

Homework 8.1 Arc Length

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1 A Comprehensive Introduction to Arc Length

The "arc length" of a function is, quite simply, the length of the curve between two points. While we can easily find the length of a straight line segment using the distance formula, calculus is needed to find the length of a curved path.

The Arc Length Formula

The concept originates from the distance formula. Imagine a tiny segment of a curve, so small that it's almost a straight line. We can call its length ds . This small segment has a horizontal change dx and a vertical change dy . Using the Pythagorean theorem, we get:

$$(ds)^2 = (dx)^2 + (dy)^2$$

To turn this into a formula we can integrate, we can factor out a $(dx)^2$:

$$(ds)^2 = (dx)^2 \left(1 + \frac{(dy)^2}{(dx)^2} \right) \implies ds = \sqrt{1 + \left(\frac{dy}{dx} \right)^2} dx$$

To find the total length L from a point $x = a$ to $x = b$, we sum up all these tiny segments by integrating:

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx} \right)^2} dx$$

This is the primary formula for arc length when the curve is given as $y = f(x)$.

Alternatively, if the curve is given as $x = g(y)$, we can factor out $(dy)^2$ instead, leading to the second form of the formula:

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy} \right)^2} dy$$

The core challenge in arc length problems is rarely the integration itself, but rather the algebraic simplification of the expression inside the square root. Most textbook problems are cleverly designed so that the expression $1 + (f'(x))^2$ simplifies into a perfect square, which cancels the square root and makes the integral solvable.

2 Arc Length Problems and Solutions

2.1 Problem 1

Find the length of the curve $y = 4x - 5$ for $-1 \leq x \leq 2$.

Solution

First, find the derivative:

$$\frac{dy}{dx} = 4$$

Set up the arc length integral:

$$L = \int_{-1}^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_{-1}^2 \sqrt{1 + (4)^2} dx = \int_{-1}^2 \sqrt{17} dx$$

Evaluate the integral:

$$L = \sqrt{17}[x]_{-1}^2 = \sqrt{17}(2 - (-1)) = 3\sqrt{17}$$

Check with Distance Formula: The endpoints are $(-1, 4(-1) - 5) = (-1, -9)$ and $(2, 4(2) - 5) = (2, 3)$.

$$D = \sqrt{(2 - (-1))^2 + (3 - (-9))^2} = \sqrt{3^2 + 12^2} = \sqrt{9 + 144} = \sqrt{153} = \sqrt{9 \cdot 17} = 3\sqrt{17}$$

The answers match. **Answer:** $3\sqrt{17}$

2.2 Problem 2

Find the length of the curve $y = \sqrt{2 - x^2}$ for $0 \leq x \leq 1$.

Solution

Note that $y^2 = 2 - x^2 \implies x^2 + y^2 = (\sqrt{2})^2$, which is a circle of radius $\sqrt{2}$. Find the derivative:

$$\frac{dy}{dx} = \frac{-2x}{2\sqrt{2 - x^2}} = \frac{-x}{\sqrt{2 - x^2}}$$

Now, find $1 + (dy/dx)^2$:

$$1 + \left(\frac{-x}{\sqrt{2 - x^2}}\right)^2 = 1 + \frac{x^2}{2 - x^2} = \frac{2 - x^2 + x^2}{2 - x^2} = \frac{2}{2 - x^2}$$

Set up the integral:

$$L = \int_0^1 \sqrt{\frac{2}{2 - x^2}} dx = \sqrt{2} \int_0^1 \frac{1}{\sqrt{(\sqrt{2})^2 - x^2}} dx$$

This is a standard inverse sine integral form:

$$L = \sqrt{2} \left[\arcsin \left(\frac{x}{\sqrt{2}} \right) \right]_0^1 = \sqrt{2} \left(\arcsin \left(\frac{1}{\sqrt{2}} \right) - \arcsin(0) \right) = \sqrt{2} \left(\frac{\pi}{4} - 0 \right) = \frac{\pi\sqrt{2}}{4}$$

Answer: $\frac{\pi\sqrt{2}}{4}$

2.3 Problem 3

Set up, but do not evaluate, an integral for the length of the curve $y = 4x^3, 0 \leq x \leq 2$.

Solution

Find the derivative:

$$\frac{dy}{dx} = 12x^2$$

Square the derivative and add 1:

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + (12x^2)^2 = 1 + 144x^4$$

Set up the integral: **Answer:** $\int_0^2 \sqrt{1 + 144x^4} dx$

2.4 Problem 4

Set up, but do not evaluate, an integral for the length of the curve $y = 7x^3, 0 \leq x \leq 2$.

Solution

This problem is structurally identical to Problem 3. Find the derivative:

$$\frac{dy}{dx} = 21x^2$$

Square the derivative and add 1:

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + (21x^2)^2 = 1 + 441x^4$$

Set up the integral: **Answer:** $\int_0^2 \sqrt{1 + 441x^4} dx$

2.5 Problem 5

Set up, but do not evaluate, an integral for the length of the curve $y = x - 9\ln(x), 1 \leq x \leq 4$.

Solution

Find the derivative:

$$\frac{dy}{dx} = 1 - \frac{9}{x}$$

Square the derivative and add 1:

$$1 + \left(1 - \frac{9}{x}\right)^2 = 1 + \left(1 - \frac{18}{x} + \frac{81}{x^2}\right) = 2 - \frac{18}{x} + \frac{81}{x^2}$$

Set up the integral: **Answer:** $\int_1^4 \sqrt{2 - \frac{18}{x} + \frac{81}{x^2}} dx$

2.6 Problem 6

Set up an integral for the length of the curve $x = \sqrt{y} - 2y, 1 \leq y \leq 4$. Then use a calculator.

Solution

The function is given as $x = g(y)$, so we will integrate with respect to y . Find the derivative dx/dy :

$$x = y^{1/2} - 2y \implies \frac{dx}{dy} = \frac{1}{2}y^{-1/2} - 2 = \frac{1}{2\sqrt{y}} - 2$$

Square the derivative and add 1:

$$1 + \left(\frac{dx}{dy}\right)^2 = 1 + \left(\frac{1}{2\sqrt{y}} - 2\right)^2 = 1 + \left(\frac{1}{4y} - \frac{4}{2\sqrt{y}} + 4\right) = 5 + \frac{1}{4y} - \frac{2}{\sqrt{y}}$$

Set up the integral: **Answer Setup:** $\int_1^4 \sqrt{5 + \frac{1}{4y} - \frac{2}{\sqrt{y}}} dy$ Using a calculator for this integral yields approximately **3.7436**.

2.7 Problem 7

Find the exact length of the curve $y = 1 + 6x^{3/2}, 0 \leq x \leq 1$.

Solution

Find the derivative:

$$\frac{dy}{dx} = 6 \cdot \frac{3}{2} x^{1/2} = 9\sqrt{x}$$

Square the derivative and add 1:

$$1 + (9\sqrt{x})^2 = 1 + 81x$$

Set up and evaluate the integral using u-substitution ($u = 1 + 81x, du = 81dx$):

$$\begin{aligned} L &= \int_0^1 \sqrt{1 + 81x} \, dx = \frac{1}{81} \int_1^{82} \sqrt{u} \, du \\ &= \frac{1}{81} \left[\frac{2}{3} u^{3/2} \right]_1^{82} = \frac{2}{243} (82^{3/2} - 1^{3/2}) \end{aligned}$$

Answer: $\frac{2}{243} (82\sqrt{82} - 1)$

2.8 Problem 8

Find the exact length of the curve $36y^2 = (x^2 - 4)^3, 2 \leq x \leq 9, y \geq 0$.

Solution

First, solve for y:

$$y = \sqrt{\frac{(x^2 - 4)^3}{36}} = \frac{1}{6} (x^2 - 4)^{3/2}$$

Find the derivative:

$$\frac{dy}{dx} = \frac{1}{6} \cdot \frac{3}{2} (x^2 - 4)^{1/2} \cdot (2x) = \frac{x}{2} \sqrt{x^2 - 4}$$

Square the derivative and add 1 (this is a classic perfect square trick):

$$1 + \left(\frac{dy}{dx} \right)^2 = 1 + \frac{x^2}{4} (x^2 - 4) = 1 + \frac{x^4 - 4x^2}{4} = \frac{4 + x^4 - 4x^2}{4} = \frac{(x^2 - 2)^2}{4}$$

Set up and evaluate the integral:

$$\begin{aligned} L &= \int_2^9 \sqrt{\frac{(x^2 - 2)^2}{4}} \, dx = \int_2^9 \frac{x^2 - 2}{2} \, dx \\ &= \frac{1}{2} \left[\frac{x^3}{3} - 2x \right]_2^9 = \frac{1}{2} \left(\left(\frac{9^3}{3} - 18 \right) - \left(\frac{2^3}{3} - 4 \right) \right) \\ &= \frac{1}{2} \left((243 - 18) - \left(\frac{8}{3} - 4 \right) \right) = \frac{1}{2} \left(225 - \left(-\frac{4}{3} \right) \right) = \frac{1}{2} \left(\frac{675 + 4}{3} \right) = \frac{679}{6} \end{aligned}$$

Answer: $\frac{679}{6}$

2.9 Problem 9

Find the exact length of the curve $y = \frac{x^3}{3} + \frac{1}{4x}, 1 \leq x \leq 2$.

Solution

This is a classic "perfect square" problem. Find the derivative:

$$y = \frac{1}{3}x^3 + \frac{1}{4}x^{-1} \implies \frac{dy}{dx} = x^2 - \frac{1}{4x^2}$$

Square the derivative and add 1:

$$\begin{aligned} 1 + \left(x^2 - \frac{1}{4x^2}\right)^2 &= 1 + \left((x^2)^2 - 2(x^2)\left(\frac{1}{4x^2}\right) + \left(\frac{1}{4x^2}\right)^2\right) \\ &= 1 + \left(x^4 - \frac{1}{2} + \frac{1}{16x^4}\right) = x^4 + \frac{1}{2} + \frac{1}{16x^4} \\ &= \left(x^2 + \frac{1}{4x^2}\right)^2 \end{aligned}$$

Set up and evaluate the integral:

$$\begin{aligned} L &= \int_1^2 \sqrt{\left(x^2 + \frac{1}{4x^2}\right)^2} dx = \int_1^2 \left(x^2 + \frac{1}{4}x^{-2}\right) dx \\ &= \left[\frac{x^3}{3} - \frac{1}{4x}\right]_1^2 = \left(\frac{8}{3} - \frac{1}{8}\right) - \left(\frac{1}{3} - \frac{1}{4}\right) \\ &= \left(\frac{64-3}{24}\right) - \left(\frac{4-3}{12}\right) = \frac{61}{24} - \frac{1}{12} = \frac{61-2}{24} = \frac{59}{24} \end{aligned}$$

Answer: $\frac{59}{24}$

2.10 Problem 10

Find the exact length of the curve $x = \frac{1}{3}\sqrt{y}(y-3)$, $16 \leq y \leq 25$.

Solution

This is an integral with respect to y . First simplify the expression for x .

$$x = \frac{1}{3}(y^{3/2} - 3y^{1/2})$$

Find the derivative dx/dy :

$$\frac{dx}{dy} = \frac{1}{3} \left(\frac{3}{2}y^{1/2} - \frac{3}{2}y^{-1/2} \right) = \frac{1}{2} \left(\sqrt{y} - \frac{1}{\sqrt{y}} \right)$$

Square the derivative and add 1:

$$\begin{aligned} 1 + \frac{1}{4} \left(\sqrt{y} - \frac{1}{\sqrt{y}} \right)^2 &= 1 + \frac{1}{4} \left(y - 2 + \frac{1}{y} \right) = 1 + \frac{y}{4} - \frac{1}{2} + \frac{1}{4y} \\ &= \frac{y}{4} + \frac{1}{2} + \frac{1}{4y} = \frac{1}{4} \left(y + 2 + \frac{1}{y} \right) = \left[\frac{1}{2} \left(\sqrt{y} + \frac{1}{\sqrt{y}} \right) \right]^2 \end{aligned}$$

Set up and evaluate the integral:

$$\begin{aligned} L &= \int_{16}^{25} \frac{1}{2} \left(y^{1/2} + y^{-1/2} \right) dy = \frac{1}{2} \left[\frac{2}{3}y^{3/2} + 2y^{1/2} \right]_{16}^{25} \\ &= \left[\frac{1}{3}y^{3/2} + y^{1/2} \right]_{16}^{25} = \left(\frac{1}{3}(25)^{3/2} + \sqrt{25} \right) - \left(\frac{1}{3}(16)^{3/2} + \sqrt{16} \right) \\ &= \left(\frac{125}{3} + 5 \right) - \left(\frac{64}{3} + 4 \right) = \frac{61}{3} + 1 = \frac{64}{3} \end{aligned}$$

Answer: $\frac{64}{3}$

2.11 Problem 11

Find the exact length of the curve $y = \ln(\sec(x)), 0 \leq x \leq \pi/4$.

Solution

Find the derivative:

$$\frac{dy}{dx} = \frac{1}{\sec(x)} \cdot \sec(x) \tan(x) = \tan(x)$$

Square the derivative and add 1, using a Pythagorean identity:

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \tan^2(x) = \sec^2(x)$$

Set up and evaluate the integral:

$$L = \int_0^{\pi/4} \sqrt{\sec^2(x)} dx = \int_0^{\pi/4} \sec(x) dx$$

The integral of $\sec(x)$ is a standard result:

$$\begin{aligned} L &= [\ln |\sec(x) + \tan(x)|]_0^{\pi/4} = \ln |\sec(\pi/4) + \tan(\pi/4)| - \ln |\sec(0) + \tan(0)| \\ &= \ln |\sqrt{2} + 1| - \ln |1 + 0| = \ln(\sqrt{2} + 1) \end{aligned}$$

Answer: $\ln(\sqrt{2} + 1)$

2.12 Problem 12

Find the exact length of the curve $y = \frac{x^2}{4} - \frac{1}{2} \ln(x), 1 \leq x \leq 8$.

Solution

This is another perfect square problem. Find the derivative:

$$\frac{dy}{dx} = \frac{2x}{4} - \frac{1}{2x} = \frac{x}{2} - \frac{1}{2x}$$

Square the derivative and add 1:

$$\begin{aligned} 1 + \left(\frac{x}{2} - \frac{1}{2x}\right)^2 &= 1 + \left(\frac{x^2}{4} - 2\left(\frac{x}{2}\right)\left(\frac{1}{2x}\right) + \frac{1}{4x^2}\right) \\ &= 1 + \left(\frac{x^2}{4} - \frac{1}{2} + \frac{1}{4x^2}\right) = \frac{x^2}{4} + \frac{1}{2} + \frac{1}{4x^2} \\ &= \left(\frac{x}{2} + \frac{1}{2x}\right)^2 \end{aligned}$$

Set up and evaluate the integral:

$$\begin{aligned} L &= \int_1^8 \sqrt{\left(\frac{x}{2} + \frac{1}{2x}\right)^2} dx = \int_1^8 \left(\frac{x}{2} + \frac{1}{2x}\right) dx \\ &= \left[\frac{x^2}{4} + \frac{1}{2} \ln|x|\right]_1^8 = \left(\frac{64}{4} + \frac{1}{2} \ln(8)\right) - \left(\frac{1}{4} + \frac{1}{2} \ln(1)\right) \\ &= 16 + \frac{1}{2} \ln(8) - \frac{1}{4} = \frac{63}{4} + \frac{1}{2} \ln(8) \end{aligned}$$

Answer: $\frac{63}{4} + \frac{1}{2} \ln(8)$

2.13 Problem 13

Find the exact length of the curve $y = \sqrt{x - x^2} + \arcsin(\sqrt{x})$.

Solution

First, find the domain. We need $x - x^2 \geq 0 \implies x(1 - x) \geq 0 \implies 0 \leq x \leq 1$. Find the derivative using the chain rule:

$$\begin{aligned}\frac{dy}{dx} &= \frac{1 - 2x}{2\sqrt{x - x^2}} + \frac{1}{\sqrt{1 - (\sqrt{x})^2}} \cdot \frac{d}{dx}(\sqrt{x}) \\&= \frac{1 - 2x}{2\sqrt{x - x^2}} + \frac{1}{\sqrt{1 - x}} \cdot \frac{1}{2\sqrt{x}} \\&= \frac{1 - 2x}{2\sqrt{x(1 - x)}} + \frac{1}{2\sqrt{x(1 - x)}} = \frac{1 - 2x + 1}{2\sqrt{x - x^2}} = \frac{2 - 2x}{2\sqrt{x - x^2}} \\&= \frac{1 - x}{\sqrt{x(1 - x)}} = \frac{\sqrt{(1 - x)^2}}{\sqrt{x}\sqrt{1 - x}} = \frac{\sqrt{1 - x}}{\sqrt{x}}\end{aligned}$$

Square the derivative and add 1:

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(\frac{\sqrt{1 - x}}{\sqrt{x}}\right)^2 = 1 + \frac{1 - x}{x} = \frac{x + 1 - x}{x} = \frac{1}{x}$$

Set up and evaluate the integral over the full domain $[0, 1]$. Note this is an improper integral since the integrand is undefined at $x = 0$, but it converges.

$$L = \int_0^1 \sqrt{\frac{1}{x}} dx = \int_0^1 x^{-1/2} dx = [2x^{1/2}]_0^1 = 2\sqrt{1} - 2\sqrt{0} = 2$$

Answer: 2

2.14 Problem 14

Find the exact length of the curve $36y^2 = (x^2 - 4)^3, 4 \leq x \leq 6, y \geq 0$.

Solution

This is the same function as Problem 8, but with different bounds. We have already found that the integrand simplifies to $\frac{x^2 - 2}{2}$.

$$\begin{aligned}L &= \int_4^6 \frac{x^2 - 2}{2} dx = \frac{1}{2} \left[\frac{x^3}{3} - 2x \right]_4^6 \\&= \frac{1}{2} \left(\left(\frac{6^3}{3} - 12 \right) - \left(\frac{4^3}{3} - 8 \right) \right) \\&= \frac{1}{2} \left((72 - 12) - \left(\frac{64}{3} - 8 \right) \right) = \frac{1}{2} \left(60 - \frac{40}{3} \right) = \frac{1}{2} \left(\frac{180 - 40}{3} \right) = \frac{1}{2} \cdot \frac{140}{3} = \frac{70}{3}\end{aligned}$$

Answer: $\frac{70}{3}$

2.15 Problem 15

Find the exact length of the curve $y = \ln(\cos(x)), 0 \leq x \leq \pi/6$.

Solution

Find the derivative:

$$\frac{dy}{dx} = \frac{1}{\cos(x)} \cdot (-\sin(x)) = -\tan(x)$$

Square the derivative and add 1:

$$1 + (-\tan(x))^2 = 1 + \tan^2(x) = \sec^2(x)$$

Set up and evaluate the integral:

$$\begin{aligned} L &= \int_0^{\pi/6} \sqrt{\sec^2(x)} dx = \int_0^{\pi/6} \sec(x) dx \\ &= [\ln |\sec(x) + \tan(x)|]_0^{\pi/6} \\ &= \ln |\sec(\pi/6) + \tan(\pi/6)| - \ln |\sec(0) + \tan(0)| \\ &= \ln \left| \frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} \right| - \ln |1 + 0| = \ln \left| \frac{3}{\sqrt{3}} \right| = \ln(\sqrt{3}) \end{aligned}$$

Answer: $\ln(\sqrt{3})$ or $\frac{1}{2} \ln(3)$

3 Analysis of Problems and Techniques

3.1 Problem Types and General Approach

1. **Geometric Shapes (Lines and Circles):** Problems 1 and 2. These serve as a good introduction, as the results can be verified with standard geometric formulas. The core process remains the same: differentiate, substitute, simplify, integrate.
2. **Setup Only:** Problems 3, 4, 5. These focus solely on the first steps: finding the derivative and correctly substituting it into the arc length formula.
3. **Simple Power Functions:** Problem 7 ($y = 1 + 6x^{3/2}$). The derivative is a simple power of x . The expression $1 + (dy/dx)^2$ often leads to a straightforward u -substitution.
4. **Integrals with respect to y :** Problems 6 and 10 ($x = g(y)$). When x is given as a function of y , it is much easier to use the formula $L = \int \sqrt{1 + (dx/dy)^2} dy$ than to solve for y and use the standard formula.
5. **The "Perfect Square" Trick:** This is the most common pattern in textbook problems designed for exact evaluation. The function is specifically constructed so that $1 + (f'(x))^2$ simplifies into a perfect square, eliminating the radical. This pattern was seen in Problems 8, 9, 10, 12, and 14.
6. **Trigonometric and Logarithmic Functions:** Problems 11, 13, and 15. These rely on knowing the derivatives of trig/log functions and then simplifying the result using identities like $1 + \tan^2(x) = \sec^2(x)$.

3.2 Key Algebraic Manipulations and Tricks

The success in solving arc length problems hinges on your ability to simplify the expression $1 + (f'(x))^2$.

- **The Perfect Square Trick Explained:** This trick is central. Many functions are of the form $y = Af(x) \pm B/f(x)$, such as $y = x^3/3 + 1/(4x)$ from Problem 9. The derivative is $y' = f'(x) - Bf'(x)/f(x)^2$. When you square this, you get a middle term of $-2B(f'(x))^2/f(x)^2$. When you add 1, the constant term magically flips the sign of the middle term, creating a new perfect square.

$$\begin{aligned} - \text{Example from Problem 9: } y' &= x^2 - \frac{1}{4x^2}. \\ - (y')^2 &= (x^2)^2 - 2(x^2)\left(\frac{1}{4x^2}\right) + \left(\frac{1}{4x^2}\right)^2 = x^4 - \frac{1}{2} + \frac{1}{16x^4}. \\ - 1 + (y')^2 &= 1 + x^4 - \frac{1}{2} + \frac{1}{16x^4} = x^4 + \frac{1}{2} + \frac{1}{16x^4}. \end{aligned}$$

- This is now the perfect square of $(x^2 + \frac{1}{4x^2})^2$.
- **Finding a Common Denominator:** In problems like 2 and 8, after squaring the derivative, getting a common denominator for the ‘ $1 + \dots$ ’ term is the key to revealing a perfect square in the numerator.
- **Using Trigonometric Identities:** The Pythagorean identities are your best friend. For problems like 11 and 15 where the derivative is $\pm \tan(x)$, the expression immediately simplifies: $1 + \tan^2(x) = \sec^2(x)$.
- **Dramatic Simplification:** Problem 13 is an example of an intimidating function whose derivative simplifies beautifully. If the derivative looks like a mess, simplify it algebraically *before* squaring it.

3.3 Essential Calculus Skills Required

- **Differentiation:** You must be proficient with the power rule, product rule, quotient rule, and especially the chain rule. You must know the derivatives of polynomial, radical, logarithmic, trigonometric, and inverse trigonometric functions.
- **Integration:** Once the setup is simplified, you will need to use basic integration rules, u-substitution, and know the integrals of standard functions like $\sec(x)$.

3.4 Cheats and Tips for Success

- **Expect Simplification:** If you are asked to find an *exact* arc length, the expression under the square root will almost certainly simplify. If it doesn’t, double-check your derivative and your algebra.
- **Look for the Trick:** Identify if the function fits the “perfect square” form. This will guide your algebraic simplification.
- **Choose the Right Variable:** If you’re given $y^2 = f(x)$, check if solving for x in terms of y might lead to an easier derivative (dx/dy) than solving for y .
- **Practice Algebra:** The calculus steps (differentiation and integration) are often the easiest part. The make-or-break step is the algebra in the middle. Practice squaring binomials and simplifying complex fractions.