Problem Set 11.10: Taylor and Maclaurin Series

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Problem Set

1 Direct Definition and Basic Concepts

1.1 Problems

1. If a function $f(x)$ is represented by the power series $\sum_{n=0}^{\infty} c_n(x-5)^n$, what is the formula for the coefficient c_7?
2. A Taylor series for a function $f(x)$ is centered at $a=-2$. If the series is given by $\sum_{n=0}^{\infty} b_n(x+2)^n$, write the formula for b_{10}.
3. Given that the Maclaurin series for a function $g(x)$ is $\sum_{n=0}^{\infty} a_n x^n$, provide the explicit formula for the coefficient a_5.
4. The third-degree Taylor polynomial for a function $f(x)$ centered at $a=1$ is $T_3(x) = 2-(x-1)+3(x-1)^2-5(x-1)^3$. What are the values of $f(1)$, $f'(1)$, $f''(1)$, and $f'''(1)$?
5. Let $f(x)$ have a Taylor series centered at $a=0$. If the series begins $3+2x-\frac{1}{2}x^2+\frac{5}{3}x^3+\ldots$, what is the value of $f^{(3)}(0)$?

2 Constructing Series from a Derivative Formula

2.1 Problems

6. Given $f^{(n)}(0) = n!$ for all $n \geq 0$, find the Maclaurin series for $f(x)$ and its radius of convergence.
7. Find the Taylor series for a function $f(x)$ centered at $a=1$, if it is known that $f^{(n)}(1) = \frac{(-1)^n n!}{2^n}$. Determine the radius of convergence.
8. If a function $g(x)$ has derivatives at $x=0$ given by $g^{(n)}(0) = (-1)^n \frac{(n+1)!}{3^n}$, find the Maclaurin series for $g(x)$ and its radius of convergence.
9. A function $h(x)$ is centered at $a=-3$ and its derivatives are given by $h^{(n)}(-3) = \frac{10}{5^n(n+2)}$. Find the Taylor series for $h(x)$ and its radius of convergence.
10. Determine the Maclaurin series and its radius of convergence for a function $f(x)$ where $f^{(n)}(0) = (2n)!$.

3 Constructing Series by Differentiating the Function

3.1 Problems

11. Find the Maclaurin series for $f(x) = \cos(x)$ and its radius of convergence.
12. Find the Taylor series for $f(x) = \ln(x)$ centered at $a=1$ and find its radius of convergence.
13. Find the Maclaurin series for $f(x) = e^{-2x}$ and its radius of convergence.

14. Find the Taylor series for $f(x) = \frac{1}{x}$ centered at $a = 2$ and determine its radius of convergence.

15. Find the Maclaurin series for $f(x) = \sinh(x) = \frac{e^x - e^{-x}}{2}$ and its radius of convergence. (Hint: Use the known series for e^x.)

16. Find the first four non-zero terms of the Taylor series for $f(x) = \sqrt{x}$ centered at $a = 4$.

17. Find the Maclaurin series for $f(x) = (1 - x)^{-1}$ (the geometric series).

18. Find the Maclaurin series for $f(x) = \sin(3x)$.

19. Find the Taylor series for $f(x) = x^3 - 2x + 4$ centered at $a = 1$.

20. Find the Maclaurin series for $f(x) = \frac{1}{(1+x)^2}$. (Hint: This is related to the derivative of a known series).

4 Constructing Series by Manipulating Known Series

4.1 Problems

21. Find the Maclaurin series for $f(x) = e^{x^2}$ by substituting into the series for e^x. What is the radius of convergence?

22. Find the Maclaurin series for $f(x) = \cos(\sqrt{x})$ for $x \geq 0$.

23. Use the series for $\frac{1}{1-x}$ to find the Maclaurin series for $f(x) = \frac{1}{1+x^2}$. What is its radius of convergence?

24. By integrating the series from the previous problem, find the Maclaurin series for $f(x) = \arctan(x)$.

25. Find the Maclaurin series for $f(x) = x^2 \sin(x)$.

26. Find the Maclaurin series for $f(x) = \frac{x}{1-2x}$.

27. By differentiating the series for $\sin(x)$, find the series for $\cos(x)$.

28. Find the Maclaurin series for $f(x) = \ln(1 - x^2)$ by integrating a known series.

29. Find the first three non-zero terms of the Maclaurin series for $f(x) = e^x \cos(x)$ by multiplying the respective series.

30. Find the Maclaurin series for $f(x) = \frac{\sin(x)}{x}$. Use this series to evaluate $\lim_{x \to 0} \frac{\sin(x)}{x}$.

31. Use a known series to evaluate the indefinite integral $\int \cos(x^3)dx$ as a power series.

32. Find the Maclaurin series for $f(x) = \cosh(x) = \frac{e^x + e^{-x}}{2}$.

5 Binomial Series

5.1 Problems

33. Use the binomial series to expand $f(x) = \frac{1}{\sqrt{1+x}}$ as a power series. State the radius of convergence.

34. Find the first four terms of the binomial series for $f(x) = \sqrt[3]{1 - x}$.

35. Find the Maclaurin series for $f(x) = (1 + x^2)^{-1/2}$.

36. Use the binomial series to find the Maclaurin series for $f(x) = \arcsin(x)$. (Hint: Recall that $\arcsin(x) = \int \frac{1}{\sqrt{1-x^2}} dx$).

6 Radius and Interval of Convergence

6.1 Problems

37. Find the radius and interval of convergence for the series $\sum_{n=0}^{\infty} \frac{(x-2)^n}{n!}$.
38. Find the radius and interval of convergence for the series $\sum_{n=1}^{\infty} \frac{(x+1)^n}{n \cdot 3^n}$.
39. Find the radius and interval of convergence for the series $\sum_{n=0}^{\infty} n!(2x-1)^n$.
40. Find the radius and interval of convergence for the series $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{\sqrt{n}}$.
41. Find the radius and interval of convergence for the series $\sum_{n=0}^{\infty} \frac{x^{2n}}{4^n}$.
42. Find the radius and interval of convergence for the series $\sum_{n=1}^{\infty} \frac{(4x-5)^n}{n^2}$.

7 Taylor's Inequality and Error Estimation

7.1 Problems

43. Use the Maclaurin polynomial of degree 3 for $f(x) = e^x$ to approximate $e^{0.1}$. Use Taylor's Inequality to estimate the accuracy of the approximation.
44. Find the degree n of the Taylor polynomial for $f(x) = \cos(x)$ centered at $a = 0$ that is needed to estimate $\cos(0.2)$ with an error of less than 0.0001.
45. Let $f(x) = \ln(1 + x)$. Use the third-degree Maclaurin polynomial to approximate $\ln(1.5)$. Use Taylor's Inequality to bound the remainder $R_3(0.5)$.
46. Prove that the Maclaurin series for $f(x) = \sin(x)$ converges to $\sin(x)$ for all x.
47. For the function $f(x) = \sqrt[3]{x}$, use the second-degree Taylor polynomial centered at $a = 8$ to approximate $\sqrt[3]{9}$. Use Taylor's Inequality to estimate the error.

8 Mixed and Advanced Problems

8.1 Problems

48. Find the sum of the series $1 - \frac{\pi^2}{2!} + \frac{\pi^4}{4!} - \frac{\pi^6}{6!} + \dots$.
49. Find the sum of the series $\sum_{n=0}^{\infty} \frac{(-1)^n (\pi/3)^{2n+1}}{(2n+1)!}$.
50. Find the sum of the series $\sum_{n=0}^{\infty} \frac{2^n}{n!}$.
51. Use series to evaluate the limit $\lim_{x\to 0} \frac{e^x - 1 - x}{x^2}$.
52. Use series to evaluate the limit $\lim_{x\to 0} \frac{x - \arctan(x)}{x^3}$.
53. Find the Maclaurin series for the indefinite integral $\int e^{-x^2} dx$.
54. Find the first three non-zero terms of the Taylor series for $f(x) = \tan(x)$ centered at $a = 0$.
55. Find the interval of convergence for the Taylor series of $f(x) = \frac{1}{3-x}$ centered at $a = 1$.
56. Use a Taylor polynomial of degree 5 to approximate the value of the definite integral $\int_0^{0.5} \frac{1}{1+x^4} dx$.
57. If the Taylor series for $f(x)$ centered at $a = 2$ is $\sum_{n=0}^{\infty} \frac{(n+1)}{3^n}(x-2)^n$, what is $f^{(4)}(2)$?
58. Find the sum of the series $1 - \ln(2) + \frac{(\ln 2)^2}{2!} - \frac{(\ln 2)^3}{3!} + \dots$.
59. Use series to solve the initial value problem $y' - y = 0$ with $y(0) = 1$.
60. Find the Maclaurin series for $f(x) = \sin^2(x)$. (Hint: Use the identity $\sin^2(x) = \frac{1-\cos(2x)}{2}$).
61. Find the first four non-zero terms of the series for $f(x) = \sec(x)$ by performing long division of 1 by the series for $\cos(x)$.

Solutions

9 Direct Definition and Basic Concepts

1. The formula for the coefficients of a Taylor series centered at a is $c_n = \frac{f^{(n)}(a)}{n!}$. Here, $a = 5$ and we need c_7. So, $c_7 = \frac{f^{(7)}(5)}{7!}$.

2. Here, the center is $a = -2$. The formula for the coefficients is $b_n = \frac{f^{(n)}(-2)}{n!}$. For $n = 10$, we have $b_{10} = \frac{f^{(10)}(-2)}{10!}$.

3. A Maclaurin series is centered at $a = 0$. The formula is $a_n = \frac{g^{(n)}(0)}{n!}$. For a_5, we get $a_5 = \frac{g^{(5)}(0)}{5!}$.

4. The general form is $T_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$. Comparing coefficients with $T_3(x) = 2 - (x-1) + 3(x-1)^2 - 5(x-1)^3$: $f(1) = 2$. $f'(1) = -1$. $\frac{f''(1)}{2!} = 3 \implies f''(1) = 3 \cdot 2! = 6$. $\frac{f'''(1)}{3!} = -5 \implies f'''(1) = -5 \cdot 3! = -30$.

5. The series is $f(x) = \sum \frac{f^{(n)}(0)}{n!}x^n$. The term with x^3 is $\frac{f^{(3)}(0)}{3!}x^3$. We are given this term is $\frac{5}{3}x^3$. So, $\frac{f^{(3)}(0)}{3!} = \frac{5}{3} \implies f^{(3)}(0) = \frac{5}{3} \cdot 3! = \frac{5}{3} \cdot 6 = 10$.

10 Constructing Series from a Derivative Formula

6. $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty} \frac{n!}{n!}x^n = \sum_{n=0}^{\infty} x^n$. This is the geometric series. Radius of convergence: Using the Ratio Test, $\lim_{n\to\infty} |\frac{x^{n+1}}{x^n}| = |x| < 1$. So $R = 1$.

7. $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n = \sum_{n=0}^{\infty} \frac{(-1)^n n!/2^n}{n!}(x-1)^n = \sum_{n=0}^{\infty} \frac{(-1)^n(x-1)^n}{2^n}$. Ratio Test: $\lim_{n\to\infty} |\frac{(x-1)^{n+1}/2^{n+1}}{(x-1)^n/2^n}| = \lim_{n\to\infty} |\frac{x-1}{2}| = \frac{|x-1|}{2} < 1 \implies |x-1| < 2$. So $R = 2$.

8. $g(x) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty} \frac{(-1)^n(n+1)!/3^n}{n!}x^n = \sum_{n=0}^{\infty} \frac{(-1)^n(n+1)}{3^n}x^n$. Ratio Test: $\lim_{n\to\infty} |\frac{(n+2)x^{n+1}/3^{n+1}}{(n+1)x^n/3^n}| = \lim_{n\to\infty} |\frac{x}{3}\frac{n+2}{n+1}| = \frac{|x|}{3} < 1 \implies |x| < 3$. So $R = 3$.

9. $h(x) = \sum_{n=0}^{\infty} \frac{h^{(n)}(-3)}{n!}(x+3)^n = \sum_{n=0}^{\infty} \frac{10/(5^n(n+2))}{n!}(x+3)^n = \sum_{n=0}^{\infty} \frac{10(x+3)^n}{5^n n!(n+2)}$. Ratio Test: $\lim_{n\to\infty} |\frac{10(x+3)^{n+1}}{5^{n+1}(n+1)!(n+3)} \cdot \frac{5^n n!(n+2)}{10(x+3)^n}| = \lim_{n\to\infty} |\frac{x+3}{5(n+1)}\frac{n+2}{n+3}| = 0$. Since the limit is $0 < 1$ for all x, the radius of convergence is $R = \infty$.

10. $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty} \frac{(2n)!}{n!}x^n$. Ratio Test: $\lim_{n\to\infty} |\frac{(2(n+1))!x^{n+1}}{(n+1)!} \cdot \frac{n!}{(2n)!x^n}| = \lim_{n\to\infty} |\frac{(2n+2)(2n+1)x}{n+1}| = \infty$. The series converges only if $x = 0$. So $R = 0$.

11 Constructing Series by Differentiating the Function

11. $f(x) = \cos(x), f(0) = 1$. $f'(x) = -\sin(x), f'(0) = 0$. $f''(x) = -\cos(x), f''(0) = -1$. $f'''(x) = \sin(x), f'''(0) = 0$. $f^{(4)}(x) = \cos(x), f^{(4)}(0) = 1$. The pattern of derivatives at 0 is $1, 0, -1, 0, \ldots$. Series: $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$. Ratio test shows $R = \infty$.

12. $f(x) = \ln(x), f(1) = 0$. $f'(x) = \frac{1}{x}, f'(1) = 1$. $f''(x) = -x^{-2}, f''(1) = -1$. $f'''(x) = 2x^{-3}, f'''(1) = 2$. $f^{(n)}(x) = (-1)^{n-1}(n-1)!x^{-n}$. So $f^{(n)}(1) = (-1)^{n-1}(n-1)!$. Series: $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!}{n!}(x-1)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x-1)^n}{n}$. Ratio test shows $R = 1$.

13. $f(x) = e^{-2x}, f(0) = 1$. $f'(x) = -2e^{-2x}, f'(0) = -2$. $f''(x) = 4e^{-2x}, f''(0) = 4$. $f^{(n)}(x) = (-2)^n e^{-2x}, f^{(n)}(0) = (-2)^n$. Series: $\sum_{n=0}^{\infty} \frac{(-2)^n}{n!}x^n = \sum_{n=0}^{\infty} \frac{(-2x)^n}{n!}$. Ratio test shows $R = \infty$.

14. $f(x) = x^{-1}, f(2) = 1/2$. $f'(x) = -x^{-2}, f'(2) = -1/4$. $f''(x) = 2x^{-3}, f''(2) = 2/8$. $f^{(n)}(2) = \frac{(-1)^n n!}{2^{n+1}}$. Series: $\sum_{n=0}^{\infty} \frac{(-1)^n n!/2^{n+1}}{n!}(x-2)^n = \sum_{n=0}^{\infty} \frac{(-1)^n(x-2)^n}{2^{n+1}}$. Ratio test shows $R = 2$.

15. Using known series: $e^x = \sum \frac{x^n}{n!}$, $e^{-x} = \sum \frac{(-x)^n}{n!}$. $\sinh(x) = \frac{1}{2}(\sum \frac{x^n}{n!} - \sum \frac{(-1)^n x^n}{n!}) = \frac{1}{2}\sum \frac{(1-(-1)^n)x^n}{n!}$. If n is even, $1-1=0$. If n is odd, $1-(-1)=2$. Series: $\sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$. $R=\infty$.

16. $f(x) = x^{1/2}, f(4) = 2$. $f'(x) = \frac{1}{2}x^{-1/2}, f'(4) = \frac{1}{4}$. $f''(x) = -\frac{1}{4}x^{-3/2}, f''(4) = -\frac{1}{32}$. $f'''(x) = \frac{3}{8}x^{-5/2}, f'''(4) = \frac{3}{256}$. $T_3(x) = 2 + \frac{1}{4}(x-4) - \frac{1/32}{2!}(x-4)^2 + \frac{3/256}{3!}(x-4)^3 = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3$.

17. This is the standard geometric series, $\sum_{n=0}^{\infty} x^n$.

18. We know $\sin(u) = \sum_{n=0}^{\infty} \frac{(-1)^n u^{2n+1}}{(2n+1)!}$. Let $u = 3x$. $\sin(3x) = \sum_{n=0}^{\infty} \frac{(-1)^n (3x)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1} x^{2n+1}}{(2n+1)!}$.

19. A polynomial is its own Taylor series. We just need to rewrite it in powers of $(x-1)$. Let $u = x-1 \implies x = u+1$. $(u+1)^3 - 2(u+1) + 4 = (u^3+3u^2+3u+1) - (2u+2) + 4 = u^3+3u^2+u+3$. $f(x) = 3+(x-1)+3(x-1)^2+(x-1)^3$.

20. We know $\frac{1}{1-u} = \sum_{n=0}^{\infty} u^n$. Let $u = -x$. $\frac{1}{1+x} = \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n$. $\frac{d}{dx}(\frac{1}{1+x}) = \frac{-1}{(1+x)^2}$. So $\frac{1}{(1+x)^2} = -\frac{d}{dx}\sum_{n=0}^{\infty}(-1)^n x^n = -\sum_{n=1}^{\infty}(-1)^n n x^{n-1} = \sum_{n=1}^{\infty}(-1)^{n+1} n x^{n-1}$. Re-index with $k = n-1$: $\sum_{k=0}^{\infty}(-1)^{k+2}(k+1)x^k = \sum_{k=0}^{\infty}(-1)^k (k+1)x^k$.

12 Constructing Series by Manipulating Known Series

21. $e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!}$. Let $u = x^2$. $e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$. Radius of convergence is $R = \infty$.

22. $\cos(u) = \sum_{n=0}^{\infty} \frac{(-1)^n u^{2n}}{(2n)!}$. Let $u = \sqrt{x}$. $\cos(\sqrt{x}) = \sum_{n=0}^{\infty} \frac{(-1)^n (\sqrt{x})^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n)!}$.

23. $\frac{1}{1-u} = \sum_{n=0}^{\infty} u^n$. Let $u = -x^2$. $\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}$. Radius of convergence is $|-x^2| < 1 \implies |x| < 1$, so $R = 1$.

24. $\arctan(x) = \int \frac{1}{1+x^2}dx = \int \sum_{n=0}^{\infty}(-1)^n x^{2n} dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$. Since $\arctan(0) = 0$, $C = 0$. So $\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$.

25. $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$. $x^2 \sin(x) = x^2 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{(2n+1)!}$.

26. $\frac{1}{1-u} = \sum_{n=0}^{\infty} u^n$. Let $u = 2x$. $\frac{1}{1-2x} = \sum_{n=0}^{\infty}(2x)^n = \sum_{n=0}^{\infty} 2^n x^n$. So, $\frac{x}{1-2x} = x\sum_{n=0}^{\infty} 2^n x^n = \sum_{n=0}^{\infty} 2^n x^{n+1}$.

27. $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$. Differentiating term-by-term: $1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} - \dots = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots = \cos(x)$.

28. $\ln(1-u) = -\sum_{n=1}^{\infty} \frac{u^n}{n}$. Let $u = x^2$. $\ln(1-x^2) = -\sum_{n=1}^{\infty} \frac{(x^2)^n}{n} = -\sum_{n=1}^{\infty} \frac{x^{2n}}{n}$.

29. $e^x = 1 + x + \frac{x^2}{2} + \dots$, $\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \dots$. $e^x\cos(x) = (1 + x + \frac{x^2}{2} + \dots)(1 - \frac{x^2}{2} + \dots) = 1(1 - \frac{x^2}{2}) + x(1) + \frac{x^2}{2}(1) + \dots = 1 + x - \frac{x^2}{2} + \frac{x^2}{2} + \dots = 1 + x + 0x^2 + \dots = 1 + x - \frac{x^3}{3} + \dots$ (Need to expand further for x^2, x^3 terms). Correct expansion: $1(1 - \frac{x^2}{2}) + x(1) + \frac{x^2}{2}(1) = 1 + x$. x^3 term: $x(-\frac{x^2}{2}) + \frac{x^3}{6}(1) = -\frac{x^3}{2} + \frac{x^3}{6} = -\frac{x^3}{3}$. Result: $1 + x - \frac{x^3}{3} + \dots$.

30. $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \dots$. $\frac{\sin(x)}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots$. $\lim_{x\to 0}(1 - \frac{x^2}{3!} + \dots) = 1$.

31. $\cos(u) = \sum_{n=0}^{\infty} \frac{(-1)^n u^{2n}}{(2n)!}$. Let $u = x^3$. $\cos(x^3) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n}}{(2n)!}$. $\int \cos(x^3)dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+1}}{(2n)!(6n+1)}$.

32. $\cosh(x) = \frac{1}{2}(e^x + e^{-x}) = \frac{1}{2}(\sum \frac{x^n}{n!} + \sum \frac{(-x)^n}{n!}) = \frac{1}{2}\sum \frac{(1+(-1)^n)x^n}{n!}$. For odd n, terms are 0. For even n, terms are $2x^n/n!$. So $\cosh(x) = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}$.

13 Binomial Series

33. $f(x) = (1+x)^{-1/2}$. Here $k = -1/2$. The series is $1 + \sum_{n=1}^{\infty} \frac{(-\frac{1}{2})(-\frac{3}{2})\cdots(-\frac{1}{2}-n+1)}{n!} x^n = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 1\cdot 3\cdots(2n-1)}{2^n n!} x^n$. $R = 1$.

34. $f(x) = (1+(-x))^{1/3}$. Here $k = 1/3$. $T_3(x) = 1 + \frac{1}{3}(-x) + \frac{\frac{1}{3}(-\frac{2}{3})}{2!}(-x)^2 + \frac{\frac{1}{3}(-\frac{2}{3})(-\frac{5}{3})}{3!}(-x)^3 = 1 - \frac{1}{3}x - \frac{1}{9}x^2 - \frac{5}{81}x^3$.

35. $f(x) = (1+x^2)^{-1/2}$. Use result from Q33, substitute x with x^2. $1 + \sum_{n=1}^{\infty} \frac{(-1)^n 1\cdot 3\cdots(2n-1)}{2^n n!}(x^2)^n = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 1\cdot 3\cdots(2n-1)}{2^n n!} x^{2n}$.

36. From Q35, the series for $(1-u^2)^{-1/2}$ is $1 + \sum_{n=1}^{\infty} \frac{1\cdot 3\cdots(2n-1)}{2^n n!} u^{2n}$. Integrate term by term: $\arcsin(x) = C + x + \sum_{n=1}^{\infty} \frac{1\cdot 3\cdots(2n-1)}{2^n n!(2n+1)} x^{2n+1}$. Since $\arcsin(0) = 0$, $C = 0$.

14 Radius and Interval of Convergence

37. Ratio test: $\lim_{n\to\infty} |\frac{(x-2)^{n+1}/(n+1)!}{(x-2)^n/n!}| = \lim_{n\to\infty} |\frac{x-2}{n+1}| = 0$. $R = \infty$, Interval: $(-\infty, \infty)$.

38. Ratio test: $\lim_{n\to\infty} |\frac{(x+1)^{n+1}/((n+1)3^{n+1})}{(x+1)^n/(n\cdot 3^n)}| = \lim_{n\to\infty} |\frac{x+1}{3}\frac{n}{n+1}| = \frac{|x+1|}{3} < 1 \implies |x+1| < 3$. $R = 3$. Interval: $(-4, 2)$. Test endpoints: $x = 2 \implies \sum \frac{3^n}{n3^n} = \sum \frac{1}{n}$ (diverges). $x = -4 \implies \sum \frac{(-3)^n}{n3^n} = \sum \frac{(-1)^n}{n}$ (converges). Interval: $[-4, 2)$.

39. Ratio test: $\lim_{n\to\infty} |\frac{(n+1)!(2x-1)^{n+1}}{n!(2x-1)^n}| = \lim_{n\to\infty} |(n+1)(2x-1)| = \infty$ unless $x = 1/2$. $R = 0$. Interval: $\{1/2\}$.

40. Ratio test: $\lim_{n\to\infty} |\frac{x^{n+1}/\sqrt{n+1}}{x^n/\sqrt{n}}| = |x| \lim_{n\to\infty} \sqrt{\frac{n}{n+1}} = |x| < 1$. $R = 1$. Interval: $(-1, 1)$. Test endpoints: $x = 1 \implies \sum \frac{(-1)^n}{\sqrt{n}}$ (converges). $x = -1 \implies \sum \frac{(-1)^n(-1)^n}{\sqrt{n}} = \sum \frac{1}{\sqrt{n}}$ (diverges, p-series). Interval: $(-1, 1]$.

41. Root test: $\lim_{n\to\infty} \sqrt[n]{|\frac{x^{2n}}{4^n}|} = \lim_{n\to\infty} \frac{|x^2|}{4} = \frac{x^2}{4} < 1 \implies x^2 < 4 \implies |x| < 2$. $R = 2$. Interval: $(-2, 2)$. Test endpoints: $x = 2 \implies \sum \frac{4^n}{4^n} = \sum 1$ (diverges). $x = -2 \implies \sum \frac{(-2)^{2n}}{4^n} = \sum \frac{4^n}{4^n} = \sum 1$ (diverges). Interval: $(-2, 2)$.

42. Ratio test: $\lim_{n\to\infty} |\frac{(4x-5)^{n+1}/(n+1)^2}{(4x-5)^n/n^2}| = |4x-5| \lim_{n\to\infty} (\frac{n}{n+1})^2 = |4x-5| < 1$. $|4(x-5/4)| < 1 \implies |x-5/4| < 1/4$. $R = 1/4$. Interval: $(1, 3/2)$. Test endpoints: $x = 3/2 \implies \sum \frac{(1)^n}{n^2}$ (converges, p-series). $x = 1 \implies \sum \frac{(-1)^n}{n^2}$ (converges). Interval: $[1, 3/2]$.

15 Taylor's Inequality and Error Estimation

43. $T_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$. $e^{0.1} \approx 1 + 0.1 + \frac{0.01}{2} + \frac{0.001}{6} \approx 1.105166\ldots$. Error: $R_3(0.1)$. $f^{(4)}(x) = e^x$. On $[0, 0.1]$, e^x is increasing, so $|f^{(4)}(x)| \leq e^{0.1} < e < 3$. Let $M = 3$. $|R_3(0.1)| \leq \frac{3}{4!}(0.1)^4 = \frac{3}{24}(0.0001) = 0.0000125$.

44. We need $|R_n(0.2)| \leq 0.0001$. For $f(x) = \cos(x)$, $|f^{(n+1)}(x)|$ is either $|\sin x|$ or $|\cos x|$, so $|f^{(n+1)}(x)| \leq 1$. Let $M = 1$. We need $\frac{1}{(n+1)!}|0.2|^{n+1} \leq 0.0001$. $n = 1 : \frac{0.2^2}{2} = 0.02$. $n = 2 : \frac{0.2^3}{6} \approx 0.0013$. $n = 3 : \frac{0.2^4}{24} \approx 0.000067 < 0.0001$. So $n = 3$ is sufficient.

45. $f(x) = \ln(1+x)$, $T_3(x) = x - \frac{x^2}{2} + \frac{x^3}{3}$. $\ln(1.5) = f(0.5) \approx 0.5 - \frac{0.5^2}{2} + \frac{0.5^3}{3} \approx 0.5 - 0.125 + 0.04166 = 0.41666$. $f^{(4)}(x) = -6(1+x)^{-4}$. On $[0, 0.5]$, $|f^{(4)}(x)| = \frac{6}{(1+x)^4}$ is decreasing. Max value is at $x = 0$, $M = 6$. $|R_3(0.5)| \leq \frac{6}{4!}|0.5|^4 = \frac{6}{24}(0.0625) = 0.015625$.

46. For $f(x) = \sin(x)$, $|f^{(n+1)}(x)| \leq 1$ for all x, n. Let $M = 1$. By Taylor's Inequality, $|R_n(x)| \leq \frac{1}{(n+1)!}|x|^{n+1}$. For any fixed x, $\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$. Since the remainder goes to 0, the series converges to $\sin(x)$.

47. $f(x) = x^{1/3}, a = 8$. $f(8) = 2, f'(8) = 1/12, f''(8) = -1/144$. $T_2(x) = 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2$. $\sqrt[3]{9} \approx T_2(9) = 2 + \frac{1}{12} - \frac{1}{288} \approx 2.080$. $f'''(x) = \frac{10}{27}x^{-8/3}$. On $[8, 9]$, $|f'''(x)| = \frac{10}{27x^{8/3}}$ is decreasing. Max is at $x = 8$, $M = \frac{10}{27 \cdot 8^{8/3}} = \frac{10}{27 \cdot 256} = \frac{5}{3456}$. $|R_2(9)| \leq \frac{5/3456}{3!}|9-8|^3 = \frac{5}{20736} \approx 0.00024$.

16 Mixed and Advanced Problems

48. This is the series for $\cos(x) = \sum \frac{(-1)^n x^{2n}}{(2n)!}$ with $x = \pi$. So the sum is $\cos(\pi) = -1$.

49. This is the series for $\sin(x)$ with $x = \pi/3$. The sum is $\sin(\pi/3) = \sqrt{3}/2$.

50. This is the series for $e^x = \sum \frac{x^n}{n!}$ with $x = 2$. The sum is e^2.

51. $e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots$. $\lim_{x\to 0} \frac{(1+x+\frac{x^2}{2}+\dots)-1-x}{x^2} = \lim_{x\to 0} \frac{\frac{x^2}{2}+\frac{x^3}{6}+\dots}{x^2} = \lim_{x\to 0}(\frac{1}{2}+\frac{x}{6}+\dots) = \frac{1}{2}$.

52. $\arctan(x) = x-\frac{x^3}{3}+\frac{x^5}{5}-\dots$. $\lim_{x\to 0} \frac{x-(x-\frac{x^3}{3}+\dots)}{x^3} = \lim_{x\to 0} \frac{\frac{x^3}{3}-\frac{x^5}{5}+\dots}{x^3} = \lim_{x\to 0}(\frac{1}{3}-\frac{x^2}{5}+\dots) = \frac{1}{3}$.

53. $e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$. $\int e^{-x^2} dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(2n+1)}$.

54. $f(x) = \tan x, f(0) = 0$. $f'(x) = \sec^2 x, f'(0) = 1$. $f''(x) = 2\sec^2 x \tan x, f''(0) = 0$. $f'''(x) = 4\sec^2 x \tan^2 x + 2\sec^4 x, f'''(0) = 2$. $T_3(x) = x + \frac{2}{3!}x^3 = x + \frac{x^3}{3}$.

55. $f(x) = \frac{1}{3-x} = \frac{1}{2-(x-1)} = \frac{1/2}{1-\frac{x-1}{2}}$. This is a geometric series with $r = \frac{x-1}{2}$. Converges for $|\frac{x-1}{2}| < 1 \implies |x-1| < 2$. Interval is $(-1, 3)$.

56. $\frac{1}{1+u} = 1 - u + u^2 - u^3 + \dots$. Let $u = x^4$. $\frac{1}{1+x^4} = 1 - x^4 + x^8 - \dots$. $\int_0^{0.5}(1-x^4)dx = [x - \frac{x^5}{5}]_0^{0.5} = 0.5 - \frac{0.5^5}{5} = 0.5 - \frac{0.03125}{5} = 0.5 - 0.00625 = 0.49375$.

57. Coefficient of $(x-2)^n$ is $c_n = \frac{f^{(n)}(2)}{n!}$. We are given $c_n = \frac{n+1}{3^n}$. For $n = 4$, $c_4 = \frac{f^{(4)}(2)}{4!} = \frac{4+1}{3^4} = \frac{5}{81}$. So $f^{(4)}(2) = \frac{5}{81} \cdot 4! = \frac{5 \cdot 24}{81} = \frac{40}{27}$.

58. This is the series for e^x with $x = -\ln(2)$. Sum is $e^{-\ln(2)} = e^{\ln(2^{-1})} = 2^{-1} = 1/2$.

59. Assume $y = \sum_{n=0}^{\infty} c_n x^n$. Then $y' = \sum_{n=1}^{\infty} nc_n x^{n-1}$. $y' - y = \sum_{n=1}^{\infty} nc_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n = 0$. Re-index first sum: $\sum_{k=0}^{\infty}(k+1)c_{k+1}x^k - \sum_{k=0}^{\infty} c_k x^k = 0$. This gives $(k+1)c_{k+1} - c_k = 0$, or $c_{k+1} = \frac{c_k}{k+1}$. $y(0) = 1 \implies c_0 = 1$. Then $c_1 = c_0/1 = 1, c_2 = c_1/2 = 1/2, c_3 = c_2/3 = 1/6$. So $c_n = 1/n!$. $y = \sum \frac{x^n}{n!} = e^x$.

60. $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$. $\cos(u) = \sum_{n=0}^{\infty} \frac{(-1)^n u^{2n}}{(2n)!}$. $\cos(2x) = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!} = 1 - \frac{4x^2}{2!} + \frac{16x^4}{4!} - \dots$. $\sin^2(x) = \frac{1}{2}(1 - (1 - \frac{4x^2}{2!} + \frac{16x^4}{4!} - \dots)) = \frac{1}{2}(\frac{4x^2}{2!} - \frac{16x^4}{4!} + \dots) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2^{2n-1} x^{2n}}{(2n)!}$.

61. $\sec(x) = \frac{1}{\cos x} = \frac{1}{1-\frac{x^2}{2}+\frac{x^4}{24}-\dots}$. Long division gives $1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \dots$

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