

# Problem Set: The Integral Test and Estimates of Sums

## Calculus II Practice

January 12, 2026

### Instructions

Use the Integral Test, p-series test, or the Test for Divergence to determine whether the following series are convergent or divergent. For problems that ask for a full evaluation, provide the value of the corresponding integral.

### Problems

1. Use the Integral Test to determine if the series  $\sum_{n=1}^{\infty} \frac{6}{3n+2}$  converges or diverges. Evaluate the corresponding integral.
2. Use the Integral Test to determine if the series  $\sum_{n=1}^{\infty} \frac{n}{n^2+9}$  converges or diverges. Evaluate the corresponding integral.
3. Use the Integral Test to determine if the series  $\sum_{n=2}^{\infty} \frac{n^2}{n^3-4}$  converges or diverges. Evaluate the corresponding integral.
4. Use the Integral Test to determine if the series  $\sum_{n=1}^{\infty} ne^{-n^2}$  converges or diverges. Evaluate the corresponding integral.
5. Use the Integral Test to determine if the series  $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$  converges or diverges. Evaluate the corresponding integral.
6. Use the Integral Test to determine if the series  $\sum_{n=0}^{\infty} \frac{1}{n^2+1}$  converges or diverges. Evaluate the corresponding integral.
7. Use the Integral Test to determine if the series  $\sum_{n=1}^{\infty} \frac{5}{n^2+25}$  converges or diverges. Evaluate the corresponding integral.
8. Use the Integral Test to determine if the series  $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$  converges or diverges. Evaluate the corresponding integral.
9. Use the Integral Test to determine if the series  $\sum_{n=3}^{\infty} \frac{1}{n\sqrt{\ln n}}$  converges or diverges. Evaluate the corresponding integral.
10. Use the Integral Test to determine if the series  $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$  converges or diverges. Evaluate the corresponding integral.
11. Determine whether the series  $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$  is convergent or divergent.

12. Determine whether the series  $\sum_{n=1}^{\infty} n^{-1.0001}$  is convergent or divergent.
13. Determine whether the series  $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n}$  is convergent or divergent.
14. Determine whether the series  $\sum_{n=1}^{\infty} \frac{1}{n^{\pi/2}}$  is convergent or divergent.
15. Determine whether the series  $\sum_{n=1}^{\infty} \frac{3n-2}{5n+1}$  is convergent or divergent.
16. Determine whether the series  $\sum_{n=1}^{\infty} \arctan(n)$  is convergent or divergent.
17. Determine whether the series  $\sum_{n=1}^{\infty} (1 + \frac{1}{n})^n$  is convergent or divergent.
18. Determine if the following series is convergent or divergent:  $1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \dots$
19. Determine if the following series is convergent or divergent:  $\frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \dots$
20. Determine if the following series is convergent or divergent:  $1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \dots$
21. Determine if the following series is convergent or divergent:  $\frac{\ln(2)}{2} + \frac{\ln(3)}{3} + \frac{\ln(4)}{4} + \dots$
22. Does the function  $f(x) = \frac{x}{x^2-1}$  satisfy the conditions of the Integral Test for the series  $\sum_{n=2}^{\infty} \frac{n}{n^2-1}$ ? Explain why or why not.
23. Does the function  $f(x) = \frac{2+\cos(x)}{x^2}$  satisfy the conditions for the Integral Test on  $[1, \infty)$ ? Explain why or why not.
24. Explain why the Integral Test cannot be used for the series  $\sum_{n=1}^{\infty} \frac{\sin^2(n)}{n^2}$ .
25. Determine if the series  $\sum_{n=1}^{\infty} \frac{n+4}{n^2+1}$  is convergent or divergent.
26. Determine if the series  $\sum_{n=1}^{\infty} \frac{1}{n^2+3n+2}$  is convergent or divergent.
27. Determine if the series  $\sum_{n=2}^{\infty} \frac{1}{n \ln(n^2)}$  is convergent or divergent.
28. Determine if the series  $\sum_{n=1}^{\infty} 5n^{-2/3}$  is convergent or divergent.
29. Determine if the series  $\sum_{n=1}^{\infty} \frac{e^{-\sqrt{n}}}{\sqrt{n}}$  is convergent or divergent.
30. Determine if the series  $\sum_{n=1}^{\infty} \frac{n}{e^n}$  is convergent or divergent.
31. Determine if the series  $\sum_{n=1}^{\infty} n \sin(1/n)$  is convergent or divergent.
32. Determine if the series  $\sum_{n=5}^{\infty} \frac{1}{(n-4)^3}$  is convergent or divergent.
33. Determine if the series  $\sum_{n=2}^{\infty} \frac{\ln(n)}{n}$  is convergent or divergent.

# Solutions

1. **Divergent.** The function  $f(x) = \frac{6}{3x+2}$  is continuous, positive, and decreasing for  $x \geq 1$ .  $\int_1^\infty \frac{6}{3x+2} dx = \lim_{t \rightarrow \infty} [2 \ln(3x+2)]_1^t = \lim_{t \rightarrow \infty} (2 \ln(3t+2) - 2 \ln(5)) = \infty$ . Since the integral diverges, the series diverges.
2. **Divergent.**  $f(x) = \frac{x}{x^2+9}$  is continuous, positive, and decreasing for  $x \geq 3$ .  $\int_1^\infty \frac{x}{x^2+9} dx = \lim_{t \rightarrow \infty} [\frac{1}{2} \ln(x^2+9)]_1^t = \lim_{t \rightarrow \infty} (\frac{1}{2} \ln(t^2+9) - \frac{1}{2} \ln(10)) = \infty$ . The integral diverges, so the series diverges.
3. **Divergent.**  $f(x) = \frac{x^2}{x^3-4}$  is continuous, positive, and decreasing for  $x \geq 2$ .  $\int_2^\infty \frac{x^2}{x^3-4} dx = \lim_{t \rightarrow \infty} [\frac{1}{3} \ln(x^3-4)]_2^t = \lim_{t \rightarrow \infty} (\frac{1}{3} \ln(t^3-4) - \frac{1}{3} \ln(4)) = \infty$ . The integral diverges, so the series diverges.
4. **Convergent.**  $f(x) = xe^{-x^2}$  is positive, continuous, and decreasing for  $x \geq 1$ . Let  $u = -x^2, du = -2xdx$ .  $\int_1^\infty xe^{-x^2} dx = \lim_{t \rightarrow \infty} [-\frac{1}{2} e^{-x^2}]_1^t = \lim_{t \rightarrow \infty} (-\frac{1}{2} e^{-t^2} - (-\frac{1}{2} e^{-1})) = 0 + \frac{1}{2e} = \frac{1}{2e}$ . The integral converges, so the series converges.
5. **Convergent.**  $f(x) = \frac{e^{1/x}}{x^2}$  is positive, continuous, and decreasing for  $x \geq 1$ . Let  $u = 1/x, du = -1/x^2 dx$ .  $\int_1^\infty \frac{e^{1/x}}{x^2} dx = \lim_{t \rightarrow \infty} [-e^{1/x}]_1^t = \lim_{t \rightarrow \infty} (-e^{1/t} - (-e^1)) = -e^0 + e = e - 1$ . The integral converges, so the series converges.
6. **Convergent.**  $f(x) = \frac{1}{x^2+1}$  is positive, continuous, and decreasing for  $x \geq 0$ .  $\int_0^\infty \frac{1}{x^2+1} dx = \lim_{t \rightarrow \infty} [\arctan(x)]_0^t = \lim_{t \rightarrow \infty} (\arctan(t) - \arctan(0)) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$ . The integral converges, so the series converges.
7. **Convergent.**  $f(x) = \frac{5}{x^2+25}$  is positive, continuous, and decreasing for  $x \geq 1$ .  $\int_1^\infty \frac{5}{x^2+25} dx = 5 \lim_{t \rightarrow \infty} [\frac{1}{5} \arctan(\frac{x}{5})]_1^t = \lim_{t \rightarrow \infty} (\arctan(\frac{t}{5}) - \arctan(\frac{1}{5})) = \frac{\pi}{2} - \arctan(\frac{1}{5})$ . The integral converges, so the series converges.
8. **Convergent.**  $f(x) = \frac{1}{x(\ln x)^3}$  is continuous, positive, and decreasing for  $x \geq 2$ . Let  $u = \ln x, du = \frac{1}{x} dx$ .  $\int_2^\infty \frac{1}{x(\ln x)^3} dx = \lim_{t \rightarrow \infty} \int_{\ln 2}^{\ln t} u^{-3} du = \lim_{t \rightarrow \infty} [-\frac{1}{2u^2}]_{\ln 2}^{\ln t} = \lim_{t \rightarrow \infty} (-\frac{1}{2(\ln t)^2} + \frac{1}{2(\ln 2)^2}) = \frac{1}{2(\ln 2)^2}$ . The integral converges, so the series converges.
9. **Divergent.**  $f(x) = \frac{1}{x\sqrt{\ln x}}$  is continuous, positive, and decreasing for  $x \geq 3$ . Let  $u = \ln x, du = \frac{1}{x} dx$ .  $\int_3^\infty \frac{1}{x\sqrt{\ln x}} dx = \lim_{t \rightarrow \infty} \int_{\ln 3}^{\ln t} u^{-1/2} du = \lim_{t \rightarrow \infty} [2\sqrt{u}]_{\ln 3}^{\ln t} = \lim_{t \rightarrow \infty} (2\sqrt{\ln t} - 2\sqrt{\ln 3}) = \infty$ . The integral diverges, so the series diverges.
10. **Convergent.**  $f(x) = \frac{\ln x}{x^2}$  is positive, continuous, and decreasing for  $x \geq 2$ . Use Integration by Parts:  $u = \ln x, dv = x^{-2} dx$ .  $\int_1^\infty \frac{\ln x}{x^2} dx = \lim_{t \rightarrow \infty} [-\frac{\ln x}{x} - \frac{1}{x}]_1^t = \lim_{t \rightarrow \infty} (-\frac{\ln t}{t} - \frac{1}{t}) - (0 - 1) = 0 - 0 + 1 = 1$ . (Note:  $\lim_{t \rightarrow \infty} \frac{\ln t}{t} = 0$  by L'Hôpital's Rule). The integral converges, so the series converges.
11. **Divergent.** This is a p-series  $\sum \frac{1}{n^p}$  with  $p = 1/5$ . Since  $p \leq 1$ , the series diverges.
12. **Convergent.** This is a p-series with  $p = 1.0001$ . Since  $p > 1$ , the series converges.
13. **Divergent.** The series is  $\sum \frac{n^{1/2}}{n} = \sum \frac{1}{n^{1/2}}$ . This is a p-series with  $p = 1/2$ . Since  $p \leq 1$ , the series diverges.

14. **Convergent.** This is a p-series with  $p = \pi/2 \approx 1.57$ . Since  $p > 1$ , the series converges.
15. **Divergent.** Use the Test for Divergence:  $\lim_{n \rightarrow \infty} \frac{3n-2}{5n+1} = \frac{3}{5} \neq 0$ . The series diverges.
16. **Divergent.** Use the Test for Divergence:  $\lim_{n \rightarrow \infty} \arctan(n) = \frac{\pi}{2} \neq 0$ . The series diverges.
17. **Divergent.** Use the Test for Divergence:  $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e \neq 0$ . The series diverges.
18. **Convergent.** The series can be written as  $\sum_{n=1}^{\infty} \frac{1}{n^3}$ . This is a p-series with  $p = 3$ . Since  $p > 1$ , the series converges.
19. **Divergent.** The series is  $\sum_{n=1}^{\infty} \frac{1}{2n+3}$ . Let  $f(x) = \frac{1}{2x+3}$ . The integral  $\int_1^{\infty} \frac{1}{2x+3} dx = \lim_{t \rightarrow \infty} [\frac{1}{2} \ln(2x+3)]_1^t = \infty$ . The series diverges. (Can also use Limit Comparison Test with  $\sum 1/n$ ).
20. **Convergent.** The series can be written as  $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ . This is a p-series with  $p = 3/2$ . Since  $p > 1$ , the series converges.
21. **Divergent.** The series is  $\sum_{n=2}^{\infty} \frac{\ln(n)}{n}$ . The function  $f(x) = \frac{\ln x}{x}$  is positive, continuous, and decreasing for  $x \geq 3$ .  $\int_2^{\infty} \frac{\ln x}{x} dx = \lim_{t \rightarrow \infty} [\frac{1}{2}(\ln x)^2]_2^t = \infty$ . The series diverges.
22. **Yes.** For  $x \geq 2$ :
- **Positive:** For  $x \geq 2$ ,  $x > 0$  and  $x^2 - 1 > 0$ , so  $f(x)$  is positive.
  - **Continuous:**  $f(x)$  is a rational function, continuous wherever the denominator is not zero ( $x \neq \pm 1$ ), so it is continuous on  $[2, \infty)$ .
  - **Decreasing:**  $f'(x) = \frac{(x^2-1)(1)-x(2x)}{(x^2-1)^2} = \frac{-x^2-1}{(x^2-1)^2}$ . Since the numerator is always negative and the denominator is always positive for  $x \geq 2$ ,  $f'(x) < 0$ , so  $f(x)$  is decreasing.
23. **Yes.** For  $x \geq 1$ :
- **Positive:** Since  $-1 \leq \cos(x) \leq 1$ , the numerator  $2 + \cos(x)$  is always between 1 and 3. The denominator  $x^2$  is positive. So  $f(x)$  is positive.
  - **Continuous:** The numerator and denominator are continuous, and the denominator is never zero on  $[1, \infty)$ , so  $f(x)$  is continuous.
  - **Decreasing:**  $f'(x) = \frac{-x \sin(x) - 4 - 2 \cos(x)}{x^3}$ . For large  $x$ , the numerator is dominated by the  $-4$  term, making  $f'(x)$  negative. The function is eventually decreasing.
24. The Integral Test requires the function  $f(x)$  to be **decreasing**. The function  $f(x) = \frac{\sin^2(x)}{x^2}$  is not decreasing on  $[1, \infty)$  because the  $\sin^2(x)$  term oscillates between 0 and 1, causing the function to have many local maxima and minima.

25. **Divergent.** Use the Limit Comparison Test with the harmonic series  $\sum \frac{1}{n}$ .  $\lim_{n \rightarrow \infty} \frac{(n+4)/(n^2+1)}{1/n} = \lim_{n \rightarrow \infty} \frac{n(n+4)}{n^2+1} = 1$ . Since the limit is a finite positive number and  $\sum \frac{1}{n}$  diverges, the series diverges. The Integral test could also be used.
26. **Convergent.** The function  $f(x) = \frac{1}{x^2+3x+2}$  is positive, continuous, and decreasing for  $x \geq 1$ . This can be compared to  $\sum \frac{1}{n^2}$ , which converges. Using the Integral Test,  $\int_1^\infty \frac{1}{(x+1)(x+2)} dx$  can be solved with partial fractions and is convergent.
27. **Divergent.** The series is  $\sum_{n=2}^\infty \frac{1}{2n \ln(n)}$ . This is a constant multiple ( $1/2$ ) of the series  $\sum_{n=2}^\infty \frac{1}{n \ln(n)}$ , which diverges by the Integral Test (p-series test for logarithms with  $p=1$ ).
28. **Divergent.** This is  $5 \sum n^{-2/3}$ . It is a constant multiple of a p-series with  $p = 2/3$ . Since  $p \leq 1$ , the series diverges.
29. **Convergent.** Use the Integral Test with  $f(x) = \frac{e^{-\sqrt{x}}}{\sqrt{x}}$ . Let  $u = -\sqrt{x}$ ,  $du = -\frac{1}{2\sqrt{x}} dx$ .  $\int_1^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}} dx = \lim_{t \rightarrow \infty} [-2e^{-\sqrt{x}}]_1^t = \lim_{t \rightarrow \infty} (-2e^{-\sqrt{t}} + 2e^{-1}) = \frac{2}{e}$ . The integral converges, so the series converges.
30. **Convergent.** Use the Integral Test.  $f(x) = xe^{-x}$  is positive, continuous, and decreasing for  $x \geq 1$ . Integrate by parts:  $\int_1^\infty xe^{-x} dx = \lim_{t \rightarrow \infty} [-xe^{-x} - e^{-x}]_1^t = (0 - 0) - (-e^{-1} - e^{-1}) = \frac{2}{e}$ . The integral converges, so the series converges.
31. **Divergent.** Use the Test for Divergence.  $\lim_{n \rightarrow \infty} n \sin(1/n) = \lim_{n \rightarrow \infty} \frac{\sin(1/n)}{1/n}$ . Let  $x = 1/n$ . As  $n \rightarrow \infty$ ,  $x \rightarrow 0$ . The limit becomes  $\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1 \neq 0$ . The series diverges.
32. **Convergent.** This is a shifted p-series. Let  $k = n - 4$ . When  $n = 5, k = 1$ . The series is  $\sum_{k=1}^\infty \frac{1}{k^3}$ . This is a p-series with  $p = 3$ . Since  $p > 1$ , it converges.
33. **Divergent.** Use the Integral Test with  $f(x) = \frac{\ln x}{x}$ . Let  $u = \ln x, du = \frac{1}{x} dx$ .  $\int_2^\infty \frac{\ln x}{x} dx = \lim_{t \rightarrow \infty} [\frac{1}{2}(\ln x)^2]_2^t = \infty$ . The integral diverges, so the series diverges.

# Concept Checklist and Problem Mapping

This checklist outlines the key concepts tested in this problem set. The numbers refer to the problems that primarily test each concept.

**C1: The Integral Test Conditions:** Verifying if a function is continuous, positive, and decreasing.

- Problems: 22, 23, 24

**C2: Applying the Integral Test for Convergence/Divergence:**

**C2a: p-Series:** Directly applying the p-series test ( $p > 1$  converges,  $p \leq 1$  diverges).

- \* Problems: 11, 12, 13, 14, 18, 20, 28, 32

**C2b: Logarithmic Functions:** Series of the form  $\sum \frac{1}{n(\ln n)^p}$ .

- \* Problems: 8, 9, 21, 27, 33

**C2c: Exponential Functions:** Series involving exponential terms.

- \* Problems: 4, 5, 29, 30

**C2d: Rational Functions:** Series where the corresponding integral is of a rational function.

- \* Problems: 1, 2, 3, 25, 26

**C2e: Inverse Trig Functions:** Series where the integral leads to an arctan function.

- \* Problems: 6, 7

**C3: Prerequisite Skill - Evaluating Improper Integrals:**

**C3a, C3b:** Basic Power and Logarithmic Rules are implicit in many problems.

**C3c: U-Substitution:** Required for integral evaluation.

- \* Problems: 1, 2, 3, 4, 5, 8, 9, 29, 33

**C3d: Integration by Parts:** Required for more complex integrals.

- \* Problems: 10, 30

**C3e: Inverse Trig Integrals:** Integrals of the form  $\int \frac{1}{x^2+a^2} dx$ .

- \* Problems: 6, 7

**C4: Test for Divergence:** Applying the test where  $\lim_{n \rightarrow \infty} a_n \neq 0$ .

- Problems: 15, 16, 17, 31

**C5: Pattern Recognition:** Deducing the general term  $a_n$  from the first few terms of a series.

- Problems: 18, 19, 20, 21