

# Homework 11.2: Infinite Series Practice Problems

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## Concept Checklist

This problem set is designed to test the following concepts related to infinite series, as detailed in the source material.

- **Core Concepts:** Understanding the difference between a sequence and a series; definition of convergence and divergence based on the limit of partial sums ( $s_n$ ).
- **Calculating from Partial Sums:** Determining the sum of a series given the formula for its  $n$ -th partial sum,  $s_n$ .
- **Analyzing Partial Sums:** Calculating the first few terms of the sequence of partial sums to form a conjecture about the convergence or divergence of a series.
- **Geometric Series:** Identifying a geometric series, its first term ( $a$ ), and its common ratio ( $r$ ). Determining convergence ( $|r| < 1$ ) and calculating its sum using the formula  $S = \frac{a}{1-r}$ .
- **Test for Divergence:** Applying the test by finding the limit of the  $n$ -th term,  $\lim_{n \rightarrow \infty} a_n$ . Understanding that if the limit is not zero, the series diverges, and if the limit is zero, the test is inconclusive.
- **Telescoping Series:** Identifying and finding the sum of a telescoping series, often requiring algebraic manipulation like partial fraction decomposition.
- **Problem Analysis ("Find the Flaw"):** Critically evaluating a given solution to identify logical or computational errors in the application of series tests and concepts.

## 1 Problems

### 1.1 Core Concepts and Partial Sums

1. Explain the fundamental difference between the sequence  $\{\frac{n}{n+1}\}_{n=1}^{\infty}$  and the series  $\sum_{n=1}^{\infty} \frac{n}{n+1}$ .
2. A series  $\sum a_n$  has a sequence of partial sums  $\{s_n\}$  where  $s_n = \frac{5n-2}{2n+8}$ . Does the series converge, and if so, to what sum?
3. The partial sums of a series  $\sum a_n$  are given by  $s_n = 4 - e^{-n}$ .

- (a) Find the sum of the series.
  - (b) Find the first term of the series,  $a_1$ .
  - (c) Find a formula for the  $n$ -th term,  $a_n$ , for  $n \geq 2$ .
4. Calculate the first eight terms of the sequence of partial sums for the series  $\sum_{n=1}^{\infty} \frac{1}{n^2}$ . Based on your calculations, does the series appear to be convergent or divergent? (Note: This is a p-series with  $p = 2 > 1$  and is known to converge).
5. Calculate the first eight terms of the sequence of partial sums for the series  $\sum_{n=1}^{\infty} \frac{n}{2n+1}$ . Does the series appear to be convergent or divergent? Use the Test for Divergence to confirm your suspicion.
6. Explain what it means to say that  $\sum_{n=1}^{\infty} a_n = 10$ .
7. The partial sums of a series  $\sum a_n$  are given by  $s_n = \frac{n^2 - \cos(n)}{3n^2 + 1}$ . Find the sum of the series.
8. Calculate the first eight terms of the sequence of partial sums for  $\sum_{n=1}^{\infty} \cos(\pi n)$ . Does it appear the series is convergent or divergent?

## 1.2 Geometric Series

For each series, determine if it is a geometric series. If it is, determine whether it converges or diverges. If it converges, find its sum.

9.  $\sum_{n=1}^{\infty} 5 \left(\frac{2}{3}\right)^{n-1}$
10.  $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n}$
11.  $\sum_{n=0}^{\infty} \frac{2^{2n}}{5^{n+1}}$
12.  $10 - 2 + 0.4 - 0.08 + \dots$
13.  $\sum_{n=1}^{\infty} \frac{e^n}{3^{n-1}}$
14.  $\sum_{n=1}^{\infty} \frac{4^n + 3^n}{5^n}$
15.  $\sum_{n=2}^{\infty} 3 \left(-\frac{1}{2}\right)^n$
16.  $\sum_{n=1}^{\infty} \frac{100}{101^n}$
17. Express the repeating decimal  $0.\overline{47} = 0.474747\dots$  as a geometric series and find its sum (as a fraction).
18. A ball is dropped from a height of 10 meters. Each time it strikes the ground, it bounces to 70% of its previous height. What is the total distance the ball travels?

### 1.3 Test for Divergence

Use the Test for Divergence to determine whether the series diverges. If the test is inconclusive, state so.

$$19. \sum_{n=1}^{\infty} \frac{n^2-1}{3n^2+n}$$

$$20. \sum_{n=1}^{\infty} \arctan(n)$$

$$21. \sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^n$$

$$22. \sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+4}}$$

$$23. \sum_{n=1}^{\infty} \frac{1}{n} \text{ (The Harmonic Series)}$$

$$24. \sum_{k=1}^{\infty} \frac{k!}{(k+2)!}$$

### 1.4 Telescoping Series

Determine whether the series converges or diverges by finding the  $n$ -th partial sum and taking the limit. If it converges, find the sum.

$$25. \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$26. \sum_{n=1}^{\infty} \frac{1}{n(n+2)}$$

$$27. \sum_{n=2}^{\infty} \frac{2}{n^2-1}$$

$$28. \sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$$

$$29. \sum_{n=1}^{\infty} \left(\cos\left(\frac{1}{n+1}\right) - \cos\left(\frac{1}{n}\right)\right)$$

$$30. \sum_{n=3}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right)$$

### 1.5 Find the Flaw

In each of the following problems, a flawed solution is presented. Identify the error in reasoning and provide a correct solution.

31. **Problem:** Determine if the series  $\sum_{n=1}^{\infty} \frac{5}{n}$  converges or diverges.

**Flawed Solution:**

- We use the Test for Divergence.
- We compute the limit of the terms:  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{5}{n} = 0$ .
- Since the limit of the terms is 0, the series converges.

32. **Problem:** Find the sum of the series  $\sum_{n=0}^{\infty} 3\left(\frac{4}{3}\right)^n$ .

**Flawed Solution:**

- This is a geometric series with first term  $a = 3(4/3)^0 = 3$ .
- The common ratio is  $r = 4/3$ .

(c) The sum is  $S = \frac{a}{1-r} = \frac{3}{1-4/3} = \frac{3}{-1/3} = -9$ .

33. **Problem:** Find the sum of the telescoping series  $\sum_{n=1}^{\infty} \left( \frac{1}{n+1} - \frac{1}{n+3} \right)$ .  
**Flawed Solution:**

(a) Let's write out the  $n$ -th partial sum,  $s_n$ .

(b)  $s_n = \left( \frac{1}{2} - \frac{1}{4} \right) + \left( \frac{1}{3} - \frac{1}{5} \right) + \cdots + \left( \frac{1}{n+1} - \frac{1}{n+3} \right)$ .

(c) The terms cancel out, leaving the first term and the last term.

(d)  $s_n = \frac{1}{2} - \frac{1}{n+3}$ .

(e) The sum is  $S = \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \left( \frac{1}{2} - \frac{1}{n+3} \right) = \frac{1}{2} - 0 = \frac{1}{2}$ .

## 2 Solutions

1. **Solution:** The sequence is an ordered list of numbers:  $\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\}$ . The series is the sum of these numbers:  $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \dots$ .
2. **Solution:** The sum of a series is the limit of its sequence of partial sums.  $S = \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \frac{5n-2}{2n+8} = \lim_{n \rightarrow \infty} \frac{5-2/n}{2+8/n} = \frac{5}{2}$ . The series converges to  $\frac{5}{2}$ .
3. **Solution:**
  - (a)  $S = \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} (4 - e^{-n}) = 4 - 0 = 4$ .
  - (b)  $a_1 = s_1 = 4 - e^{-1}$ .
  - (c)  $a_n = s_n - s_{n-1} = (4 - e^{-n}) - (4 - e^{-(n-1)}) = e^{-n+1} - e^{-n}$ .
4. **Solution:**  $s_1 = 1$ ,  $s_2 = 1.25$ ,  $s_3 \approx 1.361$ ,  $s_4 \approx 1.424$ ,  $s_5 = 1.464$ ,  $s_6 \approx 1.491$ ,  $s_7 \approx 1.512$ ,  $s_8 \approx 1.527$ . The partial sums are increasing but the amount of increase is getting smaller. It appears to be convergent.
5. **Solution:**  $s_1 \approx 0.333$ ,  $s_2 \approx 0.733$ ,  $s_3 \approx 1.162, \dots$ . The partial sums are increasing and do not appear to be leveling off. It appears to be divergent. **Test for Divergence:**  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n}{2n+1} = \frac{1}{2}$ . Since the limit is not 0, the series diverges.
6. **Solution:** This means the series is convergent, and its sum is 10. The limit of its sequence of partial sums is 10, i.e.,  $\lim_{n \rightarrow \infty} s_n = 10$ .
7. **Solution:**  $S = \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \frac{n^2 - \cos(n)}{3n^2 + 1} = \lim_{n \rightarrow \infty} \frac{1 - \cos(n)/n^2}{3 + 1/n^2} = \frac{1-0}{3+0} = \frac{1}{3}$ . The series converges to  $\frac{1}{3}$ .
8. **Solution:**  $a_n = \cos(\pi n) = (-1)^n$ . The terms are  $-1, 1, -1, 1, \dots$ . The sequence of partial sums is  $\{-1, 0, -1, 0, -1, \dots\}$ . This sequence oscillates and does not approach a single value. The series appears to be divergent.
9. **Solution:** This is a geometric series with  $a = 5$  and  $r = 2/3$ . Since  $|r| < 1$ , it converges.  $S = \frac{a}{1-r} = \frac{5}{1-2/3} = \frac{5}{1/3} = 15$ .
10. **Solution:**  $a_n = \frac{(-3)^{n-1}}{4^n} = \frac{(-3)^{n-1}}{4 \cdot 4^{n-1}} = \frac{1}{4} \left(-\frac{3}{4}\right)^{n-1}$ . This is a geometric series with  $a = 1/4$  and  $r = -3/4$ . Since  $|r| < 1$ , it converges.  $S = \frac{1/4}{1-(-3/4)} = \frac{1/4}{7/4} = \frac{1}{7}$ .
11. **Solution:**  $a_n = \frac{2^{2n}}{5^{n+1}} = \frac{(2^2)^n}{5 \cdot 5^n} = \frac{4^n}{5 \cdot 5^n} = \frac{1}{5} \left(\frac{4}{5}\right)^n$ . This is a geometric series. The first term (at  $n = 0$ ) is  $a = 1/5$ . The ratio is  $r = 4/5$ . Since  $|r| < 1$ , it converges.  $S = \frac{1/5}{1-4/5} = \frac{1/5}{1/5} = 1$ .
12. **Solution:** This is a geometric series with first term  $a = 10$  and common ratio  $r = -2/10 = -1/5$ . Since  $|r| = 1/5 < 1$ , the series converges.  $S = \frac{10}{1-(-1/5)} = \frac{10}{6/5} = \frac{50}{6} = \frac{25}{3}$ .
13. **Solution:**  $a_n = \frac{e^n}{3^{n-1}} = \frac{e \cdot e^{n-1}}{3^{n-1}} = e \left(\frac{e}{3}\right)^{n-1}$ . This is a geometric series with  $a = e$  and  $r = e/3$ . Since  $e \approx 2.718$ ,  $|r| = e/3 < 1$ . It converges.  $S = \frac{e}{1-e/3} = \frac{3e}{3-e}$ .

14. **Solution:** We can split this into two series:  $\sum_{n=1}^{\infty} \left(\frac{4}{5}\right)^n + \sum_{n=1}^{\infty} \left(\frac{3}{5}\right)^n$ . Both are convergent geometric series. For the first,  $a = 4/5, r = 4/5$ . Sum is  $\frac{4/5}{1-4/5} = 4$ . For the second,  $a = 3/5, r = 3/5$ . Sum is  $\frac{3/5}{1-3/5} = \frac{3}{2}$ . Total sum is  $4 + \frac{3}{2} = \frac{11}{2}$ .
15. **Solution:** This is a geometric series with  $r = -1/2$ . The first term is for  $n = 2$ , so  $a = 3(-1/2)^2 = 3/4$ . Since  $|r| < 1$ , it converges.  $S = \frac{a}{1-r} = \frac{3/4}{1-(-1/2)} = \frac{3/4}{3/2} = \frac{1}{2}$ .
16. **Solution:** This is a geometric series, which can be written as  $\sum_{n=1}^{\infty} 100 \left(\frac{1}{101}\right)^n$ . The first term is  $a = 100/101$  and the ratio is  $r = 1/101$ . Since  $|r| < 1$ , it converges.  $S = \frac{100/101}{1-1/101} = \frac{100/101}{100/101} = 1$ .
17. **Solution:**  $0.\overline{47} = \frac{47}{100} + \frac{47}{10000} + \frac{47}{100^3} + \dots$ . This is a geometric series with  $a = 47/100$  and  $r = 1/100$ . It converges to  $S = \frac{47/100}{1-1/100} = \frac{47/100}{99/100} = \frac{47}{99}$ .
18. **Solution:** Total distance =  $10 + 2(10 \cdot 0.7) + 2(10 \cdot 0.7^2) + \dots = 10 + \sum_{n=1}^{\infty} 20(0.7)^n$ . The sum is a geometric series with  $a = 20(0.7) = 14$  and  $r = 0.7$ . Sum =  $\frac{14}{1-0.7} = \frac{14}{0.3} = \frac{140}{3} = \frac{170}{3}$  meters.
19. **Solution:**  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n^2-1}{3n^2+n} = \frac{1}{3}$ . Since the limit is not 0, the series diverges by the Test for Divergence.
20. **Solution:**  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \arctan(n) = \frac{\pi}{2}$ . Since the limit is not 0, the series diverges by the Test for Divergence.
21. **Solution:**  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$ . Since the limit is not 0, the series diverges by the Test for Divergence.
22. **Solution:**  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n}{\sqrt{n^2+4}} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{1+4/n^2}} = 1$ . Since the limit is not 0, the series diverges by the Test for Divergence.
23. **Solution:**  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$ . The Test for Divergence is inconclusive. (Note: The harmonic series is known to diverge).
24. **Solution:**  $a_k = \frac{k!}{(k+2)!} = \frac{k!}{k!(k+1)(k+2)} = \frac{1}{(k+1)(k+2)}$ .  $\lim_{k \rightarrow \infty} a_k = 0$ . The Test for Divergence is inconclusive.
25. **Solution:**  $s_n = (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \dots + (\frac{1}{n} - \frac{1}{n+1}) = 1 - \frac{1}{n+1}$ .  $S = \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} (1 - \frac{1}{n+1}) = 1$ . The series converges to 1.
26. **Solution:** Use partial fractions:  $\frac{1}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2} \Rightarrow 1 = A(n+2) + Bn$ . Let  $n = 0 \Rightarrow A = 1/2$ . Let  $n = -2 \Rightarrow B = -1/2$ .  $a_n = \frac{1}{2} \left( \frac{1}{n} - \frac{1}{n+2} \right)$ .  $s_n = \frac{1}{2} \left[ (1 - \frac{1}{3}) + (\frac{1}{2} - \frac{1}{4}) + (\frac{1}{3} - \frac{1}{5}) + \dots + (\frac{1}{n} - \frac{1}{n+2}) \right]$ . The terms that don't cancel are 1 and 1/2 at the beginning, and  $-\frac{1}{n+1}$  and  $-\frac{1}{n+2}$  at the end.  $s_n = \frac{1}{2} \left( 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2} \right)$ .  $S = \lim_{n \rightarrow \infty} s_n = \frac{1}{2}(1 + \frac{1}{2} - 0 - 0) = \frac{3}{4}$ .
27. **Solution:** Use partial fractions:  $\frac{2}{n^2-1} = \frac{2}{(n-1)(n+1)} = \frac{1}{n-1} - \frac{1}{n+1}$ .  $s_n = \sum_{k=2}^n \left( \frac{1}{k-1} - \frac{1}{k+1} \right) = (1 - \frac{1}{3}) + (\frac{1}{2} - \frac{1}{4}) + \dots + (\frac{1}{n-1} - \frac{1}{n+1})$ . The terms that remain are 1 and 1/2 at the beginning, and  $-\frac{1}{n}$  and  $-\frac{1}{n+1}$  at the end.  $s_n = 1 + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1}$ .  $S = \lim_{n \rightarrow \infty} s_n = 1 + \frac{1}{2} = \frac{3}{2}$ .

28. **Solution:** Using logarithm properties,  $\ln(\frac{n+1}{n}) = \ln(n+1) - \ln(n)$ .  $s_n = (\ln(2) - \ln(1)) + (\ln(3) - \ln(2)) + \cdots + (\ln(n+1) - \ln(n)) = \ln(n+1) - \ln(1) = \ln(n+1)$ .  $S = \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \ln(n+1) = \infty$ . The series diverges.
29. **Solution:** This is a telescoping series. Let  $b_n = \cos(1/n)$ . The series is  $\sum(b_{n+1} - b_n)$ .  $s_n = (\cos(1/2) - \cos(1)) + (\cos(1/3) - \cos(1/2)) + \cdots + (\cos(\frac{1}{n+1}) - \cos(\frac{1}{n})) = \cos(\frac{1}{n+1}) - \cos(1)$ .  $S = \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} (\cos(\frac{1}{n+1}) - \cos(1)) = \cos(0) - \cos(1) = 1 - \cos(1)$ .
30. **Solution:** This is a telescoping series starting at  $n = 3$ .  $s_N = \sum_{n=3}^N (\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}) = (\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}}) + (\frac{1}{\sqrt{4}} - \frac{1}{\sqrt{5}}) + \cdots + (\frac{1}{\sqrt{N}} - \frac{1}{\sqrt{N+1}}) = \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{N+1}}$ .  $S = \lim_{N \rightarrow \infty} s_N = \frac{1}{\sqrt{3}} - 0 = \frac{1}{\sqrt{3}}$ .
31. **Solution: Flaw:** The conclusion in step 3 is incorrect. The Test for Divergence can only prove divergence, never convergence. If the limit of the terms is 0, the test is inconclusive. **Correct Solution:** This is the harmonic series multiplied by a constant (5). The harmonic series  $\sum \frac{1}{n}$  is a p-series with  $p = 1$ , which is known to diverge. Therefore, the series  $\sum \frac{5}{n}$  also diverges.
32. **Solution: Flaw:** Step 3 misuses the geometric series sum formula. The formula  $S = a/(1 - r)$  is only valid when the series converges, which requires  $|r| < 1$ . **Correct Solution:** This is a geometric series with ratio  $r = 4/3$ . Since  $|r| = 4/3 \geq 1$ , the series diverges. There is no finite sum.
33. **Solution: Flaw:** The cancellation pattern in step 3 is identified incorrectly. The term  $-1/4$  from the first parentheses does not cancel with  $1/3$  from the second. More terms must be written out to see the correct pattern. **Correct Solution:**  $s_n = (\frac{1}{2} - \frac{1}{4}) + (\frac{1}{3} - \frac{1}{5}) + (\frac{1}{4} - \frac{1}{6}) + \cdots + (\frac{1}{n} - \frac{1}{n+2}) + (\frac{1}{n+1} - \frac{1}{n+3})$ . The  $-1/4$  cancels with the  $1/4$ . The terms that do not cancel are  $1/2$  and  $1/3$  from the beginning, and  $-1/(n+2)$  and  $-1/(n+3)$  from the end.  $s_n = \frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3}$ .  $S = \lim_{n \rightarrow \infty} s_n = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$ .

## Concept Checklist with Problem Mapping

- **Core Concepts:** 1, 6, 8
- **Calculating from Partial Sums:** 2, 3, 7
- **Analyzing Partial Sums:** 4, 5, 8
- **Geometric Series:** 9, 10, 11, 12, 13, 14, 15, 16, 17, 18
- **Test for Divergence:** 5, 19, 20, 21, 22, 23, 24
- **Telescoping Series:** 25, 26, 27, 28, 29, 30
- **Problem Analysis ("Find the Flaw"):** 31, 32, 33