Calculus III: 3D Coordinate Systems - Practice Problem Set

Based on Homework 12.1 Concepts

Instructions

This problem set is designed to test your understanding of Three-Dimensional Coordinate Systems. It covers the following concepts found in the provided study materials:

- Coordinate Navigation and Plotting
- Distance Formulas (Points, Planes, Axes)
- Projections
- Triangle Geometry in 3D
- Spheres (Standard Form, General Form, Completing the Square)
- Surface and Region Identification (Planes, Cylinders, Inequalities)

Problem Set

1. **(Coordinate Navigation)** A particle starts at the origin $(0, 0, 0)$. It moves 5 units along the positive y-axis, then 3 units along the negative z-axis, and finally 2 units along the positive x-axis. What are the final coordinates of the particle?

2. **(Coordinate Navigation)** A point starts at $P(3, -2, 5)$. It moves 4 units in the negative y-direction and 2 units in the positive z-direction. What is the distance between the starting point and the ending point?

3. **(Distance to Planes)** Find the distance from the point $P(-4, 7, 2)$ to:
   (a) The xy-plane.
   (b) The yz-plane.
   (c) The xz-plane.

4. **(Distance to Axes)** Find the distance from the point $P(3, -4, 12)$ to:
   (a) The x-axis.
   (b) The y-axis.
   (c) The z-axis.

5. **(Projections)** Find the coordinates of the projection of the point $Q(5, -9, 8)$ onto:
   (a) The xy-plane.
   (b) The xz-plane.

6. **(Distance Between Points)** Calculate the distance between point $A(1, 5, -2)$ and point $B(4, 1, -2)$.

7. **(Closest Point)** Consider points $A(2, -5, 8)$ and $B(-3, 1, 1)$. Which point is closer to the yz-plane?

8. **(Midpoint)** Find the midpoint of the line segment connecting $P(2, -3, 8)$ and $Q(6, 5, -2)$.

9. **(Triangle Properties)** Given the points $A(1, 2, 3)$, $B(4, 6, 3)$, and $C(1, 6, 3)$:

(a) Find the lengths of the sides of triangle ABC.

(b) Determine if the triangle is a right triangle.

10. **(Isosceles Triangle)** Determine if the triangle with vertices $P(0,0,0)$, $Q(2,2,0)$, and $R(2,0,2)$ is isosceles.

11. **(Sphere Equation)** Find the standard equation of a sphere with center $C(2,-1,4)$ and radius $r = 5$.

12. **(Sphere Center/Point)** Find the equation of the sphere centered at $(0,5,-2)$ that passes through the origin.

13. **(Sphere Diameter)** Find the equation of the sphere that has the points $A(2,1,4)$ and $B(4,3,10)$ as endpoints of a diameter.

14. **(General to Standard Form)** Show that the equation $x^2 + y^2 + z^2 - 2x + 6y = 15$ represents a sphere. Find its center and radius.

15. **(General to Standard Form)** Find the center and radius of the sphere defined by $x^2 + y^2 + z^2 + 4x - 4y + 2z - 1 = 0$.

16. **(Sphere Logic)** Does the equation $x^2 + y^2 + z^2 + 2x + 2y + 2z + 10 = 0$ represent a sphere? Explain why or why not.

17. **(Surface Identification)** Describe the surface in $\mathbb{R}^3$ represented by the equation $x = 4$.

18. **(Surface Identification)** Describe the surface in $\mathbb{R}^3$ represented by the equation $y = -2$ and $z = 5$. Determine the geometric object formed by their intersection.

19. **(Cylinders)** Describe the surface defined by $x^2 + z^2 = 16$ in $\mathbb{R}^3$.

20. **(Cylinders)** Describe the surface defined by $y^2 + z^2 = 4$. Which axis does this cylinder run along?

21. **(Region Description)** Describe the region given by the inequalities $0 \leq x \leq 2$, $0 \leq y \leq 2$, and $0 \leq z \leq 2$.

22. **(Region Description)** Describe the solid region defined by $x^2 + y^2 + z^2 \leq 9$ and $z \geq 0$.

23. **(Region Description)** Describe the set of points satisfying $xy = 0$ in $\mathbb{R}^3$.

24. **(Intersection: Sphere/Plane)** Find the equation of the intersection of the sphere $(x-1)^2 + y^2 + z^2 = 25$ and the plane $z = 0$. Describe this intersection.

25. **(Intersection: DNE)** Determine the intersection of the sphere $x^2 + y^2 + (z-2)^2 = 4$ and the plane $z = 5$.

26. **(Intersection: Point)** Find the intersection of the sphere $x^2 + y^2 + z^2 = 9$ and the plane $y = 3$.

27. **(Tangency)** Find the equation of the sphere centered at $(3,-4,5)$ that is tangent to the xy-plane.

28. **(Tangency)** Find the equation of the sphere centered at $(3,-4,5)$ that is tangent to the xz-plane.

29. **(Distance Point to Sphere)** Find the shortest distance from the point $P(0,0,0)$ to the surface of the sphere $(x-3)^2 + (y-4)^2 + z^2 = 4$.

30. **(Complex Region)** Describe the region defined by $1 \leq x^2 + y^2 + z^2 \leq 4$.

Solutions

1. **Answer:** $(2, 5, -3)$. *Reasoning:* Start at $(0, 0, 0)$. $y + 5 \rightarrow (0, 5, 0)$. $z - 3 \rightarrow (0, 5, -3)$. $x + 2 \rightarrow (2, 5, -3)$.

2. **Answer:** $2\sqrt{5}$. *Reasoning:* Displacement vector is $\langle 0, -4, 2 \rangle$. Distance is magnitude: $\sqrt{0^2 + (-4)^2 + 2^2} = \sqrt{16 + 4} = \sqrt{20} = 2\sqrt{5}$.

3. (a) $|z| = |2| = 2$.
   (b) $|x| = |-4| = 4$.
   (c) $|y| = |7| = 7$.

4. (a) Distance to x-axis: $\sqrt{y^2 + z^2} = \sqrt{(-4)^2 + 12^2} = \sqrt{16 + 144} = \sqrt{160} = 4\sqrt{10}$.
   (b) Distance to y-axis: $\sqrt{x^2 + z^2} = \sqrt{3^2 + 12^2} = \sqrt{9 + 144} = \sqrt{153}$.
   (c) Distance to z-axis: $\sqrt{x^2 + y^2} = \sqrt{3^2 + (-4)^2} = \sqrt{9 + 16} = 5$.

5. (a) $(5, -9, 0)$. (Set $z = 0$).
   (b) $(5, 0, 8)$. (Set $y = 0$).

6. **Answer:** 5. *Reasoning:* $d = \sqrt{(4-1)^2 + (1-5)^2 + (-2-(-2))^2} = \sqrt{3^2 + (-4)^2 + 0} = \sqrt{9 + 16} = 5$.

7. **Answer:** Point A. *Reasoning:* Distance to yz-plane is $|x|$. For A: $|2| = 2$. For B: $|-3| = 3$. $2 < 3$.

8. **Answer:** $(4, 1, 3)$. *Reasoning:* Midpoint $= (\frac{2+6}{2}, \frac{-3+5}{2}, \frac{8-2}{2}) = (\frac{8}{2}, \frac{2}{2}, \frac{6}{2}) = (4, 1, 3)$.

9. (a) $|AB| = \sqrt{(4-1)^2 + (6-2)^2 + 0} = \sqrt{9 + 16} = 5$.
   $|BC| = \sqrt{(1-4)^2 + (6-6)^2 + 0} = \sqrt{9} = 3$.
   $|AC| = \sqrt{(1-1)^2 + (6-2)^2 + 0} = \sqrt{16} = 4$.
   (b) Yes. $3^2 + 4^2 = 9 + 16 = 25 = 5^2$. This is a 3-4-5 right triangle.

10. **Answer:** Yes. *Reasoning:* $|PQ| = \sqrt{2^2 + 2^2 + 0} = \sqrt{8}$. $|PR| = \sqrt{2^2 + 0 + 2^2} = \sqrt{8}$. Since two sides are equal, it is isosceles.

11. **Answer:** $(x - 2)^2 + (y + 1)^2 + (z - 4)^2 = 25$.

12. **Answer:** $x^2 + (y - 5)^2 + (z + 2)^2 = 29$. *Reasoning:* $r = \text{dist}(C, \text{Origin}) = \sqrt{0^2 + 5^2 + (-2)^2} = \sqrt{29}$. $r^2 = 29$.

13. **Answer:** $(x - 3)^2 + (y - 2)^2 + (z - 7)^2 = 11$. *Reasoning:* Midpoint (Center) $= (3, 2, 7)$. Radius is distance from Center to A: $\sqrt{(2-3)^2 + (1-2)^2 + (4-7)^2} = \sqrt{1 + 1 + 9} = \sqrt{11}$.

14. **Answer:** Center $(1, -3, 0)$, Radius 5. *Reasoning:* $(x^2 - 2x + 1) + (y^2 + 6y + 9) + z^2 = 15 + 1 + 9$. $(x - 1)^2 + (y + 3)^2 + z^2 = 25$.

15. **Answer:** Center $(-2, 2, -1)$, Radius $\sqrt{10}$. *Reasoning:* $(x+2)^2 + (y-2)^2 + (z+1)^2 = 1 + 4 + 4 + 1 = 10$.

16. **Answer:** No. *Reasoning:* Completing squares: $(x+1)^2 + (y+1)^2 + (z+1)^2 = -10 + 1 + 1 + 1 = -7$. Radius squared cannot be negative.

17. **Answer:** A plane parallel to the yz-plane, passing through $x = 4$.

18. **Answer:** A line parallel to the x-axis. *Reasoning:* Intersection of plane $y = -2$ and plane $z = 5$.

19. **Answer:** A circular cylinder along the y-axis with radius 4.

20. **Answer:** A circular cylinder along the x-axis with radius 2.

21. **Answer:** A cube of side length 2 in the first octant with one corner at the origin.

22. **Answer:** A solid hemisphere of radius 3 centered at the origin, lying above the xy-plane.

23. **Answer:** The union of the xz-plane ($y = 0$) and the yz-plane ($x = 0$).

24. **Answer:** Circle in the $z = 0$ plane with equation $(x-1)^2 + y^2 = 25$. Center $(1, 0, 0)$, radius 5.

25. **Answer:** DNE (Empty Set). *Reasoning:* Substitute $z = 5$: $x^2 + y^2 + (5-2)^2 = 4 \implies x^2 + y^2 + 9 = 4 \implies x^2 + y^2 = -5$. Impossible.

26. **Answer:** The point $(0, 3, 0)$. *Reasoning:* $x^2 + 3^2 + z^2 = 9 \implies x^2 + z^2 = 0 \implies x = 0, z = 0$.

27. **Answer:** $(x-3)^2 + (y+4)^2 + (z-5)^2 = 25$. *Reasoning:* Tangent to xy-plane means radius equals distance to xy-plane, which is $|z| = 5$.

28. **Answer:** $(x-3)^2 + (y+4)^2 + (z-5)^2 = 16$. *Reasoning:* Tangent to xz-plane means radius equals distance to xz-plane, which is $|y| = |-4| = 4$.

29. **Answer:** 3. *Reasoning:* Center $C(3, 4, 0)$. Radius $r = 2$. Distance from Origin to Center $d = \sqrt{3^2 + 4^2} = 5$. Distance to surface $= d - r = 5 - 2 = 3$.

30. **Answer:** A spherical shell. The solid region between the sphere of radius 1 and the sphere of radius 2 (centered at origin).

Concept Checklist Verification

This section maps the problem numbers to the concepts identified in the provided homework files.

- **Coordinate Navigation:** Problems 1, 2, 30(conceptually).

- **Distance Formula (Points):** Problems 2, 6, 8(conceptually).

- **Distance to Planes ($|x|, |y|, |z|$):** Problems 3, 7, 27, 28.

- **Distance to Axes (Pythagorean):** Problem 4.

- **Projections:** Problem 5.

- **Midpoint Formula:** Problems 8, 13.

- **Triangle Geometry (Right/Isosceles):** Problems 9, 10.

- **Sphere Equation (Center/Radius):** Problems 11, 25.

- **Sphere Equation (Center/Point):** Problem 12.

- **Sphere Equation (Diameter Endpoints):** Problem 13.

- **Completing the Square (Spheres):** Problems 14, 15, 16.

- **Surface Identification (Planes/Lines):** Problems 17, 18.

- **Surface Identification (Cylinders):** Problems 19, 20.

- **Describing Regions (Inequalities):** Problems 21, 22, 30.

- **Intersections (Sphere & Plane):** Problems 23, 24, 25.

- **Tangency Concepts:** Problems 27, 28.