int_1^5 \sqrt{u}du = \frac{\pi}{4}[\frac{2}{3}u^{3/2}]_1^5 = \frac{\pi}{6}(5\sqrt{5}-1) \end{aligned}$$

Surface Area: $\frac{\pi}{6}(5\sqrt{5}-1)$.

9. **Curve:** $y = \sin x$, $0 \le x \le \pi$; about the x-axis.

$$\begin{aligned} S &= \int_0^\pi 2\pi \sin x\sqrt{1+\cos^2 x}dx \quad (u = \cos x, du = -\sin x dx) \\ &= 2\pi \int_{-1}^1 \sqrt{1+u^2}du = 2\pi[\frac{u}{2}\sqrt{1+u^2} + \frac{1}{2}\ln|u + \sqrt{1+u^2}|]_{-1}^1 \\ &= 2\pi(\sqrt{2} + \ln(1+\sqrt{2})) \end{aligned}$$

Surface Area: $2\pi(\sqrt{2} + \ln(1+\sqrt{2}))$.

10. **Curve:** $y = \frac{1}{4}x^4 + \frac{1}{8}x^{-2}$, $1 \le x \le 2$; about the y-axis.

$$\begin{aligned} y' &= x^3 - \frac{1}{4}x^{-3} \implies 1 + (y')^2 = (x^3 + \frac{1}{4}x^{-3})^2 \\ S = \int_1^2 2\pi x\sqrt{(x^3 + \frac{1}{4}x^{-3})^2}dx &= \int_1^2 2\pi x(x^3 + \frac{1}{4}x^{-3})dx \\ &= 2\pi \int_1^2 (x^4 + \frac{1}{4x^2})dx = 2\pi[\frac{x^5}{5} - \frac{1}{4x}]_1^2 = \frac{253\pi}{20} \end{aligned}$$

Surface Area: $253\pi/20$.

4. Parametric to Cartesian Equations

1. **Equations:** $x = 3\cos t, y = 5\sin t, 0 \le t \le 2\pi$

$$(\frac{x}{3})^2 + (\frac{y}{5})^2 = \cos^2 t + \sin^2 t = 1 \implies \frac{x^2}{9} + \frac{y^2}{25} = 1$$

Path: Ellipse centered at (0,0), major vertical axis length 10, minor horizontal axis length 6. **Direction:** At $t = 0, (3, 0)$. At $t = \pi/2, (0, 5)$. Counter-clockwise.

2. **Equations:** $x = t^2, y = t^3 - 3t$

$$y = t(t^2 - 3) = \pm\sqrt{x}(x - 3) \implies y^2 = x(x - 3)^2$$

Path: A self-intersecting cubic curve. **Direction:** For $t < 0$, starts at top-left, moves to (3,0). For $t > 0$, moves away from (3,0) to top-right.

3. **Equations:** $x = 3 + 2\sec t, y = 1 + 4\tan t$

$$\sec t = \frac{x-3}{2}, \tan t = \frac{y-1}{4}. \quad \sec^2 t - \tan^2 t = 1 \implies (\frac{x-3}{2})^2 - (\frac{y-1}{4})^2 = 1$$

Path: Hyperbola centered at (3,1).

4. **Equations:** $x = e^t, y = e^{-2t}$

$$y = (e^t)^{-2} = x^{-2} = \frac{1}{x^2}. \text{ Since } x = e^t > 0, \text{ it's only the right branch.}$$

Path: Part of the reciprocal square function in the first quadrant.

5. **Equations:** $x = 2\sin t - 1, y = 3\cos t + 2, 0 \le t \le 2\pi$

$$(\frac{x+1}{2})^2 + (\frac{y-2}{3})^2 = 1$$

Path: Ellipse centered at (-1, 2). **Direction:** At $t = 0, (-1, 5)$. At $t = \pi/2, (1, 2)$. Clockwise.

6. **Equations:** $x = \sqrt{t}, y = 1 - t$

$$t = x^2 \implies y = 1 - x^2. \text{ Since } x = \sqrt{t} \ge 0, \text{ it's the right half of the parabola.}$$

Path: Parabola opening down, vertex at (0,1), for $x \ge 0$.

7. **Equations:** $x = \sin t, y = \csc t, 0 < t < \pi$

$$y = \frac{1}{\sin t} = \frac{1}{x}$$

Path: Reciprocal function for $x \in (0, 1]$.

8. **Equations:** $x = \cos(2t), y = \sin t$

$$x = 1 - 2\sin^2 t = 1 - 2y^2$$

Path: Parabola opening left, vertex at (1,0).

9. **Equations:** $x = 4t^2 - 4, y = t, -\infty < t < \infty$

$$x = 4y^2 - 4$$

Path: Parabola opening right, vertex at (-4,0).

10. **Equations:** $x = t - 1, y = t^2 - 2t + 2$

$$t = x + 1 \implies y = (x+1)^2 - 2(x+1) + 2 = x^2 + 2x + 1 - 2x - 2 + 2 = x^2 + 1$$

Path: Parabola opening up, vertex at (0,1).

5. Parametric Derivatives

1. **Equations:** $x = t^3 - 3t, y = t^2 - 2$

$$\frac{dx}{dt} = 3t^2 - 3, \frac{dy}{dt} = 2t \implies \frac{dy}{dx} = \frac{2t}{3(t^2-1)}$$

$$\frac{d}{dt}(\frac{dy}{dx}) = \frac{2(3t^2-3) - 2t(6t)}{9(t^2-1)^2} = \frac{-6t^2-6}{9(t^2-1)^2} = \frac{-2(t^2+1)}{3(t^2-1)^2}$$

$$\frac{d^2y}{dx^2} = \frac{-2(t^2+1)}{3(t^2-1)^2} \cdot \frac{1}{3(t^2-1)} = \frac{-2(t^2+1)}{9(t^2-1)^3}$$

2. **Equations:** $x = e^t, y = te^{-t}$

$$\frac{dx}{dt} = e^t, \frac{dy}{dt} = e^{-t} - te^{-t} \implies \frac{dy}{dx} = \frac{e^{-t}(1-t)}{e^t} = e^{-2t}(1-t)$$

$$\frac{d}{dt}(\frac{dy}{dx}) = -2e^{-2t}(1-t) + e^{-2t}(-1) = e^{-2t}(2t-3)$$

$$\frac{d^2y}{dx^2} = \frac{e^{-2t}(2t-3)}{e^t} = e^{-3t}(2t-3)$$

3. **Equations:** $x = 2\cos t, y = 3\sin t$

$$\frac{dx}{dt} = -2\sin t, \frac{dy}{dt} = 3\cos t \implies \frac{dy}{dx} = -\frac{3\cos t}{2\sin t} = -\frac{3}{2}\cot t$$

$$\frac{d}{dt}(\frac{dy}{dx}) = \frac{3}{2}\csc^2 t$$

$$\frac{d^2y}{dx^2} = \frac{(3/2)\csc^2 t}{-2\sin t} = -\frac{3}{4\sin^3 t} = -\frac{3}{4}\csc^3 t$$

4. **Equations:** $x = t/(1+t), y = t^2$

$$\frac{dx}{dt} = \frac{1}{(1+t)^2}, \frac{dy}{dt} = 2t \implies \frac{dy}{dx} = 2t(1+t)^2$$

$$\frac{d}{dt}(\frac{dy}{dx}) = 2(1+t)^2 + 2t(2(1+t)) = 2(1+t)(1+t+2t) = 2(1+t)(1+3t)$$

$$\frac{d^2y}{dx^2} = 2(1+t)(1+3t) \cdot (1+t)^2 = 2(1+t)^3(1+3t)$$

5. **Equations:** $x = \ln t, y = t^2 + 1$

$$\frac{dx}{dt} = 1/t, \frac{dy}{dt} = 2t \implies \frac{dy}{dx} = \frac{2t}{1/t} = 2t^2$$

$$\frac{d}{dt}(\frac{dy}{dx}) = 4t \implies \frac{d^2y}{dx^2} = \frac{4t}{1/t} = 4t^2$$

6. **Equations:** $x = a(t - \sin t), y = a(1 - \cos t)$

$$\frac{dx}{dt} = a(1-\cos t), \frac{dy}{dt} = a\sin t \implies \frac{dy}{dx} = \frac{\sin t}{1-\cos t} = \cot(t/2)$$

$$\frac{d}{dt}(\frac{dy}{dx}) = -\frac{1}{2}\csc^2(t/2)$$

$$\frac{d^2y}{dx^2} = \frac{-\frac{1}{2}\csc^2(t/2)}{a(1-\cos t)} = \frac{-\frac{1}{2}\csc^2(t/2)}{a(2\sin^2(t/2))} = -\frac{1}{4a}\csc^4(t/2)$$

7. **Equations:** $x = t^2 + 1, y = t^3 - 1$

$$\frac{dx}{dt} = 2t, \frac{dy}{dt} = 3t^2 \implies \frac{dy}{dx} = \frac{3t}{2}$$
$$\frac{d}{dt}(\frac{dy}{dx}) = \frac{3}{2} \implies \frac{d^2y}{dx^2} = \frac{3/2}{2t} = \frac{3}{4t}$$

8. **Equations:** $x = \cos^3 t, y = \sin^3 t$

$$\frac{dx}{dt} = -3\cos^2 t \sin t, \frac{dy}{dt} = 3\sin^2 t \cos t \implies \frac{dy}{dx} = -\tan t$$
$$\frac{d}{dt}(\frac{dy}{dx}) = -\sec^2 t \implies \frac{d^2y}{dx^2} = \frac{-\sec^2 t}{-3\cos^2 t \sin t} = \frac{1}{3\cos^4 t \sin t}$$

9. **Equations:** $x = \sqrt{t}, y = t$

$$\frac{dx}{dt} = \frac{1}{2\sqrt{t}}, \frac{dy}{dt} = 1 \implies \frac{dy}{dx} = 2\sqrt{t}$$
$$\frac{d}{dt}(\frac{dy}{dx}) = \frac{1}{\sqrt{t}} \implies \frac{d^2y}{dx^2} = \frac{1/\sqrt{t}}{1/(2\sqrt{t})} = 2$$

(Also clear from $y = x^2 \implies y'' = 2$).

10. **Equations:** $x = \arctan t, y = t^2$

$$\frac{dx}{dt} = \frac{1}{1+t^2}, \frac{dy}{dt} = 2t \implies \frac{dy}{dx} = 2t(1+t^2) = 2t + 2t^3$$
$$\frac{d}{dt}(\frac{dy}{dx}) = 2 + 6t^2 \implies \frac{d^2y}{dx^2} = \frac{2+6t^2}{1/(1+t^2)} = (2+6t^2)(1+t^2)$$

6. Tangent Lines to Parametric Curves

1. **Curve:** $x = t^2 + 1, y = t^3 + t$ at $t = 2$.
   - **Point:** $x(2) = 5, y(2) = 10$. Point is $(5, 10)$.
   - **Slope:** $\frac{dx}{dt} = 2t, \frac{dy}{dt} = 3t^2 + 1 \implies \frac{dy}{dx} = \frac{3t^2+1}{2t}$. At $t = 2, m = \frac{13}{4}$.
   - **Equation:** $y - 10 = \frac{13}{4}(x - 5)$.
2. **Curve:** $x = \cos t, y = \sin(2t)$ at $t = \pi/3$.
   - **Point:** $x(\pi/3) = 1/2, y(\pi/3) = \sqrt{3}/2$. Point is $(1/2, \sqrt{3}/2)$.
   - **Slope:** $\frac{dx}{dt} = -\sin t, \frac{dy}{dt} = 2\cos(2t) \implies \frac{dy}{dx} = \frac{2\cos(2t)}{-\sin t}$. At $t = \pi/3, m = \frac{2(-1/2)}{-\sqrt{3}/2} = \frac{2}{\sqrt{3}}$.
   - **Equation:** $y - \frac{\sqrt{3}}{2} = \frac{2}{\sqrt{3}}(x - \frac{1}{2})$.
3. **Curve:** $x = e^t, y = e^{-t}$ at $t = 1$.
   - **Point:** $x(1) = e, y(1) = 1/e$. Point is $(e, 1/e)$.
   - **Slope:** $\frac{dy}{dx} = \frac{-e^{-t}}{e^t} = -e^{-2t}$. At $t = 1, m = -e^{-2}$.
   - **Equation:** $y - 1/e = -e^{-2}(x - e)$.
4. **Curve:** $x = 4\sin t, y = 2\cos t$ at $t = \pi/4$.
   - **Point:** $x = 4(\sqrt{2}/2) = 2\sqrt{2}, y = 2(\sqrt{2}/2) = \sqrt{2}$. Point is $(2\sqrt{2}, \sqrt{2})$.
   - **Slope:** $\frac{dy}{dx} = \frac{-2\sin t}{4\cos t} = -\frac{1}{2}\tan t$. At $t = \pi/4, m = -1/2$.
   - **Equation:** $y - \sqrt{2} = -\frac{1}{2}(x - 2\sqrt{2})$.
5. **Curve:** $x = t^3 - 1, y = t^2 + t$ at $t = -1$.
   - **Point:** $x(-1) = -2, y(-1) = 0$. Point is $(-2, 0)$.
   - **Slope:** $\frac{dy}{dx} = \frac{2t+1}{3t^2}$. At $t = -1, m = \frac{-1}{3}$.
   - **Equation:** $y - 0 = -\frac{1}{3}(x + 2)$.
6. **Curve:** $x = \sec t, y = \csc t$ at $t = \pi/3$.
   - **Point:** $x = 2, y = 2/\sqrt{3}$. Point is $(2, 2/\sqrt{3})$.
   - **Slope:** $\frac{dy}{dx} = \frac{-\csc t \cot t}{\sec t \tan t} = -\cot^3 t$. At $t = \pi/3, m = -(1/\sqrt{3})^3 = -1/(3\sqrt{3})$.
   - **Equation:** $y - 2/\sqrt{3} = -1/(3\sqrt{3})(x - 2)$.
7. **Curve:** $x = 1 + \ln t, y = t^2 + 2$ at $t = 1$.
   - **Point:** $x = 1, y = 3$. Point is $(1, 3)$.
   - **Slope:** $\frac{dy}{dx} = \frac{2t}{1/t} = 2t^2$. At $t = 1, m = 2$.
   - **Equation:** $y - 3 = 2(x - 1)$.
8. **Curve:** $x = t\cos t, y = t\sin t$ at $t = \pi$.
   - **Point:** $x = -\pi, y = 0$. Point is $(-\pi, 0)$.
   - **Slope:** $\frac{dy}{dx} = \frac{\sin t + t\cos t}{\cos t - t\sin t}$. At $t = \pi, m = \frac{-\pi}{-1} = \pi$.
   - **Equation:** $y - 0 = \pi(x + \pi)$.

9. **Curve:** $x = t - \sin t, y = 1 - \cos t$ at $t = \pi/2$.
    - **Point:** $x = \pi/2 - 1, y = 1$. Point is $(\pi/2 - 1, 1)$.
    - **Slope:** $\frac{dy}{dx} = \frac{\sin t}{1-\cos t}$. At $t = \pi/2, m = 1/1 = 1$.
    - **Equation:** $y - 1 = 1(x - (\pi/2 - 1))$.
10. **Curve:** $x = t^2, y = t^3 - 3t$ at $t = \sqrt{3}$.
    - **Point:** $x = 3, y = 0$. Point is $(3, 0)$.
    - **Slope:** $\frac{dy}{dx} = \frac{3t^2-3}{2t}$. At $t = \sqrt{3}, m = \frac{9-3}{2\sqrt{3}} = \frac{6}{2\sqrt{3}} = \sqrt{3}$.
    - **Equation:** $y - 0 = \sqrt{3}(x - 3)$.

7. Parametric Arc Length

1. **Curve:** $x = e^t \cos t, y = e^t \sin t, 0 \le t \le \pi$.

$$\begin{aligned} (dx/dt)^2 + (dy/dt)^2 &= (e^t(\cos t - \sin t))^2 + (e^t(\sin t + \cos t))^2 \\ &= e^{2t}(\cos^2 t - 2\cos t \sin t + \sin^2 t + \sin^2 t + 2\sin t \cos t + \cos^2 \\ &= e^{2t}(2) \implies \sqrt{\dots} = \sqrt{2}e^t \end{aligned}$$

$$L = \int_0^\pi \sqrt{2}e^t dt = [\sqrt{2}e^t]_0^\pi = \sqrt{2}(e^\pi - 1)$$

Length: $\sqrt{2}(e^\pi - 1)$.

2. **Curve:** $x = \cos^3 t, y = \sin^3 t, 0 \le t \le \pi/2$.

$$\begin{aligned} (dx/dt)^2 + (dy/dt)^2 &= (-3\cos^2 t \sin t)^2 + (3\sin^2 t \cos t)^2 = 9\cos^4 t \sin^2 t + 9\sin^4 \\ &= 9\cos^2 t \sin^2 t(\cos^2 t + \sin^2 t) = (3\cos t \sin t)^2 \end{aligned}$$

$$L = \int_0^{\pi/2} 3\cos t \sin t dt = [\frac{3}{2}\sin^2 t]_0^{\pi/2} = \frac{3}{2}$$

Length: $3/2$.

3. **Curve:** $x = \frac{1}{3}t^3, y = \frac{1}{2}t^2, 0 \le t \le 1$.

$$\begin{aligned} (dx/dt)^2 + (dy/dt)^2 &= (t^2)^2 + (t)^2 = t^4 + t^2 = t^2(t^2 + 1) \\ L &= \int_0^1 t\sqrt{t^2 + 1}dt \quad (u = t^2 + 1, du = 2tdt) \\ &= \frac{1}{2}\int_1^2 \sqrt{u}du = \frac{1}{2}[\frac{2}{3}u^{3/2}]_1^2 = \frac{1}{3}(2\sqrt{2} - 1) \end{aligned}$$

Length: $\frac{1}{3}(2\sqrt{2} - 1)$.

4. **Curve:** $x = a(\cos t + t\sin t), y = a(\sin t - t\cos t), 0 \le t \le \pi$.

$$\begin{aligned} dx/dt &= a(at\cos t), dy/dt = a(at\sin t) \\ (dx/dt)^2 + (dy/dt)^2 &= (at\cos t)^2 + (at\sin t)^2 = a^2t^2 \\ L &= \int_0^\pi at\, dt = [at^2/2]_0^\pi = \frac{a\pi^2}{2} \end{aligned}$$

Length: $a\pi^2/2$.

5. **Curve:** $x = 2t, y = \frac{2}{3}t^{3/2}, 0 \le t \le 3$.

$$\begin{aligned} (dx/dt)^2 + (dy/dt)^2 &= (2)^2 + (t^{1/2})^2 = 4 + t \\ L &= \int_0^3 \sqrt{4 + t}dt = [\frac{2}{3}(4 + t)^{3/2}]_0^3 = \frac{2}{3}(7\sqrt{7} - 8) \end{aligned}$$

Length: $\frac{2}{3}(7\sqrt{7} - 8)$.

6. **Curve:** $x = t^2, y = \frac{2}{3}(2t + 1)^{3/2}, 0 \le t \le 4$. (Note: This is similar to 7a from review)

$$dx/dt = 2t, dy/dt = 2\sqrt{2t + 1} \implies (dx/dt)^2 + (dy/dt)^2 = 4t^2 + 4(2t + 1) = 4t^2 + 8t + 4 = (2t + 2)^2$$

$$L = \int_0^4 (2t + 2)dt = [t^2 + 2t]_0^4 = 16 + 8 = 24$$

Length: 24.

7. **Curve:** $x = 1 + 3t^2, y = 4 + 2t^3, 0 \leq t \leq 1.$

$$\begin{aligned}(dx/dt)^2 + (dy/dt)^2 &= (6t)^2 + (6t^2)^2 = 36t^2(1+t^2) \\ L &= \int_0^1 6t\sqrt{1+t^2}dt \quad (u = 1+t^2, du = 2tdt) \\ &= 3\int_1^2 \sqrt{u}du = 3[\frac{2}{3}u^{3/2}]_1^2 = 2(2\sqrt{2}-1)\end{aligned}$$

Length: $2(2\sqrt{2}-1)$.

8. **Curve:** $x = t - \sin t, y = 1 - \cos t, 0 \leq t \leq 2\pi.$

$$\begin{aligned}(dx/dt)^2 + (dy/dt)^2 = (1-\cos t)^2 + \sin^2 t = 1 - 2\cos t + \cos^2 t + \sin^2 t = 2 - 2\cos t \\ = 4\sin^2(t/2) \implies \sqrt{\ldots} = 2\sin(t/2) \text{ on } [0, 2\pi] \\ L = \int_0^{2\pi} 2\sin(t/2)dt = [-4\cos(t/2)]_0^{2\pi} = -4(-1) - (-4(1)) = 8\end{aligned}$$

Length: 8.

9. **Curve:** $x = 3t, y = t^3, 0 \leq t \leq 2.$

$$\begin{aligned}(dx/dt)^2 + (dy/dt)^2 = (3)^2 + (3t^2)^2 = 9(1+t^4) \\ L = \int_0^2 3\sqrt{1+t^4}dt \quad \text{(Cannot be solved with elementary functions)}\end{aligned}$$

Result: Problem is not well-posed for this level.

10. **Curve:** $x = e^t - t, y = 4e^{t/2}, 0 \leq t \leq 1.$

$$\begin{aligned}dx/dt = e^t - 1, dy/dt = 2e^{t/2} \\ (dx/dt)^2 + (dy/dt)^2 = (e^{2t} - 2e^t + 1) + 4e^t = e^{2t} + 2e^t + 1 = (e^t+1)^2 \\ L = \int_0^1 (e^t+1)dt = [e^t + t]_0^1 = (e+1) - (1+0) = e\end{aligned}$$

Length: e.

8. Vertical Tangents

1. **Curve:** $x = t^3 - 3t, y = t^2 - 3$.
   - $\frac{dx}{dt} = 3t^2 - 3 = 3(t-1)(t+1)$. $dx/dt = 0$ at $t = 1, t = -1$.
   - $\frac{dy}{dt} = 2t$.
   - At $t = 1, dy/dt = 2 \neq 0$. At $t = -1, dy/dt = -2 \neq 0$.

   Result: Vertical tangents at $t = 1$ and $t = -1$.

2. **Curve:** $x = t^2 - t, y = t^3 - 3t$.
   - $\frac{dx}{dt} = 2t - 1$. $dx/dt = 0$ at $t = 1/2$.
   - $\frac{dy}{dt} = 3t^2 - 3$. At $t = 1/2, dy/dt = 3/4 - 3 \neq 0$.

   Result: Vertical tangent at $t = 1/2$.

3. **Curve:** $x = 2\cos t, y = \sin(2t)$.
   - $\frac{dx}{dt} = -2\sin t$. $dx/dt = 0$ at $t = k\pi$ for integer k.
   - $\frac{dy}{dt} = 2\cos(2t)$. At $t = k\pi, \cos(2k\pi) = 1, dy/dt = 2 \neq 0$.

   Result: Vertical tangents at $t = k\pi$.

4. **Curve:** $x = t - \sin t, y = 1 - \cos t$.
   - $\frac{dx}{dt} = 1 - \cos t$. $dx/dt = 0$ at $t = 2k\pi$.
   - $\frac{dy}{dt} = \sin t$. At $t = 2k\pi, dy/dt = 0$. The slope is indeterminate (0/0).

   Result: No vertical tangents (cusps at these points).

5. **Curve:** $x = t^4 - 2t^2, y = t^3 - t$.
   - $\frac{dx}{dt} = 4t^3 - 4t = 4t(t-1)(t+1)$. $dx/dt = 0$ at $t = 0, 1, -1$.
   - $\frac{dy}{dt} = 3t^2 - 1$. At $t = 0, dy/dt = -1 \neq 0$. At $t = \pm 1, dy/dt = 2 \neq 0$.

   Result: Vertical tangents at $t = 0, 1, -1$.

6. **Curve:** $x = te^t, y = t^2 - t$.
   - $\frac{dx}{dt} = e^t + te^t = e^t(1+t)$. $dx/dt = 0$ at $t = -1$.
   - $\frac{dy}{dt} = 2t - 1$. At $t = -1, dy/dt = -3 \neq 0$.

   Result: Vertical tangent at $t = -1$.

7. **Curve:** $x = \sin t, y = \cos t$.
   - $\frac{dx}{dt} = \cos t$. $dx/dt = 0$ at $t = \pi/2 + k\pi$.
   - $\frac{dy}{dt} = -\sin t$. At these t, $|\sin t| = 1, dy/dt \neq 0$.

   Result: Vertical tangents at $t = \pi/2 + k\pi$.

8. **Curve:** $x = t^2, y = t^3 - 3t$.

- $\frac{dx}{dt} = 2t$. $dx/dt = 0$ at $t = 0$.
- $\frac{dy}{dt} = 3t^2 - 3$. At $t = 0, dy/dt = -3 \neq 0$.

Result: Vertical tangent at $t = 0$.

9. **Curve:** $x = \ln(t^2 + 1), y = t - 2$.

- $\frac{dx}{dt} = \frac{2t}{t^2+1}$. $dx/dt = 0$ at $t = 0$.
- $\frac{dy}{dt} = 1 \neq 0$ for all t.

Result: Vertical tangent at $t = 0$.

10. **Curve:** $x = t^3 - 12t, y = t^2 - 1$.

- $\frac{dx}{dt} = 3t^2 - 12 = 3(t-2)(t+2)$. $dx/dt = 0$ at $t = \pm 2$.
- $\frac{dy}{dt} = 2t$. At $t = \pm 2, dy/dt = \pm 4 \neq 0$.

Result: Vertical tangents at $t = 2$ and $t = -2$.

9. Particle at Rest

1. **Curve:** $x = t^3 - 3t^2, y = t^3 - 3t$.
   - $\frac{dx}{dt} = 3t^2 - 6t = 3t(t-2)$. Roots: $t = 0, 2$.
   - $\frac{dy}{dt} = 3t^2 - 3 = 3(t-1)(t+1)$. Roots: $t = 1, -1$.

   Result: No common roots. Particle is never at rest.

2. **Curve:** $x = \cos t, y = \sin(2t)$.
   - $\frac{dx}{dt} = -\sin t$. Roots: $t = k\pi$.
   - $\frac{dy}{dt} = 2\cos(2t)$. At $t = k\pi, \cos(2k\pi) = 1, dy/dt \neq 0$.

   Result: Never at rest.

3. **Curve:** $x = t^2 - 4t, y = t^3 - 12t$.
   - $\frac{dx}{dt} = 2t - 4$. Root: $t = 2$.
   - $\frac{dy}{dt} = 3t^2 - 12 = 3(t-2)(t+2)$. Roots: $t = 2, -2$.

   Result: Common root is $t = 2$. Particle is at rest at $t = 2$.

4. **Curve:** $x = \sin t, y = \sin t$.
   - $\frac{dx}{dt} = \cos t$, $\frac{dy}{dt} = \cos t$. Both are zero at $t = \pi/2 + k\pi$.

   Result: At rest at $t = \pi/2 + k\pi$.

5. **Curve:** $x = t^4 - 2t^2, y = t^3 - 3t^2$.
   - $\frac{dx}{dt} = 4t^3 - 4t = 4t(t-1)(t+1)$. Roots: $t = 0, 1, -1$.
   - $\frac{dy}{dt} = 3t^2 - 6t = 3t(t-2)$. Roots: $t = 0, 2$.

   Result: Common root is $t = 0$. At rest at $t = 0$.

6. **Curve:** $x = t^2 - 1, y = t^3 - t$.
   - $\frac{dx}{dt} = 2t$. Root: $t = 0$.
   - $\frac{dy}{dt} = 3t^2 - 1$. At $t = 0, dy/dt = -1 \neq 0$.

   Result: Never at rest.

7. **Curve:** $x = \sin t - t, y = \cos t - 1$.
   - $\frac{dx}{dt} = \cos t - 1$. Roots: $t = 2k\pi$.
   - $\frac{dy}{dt} = -\sin t$. Roots: $t = k\pi$.

   Result: Common roots are $t = 2k\pi$. At rest at $t = 2k\pi$.

8. **Curve:** $x = t^3/3 - t, y = t^2 - 1$.
   - $\frac{dx}{dt} = t^2 - 1 = (t-1)(t+1)$. Roots: $t = 1, -1$.
   - $\frac{dy}{dt} = 2t$. Root: $t = 0$.

Result: Never at rest.

9. **Curve:** $x = t^3 - 3t, y = t^3 - 12t$.

   - $\frac{dx}{dt} = 3t^2 - 3 = 3(t-1)(t+1)$. Roots: $t = 1, -1$.
   - $\frac{dy}{dt} = 3t^2 - 12 = 3(t-2)(t+2)$. Roots: $t = 2, -2$.

   Result: Never at rest.

10. **Curve:** $x = t^2 - 2t, y = t^3 - 3t^2 + 2t$.

    - $\frac{dx}{dt} = 2t - 2 = 2(t-1)$. Root: $t = 1$.
    - $\frac{dy}{dt} = 3t^2 - 6t + 2$. At $t = 1, dy/dt = 3 - 6 + 2 = -1 \neq 0$.

    Result: Never at rest.

Concept Check List

This list summarizes the concepts tested. The numbers refer to the problem sets generated in this document.

- **Improper Integrals**
  - Type 1 (Infinite Limit) requiring partial fractions: **1.a** (1-10)
  - Type 2 (Discontinuity) requiring u-substitution: **1.b** (1-10)
  - Type 2 (Discontinuity) from a vertical asymptote: **1.c** (1-10)
  - Type 2 (Discontinuity) solved with standard forms (arcsin, etc.): **1.d** (1-10)
  - Type 1 (Double Infinite Limits), possibly using symmetry: **1.e** (1-10)
- **Arc Length (Cartesian)**
  - Integrand $\sqrt{1+(y')^2}$ simplifies to a perfect square: **2** (1-10)
- **Surface Area of Revolution (Cartesian)**
  - Calculating surface area for various curves, requiring algebraic simplification and u-substitution: **3** (1-10)
- **Parametric Equations**
  - Eliminating the parameter to find the Cartesian equation (circles, ellipses, parabolas, hyperbolas, etc.): **4** (1-10)
- **Calculus with Parametric curves**
  - Finding first and second derivatives ($\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$): **5** (1-10)
  - Finding the equation of a tangent line at a point: **6** (1-10)
  - Calculating arc length, often involving perfect squares or u-substitution: **7** (1-10)
  - Finding points of vertical tangents ($\frac{dx}{dt} = 0, \frac{dy}{dt} \neq 0$): **8** (1-10)
  - Finding when a particle is at rest ($\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$): **9** (1-10)