Deconstruction of Option Skewness & Kelly Criterion

Mathematical First Principles & Stress Testing

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Contents

1 Mathematical First Principles

1.1 Variable Dictionary

Underlying Dynamics

- S_t: The spot price of the underlying asset at time t.
- μ_{drift}: The real-world drift (expected return) of the underlying asset.
- r: The risk-free interest rate.
- σ: The constant volatility of the underlying asset returns (Geometric Brownian Motion assumption).
- dW_t: Increment of a Wiener process (Brownian motion).

Option Parameters

- C, P: The value of a European Call and Put respectively.
- X (or K): The strike price.
- $\tau = T - t$: Time to expiration.
- d_1, d_2: Standard Black-Scholes-Merton standardized moneyness terms.
- $\Phi(\cdot)$ or $N(\cdot)$: The Cumulative Normal Distribution Function.
- $\phi(\cdot)$: The Standard Normal Probability Density Function.

Kelly / Moments Notation

- B_n: Bankroll after n trades.
- f: The fraction of the bankroll wagered (The Kelly Fraction).
- $g(x)$: The return payoff function of the trade.
- μ: The expected return (first moment) of the *option position* (distinct from μ_{drift} of spot).
- σ^2_{opt}: The variance of the option position returns.
- λ_k: The k-th raw moment of the option return distribution ($E[r^k]$).
- γ_1: Skewness (Standardized third moment).
- κ: Kurtosis (Standardized fourth moment).

1.2 Core Dynamics (SDEs)

The paper builds its foundation on the standard Black-Scholes-Merton framework to isolate the skewness inherent in the *contract structure* rather than the underlying. Therefore, the underlying spot price S_t follows **Geometric Brownian Motion (GBM)**.

The SDE under the physical measure $\mathbb{P}$ is:

$$dS_t = \mu_{drift} S_t dt + \sigma S_t dW_t$$

The SDE under the risk-neutral measure $\mathbb{Q}$ (used for pricing the moments) is:

$$dS_t = r S_t dt + \sigma S_t dW_t^{\mathbb{Q}}$$

1.3 The Intuition

The Generator of Skewness:
Usually, skewness in finance arises from the "Leverage Effect" (spot down, vol up) modeled by stochastic volatility (e.g., Heston). However, this paper purposefully sets standard volatility σ to a constant. The skewness here arises purely from the **non-linear transformation** of the log-normal probability density function (PDF) of S_T by the option payoff function $\max(S_T - K, 0)$.

- **Long Call (C):** The payoff is convex. It truncates the left tail (limited loss) and extends the right tail (unlimited gain). This forces positive skewness ($\gamma_1 > 0$).
- **Short Call ($-C$):** The payoff is concave. It limits the gain (premium collected) and exposes the trader to unlimited downside. This forces negative skewness ($\gamma_1 < 0$).

The paper demonstrates that even in a "perfect" BSM world, a Mean-Variance optimizer (Standard Kelly) fails because it assumes the return distribution of the bet is symmetric (Gaussian), whereas the option payoff transforms a Log-Gaussian input into a highly skewed output.

1.4 The "Trick": Analytic Moment Integration & Taylor Series Utility

The authors utilize two distinct mathematical techniques to close the system:

Trick A: Analytical Integration of Moments

Instead of using Characteristic Functions (Fourier Transforms), they exploit the properties of the Log-Normal distribution. Since S_T is Log-Normal, S_T^n is also Log-Normal. They derive the n-th raw moment of the Call option $E[C^n]$ directly:

$$E[C^n] = \int_0^\infty (S - X)^n \cdot p_{lognorm}(S)\, dS$$

By expanding $(S - X)^n$ using the Binomial Theorem, they obtain a sum of integrals that can be solved in closed form using modified BSM terms (shifting the d_1, d_2 terms by $n\sigma\sqrt{T}$).

Trick B: Taylor Expansion of the Kelly Criterion

The objective is to maximize the expected geometric growth rate $g(f)$:

$$g(f) = E\left[\ln(1 + fr)\right]$$

Since the PDF of returns r is non-trivial, numerical integration is usually required. The authors apply a Taylor Maclaurin series expansion to the function $H(f) = \ln(1 + fr)$ around $f = 0$:

$$\ln(1 + fr) \approx fr - \frac{1}{2}f^2r^2 + \frac{1}{3}f^3r^3 - \dots$$

Taking the expectation $E[\cdot]$ of both sides allows them to express the utility function in terms of the raw moments $\lambda_k = E[r^k]$:

$$E[\ln(1 + fr)] \approx f\mu - \frac{1}{2}f^2\lambda_2 + \frac{1}{3}f^3\lambda_3 - \frac{1}{4}f^4\lambda_4$$

Differentiation with respect to f and setting to 0 yields a polynomial (Cubic in the 4th order expansion) for the optimal fraction f:

$$0 = \mu - f\lambda_2 + f^2\lambda_3 - f^3\lambda_4$$

2 "Find the Flaw" (Stress Testing)

2.1 Corner Cases

Case A: The "Singularity" of Zero Skew

The paper notes that the quadratic approximation (truncating at 3rd moment) for the optimal fraction is:

$$f_{opt} \approx \frac{\lambda_2 \pm \sqrt{\lambda_2^2 - 4\lambda_3\mu}}{2\lambda_3}$$

As the skewness $\lambda_3 \to 0$, the denominator vanishes, creating a singularity. While the limit exists (converging to μ/λ_2), a numerical solver in a production engine could crash or output `NaN` if the skew passes through zero (e.g., an ATM option exactly at the money where skewness transitions).

Case B: The Taylor Divergence (Large Drawdowns)

The Taylor expansion of $\ln(1+x)$ is only valid for $|x| < 1$, and converges slowly as x approaches -1.

- In a "Short Volatility" strategy (e.g., Short Straddles), the return r in a crash scenario is often -100% or worse (if leveraged).
- When $f \cdot r \approx -1$ (ruin), the logarithm tends to $-\infty$.
- The polynomial approximation (Eq 34) sees this merely as a large negative number in the power series, completely failing to capture the **barrier of ruin** (logarithmic singularity).
- **Result:** The model may suggest a bet size that is "safe" by moment standards but mathematically allows for bankruptcy because the Taylor approximation smoothed out the "wall of death" at -100%.

2.2 Parameter Stability

The "Edge" (μ) Sensitivity

The optimal fraction f is linearly (or heavily) dependent on μ (Expected Return).

$$\mu = \text{Theoretical Price} - \text{Market Price}$$

In options, the "Expected Return" is derived from the difference between *Realized Volatility* (future) and *Implied Volatility* (current).

- **Flaw:** Realized Volatility is a stochastic variable. The paper assumes a fixed "Edge" (e.g., \$5).
- If Realized Vol spikes momentarily, the Edge μ becomes negative. The polynomial root will flip signs, suggesting a *short* position becomes a *long* position instantaneously. This creates "flickering" alpha that generates excessive transaction costs.

Implicit Normal Distribution of Underlying

The derivation assumes S_T is Log-Normal.

- **Reality:** Market returns exhibit excess kurtosis (Fat Tails) and negative skewness independent of the option structure.

- **Impact:** The calculated λ_3 (Skew) and λ_4 (Kurtosis) in the paper are **underestimates** of the true risk. The "Intrinsic Option Skew" is compounded by the "Underlying Jump Skew."

- Using this model to size Short Put positions will result in over-betting, as it assumes the underlying asset cannot gap down (Diffusion only).

3 Implementation Guide

3.1 Prerequisites & Mathematical Requirements

To implement the *Skew-Adjusted Kelly Criterion* for options, the quantitative developer requires the following foundational knowledge and libraries:

- **Mathematical Frameworks:**
  - **Black-Scholes-Merton (BSM) Mechanics:** Understanding d_1, d_2 and the pricing of Vanilla Europeans.
  - **Moment Calculus:** Ability to manipulate raw moments $E[x^n]$ and convert them to central moments (Variance, Skewness, Kurtosis).
  - **Numerical Root Finding:** Newton-Raphson or Brent's Method to solve cubic polynomials.

- **Software Stack:**
  - Python (NumPy/SciPy) or C++ (QuantLib/Boost).
  - A root-finding algorithm (e.g., `scipy.optimize.brentq`).

3.2 Step-by-Step Recreation Logic

Step 1: The Input Vector

Define the state of the market. The model requires two distinct volatility inputs to calculate the "Edge" (μ).

- **Market Data:** Spot (S), Strike (K), Time to Maturity (T), Risk-Free Rate (r), Market Price (P_{mkt}).

- **Implied Volatility (σ_{imp}):** Backed out from P_{mkt}.

- **Forecast Volatility (σ_{fcst}):** The trader's estimate of future realized volatility (e.g., from GARCH or realized estimator).

Step 2: The Moment Engine (Appendix A Implementation)

This is the computational core. You must implement the closed-form analytical equations for the moments of option value as derived in the paper's Appendix.

Pseudocode Logic:

1. Calculate baseline BSM parameters d_1, d_2 using σ_{fcst}.

2. **First Moment (μ_{opt}):** Calculate Expected Option Payoff at expiry using BSM logic but with drift $\mu_{drift} = 0$ (if assuming driftless) or trader's drift.

3. **Higher Moments (λ_k):** Implement the raw moment integrals.

$$\lambda_n = E[\text{Payoff}^n]$$

Note: For a Call option, calculating λ_3 involves terms like $N(3d_1 - 2d_2)$. Ensure your Normal CDF function has high precision.

4. **Standardization:** Convert raw payoff moments into return moments:

$$\mu_{return} = \frac{E[\text{Payoff}] - P_{mkt}}{P_{mkt}}, \quad \lambda_{n,return} = E\left[\left(\frac{\text{Payoff} - P_{mkt}}{P_{mkt}}\right)^n\right]$$

Step 3: The Polynomial Solver

Construct the Taylor-expanded utility function derivative (Equation 34 in the paper).

$$P(f) = \mu_{return} - f\lambda_2 + f^2\lambda_3 - f^3\lambda_4 = 0$$

- Use a numerical solver to find roots for f.
- **Filter Roots:** Discard complex roots. Discard roots where $f < 0$ (unless shorting) or $f > 1$ (unrealistic leverage constraints).
- **Selection:** If multiple real positive roots exist, select the smallest positive root to remain conservative.

3.3 Calibration Strategy

- **The "Edge" (μ):** This is the most sensitive parameter.

$$\mu \approx \text{BSM}(S, K, T, \sigma_{fcst}) - P_{mkt}(\sigma_{imp})$$

  Do *not* use historical mean returns of options. Use the spread between your volatility forecast and the market's implied volatility.

- **Safety Caps:** The Taylor approximation diverges if returns approach -100%. *Heuristic:* Hard-code a "Ruin Constraint."

$$f_{final} = \min(f_{poly}, \frac{1}{\text{Max Loss Scenario}})$$

4 Critical Analysis: Assumptions & Flaws

4.1 The "Hidden" Assumptions

Assumption A: The Taylor Series Validity Domain

The derivation relies on the expansion $\ln(1 + x) \approx x - x^2/2 + x^3/3$.

- **The Flaw:** This series converges only for $|x| < 1$.
- **Reality Check:** In short option strategies (e.g., Short Straddles), a tail event can cause a loss of 500% or more (if unhedged or leveraged). In these regions ($x \ll -1$), the Taylor approximation is mathematically invalid and underestimates the penalty for ruin.
- **Risk:** The model might suggest a bet size that allows for a -100% portfolio wipeout because the polynomial "smooths over" the singularity of $\ln(0)$.

Assumption B: Log-Normality of the Underlying

The paper derives option skewness assuming the underlying stock follows Geometric Brownian Motion (GBM).

- **The Flaw:** It assumes the underlying asset has zero skew and excess kurtosis of 0.
- **Reality Check:** Equity indices have intrinsic negative skew (crashes are faster than rallies) and fat tails.
- **Consequence:** The paper calculates the skewness *generated by the contract*, but ignores the skewness *inherited from the asset*. This leads to a systematic underestimation of risk for Short Put strategies.

4.2 Practitioner Reality

- **Discrete Hedging vs. Continuous Theory:** The model assumes a "Buy and Hold" or single-period bet. It does not account for the path-dependency of delta-hedging or margin calls that occur *during* the life of the option.
- **Liquidity Gaps:** The math assumes a continuous probability density function. It cannot price "Gap Risk"—the scenario where the market closes at \$100 and opens at \$80. In such a scenario, the "Limited Loss" assumption of a calendar spread or the "Dynamic Hedge" capability vanishes.
- **Parameter Instability:** The optimal f is highly sensitive to the difference between σ_{imp} and σ_{real}. Since σ_{real} is an estimate, estimation error is leveraged. A small error in forecasting volatility can flip the sign of the Kelly fraction (telling you to go Long instead of Short).