

Part 1: Comprehensive Introduction, Context, and Prerequisites

Core Concepts

The topic of Arc Length in calculus deals with finding the exact length of a smooth curve in a plane. Unlike straight line segments, whose lengths can be easily found using the distance formula, curves require a more sophisticated approach. The central idea is to approximate the curve with a series of small, straight line segments and then use the concept of a definite integral to sum up the lengths of these infinitesimally small segments. This process is sometimes referred to as the rectification of a curve.

Intuition and Derivation

The formula for arc length is elegantly derived from the Pythagorean theorem. Imagine a small segment of a curve. If we zoom in close enough, this tiny segment appears almost straight. We can think of this small segment of arc length, denoted as ds , as the hypotenuse of a tiny right triangle with base dx (a small change in x) and height dy (a small change in y).

According to the Pythagorean theorem:

$$(ds)^2 = (dx)^2 + (dy)^2$$

Taking the square root of both sides gives:

$$ds = \sqrt{(dx)^2 + (dy)^2}$$

To turn this into a form we can integrate, we can factor out a $(dx)^2$ from under the square root:

$$\begin{aligned} ds &= \sqrt{dx^2(1 + \frac{(dy)^2}{(dx)^2})} \\ ds &= \sqrt{1 + (\frac{dy}{dx})^2} dx \end{aligned}$$

To find the total length of the curve from $x = a$ to $x = b$, we integrate this infinitesimal length ds over the interval $[a, b]$:

$$L = \int_a^b \sqrt{1 + (\frac{dy}{dx})^2} dx$$

Historical Context and Motivation

The problem of finding the length of a curve, or rectification, has a long history in mathematics. Ancient Greek mathematicians, such as Archimedes, were able to approximate the circumference of a circle (the arc length of a circle) by inscribing and circumscribing polygons with an increasing number of sides. However, for more general curves, a solution remained elusive for centuries.

The development of calculus in the 17th century by Isaac Newton and Gottfried Leibniz provided the necessary tools to solve the arc length problem for a wide variety of curves. The ability to conceptualize a curve as an infinite number of infinitesimally small straight line segments and to sum them up using integration was a monumental leap forward. This breakthrough was not just a mathematical curiosity; it was driven by the need to solve problems in physics, astronomy, and engineering, such as calculating the distance traveled by a planet in its orbit or the length of a cable needed for a suspension bridge.

Key Formulas

The primary formulas for arc length depend on how the curve is defined:

- For a function $y = f(x)$ from $x = a$ to $x = b$: The derivative $f'(x)$ must be continuous on $[a, b]$.

$$L = \int_a^b \sqrt{1 + (f'(x))^2} dx$$

- For a function $x = g(y)$ from $y = c$ to $y = d$: The derivative $g'(y)$ must be continuous on $[c, d]$.

$$L = \int_c^d \sqrt{1 + (g'(y))^2} dy$$

- For a parametric curve $x = f(t)$, $y = g(t)$ from $t = \alpha$ to $t = \beta$: The derivatives $f'(t)$ and $g'(t)$ must be continuous on $[\alpha, \beta]$.

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

- For a polar curve $r = f(\theta)$ from $\theta = \alpha$ to $\theta = \beta$:

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

Prerequisites

To successfully tackle arc length problems, a solid foundation in the following areas is essential:

Algebra:

- Simplifying complex algebraic expressions.

- Factoring and expanding polynomials, especially recognizing perfect squares.
- Working with radicals and exponents.

Trigonometry:

- Knowledge of fundamental trigonometric identities (e.g., $\sin^2 x + \cos^2 x = 1$, $\sec^2 x - \tan^2 x = 1$).

- Derivatives and integrals of trigonometric functions.

Calculus I:

- Differentiation: Proficiency with all differentiation rules (power rule, product rule, quotient rule, chain rule).

- Integration: Mastery of basic integration techniques, including u-substitution and integration of common functions.
- The Fundamental Theorem of Calculus: Understanding how to evaluate definite integrals.

Part 2: Detailed Homework Solutions

Here are the detailed, step-by-step solutions for every problem in the homework assignment.

Problem 1

Use the arc length formula to find the length of the curve $y = 4x - 5$, $-1 \leq x \leq 2$. Check your answer by noting that the curve is a line segment and calculating its length by the distance formula.

Step 1: Find the derivative. $y = 4x - 5$

$$\frac{dy}{dx} = 4$$

Step 2: Substitute into the arc length formula.

$$L = \int_{-1}^2 \sqrt{1 + (4)^2} dx = \int_{-1}^2 \sqrt{1 + 16} dx = \int_{-1}^2 \sqrt{17} dx$$

Step 3: Evaluate the integral.

$$L = [\sqrt{17} \cdot x]_{-1}^2$$
$$L = (\sqrt{17} \cdot 2) - (\sqrt{17} \cdot -1) = 2\sqrt{17} + \sqrt{17} = 3\sqrt{17}$$

Check with distance formula: The endpoints are $(-1, -9)$ and $(2, 3)$.

$$D = \sqrt{((2 - (-1))^2 + (3 - (-9))^2)} = \sqrt{(3)^2 + (12)^2} = \sqrt{9 + 144} = \sqrt{153} = \sqrt{9 \cdot 17} = 3\sqrt{17}.$$

The answers match.

Final Answer: $3\sqrt{17}$

Problem 2

Use the arc length formula to find the length of the curve $y = \sqrt{2 - x^2}$, $0 \leq x \leq 1$. Check your answer by noting that the curve is part of a circle.

Step 1: Find the derivative. $y = (2 - x^2)^{1/2}$

$$\frac{dy}{dx} = \frac{1}{2}(2 - x^2)^{-1/2} \cdot (-2x) = \frac{-x}{\sqrt{2 - x^2}}$$

Step 2: Substitute and simplify.

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{x^2}{2 - x^2} = \frac{2 - x^2 + x^2}{2 - x^2} = \frac{2}{2 - x^2}$$

$$L = \int_0^1 \sqrt{\frac{2}{2 - x^2}} dx = \sqrt{2} \int_0^1 \frac{1}{\sqrt{2 - x^2}} dx$$

Step 3: Evaluate the integral. This is an arcsin form: $\int \frac{1}{\sqrt{a^2 - u^2}} du = \arcsin(\frac{u}{a})$. Here, $a = \sqrt{2}$.

$$L = \sqrt{2} \left[\arcsin\left(\frac{x}{\sqrt{2}}\right) \right]_0^1$$

$$L = \sqrt{2} \left(\arcsin\left(\frac{1}{\sqrt{2}}\right) - \arcsin(0) \right) = \sqrt{2} \left(\frac{\pi}{4} - 0 \right) = \frac{\pi\sqrt{2}}{4}$$

Final Answer: $\frac{\pi\sqrt{2}}{4}$

Problem 3

Set up, but do not evaluate, an integral for the length of the curve $y = 4x^3$, $0 \leq x \leq 2$.

Step 1: Find the derivative.

$$\frac{dy}{dx} = 12x^2$$

Step 2: Substitute into the formula.

$$L = \int_0^2 \sqrt{1 + (12x^2)^2} dx = \int_0^2 \sqrt{1 + 144x^4} dx$$

Final Answer (Integral Setup): $\int_0^2 \sqrt{1 + 144x^4} dx$

Problem 4

Set up, but do not evaluate, an integral for the length of the curve $y = 7x^3$, $0 \leq x \leq 2$.

Step 1: Find the derivative.

$$\frac{dy}{dx} = 21x^2$$

Step 2: Substitute into the formula.

$$L = \int_0^2 \sqrt{1 + (21x^2)^2} dx = \int_0^2 \sqrt{1 + 441x^4} dx$$

Final Answer (Integral Setup): $\int_0^2 \sqrt{1 + 441x^4} dx$

Problem 5

Set up, but do not evaluate, an integral for the length of the curve $y = x - 9 \ln(x)$, $1 \leq x \leq 4$.

Step 1: Find the derivative.

$$\frac{dy}{dx} = 1 - \frac{9}{x}$$

Step 2: Substitute into the formula.

$$L = \int_1^4 \sqrt{1 + (1 - \frac{9}{x})^2} dx$$

Final Answer (Integral Setup): $\int_1^4 \sqrt{1 + (1 - \frac{9}{x})^2} dx$

Problem 6

Set up an integral that represents the length of the curve. Then use your calculator to find the length correct to four decimal places. $x = \sqrt{y} - 2y$, $1 \leq y \leq 4$.

Step 1: Find the derivative with respect to y. $x = y^{1/2} - 2y$

$$\frac{dx}{dy} = \frac{1}{2}y^{-1/2} - 2 = \frac{1}{2\sqrt{y}} - 2$$

Step 2: Set up the integral.

$$L = \int_1^4 \sqrt{1 + (\frac{1}{2\sqrt{y}} - 2)^2} dy$$

Step 3: Use a calculator to evaluate.

$$L \approx 3.7385$$

Final Answer: Integral: $\int_1^4 \sqrt{1 + (\frac{1}{2\sqrt{y}} - 2)^2} dy$; Length: ≈ 3.7385

Problem 7

Find the exact length of the curve $y = 1 + 6x^{3/2}$, $0 \leq x \leq 1$.

Step 1: Find the derivative.

$$\frac{dy}{dx} = 6 \cdot \frac{3}{2} x^{1/2} = 9\sqrt{x}$$

Step 2: Substitute into the formula.

$$L = \int_0^1 \sqrt{1 + (9\sqrt{x})^2} dx = \int_0^1 \sqrt{1 + 81x} dx$$

Step 3: Evaluate using u-substitution. Let $u = 1 + 81x$, so $du = 81 dx$. Bounds become $u(0) = 1$ and $u(1) = 82$.

$$\begin{aligned} L &= \int_1^{82} \sqrt{u} \frac{du}{81} = \frac{1}{81} \left[\frac{2}{3} u^{3/2} \right]_1^{82} \\ L &= \frac{2}{243} [82^{3/2} - 1^{3/2}] = \frac{2}{243} (82\sqrt{82} - 1) \end{aligned}$$

Final Answer: $\frac{2}{243} (82\sqrt{82} - 1)$

Problem 8

Find the exact length of the curve $36y^2 = (x^2 - 4)^3$, $2 \leq x \leq 9$, $y \geq 0$.

Step 1: Solve for y and find the derivative. $y = \frac{1}{6}(x^2 - 4)^{3/2}$

$$\frac{dy}{dx} = \frac{1}{6} \cdot \frac{3}{2}(x^2 - 4)^{1/2} \cdot (2x) = \frac{x}{2} \sqrt{x^2 - 4}$$

Step 2: Square the derivative and add 1 (Perfect Square Trick).

$$\begin{aligned} \left(\frac{dy}{dx}\right)^2 &= \frac{x^2}{4}(x^2 - 4) = \frac{x^4}{4} - x^2 \\ 1 + \left(\frac{dy}{dx}\right)^2 &= 1 + \frac{x^4}{4} - x^2 = \frac{4 + x^4 - 4x^2}{4} = \frac{x^4 - 4x^2 + 4}{4} = \frac{(x^2 - 2)^2}{4} \end{aligned}$$

Step 3: Set up and evaluate the integral.

$$L = \int_2^9 \sqrt{\frac{(x^2 - 2)^2}{4}} dx = \int_2^9 \left| \frac{x^2 - 2}{2} \right| dx$$

Since $x \geq 2$, $x^2 - 2$ is positive.

$$\begin{aligned} L &= \frac{1}{2} \int_2^9 (x^2 - 2) dx = \frac{1}{2} \left[\frac{x^3}{3} - 2x \right]_2^9 \\ L &= \frac{1}{2} \left[\left(\frac{9^3}{3} - 18 \right) - \left(\frac{2^3}{3} - 4 \right) \right] = \frac{1}{2} \left[(243 - 18) - \left(\frac{8}{3} - \frac{12}{3} \right) \right] \\ L &= \frac{1}{2} [225 - (-\frac{4}{3})] = \frac{1}{2} \left[\frac{675}{3} + \frac{4}{3} \right] = \frac{1}{2} \left(\frac{679}{3} \right) = \frac{679}{6} \end{aligned}$$

Final Answer: $\frac{679}{6}$

Problem 9

Find the exact length of the curve $y = \frac{x^3}{3} + \frac{1}{4x}$, $1 \leq x \leq 2$.

Step 1: Find the derivative.

$$\frac{dy}{dx} = x^2 - \frac{1}{4x^2}$$

Step 2: Square the derivative and add 1 (Perfect Square Trick).

$$\begin{aligned} \left(\frac{dy}{dx}\right)^2 &= x^4 - 2(x^2)\left(\frac{1}{4x^2}\right) + \frac{1}{16x^4} = x^4 - \frac{1}{2} + \frac{1}{16x^4} \\ 1 + \left(\frac{dy}{dx}\right)^2 &= 1 + x^4 - \frac{1}{2} + \frac{1}{16x^4} = x^4 + \frac{1}{2} + \frac{1}{16x^4} = \left(x^2 + \frac{1}{4x^2}\right)^2 \end{aligned}$$

Step 3: Set up and evaluate the integral.

$$\begin{aligned} L &= \int_1^2 \sqrt{\left(x^2 + \frac{1}{4x^2}\right)^2} dx = \int_1^2 \left(x^2 + \frac{1}{4x^2}\right) dx \\ L &= \left[\frac{x^3}{3} - \frac{1}{4x}\right]_1^2 \\ L &= \left(\left(\frac{8}{3} - \frac{1}{8}\right) - \left(\frac{1}{3} - \frac{1}{4}\right)\right) = \left(\left(\frac{64-3}{24}\right) - \left(\frac{4-3}{12}\right)\right) = \frac{61}{24} - \frac{1}{12} = \frac{59}{24} \end{aligned}$$

Final Answer: $\frac{59}{24}$

Problem 10

Find the exact length of the curve $x = \frac{1}{3}\sqrt{y}(y-3)$, $16 \leq y \leq 25$.

Step 1: Expand and find the derivative w.r.t. y. $x = \frac{1}{3}(y^{3/2} - 3y^{1/2})$

$$\frac{dx}{dy} = \frac{1}{3}\left(\frac{3}{2}y^{1/2} - \frac{3}{2}y^{-1/2}\right) = \frac{1}{2}\left(\sqrt{y} - \frac{1}{\sqrt{y}}\right)$$

Step 2: Square the derivative and add 1 (Perfect Square Trick).

$$\begin{aligned} \left(\frac{dx}{dy}\right)^2 &= \frac{1}{4}\left(y - 2 + \frac{1}{y}\right) = \frac{y}{4} - \frac{1}{2} + \frac{1}{4y} \\ 1 + \left(\frac{dx}{dy}\right)^2 &= 1 + \frac{y}{4} - \frac{1}{2} + \frac{1}{4y} = \frac{y}{4} + \frac{1}{2} + \frac{1}{4y} = \left(\frac{1}{2}\left(\sqrt{y} + \frac{1}{\sqrt{y}}\right)\right)^2 \end{aligned}$$

Step 3: Set up and evaluate the integral.

$$\begin{aligned} L &= \int_{16}^{25} \frac{1}{2}\left(\sqrt{y} + \frac{1}{\sqrt{y}}\right) dy = \frac{1}{2}\left[\frac{2}{3}y^{3/2} + 2\sqrt{y}\right]_{16}^{25} \\ L &= \left[\frac{1}{3}y^{3/2} + \sqrt{y}\right]_{16}^{25} \\ L &= \left(\left(\frac{1}{3}(125) + 5\right) - \left(\frac{1}{3}(64) + 4\right)\right) = \left(\frac{140}{3}\right) - \left(\frac{76}{3}\right) = \frac{64}{3} \end{aligned}$$

Final Answer: $\frac{64}{3}$

Problem 11

Find the exact length of the curve $y = \ln(\sec(x))$, $0 \leq x \leq \pi/4$.

Step 1: Find the derivative.

$$\frac{dy}{dx} = \frac{1}{\sec(x)} \cdot (\sec(x) \tan(x)) = \tan(x)$$

Step 2: Substitute and use a trig identity.

$$L = \int_0^{\pi/4} \sqrt{1 + \tan^2(x)} dx = \int_0^{\pi/4} \sqrt{\sec^2(x)} dx = \int_0^{\pi/4} \sec(x) dx$$

Step 3: Evaluate the integral.

$$L = [\ln |\sec(x) + \tan(x)|]_0^{\pi/4}$$

$$L = (\ln |\sec(\pi/4) + \tan(\pi/4)|) - (\ln |\sec(0) + \tan(0)|)$$

$$L = \ln(\sqrt{2} + 1) - \ln(1 + 0) = \ln(\sqrt{2} + 1)$$

Final Answer: $\ln(\sqrt{2} + 1)$

Problem 12

Find the exact length of the curve $y = \frac{1}{4}x^2 - \frac{1}{2}\ln(x)$, $1 \leq x \leq 8$.

Step 1: Find the derivative.

$$\frac{dy}{dx} = \frac{x}{2} - \frac{1}{2x}$$

Step 2: Square the derivative and add 1 (Perfect Square Trick).

$$\begin{aligned} \left(\frac{dy}{dx}\right)^2 &= \frac{x^2}{4} - \frac{1}{2} + \frac{1}{4x^2} \\ 1 + \left(\frac{dy}{dx}\right)^2 &= 1 + \frac{x^2}{4} - \frac{1}{2} + \frac{1}{4x^2} = \frac{x^2}{4} + \frac{1}{2} + \frac{1}{4x^2} = \left(\frac{x}{2} + \frac{1}{2x}\right)^2 \end{aligned}$$

Step 3: Set up and evaluate the integral.

$$\begin{aligned} L &= \int_1^8 \left(\frac{x}{2} + \frac{1}{2x}\right) dx = \frac{1}{2} \left[\frac{x^2}{2} + \ln|x|\right]_1^8 \\ L &= \frac{1}{2} \left[\left(\frac{64}{2} + \ln(8)\right) - \left(\frac{1}{2} + \ln(1)\right) \right] = \frac{1}{2} [32 + 3\ln(2) - \frac{1}{2}] \\ L &= \frac{1}{2} \left[\frac{63}{2} + 3\ln(2) \right] = \frac{63}{4} + \frac{3}{2}\ln(2) \end{aligned}$$

Final Answer: $\frac{63}{4} + \frac{3}{2}\ln(2)$

Problem 13

Find the exact length of the curve $y = \sqrt{x-x^2} + \arcsin(\sqrt{x})$. (Domain is $0 \leq x \leq 1$)

Step 1: Find the derivative and simplify.

$$\frac{dy}{dx} = \frac{1-2x}{2\sqrt{x-x^2}} + \frac{1}{\sqrt{1-(\sqrt{x})^2}} \cdot \frac{1}{2\sqrt{x}}$$

$$\frac{dy}{dx} = \frac{1-2x}{2\sqrt{x}\sqrt{1-x}} + \frac{1}{2\sqrt{x}\sqrt{1-x}}$$

$$\frac{dy}{dx} = \frac{1-2x+1}{2\sqrt{x}\sqrt{1-x}} = \frac{2-2x}{2\sqrt{x}\sqrt{1-x}} = \frac{\sqrt{1-x}}{\sqrt{x}}$$

Step 2: Substitute and simplify.

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{1-x}{x} = \frac{x+1-x}{x} = \frac{1}{x}$$

Step 3: Evaluate the improper integral.

$$\begin{aligned} L &= \int_0^1 \sqrt{\frac{1}{x}} dx = \int_0^1 x^{-1/2} dx \\ L &= \lim_{a \rightarrow 0^+} \int_a^1 x^{-1/2} dx = \lim_{a \rightarrow 0^+} [2\sqrt{x}]_a^1 \\ L &= \lim_{a \rightarrow 0^+} [2\sqrt{1} - 2\sqrt{a}] = 2 - 0 = 2 \end{aligned}$$

Final Answer: 2

Problem 14

Find the exact length of the curve $36y^2 = (x^2 - 4)^3$, $4 \leq x \leq 6$, $y \geq 0$. This problem has the same function as Problem 8, just with different bounds. The setup is identical.

Step 1: Derivative and Perfect Square.

$$\begin{aligned} \frac{dy}{dx} &= \frac{x}{2} \sqrt{x^2 - 4} \\ \sqrt{1 + \left(\frac{dy}{dx}\right)^2} &= \frac{x^2 - 2}{2} \quad (\text{since } x^2 - 2 \text{ is positive on } [4, 6]) \end{aligned}$$

Step 2: Evaluate the integral.

$$\begin{aligned} L &= \frac{1}{2} \int_4^6 (x^2 - 2) dx = \frac{1}{2} \left[\frac{x^3}{3} - 2x \right]_4^6 \\ L &= \frac{1}{2} \left[\left(\frac{6^3}{3} - 12 \right) - \left(\frac{4^3}{3} - 8 \right) \right] = \frac{1}{2} \left[(72 - 12) - \left(\frac{64}{3} - \frac{24}{3} \right) \right] \\ L &= \frac{1}{2} [60 - \frac{40}{3}] = \frac{1}{2} \left[\frac{180}{3} - \frac{40}{3} \right] = \frac{1}{2} \left(\frac{140}{3} \right) = \frac{70}{3} \end{aligned}$$

Final Answer: $\frac{70}{3}$

Problem 15

Find the exact length of the curve $y = \ln(\cos(x))$, $0 \leq x \leq \pi/6$.

Step 1: Find the derivative.

$$\frac{dy}{dx} = \frac{1}{\cos(x)} \cdot (-\sin(x)) = -\tan(x)$$

Step 2: Substitute and use a trig identity.

$$L = \int_0^{\pi/6} \sqrt{1 + (-\tan(x))^2} dx = \int_0^{\pi/6} \sqrt{\sec^2(x)} dx = \int_0^{\pi/6} \sec(x) dx$$

Step 3: Evaluate the integral.

$$L = [\ln |\sec(x) + \tan(x)|]_0^{\pi/6}$$

$$L = (\ln |\sec(\pi/6) + \tan(\pi/6)|) - (\ln |\sec(0) + \tan(0)|)$$

$$L = \ln\left(\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right) - \ln(1+0) = \ln\left(\frac{3}{\sqrt{3}}\right) = \ln(\sqrt{3})$$

Final Answer: $\ln(\sqrt{3})$ or $\frac{1}{2}\ln(3)$

Part 3: In-Depth Analysis of Problems and Techniques

A) Problem Types and General Approach

The homework problems can be categorized into several distinct types:

Direct Integration Problems (Problems 1, 7): Description: After finding the derivative and plugging it into the formula, the resulting integral can be solved using basic techniques like the power rule or a simple u-substitution.

Strategy: Differentiate, substitute, and integrate directly.

"Setup Only" Problems (Problems 3, 4, 5, 6): Description: These problems ask you to form the definite integral but not evaluate it, usually because the integral is very difficult or impossible to solve analytically.

Strategy: Differentiate, substitute into the formula, and present the final integral with the correct bounds.

Perfect Square Trick Problems (Problems 8, 9, 10, 12, 14): Description: This is the most common "trick." The expression $1 + (dy/dx)^2$ is carefully constructed to simplify into a perfect square, which eliminates the radical and makes the integral solvable.

Strategy: Differentiate (often getting a term like $A - B$). Square it to get $A^2 - 2AB + B^2$. Adding 1 changes the middle term, resulting in $A^2 + 2AB + B^2$, which is $(A + B)^2$.

Trigonometric Identity Problems (Problems 2, 11, 15): Description: After substitution, a trigonometric identity (usually $1 + \tan^2 x = \sec^2 x$) is needed to simplify the expression under the square root.

Strategy: Differentiate, substitute, and apply a Pythagorean trig identity to eliminate the square root.

Advanced Simplification Problems (Problem 13): Description: This problem required careful differentiation of a complex function, followed by significant algebraic simplification which led to an unexpectedly simple integrand. It also resulted in an improper integral.

Strategy: Be meticulous with differentiation and algebra. Recognize and correctly evaluate any improper integrals using limits.

B) Key Algebraic and Calculus Manipulations

The Perfect Square Trick: This is the most critical algebraic manipulation in this set.

Example (Problem 9): The expression $1 + (x^2 - 1/(4x^2))^2$ simplifies from $1 + x^4 - 1/2 + 1/(16x^4)$ to $x^4 + 1/2 + 1/(16x^4)$, which is then factored into $(x^2 + 1/(4x^2))^2$. This step was crucial because it eliminated the square root.

Simplifying Radicals with Trigonometric Identities: Example (Problem 11): The integrand was $\sqrt{1 + \tan^2(x)}$. Using the identity $1 + \tan^2(x) = \sec^2(x)$ simplified this to $\sqrt{\sec^2(x)} = \sec(x)$, making the integral solvable.

U-Substitution: Example (Problem 7): The integral was $\int \sqrt{1+81x} dx$. The substitution $u = 1+81x$ was required to evaluate this.

Recognizing Inverse Trigonometric Integral Forms: Example (Problem 2): The integral was simplified to $\int 1/\sqrt{2-x^2} dx$. Recognizing this as the form for $\arcsin(x/a)$ was essential.

Evaluating Improper Integrals: Example (Problem 13): The integral $\int_0^1 1/\sqrt{x} dx$ is improper at $x = 0$. It must be evaluated using a limit: $\lim_{a \rightarrow 0^+} \int_a^1 1/\sqrt{x} dx$.

Part 4: "Cheatsheet" and Tips for Success

Summary of Important Formulas

$$\text{For } y = f(x): L = \int_a^b \sqrt{1 + (dy/dx)^2} dx$$

$$\text{For } x = g(y): L = \int_c^d \sqrt{1 + (dx/dy)^2} dy$$

Cheats, Tricks, and Shortcuts

- **The Perfect Square Signal:** If you have a function of the form $y = A \cdot f(x) \pm B/[f(x)]$, it is a very strong indicator that the "Perfect Square Trick" will be used.
- **Trig Functions = Trig Identities:** If the original function involves $\ln(\sec x)$, $\ln(\cos x)$, or other trig functions, your first thought for simplifying the integral should be the Pythagorean identities.
- **Check the Algebra:** The vast majority of errors in arc length problems are algebraic, not calculus-based. Double-check your expansion of $(dy/dx)^2$ and the addition of 1.

Common Pitfalls and Mistakes

- **Forgetting to Square:** Don't evaluate $\int \sqrt{1+dy/dx} dx$. You must square the derivative: $\int \sqrt{1 + (dy/dx)^2} dx$.
- **Illegal Square Root Simplification:** Remember that $\sqrt{a^2 + b^2} \neq a + b$. The radical can only be removed if the entire expression inside is a single perfect square.
- **Mixing up dx and dy:** If your function is $x = g(y)$, the integral, derivative, and bounds must all be in terms of y.
- **Forgetting to Change Bounds:** When using u-substitution on a definite integral, you must change the integration bounds to be in terms of u.

How to Recognize Problem Types

- **Polynomial/Rational Function:** Especially one with a term $1/x$, it's likely a Perfect Square Trick problem. (e.g., Problem 9).
- **Logarithm of a Trig Function:** Almost certainly a Trigonometric Identity problem. (e.g., Problems 11, 15).
- **Simple Linear Function:** A basic check of the formula, verifiable with the distance formula. (e.g., Problem 1).
- **Messy Radical that Doesn't Simplify:** Likely a "Setup Only" or calculator problem. (e.g., Problem 5).

Part 5: Conceptual Synthesis and The "Big Picture"

A) Thematic Connections

The central theme of arc length is using integration to sum up an infinite number of infinitesimally small pieces to find a total quantity. This is a cornerstone concept of integral calculus. We saw this same theme when we learned about:

- **Area Under a Curve:** Summing the areas of infinite, infinitesimally thin rectangles.
- **Volume of Solids of Revolution:** Summing the volumes of infinite, infinitesimally thin disks or washers.
- **Work Done by a Variable Force:** Summing the work done over infinite, infinitesimally small displacements.

In each case, the process is the same: break a complex object into simple infinitesimal pieces, write a formula for one piece, and use the integral to sum them all up. Arc length is a direct geometric application of this powerful idea.

B) Forward and Backward Links

Backward Links (Foundations): • **Pythagorean Theorem:** The entire arc length formula is a direct application of $a^2 + b^2 = c^2$ to an infinitesimal triangle (dx, dy, ds).

- **Riemann Sums:** The concept of summing up pieces ($\sum f(x_i)\Delta x$) and taking a limit is the formal definition of the integral. Our arc length integral is the result of this process.

Forward Links (Future Applications): • **Surface Area of Revolution:** This is the very next topic, and its formula, $S = \int 2\pi r ds$, directly uses the infinitesimal arc length element, ds , that we derived here.

- **Vector Calculus and Line Integrals:** In multivariable calculus, arc length is generalized to find the length of curves in 3D space. More importantly, ds becomes the basis for line integrals, used to calculate quantities like the work done by a force field along a curved path or the mass of a curved wire.

Part 6: Real-World Application and Modeling

A) Concrete Scenarios in Finance, Economics, and Statistics

Measuring Path-Dependent Option Payoffs (Financial Engineering): Some complex financial derivatives have payoffs that depend on the path a stock price took, not just its final value. A model of the stock price over time is a curve, $P(t)$. The arc length of this curve is a measure of its total movement or volatility. A highly volatile stock has a longer arc length. This "path length" can be a critical input in a derivative pricing model, especially for options sensitive to volatility.

Quantifying Volatility as Risk (Econometrics/Statistics): In time series analysis, volatility is a key measure of risk. The arc length of a stock's price history provides an intuitive and direct index of its total volatility. Analysts can use this to compare the riskiness of different assets by comparing their price-path lengths over the same period.

Modeling Economic Growth Paths (Economics): The relationship between two economic variables, like inflation and unemployment (the Phillips Curve), can be plotted as a curve. The arc length of this curve over a specific range could represent the total "economic distance" traveled during a policy change, giving a measure of the magnitude and cost of the economic shift.

B) Model Problem Setup

Let's set up a simplified model to compare the volatility of two stocks.

Scenario: A quantitative analyst wants to compare the total price volatility of Stock A (a stable utility) and Stock B (a volatile tech stock) over the last trading month (22 days).

Model Setup:

- **Variables:** t = time in days (0 to 22), $P_A(t)$ = Price of Stock A, $P_B(t)$ = Price of Stock B.
- **Formulation:** Using regression, the analyst models the price functions:
 - Stock A (stable): $P_A(t) = 50 + 0.1t + 0.5 \cos(\pi t/11)$ (slow growth, low-frequency oscillation)
 - Stock B (volatile): $P_B(t) = 200 + 0.5t + 5 \cos(4\pi t)$ (faster growth, high-frequency, high-amplitude oscillation)
- **Equation to Solve:** The total volatility can be quantified by the arc length of each price curve.

Volatility of Stock A (L_A):

- Find the derivative: $P'_A(t) = 0.1 - (0.5\pi/11) \sin(\pi t/11)$
- Set up the integral: $L_A = \int_0^{22} \sqrt{1 + (0.1 - (0.5\pi/11) \sin(\pi t/11))^2} dt$

Volatility of Stock B (L_B):

- Find the derivative: $P'_B(t) = 0.5 - 20\pi \sin(4\pi t)$
- Set up the integral: $L_B = \int_0^{22} \sqrt{1 + (0.5 - 20\pi \sin(4\pi t))^2} dt$

Conclusion: The analyst would solve these integrals numerically. The stock with the larger arc length (L_B would be significantly larger) is demonstrably more volatile and represents a higher-risk asset over that period.

Part 7: Common Variations and Untested Concepts

Your homework provided excellent coverage of Cartesian functions ($y = f(x)$ and $x = g(y)$), but it did not test curves defined in other common coordinate systems.

Arc Length for Parametric Curves:

Explanation: Many curves are best described by expressing x and y as separate functions of a parameter, t. The arc length formula is derived directly from the Cartesian form.

Formula: $L = \int_{\alpha}^{\beta} \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} dt$

Worked-Out Example: Find the length of one arch of the cycloid $x = t - \sin(t)$, $y = 1 - \cos(t)$ for $0 \leq t \leq 2\pi$.

- **Derivatives:** $dx/dt = 1 - \cos(t)$, $dy/dt = \sin(t)$.
- **Square and add:** $(1 - \cos(t))^2 + \sin^2(t) = 1 - 2\cos(t) + \cos^2(t) + \sin^2(t) = 2 - 2\cos(t)$.
- **Use half-angle identity:** $2 - 2\cos(t) = 2(1 - \cos(t)) = 2(2\sin^2(t/2)) = 4\sin^2(t/2)$.
- **Integrate:** $L = \int_0^{2\pi} \sqrt{4\sin^2(t/2)} dt = \int_0^{2\pi} 2\sin(t/2) dt$
- $L = [-4\cos(t/2)]_0^{2\pi} = -4[\cos(\pi) - \cos(0)] = -4[-1 - 1] = 8$.

Arc Length for Polar Curves:

Explanation: For curves defined by a radius r as a function of an angle θ .

Formula: $L = \int_{\alpha}^{\beta} \sqrt{r^2 + (\frac{dr}{d\theta})^2} d\theta$

Worked-Out Example: Find the length of the cardioid $r = 1 + \cos(\theta)$.

- **Derivative:** $dr/d\theta = -\sin(\theta)$.
- **Square and add:** $r^2 + (dr/d\theta)^2 = (1 + \cos(\theta))^2 + (-\sin(\theta))^2 = 1 + 2\cos(\theta) + \cos^2(\theta) + \sin^2(\theta) = 2 + 2\cos(\theta)$.
- **Use half-angle identity:** $2(1 + \cos(\theta)) = 2(2\cos^2(\theta/2)) = 4\cos^2(\theta/2)$.
- **Integrate (using symmetry):** $L = 2 \int_0^{\pi} \sqrt{4\cos^2(\theta/2)} d\theta = 4 \int_0^{\pi} \cos(\theta/2) d\theta$
- $L = 4[2\sin(\theta/2)]_0^{\pi} = 8[\sin(\pi/2) - \sin(0)] = 8[1 - 0] = 8$.

Part 8: Advanced Diagnostic Testing: "Find the Flaw"

Here are five problems with flawed solutions. Your task is to find the single critical error, explain it, and provide the correct step and final solution.

Problem #1

Find the exact length of the curve $y = (2/3)x^{3/2}$ from $x = 0$ to $x = 3$.

Flawed Solution: $y' = (2/3) \cdot (3/2)x^{1/2} = x^{1/2}$
 $1 + (y')^2 = 1 + (x^{1/2})^2 = 1 + x^2$

$L = \int_0^3 \sqrt{1+x^2} dx$

The solution proceeds to say this integral is difficult.

Problem #2

Find the exact length of the curve $x = y^3/6 + 1/(2y)$ from $y = 1$ to $y = 2$.

Flawed Solution: $dx/dy = y^2/2 - 1/(2y^2)$
 $(dx/dy)^2 = y^4/4 - 1/2 + 1/(4y^4)$
 $1 + (dx/dy)^2 = y^4/4 + 1/2 + 1/(4y^4) = (y^2/2 - 1/(2y^2))^2$
 $L = \int_1^2 (y^2/2 - 1/(2y^2)) dy$

The solution proceeds to integrate this expression.

Problem #3

Find the arc length of $x = \cos^3(t)$, $y = \sin^3(t)$ for $0 \leq t \leq \pi/2$.

Flawed Solution: $dx/dt = -3\cos^2(t)\sin(t)$ and $dy/dt = 3\sin^2(t)\cos(t)$
 $(dx/dt)^2 + (dy/dt)^2 = 9\sin^2(t)\cos^2(t)$
 $L = \int_0^{\pi/2} 3\sin(t)\cos(t) dt = (3/2) \int_0^{\pi/2} \sin(2t) dt$
 $L = (3/2)[-1/2\cos(2t)]_0^{\pi/2}$
 $L = (-3/4)[\cos(\pi) - \cos(0)] = (-3/4)[1 - 1] = 0$

Problem #4

Set up the integral for the arc length of $y = e^x$ from $x = 0$ to $x = 1$.

Flawed Solution: $y' = e^x$

$$(y')^2 = (e^x)^2 = e^{(x^2)}$$

$$L = \int_0^1 \sqrt{1 + e^{(x^2)}} dx$$

Problem #5

Find the exact length of the curve by switching variables. $y = \ln(x)$ from $x = 1$ to $x = e$.

Flawed Solution: The integral in x is $\int_1^e \sqrt{1 + 1/x^2} dx$, which is difficult.

Switch variables: $x = e^y$.

Find new bounds: $y(1) = \ln(1) = 0$ and $y(e) = \ln(e) = 1$.

Find derivative: $dx/dy = e^y$.

$$L = \int_1^e \sqrt{1 + (e^y)^2} dy$$