Calculus III - Section 14.1
Functions of Several Variables
Problem Set

Comprehensive Practice Problems

January 2026

Instructions

This problem set contains 30 problems covering all concepts from Section 14.1: Functions of Several Variables. Work through each problem carefully, showing all steps. A concept reference guide is provided at the end.

1 Function Evaluation

Problem 1. Given $f(x, y) = \frac{x^2-3y}{2x+y^2}$, evaluate:

(a) $f(2, 1)$

(b) $f(-1, 3)$

(c) $f(0, -2)$

Problem 2. Given $g(x, y) = x^3y - 2xy^2 + 5$, find:

(a) $g(1, 2)$

(b) $g(-2, -1)$

(c) $g(x + h, y)$

(d) $g(2t, t)$

Problem 3. For $h(x, y, z) = e^{xy} + \ln(z - x)$, evaluate:

(a) $h(0, 5, 3)$

(b) $h(1, 2, 5)$

(c) $h(-1, 3, 2)$

Problem 4. Given $f(x,y) = \sqrt{x^2+y^2} \cdot \sin(\frac{\pi x}{y})$, find:

(a) $f(3,4)$

(b) $f(1,2)$

(c) $f(y,y)$

2 Domain Problems - Two Variables

Problem 5. Find and sketch the domain of $f(x,y) = \ln(x+2y-4)$.

Problem 6. Find and sketch the domain of $g(x,y) = \sqrt{16-x^2-y^2}$.

Problem 7. Determine the domain of $h(x,y) = \frac{1}{\sqrt{y-x^2}}$ and sketch it in the xy-plane.

Problem 8. Find the domain of $f(x,y) = \frac{x-y}{x^2+y^2-9}$ and describe it using inequalities.

Problem 9. Find and sketch the domain of $k(x,y) = \sqrt[3]{2x+5y}$. (Hint: Odd roots have different restrictions than even roots)

Problem 10. Determine the domain of $f(x,y) = \ln(x^2-y) + \sqrt{y-1}$.

Problem 11. Find the domain of $g(x,y) = \frac{\sqrt{x}}{y-3}$.

3 Domain and Range - Combined

Problem 12. For $f(x,y) = e^{\sqrt{4-x^2-y^2}}$:

(a) Find and sketch the domain

(b) Determine the range

Problem 13. For $g(x,y) = \frac{xy}{x^2+y^2+1}$:

(a) State the domain

(b) Find the range (Hint: Consider extreme values)

Problem 14. For $h(x,y) = \arcsin(x+y)$:

(a) Find the domain

(b) State the range

4 Three Variable Functions

Problem 15. Find the domain of $f(x,y,z) = \ln(16-x^2-y^2-4z^2)$ and describe the geometric shape.

Problem 16. Determine the domain of $g(x,y,z) = \frac{1}{\sqrt{z-\sqrt{x^2+y^2}}}$ and write the inequality that defines it.

Problem 17. For $h(x,y,z) = \sqrt{x^2+y^2+z^2-25}$:

(a) Evaluate $h(3, 4, 0)$

(b) Find and describe the domain geometrically

Problem 18. Find the domain of $f(x, y, z) = \ln(z) + \sqrt{9 - x^2 - y^2}$ and describe it in words.

5 Level Curves and Contour Maps

Problem 19. For $f(x, y) = 4x^2 + 9y^2$, find the level curves for $k = 0, 36, 144$ and identify the type of curves.

Problem 20. Sketch several level curves of $f(x, y) = xy$ for $k = -4, -1, 0, 1, 4$. What type of curves are these?

Problem 21. For $g(x, y) = y - x^3$, sketch the level curves for $k = -1, 0, 1, 2$.

Problem 22. Match the function $f(x, y) = \cos(x)\sin(y)$ with its contour map description:

(a) Concentric circles

(b) Parallel lines

(c) Rectangular grid pattern

(d) Hyperbolas

Problem 23. Two contour maps are shown for $z = \sqrt{x^2 + y^2}$ (cone) and $z = \frac{1}{\sqrt{x^2+y^2}}$ (inverted cone). Explain how the spacing of level curves differs between these two surfaces.

6 Cylindrical Surfaces and Cross Sections

Problem 24. For $f(x, y) = \cos(y)$:

(a) Describe the surface in words

(b) Find the cross section in the plane $x = 1$

(c) Find the cross section in the plane $y = \pi/4$

Problem 25. For $g(x, y) = 4 - y^2$:

(a) Sketch the surface

(b) Find equations for the cross sections at $x = -2, 0, 2$

(c) Find the cross section in the xz-plane (where $y = 0$)

Problem 26. Consider $f(x, y) = e^x$. Find the cross sections:

(a) In planes parallel to the xz-plane (fix $y = k$)

(b) In planes parallel to the yz-plane (fix $x = 0, 1, -1$)

7 Table Interpretation

Problem 27. The table below shows the wind chill temperature $W = f(T, v)$ where T is the actual temperature in °F and v is the wind speed in mph.

$T\backslash v$	5	10	15	20	25
30	27	21	17	15	13
20	16	9	4	2	0
10	4	-4	-9	-12	-15
0	-7	-16	-22	-26	-29

(a) Find and interpret $f(20, 15)$

(b) For what value of v is $f(10, v) = -4$?

(c) For what value of T is $f(T, 20) = 2$?

(d) Compare how $f(30, v)$ and $f(0, v)$ change as v increases

8 Advanced Domain Problems

Problem 28. Find the domain of $f(x, y) = \sqrt{\ln(x + y)}$ and explain your reasoning carefully.

Problem 29. Determine the domain of $g(x, y) = \arctan\left(\frac{y}{x}\right) + \sqrt{1 - x^2 - y^2}$.

Problem 30. For the function $h(x, y, z) = \frac{\sqrt{z^2 - x^2 - y^2}}{x^2 + y^2 - 4}$:

(a) Find the domain

(b) Describe the domain geometrically as the region between two surfaces

Solutions

Problem 1

(a) $f(2,1) = \frac{(2)^2-3(1)}{2(2)+(1)^2} = \frac{4-3}{4+1} = \frac{1}{5}$

(b) $f(-1,3) = \frac{(-1)^2-3(3)}{2(-1)+(3)^2} = \frac{1-9}{-2+9} = \frac{-8}{7} = -\frac{8}{7}$

(c) $f(0,-2) = \frac{(0)^2-3(-2)}{2(0)+(-2)^2} = \frac{0+6}{0+4} = \frac{6}{4} = \frac{3}{2}$

Problem 2

(a) $g(1,2) = (1)^3(2) - 2(1)(2)^2 + 5 = 2 - 8 + 5 = -1$

(b) $g(-2,-1) = (-2)^3(-1) - 2(-2)(-1)^2 + 5 = 8 - 4 + 5 = 9$

(c) $g(x+h,y) = (x+h)^3y - 2(x+h)y^2 + 5$

(d) $g(2t,t) = (2t)^3(t) - 2(2t)(t)^2 + 5 = 8t^4 - 4t^3 + 5$

Problem 3

(a) $h(0,5,3) = e^{(0)(5)} + \ln(3-0) = e^0 + \ln(3) = 1 + \ln(3)$

(b) $h(1,2,5) = e^{(1)(2)} + \ln(5-1) = e^2 + \ln(4) = e^2 + 2\ln(2)$

(c) $h(-1,3,2) = e^{(-1)(3)} + \ln(2-(-1)) = e^{-3} + \ln(3)$

Problem 4

(a) $f(3,4) = \sqrt{3^2+4^2} \cdot \sin\left(\frac{\pi \cdot 3}{4}\right) = 5 \cdot \sin\left(\frac{3\pi}{4}\right) = 5 \cdot \frac{\sqrt{2}}{2} = \frac{5\sqrt{2}}{2}$

(b) $f(1,2) = \sqrt{1^2+2^2} \cdot \sin\left(\frac{\pi \cdot 1}{2}\right) = \sqrt{5} \cdot \sin\left(\frac{\pi}{2}\right) = \sqrt{5} \cdot 1 = \sqrt{5}$

(c) $f(y,y) = \sqrt{y^2+y^2} \cdot \sin\left(\frac{\pi y}{y}\right) = \sqrt{2y^2} \cdot \sin(\pi) = |y|\sqrt{2} \cdot 0 = 0$

Problem 5

For $\ln(x+2y-4)$ to be defined: $x + 2y - 4 > 0 \Rightarrow 2y > 4 - x \Rightarrow y > 2 - \frac{x}{2}$

Domain: All points strictly above the line $y = 2 - \frac{x}{2}$.

Sketch: Line with slope $-\frac{1}{2}$ and y-intercept 2, dashed boundary, shade above.

Problem 6

For $\sqrt{16-x^2-y^2}$ to be defined: $16 - x^2 - y^2 \geq 0 \Rightarrow x^2 + y^2 \leq 16$

Domain: All points on and inside the circle of radius 4 centered at origin.

Sketch: Solid circle with radius 4, shaded interior.

Problem 7

Two conditions:

- Denominator $\neq 0$: $y - x^2 \neq 0$
- Inside square root > 0: $y - x^2 > 0 \Rightarrow y > x^2$

Domain: All points strictly above the parabola $y = x^2$.
Sketch: Dashed parabola opening upward, shade region above (inside the cup).

Problem 8

Denominator cannot be zero: $x^2 + y^2 - 9 \neq 0 \Rightarrow x^2 + y^2 \neq 9$
Domain: All points in the xy-plane except those on the circle $x^2 + y^2 = 9$.
In inequality form: $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \neq 9\}$

Problem 9

Cube roots (odd roots) are defined for all real numbers.
Domain: All of $\mathbb{R}^2$ (the entire xy-plane).
Sketch: The entire plane with no restrictions.

Problem 10

Two conditions:

- $\ln(x^2 - y)$: Need $x^2 - y > 0 \Rightarrow y < x^2$
- $\sqrt{y-1}$: Need $y - 1 \geq 0 \Rightarrow y \geq 1$

Domain: $\{(x, y) : y < x^2 \text{ and } y \geq 1\} = \{(x, y) : 1 \leq y < x^2\}$
This requires $x^2 > 1$, so $|x| > 1$.

Problem 11

Two conditions:

- $\sqrt{x}$: Need $x \geq 0$
- Denominator: $y - 3 \neq 0 \Rightarrow y \neq 3$

Domain: $\{(x, y) : x \geq 0 \text{ and } y \neq 3\}$
Half-plane $x \geq 0$ with the horizontal line $y = 3$ removed.

Problem 12

(a) Domain: $4 - x^2 - y^2 \geq 0 \Rightarrow x^2 + y^2 \leq 4$.

Disk of radius 2 centered at origin (solid boundary).

(b) Range: Let $u = \sqrt{4 - x^2 - y^2}$. Since $0 \leq x^2 + y^2 \leq 4$, we have $0 \leq u \leq 2$.

Thus $z = e^u$ where $0 \leq u \leq 2$, giving $e^0 \leq z \leq e^2$.

Range: $[1, e^2]$

Problem 13

(a) Domain: Denominator $x^2 + y^2 + 1 > 0$ for all (x, y).

Domain: $\mathbb{R}^2$ (entire plane)

(b) For range, note that by substituting $x = r\cos\theta$, $y = r\sin\theta$:

$g = \frac{r^2 \cos\theta \sin\theta}{r^2+1} = \frac{r^2 \sin(2\theta)/2}{r^2+1}$

As $r \to \infty$, $g \to \frac{\sin(2\theta)}{2}$, and as $r \to 0$, $g \to 0$.

Maximum value approaches $\frac{1}{2}$, minimum approaches $-\frac{1}{2}$.

Range: $\left(-\frac{1}{2}, \frac{1}{2}\right)$

Problem 14

(a) Domain: For arcsin, need $-1 \leq x + y \leq 1$.

Domain: $\{(x, y) : -1 \leq x + y \leq 1\}$ (strip between parallel lines)

(b) Range: The range of arcsin is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Problem 15

Need: $16 - x^2 - y^2 - 4z^2 > 0 \Rightarrow x^2 + y^2 + 4z^2 < 16$

Dividing by 16: $\frac{x^2}{16} + \frac{y^2}{16} + \frac{z^2}{4} < 1$

Geometric shape: Interior of an ellipsoid with semi-axes $a = 4$, $b = 4$, $c = 2$.

Problem 16

Two conditions:

- Denominator $\neq 0$: $z - \sqrt{x^2 + y^2} \neq 0$
- Inside square root > 0: $z - \sqrt{x^2 + y^2} > 0$

Domain inequality: $z > \sqrt{x^2 + y^2}$

This is the region strictly above (outside) the cone $z = \sqrt{x^2 + y^2}$.

Problem 17

(a) $h(3,4,0) = \sqrt{3^2 + 4^2 + 0^2 - 25} = \sqrt{9 + 16 - 25} = \sqrt{0} = 0$

(b) Domain: $x^2 + y^2 + z^2 - 25 \geq 0 \Rightarrow x^2 + y^2 + z^2 \geq 25$

Geometric description: Points on and outside the sphere of radius 5 centered at origin.

Problem 18

Two conditions:

- $\ln(z)$: Need $z > 0$
- $\sqrt{9 - x^2 - y^2}$: Need $9 - x^2 - y^2 \geq 0 \Rightarrow x^2 + y^2 \leq 9$

Domain: $\{(x, y, z) : x^2 + y^2 \leq 9 \text{ and } z > 0\}$
Description: A solid cylinder of radius 3 along the z-axis, for all positive z values.

Problem 19

Level curves: $4x^2 + 9y^2 = k$
For $k = 0$: $4x^2 + 9y^2 = 0 \Rightarrow x = 0, y = 0$ (single point, the origin)
For $k = 36$: $4x^2 + 9y^2 = 36 \Rightarrow \frac{x^2}{9} + \frac{y^2}{4} = 1$ (ellipse with $a = 3$, $b = 2$)
For $k = 144$: $\frac{x^2}{36} + \frac{y^2}{16} = 1$ (ellipse with $a = 6$, $b = 4$)
Type: Concentric ellipses centered at origin.

Problem 20

Level curves: $xy = k \Rightarrow y = \frac{k}{x}$
These are rectangular hyperbolas:

- $k = -4$: $y = -\frac{4}{x}$ (hyperbola in quadrants II and IV)
- $k = -1$: $y = -\frac{1}{x}$
- $k = 0$: $xy = 0$ (the coordinate axes)
- $k = 1$: $y = \frac{1}{x}$ (hyperbola in quadrants I and III)
- $k = 4$: $y = \frac{4}{x}$

Type: Rectangular hyperbolas.

Problem 21

Level curves: $y - x^3 = k \Rightarrow y = x^3 + k$

These are cubic curves (vertical translations of $y = x^3$):

- $k = -1$: $y = x^3 - 1$
- $k = 0$: $y = x^3$
- $k = 1$: $y = x^3 + 1$
- $k = 2$: $y = x^3 + 2$

Sketch shows family of S-shaped curves.

Problem 22

Answer: (c) Rectangular grid pattern

The level curves $\cos(x)\sin(y) = k$ form a grid-like pattern because:

- When $x = \frac{\pi}{2} + n\pi$, $\cos(x) = 0$, so $f = 0$ (vertical lines)
- When $y = n\pi$, $\sin(y) = 0$, so $f = 0$ (horizontal lines)
- Other level curves form rounded rectangular shapes

Problem 23

For $z = \sqrt{x^2 + y^2}$ (cone):

- Level curves: $\sqrt{x^2 + y^2} = k \Rightarrow x^2 + y^2 = k^2$
- Circles with radii k
- For $k = 1, 2, 3$: radii are 1, 2, 3 (equally spaced)

For $z = \frac{1}{\sqrt{x^2+y^2}}$ (inverted cone):

- Level curves: $x^2 + y^2 = \frac{1}{k^2}$
- For $k = 1, 2, 3$: radii are 1, $\frac{1}{2}$, $\frac{1}{3}$
- Circles get closer together as k increases

The cone has evenly-spaced level curves; the inverted cone has level curves that bunch together near the origin.

Problem 24

(a) The surface is a cylindrical wave extending along the x-axis, with sinusoidal variation in the y-direction.

(b) Cross section at $x = 1$: $z = \cos(y)$ (same for any x value)

(c) Cross section at $y = \frac{\pi}{4}$: $z = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$ (horizontal line)

Problem 25

(a) Surface is a parabolic cylinder opening downward along the x-axis

(b) Cross sections at any x value: $z = 4 - y^2$ (same parabola)

At $x = -2, 0, 2$: all give $z = 4 - y^2$

(c) Cross section at $y = 0$: $z = 4 - 0^2 = 4$ (horizontal line)

Problem 26

(a) For any fixed $y = k$: $z = e^x$ (same exponential curve)

(b) For fixed x values:

- $x = 0$: $z = e^0 = 1$ (horizontal line)
- $x = 1$: $z = e^1 = e$ (horizontal line)
- $x = -1$: $z = e^{-1} = \frac{1}{e}$ (horizontal line)

Problem 27

(a) $f(20, 15) = 4$.

Interpretation: When the actual temperature is 20°F and the wind speed is 15 mph, the perceived temperature (wind chill) is 4°F.

(b) Look at row $T = 10$, find where value equals -4.

This occurs at $v = 10$ mph.

(c) Look at column $v = 20$, find where value equals 2.

This occurs at $T = 20$°F.

(d) $f(30, v)$: As v increases from 5 to 25, values go from 27 to 13, decreasing by about 3-4° per 5 mph increment (relatively steady decrease).

$f(0, v)$: As v increases from 5 to 25, values go from -7 to -29, decreasing more rapidly (about 5-6° per 5 mph increment at higher speeds).

The wind chill effect is more pronounced at lower actual temperatures.

Problem 28

Two conditions:

- Inside ln: Need $x + y > 0 \Rightarrow y > -x$
- Inside $\sqrt{\ }$: Need $\ln(x + y) \geq 0 \Rightarrow x + y \geq 1$

Domain: $\{(x, y) : x + y \geq 1\}$
This is the region on and above the line $y = 1 - x$ (solid boundary).

Problem 29

Two conditions:

- $\arctan(y/x)$: Defined for all (x, y) except where $x = 0$
- $\sqrt{1 - x^2 - y^2}$: Need $1 - x^2 - y^2 \geq 0 \Rightarrow x^2 + y^2 \leq 1$

Domain: $\{(x, y) : x^2 + y^2 \leq 1 \text{ and } x \neq 0\}$
The closed unit disk with the y-axis removed.

Problem 30

(a) Three conditions:

- Numerator: $z^2 - x^2 - y^2 \geq 0 \Rightarrow z^2 \geq x^2 + y^2$
- Denominator: $x^2 + y^2 - 4 \neq 0 \Rightarrow x^2 + y^2 \neq 4$
- Combined with numerator: $z^2 \geq x^2 + y^2$ and $x^2 + y^2 \neq 4$

Domain: $\{(x, y, z) : z^2 \geq x^2 + y^2 \text{ and } x^2 + y^2 \neq 4\}$

(b) Geometric description: Points on and outside the double cone $z^2 = x^2 + y^2$, excluding points on the cylinder $x^2 + y^2 = 4$.

Concept Reference Guide

This reference links each concept/problem type to the relevant question numbers.

1. **Function Evaluation - Direct Substitution**
   - Two variables: Problems 1, 2
   - Three variables: Problem 3
   - With special functions: Problem 4
2. **Function Evaluation - Algebraic Expressions**
   - $f(x+h, y)$ form: Problem 2(c)
   - $f(at, bt)$ form: Problem 2(d)
   - $f(y, y)$ form: Problem 4(c)
3. **Domain - Logarithmic Functions**
   - Basic log domain: Problem 5
   - Combined with other restrictions: Problems 10, 28
   - Three variables: Problems 15, 18
4. **Domain - Even Root Functions**
   - Basic square root: Problem 6
   - Square root in denominator: Problem 7
   - Combined restrictions: Problems 10, 11, 12, 29
   - Three variables: Problems 16, 17, 30
5. **Domain - Odd Root Functions**
   - Cube root (no restriction): Problem 9
6. **Domain - Rational Functions**
   - Simple denominator restriction: Problem 8
   - Combined with other restrictions: Problems 11, 30
7. **Domain - Inverse Trigonometric Functions**
   - Arcsine: Problem 14
   - Arctangent: Problem 29
8. **Range Determination**
   - With exponential: Problem 12

- With rational function: Problem 13
- With inverse trig: Problem 14

9. **Geometric Recognition**
   - Circle: Problems 6, 8
   - Parabola: Problem 7
   - Sphere: Problem 17
   - Ellipsoid: Problem 15
   - Cone: Problems 16, 30
   - Cylinder: Problem 18
   - Ellipse: Problem 19
   - Hyperbola: Problem 20

10. **Domain Sketching**
    - Linear boundaries: Problem 5
    - Circular regions: Problems 6, 8, 12, 29
    - Parabolic regions: Problem 7
    - Combined regions: Problems 10, 11, 14, 28, 29

11. **Three-Variable Domains**
    - Ellipsoid interior: Problem 15
    - Cone exterior: Problem 16
    - Sphere exterior: Problem 17
    - Cylindrical region: Problem 18
    - Combined surfaces: Problem 30

12. **Level Curves - Identification**
    - Ellipses: Problem 19
    - Hyperbolas: Problem 20
    - Cubic curves: Problem 21
    - Grid patterns: Problem 22

13. **Level Curves - Spacing Analysis**
    - Cone vs. inverted cone: Problem 23

14. **Cylindrical Surfaces**
    - $f(x, y) = g(y)$ only: Problems 24, 25

- $f(x, y) = g(x)$ only: Problem 26

15. **Cross Sections**
    - Fixing x: Problems 24, 25
    - Fixing y: Problems 24, 25, 26
    - Different values: Problems 24, 25, 26

16. **Table/Data Interpretation**
    - Reading values: Problem 27(a)
    - Finding inputs for given outputs: Problems 27(b), 27(c)
    - Comparing rates of change: Problem 27(d)

17. **Advanced Domain Analysis**
    - Nested functions: Problem 28
    - Multiple restrictions: Problems 29, 30
    - Geometric interpretation: Problem 30(b)

18. **Inequality Manipulation**
    - Converting to standard form: Problems 6, 8, 15, 17
    - Recognizing geometric shapes: Problems 15, 16, 17, 18, 30

Study Tips

- **The Big Three Domain Restrictions:**
  1. Denominators cannot be zero
  2. Even roots require non-negative arguments
  3. Logarithms require strictly positive arguments

- **Boundary Line Rules:**
  - Strict inequality ($>$ or $<$): Dashed line
  - Non-strict inequality ($\geq$ or $\leq$): Solid line

- **Missing Variable Pattern:**
  - If y is missing from $f(x, y)$, the surface is a cylinder along the y-axis
  - If x is missing, the cylinder extends along the x-axis

- **Common Level Curve Patterns:**

- $x^2 + y^2 = k$: Circles (cone/paraboloid)
- $xy = k$: Hyperbolas (saddle surface)
- $ax + by = k$: Parallel lines (plane)
- $y = x^n + k$: Power curves (generalized cylinders)

- **Three-Variable Shapes:**
  - $x^2 + y^2 + z^2 = r^2$: Sphere
  - $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$: Ellipsoid
  - $z = \sqrt{x^2 + y^2}$: Cone
  - $x^2 + y^2 = r^2$: Cylinder (no z restriction)