

# Integration Techniques Revision

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## 1 Integration Problems and Solutions

### 1.1 Problem 1

Evaluate the integral:  $\int \ln(\cos(x)) \tan(x) dx$

#### Solution

This integral is solved using a clever u-substitution. Let  $u = \ln(\cos(x))$ . Then  $du = \frac{1}{\cos(x)}(-\sin(x)) dx = -\tan(x) dx$ . Substituting these into the integral gives:

$$\int u(-du) = -\int u du = -\frac{u^2}{2} + C$$

Substituting back for u: **Answer:**  $-\frac{(\ln(\cos(x)))^2}{2} + C$

### 1.2 Problem 2

Evaluate the integral:  $\int e^x \sin(x) dx$

#### Solution

This requires "looping" integration by parts. Let  $I = \int e^x \sin(x) dx$ . First IBP ( $u = \sin(x), dv = e^x dx$ ):  $I = e^x \sin(x) - \int e^x \cos(x) dx$ . Second IBP ( $u = \cos(x), dv = e^x dx$ ):  $\int e^x \cos(x) dx = e^x \cos(x) + \int e^x \sin(x) dx = e^x \cos(x) + I$ . Substitute back:  $I = e^x \sin(x) - (e^x \cos(x) + I) \implies 2I = e^x(\sin(x) - \cos(x))$ . **Answer:**  $\frac{e^x(\sin(x) - \cos(x))}{2} + C$

### 1.3 Problem 3

Evaluate the integral:  $\int \arctan(x) dx$

#### Solution

This integral requires the "Stealth dx" IBP trick. Let  $u = \arctan(x)$  and  $dv = dx$ . Then  $du = \frac{1}{1+x^2} dx$  and  $v = x$ . The integral becomes:  $x \arctan(x) - \int \frac{x}{1+x^2} dx$ . The second integral is a u-substitution with  $w = 1 + x^2$ , giving  $\frac{1}{2} \ln(1 + x^2)$ . **Answer:**  $x \arctan(x) - \frac{1}{2} \ln(1 + x^2) + C$

### 1.4 Problem 4

Evaluate the integral:  $\int \ln(x) dx$

#### Solution

This uses the "Stealth dx" IBP trick. Let  $u = \ln(x)$  and  $dv = dx$ . Then  $du = \frac{1}{x} dx$  and  $v = x$ . The integral becomes:  $x \ln(x) - \int x \cdot \frac{1}{x} dx = x \ln(x) - \int 1 dx$ . **Answer:**  $x \ln(x) - x + C$

### 1.5 Problem 5

Evaluate the integral:  $\int \ln(x^2 + 4) dx$

#### Solution

Use IBP with  $u = \ln(x^2 + 4)$ ,  $dv = dx$ . This gives  $x \ln(x^2 + 4) - \int \frac{2x^2}{x^2 + 4} dx$ . The second integral requires algebraic manipulation:

$$\int \frac{2x^2}{x^2 + 4} dx = 2 \int \frac{x^2 + 4 - 4}{x^2 + 4} dx = 2 \int \left(1 - \frac{4}{x^2 + 4}\right) dx = 2\left(x - 2 \arctan\left(\frac{x}{2}\right)\right)$$

**Answer:**  $x \ln(x^2 + 4) - 2x + 4 \arctan\left(\frac{x}{2}\right) + C$

### 1.6 Problem 6

Evaluate the integral:  $\int \ln(x^2 + 5) dx$

#### Solution

Use IBP with  $u = \ln(x^2 + 5)$ ,  $dv = dx$ . This gives  $x \ln(x^2 + 5) - \int \frac{2x^2}{x^2 + 5} dx$ . The second integral requires algebraic manipulation:

$$2 \int \frac{x^2 + 5 - 5}{x^2 + 5} dx = 2 \int \left(1 - \frac{5}{x^2 + 5}\right) dx = 2 \left(x - \frac{5}{\sqrt{5}} \arctan\left(\frac{x}{\sqrt{5}}\right)\right)$$

**Answer:**  $x \ln(x^2 + 5) - 2x + 2\sqrt{5} \arctan\left(\frac{x}{\sqrt{5}}\right) + C$

### 1.7 Problem 7

Evaluate the integral:  $\int \frac{1}{x^2 + 6x + 13} dx$

#### Solution

This integral requires completing the square in the denominator.  $x^2 + 6x + 13 = (x^2 + 6x + 9) + 4 = (x + 3)^2 + 2^2$ . The integral becomes  $\int \frac{1}{(x+3)^2 + 2^2} dx$ . This is the standard arctangent form with  $u = x + 3$  and  $a = 2$ . **Answer:**  $\frac{1}{2} \arctan\left(\frac{x+3}{2}\right) + C$

### 1.8 Problem 8

Evaluate the integral:  $\int x \arctan(x) dx$

#### Solution

Use IBP with  $u = \arctan(x)$ ,  $dv = x dx$ . This gives  $\frac{x^2}{2} \arctan(x) - \frac{1}{2} \int \frac{x^2}{1+x^2} dx$ . The second integral requires algebraic manipulation:

$$\int \frac{x^2}{1+x^2} dx = \int \frac{1+x^2-1}{1+x^2} dx = \int \left(1 - \frac{1}{1+x^2}\right) dx = x - \arctan(x)$$

**Answer:**  $\frac{x^2}{2} \arctan(x) - \frac{1}{2}(x - \arctan(x)) + C$

### 1.9 Problem 9

Evaluate the integral:  $\int_0^\pi e^{\cos(t)} \sin(2t) dt$

### Solution

Use identity  $\sin(2t) = 2\sin(t)\cos(t)$ . Let  $u = \cos(t)$ , so  $du = -\sin(t)dt$ . Limits:  $t = 0 \rightarrow u = 1$ ,  $t = \pi \rightarrow u = -1$ . The integral becomes  $\int_1^{-1} e^u \cdot 2u \cdot (-du) = 2 \int_{-1}^1 ue^u du$ . Using IBP gives  $2[ue^u - e^u]_{-1}^1 = 2[(e - e) - (-e^{-1} - e^{-1})] = 2(2/e)$ .

**Answer:**  $\frac{4}{e}$

### 1.10 Problem 10

Evaluate the integral:  $\int_{\pi/6}^{\pi/2} \sin(2x) \ln(\sin(x)) dx$

### Solution

Use identity  $\sin(2x) = 2\sin(x)\cos(x)$ . Let  $u = \sin(x)$ , so  $du = \cos(x)dx$ . Limits:  $x = \pi/6 \rightarrow u = 1/2$ ,  $x = \pi/2 \rightarrow u = 1$ . The integral becomes  $\int_{1/2}^1 2u \ln(u) du$ . Using IBP gives  $2[\frac{u^2}{2} \ln(u) - \frac{u^2}{4}]_{1/2}^1 = [\frac{1}{2} \ln(1) - \frac{1}{4}] - [\frac{(1/4)}{2} \ln(1/2) - \frac{1/4}{4}]$ .

**Answer:**  $\frac{\ln(2)}{4} - \frac{3}{8}$

### 1.11 Problem 11

Evaluate the integral:  $\int_0^{\pi/2} e^{\cos(x)} \sin(2x) dx$

### Solution

Use identity  $\sin(2x) = 2\sin(x)\cos(x)$ . Let  $u = \cos(x)$ , so  $du = -\sin(x)dx$ . Limits:  $x = 0 \rightarrow u = 1$ ,  $x = \pi/2 \rightarrow u = 0$ . The integral becomes  $\int_1^0 e^u \cdot 2u \cdot (-du) = 2 \int_0^1 ue^u du$ . Using IBP gives  $2[ue^u - e^u]_0^1 = 2[(e - e) - (0 - e^0)] = 2(1)$ .

**Answer:** 2

### 1.12 Problem 12

Evaluate the integral:  $\int_{\pi/3}^{\pi/2} \sin(2x) \cos(\cos(x)) dx$

### Solution

Use identity  $\sin(2x) = 2\sin(x)\cos(x)$ . Let  $u = \cos(x)$ , so  $du = -\sin(x)dx$ . Limits:  $x = \pi/3 \rightarrow u = 1/2$ ,  $x = \pi/2 \rightarrow u = 0$ . The integral becomes  $\int_{1/2}^0 2u \cos(u) (-du) = 2 \int_0^{1/2} u \cos(u) du$ . Using IBP gives  $2[u \sin(u) + \cos(u)]_0^{1/2} = 2[(\frac{1}{2} \sin(\frac{1}{2}) + \cos(\frac{1}{2})) - (0 + \cos(0))]$ . **Answer:**  $\sin(\frac{1}{2}) + 2\cos(\frac{1}{2}) - 2$

### 1.13 Problem 13

Evaluate the integral:  $\int_0^{\pi/2} \sin(2x) e^{\sin(x)} dx$

### Solution

Use identity  $\sin(2x) = 2\sin(x)\cos(x)$ . Let  $u = \sin(x)$ , so  $du = \cos(x)dx$ . Limits:  $x = 0 \rightarrow u = 0$ ,  $x = \pi/2 \rightarrow u = 1$ . The integral becomes  $\int_0^1 2ue^u du$ . Using IBP gives  $2[ue^u - e^u]_0^1 = 2[(e - e) - (0 - e^0)] = 2(1)$ . **Answer:** 2

### 1.14 Problem 14

Evaluate the integral:  $\int_0^{\pi} \sin(2x) \cos(\cos(x)) dx$

### Solution

Use identity  $\sin(2x) = 2\sin(x)\cos(x)$ . Let  $u = \cos(x)$ , so  $du = -\sin(x)dx$ . Limits:  $x = 0 \rightarrow u = 1$ ,  $x = \pi \rightarrow u = -1$ . The integral becomes  $\int_1^{-1} 2u \cos(u) (-du) = 2 \int_{-1}^1 u \cos(u) du$ . Using IBP gives  $2[u \sin(u) + \cos(u)]_{-1}^1 = 2[(\sin(1) + \cos(1)) - (-\sin(-1) + \cos(-1))]$ . Since sine is odd and cosine is even, this simplifies to  $2[(\sin(1) + \cos(1)) - (\sin(1) + \cos(1))]$ . **Answer:** 0

### 1.15 Problem 15

Evaluate the definite integral:  $\int_0^{\sqrt{\ln(2)}} xe^{x^2} \cos(e^{x^2}) dx$

#### Solution

This integral can be solved with a u-substitution. Let  $u = e^{x^2}$ . Then  $du = 2xe^{x^2} dx$ , which means  $xe^{x^2} dx = \frac{1}{2} du$ . We must also change the bounds of integration. When  $x = 0$ ,  $u = e^{0^2} = 1$ . When  $x = \sqrt{\ln(2)}$ ,  $u = e^{(\sqrt{\ln(2)})^2} = e^{\ln(2)} = 2$ . The integral becomes  $\int_1^2 \cos(u) \cdot \frac{1}{2} du = \frac{1}{2} \int_1^2 \cos(u) du$ . Integrating gives  $\frac{1}{2} [\sin(u)]_1^2$ . **Answer:**  $\frac{1}{2}(\sin(2) - \sin(1))$

### 1.16 Problem 16

Evaluate the integral:  $\int \frac{x}{1+\sqrt{x+4}} dx$

#### Solution

This requires a "clever" u-substitution to rationalize the expression. Let  $u = \sqrt{x+4}$ . Then  $u^2 = x+4$ , so  $x = u^2 - 4$ . The differential is  $dx = 2u du$ . Substitute everything into the integral:  $\int \frac{u^2-4}{1+u} (2u) du = \int \frac{2u^3-8u}{u+1} du$ . Using polynomial long division, we get  $\int \left(2u^2 - 2u - 6 + \frac{6}{u+1}\right) du$ . Integrating term by term yields  $\frac{2}{3}u^3 - u^2 - 6u + 6 \ln|u+1| + C$ . Finally, substitute back  $u = \sqrt{x+4}$ . **Answer:**  $\frac{2}{3}(x+4)^{3/2} - (x+4) - 6\sqrt{x+4} + 6 \ln|\sqrt{x+4}+1| + C$

### 1.17 Problem 17

Evaluate the integral:  $\int \sin^2(x) \cos^4(x) dx$

#### Solution

This integral requires power-reducing trigonometric identities. Rewrite the integrand:  $\int (\sin(x) \cos(x))^2 \cos^2(x) dx = \int \left(\frac{1}{2} \sin(2x)\right)^2 \left(\frac{1+\cos(2x)}{2}\right) dx$ . This simplifies to  $\frac{1}{8} \int \sin^2(2x)(1+\cos(2x)) dx = \frac{1}{8} \int \sin^2(2x) dx + \frac{1}{8} \int \sin^2(2x) \cos(2x) dx$ . For the first integral, use the identity  $\sin^2(\theta) = \frac{1-\cos(2\theta)}{2}$ :  $\frac{1}{8} \int \frac{1-\cos(4x)}{2} dx = \frac{1}{16} \left(x - \frac{1}{4} \sin(4x)\right)$ . For the second integral, let  $u = \sin(2x)$ ,  $du = 2 \cos(2x) dx$ :  $\frac{1}{16} \int u^2 du = \frac{1}{16} \frac{u^3}{3} = \frac{1}{48} \sin^3(2x)$ . Combining the parts gives the final answer. **Answer:**  $\frac{1}{16}x - \frac{1}{64} \sin(4x) + \frac{1}{48} \sin^3(2x) + C$

### 1.18 Problem 18

Evaluate the integral:  $\int \frac{2x+3}{x^2-4x+20} dx$

#### Solution

First, complete the square in the denominator:  $x^2 - 4x + 20 = (x^2 - 4x + 4) + 16 = (x-2)^2 + 16$ . Next, split the numerator to create the derivative of the denominator's quadratic part, which is  $2x-4$ .  $2x+3 = (2x-4) + 7$ . The integral becomes  $\int \frac{2x-4}{x^2-4x+20} dx + \int \frac{7}{(x-2)^2+16} dx$ . The first integral is a logarithm:  $\ln(x^2 - 4x + 20)$ . The second integral is an arctangent form:  $7 \int \frac{1}{(x-2)^2+4^2} dx = \frac{7}{4} \arctan\left(\frac{x-2}{4}\right)$ . **Answer:**  $\ln(x^2 - 4x + 20) + \frac{7}{4} \arctan\left(\frac{x-2}{4}\right) + C$

### 1.19 Problem 19

Evaluate the integral:  $\int x\sqrt{x+1} dx$

#### Solution

Use the substitution  $u = x+1$ . This implies  $x = u-1$  and  $dx = du$ . The integral transforms to  $\int (u-1)\sqrt{u} du = \int (u^{3/2} - u^{1/2}) du$ . Integrating term by term:  $\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + C$ . Substitute back for  $u = x+1$ . **Answer:**  $\frac{2}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2} + C$

### 1.20 Problem 20

Evaluate the integral:  $\int \frac{1}{e^x + e^{-x}} dx$

#### Solution

Multiply the numerator and denominator by  $e^x$  to simplify the expression.  $\int \frac{e^x}{(e^x)(e^x + e^{-x})} dx = \int \frac{e^x}{e^{2x} + 1} dx$ . Now, perform a u-substitution. Let  $u = e^x$ , so  $du = e^x dx$ . The integral becomes  $\int \frac{1}{u^2 + 1} du$ , which is the standard form for arctangent. **Answer:**  $\arctan(e^x) + C$

### 1.21 Problem 21

Evaluate the integral:  $\int \frac{1}{\sqrt{x} + \sqrt[3]{x}} dx$

#### Solution

To handle the different roots, substitute using the least common multiple of the denominators of the fractional exponents (2 and 3), which is 6. Let  $u = x^{1/6}$ . Then  $x = u^6$  and  $dx = 6u^5 du$ . Also,  $\sqrt{x} = u^3$  and  $\sqrt[3]{x} = u^2$ . The integral becomes  $\int \frac{1}{u^3 + u^2} (6u^5) du = \int \frac{6u^5}{u^2(u+1)} du = 6 \int \frac{u^3}{u+1} du$ . By polynomial long division, this is  $6 \int \left( u^2 - u + 1 - \frac{1}{u+1} \right) du$ . Integrating yields  $6 \left( \frac{u^3}{3} - \frac{u^2}{2} + u - \ln|u+1| \right) + C$ . Substitute back  $u = x^{1/6}$ . **Answer:**  $2\sqrt{x} - 3\sqrt[3]{x} + 6\sqrt[6]{x} - 6\ln(x^{1/6} + 1) + C$

### 1.22 Problem 22

Evaluate the integral:  $\int \frac{x^3}{(x^2+1)^2} dx$

#### Solution

Rewrite the integrand as  $\int \frac{x^2 \cdot x}{(x^2+1)^2} dx$ . Let  $u = x^2 + 1$ . Then  $du = 2x dx$ , so  $x dx = \frac{1}{2} du$ . Also,  $x^2 = u - 1$ . Substituting gives  $\int \frac{u-1}{u^2} \cdot \frac{1}{2} du = \frac{1}{2} \int \left( \frac{1}{u} - \frac{1}{u^2} \right) du$ . Integrating gives  $\frac{1}{2} \left( \ln|u| + \frac{1}{u} \right) + C$ . Substitute back  $u = x^2 + 1$ . **Answer:**  $\frac{1}{2} \ln(x^2 + 1) + \frac{1}{2(x^2 + 1)} + C$

### 1.23 Problem 23

Evaluate the definite integral:  $\int_0^7 \frac{x}{\sqrt{x+1}} dx$

#### Solution

Let  $u = x + 1$ . Then  $x = u - 1$  and  $dx = du$ . Change the bounds: when  $x = 0$ ,  $u = 1$ ; when  $x = 7$ ,  $u = 8$ . The integral becomes  $\int_1^8 \frac{u-1}{\sqrt{u}} du = \int_1^8 (u^{1/2} - u^{-1/2}) du$ . The antiderivative is  $\left[ \frac{2}{3} u^{3/2} - 2u^{1/2} \right]_1^8$ . Evaluate at the bounds:  $\left( \frac{2}{3} 8^{3/2} - 2 \cdot 8^{1/2} \right) - \left( \frac{2}{3} - 2 \right) = \left( \frac{32\sqrt{2}}{3} - 4\sqrt{2} \right) - \left( -\frac{4}{3} \right)$ . **Answer:**  $\frac{20\sqrt{2}+4}{3}$

### 1.24 Problem 24

Evaluate the integral:  $\int \frac{e^x}{9 + e^{2x}} dx$

#### Solution

This integral is in a form that leads to arctangent. Rewrite as  $\int \frac{e^x}{3^2 + (e^x)^2} dx$ . Let  $u = e^x$ , so  $du = e^x dx$ . The integral becomes  $\int \frac{1}{3^2 + u^2} du$ . **Answer:**  $\frac{1}{3} \arctan\left(\frac{e^x}{3}\right) + C$

### 1.25 Problem 25

Evaluate the integral:  $\int \frac{e^{2x}}{\sqrt{1-e^{4x}}} dx$

### Solution

This integral is in a form that leads to arcsine. Rewrite as  $\int \frac{e^{2x}}{\sqrt{1^2 - (e^{2x})^2}} dx$ . Let  $u = e^{2x}$ , so  $du = 2e^{2x} dx$ , which means  $e^{2x} dx = \frac{1}{2} du$ . The integral becomes  $\frac{1}{2} \int \frac{1}{\sqrt{1-u^2}} du$ . **Answer:**  $\frac{1}{2} \arcsin(e^{2x}) + C$

### 1.26 Problem 26

Evaluate the integral:  $\int \frac{e^x + 4}{e^{2x} + 2e^x + 5} dx$

### Solution

Split the integral:  $\int \frac{e^x}{e^{2x} + 2e^x + 5} dx + \int \frac{4}{e^{2x} + 2e^x + 5} dx$ . Complete the square on the denominator:  $(e^x + 1)^2 + 4$ . The first integral  $I_1 = \int \frac{e^x}{(e^x + 1)^2 + 4} dx$ . Let  $u = e^x + 1$ ,  $du = e^x dx$ . This becomes  $\int \frac{du}{u^2 + 4} = \frac{1}{2} \arctan(\frac{u}{2}) = \frac{1}{2} \arctan(\frac{e^x + 1}{2})$ . The second integral  $I_2 = 4 \int \frac{dx}{(e^x + 1)^2 + 4}$ . Let  $u = e^x$ ,  $dx = du/u$ . This is  $4 \int \frac{du}{u((u+1)^2 + 4)}$ . Use partial fractions:  $\frac{4}{u(u^2 + 2u + 5)} = \frac{4}{5u} - \frac{4u+8}{5(u^2 + 2u + 5)}$ . This integrates to  $\frac{4}{5} \ln|u| - \frac{2}{5} \ln(u^2 + 2u + 5) - \frac{2}{5} \arctan(\frac{u+1}{2})$ . Substituting back  $u = e^x$  and combining with  $I_1$  gives the final answer. **Answer:**  $\frac{1}{10} \arctan(\frac{e^x + 1}{2}) - \frac{2}{5} \ln(e^{2x} + 2e^x + 5) + \frac{4}{5} x + C$

### 1.27 Problem 27

Evaluate the integral:  $\int \frac{e^x}{e^x \sqrt{4e^{2x} - 1}} dx$

### Solution

First, cancel the  $e^x$  terms:  $\int \frac{1}{\sqrt{4e^{2x} - 1}} dx$ . To transform this into a standard form, let  $u = 2e^x$ . Then  $du = 2e^x dx = u dx$ , so  $dx = \frac{du}{u}$ . The integral becomes  $\int \frac{1}{\sqrt{u^2 - 1}} \cdot \frac{du}{u} = \int \frac{1}{u\sqrt{u^2 - 1}} du$ . This is the standard form for the arcsecant function. **Answer:**  $\operatorname{arcsec}(2e^x) + C$

### 1.28 Problem 28

Evaluate the definite integral:  $\int_0^{\ln(2)} \frac{1}{\sqrt{4 - e^{2x}}} dx$

### Solution

Let  $u = e^x$ . Then  $dx = du/u$ . The bounds become  $u = 1$  and  $u = 2$ . The integral is  $\int_1^2 \frac{1}{\sqrt{4 - u^2}} \frac{du}{u} = \int_1^2 \frac{du}{u\sqrt{4 - u^2}}$ . Use a trigonometric substitution. Let  $u = 2 \sin \theta$ , so  $du = 2 \cos \theta d\theta$ . The integral becomes  $\int \frac{2 \cos \theta d\theta}{2 \sin \theta \sqrt{4 - 4 \sin^2 \theta}} = \int \frac{\cos \theta d\theta}{2 \sin \theta \cos \theta} = \frac{1}{2} \int \csc \theta d\theta$ . The antiderivative is  $-\frac{1}{2} \ln |\csc \theta + \cot \theta|$ . Convert back to  $u$ :  $\sin \theta = u/2$ ,  $\csc \theta = 2/u$ ,  $\cot \theta = \sqrt{4 - u^2}/u$ . Antiderivative in terms of  $u$ :  $-\frac{1}{2} \ln \left| \frac{2 + \sqrt{4 - u^2}}{u} \right|$ . Evaluate from  $u = 1$  to  $u = 2$ :  $\left[ -\frac{1}{2} \ln \left| \frac{2 + \sqrt{4 - u^2}}{u} \right| \right]_1^2 = -\frac{1}{2} (\ln(1) - \ln(2 + \sqrt{3}))$ . **Answer:**  $\frac{1}{2} \ln(2 + \sqrt{3})$

### 1.29 Problem 29

Find the area of the region enclosed by the curves:  $y = 5x - x^2$  and  $y = x$ .

### Solution

First, find the points of intersection by setting the equations equal to each other:  $5x - x^2 = x \implies 4x - x^2 = 0 \implies x(4 - x) = 0$ . The intersection points are at  $x = 0$  and  $x = 4$ . These are the limits of integration. In the interval  $[0, 4]$ , the curve  $y = 5x - x^2$  is above the line  $y = x$ . The area  $A$  is given by the integral of the upper curve minus the lower curve:

$$A = \int_0^4 ((5x - x^2) - x) dx = \int_0^4 (4x - x^2) dx$$

Now, evaluate the integral:

$$A = \left[ 2x^2 - \frac{x^3}{3} \right]_0^4 = \left( 2(4)^2 - \frac{4^3}{3} \right) - (0) = 32 - \frac{64}{3} = \frac{96 - 64}{3} = \frac{32}{3}$$

**Answer:**  $\frac{32}{3}$

### 1.30 Problem 30

Find the area of the region enclosed by the curves  $y = e^x$ ,  $y = x^6$ , from  $x = 0$  to  $x = 1$ .

#### Solution

The limits of integration are given as  $x = 0$  and  $x = 1$ . In this interval,  $y = e^x$  is the upper curve and  $y = x^6$  is the lower curve. Set up the integral for the area  $A$ :

$$A = \int_0^1 (e^x - x^6) dx$$

Evaluate the integral:

$$A = \left[ e^x - \frac{x^7}{7} \right]_0^1 = \left( e^1 - \frac{1^7}{7} \right) - \left( e^0 - \frac{0^7}{7} \right) = \left( e - \frac{1}{7} \right) - (1 - 0) = e - \frac{8}{7}$$

**Answer:**  $e - \frac{8}{7}$

### 1.31 Problem 31

Find the area of the region enclosed by the curves  $x = y^2$ ,  $x = y^2 - 5$ , from  $y = -1$  to  $y = 1$ .

#### Solution

The region is bounded by functions of  $y$ , so we integrate with respect to  $y$ . The limits are given as  $y = -1$  and  $y = 1$ . The right boundary is  $x_{\text{right}} = y^2$  and the left boundary is  $x_{\text{left}} = y^2 - 5$ . Set up the integral for the area  $A$ :

$$A = \int_{-1}^1 (x_{\text{right}} - x_{\text{left}}) dy = \int_{-1}^1 (y^2 - (y^2 - 5)) dy = \int_{-1}^1 5 dy$$

Evaluate the integral:

$$A = [5y]_{-1}^1 = 5(1) - 5(-1) = 5 + 5 = 10$$

**Answer:** 10

### 1.32 Problem 32

Find the area of the region enclosed by the curves  $x = 2y - y^2$  and  $x = y^2 - 4y$ .

#### Solution

First, find the points of intersection:  $2y - y^2 = y^2 - 4y \implies 2y^2 - 6y = 0 \implies 2y(y - 3) = 0$ . The intersection points are at  $y = 0$  and  $y = 3$ . In the interval  $[0, 3]$ , the curve  $x = 2y - y^2$  is the right boundary. Set up the integral with respect to  $y$ :

$$A = \int_0^3 ((2y - y^2) - (y^2 - 4y)) dy = \int_0^3 (6y - 2y^2) dy$$

Evaluate the integral:

$$A = \left[ 3y^2 - \frac{2y^3}{3} \right]_0^3 = \left( 3(3)^2 - \frac{2(3)^3}{3} \right) - (0) = 27 - 18 = 9$$

**Answer:** 9

### 1.33 Problem 33

Find the area of the region enclosed by the curves  $y = x^3 - 15x$  and  $y = x$ .

#### Solution

Find intersections:  $x^3 - 15x = x \implies x^3 - 16x = 0 \implies x(x-4)(x+4) = 0$ . Intersections are at  $x = -4, 0, 4$ . This defines two regions. Region 1 (from -4 to 0): The curve  $y = x^3 - 15x$  is above  $y = x$ .

$$A_1 = \int_{-4}^0 (x^3 - 16x) dx = \left[ \frac{x^4}{4} - 8x^2 \right]_{-4}^0 = 0 - (64 - 128) = 64$$

Region 2 (from 0 to 4): The line  $y = x$  is above  $y = x^3 - 15x$ .

$$A_2 = \int_0^4 (16x - x^3) dx = \left[ 8x^2 - \frac{x^4}{4} \right]_0^4 = (128 - 64) - 0 = 64$$

Total Area:  $A = A_1 + A_2 = 64 + 64 = 128$ . **Answer:** 128

### 1.34 Problem 34

Find the area of the region enclosed by the curves  $y = x^2$ ,  $y = \frac{2}{3}x + \frac{16}{3}$ , and  $y = 8 - 2x$ .

#### Solution

The region must be split into two parts. The intersection points are at  $x = -2, 1, 2$ . Region 1 (from -2 to 1): The upper boundary is  $y = \frac{2}{3}x + \frac{16}{3}$  and the lower is  $y = x^2$ .

$$A_1 = \int_{-2}^1 \left( \frac{2}{3}x + \frac{16}{3} - x^2 \right) dx = \left[ \frac{x^2}{3} + \frac{16x}{3} - \frac{x^3}{3} \right]_{-2}^1 = \left( \frac{16}{3} \right) - \left( -\frac{20}{3} \right) = 12$$

Region 2 (from 1 to 2): The upper boundary is  $y = 8 - 2x$  and the lower is  $y = x^2$ .

$$A_2 = \int_1^2 (8 - 2x - x^2) dx = \left[ 8x - x^2 - \frac{x^3}{3} \right]_1^2 = \left( 12 - \frac{8}{3} \right) - \left( 7 - \frac{1}{3} \right) = \frac{8}{3}$$

Total Area:  $A = A_1 + A_2 = 12 + \frac{8}{3} = \frac{44}{3}$ . **Answer:**  $\frac{44}{3}$

### 1.35 Problem 35

Set up an integral representing the area A of the region enclosed by the curves  $x = y^4$  and  $x = 2 - y^2$ .

#### Solution

Find intersections:  $y^4 = 2 - y^2 \implies y^4 + y^2 - 2 = 0 \implies (y^2 + 2)(y^2 - 1) = 0$ . The real solutions are  $y = \pm 1$ , which are the limits of integration. In the interval  $[-1, 1]$ , the curve  $x = 2 - y^2$  is the right boundary. The integral for the area A is:

$$A = \int_{-1}^1 ((2 - y^2) - y^4) dy = \int_{-1}^1 (2 - y^2 - y^4) dy$$

**Answer:**  $A = \int_{-1}^1 (2 - y^2 - y^4) dy$

### 1.36 Problem 36

Find the area of the region enclosed by the curves  $y = 3 + x^3$ ,  $y = 5 - x$ , for  $x = -1$  to  $x = 0$ .



**Solution**

The limits are given as  $x = -1$  and  $x = 0$ . In this interval, the line  $y = 5 - x$  is the upper boundary. Set up the integral for the area  $A$ :

$$A = \int_{-1}^0 ((5 - x) - (3 + x^3)) dx = \int_{-1}^0 (2 - x - x^3) dx$$

Evaluate the integral:

$$A = \left[ 2x - \frac{x^2}{2} - \frac{x^4}{4} \right]_{-1}^0 = (0) - \left( -2 - \frac{1}{2} - \frac{1}{4} \right) = \frac{11}{4}$$

**Answer:**  $\frac{11}{4}$

**1.37 Problem 37**

Find the area of the region enclosed by the curves  $y = 4 \cos(x)$ ,  $y = 4e^x$ , and  $x = \frac{\pi}{2}$ .

**Solution**

The curves intersect when  $4 \cos(x) = 4e^x$ , which occurs at  $x = 0$ . The limits of integration are  $[0, \pi/2]$ . In this interval,  $y = 4e^x$  is the upper boundary.

$$A = \int_0^{\pi/2} (4e^x - 4 \cos(x)) dx = 4 \int_0^{\pi/2} (e^x - \cos(x)) dx$$

$$A = 4 [e^x - \sin(x)]_0^{\pi/2} = 4 \left( (e^{\pi/2} - \sin(\pi/2)) - (e^0 - \sin(0)) \right) = 4(e^{\pi/2} - 1 - 1)$$

**Answer:**  $4e^{\pi/2} - 8$

**1.38 Problem 38**

Find the area of the region enclosed by the curves  $y = x^2 - 4x$  and  $y = 4x$ .

**Solution**

Find intersections:  $x^2 - 4x = 4x \implies x^2 - 8x = 0 \implies x(x - 8) = 0$ . The limits are  $x = 0$  and  $x = 8$ . In the interval  $[0, 8]$ , the line  $y = 4x$  is the upper boundary.

$$A = \int_0^8 (4x - (x^2 - 4x)) dx = \int_0^8 (8x - x^2) dx$$

$$A = \left[ 4x^2 - \frac{x^3}{3} \right]_0^8 = \left( 4(8)^2 - \frac{8^3}{3} \right) - 0 = 256 - \frac{512}{3} = \frac{768 - 512}{3} = \frac{256}{3}$$

**Answer:**  $\frac{256}{3}$

**1.39 Problem 39**

Find the area of the region enclosed by the curves  $x = 4 - y^2$  and  $x = y^2 - 4$ .

**Solution**

Find intersections:  $4 - y^2 = y^2 - 4 \implies 8 = 2y^2 \implies y^2 = 4$ . The limits are  $y = -2$  and  $y = 2$ . The right boundary is  $x = 4 - y^2$ .

$$A = \int_{-2}^2 ((4 - y^2) - (y^2 - 4)) dy = \int_{-2}^2 (8 - 2y^2) dy$$

$$A = \left[ 8y - \frac{2y^3}{3} \right]_{-2}^2 = \left( 16 - \frac{16}{3} \right) - \left( -16 + \frac{16}{3} \right) = 32 - \frac{32}{3} = \frac{64}{3}$$

**Answer:**  $\frac{64}{3}$

### 1.40 Problem 40

Find the area of the region enclosed by the curves  $2x + y^2 = 8$  and  $x = y$ .

#### Solution

Solve for  $x$ :  $x = 4 - \frac{1}{2}y^2$ . Find intersections:  $y = 4 - \frac{1}{2}y^2 \implies y^2 + 2y - 8 = 0 \implies (y + 4)(y - 2) = 0$ . The limits are  $y = -4$  and  $y = 2$ . The right boundary is  $x = 4 - \frac{1}{2}y^2$ .

$$A = \int_{-4}^2 \left( \left( 4 - \frac{1}{2}y^2 \right) - y \right) dy$$

$$A = \left[ 4y - \frac{y^2}{2} - \frac{y^3}{6} \right]_{-4}^2 = \left( 8 - 2 - \frac{8}{6} \right) - \left( -16 - 8 + \frac{64}{6} \right) = \left( 6 - \frac{4}{3} \right) - \left( -24 + \frac{32}{3} \right) = \frac{14}{3} - \left( -\frac{40}{3} \right) = \frac{54}{3}$$

**Answer:** 18

### 1.41 Problem 41

Find the area of the region enclosed by the curves  $x = 8y^2$  and  $x = 28 + y^2$ .

#### Solution

Find intersections:  $8y^2 = 28 + y^2 \implies 7y^2 = 28 \implies y^2 = 4$ . The limits are  $y = -2$  and  $y = 2$ . The right boundary is  $x = 28 + y^2$ .

$$A = \int_{-2}^2 ((28 + y^2) - 8y^2) dy = \int_{-2}^2 (28 - 7y^2) dy$$

$$A = \left[ 28y - \frac{7y^3}{3} \right]_{-2}^2 = \left( 56 - \frac{56}{3} \right) - \left( -56 + \frac{56}{3} \right) = 112 - \frac{112}{3} = \frac{224}{3}$$

**Answer:**  $\frac{224}{3}$

### 1.42 Problem 42

Find the area of the region enclosed by the curves  $x = y^2 - 5$  and  $x = e^y$ , from  $y = -1$  to  $y = 1$ .

#### Solution

The limits are given as  $y = -1$  and  $y = 1$ . The right boundary is  $x = e^y$ .

$$A = \int_{-1}^1 (e^y - (y^2 - 5)) dy = \int_{-1}^1 (e^y - y^2 + 5) dy$$

$$A = \left[ e^y - \frac{y^3}{3} + 5y \right]_{-1}^1 = \left( e - \frac{1}{3} + 5 \right) - \left( e^{-1} + \frac{1}{3} - 5 \right) = e - e^{-1} - \frac{2}{3} + 10 = e - \frac{1}{e} + \frac{28}{3}$$

**Answer:**  $e - \frac{1}{e} + \frac{28}{3}$

### 1.43 Problem 43

Find the area of the region enclosed by the curves  $y = \sqrt{x}$  and  $y = \frac{1}{5}x$ , for  $0 \leq x \leq 36$ .

### Solution

Find intersection:  $\sqrt{x} = \frac{1}{5}x \implies x = \frac{1}{25}x^2 \implies x(x - 25) = 0$ . Intersection is at  $x = 25$ . The region is split.  
 Region 1 ( $0 \leq x \leq 25$ ):  $y = \sqrt{x}$  is upper.  $A_1 = \int_0^{25} (\sqrt{x} - \frac{1}{5}x) dx = [\frac{2}{3}x^{3/2} - \frac{x^2}{10}]_0^{25} = \frac{250}{3} - \frac{625}{10} = \frac{125}{6}$ . Region 2 ( $25 \leq x \leq 36$ ):  $y = \frac{1}{5}x$  is upper.  $A_2 = \int_{25}^{36} (\frac{1}{5}x - \sqrt{x}) dx = [\frac{x^2}{10} - \frac{2}{3}x^{3/2}]_{25}^{36} = (\frac{1296}{10} - \frac{2}{3}(216)) - (\frac{625}{10} - \frac{250}{3}) = (129.6 - 144) - (62.5 - \frac{250}{3}) = -14.4 - (-\frac{125}{6}) = \frac{193}{30}$ . Total Area:  $A = \frac{125}{6} + \frac{193}{30} = \frac{625+193}{30} = \frac{818}{30} = \frac{409}{15}$ . **Answer:**  $\frac{409}{15}$

### 1.44 Problem 44

Find the area of the region enclosed by the curves  $y = \cos(x)$  and  $y = 2 - \cos(x)$ , for  $0 \leq x \leq 2\pi$ .

### Solution

In the interval  $[0, 2\pi]$ , the curve  $y = 2 - \cos(x)$  is always above  $y = \cos(x)$ . They touch at  $x = 0$  and  $x = 2\pi$ .

$$A = \int_0^{2\pi} ((2 - \cos(x)) - \cos(x)) dx = \int_0^{2\pi} (2 - 2\cos(x)) dx$$

$$A = [2x - 2\sin(x)]_0^{2\pi} = (4\pi - 2\sin(2\pi)) - (0 - 2\sin(0)) = 4\pi$$

**Answer:**  $4\pi$

### 1.45 Problem 45

Find the area of the region enclosed by the curves  $y = \cos(x)$  and  $y = \sin(2x)$ , for  $0 \leq x \leq \frac{\pi}{2}$ .

### Solution

Find intersection:  $\cos(x) = \sin(2x) = 2\sin(x)\cos(x) \implies \cos(x)(1 - 2\sin(x)) = 0$ . Intersections at  $x = \pi/6$  and  $x = \pi/2$ . Region 1 ( $0 \leq x \leq \pi/6$ ):  $y = \cos(x)$  is upper.  $A_1 = \int_0^{\pi/6} (\cos(x) - \sin(2x)) dx = [\sin(x) + \frac{1}{2}\cos(2x)]_0^{\pi/6} = (\frac{1}{2} + \frac{1}{4}) - (\frac{1}{2}) = \frac{1}{4}$ . Region 2 ( $\pi/6 \leq x \leq \pi/2$ ):  $y = \sin(2x)$  is upper.  $A_2 = \int_{\pi/6}^{\pi/2} (\sin(2x) - \cos(x)) dx = [-\frac{1}{2}\cos(2x) - \sin(x)]_{\pi/6}^{\pi/2} = (\frac{1}{2} - 1) - (-\frac{1}{4} - \frac{1}{2}) = -\frac{1}{2} - (-\frac{3}{4}) = \frac{1}{4}$ . Total Area:  $A = A_1 + A_2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$ . **Answer:**  $\frac{1}{2}$

### 1.46 Problem 46

Evaluate the integral. (Remember the constant of integration.)

$$\int 2\sin^2(x)\cos^3(x) dx$$

### Solution

We have an odd power of cosine, so we save one cosine factor and convert the rest to sines using  $\cos^2(x) = 1 - \sin^2(x)$ .

$$\int 2\sin^2(x)\cos^2(x)\cos(x) dx = \int 2\sin^2(x)(1 - \sin^2(x))\cos(x) dx$$

Let  $u = \sin(x)$ , so  $du = \cos(x) dx$ . The integral becomes:

$$\begin{aligned} \int 2u^2(1 - u^2) du &= \int (2u^2 - 2u^4) du \\ &= \frac{2}{3}u^3 - \frac{2}{5}u^5 + C \end{aligned}$$

Substituting back  $u = \sin(x)$ :

$$= \frac{2}{3} \sin^3(x) - \frac{2}{5} \sin^5(x) + C$$

**Answer:**  $\frac{2}{3} \sin^3(x) - \frac{2}{5} \sin^5(x) + C$

### 1.47 Problem 47

Evaluate the integral. (Remember the constant of integration.)

$$\int \sin^3(y) \cos^4(y) dy$$

#### Solution

We have an odd power of sine, so we save one sine factor and convert the rest to cosines using  $\sin^2(y) = 1 - \cos^2(y)$ .

$$\int \sin^2(y) \cos^4(y) \sin(y) dy = \int (1 - \cos^2(y)) \cos^4(y) \sin(y) dy$$

Let  $u = \cos(y)$ , so  $du = -\sin(y) dy$ . The integral becomes:

$$\begin{aligned} \int (1 - u^2) u^4 (-du) &= - \int (u^4 - u^6) du \\ &= - \left( \frac{u^5}{5} - \frac{u^7}{7} \right) + C = \frac{1}{7} u^7 - \frac{1}{5} u^5 + C \end{aligned}$$

Substituting back  $u = \cos(y)$ :

$$= \frac{1}{7} \cos^7(y) - \frac{1}{5} \cos^5(y) + C$$

**Answer:**  $\frac{1}{7} \cos^7(y) - \frac{1}{5} \cos^5(y) + C$

### 1.48 Problem 48

Evaluate the integral.

$$\int_0^{\pi/2} \cos^{13}(x) \sin^5(x) dx$$

#### Solution

The power of sine is odd. We save a  $\sin(x)$  factor and convert the rest.

$$\int_0^{\pi/2} \cos^{13}(x) \sin^4(x) \sin(x) dx = \int_0^{\pi/2} \cos^{13}(x) (1 - \cos^2(x))^2 \sin(x) dx$$

Let  $u = \cos(x)$ , so  $du = -\sin(x) dx$ . The bounds change:  $x = 0 \implies u = 1$ , and  $x = \pi/2 \implies u = 0$ .

$$\begin{aligned} \int_1^0 u^{13} (1 - u^2)^2 (-du) &= \int_0^1 u^{13} (1 - 2u^2 + u^4) du \\ &= \int_0^1 (u^{13} - 2u^{15} + u^{17}) du = \left[ \frac{u^{14}}{14} - \frac{2u^{16}}{16} + \frac{u^{18}}{18} \right]_0^1 \\ &= \left( \frac{1}{14} - \frac{1}{8} + \frac{1}{18} \right) - (0) = \frac{36 - 63 + 28}{504} = \frac{1}{504} \end{aligned}$$

**Answer:**  $\frac{1}{504}$

### 1.49 Problem 49

Evaluate the integral.

$$\int_0^{\pi/2} 9 \sin^2(x) \cos^2(x) dx$$

#### Solution

We use the identity  $\sin(x) \cos(x) = \frac{1}{2} \sin(2x)$ , and then  $\sin^2(\theta) = \frac{1 - \cos(2\theta)}{2}$ .

$$\begin{aligned} \int_0^{\pi/2} 9(\sin(x) \cos(x))^2 dx &= \int_0^{\pi/2} 9 \left( \frac{1}{2} \sin(2x) \right)^2 dx = \frac{9}{4} \int_0^{\pi/2} \sin^2(2x) dx \\ &= \frac{9}{4} \int_0^{\pi/2} \frac{1 - \cos(4x)}{2} dx = \frac{9}{8} \int_0^{\pi/2} (1 - \cos(4x)) dx \\ &= \frac{9}{8} \left[ x - \frac{1}{4} \sin(4x) \right]_0^{\pi/2} = \frac{9}{8} \left[ \left( \frac{\pi}{2} - \frac{1}{4} \sin(2\pi) \right) - \left( 0 - \frac{1}{4} \sin(0) \right) \right] \\ &= \frac{9}{8} \left( \frac{\pi}{2} \right) = \frac{9\pi}{16} \end{aligned}$$

**Answer:**  $\frac{9\pi}{16}$

### 1.50 Problem 50

Evaluate the integral.

$$\int_0^{\pi/2} 5 \cos^2(\theta) d\theta$$

#### Solution

Using the half-angle identity  $\cos^2(\theta) = \frac{1 + \cos(2\theta)}{2}$ .

$$\begin{aligned} \int_0^{\pi/2} 5 \left( \frac{1 + \cos(2\theta)}{2} \right) d\theta &= \frac{5}{2} \int_0^{\pi/2} (1 + \cos(2\theta)) d\theta \\ &= \frac{5}{2} \left[ \theta + \frac{1}{2} \sin(2\theta) \right]_0^{\pi/2} = \frac{5}{2} \left[ \left( \frac{\pi}{2} + \frac{1}{2} \sin(\pi) \right) - \left( 0 + \frac{1}{2} \sin(0) \right) \right] \\ &= \frac{5}{2} \left( \frac{\pi}{2} \right) = \frac{5\pi}{4} \end{aligned}$$

**Answer:**  $\frac{5\pi}{4}$

### 1.51 Problem 51

Evaluate the integral.

$$\int \sqrt{\cos(\theta)} \sin^3(\theta) d\theta$$

**Solution**

The power of sine is odd. We save a  $\sin(\theta)$  factor.

$$\int \sqrt{\cos(\theta)} \sin^2(\theta) \sin(\theta) d\theta = \int \sqrt{\cos(\theta)} (1 - \cos^2(\theta)) \sin(\theta) d\theta$$

Let  $u = \cos(\theta)$ , so  $du = -\sin(\theta) d\theta$ .

$$\begin{aligned} \int \sqrt{u} (1 - u^2) (-du) &= - \int (u^{1/2} - u^{5/2}) du \\ &= - \left( \frac{u^{3/2}}{3/2} - \frac{u^{7/2}}{7/2} \right) + C = -\frac{2}{3} u^{3/2} + \frac{2}{7} u^{7/2} + C \\ &= \frac{2}{7} \cos^{7/2}(\theta) - \frac{2}{3} \cos^{3/2}(\theta) + C \end{aligned}$$

**Answer:**  $\frac{2}{7} \cos^{7/2}(\theta) - \frac{2}{3} \cos^{3/2}(\theta) + C$

**1.52 Problem 52**

Evaluate the integral.

$$\int \sin(3x) \sec^5(3x) dx$$

**Solution**

Rewrite  $\sec(3x)$  as  $1/\cos(3x)$ .

$$\int \sin(3x) \frac{1}{\cos^5(3x)} dx = \int \frac{\sin(3x)}{\cos^5(3x)} dx$$

Let  $u = \cos(3x)$ , so  $du = -3 \sin(3x) dx$ , which means  $\sin(3x) dx = -du/3$ .

$$\begin{aligned} \int \frac{1}{u^5} \left( -\frac{du}{3} \right) &= -\frac{1}{3} \int u^{-5} du \\ &= -\frac{1}{3} \frac{u^{-4}}{-4} + C = \frac{1}{12} u^{-4} + C = \frac{1}{12 \cos^4(3x)} + C \\ &= \frac{1}{12} \sec^4(3x) + C \end{aligned}$$

**Answer:**  $\frac{1}{12} \sec^4(3x) + C$

**1.53 Problem 53**

Evaluate the integral.

$$\int 4 \tan(x) \sec^3(x) dx$$

**Solution**

We can rewrite the integrand to isolate a  $\sec(x) \tan(x)$  factor.

$$4 \int \sec^2(x) (\sec(x) \tan(x)) dx$$

Let  $u = \sec(x)$ , so  $du = \sec(x) \tan(x) dx$ .

$$4 \int u^2 du = 4 \left( \frac{u^3}{3} \right) + C = \frac{4}{3} u^3 + C$$

Substituting back  $u = \sec(x)$ :

$$= \frac{4}{3} \sec^3(x) + C$$

**Answer:**  $\frac{4}{3} \sec^3(x) + C$

### 1.54 Problem 54

Evaluate the integral.

$$\int 5 \tan^2(x) dx$$

**Solution**

Using the identity  $\tan^2(x) = \sec^2(x) - 1$ .

$$\begin{aligned} 5 \int (\sec^2(x) - 1) dx &= 5 \left( \int \sec^2(x) dx - \int 1 dx \right) \\ &= 5(\tan(x) - x) + C = 5 \tan(x) - 5x + C \end{aligned}$$

**Answer:**  $5 \tan(x) - 5x + C$

### 1.55 Problem 55

Evaluate the integral.

$$\int 11 \tan^4(x) \sec^6(x) dx$$

**Solution**

The power of secant is even. We save a  $\sec^2(x)$  factor and convert the rest to tangents using  $\sec^2(x) = 1 + \tan^2(x)$ .

$$11 \int \tan^4(x) \sec^4(x) \sec^2(x) dx = 11 \int \tan^4(x) (1 + \tan^2(x))^2 \sec^2(x) dx$$

Let  $u = \tan(x)$ , so  $du = \sec^2(x) dx$ .

$$\begin{aligned} 11 \int u^4 (1 + u^2)^2 du &= 11 \int u^4 (1 + 2u^2 + u^4) du \\ &= 11 \int (u^4 + 2u^6 + u^8) du = 11 \left( \frac{u^5}{5} + \frac{2u^7}{7} + \frac{u^9}{9} \right) + C \\ &= \frac{11}{5} \tan^5(x) + \frac{22}{7} \tan^7(x) + \frac{11}{9} \tan^9(x) + C \end{aligned}$$

**Answer:**  $\frac{11}{9} \tan^9(x) + \frac{22}{7} \tan^7(x) + \frac{11}{5} \tan^5(x) + C$

### 1.56 Problem 56

Evaluate the integral.

$$\int \tan^3(x) \sec(x) dx$$

**Solution**

The power of tangent is odd. We save a  $\sec(x) \tan(x)$  factor and convert the rest to secants.

$$\int \tan^2(x) (\sec(x) \tan(x)) dx = \int (\sec^2(x) - 1) (\sec(x) \tan(x)) dx$$

Let  $u = \sec(x)$ , so  $du = \sec(x) \tan(x) dx$ .

$$\begin{aligned} \int (u^2 - 1) du &= \frac{u^3}{3} - u + C \\ &= \frac{1}{3} \sec^3(x) - \sec(x) + C \end{aligned}$$

**Answer:**  $\frac{1}{3} \sec^3(x) - \sec(x) + C$

**1.57 Problem 57**

Evaluate the integral.

$$\int \tan^3(x) \sec^6(x) dx$$

**Solution**

The power of tangent is odd, so we can let  $u = \sec(x)$ .

$$\int \tan^2(x) \sec^5(x) (\sec(x) \tan(x)) dx = \int (\sec^2(x) - 1) \sec^5(x) (\sec(x) \tan(x)) dx$$

Let  $u = \sec(x)$ , so  $du = \sec(x) \tan(x) dx$ .

$$\begin{aligned} \int (u^2 - 1) u^5 du &= \int (u^7 - u^5) du \\ &= \frac{u^8}{8} - \frac{u^6}{6} + C = \frac{1}{8} \sec^8(x) - \frac{1}{6} \sec^6(x) + C \end{aligned}$$

**Answer:**  $\frac{1}{8} \sec^8(x) - \frac{1}{6} \sec^6(x) + C$

**1.58 Problem 58**

Evaluate the integral.

$$\int_0^{\pi/6} \tan^4(t) dt$$

**Solution**

We reduce the power of tangent using  $\tan^2(t) = \sec^2(t) - 1$ .

$$\int_0^{\pi/6} \tan^2(t) (\sec^2(t) - 1) dt = \int_0^{\pi/6} \tan^2(t) \sec^2(t) dt - \int_0^{\pi/6} \tan^2(t) dt$$

The first integral is  $\int u^2 du$  with  $u = \tan(t)$ . The second integral becomes  $\int (\sec^2(t) - 1) dt$ .

$$\begin{aligned} &\left[ \frac{1}{3} \tan^3(t) \right]_0^{\pi/6} - \int_0^{\pi/6} (\sec^2(t) - 1) dt \\ &= \left[ \frac{1}{3} \tan^3(t) \right]_0^{\pi/6} - [\tan(t) - t]_0^{\pi/6} = \left[ \frac{1}{3} \tan^3(t) - \tan(t) + t \right]_0^{\pi/6} \end{aligned}$$



Since  $\tan(\pi/6) = 1/\sqrt{3}$ :

$$\begin{aligned} &= \left( \frac{1}{3} \left( \frac{1}{\sqrt{3}} \right)^3 - \frac{1}{\sqrt{3}} + \frac{\pi}{6} \right) - (0) = \frac{1}{9\sqrt{3}} - \frac{1}{\sqrt{3}} + \frac{\pi}{6} \\ &= \frac{1-9}{9\sqrt{3}} + \frac{\pi}{6} = -\frac{8}{9\sqrt{3}} + \frac{\pi}{6} = \frac{\pi}{6} - \frac{8\sqrt{3}}{27} \end{aligned}$$

**Answer:**  $\frac{\pi}{6} - \frac{8\sqrt{3}}{27}$

## 1.59 Problem 59

Evaluate the integral.

$$\int \tan^5(x) dx$$

**Solution**

We reduce the power of tangent.

$$\begin{aligned} \int \tan^3(x) \tan^2(x) dx &= \int \tan^3(x) (\sec^2(x) - 1) dx \\ &= \int \tan^3(x) \sec^2(x) dx - \int \tan^3(x) dx \end{aligned}$$

The first integral is  $\frac{1}{4} \tan^4(x)$ . For the second integral:

$$\begin{aligned} \int \tan^3(x) dx &= \int \tan(x) (\sec^2(x) - 1) dx = \int \tan(x) \sec^2(x) dx - \int \tan(x) dx \\ &= \frac{1}{2} \tan^2(x) - \ln |\sec(x)| \end{aligned}$$

Combining the results:

$$\frac{1}{4} \tan^4(x) - \left( \frac{1}{2} \tan^2(x) - \ln |\sec(x)| \right) + C$$

**Answer:**  $\frac{1}{4} \tan^4(x) - \frac{1}{2} \tan^2(x) + \ln |\sec(x)| + C$

## 1.60 Problem 60

Evaluate the integral.

$$\int \frac{\tan(x) \sec^2(x)}{\cos(x)} dx$$

**Solution**

Since  $1/\cos(x) = \sec(x)$ , the integral becomes:

$$\int \tan(x) \sec^3(x) dx = \int \sec^2(x) (\sec(x) \tan(x)) dx$$

Let  $u = \sec(x)$ , so  $du = \sec(x) \tan(x) dx$ .

$$\begin{aligned} \int u^2 du &= \frac{u^3}{3} + C \\ &= \frac{1}{3} \sec^3(x) + C \end{aligned}$$

**Answer:**  $\frac{1}{3} \sec^3(x) + C$

### 1.61 Problem 61

Evaluate the integral.

$$\int_{\pi/6}^{\pi/2} 5 \cot^2(x) dx$$

#### Solution

Using the identity  $\cot^2(x) = \csc^2(x) - 1$ .

$$\begin{aligned} 5 \int_{\pi/6}^{\pi/2} (\csc^2(x) - 1) dx &= 5 [-\cot(x) - x]_{\pi/6}^{\pi/2} \\ &= 5 \left[ \left(-\cot\left(\frac{\pi}{2}\right) - \frac{\pi}{2}\right) - \left(-\cot\left(\frac{\pi}{6}\right) - \frac{\pi}{6}\right) \right] \end{aligned}$$

Since  $\cot(\pi/2) = 0$  and  $\cot(\pi/6) = \sqrt{3}$ :

$$\begin{aligned} &= 5 \left[ \left(0 - \frac{\pi}{2}\right) - \left(-\sqrt{3} - \frac{\pi}{6}\right) \right] = 5 \left[ -\frac{\pi}{2} + \sqrt{3} + \frac{\pi}{6} \right] \\ &= 5 \left[ \sqrt{3} - \frac{3\pi}{6} + \frac{\pi}{6} \right] = 5 \left( \sqrt{3} - \frac{2\pi}{6} \right) = 5 \left( \sqrt{3} - \frac{\pi}{3} \right) \end{aligned}$$

**Answer:**  $5 \left( \sqrt{3} - \frac{\pi}{3} \right)$

### 1.62 Problem 62

Evaluate the integral.

$$\int \sin(8x) \cos(5x) dx$$

#### Solution

We use the product-to-sum identity  $\sin(A) \cos(B) = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$ .

$$\begin{aligned} \int \frac{1}{2} [\sin(8x+5x) + \sin(8x-5x)] dx &= \frac{1}{2} \int (\sin(13x) + \sin(3x)) dx \\ &= \frac{1}{2} \left( -\frac{1}{13} \cos(13x) - \frac{1}{3} \cos(3x) \right) + C \\ &= -\frac{1}{26} \cos(13x) - \frac{1}{6} \cos(3x) + C \end{aligned}$$

**Answer:**  $-\frac{1}{26} \cos(13x) - \frac{1}{6} \cos(3x) + C$

### 1.63 Problem 63

Evaluate the integral.

$$\int 5 \tan^2(x) \sec(x) dx$$

**Solution**

Using the identity  $\tan^2(x) = \sec^2(x) - 1$ .

$$5 \int (\sec^2(x) - 1) \sec(x) \, dx = 5 \int (\sec^3(x) - \sec(x)) \, dx$$

We use the standard integrals for  $\sec^3(x)$  and  $\sec(x)$ .

$$\int \sec(x) \, dx = \ln |\sec(x) + \tan(x)| + C$$

$$\int \sec^3(x) \, dx = \frac{1}{2}(\sec(x) \tan(x) + \ln |\sec(x) + \tan(x)|) + C$$

So, the integral is:

$$\begin{aligned} 5 \left[ \frac{1}{2}(\sec(x) \tan(x) + \ln |\sec(x) + \tan(x)|) - \ln |\sec(x) + \tan(x)| \right] + C \\ = 5 \left[ \frac{1}{2} \sec(x) \tan(x) - \frac{1}{2} \ln |\sec(x) + \tan(x)| \right] + C \end{aligned}$$

**Answer:**  $\frac{5}{2}(\sec(x) \tan(x) - \ln |\sec(x) + \tan(x)|) + C$

## 2 Summary of Rules, Formulas, and Tricks

This document covers a wide range of integration techniques, from basic substitutions to multi-stage problems requiring a sequence of tricks.

### Category 1: U-Substitution

- **”Clever” First-Step Substitution:** Recognizing a non-obvious substitution that dramatically simplifies the integral from the start (e.g., Problem 1).
- **Standard Second-Step Substitution:** A more routine substitution that becomes necessary *after* an initial step like IBP has been applied.

### Category 2: Integration by Parts (IBP)

- **The ”Stealth dx” Trick:** Creating a product by setting  $dv = dx$  to integrate single functions like  $\ln(x)$  or  $\arctan(x)$  (e.g., Problems 3, 4, 5, 6).
- **Looping / Circular Integration:** Applying IBP twice to problems like  $\int e^x \sin(x) dx$  to solve for the original integral algebraically (e.g., Problem 2).
- **Standard IBP:** The direct application of the formula, often used as a key step in a larger problem (e.g., Problem 8 and the definite integral sequence).

### Category 3: Algebraic & Trigonometric Manipulation

- **Trigonometric Identities:** Using identities like  $\sin(2x) = 2\sin(x)\cos(x)$  to unlock the problem (e.g., Problems 9-14).
- **The ”Add and Subtract” Trick:** A shortcut for polynomial long division to simplify rational functions where the numerator’s degree is equal to the denominator’s (e.g., Problems 5, 6, 8).
- **Completing the Square:** Rewriting a quadratic denominator to fit the standard arctangent form (e.g., Problem 7).

### Category 4: Multi-Stage ”Grand Challenge” Problems

- **The ”Trig Identity  $\rightarrow$  U-Sub  $\rightarrow$  IBP” Sequence:** A powerful pattern for solving complex definite integrals (e.g., Problems 9-14).
- **The ”IBP  $\rightarrow$  Algebraic Trick  $\rightarrow$  Standard Form” Sequence:** A common pattern for integrating logarithmic and inverse trig functions (e.g., Problems 5, 6, 8).

### Category 5: Definite Integral Skills

- **Changing Limits of Integration:** A mandatory step when performing u-substitution on a definite integral to avoid having to substitute back.
- **Flipping Limits of Integration:** Using the property  $\int_a^b f(x) dx = -\int_b^a f(x) dx$  to cancel negative signs and simplify calculations.
- **The F(b) - F(a) Evaluation:** The final, crucial arithmetic step of the Fundamental Theorem of Calculus.
- **Understanding Function Properties (Even/Odd):** Using symmetry, like  $\cos(-x) = \cos(x)$ , to simplify the final evaluation (e.g., Problem 14).

## Untouched Tricks & Problem Types

The following are integration strategies that were identified but not yet practiced.

- **The DI Method (Tabular Method):** A shortcut for repeated IBP.
- **Reduction Formulas:** Using IBP to create a formula relating an integral to a simpler version of itself.
- **Partial Fraction Decomposition:** The main algebraic technique for integrating more complex rational functions.
- **The "King Property" Connection:** A specific trick for definite integrals.
- **Feynman's Trick (Differentiation Under the Integral Sign):** A powerful method where you introduce a parameter into the integral, differentiate with respect to that parameter, and then integrate back.
- **Weierstrass Substitution (Tangent Half-Angle Substitution):** A substitution that can convert any rational function of trigonometric functions into an algebraic rational function, which can then be solved using partial fractions.
- **The "King Property" Connection:** A specific trick for definite integrals.
- **Contour Integration and the Residue Theorem:** These are techniques from complex analysis that can be used to solve very difficult real-valued definite integrals.
- **Meijer G-Function:** A highly generalized function that can represent most elementary and many special functions. There are rules for finding the antiderivative of a Meijer G-function, making it a powerful, though complex, symbolic integration method.