

Calculus II - Test 2 Additional Problems

Based on Review by Prof. Ibrahim El Haitami - MAC 2312

1.a. Improper Integrals: Infinite Limit with Partial Fractions

1. **Integral:** $\int_3^{\infty} \frac{4}{x^2 - 4} dx$

$$\frac{4}{x^2 - 4} = \frac{4}{(x - 2)(x + 2)} = \frac{A}{x - 2} + \frac{B}{x + 2} \implies A = 1, B = -1$$

$$\begin{aligned} \int_3^{\infty} \left(\frac{1}{x - 2} - \frac{1}{x + 2} \right) dx &= \lim_{b \rightarrow \infty} [\ln |x - 2| - \ln |x + 2|]_3^b \\ &= \lim_{b \rightarrow \infty} \left[\ln \left| \frac{x - 2}{x + 2} \right| \right]_3^b \\ &= \lim_{b \rightarrow \infty} \left(\ln \left| \frac{b - 2}{b + 2} \right| - \ln \left| \frac{1}{5} \right| \right) = \ln(1) - \ln(1/5) = \ln 5 \end{aligned}$$

Result: The integral **converges** to $\ln 5$.

2. **Integral:** $\int_2^{\infty} \frac{6}{x^2 + 2x - 3} dx$

$$\frac{6}{(x + 3)(x - 1)} = \frac{A}{x + 3} + \frac{B}{x - 1} \implies A = -3/2, B = 3/2$$

$$\begin{aligned} \frac{3}{2} \int_2^{\infty} \left(\frac{1}{x - 1} - \frac{1}{x + 3} \right) dx &= \frac{3}{2} \lim_{b \rightarrow \infty} \left[\ln \left| \frac{x - 1}{x + 3} \right| \right]_2^b \\ &= \frac{3}{2} \left(\ln(1) - \ln \left| \frac{1}{5} \right| \right) = \frac{3}{2} \ln 5 \end{aligned}$$

Result: The integral **converges** to $\frac{3}{2} \ln 5$.

3. **Integral:** $\int_0^{\infty} \frac{1}{x^2 + 3x + 2} dx$

$$\frac{1}{(x + 1)(x + 2)} = \frac{1}{x + 1} - \frac{1}{x + 2}$$

$$\begin{aligned} \int_0^{\infty} \left(\frac{1}{x + 1} - \frac{1}{x + 2} \right) dx &= \lim_{b \rightarrow \infty} \left[\ln \left| \frac{x + 1}{x + 2} \right| \right]_0^b \\ &= \ln(1) - \ln(1/2) = \ln 2 \end{aligned}$$

Result: The integral **converges** to $\ln 2$.

4. **Integral:** $\int_1^{\infty} \frac{1}{x(x+1)} dx$

$$\begin{aligned} \frac{1}{x(x+1)} &= \frac{1}{x} - \frac{1}{x+1} \\ \int_1^{\infty} \left(\frac{1}{x} - \frac{1}{x+1} \right) dx &= \lim_{b \rightarrow \infty} \left[\ln \left| \frac{x}{x+1} \right| \right]_1^b \\ &= \ln(1) - \ln(1/2) = \ln 2 \end{aligned}$$

Result: The integral **converges** to $\ln 2$.

5. **Integral:** $\int_4^{\infty} \frac{3}{x^2 - 9} dx$

$$\begin{aligned} \frac{3}{(x-3)(x+3)} &= \frac{1/2}{x-3} - \frac{1/2}{x+3} \\ \frac{1}{2} \int_4^{\infty} \left(\frac{1}{x-3} - \frac{1}{x+3} \right) dx &= \frac{1}{2} \lim_{b \rightarrow \infty} \left[\ln \left| \frac{x-3}{x+3} \right| \right]_4^b \\ &= \frac{1}{2} (\ln(1) - \ln(1/7)) = \frac{1}{2} \ln 7 \end{aligned}$$

Result: The integral **converges** to $\frac{1}{2} \ln 7$.

6. **Integral:** $\int_1^{\infty} \frac{4}{x(x+2)} dx$

$$\begin{aligned} \frac{4}{x(x+2)} &= \frac{2}{x} - \frac{2}{x+2} \\ \int_1^{\infty} \left(\frac{2}{x} - \frac{2}{x+2} \right) dx &= \lim_{b \rightarrow \infty} \left[2 \ln \left| \frac{x}{x+2} \right| \right]_1^b \\ &= 2(\ln(1) - \ln(1/3)) = 2 \ln 3 \end{aligned}$$

Result: The integral **converges** to $2 \ln 3$.

7. **Integral:** $\int_2^{\infty} \frac{x-1}{x^3 + x^2 - 2x} dx$

$$\begin{aligned} \frac{x-1}{x(x+2)(x-1)} &= \frac{1}{x(x+2)} = \frac{1/2}{x} - \frac{1/2}{x+2} \\ \frac{1}{2} \int_2^{\infty} \left(\frac{1}{x} - \frac{1}{x+2} \right) dx &= \frac{1}{2} \lim_{b \rightarrow \infty} \left[\ln \left| \frac{x}{x+2} \right| \right]_2^b \\ &= \frac{1}{2} (\ln(1) - \ln(2/4)) = \frac{1}{2} \ln 2 \end{aligned}$$

Result: The integral **converges** to $\frac{1}{2} \ln 2$.

8. **Integral:** $\int_0^\infty \frac{1}{(x+1)(x+2)(x+3)} dx$

$$\frac{1}{(x+1)(x+2)(x+3)} = \frac{1/2}{x+1} - \frac{1}{x+2} + \frac{1/2}{x+3}$$

$$\begin{aligned} \int_0^\infty \dots dx &= \lim_{b \rightarrow \infty} \left[\frac{1}{2} \ln|x+1| - \ln|x+2| + \frac{1}{2} \ln|x+3| \right]_0^b \\ &= \lim_{b \rightarrow \infty} \left[\frac{1}{2} \ln \left| \frac{(x+1)(x+3)}{(x+2)^2} \right| \right]_0^b \\ &= \frac{1}{2} \left(\ln(1) - \ln\left(\frac{3}{4}\right) \right) = \frac{1}{2} \ln(4/3) \end{aligned}$$

Result: The integral **converges** to $\frac{1}{2} \ln(4/3)$.

9. **Integral:** $\int_3^\infty \frac{2x}{x^2-1} dx$

$$\int_3^\infty \frac{2x}{x^2-1} dx = \lim_{b \rightarrow \infty} [\ln|x^2-1|]_3^b = \lim_{b \rightarrow \infty} (\ln|b^2-1| - \ln 8) = \infty$$

Result: The integral **diverges**.

10. **Integral:** $\int_1^\infty \frac{x+1}{x^2+4x+3} dx$

$$\frac{x+1}{(x+1)(x+3)} = \frac{1}{x+3}$$

$$\int_1^\infty \frac{1}{x+3} dx = \lim_{b \rightarrow \infty} [\ln|x+3|]_1^b = \lim_{b \rightarrow \infty} (\ln|b+3| - \ln 4) = \infty$$

Result: The integral **diverges**.

1.b. Improper Integrals: Discontinuity with U-Substitution

1. **Integral:** $\int_0^2 \frac{x}{\sqrt{4-x^2}} dx$ Discontinuity at $x = 2$. Let $u = 4 - x^2$, $du = -2x dx$.

$$\lim_{b \rightarrow 2^-} \int_0^b \frac{x}{\sqrt{4-x^2}} dx = \lim_{b \rightarrow 2^-} [-\sqrt{4-x^2}]_0^b = 0 - (-\sqrt{4}) = 2$$

Result: The integral **converges** to 2.

2. **Integral:** $\int_1^2 \frac{1}{(x-1)^{1/3}} dx$ Discontinuity at $x = 1$.

$$\lim_{a \rightarrow 1^+} \int_a^2 (x-1)^{-1/3} dx = \lim_{a \rightarrow 1^+} \left[\frac{3}{2} (x-1)^{2/3} \right]_a^2 = \frac{3}{2} - 0 = \frac{3}{2}$$

Result: The integral **converges** to $3/2$.

3. **Integral:** $\int_1^e \frac{1}{x\sqrt{\ln x}} dx$ Discontinuity at $x = 1$. Let $u = \ln x$, $du = (1/x)dx$.

$$\lim_{a \rightarrow 1^+} \int_a^e \frac{1}{x\sqrt{\ln x}} dx = \lim_{a \rightarrow 1^+} [2\sqrt{\ln x}]_a^e = 2\sqrt{1} - 0 = 2$$

Result: The integral **converges** to 2.

4. **Integral:** $\int_0^4 \frac{1}{\sqrt{x}} dx$ Discontinuity at $x = 0$.

$$\lim_{a \rightarrow 0^+} \int_a^4 x^{-1/2} dx = \lim_{a \rightarrow 0^+} [2\sqrt{x}]_a^4 = 4 - 0 = 4$$

Result: The integral **converges** to 4.

5. **Integral:** $\int_0^{\pi/2} \frac{\cos x}{\sqrt{\sin x}} dx$ Discontinuity at $x = 0$. Let $u = \sin x$, $du = \cos x dx$.

$$\lim_{a \rightarrow 0^+} \int_a^{\pi/2} \frac{\cos x}{\sqrt{\sin x}} dx = \lim_{a \rightarrow 0^+} [2\sqrt{\sin x}]_a^{\pi/2} = 2\sqrt{1} - 0 = 2$$

Result: The integral **converges** to 2.

6. **Integral:** $\int_{-1}^0 \frac{x^2}{(x^3+1)^{1/3}} dx$ Discontinuity at $x = -1$. Let $u = x^3 + 1$, $du = 3x^2 dx$.

$$\lim_{a \rightarrow -1^+} \int_a^0 \dots dx = \lim_{a \rightarrow -1^+} \left[\frac{1}{2} (x^3+1)^{2/3} \right]_a^0 = \frac{1}{2} - 0 = \frac{1}{2}$$

Result: The integral **converges** to $1/2$.

7. **Integral:** $\int_0^{\ln 2} \frac{e^x}{e^x - 1} dx$ Discontinuity at $x = 0$. Let $u = e^x - 1$, $du = e^x dx$.

$$\lim_{a \rightarrow 0^+} \int_a^{\ln 2} \frac{e^x}{e^x - 1} dx = \lim_{a \rightarrow 0^+} [\ln |e^x - 1|]_a^{\ln 2} = \ln(1) - \lim_{a \rightarrow 0^+} \ln |e^a - 1| = \infty$$

Result: The integral **diverges**.

8. **Integral:** $\int_2^3 \frac{3x^2}{\sqrt{x^3 - 8}} dx$ Discontinuity at $x = 2$. Let $u = x^3 - 8$, $du = 3x^2 dx$.

$$\lim_{a \rightarrow 2^+} \int_a^3 \dots dx = \lim_{a \rightarrow 2^+} [2\sqrt{x^3 - 8}]_a^3 = 2\sqrt{19} - 0 = 2\sqrt{19}$$

Result: The integral **converges** to $2\sqrt{19}$.

9. **Integral:** $\int_0^1 \frac{\arcsin x}{\sqrt{1 - x^2}} dx$ Discontinuity at $x = 1$. Let $u = \arcsin x$, $du = dx/\sqrt{1 - x^2}$.

$$\begin{aligned} \lim_{b \rightarrow 1^-} \int_0^b \frac{\arcsin x}{\sqrt{1 - x^2}} dx &= \lim_{b \rightarrow 1^-} \left[\frac{1}{2} (\arcsin x)^2 \right]_0^b \\ &= \frac{1}{2} (\arcsin 1)^2 - 0 = \frac{1}{2} (\pi/2)^2 = \frac{\pi^2}{8} \end{aligned}$$

Result: The integral **converges** to $\pi^2/8$.

10. **Integral:** $\int_0^1 \frac{1}{(1 - x)^{2/3}} dx$ Discontinuity at $x = 1$. Let $u = 1 - x$, $du = -dx$.

$$\lim_{b \rightarrow 1^-} \int_0^b (1 - x)^{-2/3} dx = \lim_{b \rightarrow 1^-} [-3(1 - x)^{1/3}]_0^b = 0 - (-3) = 3$$

Result: The integral **converges** to 3.

1.c. Improper Integrals: Vertical Asymptote

1. **Integral:** $\int_{\pi/2}^{\pi} \cot \theta \, d\theta$ Asymptote at $\theta = \pi$.

$$\lim_{b \rightarrow \pi^-} \int_{\pi/2}^b \cot \theta \, d\theta = \lim_{b \rightarrow \pi^-} [\ln |\sin \theta|]_{\pi/2}^b = (-\infty) - \ln(1) = -\infty$$

Result: The integral **diverges**.

2. **Integral:** $\int_0^{\pi/2} \sec \theta \, d\theta$ Asymptote at $\theta = \pi/2$.

$$\lim_{b \rightarrow (\pi/2)^-} \int_0^b \sec \theta \, d\theta = \lim_{b \rightarrow (\pi/2)^-} [\ln |\sec \theta + \tan \theta|]_0^b = \infty - 0 = \infty$$

Result: The integral **diverges**.

3. **Integral:** $\int_0^1 \ln x \, dx$ Asymptote at $x = 0$. Integration by parts.

$$\lim_{a \rightarrow 0^+} \int_a^1 \ln x \, dx = \lim_{a \rightarrow 0^+} [x \ln x - x]_a^1 = (-1) - \lim_{a \rightarrow 0^+} (a \ln a - a) = -1$$

Result: The integral **converges** to -1 .

4. **Integral:** $\int_0^1 \frac{1}{x-1} \, dx$ Asymptote at $x = 1$.

$$\lim_{b \rightarrow 1^-} \int_0^b \frac{1}{x-1} \, dx = \lim_{b \rightarrow 1^-} [\ln |x-1|]_0^b = (-\infty) - 0 = -\infty$$

Result: The integral **diverges**.

5. **Integral:** $\int_{-\pi/2}^0 \tan \theta \, d\theta$ Asymptote at $\theta = -\pi/2$.

$$\lim_{a \rightarrow (-\pi/2)^+} \int_a^0 \tan \theta \, d\theta = \lim_{a \rightarrow (-\pi/2)^+} [-\ln |\cos \theta|]_a^0 = 0 - (-\infty) = \infty$$

Result: The integral **diverges**.

6. **Integral:** $\int_0^2 \frac{1}{(x-2)^2} \, dx$ Asymptote at $x = 2$.

$$\lim_{b \rightarrow 2^-} \int_0^b (x-2)^{-2} \, dx = \lim_{b \rightarrow 2^-} \left[-\frac{1}{x-2} \right]_0^b = \infty - \frac{1}{2} = \infty$$

Result: The integral **diverges**.

7. **Integral:** $\int_1^2 \frac{1}{\sqrt[3]{x-1}} \, dx$ Asymptote at $x = 1$.

$$\lim_{a \rightarrow 1^+} \int_a^2 (x-1)^{-1/3} \, dx = \lim_{a \rightarrow 1^+} \left[\frac{3}{2} (x-1)^{2/3} \right]_a^2 = \frac{3}{2}$$

Result: The integral **converges** to $3/2$.

8. **Integral:** $\int_0^\pi \csc \theta \, d\theta$ *Asymptotes at $\theta = 0, \pi$. Split at $\pi/2$.*

$$\int_0^{\pi/2} \csc \theta \, d\theta = \lim_{a \rightarrow 0^+} [-\ln |\csc \theta + \cot \theta|]_a^{\pi/2} = \infty$$

Result: The integral **diverges**.

9. **Integral:** $\int_0^1 \frac{1}{x} \, dx$ *Asymptote at $x = 0$.*

$$\lim_{a \rightarrow 0^+} \int_a^1 \frac{1}{x} \, dx = \lim_{a \rightarrow 0^+} [\ln |x|]_a^1 = 0 - (-\infty) = \infty$$

Result: The integral **diverges**.

10. **Integral:** $\int_1^3 \frac{1}{x-3} \, dx$ *Asymptote at $x = 3$.*

$$\lim_{b \rightarrow 3^-} \int_1^b \frac{1}{x-3} \, dx = \lim_{b \rightarrow 3^-} [\ln |x-3|]_1^b = -\infty$$

Result: The integral **diverges**.

1.d. Improper Integrals: Discontinuity and Standard Forms

1. **Integral:** $\int_{-1}^0 \frac{1}{\sqrt{1-x^2}} dx$ *Discontinuity at $x = -1$.*

$$\lim_{a \rightarrow -1^+} \int_a^0 \frac{1}{\sqrt{1-x^2}} dx = \lim_{a \rightarrow -1^+} [\arcsin(x)]_a^0 = 0 - (-\pi/2) = \frac{\pi}{2}$$

Result: The integral **converges** to $\pi/2$.

2. **Integral:** $\int_0^2 \frac{1}{\sqrt{4-x^2}} dx$ *Discontinuity at $x = 2$.*

$$\lim_{b \rightarrow 2^-} \int_0^b \frac{1}{\sqrt{4-x^2}} dx = \lim_{b \rightarrow 2^-} [\arcsin(x/2)]_0^b = \pi/2 - 0 = \frac{\pi}{2}$$

Result: The integral **converges** to $\pi/2$.

3. **Integral:** $\int_0^\infty \frac{1}{1+x^2} dx$ *Infinite upper limit.*

$$\lim_{b \rightarrow \infty} \int_0^b \frac{1}{1+x^2} dx = \lim_{b \rightarrow \infty} [\arctan(x)]_0^b = \pi/2 - 0 = \frac{\pi}{2}$$

Result: The integral **converges** to $\pi/2$.

4. **Integral:** $\int_0^2 \frac{1}{x^2+4} dx$ *Standard integral, no discontinuity.*

$$\left[\frac{1}{2} \arctan(x/2) \right]_0^2 = \frac{1}{2} \arctan(1) - 0 = \frac{1}{2} (\pi/4) = \frac{\pi}{8}$$

Result: This is a proper integral. The value is $\pi/8$.

5. **Integral:** $\int_0^3 \frac{1}{\sqrt{3-x}} dx$ *Discontinuity at $x = 3$.*

$$\lim_{b \rightarrow 3^-} \int_0^b (3-x)^{-1/2} dx = \lim_{b \rightarrow 3^-} [-2\sqrt{3-x}]_0^b = 0 - (-2\sqrt{3}) = 2\sqrt{3}$$

Result: The integral **converges** to $2\sqrt{3}$.

6. **Integral:** $\int_{-3}^0 \frac{1}{\sqrt{9-x^2}} dx$ *Discontinuity at $x = -3$.*

$$\lim_{a \rightarrow -3^+} \int_a^0 \frac{1}{\sqrt{9-x^2}} dx = \lim_{a \rightarrow -3^+} [\arcsin(x/3)]_a^0 = 0 - (-\pi/2) = \frac{\pi}{2}$$

Result: The integral **converges** to $\pi/2$.

7. **Integral:** $\int_{-\infty}^0 e^x dx$ *Infinite lower limit.*

$$\lim_{a \rightarrow -\infty} \int_a^0 e^x dx = \lim_{a \rightarrow -\infty} [e^x]_a^0 = 1 - 0 = 1$$

Result: The integral **converges** to 1.

8. **Integral:** $\int_1^2 \frac{1}{(x-1)^{2/3}} dx$ *Discontinuity at $x = 1$.*

$$\lim_{a \rightarrow 1^+} \int_a^2 (x-1)^{-2/3} dx = \lim_{a \rightarrow 1^+} [3(x-1)^{1/3}]_a^2 = 3 - 0 = 3$$

Result: The integral **converges** to 3.

9. **Integral:** $\int_2^4 \frac{1}{\sqrt{x-2}} dx$ *Discontinuity at $x = 2$.*

$$\lim_{a \rightarrow 2^+} \int_a^4 (x-2)^{-1/2} dx = \lim_{a \rightarrow 2^+} [2\sqrt{x-2}]_a^4 = 2\sqrt{2} - 0 = 2\sqrt{2}$$

Result: The integral **converges** to $2\sqrt{2}$.

10. **Integral:** $\int_0^\infty \frac{e^{-x}}{2} dx$ *Infinite upper limit.*

$$\lim_{b \rightarrow \infty} \int_0^b \frac{1}{2} e^{-x} dx = \lim_{b \rightarrow \infty} [-\frac{1}{2} e^{-x}]_0^b = 0 - (-\frac{1}{2}) = \frac{1}{2}$$

Result: The integral **converges** to $1/2$.

1.e. Improper Integrals: Infinite Limits in Both Directions

1. **Integral:** $\int_{-\infty}^{\infty} xe^{-x^2} dx$ *Odd function over a symmetric interval.*

$$\int_0^{\infty} xe^{-x^2} dx = \lim_{b \rightarrow \infty} \left[-\frac{1}{2}e^{-x^2}\right]_0^b = 1/2. \quad \int_{-\infty}^0 \dots = -1/2.$$

Result: The integral **converges** to $1/2 - 1/2 = 0$.

2. **Integral:** $\int_{-\infty}^{\infty} \frac{1}{x^2 + 9} dx$

$$\int_{-\infty}^{\infty} \frac{1}{x^2 + 9} dx = \left[\frac{1}{3} \arctan(x/3)\right]_{-\infty}^{\infty} = \frac{1}{3}(\pi/2 - (-\pi/2)) = \frac{\pi}{3}$$

Result: The integral **converges** to $\pi/3$.

3. **Integral:** $\int_{-\infty}^{\infty} x^3 e^{-x^4} dx$ *Odd function. Converges to 0.*

$$\int_0^{\infty} x^3 e^{-x^4} dx = \lim_{b \rightarrow \infty} [-e^{-x^4}/4]_0^b = 1/4. \quad \int_{-\infty}^0 \dots = -1/4.$$

Result: The integral **converges** to 0.

4. **Integral:** $\int_{-\infty}^{\infty} \frac{e^x}{1 + e^{2x}} dx$ *Let $u = e^x$. Limits become 0 to ∞ .*

$$\int_0^{\infty} \frac{1}{1 + u^2} du = [\arctan u]_0^{\infty} = \pi/2$$

Result: The integral **converges** to $\pi/2$.

5. **Integral:** $\int_{-\infty}^{\infty} \frac{1}{x^2 + 2x + 2} dx$ *Complete the square: $(x + 1)^2 + 1$.*

$$\int_{-\infty}^{\infty} \frac{1}{(x + 1)^2 + 1} dx = [\arctan(x + 1)]_{-\infty}^{\infty} = \pi/2 - (-\pi/2) = \pi$$

Result: The integral **converges** to π .

6. **Integral:** $\int_{-\infty}^{\infty} \frac{1}{e^x + e^{-x}} dx$ *Multiply by e^x/e^x : $\int \frac{e^x}{e^{2x} + 1} dx$. Same as #4. **Result:** The integral **converges** to $\pi/2$.*

7. **Integral:** $\int_{-\infty}^{\infty} \frac{x^2}{(x^3 - 1)^2} dx$ *Discontinuity at $x = 1$. Let's check convergence near 1.*

$$\lim_{a \rightarrow 1^+} \int_a^2 \frac{x^2}{(x^3 - 1)^2} dx = \lim_{a \rightarrow 1^+} \left[-\frac{1}{3(x^3 - 1)}\right]_a^2 = -\frac{1}{21} - (-\infty) = \infty$$

Result: The integral **diverges**.

8. **Integral:** $\int_{-\infty}^{\infty} \frac{x}{(x^2 + 1)^2} dx$ *Odd function. Let's check convergence.*

$$\int_0^{\infty} \frac{x}{(x^2 + 1)^2} dx = \lim_{b \rightarrow \infty} \left[-\frac{1}{2(x^2 + 1)} \right]_0^b = 0 - \left(-\frac{1}{2} \right) = \frac{1}{2}.$$

Result: The integral **converges** to $1/2 - 1/2 = 0$.

9. **Integral:** $\int_{-\infty}^{\infty} \frac{x}{x^4 + 1} dx$ *Odd function. Let's check convergence.*

$$\int_0^{\infty} \frac{x}{x^4 + 1} dx = \lim_{b \rightarrow \infty} \left[\frac{1}{2} \arctan(x^2) \right]_0^b = \frac{1}{2} (\pi/2 - 0) = \pi/4.$$

Result: The integral **converges** to $\pi/4 - \pi/4 = 0$.

10. **Integral:** $\int_{-\infty}^{\infty} x dx$

$$\int_0^{\infty} x dx = \lim_{b \rightarrow \infty} [x^2/2]_0^b = \infty.$$

Result: The integral **diverges**.

2. Arc Length: Perfect Square Integrands

1. **Curve:** $y = \frac{x^3}{6} + \frac{1}{2x}$ from $x = 1$ to $x = 2$

$$\begin{aligned} y' = \frac{x^2}{2} - \frac{1}{2x^2} &\implies 1 + (y')^2 = 1 + \left(\frac{x^4}{4} - \frac{1}{2} + \frac{1}{4x^4} \right) \\ &= \frac{x^4}{4} + \frac{1}{2} + \frac{1}{4x^4} = \left(\frac{x^2}{2} + \frac{1}{2x^2} \right)^2 \\ L = \int_1^2 \left(\frac{x^2}{2} + \frac{1}{2x^2} \right) dx &= \left[\frac{x^3}{6} - \frac{1}{2x} \right]_1^2 = \left(\frac{8}{6} - \frac{1}{4} \right) - \left(\frac{1}{6} - \frac{1}{2} \right) = \frac{17}{12} \end{aligned}$$

Length: $17/12$.

2. **Curve:** $y = \frac{2}{3}(x-1)^{3/2}$ from $x = 1$ to $x = 5$

$$\begin{aligned} y' = (x-1)^{1/2} &\implies 1 + (y')^2 = 1 + (x-1) = x \\ L = \int_1^5 \sqrt{x} dx &= \left[\frac{2}{3}x^{3/2} \right]_1^5 = \frac{2}{3}(5\sqrt{5} - 1) \end{aligned}$$

Length: $\frac{2}{3}(5\sqrt{5} - 1)$.

3. **Curve:** $y = \ln(\cos x)$ from $x = 0$ to $x = \pi/4$

$$\begin{aligned} y' = \frac{-\sin x}{\cos x} = -\tan x &\implies 1 + (y')^2 = 1 + \tan^2 x = \sec^2 x \\ L = \int_0^{\pi/4} \sec x dx &= [\ln |\sec x + \tan x|]_0^{\pi/4} = \ln(\sqrt{2} + 1) \end{aligned}$$

Length: $\ln(\sqrt{2} + 1)$.

4. **Curve:** $y = \frac{x^2}{4} - \frac{\ln x}{2}$ from $x = 1$ to $x = e$

$$\begin{aligned} y' = \frac{x}{2} - \frac{1}{2x} &\implies 1 + (y')^2 = 1 + \left(\frac{x^2}{4} - \frac{1}{2} + \frac{1}{4x^2} \right) = \left(\frac{x}{2} + \frac{1}{2x} \right)^2 \\ L = \int_1^e \left(\frac{x}{2} + \frac{1}{2x} \right) dx &= \left[\frac{x^2}{4} + \frac{\ln x}{2} \right]_1^e = \left(\frac{e^2}{4} + \frac{1}{2} \right) - \left(\frac{1}{4} + 0 \right) = \frac{e^2 + 1}{4} \end{aligned}$$

Length: $(e^2 + 1)/4$.

5. **Curve:** $y = \frac{2}{3}x^{3/2}$ from $x = 0$ to $x = 8$

$$\begin{aligned} y' = x^{1/2} &\implies 1 + (y')^2 = 1 + x \\ L = \int_0^8 \sqrt{1+x} dx &= \left[\frac{2}{3}(1+x)^{3/2} \right]_0^8 = \frac{2}{3}(9^{3/2} - 1^{3/2}) = \frac{2}{3}(27 - 1) = \frac{52}{3} \end{aligned}$$

Length: $52/3$.

6. **Curve:** $y = \cosh(x)$ from $x = 0$ to $x = \ln 2$

$$\begin{aligned} y' = \sinh(x) &\implies 1 + (y')^2 = 1 + \sinh^2 x = \cosh^2 x \\ L = \int_0^{\ln 2} \cosh x dx &= [\sinh x]_0^{\ln 2} = \sinh(\ln 2) - 0 = \frac{e^{\ln 2} - e^{-\ln 2}}{2} = \frac{2 - 1/2}{2} = \frac{3}{4} \end{aligned}$$

Length: $3/4$.

7. **Curve:** $y = \frac{x^4}{4} + \frac{1}{8x^2}$ from $x = 1$ to $x = 2$

$$y' = x^3 - \frac{1}{4x^3} \implies 1 + (y')^2 = 1 + (x^6 - \frac{1}{2} + \frac{1}{16x^6}) = (x^3 + \frac{1}{4x^3})^2$$

$$L = \int_1^2 (x^3 + \frac{1}{4x^3}) dx = [\frac{x^4}{4} - \frac{1}{8x^2}]_1^2 = (4 - \frac{1}{32}) - (\frac{1}{4} - \frac{1}{8}) = \frac{123}{32}$$

Length: $123/32$.

8. **Curve:** $y = \frac{1}{3}(x^2 - 2)^{3/2}$ from $x = \sqrt{2}$ to $x = 3$

$$y' = x\sqrt{x^2 - 2} \implies 1 + (y')^2 = 1 + x^2(x^2 - 2) = x^4 - 2x^2 + 1 = (x^2 - 1)^2$$

$$L = \int_{\sqrt{2}}^3 (x^2 - 1) dx = [\frac{x^3}{3} - x]_{\sqrt{2}}^3 = (9 - 3) - (\frac{2\sqrt{2}}{3} - \sqrt{2}) = 6 + \frac{\sqrt{2}}{3}$$

Length: $6 + \sqrt{2}/3$.

9. **Curve:** $y = \ln(\csc x - \cot x)$ from $x = \pi/6$ to $x = \pi/2$

$$y' = \frac{-\csc x \cot x + \csc^2 x}{\csc x - \cot x} = \frac{\csc x(\csc x - \cot x)}{\csc x - \cot x} = \csc x$$

$1 + (y')^2 = 1 + \csc^2 x$ This does not simplify well. Typo in problem design.

Corrected Curve: $y = \ln(\sin x)$ from $x = \pi/4$ to $x = \pi/2$.

$$y' = \frac{\cos x}{\sin x} = \cot x \implies 1 + (y')^2 = 1 + \cot^2 x = \csc^2 x$$

$$L = \int_{\pi/4}^{\pi/2} \csc x dx = [-\ln |\csc x + \cot x|]_{\pi/4}^{\pi/2}$$

$$= (-\ln |1 + 0|) - (-\ln |\sqrt{2} + 1|) = \ln(\sqrt{2} + 1)$$

Length: $\ln(\sqrt{2} + 1)$.

10. **Curve:** $y = \frac{x^5}{10} + \frac{1}{6x^3}$ from $x = 1$ to $x = 2$

$$y' = \frac{x^4}{2} - \frac{1}{2x^4} \implies 1 + (y')^2 = 1 + (\frac{x^8}{4} - \frac{1}{2} + \frac{1}{4x^8}) = (\frac{x^4}{2} + \frac{1}{2x^4})^2$$

$$L = \int_1^2 (\frac{x^4}{2} + \frac{1}{2x^4}) dx = [\frac{x^5}{10} - \frac{1}{6x^3}]_1^2 = (\frac{32}{10} - \frac{1}{48}) - (\frac{1}{10} - \frac{1}{6}) = \frac{779}{240}$$

Length: $779/240$.

3. Surface Area of Revolution

1. **Curve:** $y = \sqrt{9 - x^2}, 0 \leq x \leq 3$; about the x-axis.

$$y' = \frac{-x}{\sqrt{9 - x^2}} \implies 1 + (y')^2 = 1 + \frac{x^2}{9 - x^2} = \frac{9}{9 - x^2}$$

$$S = \int_0^3 2\pi \sqrt{9 - x^2} \sqrt{\frac{9}{9 - x^2}} dx = \int_0^3 2\pi(3) dx = [6\pi x]_0^3 = 18\pi$$

Surface Area: 18π . (Surface of a hemisphere).

2. **Curve:** $y = x^3$ from $x = 0$ to $x = 1$; about the x-axis.

$$y' = 3x^2 \implies 1 + (y')^2 = 1 + 9x^4$$

$$S = \int_0^1 2\pi x^3 \sqrt{1 + 9x^4} dx \quad (u = 1 + 9x^4, du = 36x^3 dx)$$

$$= \frac{2\pi}{36} \int_1^{10} \sqrt{u} du = \frac{\pi}{18} \left[\frac{2}{3} u^{3/2} \right]_1^{10} = \frac{\pi}{27} (10\sqrt{10} - 1)$$

Surface Area: $\frac{\pi}{27} (10\sqrt{10} - 1)$.

3. **Curve:** $y = \sqrt{x}, 1 \leq x \leq 4$; about the x-axis.

$$y' = \frac{1}{2\sqrt{x}} \implies 1 + (y')^2 = 1 + \frac{1}{4x} = \frac{4x + 1}{4x}$$

$$S = \int_1^4 2\pi \sqrt{x} \sqrt{\frac{4x + 1}{4x}} dx = \int_1^4 \pi \sqrt{4x + 1} dx \quad (u = 4x + 1)$$

$$= \frac{\pi}{4} \int_5^{17} \sqrt{u} du = \frac{\pi}{4} \left[\frac{2}{3} u^{3/2} \right]_5^{17} = \frac{\pi}{6} (17\sqrt{17} - 5\sqrt{5})$$

Surface Area: $\frac{\pi}{6} (17\sqrt{17} - 5\sqrt{5})$.

4. **Curve:** $y = \cosh(x), 0 \leq x \leq 1$; about the x-axis.

$$y' = \sinh(x) \implies 1 + (y')^2 = \cosh^2(x)$$

$$S = \int_0^1 2\pi \cosh(x) \sqrt{\cosh^2(x)} dx = 2\pi \int_0^1 \cosh^2(x) dx$$

$$= \pi \int_0^1 (1 + \cosh(2x)) dx = \pi \left[x + \frac{1}{2} \sinh(2x) \right]_0^1 = \pi \left(1 + \frac{1}{2} \sinh 2 \right)$$

Surface Area: $\pi \left(1 + \frac{1}{2} \sinh 2 \right)$.

5. **Curve:** $y = e^{-x}, 0 \leq x \leq \infty$; about the x-axis.

$$y' = -e^{-x} \implies 1 + (y')^2 = 1 + e^{-2x}$$

$$S = \int_0^\infty 2\pi e^{-x} \sqrt{1 + e^{-2x}} dx \quad (u = e^{-x}, du = -e^{-x} dx)$$

$$= 2\pi \int_1^0 \sqrt{1 + u^2} (-du) = 2\pi \int_0^1 \sqrt{1 + u^2} du \quad (\text{Trig sub } u = \tan \theta)$$

$$= 2\pi \left[\frac{u}{2} \sqrt{1 + u^2} + \frac{1}{2} \ln |u + \sqrt{1 + u^2}| \right]_0^1 = \pi (\sqrt{2} + \ln(1 + \sqrt{2}))$$

Surface Area: $\pi (\sqrt{2} + \ln(1 + \sqrt{2}))$.

6. **Curve:** $y = \frac{x^3}{6} + \frac{1}{2x}$, $1 \leq x \leq 2$; about the x-axis.

$$y' = \frac{x^2}{2} - \frac{1}{2x^2} \implies 1 + (y')^2 = \left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2$$

$$\begin{aligned} S &= \int_1^2 2\pi \left(\frac{x^3}{6} + \frac{1}{2x}\right) \left(\frac{x^2}{2} + \frac{1}{2x^2}\right) dx \\ &= 2\pi \int_1^2 \left(\frac{x^5}{12} + \frac{x}{12} + \frac{x}{4} + \frac{1}{4x^3}\right) dx = 2\pi \left[\frac{x^6}{72} + \frac{x^2}{6} - \frac{1}{8x^2}\right]_1^2 \end{aligned}$$

Surface Area: $47\pi/16$.

7. **Curve:** $x = \frac{1}{3}(y^2 + 2)^{3/2}$, $1 \leq y \leq 2$; about the y-axis.

$$\begin{aligned} \frac{dx}{dy} &= y\sqrt{y^2 + 2} \implies 1 + (dx/dy)^2 = (y^2 + 1)^2 \\ S &= \int_1^2 2\pi y \sqrt{(y^2 + 1)^2} dy = \int_1^2 2\pi y (y^2 + 1) dy \\ &= 2\pi \left[\frac{y^4}{4} + \frac{y^2}{2}\right]_1^2 = 2\pi \left[(4 + 2) - \left(\frac{1}{4} + \frac{1}{2}\right)\right] = \frac{21\pi}{2} \end{aligned}$$

Surface Area: $21\pi/2$.

8. **Curve:** $y = 1 - x^2$, $0 \leq x \leq 1$; about the y-axis.

$$\begin{aligned} S &= \int_0^1 2\pi x \sqrt{1 + (-2x)^2} dx = 2\pi \int_0^1 x \sqrt{1 + 4x^2} dx \quad (u = 1 + 4x^2) \\ &= \frac{2\pi}{8} \int_1^5 \sqrt{u} du = \frac{\pi}{4} \left[\frac{2}{3} u^{3/2}\right]_1^5 = \frac{\pi}{6} (5\sqrt{5} - 1) \end{aligned}$$

Surface Area: $\frac{\pi}{6}(5\sqrt{5} - 1)$.

9. **Curve:** $y = \sin x$, $0 \leq x \leq \pi$; about the x-axis.

$$\begin{aligned} S &= \int_0^\pi 2\pi \sin x \sqrt{1 + \cos^2 x} dx \quad (u = \cos x, du = -\sin x dx) \\ &= 2\pi \int_{-1}^1 \sqrt{1 + u^2} du = 2\pi \left[\frac{u}{2} \sqrt{1 + u^2} + \frac{1}{2} \ln |u + \sqrt{1 + u^2}|\right]_{-1}^1 \\ &= 2\pi (\sqrt{2} + \ln(1 + \sqrt{2})) \end{aligned}$$

Surface Area: $2\pi(\sqrt{2} + \ln(1 + \sqrt{2}))$.

10. **Curve:** $y = \frac{1}{4}x^4 + \frac{1}{8}x^{-2}$, $1 \leq x \leq 2$; about the y-axis.

$$\begin{aligned} y' &= x^3 - \frac{1}{4}x^{-3} \implies 1 + (y')^2 = \left(x^3 + \frac{1}{4}x^{-3}\right)^2 \\ S &= \int_1^2 2\pi x \sqrt{\left(x^3 + \frac{1}{4}x^{-3}\right)^2} dx = \int_1^2 2\pi x \left(x^3 + \frac{1}{4}x^{-3}\right) dx \\ &= 2\pi \int_1^2 \left(x^4 + \frac{1}{4x^2}\right) dx = 2\pi \left[\frac{x^5}{5} - \frac{1}{4x}\right]_1^2 = \frac{253\pi}{20} \end{aligned}$$

Surface Area: $253\pi/20$.

4. Parametric to Cartesian Equations

1. **Equations:** $x = 3 \cos t, y = 5 \sin t, 0 \leq t \leq 2\pi$

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{5}\right)^2 = \cos^2 t + \sin^2 t = 1 \implies \frac{x^2}{9} + \frac{y^2}{25} = 1$$

Path: Ellipse centered at (0,0), major vertical axis length 10, minor horizontal axis length 6. **Direction:** At $t = 0$, (3, 0). At $t = \pi/2$, (0, 5). Counter-clockwise.

2. **Equations:** $x = t^2, y = t^3 - 3t$

$$y = t(t^2 - 3) = \pm\sqrt{x}(x - 3) \implies y^2 = x(x - 3)^2$$

Path: A self-intersecting cubic curve. **Direction:** For $t < 0$, starts at top-left, moves to (3,0). For $t > 0$, moves away from (3,0) to top-right.

3. **Equations:** $x = 3 + 2 \sec t, y = 1 + 4 \tan t$

$$\sec t = \frac{x-3}{2}, \tan t = \frac{y-1}{4}. \quad \sec^2 t - \tan^2 t = 1 \implies \left(\frac{x-3}{2}\right)^2 - \left(\frac{y-1}{4}\right)^2 = 1$$

Path: Hyperbola centered at (3,1).

4. **Equations:** $x = e^t, y = e^{-2t}$

$$y = (e^t)^{-2} = x^{-2} = \frac{1}{x^2}. \text{ Since } x = e^t > 0, \text{ it's only the right branch.}$$

Path: Part of the reciprocal square function in the first quadrant.

5. **Equations:** $x = 2 \sin t - 1, y = 3 \cos t + 2, 0 \leq t \leq 2\pi$

$$\left(\frac{x+1}{2}\right)^2 + \left(\frac{y-2}{3}\right)^2 = 1$$

Path: Ellipse centered at (-1, 2). **Direction:** At $t = 0$, (-1, 5). At $t = \pi/2$, (1, 2). Clockwise.

6. **Equations:** $x = \sqrt{t}, y = 1 - t$

$$t = x^2 \implies y = 1 - x^2. \text{ Since } x = \sqrt{t} \geq 0, \text{ it's the right half of the parabola.}$$

Path: Parabola opening down, vertex at (0,1), for $x \geq 0$.

7. **Equations:** $x = \sin t, y = \csc t, 0 < t < \pi$

$$y = \frac{1}{\sin t} = \frac{1}{x}$$

Path: Reciprocal function for $x \in (0, 1]$.

8. **Equations:** $x = \cos(2t), y = \sin t$

$$x = 1 - 2 \sin^2 t = 1 - 2y^2$$

Path: Parabola opening left, vertex at (1,0).

9. **Equations:** $x = 4t^2 - 4, y = t, -\infty < t < \infty$

$$x = 4y^2 - 4$$

Path: Parabola opening right, vertex at (-4,0).

10. **Equations:** $x = t - 1, y = t^2 - 2t + 2$

$$t = x + 1 \implies y = (x + 1)^2 - 2(x + 1) + 2 = x^2 + 2x + 1 - 2x - 2 + 2 = x^2 + 1$$

Path: Parabola opening up, vertex at (0,1).

5. Parametric Derivatives

1. **Equations:** $x = t^3 - 3t, y = t^2 - 2$

$$\begin{aligned}\frac{dx}{dt} &= 3t^2 - 3, \frac{dy}{dt} = 2t \implies \frac{dy}{dx} = \frac{2t}{3(t^2 - 1)} \\ \frac{d}{dt}\left(\frac{dy}{dx}\right) &= \frac{2(3t^2 - 3) - 2t(6t)}{9(t^2 - 1)^2} = \frac{-6t^2 - 6}{9(t^2 - 1)^2} = \frac{-2(t^2 + 1)}{3(t^2 - 1)^2} \\ \frac{d^2y}{dx^2} &= \frac{-2(t^2 + 1)}{3(t^2 - 1)^2} \cdot \frac{1}{3(t^2 - 1)} = \frac{-2(t^2 + 1)}{9(t^2 - 1)^3}\end{aligned}$$

2. **Equations:** $x = e^t, y = te^{-t}$

$$\begin{aligned}\frac{dx}{dt} &= e^t, \frac{dy}{dt} = e^{-t} - te^{-t} \implies \frac{dy}{dx} = \frac{e^{-t}(1 - t)}{e^t} = e^{-2t}(1 - t) \\ \frac{d}{dt}\left(\frac{dy}{dx}\right) &= -2e^{-2t}(1 - t) + e^{-2t}(-1) = e^{-2t}(2t - 3) \\ \frac{d^2y}{dx^2} &= \frac{e^{-2t}(2t - 3)}{e^t} = e^{-3t}(2t - 3)\end{aligned}$$

3. **Equations:** $x = 2 \cos t, y = 3 \sin t$

$$\begin{aligned}\frac{dx}{dt} &= -2 \sin t, \frac{dy}{dt} = 3 \cos t \implies \frac{dy}{dx} = -\frac{3 \cos t}{2 \sin t} = -\frac{3}{2} \cot t \\ \frac{d}{dt}\left(\frac{dy}{dx}\right) &= \frac{3}{2} \csc^2 t \\ \frac{d^2y}{dx^2} &= \frac{(3/2) \csc^2 t}{-2 \sin t} = -\frac{3}{4 \sin^3 t} = -\frac{3}{4} \csc^3 t\end{aligned}$$

4. **Equations:** $x = t/(1 + t), y = t^2$

$$\begin{aligned}\frac{dx}{dt} &= \frac{1}{(1 + t)^2}, \frac{dy}{dt} = 2t \implies \frac{dy}{dx} = 2t(1 + t)^2 \\ \frac{d}{dt}\left(\frac{dy}{dx}\right) &= 2(1 + t)^2 + 2t(2(1 + t)) = 2(1 + t)(1 + t + 2t) = 2(1 + t)(1 + 3t) \\ \frac{d^2y}{dx^2} &= 2(1 + t)(1 + 3t) \cdot (1 + t)^2 = 2(1 + t)^3(1 + 3t)\end{aligned}$$

5. **Equations:** $x = \ln t, y = t^2 + 1$

$$\begin{aligned}\frac{dx}{dt} &= 1/t, \frac{dy}{dt} = 2t \implies \frac{dy}{dx} = \frac{2t}{1/t} = 2t^2 \\ \frac{d}{dt}\left(\frac{dy}{dx}\right) &= 4t \implies \frac{d^2y}{dx^2} = \frac{4t}{1/t} = 4t^2\end{aligned}$$

6. **Equations:** $x = a(t - \sin t), y = a(1 - \cos t)$

$$\begin{aligned}\frac{dx}{dt} &= a(1 - \cos t), \frac{dy}{dt} = a \sin t \implies \frac{dy}{dx} = \frac{\sin t}{1 - \cos t} = \cot(t/2) \\ \frac{d}{dt}\left(\frac{dy}{dx}\right) &= -\frac{1}{2} \csc^2(t/2) \\ \frac{d^2y}{dx^2} &= \frac{-\frac{1}{2} \csc^2(t/2)}{a(1 - \cos t)} = \frac{-\frac{1}{2} \csc^2(t/2)}{a(2 \sin^2(t/2))} = -\frac{1}{4a} \csc^4(t/2)\end{aligned}$$

7. **Equations:** $x = t^2 + 1, y = t^3 - 1$

$$\begin{aligned}\frac{dx}{dt} &= 2t, \frac{dy}{dt} = 3t^2 \implies \frac{dy}{dx} = \frac{3t}{2} \\ \frac{d}{dt}\left(\frac{dy}{dx}\right) &= \frac{3}{2} \implies \frac{d^2y}{dx^2} = \frac{3/2}{2t} = \frac{3}{4t}\end{aligned}$$

8. **Equations:** $x = \cos^3 t, y = \sin^3 t$

$$\begin{aligned}\frac{dx}{dt} &= -3\cos^2 t \sin t, \frac{dy}{dt} = 3\sin^2 t \cos t \implies \frac{dy}{dx} = -\tan t \\ \frac{d}{dt}\left(\frac{dy}{dx}\right) &= -\sec^2 t \implies \frac{d^2y}{dx^2} = \frac{-\sec^2 t}{-3\cos^2 t \sin t} = \frac{1}{3\cos^4 t \sin t}\end{aligned}$$

9. **Equations:** $x = \sqrt{t}, y = t$

$$\begin{aligned}\frac{dx}{dt} &= \frac{1}{2\sqrt{t}}, \frac{dy}{dt} = 1 \implies \frac{dy}{dx} = 2\sqrt{t} \\ \frac{d}{dt}\left(\frac{dy}{dx}\right) &= \frac{1}{\sqrt{t}} \implies \frac{d^2y}{dx^2} = \frac{1/\sqrt{t}}{1/(2\sqrt{t})} = 2\end{aligned}$$

(Also clear from $y = x^2 \implies y'' = 2$).

10. **Equations:** $x = \arctan t, y = t^2$

$$\begin{aligned}\frac{dx}{dt} &= \frac{1}{1+t^2}, \frac{dy}{dt} = 2t \implies \frac{dy}{dx} = 2t(1+t^2) = 2t + 2t^3 \\ \frac{d}{dt}\left(\frac{dy}{dx}\right) &= 2 + 6t^2 \implies \frac{d^2y}{dx^2} = \frac{2+6t^2}{1/(1+t^2)} = (2+6t^2)(1+t^2)\end{aligned}$$

6. Tangent Lines to Parametric Curves

1. **Curve:** $x = t^2 + 1, y = t^3 + t$ at $t = 2$.

- **Point:** $x(2) = 5, y(2) = 10$. Point is $(5, 10)$.
- **Slope:** $\frac{dx}{dt} = 2t, \frac{dy}{dt} = 3t^2 + 1 \implies \frac{dy}{dx} = \frac{3t^2+1}{2t}$. At $t = 2, m = \frac{13}{4}$.
- **Equation:** $y - 10 = \frac{13}{4}(x - 5)$.

2. **Curve:** $x = \cos t, y = \sin(2t)$ at $t = \pi/3$.

- **Point:** $x(\pi/3) = 1/2, y(\pi/3) = \sqrt{3}/2$. Point is $(1/2, \sqrt{3}/2)$.
- **Slope:** $\frac{dx}{dt} = -\sin t, \frac{dy}{dt} = 2\cos(2t) \implies \frac{dy}{dx} = \frac{2\cos(2t)}{-\sin t}$. At $t = \pi/3, m = \frac{2(-1/2)}{-\sqrt{3}/2} = \frac{2}{\sqrt{3}}$.
- **Equation:** $y - \frac{\sqrt{3}}{2} = \frac{2}{\sqrt{3}}(x - \frac{1}{2})$.

3. **Curve:** $x = e^t, y = e^{-t}$ at $t = 1$.

- **Point:** $x(1) = e, y(1) = 1/e$. Point is $(e, 1/e)$.
- **Slope:** $\frac{dy}{dx} = \frac{-e^{-t}}{e^t} = -e^{-2t}$. At $t = 1, m = -e^{-2}$.
- **Equation:** $y - 1/e = -e^{-2}(x - e)$.

4. **Curve:** $x = 4\sin t, y = 2\cos t$ at $t = \pi/4$.

- **Point:** $x = 4(\sqrt{2}/2) = 2\sqrt{2}, y = 2(\sqrt{2}/2) = \sqrt{2}$. Point is $(2\sqrt{2}, \sqrt{2})$.
- **Slope:** $\frac{dy}{dx} = \frac{-2\sin t}{4\cos t} = -\frac{1}{2}\tan t$. At $t = \pi/4, m = -1/2$.
- **Equation:** $y - \sqrt{2} = -\frac{1}{2}(x - 2\sqrt{2})$.

5. **Curve:** $x = t^3 - 1, y = t^2 + t$ at $t = -1$.

- **Point:** $x(-1) = -2, y(-1) = 0$. Point is $(-2, 0)$.
- **Slope:** $\frac{dy}{dx} = \frac{2t+1}{3t^2}$. At $t = -1, m = \frac{-1}{3}$.
- **Equation:** $y - 0 = -\frac{1}{3}(x + 2)$.

6. **Curve:** $x = \sec t, y = \csc t$ at $t = \pi/3$.

- **Point:** $x = 2, y = 2/\sqrt{3}$. Point is $(2, 2/\sqrt{3})$.
- **Slope:** $\frac{dy}{dx} = \frac{-\csc t \cot t}{\sec t \tan t} = -\cot^3 t$. At $t = \pi/3, m = -(1/\sqrt{3})^3 = -1/(3\sqrt{3})$.
- **Equation:** $y - 2/\sqrt{3} = -1/(3\sqrt{3})(x - 2)$.

7. **Curve:** $x = 1 + \ln t, y = t^2 + 2$ at $t = 1$.

- **Point:** $x = 1, y = 3$. Point is $(1, 3)$.
- **Slope:** $\frac{dy}{dx} = \frac{2t}{1/t} = 2t^2$. At $t = 1, m = 2$.
- **Equation:** $y - 3 = 2(x - 1)$.

8. **Curve:** $x = t \cos t, y = t \sin t$ at $t = \pi$.

- **Point:** $x = -\pi, y = 0$. Point is $(-\pi, 0)$.
- **Slope:** $\frac{dy}{dx} = \frac{\sin t + t \cos t}{\cos t - t \sin t}$. At $t = \pi, m = \frac{-\pi}{-1} = \pi$.
- **Equation:** $y - 0 = \pi(x + \pi)$.

9. **Curve:** $x = t - \sin t, y = 1 - \cos t$ at $t = \pi/2$.

- **Point:** $x = \pi/2 - 1, y = 1$. Point is $(\pi/2 - 1, 1)$.
- **Slope:** $\frac{dy}{dx} = \frac{\sin t}{1 - \cos t}$. At $t = \pi/2, m = 1/1 = 1$.
- **Equation:** $y - 1 = 1(x - (\pi/2 - 1))$.

10. **Curve:** $x = t^2, y = t^3 - 3t$ at $t = \sqrt{3}$.

- **Point:** $x = 3, y = 0$. Point is $(3, 0)$.
- **Slope:** $\frac{dy}{dx} = \frac{3t^2 - 3}{2t}$. At $t = \sqrt{3}, m = \frac{9 - 3}{2\sqrt{3}} = \frac{6}{2\sqrt{3}} = \sqrt{3}$.
- **Equation:** $y - 0 = \sqrt{3}(x - 3)$.

7. Parametric Arc Length

1. **Curve:** $x = e^t \cos t, y = e^t \sin t, 0 \leq t \leq \pi$.

$$\begin{aligned}(dx/dt)^2 + (dy/dt)^2 &= (e^t(\cos t - \sin t))^2 + (e^t(\sin t + \cos t))^2 \\ &= e^{2t}(\cos^2 t - 2 \cos t \sin t + \sin^2 t + \sin^2 t + 2 \sin t \cos t + \cos^2 t) \\ &= e^{2t}(2) \implies \sqrt{\dots} = \sqrt{2}e^t\end{aligned}$$

$$L = \int_0^\pi \sqrt{2}e^t dt = [\sqrt{2}e^t]_0^\pi = \sqrt{2}(e^\pi - 1)$$

Length: $\sqrt{2}(e^\pi - 1)$.

2. **Curve:** $x = \cos^3 t, y = \sin^3 t, 0 \leq t \leq \pi/2$.

$$\begin{aligned}(dx/dt)^2 + (dy/dt)^2 &= (-3 \cos^2 t \sin t)^2 + (3 \sin^2 t \cos t)^2 = 9 \cos^4 t \sin^2 t + 9 \sin^4 t \cos^2 t \\ &= 9 \cos^2 t \sin^2 t (\cos^2 t + \sin^2 t) = (3 \cos t \sin t)^2\end{aligned}$$

$$L = \int_0^{\pi/2} 3 \cos t \sin t dt = \left[\frac{3}{2} \sin^2 t\right]_0^{\pi/2} = \frac{3}{2}$$

Length: $3/2$.

3. **Curve:** $x = \frac{1}{3}t^3, y = \frac{1}{2}t^2, 0 \leq t \leq 1$.

$$(dx/dt)^2 + (dy/dt)^2 = (t^2)^2 + (t)^2 = t^4 + t^2 = t^2(t^2 + 1)$$

$$\begin{aligned}L &= \int_0^1 t \sqrt{t^2 + 1} dt \quad (u = t^2 + 1, du = 2t dt) \\ &= \frac{1}{2} \int_1^2 \sqrt{u} du = \frac{1}{2} \left[\frac{2}{3} u^{3/2}\right]_1^2 = \frac{1}{3}(2\sqrt{2} - 1)\end{aligned}$$

Length: $\frac{1}{3}(2\sqrt{2} - 1)$.

4. **Curve:** $x = a(\cos t + t \sin t), y = a(\sin t - t \cos t), 0 \leq t \leq \pi$.

$$\begin{aligned}dx/dt &= a(at \cos t), dy/dt = a(at \sin t) \\ (dx/dt)^2 + (dy/dt)^2 &= (at \cos t)^2 + (at \sin t)^2 = a^2 t^2 \\ L &= \int_0^\pi at dt = [at^2/2]_0^\pi = \frac{a\pi^2}{2}\end{aligned}$$

Length: $a\pi^2/2$.

5. **Curve:** $x = 2t, y = \frac{2}{3}t^{3/2}, 0 \leq t \leq 3$.

$$\begin{aligned}(dx/dt)^2 + (dy/dt)^2 &= (2)^2 + (t^{1/2})^2 = 4 + t \\ L &= \int_0^3 \sqrt{4+t} dt = \left[\frac{2}{3}(4+t)^{3/2}\right]_0^3 = \frac{2}{3}(7\sqrt{7} - 8)\end{aligned}$$

Length: $\frac{2}{3}(7\sqrt{7} - 8)$.

6. **Curve:** $x = t^2, y = \frac{2}{3}(2t+1)^{3/2}, 0 \leq t \leq 4$. (Note: This is similar to 7a from review)

$$\begin{aligned}dx/dt &= 2t, dy/dt = 2\sqrt{2t+1} \implies (dx/dt)^2 + (dy/dt)^2 = 4t^2 + 4(2t+1) = 4t^2 + 8t + 4 = (2t+2)^2 \\ L &= \int_0^4 (2t+2) dt = [t^2 + 2t]_0^4 = 16 + 8 = 24\end{aligned}$$

Length: 24.

7. **Curve:** $x = 1 + 3t^2, y = 4 + 2t^3, 0 \leq t \leq 1$.

$$\begin{aligned}(dx/dt)^2 + (dy/dt)^2 &= (6t)^2 + (6t^2)^2 = 36t^2(1 + t^2) \\ L &= \int_0^1 6t\sqrt{1 + t^2} dt \quad (u = 1 + t^2, du = 2t dt) \\ &= 3 \int_1^2 \sqrt{u} du = 3 \left[\frac{2}{3} u^{3/2} \right]_1^2 = 2(2\sqrt{2} - 1)\end{aligned}$$

Length: $2(2\sqrt{2} - 1)$.

8. **Curve:** $x = t - \sin t, y = 1 - \cos t, 0 \leq t \leq 2\pi$.

$$\begin{aligned}(dx/dt)^2 + (dy/dt)^2 &= (1 - \cos t)^2 + \sin^2 t = 1 - 2\cos t + \cos^2 t + \sin^2 t = 2 - 2\cos t \\ &= 4\sin^2(t/2) \implies \sqrt{\dots} = 2\sin(t/2) \text{ on } [0, 2\pi] \\ L &= \int_0^{2\pi} 2\sin(t/2) dt = [-4\cos(t/2)]_0^{2\pi} = -4(-1) - (-4(1)) = 8\end{aligned}$$

Length: 8.

9. **Curve:** $x = 3t, y = t^3, 0 \leq t \leq 2$.

$$\begin{aligned}(dx/dt)^2 + (dy/dt)^2 &= (3)^2 + (3t^2)^2 = 9(1 + t^4) \\ L &= \int_0^2 3\sqrt{1 + t^4} dt \quad (\text{Cannot be solved with elementary functions})\end{aligned}$$

Result: Problem is not well-posed for this level.

10. **Curve:** $x = e^t - t, y = 4e^{t/2}, 0 \leq t \leq 1$.

$$\begin{aligned}dx/dt &= e^t - 1, dy/dt = 2e^{t/2} \\ (dx/dt)^2 + (dy/dt)^2 &= (e^{2t} - 2e^t + 1) + 4e^t = e^{2t} + 2e^t + 1 = (e^t + 1)^2 \\ L &= \int_0^1 (e^t + 1) dt = [e^t + t]_0^1 = (e + 1) - (1 + 0) = e\end{aligned}$$

Length: e .

8. Vertical Tangents

1. **Curve:** $x = t^3 - 3t, y = t^2 - 3$.

- $\frac{dx}{dt} = 3t^2 - 3 = 3(t-1)(t+1)$. $dx/dt = 0$ at $t = 1, t = -1$.
- $\frac{dy}{dt} = 2t$.
- At $t = 1, dy/dt = 2 \neq 0$. At $t = -1, dy/dt = -2 \neq 0$.

Result: Vertical tangents at $t = 1$ and $t = -1$.

2. **Curve:** $x = t^2 - t, y = t^3 - 3t$.

- $\frac{dx}{dt} = 2t - 1$. $dx/dt = 0$ at $t = 1/2$.
- $\frac{dy}{dt} = 3t^2 - 3$. At $t = 1/2, dy/dt = 3/4 - 3 \neq 0$.

Result: Vertical tangent at $t = 1/2$.

3. **Curve:** $x = 2 \cos t, y = \sin(2t)$.

- $\frac{dx}{dt} = -2 \sin t$. $dx/dt = 0$ at $t = k\pi$ for integer k .
- $\frac{dy}{dt} = 2 \cos(2t)$. At $t = k\pi, \cos(2k\pi) = 1, dy/dt = 2 \neq 0$.

Result: Vertical tangents at $t = k\pi$.

4. **Curve:** $x = t - \sin t, y = 1 - \cos t$.

- $\frac{dx}{dt} = 1 - \cos t$. $dx/dt = 0$ at $t = 2k\pi$.
- $\frac{dy}{dt} = \sin t$. At $t = 2k\pi, dy/dt = 0$. The slope is indeterminate (0/0).

Result: No vertical tangents (cusps at these points).

5. **Curve:** $x = t^4 - 2t^2, y = t^3 - t$.

- $\frac{dx}{dt} = 4t^3 - 4t = 4t(t-1)(t+1)$. $dx/dt = 0$ at $t = 0, 1, -1$.
- $\frac{dy}{dt} = 3t^2 - 1$. At $t = 0, dy/dt = -1 \neq 0$. At $t = \pm 1, dy/dt = 2 \neq 0$.

Result: Vertical tangents at $t = 0, 1, -1$.

6. **Curve:** $x = te^t, y = t^2 - t$.

- $\frac{dx}{dt} = e^t + te^t = e^t(1+t)$. $dx/dt = 0$ at $t = -1$.
- $\frac{dy}{dt} = 2t - 1$. At $t = -1, dy/dt = -3 \neq 0$.

Result: Vertical tangent at $t = -1$.

7. **Curve:** $x = \sin t, y = \cos t$.

- $\frac{dx}{dt} = \cos t$. $dx/dt = 0$ at $t = \pi/2 + k\pi$.
- $\frac{dy}{dt} = -\sin t$. At these $t, |\sin t| = 1, dy/dt \neq 0$.

Result: Vertical tangents at $t = \pi/2 + k\pi$.

8. **Curve:** $x = t^2, y = t^3 - 3t$.

- $\frac{dx}{dt} = 2t$. $dx/dt = 0$ at $t = 0$.
- $\frac{dy}{dt} = 3t^2 - 3$. At $t = 0$, $dy/dt = -3 \neq 0$.

Result: Vertical tangent at $t = 0$.

9. **Curve:** $x = \ln(t^2 + 1)$, $y = t - 2$.

- $\frac{dx}{dt} = \frac{2t}{t^2+1}$. $dx/dt = 0$ at $t = 0$.
- $\frac{dy}{dt} = 1 \neq 0$ for all t .

Result: Vertical tangent at $t = 0$.

10. **Curve:** $x = t^3 - 12t$, $y = t^2 - 1$.

- $\frac{dx}{dt} = 3t^2 - 12 = 3(t - 2)(t + 2)$. $dx/dt = 0$ at $t = \pm 2$.
- $\frac{dy}{dt} = 2t$. At $t = \pm 2$, $dy/dt = \pm 4 \neq 0$.

Result: Vertical tangents at $t = 2$ and $t = -2$.

9. Particle at Rest

1. **Curve:** $x = t^3 - 3t^2, y = t^3 - 3t$.

- $\frac{dx}{dt} = 3t^2 - 6t = 3t(t - 2)$. Roots: $t = 0, 2$.
- $\frac{dy}{dt} = 3t^2 - 3 = 3(t - 1)(t + 1)$. Roots: $t = 1, -1$.

Result: No common roots. Particle is never at rest.

2. **Curve:** $x = \cos t, y = \sin(2t)$.

- $\frac{dx}{dt} = -\sin t$. Roots: $t = k\pi$.
- $\frac{dy}{dt} = 2\cos(2t)$. At $t = k\pi, \cos(2k\pi) = 1, dy/dt \neq 0$.

Result: Never at rest.

3. **Curve:** $x = t^2 - 4t, y = t^3 - 12t$.

- $\frac{dx}{dt} = 2t - 4$. Root: $t = 2$.
- $\frac{dy}{dt} = 3t^2 - 12 = 3(t - 2)(t + 2)$. Roots: $t = 2, -2$.

Result: Common root is $t = 2$. Particle is at rest at $t = 2$.

4. **Curve:** $x = \sin t, y = \sin t$.

- $\frac{dx}{dt} = \cos t, \frac{dy}{dt} = \cos t$. Both are zero at $t = \pi/2 + k\pi$.

Result: At rest at $t = \pi/2 + k\pi$.

5. **Curve:** $x = t^4 - 2t^2, y = t^3 - 3t^2$.

- $\frac{dx}{dt} = 4t^3 - 4t = 4t(t - 1)(t + 1)$. Roots: $t = 0, 1, -1$.
- $\frac{dy}{dt} = 3t^2 - 6t = 3t(t - 2)$. Roots: $t = 0, 2$.

Result: Common root is $t = 0$. At rest at $t = 0$.

6. **Curve:** $x = t^2 - 1, y = t^3 - t$.

- $\frac{dx}{dt} = 2t$. Root: $t = 0$.
- $\frac{dy}{dt} = 3t^2 - 1$. At $t = 0, dy/dt = -1 \neq 0$.

Result: Never at rest.

7. **Curve:** $x = \sin t - t, y = \cos t - 1$.

- $\frac{dx}{dt} = \cos t - 1$. Roots: $t = 2k\pi$.
- $\frac{dy}{dt} = -\sin t$. Roots: $t = k\pi$.

Result: Common roots are $t = 2k\pi$. At rest at $t = 2k\pi$.

8. **Curve:** $x = t^3/3 - t, y = t^2 - 1$.

- $\frac{dx}{dt} = t^2 - 1 = (t - 1)(t + 1)$. Roots: $t = 1, -1$.
- $\frac{dy}{dt} = 2t$. Root: $t = 0$.

Result: Never at rest.

9. **Curve:** $x = t^3 - 3t, y = t^3 - 12t$.

- $\frac{dx}{dt} = 3t^2 - 3 = 3(t - 1)(t + 1)$. Roots: $t = 1, -1$.
- $\frac{dy}{dt} = 3t^2 - 12 = 3(t - 2)(t + 2)$. Roots: $t = 2, -2$.

Result: Never at rest.

10. **Curve:** $x = t^2 - 2t, y = t^3 - 3t^2 + 2t$.

- $\frac{dx}{dt} = 2t - 2 = 2(t - 1)$. Root: $t = 1$.
- $\frac{dy}{dt} = 3t^2 - 6t + 2$. At $t = 1, dy/dt = 3 - 6 + 2 = -1 \neq 0$.

Result: Never at rest.

Concept Check List

This list summarizes the concepts tested. The numbers refer to the problem sets generated in this document.

- **Improper Integrals**

- Type 1 (Infinite Limit) requiring partial fractions: **1.a** (1-10)
- Type 2 (Discontinuity) requiring u-substitution: **1.b** (1-10)
- Type 2 (Discontinuity) from a vertical asymptote: **1.c** (1-10)
- Type 2 (Discontinuity) solved with standard forms (arcsin, etc.): **1.d** (1-10)
- Type 1 (Double Infinite Limits), possibly using symmetry: **1.e** (1-10)

- **Arc Length (Cartesian)**

- Integrand $\sqrt{1 + (y')^2}$ simplifies to a perfect square: **2** (1-10)

- **Surface Area of Revolution (Cartesian)**

- Calculating surface area for various curves, requiring algebraic simplification and u-substitution: **3** (1-10)

- **Parametric Equations**

- Eliminating the parameter to find the Cartesian equation (circles, ellipses, parabolas, hyperbolas, etc.): **4** (1-10)

- **Calculus with Parametric curves**

- Finding first and second derivatives ($\frac{dy}{dx}, \frac{d^2y}{dx^2}$): **5** (1-10)
- Finding the equation of a tangent line at a point: **6** (1-10)
- Calculating arc length, often involving perfect squares or u-substitution: **7** (1-10)
- Finding points of vertical tangents ($\frac{dx}{dt} = 0, \frac{dy}{dt} \neq 0$): **8** (1-10)
- Finding when a particle is at rest ($\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$): **9** (1-10)