Homework 14.1: Functions of Several Variables

Comprehensive Study Guide and Solutions

Tashfeen Omran

January 2026

Contents

1 Part 1: Introduction, Context, and Prerequisites

1.1 Core Concepts

In single-variable calculus, we studied functions of the form $y = f(x)$, which map a single real number to another real number. This chapter introduces **functions of several variables**, specifically functions mapping $\mathbb{R}^n \to \mathbb{R}$.

- **Notation:** $z = f(x, y)$ or $w = f(x, y, z)$.
- **Input:** An ordered pair (x, y) or triple (x, y, z).
- **Output:** A single real number (z or w).
- **Graphing:** The graph of a function of two variables, $f(x, y)$, is a **surface** in 3D space defined by the set of points (x, y, z) where $z = f(x, y)$.
- **Domain:** The set of all inputs (x, y) for which the function is defined.
- **Range:** The set of all possible output values z.

1.2 Intuition and Visualizing: Level Curves

Visualizing 3D surfaces on 2D paper is difficult. To solve this, we use **Level Curves** (or Contour Maps). A level curve is the set of points (x, y) in the domain where the function has a constant value k:

$$f(x, y) = k$$

Geometrically, this is the result of slicing the 3D surface with a horizontal plane at height $z = k$ and projecting the intersection down onto the xy-plane.

1.3 Historical Context and Motivation

Historically, the shift from single-variable to multivariable calculus was driven by physics and celestial mechanics in the 18th and 19th centuries (Euler, Lagrange, Laplace). Real-world phenomena rarely depend on just one factor.

- **Example:** The temperature in a room depends on position (x, y, z) and time t.
- **Motivation:** To model fluid flow, gravitational fields, or economic systems, we must understand how a quantity changes when multiple input variables change simultaneously.

1.4 Key Formulas

1. **Function Notation:** $z = f(x, y)$.
2. **Domain Restrictions (The "Big Three"):**
   - Denominators cannot be zero: $Q(x, y) \neq 0$ in $\frac{P(x,y)}{Q(x,y)}$.
   - Even roots must be non-negative: $A(x, y) \geq 0$ in $\sqrt[n]{A(x, y)}$ (if n is even).
   - Logarithm arguments must be strictly positive: $B(x, y) > 0$ in $\ln(B(x, y))$.
3. **Equation of a Sphere:** $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$.

1.5 Prerequisites

To succeed in this chapter, you must master:

- **Inequalities:** Solving inequalities like $9 - x^2 - y^2 \geq 0$.
- **Conic Sections:** Recognizing equations for circles ($x^2 + y^2 = r^2$), ellipses, parabolas, and hyperbolas.
- **Interval Notation:** correctly writing sets like $(-\infty, 3] \cup [5, \infty)$.

2 Part 2: Detailed Homework Solutions

Problem 1: Evaluating a Function

Given: $f(x,y) = \frac{x^2y}{(3x-y^2)^2}$

(a) Find $f(1,5)$. Substitute $x = 1$ and $y = 5$:

$$f(1,5) = \frac{(1)^2(5)}{(3(1)-(5)^2)^2} = \frac{5}{(3-25)^2} = \frac{5}{(-22)^2} = \frac{5}{484}$$

(b) Find $f(-3,-1)$. Substitute $x = -3$ and $y = -1$:

$$f(-3,-1) = \frac{(-3)^2(-1)}{(3(-3)-(-1)^2)^2} = \frac{9(-1)}{(-9-1)^2} = \frac{-9}{(-10)^2} = \frac{-9}{100} = -0.09$$

(c) Find $f(x+h,y)$. Replace every x with $(x+h)$. y remains unchanged.

$$f(x+h,y) = \frac{(x+h)^2y}{(3(x+h)-y^2)^2}$$

(d) Find $f(x,x)$. Replace every y with x.

$$f(x,x) = \frac{x^2(x)}{(3x-(x)^2)^2} = \frac{x^3}{(3x-x^2)^2}$$

Problem 2: Domain and Range of Log Function

Given: $g(x,y) = x^2\ln(x+y)$.

(b) Find and sketch the domain. For the natural logarithm $\ln(u)$ to be defined, the argument u must be strictly positive.

$$x+y>0 \implies y>-x$$

The domain is all points strictly above the line $y=-x$.

- **Boundary:** The line $y=-x$ is dashed (dotted).
- **Shading:** Shade the region above the line.

(a) Evaluate $g(7,1)$.

$$g(7,1) = 7^2\ln(7+1) = 49\ln(8)$$

Using $\ln(8) = \ln(2^3) = 3\ln 2$, answer is $49\ln 8$ or $147\ln 2$.

(c) Find the range. The term $\ln(x+y)$ can take any real value from $-\infty$ to ∞ as $(x+y)$ goes from 0 to ∞. The term x^2 is non-negative. By choosing appropriate x and y, $g(x,y)$ can result in any real number. Range: $(-\infty,\infty)$.

Problem 3: Domain and Range of Exponential Root

Given: $h(x, y) = e^{\sqrt{y-x^2}}$.

(a) Evaluate $h(-1, 2)$**.**

$$h(-1, 2) = e^{\sqrt{2-(-1)^2}} = e^{\sqrt{2-1}} = e^{\sqrt{1}} = e^1 = e$$

(b) Domain. The expression inside the square root must be non-negative:

$$y - x^2 \geq 0 \implies y \geq x^2$$

This describes the region on and inside (above) the parabola $y = x^2$.

- **Boundary:** Solid parabola $y = x^2$.
- **Shading:** Inside the parabola cup (where y is larger).

(c) Range. Let $u = \sqrt{y - x^2}$. Since square roots yield non-negative numbers, $u \geq 0$. The function becomes $z = e^u$ for $u \geq 0$. Since $e^0 = 1$ and $e^u \to \infty$ as $u \to \infty$: Range: $[1, \infty)$.

Problem 4: Function of Three Variables

Given: $f(x, y, z) = \ln(z - \sqrt{x^2 + y^2})$.

(a) Evaluate $f(3, -4, 9)$**.**

$$f(3, -4, 9) = \ln(9 - \sqrt{3^2 + (-4)^2}) = \ln(9 - \sqrt{9 + 16}) = \ln(9 - \sqrt{25}) = \ln(9 - 5) = \ln(4)$$

(b) Domain. Argument of ln must be positive:

$$z - \sqrt{x^2 + y^2} > 0 \implies z > \sqrt{x^2 + y^2}$$

Geometrically, $z = \sqrt{x^2 + y^2}$ is the top half of a cone. The domain is the set of points strictly **inside** the cone (above the surface of the cone). Inequality: $z > \sqrt{x^2 + y^2}$.

Problem 5: Domain with Even Root

Given: $f(x, y) = \sqrt[4]{x - 8y}$. For the 4th root to be real, the radicand must be non-negative:

$$x - 8y \geq 0 \implies x \geq 8y \implies y \leq \frac{1}{8}x$$

- Boundary line: $y = \frac{1}{8}x$ (Solid line).
- Shading: Below the line.

Looking at the graphs in the PDF, select the one showing the line passing through origin with a shallow positive slope ($m = 1/8$), shaded below.

Problem 6: Rational Function Domain

Given: $g(x, y) = \frac{x-y}{x+y}$. Denominator cannot be zero:

$$x + y \neq 0 \implies y \neq -x$$

The domain is the entire xy-plane **except** the line $y = -x$. Select the graph showing the whole plane with a dotted line along $y = -x$.

Problem 7: Domain of 3 Variables (Log)

Given: $f(x, y, z) = \ln(36 - 9x^2 - 4y^2 - z^2)$. Argument > 0:

$$36 - 9x^2 - 4y^2 - z^2 > 0 \implies 9x^2 + 4y^2 + z^2 < 36$$

Divide by 36:

$$\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{36} < 1$$

This describes the **interior** of an ellipsoid. Select the graph depicting a solid, football-like shape (ellipsoid) centered at the origin.

Problem 8: Interpreting Table Data

Given: Table for Humidex $I = f(T, h)$.

(a) Value and Meaning of $f(85, 70)$. Locate Row $T = 85$, Column $h = 70$. The value is 93. **Meaning:** When the actual temperature is 85°F and relative humidity is 70%, the perceived air temperature is approximately 93°F.

(b) For what value of h is $f(90, h) = 100$? Look at Row $T = 90$. Scan across until you find the value 100. This occurs at column $h = 60$. Answer: 60%.

(c) For what value of T is $f(T, 40) = 86$? Look at Column $h = 40$. Scan down until you find 86. This occurs at row $T = 85$. Answer: 85°F.

(d) Meanings and Behavior. $I = f(80, h)$: Fixed $T = 80$, h varies. Looking at the row: $77, 78, 79, 81, 82, 83$. Rate of change: It increases by about 1 degree for every 10% humidity. Linear-ish. $I = f(95, h)$: Fixed $T = 95$, h varies. Row: $93, 96, 101, 107, 114, 124$. Rate of change: Increases slowly at first (3 units), then rapidly (10 units). Answer selection:

- $I = f(80, h)$ increases at a relatively constant rate.
- $f(95, h)$ increases more quickly.
- At an increasing rate.

Problem 9: Sketching a Parabolic Cylinder

Given: $f(x, y) = x^2$. Notice y is missing. This means for any y, the cross-section is the same parabola $z = x^2$. This forms a "trough" or cylinder shape extending along the y-axis. **Table:** If $x = -2, z = 4$. If $x = -1, z = 1$. If $x = 0, z = 0$. Sketch: A parabolic "half-pipe" aligned with the y-axis.

Problem 10: Cross Sections of Sine Surface

Given: $f(x, y) = \sin(x)$. Variable y is missing. The surface is a wave extending infinitely along the y-axis.

Cross Sections using x and z: (Fixing y) Since z does not depend on y, for any fixed y (like $y = 0$ or $y = -\pi/2$), the cross section is simply the sine curve:

$$z = \sin(x)$$

Answer for first 3 boxes: $z = \sin(x)$.

Cross Sections using y and z: (Fixing x)

- At $x = -\pi/2$, $z = \sin(-\pi/2) = -1$. Equation: $z = -1$.
- At $x = 0$, $z = \sin(0) = 0$. Equation: $z = 0$.
- At $x = \pi/2$, $z = \sin(\pi/2) = 1$. Equation: $z = 1$.

Problem 11 12: Matching Contour Maps (Cone vs. Paraboloid)

Comparison:

- **Cone:** $z = \sqrt{x^2 + y^2}$. Slope is constant. The change in z is proportional to change in distance r. Level curves for $z = 1, 2, 3$ are circles with radii $1, 2, 3$. They are **equally spaced**.
- **Paraboloid:** $z = x^2 + y^2$. Slope increases as we go out. Level curves for $z = 1, 2, 3$ are circles with radii $1, \sqrt{2} \approx 1.41, \sqrt{3} \approx 1.73$. The gaps between circles get **smaller** as z increases (circles get closer together).

Conclusion:

- Map I (Equally spaced circles) is the **Cone**.
- Map II (Circles getting closer together) is the **Paraboloid**.

Selection: "Map II is the paraboloid. Map I is the cone. The cone's z-values change at a constant rate."

Problem 13: Reading a Contour Map

Given: Contour map with concentric circles. **Cross section at $z = 1$ using x and y:** Looking at the map, the innermost circle is labeled 1. It looks to have a radius of roughly 2 units (spanning from -2 to 2). Equation: $x^2 + y^2 = 2^2 \implies x^2 + y^2 = 4$. (Or match visual cues). *Note: The provided answer key in the OCR hints at 4.8^2. We must read the graph carefully.* Let's check the axis labels. The tick marks are every 1 unit? No, label is 5, 10. Inner circle 1 seems to have radius approx 1.5? **Wait, look at provided text:** "Given equation for $z = 3$ is $x^2 + y^2 = 4.8^2$." This implies r^2 is scaling. If $z = 3 \to r = 4.8$, and the graph shows circles getting closer (paraboloid-ish inverted? or cone?), we just read the graph. Let's assume the question asks to read the radius r from the grid for specific z values. Form: $x^2 + y^2 = r^2$. Simply estimate r for the rings labeled 1, 3, 5, 7, 9.

Problem 14: Sketch from Contour Map

Map: Vertical parabolas opening upward. $y = x^2 + k$. This implies the surface is a "valley" or "trough" with a parabolic floor rising as y increases. Graph: A parabolic cylinder where the "u" shape is in the xy plane. Actually, if contours are $f(x, y) = k$, and lines are parabolas $y - x^2 = k$, then $z = y - x^2$. This is a hyperbolic paraboloid (saddle) or a slanted trough. Look for a surface that corresponds to parabolas in the top-down view.

Problem 15: Sketch from Diamond Contours

Map: Diamond shapes (squares rotated 45 degrees). Equation: $|x| + |y| = k$. This corresponds to a pyramid structure. Since the values $(3, 2, 1, 0, -1 \dots)$ decrease as we go outward (or inward?), check labels. Center is 3. Moving out: 2, 1, 0. This is a pyramid with a peak at $z = 3$ at the origin, sloping down. Select the graph of a square pyramid.

Problem 16: Contour Map of $f(x, y) = x^2 - y^2$

Level curves: $x^2 - y^2 = k$. These are **hyperbolas**.

- If $k > 0$, $x^2 - y^2 = k$ (Hyperbolas opening left/right).
- If $k < 0$, $y^2 - x^2 = -k$ (Hyperbolas opening up/down).
- If $k = 0$, $y = \pm x$ (Diagonal lines).

Correct Graph: A "saddle" contour map. An X shape in the middle, hyperbolas in the 4 quadrants. (Matches the top-right image in typical textbook sets, or the "cross" shape image).

Problem 17: Contour Map of $f(x, y) = ye^x$

Level curves: $ye^x = k \implies y = ke^{-x}$. These are exponential decay curves (if $k > 0$) or flipped (if $k < 0$).

- As $x \to \infty$, $y \to 0$.
- As $x \to -\infty$, $|y| \to \infty$.

Correct Graph: Curves that "funnel" towards the positive x-axis ($y = 0$). Looks like a fan of curves pinching on the right.

Problem 18: Contour Map of $f(x, y) = \sqrt[3]{x^2 + y^2}$ (Assume)

Or $x^3 + y^3$? No, typical problem is usually $f(x, y) = \frac{1}{x^2+y^2}$ or similar. Wait, let's check the OCR/Image. Problem 18 text says $f(x, y) = \sqrt[3]{x^2 + y^2}$ (it's hard to read but looks like radical). If $z = (x^2 + y^2)^{1/3}$, then $k^3 = x^2 + y^2$. Level curves are **circles**. However, the spacing changes. If the function is different, e.g., y/x, we get lines. Based on the image options usually provided:

- If circles: Concentric.
- If lines: Radial fan.

Given the previous problem types, if the function is indeed a radial function $\sqrt[3]{x^2 + y^2}$, the answer is the concentric circles map.

Problem 19: Matching Surface $z = \sin(xy)$

Analysis:

- If $x = 0$ or $y = 0$, then $z = \sin(0) = 0$. The surface must be flat (zero height) along both the x and y axes.
- In the first quadrant $(x, y > 0)$, as we move away from the origin, xy increases, so $\sin(xy)$ oscillates.
- The "hills" get thinner because xy changes faster for large x, y.

Match: Look for the graph that is flat on the axes and has curved hills in the corners. usually labeled **A** or similar in standard sets.

Problem 20: Matching Surface $z = \sin(x - y)$

Analysis:

- Let $u = x - y$. $z = \sin(u)$.
- This is a sine wave traveling in the direction perpendicular to the lines $x - y = k$.
- Along the line $y = x$ $(k = 0)$, $z = \sin(0) = 0$.
- The wave ridges run diagonally (parallel to $y = x$).

Match: Look for the graph with diagonal ripples/waves. Usually labeled **F**.

Part (b): Contour Matching

- $z = \sin(xy)$: Hyperbolic-shaped regions. Matches contours that look like hyperbolas (Map II in standard sets).
- $z = \sin(x - y)$: Linear contours $y = x - k$. Matches diagonal parallel lines (Map I).

3 Part 3: In-Depth Analysis

3.1 A) Problem Types and Approaches

1. **Evaluation Problems (Q1, Q8):** *Strategy:* Simple substitution. Be careful with signs and order of operations. For tables, treat rows/columns as single-variable functions.

2. **Domain Problems (Q2-Q7):** *Strategy:* Identify the "Big Three" restrictions:

   - Denominator $\neq 0$.
   - EvenRoot$(\dots) \geq 0$.
   - $\ln(\dots) > 0$.

   Set up the inequality and sketch the region. Use a test point (like $(0, 0)$) to determine shading.

3. **Level Curve/Contour Map Problems (Q11-Q18):** *Strategy:* Set $z = k$. Rearrange the equation to recognize a known 2D curve (circle, line, parabola, hyperbola). Analyze the spacing: equal spacing = linear growth; decreasing spacing = accelerating growth (getting steeper).

4. **Surface Matching (Q19-Q20):** *Strategy:* Check the axes (intercepts). Check traces (set $x = 0$ or $y = 0$). Check symmetry. For $\sin(xy)$, axes are zero. For $\sin(x - y)$, diagonals are constant.

3.2 B) Key Techniques

- **The "Slice" Method (Traces):** To visualize $z = f(x, y)$, set $x = c$ (slice parallel to y-axis) and $y = c$ (slice parallel to x-axis). This reduces the problem to single-variable calculus graphs. Used in Q9, Q10.

- **Inequality Manipulation:** Converting $9 - x^2 - y^2 - z^2 > 0$ into $x^2 + y^2 + z^2 < 9$ to recognize the interior of a sphere/ellipsoid (Q7).

- **Exponential Domain Analysis:** Knowing that $e^{\cdots}$ is defined everywhere, so the restriction comes entirely from the exponent (Q3).

4 Part 4: Cheatsheet and Tips

Formulas

- **Plane:** $ax + by + cz = d$
- **Sphere:** $(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$
- **Ellipsoid:** $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
- **Cone:** $z^2 = x^2 + y^2$ or $z = \sqrt{x^2 + y^2}$
- **Paraboloid:** $z = x^2 + y^2$

Rapid Recognition Tricks

- **Missing Variable?** If a variable is missing (e.g., $z = x^2$), the graph is a **cylinder** extending along the missing axis.
- **Sum of Squares ($x^2 + y^2$)?** The graph has radial symmetry (circles). Think Paraboloids, Cones, Spheres.
- **Difference of Squares ($x^2 - y^2$)?** The graph is a **Saddle** (Hyperbolic Paraboloid). Contours are hyperbolas.
- **Product (xy)?** Contours are hyperbolas $y = k/x$. Surface is a saddle.
- **Linear Argument ($ax + by$)?** The graph is a wave or plane traveling in a specific direction. Contours are straight parallel lines.

Common Pitfalls

- **Strict vs. Non-Strict:** Logarithms are strictly > 0 (dashed boundary). Roots are ≥ 0 (solid boundary).
- **Shading:** Always test the point $(0, 0)$ or $(1, 1)$ to confirm which side of the line/curve to shade.
- **Range of e^x:** Remember $e^{\text{anything}} > 0$. It is never negative and never zero.

5 Part 5: Conceptual Synthesis

5.1 A) Thematic Connections

The core theme of this topic is **Dimensional Expansion**. Just as we moved from a number line (1D) to a coordinate plane (2D) in algebra, we are now moving to 3D space. However, the fundamental tool—limits and analysis—remains the same. We are simply analyzing how a system reacts to *multiple* simultaneous inputs.

5.2 B) Forward and Backward Links

- **Backward:** Relies heavily on Conic Sections (Algebra II/Pre-Calc) and Domain rules (Calc I).
- **Forward:** This is the foundation for **Partial Derivatives** (14.3). Just as we took slices (y = constant) to find cross-sections, we will take derivatives along those slices to find rates of change. It also leads to **Double/Triple Integrals** (Chapter 15), which calculate the volume under these surfaces.

6 Part 6: Real-World Application (Finance Focus)

6.1 A) Scenario: Option Pricing (The Black-Scholes Model)

In quantitative finance, the price of a European call option, C, is a function of five variables:

1. Current stock price (S)
2. Strike price (K)
3. Time to maturity (T)
4. Risk-free interest rate (r)
5. Volatility (σ)

Function: $C = f(S, K, T, r, \sigma)$. Multivariable calculus is used to manage risk. For example, the partial derivative $\frac{\partial C}{\partial S}$ is called **Delta** (Δ). It tells traders how much the option price changes when the stock price moves.

6.2 B) Model Problem Setup

Problem: A portfolio manager wants to estimate the change in value of a bond portfolio based on changes in interest rates (r) and inflation (i). **Model:** Let $V(r, i)$ be the value of the portfolio.

$$V(r, i) = 1000e^{-2r} + 500e^{-3(r+i)}$$

To find the impact of a rate hike, we would analyze the surface defined by $z = V(r, i)$ and look at the slopes (derivatives) at the current market rates.

7 Part 7: Common Variations and Untested Concepts

Concept Not Covered: Limits of Multivariable Functions. Your homework focuses on domains and graphs. A standard next step is proving limits exist (or don't). **Example:** $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$. *Method:* Approach along $y = 0$ (limit is 1). Approach along $x = 0$ (limit is -1). Since $1 \neq -1$, the limit DNE.

Concept Not Covered: Graphing Elliptic Paraboloids. You saw circular paraboloids ($z = x^2 + y^2$). An elliptic paraboloid is $z = 2x^2 + 5y^2$. The contours are ellipses, not circles.

8 Part 8: Advanced Diagnostic Testing ("Find the Flaw")

Flawed Problem 1: Domain of Log-Root

Problem: Find domain of $f(x, y) = \ln(y - \sqrt{x})$. **Flawed Solution:** Inside log must be positive: $y - \sqrt{x} > 0 \implies y > \sqrt{x}$. Also, inside sqrt must be positive: $x \geq 0$. Sketch: Graph $y = \sqrt{x}$, shade above. Solid line because of $x \geq 0$. **The Error:** The student drew a solid line for $y = \sqrt{x}$. **Correction:** Because the log argument must be **strictly** positive (>), the boundary curve $y = \sqrt{x}$ must be **dashed**.

Flawed Problem 2: Range of Square Sum

Problem: Find range of $f(x, y) = \sqrt{9 - x^2 - y^2}$. **Flawed Solution:** $9 - x^2 - y^2 \geq 0$. Max value is when $x = 0, y = 0$, so $\sqrt{9} = 3$. Min value is when $x, y \to \infty$, so $\sqrt{\text{negative}} \to$ undefined. Range is $(-\infty, 3]$. **The Error:** Square roots cannot yield negative numbers. **Correction:** The term inside the root cannot be negative. The smallest real value for the root is 0 (on the boundary circle). Range is $[0, 3]$.

Flawed Problem 3: Contour Map Identification

Problem: Identify contours of $z = y/x$. **Flawed Solution:** $y/x = k \implies y = kx$. These are lines passing through the origin. Therefore, the graph is a plane. **The Error:** While the contours are lines, the surface is not a plane. A plane $ax + by + cz = d$ has parallel, equally spaced linear contours. Here, the lines $y = kx$ all intersect at the origin like spokes on a wheel. **Correction:** This is a "spiral staircase" or helicoid-like singularity at 0. It is not a plane.

Flawed Problem 4: Level Curves of Sphere

Problem: Describe level curves of $x^2 + y^2 + z^2 = 1$. **Flawed Solution:** Set $z = k$. $x^2 + y^2 + k^2 = 1 \implies x^2 + y^2 = 1 - k^2$. These are circles for any k. **The Error:** Failed to specify the range of k. **Correction:** These are circles only if $-1 < k < 1$. If $|k| > 1$, there is no curve. If $k = 1$, it's a point.

Flawed Problem 5: Evaluating Functions

Problem: $f(x, y) = x^2 + y$. Find $f(t^2, t)$. **Flawed Solution:** $f(t^2, t) = (t^2)^2 + t^2 = t^4 + t^2$. Wait, I substituted t for y, but then squared it? **The Error:** In the problem description, y is just y, not y^2. The student might accidentally square the second term if confused with a circle equation. **Correction:** $f(t^2, t) = (t^2)^2 + (t) = t^4 + t$.