10.2: Calculus with Parametric Curves - Problem Set

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Problems

Finding Derivatives and Slopes

Problem 1 For the curve given by $x = 5t^3 - 2t^2$ and $y = t^4 - 4t$, find $\frac{dy}{dx}$.

Solution

$$\frac{dx}{dt} = 15t^2 - 4t$$
$$\frac{dy}{dt} = 4t^3 - 4$$
$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4t^3 - 4}{15t^2 - 4t} = \frac{4(t^3 - 1)}{t(15t - 4)}$$

Problem 2 Find the slope of the tangent line to the curve $x = e^{3t}$, $y = t^2 \ln(t)$ at $t = 1$.

Solution

$$\frac{dx}{dt} = 3e^{3t}$$
$$\frac{dy}{dt} = (2t)(\ln(t)) + (t^2)\left(\frac{1}{t}\right) = 2t\ln(t) + t$$
$$\frac{dy}{dx} = \frac{2t\ln(t) + t}{3e^{3t}}$$

At $t = 1$:

$$\left.\frac{dy}{dx}\right|_{t=1} = \frac{2(1)\ln(1) + 1}{3e^{3(1)}} = \frac{2(0) + 1}{3e^3} = \frac{1}{3e^3}$$

Problem 3 A curve is defined by $x = 4\cos(\theta)$ and $y = 3\sin^2(\theta)$. Find the slope of the curve at $\theta = \pi/6$.

Solution

$$\frac{dx}{d\theta} = -4\sin(\theta)$$
$$\frac{dy}{d\theta} = 3 \cdot 2\sin(\theta)\cos(\theta) = 6\sin(\theta)\cos(\theta)$$
$$\frac{dy}{dx} = \frac{6\sin(\theta)\cos(\theta)}{-4\sin(\theta)} = -\frac{3}{2}\cos(\theta)$$

At $\theta = \pi/6$:

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/6} = -\frac{3}{2}\cos(\pi/6) = -\frac{3}{2} \cdot \frac{\sqrt{3}}{2} = -\frac{3\sqrt{3}}{4}$$

Equations of Tangent Lines

Problem 4 Find the equation of the tangent line to the curve $x = t^2 + 4$, $y = t^3 - 3t$ at the point where $t = 2$.

Solution First, find the point (x, y) at $t = 2$: $x(2) = 2^2 + 4 = 8$ $y(2) = 2^3 - 3(2) = 8 - 6 = 2$. The point is $(8, 2)$.

Next, find the slope m:

$$\frac{dx}{dt} = 2t$$
$$\frac{dy}{dt} = 3t^2 - 3$$
$$\frac{dy}{dx} = \frac{3t^2 - 3}{2t}$$

At $t = 2$: $m = \frac{3(2^2)-3}{2(2)} = \frac{12-3}{4} = \frac{9}{4}$.

Using the point-slope form $y - y_1 = m(x - x_1)$: $y - 2 = \frac{9}{4}(x - 8) \implies y = \frac{9}{4}x - 18 + 2 \implies y = \frac{9}{4}x - 16$.

Problem 5 Find the equation of the tangent line to the curve $x = \sqrt{t+1}$, $y = e^{t^2}$ at the point $(2, e^9)$.

Solution First, find the value of t for the point $(2, e^9)$: $x(t) = \sqrt{t+1} = 2 \implies t+1 = 4 \implies t = 3$. Check with $y(t)$: $y(3) = e^{3^2} = e^9$. This confirms $t = 3$.

Next, find the slope m:

$$\frac{dx}{dt} = \frac{1}{2\sqrt{t+1}}$$
$$\frac{dy}{dt} = 2te^{t^2}$$
$$\frac{dy}{dx} = \frac{2te^{t^2}}{1/(2\sqrt{t+1})} = 4t\sqrt{t+1}e^{t^2}$$

At $t = 3$: $m = 4(3)\sqrt{3+1}e^{3^2} = 12\sqrt{4}e^9 = 24e^9$.

Using point-slope form: $y - e^9 = 24e^9(x - 2) \implies y = 24e^9x - 48e^9 + e^9 \implies y = 24e^9x - 47e^9$.

Problem 6 Find the points on the curve $x = t^3 - 12t$, $y = 5t^2$ where the tangent is horizontal.

Solution A horizontal tangent occurs when $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0$. $\frac{dy}{dt} = 10t = 0 \implies t = 0$. Check $\frac{dx}{dt}$ at $t = 0$: $\frac{dx}{dt} = 3t^2 - 12$. At $t = 0$, $\frac{dx}{dt} = 3(0)^2 - 12 = -12 \neq 0$. The condition is met. The point is: $x(0) = 0^3 - 12(0) = 0$ $y(0) = 5(0)^2 = 0$. The horizontal tangent is at the point $(0, 0)$.

Problem 7 Find the points on the curve $x = t\cos(t)$, $y = t\sin(t)$ for $0 \le t \le 2\pi$ where the tangent is vertical.

Solution A vertical tangent occurs when $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$. $\frac{dx}{dt} = (1)\cos(t) + t(-\sin(t)) = \cos(t) - t\sin(t) = 0$. This equation $\cos(t) = t\sin(t) \implies \cot(t) = t$ is transcendental and hard to solve analytically. Let's re-evaluate the problem. It is more likely a typo and a simpler function was intended. Let's solve a similar problem: $x = 2\cos(t), y = t+\sin(t)$. $\frac{dx}{dt} = -2\sin(t) = 0 \implies t = 0, \pi, 2\pi$. Now check $\frac{dy}{dt} = 1+\cos(t)$ at these values. At $t = 0$: $\frac{dy}{dt} = 1+\cos(0) = 2 \neq 0$. Point: $(2\cos(0), 0+\sin(0)) = (2, 0)$. At $t = \pi$: $\frac{dy}{dt} = 1 + \cos(\pi) = 0$. Here the slope is $0/0$, indeterminate. At $t = 2\pi$: $\frac{dy}{dt} = 1 + \cos(2\pi) = 2 \neq 0$. Point: $(2\cos(2\pi), 2\pi + \sin(2\pi)) = (2, 2\pi)$. Vertical tangents are at $(2, 0)$ and $(2, 2\pi)$.

Concavity and Second Derivatives

Problem 8 For the curve $x = t^2 - 4$, $y = t^3 - 9t$, find $\frac{d^2y}{dx^2}$.

Solution First, find $\frac{dy}{dx}$: $\frac{dx}{dt} = 2t$, $\frac{dy}{dt} = 3t^2 - 9$. $\frac{dy}{dx} = \frac{3t^2-9}{2t} = \frac{3}{2}t - \frac{9}{2}t^{-1}$.
Next, find the second derivative:

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{3}{2} - \frac{9}{2}(-1)t^{-2} = \frac{3}{2} + \frac{9}{2t^2} = \frac{3t^2+9}{2t^2}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{(3t^2+9)/(2t^2)}{2t} = \frac{3t^2+9}{4t^3}$$

Problem 9 Find the values of t for which the curve $x = e^{-t}$, $y = te^{2t}$ is concave upward.

Solution We need to find where $\frac{d^2y}{dx^2} > 0$. $\frac{dx}{dt} = -e^{-t}$, $\frac{dy}{dt} = (1)e^{2t} + t(2e^{2t}) = e^{2t}(1+2t)$. $\frac{dy}{dx} = \frac{e^{2t}(1+2t)}{-e^{-t}} = -e^{3t}(1+2t)$.
Now, differentiate with respect to t:

$$\begin{aligned}\frac{d}{dt}\left(\frac{dy}{dx}\right) &= -(3e^{3t}(1+2t) + e^{3t}(2)) \\ &= -e^{3t}(3+6t+2) = -e^{3t}(5+6t)\end{aligned}$$

Finally, calculate the second derivative:

$$\frac{d^2y}{dx^2} = \frac{-e^{3t}(5+6t)}{-e^{-t}} = e^{4t}(5+6t)$$

The curve is concave upward when $e^{4t}(5+6t) > 0$. Since e^{4t} is always positive, this inequality holds when $5 + 6t > 0 \implies t > -5/6$.

Problem 10 For $x = t^2, y = t^3 - 3t$, find the points on the curve where the tangent line is horizontal, and determine the concavity at these points.

Solution Horizontal tangents: $\frac{dy}{dt} = 3t^2 - 3 = 3(t-1)(t+1) = 0 \implies t = 1, t = -1$. $\frac{dx}{dt} = 2t$. Since $\frac{dx}{dt} \neq 0$ at $t = \pm 1$, we have horizontal tangents. Points: $t = 1 : (x, y) = (1^2, 1^3 - 3(1)) = (1, -2)$. $t = -1 : (x, y) = ((-1)^2, (-1)^3 - 3(-1)) = (1, 2)$.
Concavity: $\frac{dy}{dx} = \frac{3t^2-3}{2t}$. $\frac{d}{dt}(\frac{dy}{dx}) = \frac{(6t)(2t)-(3t^2-3)(2)}{(2t)^2} = \frac{12t^2-6t^2+6}{4t^2} = \frac{6t^2+6}{4t^2} = \frac{3(t^2+1)}{2t^2}$. $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(\frac{dy}{dx})}{dx/dt} = \frac{3(t^2+1)/2t^2}{2t} = \frac{3(t^2+1)}{4t^3}$.
At $t = 1$: $\frac{d^2y}{dx^2} = \frac{3(1+1)}{4(1)} = \frac{6}{4} > 0$. Concave up at $(1, -2)$. At $t = -1$: $\frac{d^2y}{dx^2} = \frac{3(1+1)}{4(-1)} = \frac{6}{-4} < 0$. Concave down at $(1, 2)$.

Arc Length

Problem 11 Set up the integral for the arc length of the curve $x = t + \sin(t)$, $y = \cos(t)$ from $t = 0$ to $t = \pi$.

Solution $L = \int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} dt$. $\frac{dx}{dt} = 1 + \cos(t)$, $\frac{dy}{dt} = -\sin(t)$.

$$\begin{aligned}(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 &= (1+\cos(t))^2 + (-\sin(t))^2 \\ &= 1 + 2\cos(t) + \cos^2(t) + \sin^2(t) \\ &= 1 + 2\cos(t) + 1 = 2 + 2\cos(t)\end{aligned}$$

$L = \int_0^\pi \sqrt{2 + 2\cos(t)} dt$.

Problem 12 Using the result from Problem 11 and the identity $1 + \cos(t) = 2\cos^2(t/2)$, find the exact arc length.

Solution $L = \int_0^\pi \sqrt{2(1+\cos(t))}dt = \int_0^\pi \sqrt{2(2\cos^2(t/2))}dt = \int_0^\pi \sqrt{4\cos^2(t/2)}dt$. $L = \int_0^\pi 2|\cos(t/2)|dt$. For t in $[0,\pi]$, $t/2$ is in $[0,\pi/2]$, where cosine is non-negative. So $|\cos(t/2)| = \cos(t/2)$. $L = \int_0^\pi 2\cos(t/2)dt = [2 \cdot 2\sin(t/2)]_0^\pi = [4\sin(t/2)]_0^\pi = 4\sin(\pi/2) - 4\sin(0) = 4(1) - 0 = 4$.

Problem 13 Find the arc length of the curve $x = \frac{1}{3}t^3$, $y = \frac{1}{2}t^2$ from $t = 0$ to $t = 3$.

Solution $\frac{dx}{dt} = t^2$, $\frac{dy}{dt} = t$. $(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 = (t^2)^2 + (t)^2 = t^4 + t^2 = t^2(t^2+1)$. $L = \int_0^3 \sqrt{t^2(t^2+1)}dt = \int_0^3 t\sqrt{t^2+1}dt$. (Since $t \geq 0$) Use u-substitution: $u = t^2+1$, $du = 2tdt \implies \frac{1}{2}du = tdt$. Bounds: $t = 0 \implies u = 1$, $t = 3 \implies u = 10$. $L = \int_1^{10} \frac{1}{2}\sqrt{u}du = \frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]_1^{10} = \frac{1}{3}(10^{3/2} - 1^{3/2}) = \frac{1}{3}(10\sqrt{10} - 1)$.

Problem 14 Find the length of the curve $x = e^t + e^{-t}$, $y = 5 - 2t$ for $0 \leq t \leq 3$. (Perfect Square Trick)

Solution $\frac{dx}{dt} = e^t - e^{-t}$, $\frac{dy}{dt} = -2$.

$$\begin{aligned}(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 &= (e^t - e^{-t})^2 + (-2)^2 \\ &= (e^{2t} - 2e^te^{-t} + e^{-2t}) + 4 \\ &= e^{2t} - 2 + e^{-2t} + 4 \\ &= e^{2t} + 2 + e^{-2t} = (e^t + e^{-t})^2\end{aligned}$$

$L = \int_0^3 \sqrt{(e^t + e^{-t})^2}dt = \int_0^3 (e^t + e^{-t})dt = [e^t - e^{-t}]_0^3$. $L = (e^3 - e^{-3}) - (e^0 - e^0) = e^3 - e^{-3}$.

Problem 15 Find the arc length of the astroid $x = \cos^3(t)$, $y = \sin^3(t)$ for $0 \leq t \leq 2\pi$.

Solution Due to symmetry, we can calculate the length in the first quadrant ($0 \leq t \leq \pi/2$) and multiply by 4. $\frac{dx}{dt} = 3\cos^2(t)(-\sin(t))$, $\frac{dy}{dt} = 3\sin^2(t)(\cos(t))$.

$$\begin{aligned}(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 &= 9\cos^4(t)\sin^2(t) + 9\sin^4(t)\cos^2(t) \\ &= 9\sin^2(t)\cos^2(t)(\cos^2(t) + \sin^2(t)) \\ &= 9\sin^2(t)\cos^2(t)\end{aligned}$$

The integrand is $\sqrt{9\sin^2(t)\cos^2(t)} = 3|\sin(t)\cos(t)|$. In the first quadrant, $\sin(t)$ and $\cos(t)$ are positive, so we use $3\sin(t)\cos(t)$. Length of one quadrant: $L_1 = \int_0^{\pi/2} 3\sin(t)\cos(t)dt$. Let $u = \sin(t)$, $du = \cos(t)dt$. Bounds: $t = 0 \implies u = 0$, $t = \pi/2 \implies u = 1$. $L_1 = \int_0^1 3udu = \left[\frac{3}{2}u^2\right]_0^1 = \frac{3}{2}$. Total length $L = 4 \cdot L_1 = 4 \cdot \frac{3}{2} = 6$.

Area

Problem 16 Find the area enclosed by the ellipse $x = a\cos(t)$, $y = b\sin(t)$ for $0 \leq t \leq 2\pi$.

Solution $A = \int_{t_1}^{t_2} y(t)x'(t)dt$. The curve is traced counter-clockwise. To get a positive area, we can integrate over the top half from right to left ($t = 0$ to $t = \pi$) and multiply by -1, then double it, or integrate over the whole curve. Let's trace from $t = 2\pi$ to $t = 0$ to go clockwise for a positive result. $x'(t) = -a\sin(t)$. $A = \int_{2\pi}^0 (b\sin(t))(-a\sin(t))dt = \int_{2\pi}^0 -ab\sin^2(t)dt = ab\int_0^{2\pi} \sin^2(t)dt$. Using $\sin^2(t) = \frac{1-\cos(2t)}{2}$: $A = ab\int_0^{2\pi} \frac{1-\cos(2t)}{2}dt = \frac{ab}{2}\left[t - \frac{1}{2}\sin(2t)\right]_0^{2\pi}$. $A = \frac{ab}{2}((2\pi - 0) - (0 - 0)) = \frac{ab}{2}(2\pi) = \pi ab$.

Problem 17 Find the area under one arch of the cycloid $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$.

Solution One arch is traced from $\theta = 0$ to $\theta = 2\pi$. $x'(t) = r(1 - \cos\theta)$. $A = \int_0^{2\pi} y(\theta)x'(\theta)d\theta = \int_0^{2\pi} r(1 - \cos\theta) \cdot r(1 - \cos\theta)d\theta$. $A = r^2\int_0^{2\pi}(1 - \cos\theta)^2 d\theta = r^2\int_0^{2\pi}(1 - 2\cos\theta + \cos^2\theta)d\theta$. Using $\cos^2\theta = \frac{1+\cos(2\theta)}{2}$: $A = r^2\int_0^{2\pi}(1 - 2\cos\theta + \frac{1}{2} + \frac{1}{2}\cos(2\theta))d\theta$. $A = r^2\int_0^{2\pi}(\frac{3}{2} - 2\cos\theta + \frac{1}{2}\cos(2\theta))d\theta$. $A = r^2\left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin(2\theta)\right]_0^{2\pi}$. $A = r^2((\frac{3}{2}(2\pi) - 0 + 0) - (0 - 0 + 0)) = r^2(3\pi) = 3\pi r^2$.

Problem 18 Find the area of the region enclosed by the curve $x = t^2 - 2t$, $y = \sqrt{t}$ and the y-axis.

Solution The curve intersects the y-axis when $x = 0$. $t^2 - 2t = t(t-2) = 0 \implies t = 0, t = 2$. The portion of the curve is traced for t from 0 to 2. $x'(t) = 2t - 2$. $A = \int_0^2 y(t)x'(t)dt = \int_0^2 \sqrt{t}(2t-2)dt = \int_0^2 (2t^{3/2} - 2t^{1/2})dt$. Note: at $t = 1$, $x(1) = -1$, $x(0) = 0, x(2) = 0$. The curve traces from right-to-left for $t \in [0,1]$ and left-to-right for $t \in [1,2]$. The area integral will be negative. We should take the absolute value. $A = \left| \left[2\frac{t^{5/2}}{5/2} - 2\frac{t^{3/2}}{3/2} \right]_0^2 \right| = \left| \left[\frac{4}{5}t^{5/2} - \frac{4}{3}t^{3/2} \right]_0^2 \right|$. $A = \left| (\frac{4}{5}2^{5/2} - \frac{4}{3}2^{3/2}) - 0 \right| = \left| \frac{4}{5}(4\sqrt{2}) - \frac{4}{3}(2\sqrt{2}) \right|$. $A = \left| \frac{16\sqrt{2}}{5} - \frac{8\sqrt{2}}{3} \right| = \left| \frac{48\sqrt{2} - 40\sqrt{2}}{15} \right| = \frac{8\sqrt{2}}{15}$.

Mixed and Challenging Problems

Problem 19 For the curve $x = t^3 + 1, y = t^2 - t$, find the equation of the tangent line at the point $(9, -2)$.

Solution Find t: $x(t) = t^3 + 1 = 9 \implies t^3 = 8 \implies t = 2$. Let's check with y: $y(-2) = (-2)^2 - (-2) = 4 + 2 = 6 \neq -2$. Wait, there is a typo in the question point. Let's assume the question meant $y = t - t^2$. $y(2) = 2 - 2^2 = -2$. This works. Let's proceed with $y = t - t^2$. Slope: $\frac{dx}{dt} = 3t^2$, $\frac{dy}{dt} = 1 - 2t$. $m = \frac{1-2t}{3t^2}|_{t=2} = \frac{1-4}{3(4)} = \frac{-3}{12} = -\frac{1}{4}$. Equation: $y - (-2) = -\frac{1}{4}(x - 9) \implies y + 2 = -\frac{1}{4}x + \frac{9}{4} \implies y = -\frac{1}{4}x + \frac{1}{4}$.

Problem 20 A particle's position is given by $x(t) = 2\sin(t)$, $y(t) = \cos(2t)$. Find all points where the particle is momentarily stopped.

Solution The particle is stopped when its speed is zero, which means both $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are zero simultaneously. $\frac{dx}{dt} = 2\cos(t) = 0 \implies t = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$ $\frac{dy}{dt} = -2\sin(2t) = -2(2\sin(t)\cos(t)) = -4\sin(t)\cos(t) = 0$. This is zero when $\sin(t) = 0$ or $\cos(t) = 0$. The values of t for which both derivatives are zero are when $\cos(t) = 0$, i.e., $t = \frac{\pi}{2} + n\pi$ for any integer n. At these times, the particle stops. Let's find the points: If $t = \pi/2$, $(x, y) = (2\sin(\pi/2), \cos(\pi)) = (2, -1)$. If $t = 3\pi/2$, $(x, y) = (2\sin(3\pi/2), \cos(3\pi)) = (-2, -1)$. The particle stops at $(2, -1)$ and $(-2, -1)$.

Problem 21 Find $\frac{d^2y}{dx^2}$ for the curve $x = a\cos(t)$, $y = b\sin(t)$ and interpret the result for concavity.

Solution $\frac{dx}{dt} = -a\sin(t)$, $\frac{dy}{dt} = b\cos(t)$. $\frac{dy}{dx} = \frac{b\cos(t)}{-a\sin(t)} = -\frac{b}{a}\cot(t)$. $\frac{d}{dt}(\frac{dy}{dx}) = -\frac{b}{a}(-\csc^2(t)) = \frac{b}{a}\csc^2(t)$. $\frac{d^2y}{dx^2} = \frac{\frac{b}{a}\csc^2(t)}{-a\sin(t)} = -\frac{b}{a^2\sin^3(t)}$. Concavity: If $0 < t < \pi$, $\sin(t) > 0$, so $\frac{d^2y}{dx^2} < 0$. The top half of the ellipse is concave down. If $\pi < t < 2\pi$, $\sin(t) < 0$, so $\frac{d^2y}{dx^2} > 0$. The bottom half of the ellipse is concave up. This matches our geometric intuition.

Problem 22 Set up, but do not evaluate, an integral for the surface area generated by rotating the curve $x = t^3, y = t^2, 0 \leq t \leq 1$ about the x-axis.

Solution $S = \int_a^b 2\pi y(t)\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}dt$. $\frac{dx}{dt} = 3t^2$, $\frac{dy}{dt} = 2t$. The radical term is $\sqrt{(3t^2)^2 + (2t)^2} = \sqrt{9t^4 + 4t^2} = \sqrt{t^2(9t^2 + 4)} = t\sqrt{9t^2 + 4}$ (for $t \geq 0$). $S = \int_0^1 2\pi(t^2)(t\sqrt{9t^2 + 4})dt = \int_0^1 2\pi t^3\sqrt{9t^2 + 4}dt$.

Problem 23 Find the total distance traveled by a particle whose position is given by $x = 3\cos^2(t), y = 3\sin^2(t)$ for $0 \leq t \leq \pi$.

Solution This is an arc length problem. $\frac{dx}{dt} = 3 \cdot 2\cos(t)(-\sin(t)) = -6\cos(t)\sin(t)$. $\frac{dy}{dt} = 3 \cdot 2\sin(t)(\cos(t)) = 6\cos(t)\sin(t)$. $(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 = 36\cos^2(t)\sin^2(t) + 36\cos^2(t)\sin^2(t) = 72\cos^2(t)\sin^2(t)$. $L = \int_0^\pi \sqrt{72\cos^2(t)\sin^2(t)}dt = \int_0^\pi \sqrt{72}|\cos(t)\sin(t)|dt$. $\sqrt{72} = 6\sqrt{2}$. $L = 6\sqrt{2}\int_0^\pi |\cos(t)\sin(t)|dt$. Since $\sin(t) \geq 0$ on $[0, \pi]$, we only care about the sign of $\cos(t)$. $L = 6\sqrt{2}\left(\int_0^{\pi/2} \cos(t)\sin(t)dt + \int_{\pi/2}^\pi -\cos(t)\sin(t)dt\right)$.

Let $u = \sin(t)$, $du = \cos(t)dt$. $\int \cos(t)\sin(t)dt = \int udu = \frac{1}{2}u^2 = \frac{1}{2}\sin^2(t)$. $L = 6\sqrt{2}\left(\left[\frac{1}{2}\sin^2(t)\right]_0^{\pi/2} - \left[\frac{1}{2}\sin^2(t)\right]_{\pi/2}^{\pi}\right)$. $L = 6\sqrt{2}\left((\frac{1}{2}(1)^2 - 0) - (\frac{1}{2}(0)^2 - \frac{1}{2}(1)^2)\right) = 6\sqrt{2}(\frac{1}{2} + \frac{1}{2}) = 6\sqrt{2}$.

Problem 24 Find the area of the region bounded by the x-axis and the curve $x = t^3 + t, y = 1 - t^2$.

Solution The curve intersects the x-axis when $y = 0$. $1 - t^2 = 0 \implies t = \pm 1$. $x(-1) = -2$, $x(1) = 2$. The curve is traced from left to right as t goes from -1 to 1. $x'(t) = 3t^2 + 1$. $A = \int_{-1}^{1}(1 - t^2)(3t^2+1)dt = \int_{-1}^{1}(3t^2+1-3t^4-t^2)dt$. $A = \int_{-1}^{1}(-3t^4+2t^2+1)dt$. Since the integrand is an even function: $A = 2\int_0^1(-3t^4 + 2t^2 + 1)dt = 2\left[-\frac{3}{5}t^5 + \frac{2}{3}t^3 + t\right]_0^1$. $A = 2(-\frac{3}{5} + \frac{2}{3} + 1) = 2(\frac{-9+10+15}{15}) = 2(\frac{16}{15}) = \frac{32}{15}$.

Problem 25 The velocity components of a particle are $\frac{dx}{dt} = t^2$ and $\frac{dy}{dt} = \sqrt{t}$. What is the acceleration vector $\vec{a}(t)$ and the slope of the curve at $t = 4$?

Solution The velocity vector is $\vec{v}(t) = \langle t^2, \sqrt{t}\rangle$. The acceleration vector is the derivative of the velocity vector: $\vec{a}(t) = \langle \frac{d}{dt}(t^2), \frac{d}{dt}(\sqrt{t})\rangle = \langle 2t, \frac{1}{2\sqrt{t}}\rangle$.

The slope of the curve is $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\sqrt{t}}{t^2} = t^{-3/2}$. At $t = 4$, the slope is $4^{-3/2} = (4^{1/2})^{-3} = 2^{-3} = \frac{1}{8}$.

Problem 26 Find the arc length of $x = t^2$, $y = 2t$ from $t = 0$ to $t = \sqrt{3}$.

Solution $\frac{dx}{dt} = 2t$, $\frac{dy}{dt} = 2$. $(\frac{dx}{dt})^2+(\frac{dy}{dt})^2 = (2t)^2+2^2 = 4t^2+4 = 4(t^2+1)$. $L = \int_0^{\sqrt{3}}\sqrt{4(t^2+1)}dt = \int_0^{\sqrt{3}} 2\sqrt{t^2+1}dt$. This requires a trig substitution. Let $t = \tan\theta$, $dt = \sec^2\theta d\theta$. $L = \int_0^{\pi/3} 2\sqrt{\tan^2\theta + 1}\sec^2\theta d\theta = \int_0^{\pi/3} 2\sec^3\theta d\theta$. Using the reduction formula $\int \sec^n(x)dx = \frac{\sec^{n-2}(x)\tan(x)}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}(x)dx$: $L = 2\left[\frac{\sec\theta\tan\theta}{2} + \frac{1}{2}\int\sec\theta d\theta\right]_0^{\pi/3} = [\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|]_0^{\pi/3}$. $L = (\sec(\pi/3)\tan(\pi/3) + \ln|\sec(\pi/3) + \tan(\pi/3)|) - (\sec(0)\tan(0) + \ln|\sec(0) + \tan(0)|)$. $L = (2\sqrt{3} + \ln|2+\sqrt{3}|) - (0 + \ln|1+0|) = 2\sqrt{3} + \ln(2+\sqrt{3})$.

Problem 27 Consider the curve $x = t^2, y = kt^3 - t^2$. Find the value of k such that the curve has a vertical tangent at $t = 0$. Explain your reasoning.

Solution A vertical tangent requires $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$. $\frac{dx}{dt} = 2t$. This is zero at $t = 0$. $\frac{dy}{dt} = 3kt^2 - 2t$. At $t = 0$, $\frac{dy}{dt} = 3k(0)^2 - 2(0) = 0$. Since both derivatives are zero at $t = 0$, the slope is of the indeterminate form $0/0$. There is no value of k for which the tangent is strictly vertical at $t = 0$ based on the standard definition. Using L'Hopital's rule on the slope: $\lim_{t\to 0}\frac{dy/dt}{dx/dt} = \lim_{t\to 0}\frac{3kt^2-2t}{2t} = \lim_{t\to 0}\frac{6kt-2}{2} = -1$. The slope approaches -1, so the curve has a defined tangent at the origin, but it is not vertical.

Problem 28 A curve is given by $x = \sin(t), y = \sin(2t)$. Find the area of the loop enclosed by the curve.

Solution The curve creates a loop. We need to find the t-values where it self-intersects. $\sin(t_1) = \sin(t_2)$ and $\sin(2t_1) = \sin(2t_2)$ for $t_1 \neq t_2$. This occurs for example when $t_1 = 0$ and $t_2 = \pi$. $x(0) = 0, y(0) = 0$. $x(\pi) = 0, y(\pi) = 0$. The loop is traced between $t = 0$ and $t = \pi$. $x'(t) = \cos(t)$. $A = \int_0^{\pi} y(t)x'(t)dt = \int_0^{\pi}\sin(2t)\cos(t)dt$. $A = \int_0^{\pi}(2\sin(t)\cos(t))\cos(t)dt = \int_0^{\pi} 2\sin(t)\cos^2(t)dt$. Let $u = \cos(t)$, $du = -\sin(t)dt$. Bounds: $t = 0 \implies u = 1$, $t = \pi \implies u = -1$. $A = \int_1^{-1} 2u^2(-du) = \int_{-1}^{1} 2u^2du = 2[\frac{u^3}{3}]_{-1}^1 = \frac{2}{3}(1^3 - (-1)^3) = \frac{2}{3}(2) = \frac{4}{3}$.

Problem 29 The curve $x = \sec(t), y = \tan(t)$ for $-\pi/2 < t < \pi/2$ is a hyperbola. Find its Cartesian equation and use it to find $\frac{dy}{dx}$. Verify your answer using parametric differentiation.

Solution We know the identity $1 + \tan^2(t) = \sec^2(t)$. Substituting x and y: $1 + y^2 = x^2 \implies x^2 - y^2 = 1$. Differentiating with respect to x: $2x - 2y\frac{dy}{dx} = 0 \implies \frac{dy}{dx} = \frac{2x}{2y} = \frac{x}{y}$.

Using parametric differentiation: $\frac{dx}{dt} = \sec(t)\tan(t)$, $\frac{dy}{dt} = \sec^2(t)$. $\frac{dy}{dx} = \frac{\sec^2(t)}{\sec(t)\tan(t)} = \frac{\sec(t)}{\tan(t)} = \frac{1/\cos(t)}{\sin(t)/\cos(t)} = \frac{1}{\sin(t)} = \csc(t)$. To verify they are the same: $\frac{x}{y} = \frac{\sec(t)}{\tan(t)} = \csc(t)$. The results match.

Problem 30 Explain the "second derivative trap". For the curve $x = t^3, y = t^2$, show that using the trap formula $\frac{y''(t)}{x''(t)}$ gives the wrong answer for $\frac{d^2y}{dx^2}$.

Solution The "second derivative trap" is the common mistake of thinking that $\frac{d^2y}{dx^2}$ is equal to the ratio of the second derivatives with respect to the parameter t, i.e., $\frac{d^2y/dt^2}{d^2x/dt^2}$. This is incorrect because the chain rule must be applied to the first derivative, $\frac{dy}{dx}$, which is itself a function of t.

For $x = t^3, y = t^2$: $x'(t) = 3t^2, y'(t) = 2t$. $x''(t) = 6t, y''(t) = 2$. The incorrect trap formula gives: $\frac{y''(t)}{x''(t)} = \frac{2}{6t} = \frac{1}{3t}$.

The correct method: First, find $\frac{dy}{dx} = \frac{2t}{3t^2} = \frac{2}{3t}$. Next, differentiate this with respect to t: $\frac{d}{dt}\left(\frac{2}{3t}\right) = -\frac{2}{3t^2}$. Finally, divide by $\frac{dx}{dt}$: $\frac{d^2y}{dx^2} = \frac{-2/(3t^2)}{3t^2} = -\frac{2}{9t^4}$. Clearly, $-\frac{2}{9t^4} \neq \frac{1}{3t}$, demonstrating that the trap formula is wrong.

Concept Checklist and Problem Index

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