Homework 11.2: Infinite Series Practice Problems

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Concept Checklist

This problem set is designed to test the following concepts related to infinite series, as detailed in the source material.

- **Core Concepts:** Understanding the difference between a sequence and a series; definition of convergence and divergence based on the limit of partial sums (s_n).
- **Calculating from Partial Sums:** Determining the sum of a series given the formula for its n-th partial sum, s_n.
- **Analyzing Partial Sums:** Calculating the first few terms of the sequence of partial sums to form a conjecture about the convergence or divergence of a series.
- **Geometric Series:** Identifying a geometric series, its first term (a), and its common ratio (r). Determining convergence ($|r| < 1$) and calculating its sum using the formula $S = \frac{a}{1-r}$.
- **Test for Divergence:** Applying the test by finding the limit of the n-th term, $\lim_{n\to\infty} a_n$. Understanding that if the limit is not zero, the series diverges, and if the limit is zero, the test is inconclusive.
- **Telescoping Series:** Identifying and finding the sum of a telescoping series, often requiring algebraic manipulation like partial fraction decomposition.
- **Problem Analysis ("Find the Flaw"):** Critically evaluating a given solution to identify logical or computational errors in the application of series tests and concepts.

1 Problems

1.1 Core Concepts and Partial Sums

1. Explain the fundamental difference between the sequence $\{\frac{n}{n+1}\}_{n=1}^{\infty}$ and the series $\sum_{n=1}^{\infty} \frac{n}{n+1}$.
2. A series $\sum a_n$ has a sequence of partial sums $\{s_n\}$ where $s_n = \frac{5n-2}{2n+8}$. Does the series converge, and if so, to what sum?
3. The partial sums of a series $\sum a_n$ are given by $s_n = 4 - e^{-n}$.

(a) Find the sum of the series.

(b) Find the first term of the series, a_1.

(c) Find a formula for the n-th term, a_n, for $n \geq 2$.

4. Calculate the first eight terms of the sequence of partial sums for the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Based on your calculations, does the series appear to be convergent or divergent? (Note: This is a p-series with $p = 2 > 1$ and is known to converge).

5. Calculate the first eight terms of the sequence of partial sums for the series $\sum_{n=1}^{\infty} \frac{n}{2n+1}$. Does the series appear to be convergent or divergent? Use the Test for Divergence to confirm your suspicion.

6. Explain what it means to say that $\sum_{n=1}^{\infty} a_n = 10$.

7. The partial sums of a series $\sum a_n$ are given by $s_n = \frac{n^2 - \cos(n)}{3n^2+1}$. Find the sum of the series.

8. Calculate the first eight terms of the sequence of partial sums for $\sum_{n=1}^{\infty} \cos(\pi n)$. Does it appear the series is convergent or divergent?

1.2 Geometric Series

For each series, determine if it is a geometric series. If it is, determine whether it converges or diverges. If it converges, find its sum.

9. $\sum_{n=1}^{\infty} 5 \left(\frac{2}{3}\right)^{n-1}$

10. $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n}$

11. $\sum_{n=0}^{\infty} \frac{2^{2n}}{5^{n+1}}$

12. $10 - 2 + 0.4 - 0.08 + \dots$

13. $\sum_{n=1}^{\infty} \frac{e^n}{3^{n-1}}$

14. $\sum_{n=1}^{\infty} \frac{4^n + 3^n}{5^n}$

15. $\sum_{n=2}^{\infty} 3 \left(-\frac{1}{2}\right)^n$

16. $\sum_{n=1}^{\infty} \frac{100}{101^n}$

17. Express the repeating decimal $0.\overline{47} = 0.474747\dots$ as a geometric series and find its sum (as a fraction).

18. A ball is dropped from a height of 10 meters. Each time it strikes the ground, it bounces to 70% of its previous height. What is the total distance the ball travels?

1.3 Test for Divergence

Use the Test for Divergence to determine whether the series diverges. If the test is inconclusive, state so.

19. $\sum_{n=1}^{\infty} \frac{n^2-1}{3n^2+n}$
20. $\sum_{n=1}^{\infty} \arctan(n)$
21. $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n$
22. $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+4}}$
23. $\sum_{n=1}^{\infty} \frac{1}{n}$ (The Harmonic Series)
24. $\sum_{k=1}^{\infty} \frac{k!}{(k+2)!}$

1.4 Telescoping Series

Determine whether the series converges or diverges by finding the n-th partial sum and taking the limit. If it converges, find the sum.

25. $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$
26. $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$
27. $\sum_{n=2}^{\infty} \frac{2}{n^2-1}$
28. $\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$
29. $\sum_{n=1}^{\infty} (\cos(\frac{1}{n+1}) - \cos(\frac{1}{n}))$
30. $\sum_{n=3}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right)$

1.5 Find the Flaw

In each of the following problems, a flawed solution is presented. Identify the error in reasoning and provide a correct solution.

31. **Problem:** Determine if the series $\sum_{n=1}^{\infty} \frac{5}{n}$ converges or diverges.
    Flawed Solution:
    (a) We use the Test for Divergence.
    (b) We compute the limit of the terms: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{5}{n} = 0$.
    (c) Since the limit of the terms is 0, the series converges.
32. **Problem:** Find the sum of the series $\sum_{n=0}^{\infty} 3\left(\frac{4}{3}\right)^n$.
    Flawed Solution:
    (a) This is a geometric series with first term $a = 3(4/3)^0 = 3$.
    (b) The common ratio is $r = 4/3$.

(c) The sum is $S = \frac{a}{1-r} = \frac{3}{1-4/3} = \frac{3}{-1/3} = -9$.

33. **Problem:** Find the sum of the telescoping series $\sum_{n=1}^{\infty}(\frac{1}{n+1} - \frac{1}{n+3})$.
Flawed Solution:

(a) Let's write out the n-th partial sum, s_n.

(b) $s_n = (\frac{1}{2} - \frac{1}{4}) + (\frac{1}{3} - \frac{1}{5}) + \cdots + (\frac{1}{n+1} - \frac{1}{n+3})$.

(c) The terms cancel out, leaving the first term and the last term.

(d) $s_n = \frac{1}{2} - \frac{1}{n+3}$.

(e) The sum is $S = \lim_{n\to\infty} s_n = \lim_{n\to\infty}(\frac{1}{2} - \frac{1}{n+3}) = \frac{1}{2} - 0 = \frac{1}{2}$.

2 Solutions

1. **Solution:** The sequence is an ordered list of numbers: $\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\}$. The series is the sum of these numbers: $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \dots$.

2. **Solution:** The sum of a series is the limit of its sequence of partial sums. $S = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{5n-2}{2n+8} = \lim_{n\to\infty} \frac{5-2/n}{2+8/n} = \frac{5}{2}$. The series converges to $\frac{5}{2}$.

3. **Solution:**

   (a) $S = \lim_{n\to\infty} s_n = \lim_{n\to\infty}(4 - e^{-n}) = 4 - 0 = 4$.

   (b) $a_1 = s_1 = 4 - e^{-1}$.

   (c) $a_n = s_n - s_{n-1} = (4 - e^{-n}) - (4 - e^{-(n-1)}) = e^{-n+1} - e^{-n}$.

4. **Solution:** $s_1 = 1$, $s_2 = 1.25$, $s_3 \approx 1.361$, $s_4 \approx 1.424$, $s_5 = 1.464$, $s_6 \approx 1.491$, $s_7 \approx 1.512$, $s_8 \approx 1.527$. The partial sums are increasing but the amount of increase is getting smaller. It appears to be convergent.

5. **Solution:** $s_1 \approx 0.333$, $s_2 \approx 0.733$, $s_3 \approx 1.162$,... The partial sums are increasing and do not appear to be leveling off. It appears to be divergent. **Test for Divergence:** $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2}$. Since the limit is not 0, the series diverges.

6. **Solution:** This means the series is convergent, and its sum is 10. The limit of its sequence of partial sums is 10, i.e., $\lim_{n\to\infty} s_n = 10$.

7. **Solution:** $S = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{n^2-\cos(n)}{3n^2+1} = \lim_{n\to\infty} \frac{1-\cos(n)/n^2}{3+1/n^2} = \frac{1-0}{3+0} = \frac{1}{3}$. The series converges to $\frac{1}{3}$.

8. **Solution:** $a_n = \cos(\pi n) = (-1)^n$. The terms are $-1, 1, -1, 1, \dots$. The sequence of partial sums is $\{-1, 0, -1, 0, -1, \dots\}$. This sequence oscillates and does not approach a single value. The series appears to be divergent.

9. **Solution:** This is a geometric series with $a = 5$ and $r = 2/3$. Since $|r| < 1$, it converges. $S = \frac{a}{1-r} = \frac{5}{1-2/3} = \frac{5}{1/3} = 15$.

10. **Solution:** $a_n = \frac{(-3)^{n-1}}{4^n} = \frac{(-3)^{n-1}}{4\cdot 4^{n-1}} = \frac{1}{4}\left(-\frac{3}{4}\right)^{n-1}$. This is a geometric series with $a = 1/4$ and $r = -3/4$. Since $|r| < 1$, it converges. $S = \frac{1/4}{1-(-3/4)} = \frac{1/4}{7/4} = \frac{1}{7}$.

11. **Solution:** $a_n = \frac{2^{2n}}{5^{n+1}} = \frac{(2^2)^n}{5\cdot 5^n} = \frac{4^n}{5\cdot 5^n} = \frac{1}{5}\left(\frac{4}{5}\right)^n$. This is a geometric series. The first term (at $n = 0$) is $a = 1/5$. The ratio is $r = 4/5$. Since $|r| < 1$, it converges. $S = \frac{1/5}{1-4/5} = \frac{1/5}{1/5} = 1$.

12. **Solution:** This is a geometric series with first term $a = 10$ and common ratio $r = -2/10 = -1/5$. Since $|r| = 1/5 < 1$, the series converges. $S = \frac{10}{1-(-1/5)} = \frac{10}{6/5} = \frac{50}{6} = \frac{25}{3}$.

13. **Solution:** $a_n = \frac{e^n}{3^{n-1}} = \frac{e\cdot e^{n-1}}{3^{n-1}} = e\left(\frac{e}{3}\right)^{n-1}$. This is a geometric series with $a = e$ and $r = e/3$. Since $e \approx 2.718$, $|r| = e/3 < 1$. It converges. $S = \frac{e}{1-e/3} = \frac{3e}{3-e}$.

14. **Solution:** We can split this into two series: $\sum_{n=1}^{\infty}\left(\frac{4}{5}\right)^n + \sum_{n=1}^{\infty}\left(\frac{3}{5}\right)^n$. Both are convergent geometric series. For the first, $a = 4/5, r = 4/5$. Sum is $\frac{4/5}{1-4/5} = 4$. For the second, $a = 3/5, r = 3/5$. Sum is $\frac{3/5}{1-3/5} = \frac{3}{2}$. Total sum is $4 + \frac{3}{2} = \frac{11}{2}$.

15. **Solution:** This is a geometric series with $r = -1/2$. The first term is for $n = 2$, so $a = 3(-1/2)^2 = 3/4$. Since $|r| < 1$, it converges. $S = \frac{a}{1-r} = \frac{3/4}{1-(-1/2)} = \frac{3/4}{3/2} = \frac{1}{2}$.

16. **Solution:** This is a geometric series, which can be written as $\sum_{n=1}^{\infty} 100\left(\frac{1}{101}\right)^n$. The first term is $a = 100/101$ and the ratio is $r = 1/101$. Since $|r| < 1$, it converges. $S = \frac{100/101}{1-1/101} = \frac{100/101}{100/101} = 1$.

17. **Solution:** $0.\overline{47} = \frac{47}{100} + \frac{47}{10000} + \frac{47}{100^3} + \ldots$. This is a geometric series with $a = 47/100$ and $r = 1/100$. It converges to $S = \frac{47/100}{1-1/100} = \frac{47/100}{99/100} = \frac{47}{99}$.

18. **Solution:** Total distance $= 10 + 2(10\cdot 0.7) + 2(10\cdot 0.7^2) + \cdots = 10 + \sum_{n=1}^{\infty} 20(0.7)^n$. The sum is a geometric series with $a = 20(0.7) = 14$ and $r = 0.7$. Sum $= \frac{14}{1-0.7} = \frac{14}{0.3} = \frac{140}{3}$. Total distance $= 10 + \frac{140}{3} = \frac{170}{3}$ meters.

19. **Solution:** $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2-1}{3n^2+n} = \frac{1}{3}$. Since the limit is not 0, the series diverges by the Test for Divergence.

20. **Solution:** $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \arctan(n) = \frac{\pi}{2}$. Since the limit is not 0, the series diverges by the Test for Divergence.

21. **Solution:** $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$. Since the limit is not 0, the series diverges by the Test for Divergence.

22. **Solution:** $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+4}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+4/n^2}} = 1$. Since the limit is not 0, the series diverges by the Test for Divergence.

23. **Solution:** $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$. The Test for Divergence is inconclusive. (Note: The harmonic series is known to diverge).

24. **Solution:** $a_k = \frac{k!}{(k+2)!} = \frac{k!}{k!(k+1)(k+2)} = \frac{1}{(k+1)(k+2)}$. $\lim_{k\to\infty} a_k = 0$. The Test for Divergence is inconclusive.

25. **Solution:** $s_n = (1-\frac{1}{2}) + (\frac{1}{2}-\frac{1}{3}) + \cdots + (\frac{1}{n}-\frac{1}{n+1}) = 1 - \frac{1}{n+1}$. $S = \lim_{n\to\infty} s_n = \lim_{n\to\infty}(1-\frac{1}{n+1}) = 1$. The series converges to 1.

26. **Solution:** Use partial fractions: $\frac{1}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2} \Rightarrow 1 = A(n+2) + Bn$. Let $n = 0 \Rightarrow A = 1/2$. Let $n = -2 \Rightarrow B = -1/2$. $a_n = \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+2}\right)$. $s_n = \frac{1}{2}\left[(1-\frac{1}{3}) + (\frac{1}{2}-\frac{1}{4}) + (\frac{1}{3}-\frac{1}{5}) + \cdots + (\frac{1}{n}-\frac{1}{n+2})\right]$. The terms that don't cancel are 1 and 1/2 at the beginning, and $-\frac{1}{n+1}$ and $-\frac{1}{n+2}$ at the end. $s_n = \frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right)$. $S = \lim_{n\to\infty} s_n = \frac{1}{2}(1 + \frac{1}{2} - 0 - 0) = \frac{3}{4}$.

27. **Solution:** Use partial fractions: $\frac{2}{n^2-1} = \frac{2}{(n-1)(n+1)} = \frac{1}{n-1} - \frac{1}{n+1}$. $s_n = \sum_{k=2}^{n}(\frac{1}{k-1} - \frac{1}{k+1}) = (1-\frac{1}{3}) + (\frac{1}{2}-\frac{1}{4}) + \cdots + (\frac{1}{n-1}-\frac{1}{n+1})$. The terms that remain are 1 and 1/2 at the beginning, and $-\frac{1}{n}$ and $-\frac{1}{n+1}$ at the end. $s_n = 1 + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1}$. $S = \lim_{n\to\infty} s_n = 1 + \frac{1}{2} = \frac{3}{2}$.

28. **Solution:** Using logarithm properties, $\ln(\frac{n+1}{n}) = \ln(n+1) - \ln(n)$. $s_n = (\ln(2) - \ln(1)) + (\ln(3) - \ln(2)) + \cdots + (\ln(n+1) - \ln(n)) = \ln(n+1) - \ln(1) = \ln(n+1)$. $S = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \ln(n+1) = \infty$. The series diverges.

29. **Solution:** This is a telescoping series. Let $b_n = \cos(1/n)$. The series is $\sum(b_{n+1} - b_n)$. $s_n = (\cos(1/2) - \cos(1)) + (\cos(1/3) - \cos(1/2)) + \cdots + (\cos(\frac{1}{n+1}) - \cos(\frac{1}{n})) = \cos(\frac{1}{n+1}) - \cos(1)$. $S = \lim_{n\to\infty} s_n = \lim_{n\to\infty}(\cos(\frac{1}{n+1}) - \cos(1)) = \cos(0) - \cos(1) = 1 - \cos(1)$.

30. **Solution:** This is a telescoping series starting at $n = 3$. $s_N = \sum_{n=3}^{N}(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}) = (\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}}) + (\frac{1}{\sqrt{4}} - \frac{1}{\sqrt{5}}) + \cdots + (\frac{1}{\sqrt{N}} - \frac{1}{\sqrt{N+1}}) = \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{N+1}}$. $S = \lim_{N\to\infty} s_N = \frac{1}{\sqrt{3}} - 0 = \frac{1}{\sqrt{3}}$.

31. **Solution: Flaw:** The conclusion in step 3 is incorrect. The Test for Divergence can only prove divergence, never convergence. If the limit of the terms is 0, the test is inconclusive. **Correct Solution:** This is the harmonic series multiplied by a constant (5). The harmonic series $\sum \frac{1}{n}$ is a p-series with $p = 1$, which is known to diverge. Therefore, the series $\sum \frac{5}{n}$ also diverges.

32. **Solution: Flaw:** Step 3 misuses the geometric series sum formula. The formula $S = a/(1-r)$ is only valid when the series converges, which requires $|r| < 1$. **Correct Solution:** This is a geometric series with ratio $r = 4/3$. Since $|r| = 4/3 \geq 1$, the series diverges. There is no finite sum.

33. **Solution: Flaw:** The cancellation pattern in step 3 is identified incorrectly. The term $-1/4$ from the first parentheses does not cancel with $1/3$ from the second. More terms must be written out to see the correct pattern. **Correct Solution:** $s_n = (\frac{1}{2} - \frac{1}{4}) + (\frac{1}{3} - \frac{1}{5}) + (\frac{1}{4} - \frac{1}{6}) + \cdots + (\frac{1}{n} - \frac{1}{n+2}) + (\frac{1}{n+1} - \frac{1}{n+3})$. The $-1/4$ cancels with the $1/4$. The terms that do not cancel are $1/2$ and $1/3$ from the beginning, and $-1/(n+2)$ and $-1/(n+3)$ from the end. $s_n = \frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3}$. $S = \lim_{n\to\infty} s_n = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$.

Concept Checklist with Problem Mapping

- **Core Concepts:** 1, 6, 8
- **Calculating from Partial Sums:** 2, 3, 7
- **Analyzing Partial Sums:** 4, 5, 8
- **Geometric Series:** 9, 10, 11, 12, 13, 14, 15, 16, 17, 18
- **Test for Divergence:** 5, 19, 20, 21, 22, 23, 24
- **Telescoping Series:** 25, 26, 27, 28, 29, 30
- **Problem Analysis ("Find the Flaw"):** 31, 32, 33