

Extensive Problem Set for Chapter 11.9: Representation of Functions as Power Series

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Practice Problems

This problem set contains 65 problems designed to test all concepts related to representing functions as power series, as detailed in your study guide. The problems gradually increase in complexity.

Problem 1

Find a power series representation for the function $f(x) = \frac{1}{1+x}$ and determine its interval of convergence.

Problem 2

Find a power series representation for the function $f(x) = \frac{3}{1-x}$ and determine its interval of convergence.

Problem 3

Find a power series representation for the function $f(x) = \frac{1}{2-x}$ and determine its interval of convergence.

Problem 4

Find a power series representation for the function $f(x) = \frac{5}{3+x}$ and determine its interval of convergence.

Problem 5

Find a power series representation for the function $f(x) = \frac{1}{1-x^3}$ and determine its interval of convergence.

Problem 6

Find a power series representation for the function $f(x) = \frac{1}{1+x^4}$ and determine its interval of convergence.

Problem 7

Find a power series representation for the function $f(x) = \frac{1}{4-x^2}$ and determine its interval of convergence.

Problem 8

Find a power series representation for the function $f(x) = \frac{x}{1-x}$ and determine its interval of convergence.

Problem 9

Find a power series representation for the function $f(x) = \frac{x^2}{1+x}$ and determine its interval of convergence.

Problem 10

Find a power series representation for the function $f(x) = \frac{3x^4}{1-x^2}$ and determine its interval of convergence.

Problem 11

Find a power series representation for the function $f(x) = \frac{x}{9+x^2}$ and determine its interval of convergence.

Problem 12

Find a power series representation for the function $f(x) = \frac{x^3}{2-x^2}$ and determine its interval of convergence.

Problem 13

Use differentiation to find a power series representation for $f(x) = \frac{1}{(1-x)^2}$. What is the radius of convergence?

Problem 14

Use differentiation to find a power series representation for $f(x) = \frac{1}{(1+x)^2}$. What is the radius of convergence?

Problem 15

Use differentiation to find a power series representation for $f(x) = \frac{2}{(5-x)^2}$. What is the radius of convergence?

Problem 16

Use the result from Problem 13 to find a power series for $f(x) = \frac{2x}{(1-x)^2}$.

Problem 17

Use differentiation to find a power series representation for $f(x) = \frac{x}{(1+x^2)^2}$. What is the radius of convergence?

Problem 18

Use differentiation twice to find a power series for $f(x) = \frac{2}{(1-x)^3}$. What is the radius of convergence?

Problem 19

Use integration to find a power series representation for $f(x) = \ln(1+x)$. What is the radius of convergence?

Problem 20

Use integration to find a power series representation for $f(x) = \ln(1-x)$. What is the radius of convergence?

Problem 21

Use integration to find a power series representation for $f(x) = \ln(3+x)$. What is the radius of convergence?

Problem 22

Use integration to find a power series representation for $f(x) = \ln(1+x^2)$. What is the radius of convergence?

Problem 23

Find a power series representation for the function $f(x) = x \ln(1+x)$.

Problem 24

Find a power series representation for the function $f(x) = x^2 \ln(1 - x^2)$.

Problem 25

Find a power series representation for the function $f(x) = \arctan(x)$. What is the radius of convergence?

Problem 26

Find a power series representation for the function $f(x) = \arctan(x^3)$.

Problem 27

Find a power series representation for the function $f(x) = x \arctan(x)$.

Problem 28

Find a power series representation for the function $f(x) = \frac{x+2}{x-1}$. (Hint: Use algebraic manipulation first).

Problem 29

Find a power series representation for the function $f(x) = \frac{x^2}{x+3}$. (Hint: Use algebraic manipulation first).

Problem 30

Evaluate the indefinite integral $\int \frac{1}{1+x^5} dx$ as a power series. What is the radius of convergence?

Problem 31

Evaluate the indefinite integral $\int \frac{x}{1-x^4} dx$ as a power series. What is the radius of convergence?

Problem 32

Evaluate the indefinite integral $\int \ln(1 - x) dx$ as a power series.

Problem 33

Evaluate the indefinite integral $\int x \arctan(x^2) dx$ as a power series.

Problem 34

Use the first three non-zero terms of the series for $\arctan(x)$ to approximate $\int_0^{0.5} \arctan(x) dx$.

Problem 35

Find the sum of the series $\sum_{n=1}^{\infty} nx^{n-1}$ for $|x| < 1$.

Problem 36

Find the sum of the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ by recognizing it as the Maclaurin series for a known function.

Problem 37

Find the sum of the series $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$.

Problem 38

Find the sum of the series $\sum_{n=0}^{\infty} \frac{3^n}{5^n n!}$.

Problem 39

Find the sum of the series $1 - \ln(2) + \frac{(\ln 2)^2}{2!} - \frac{(\ln 2)^3}{3!} + \dots$.

Problem 40

Find the sum of the series $\sum_{n=2}^{\infty} \frac{x^n}{n}$. (Hint: Differentiate the series).

Problem 41

Use the binomial series to find the Maclaurin series for $f(x) = \sqrt{1+x}$. What is the radius of convergence?

Problem 42

Use the binomial series to find the Maclaurin series for $f(x) = \frac{1}{\sqrt{1+x}}$. What is the radius of convergence?

Problem 43

Use the binomial series to find the Maclaurin series for $f(x) = \sqrt[3]{1+x}$. What is the radius of convergence?

Problem 44

Use the binomial series to find the Maclaurin series for $f(x) = \frac{1}{\sqrt{1-x^2}}$. What is the radius of convergence?

Problem 45

Use the binomial series to find the Maclaurin series for $f(x) = (1+x)^{3/2}$.

Problem 46

Find a power series representation for $f(x) = \frac{\ln(1+x)}{x}$.

Problem 47

Find a power series for $f(x) = \frac{d}{dx} \left(\frac{1}{1+x^3} \right)$.

Problem 48

Find a power series for $f(x) = \frac{x^2-1}{x-2}$.

Problem 49

Find a power series representation for $f(x) = \ln \left(\frac{1+x}{1-x} \right)$. (Hint: Use properties of logarithms).

Problem 50

Find a power series representation for $f(x) = \frac{1}{(4-x)^3}$. What is the radius of convergence?

Problem 51

Evaluate $\int \frac{e^x - 1}{x} dx$ as a power series.

Problem 52

Find a power series for $f(x) = \sin(x^2)$.

Problem 53

Use the series for e^x to find the series for $f(x) = e^{-x^2}$.

Problem 54

Evaluate $\int e^{-x^2} dx$ as a power series. This integral is fundamental in statistics.

Problem 55

Find a power series for $f(x) = \cos(\sqrt{x})$. (Assume $x \geq 0$).

Problem 56

Find a power series for $f(x) = \frac{1+x^2}{1-x^2}$. (Hint: Split the fraction).

Problem 57

Use the binomial series to find the first four terms of the Maclaurin series for $f(x) = \sqrt[4]{16-x}$.

Problem 58

Find a power series for the function $f(x) = \frac{5x-1}{x^2-x-2}$. (Hint: Use partial fraction decomposition).

Problem 59

Find the sum of the series $\sum_{n=1}^{\infty} \frac{n}{2^n}$. (Hint: Consider a function $f(x) = \sum_{n=1}^{\infty} nx^n$).

Problem 60

Find the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n \cdot 3^n}$. (Hint: Recognize the series for $\ln(1+x)$).

Problem 61

Find a power series for $f(x) = \frac{x^3}{(1-2x)^2}$.

Problem 62

Find a power series for $f(x) = \ln(x^2 + 4)$.

Problem 63

Find the sum of the series $\frac{\pi}{2} - \frac{\pi^3}{2^3 \cdot 3!} + \frac{\pi^5}{2^5 \cdot 5!} - \dots$.

Problem 64

Evaluate the indefinite integral $\int \frac{\arctan(x)}{x} dx$ as a power series.

Problem 65

Use a power series to find the limit $\lim_{x \rightarrow 0} \frac{x - \ln(1+x)}{x^2}$.

Solutions

Solution to Problem 1

The function is in the form $\frac{1}{1-r}$ with $r = -x$.

$$f(x) = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$$

The series converges for $|r| < 1 \implies |-x| < 1 \implies |x| < 1$. The interval of convergence is $(-1, 1)$.

Solution to Problem 2

The function is in the form $\frac{a}{1-r}$ with $a = 3, r = x$.

$$f(x) = \frac{3}{1-x} = 3 \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} 3x^n$$

Converges for $|x| < 1$. Interval: $(-1, 1)$.

Solution to Problem 3

Factor out 2 from the denominator:

$$f(x) = \frac{1}{2(1-x/2)} = \frac{1}{2} \cdot \frac{1}{1-(x/2)}$$

This is a geometric series with $a = 1/2, r = x/2$.

$$f(x) = \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{x^n}{2^{n+1}}$$

Converges for $|x/2| < 1 \implies |x| < 2$. Interval: $(-2, 2)$.

Solution to Problem 4

Factor out 3:

$$f(x) = \frac{5}{3(1+x/3)} = \frac{5}{3} \cdot \frac{1}{1-(-x/3)}$$

Geometric series with $a = 5/3, r = -x/3$.

$$f(x) = \frac{5}{3} \sum_{n=0}^{\infty} \left(-\frac{x}{3}\right)^n = \sum_{n=0}^{\infty} \frac{5(-1)^n x^n}{3^{n+1}}$$

Converges for $|-x/3| < 1 \implies |x| < 3$. Interval: $(-3, 3)$.

Solution to Problem 5

This is a geometric series with $r = x^3$.

$$f(x) = \frac{1}{1-x^3} = \sum_{n=0}^{\infty} (x^3)^n = \sum_{n=0}^{\infty} x^{3n}$$

Converges for $|x^3| < 1 \implies |x| < 1$. Interval: $(-1, 1)$.

Solution to Problem 6

Geometric series with $r = -x^4$.

$$f(x) = \frac{1}{1 - (-x^4)} = \sum_{n=0}^{\infty} (-x^4)^n = \sum_{n=0}^{\infty} (-1)^n x^{4n}$$

Converges for $| -x^4 | < 1 \implies |x| < 1$. Interval: $(-1, 1)$.

Solution to Problem 7

Factor out 4:

$$f(x) = \frac{1}{4(1 - x^2/4)} = \frac{1}{4} \cdot \frac{1}{1 - (x^2/4)}$$

Geometric series with $a = 1/4, r = x^2/4$.

$$f(x) = \frac{1}{4} \sum_{n=0}^{\infty} \left(\frac{x^2}{4}\right)^n = \sum_{n=0}^{\infty} \frac{x^{2n}}{4^{n+1}}$$

Converges for $|x^2/4| < 1 \implies x^2 < 4 \implies |x| < 2$. Interval: $(-2, 2)$.

Solution to Problem 8

$$f(x) = x \cdot \frac{1}{1-x} = x \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} x^{n+1}$$

Converges for $|x| < 1$. Interval: $(-1, 1)$.

Solution to Problem 9

$$f(x) = x^2 \cdot \frac{1}{1 - (-x)} = x^2 \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^{n+2}$$

Converges for $| -x | < 1 \implies |x| < 1$. Interval: $(-1, 1)$.

Solution to Problem 10

$$f(x) = 3x^4 \cdot \frac{1}{1 - x^2} = 3x^4 \sum_{n=0}^{\infty} (x^2)^n = \sum_{n=0}^{\infty} 3x^{2n+4}$$

Converges for $|x^2| < 1 \implies |x| < 1$. Interval: $(-1, 1)$.

Solution to Problem 11

$$f(x) = x \cdot \frac{1}{9(1 + x^2/9)} = \frac{x}{9} \cdot \frac{1}{1 - (-x^2/9)} = \frac{x}{9} \sum_{n=0}^{\infty} \left(-\frac{x^2}{9}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{9^{n+1}}$$

Converges for $| -x^2/9 | < 1 \implies x^2 < 9 \implies |x| < 3$. Interval: $(-3, 3)$.

Solution to Problem 12

$$f(x) = x^3 \cdot \frac{1}{2(1 - x^2/2)} = \frac{x^3}{2} \sum_{n=0}^{\infty} \left(\frac{x^2}{2}\right)^n = \sum_{n=0}^{\infty} \frac{x^{2n+3}}{2^{n+1}}$$

Converges for $|x^2/2| < 1 \implies x^2 < 2 \implies |x| < \sqrt{2}$. Interval: $(-\sqrt{2}, \sqrt{2})$.

Solution to Problem 13

Note that $\frac{d}{dx} \left(\frac{1}{1-x} \right) = \frac{1}{(1-x)^2}$.

$$f(x) = \frac{d}{dx} \left(\sum_{n=0}^{\infty} x^n \right) = \sum_{n=1}^{\infty} nx^{n-1}$$

The radius of convergence remains $R = 1$.

Solution to Problem 14

Note that $\frac{d}{dx} \left(\frac{-1}{1+x} \right) = \frac{1}{(1+x)^2}$. The series for $\frac{-1}{1+x} = -\sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^{n+1} x^n$.

$$f(x) = \frac{d}{dx} \left(\sum_{n=0}^{\infty} (-1)^{n+1} x^n \right) = \sum_{n=1}^{\infty} (-1)^{n+1} nx^{n-1}$$

The radius of convergence remains $R = 1$.

Solution to Problem 15

Note that $\frac{d}{dx} \left(\frac{1}{5-x} \right) = \frac{1}{(5-x)^2}$. The series for $\frac{1}{5-x} = \frac{1}{5(1-x/5)} = \sum_{n=0}^{\infty} \frac{x^n}{5^{n+1}}$. So, $\frac{1}{(5-x)^2} = \frac{d}{dx} \left(\sum_{n=0}^{\infty} \frac{x^n}{5^{n+1}} \right) = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{5^{n+1}}$.

$$f(x) = 2 \cdot \frac{1}{(5-x)^2} = \sum_{n=1}^{\infty} \frac{2nx^{n-1}}{5^{n+1}}$$

Radius of convergence remains $R = 5$.

Solution to Problem 16

Take the series from Problem 13 and multiply by $2x$:

$$f(x) = 2x \sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=1}^{\infty} 2nx^n$$

Solution to Problem 17

Note that $\frac{d}{dx} \left(\frac{-1}{2(1+x^2)} \right) = \frac{x}{(1+x^2)^2}$. The series for $\frac{-1}{2(1+x^2)} = -\frac{1}{2} \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2} x^{2n}$.

$$f(x) = \frac{d}{dx} \left(\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2} x^{2n} \right) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2} (2n)x^{2n-1} = \sum_{n=1}^{\infty} (-1)^{n+1} nx^{2n-1}$$

Radius of convergence remains $R = 1$.

Solution to Problem 18

Note that $\frac{d^2}{dx^2} \left(\frac{1}{1-x} \right) = \frac{2}{(1-x)^3}$. We differentiate the series for $\frac{1}{1-x}$ twice. First derivative: $\sum_{n=1}^{\infty} nx^{n-1}$. Second derivative: $\frac{d}{dx} \left(\sum_{n=1}^{\infty} nx^{n-1} \right) = \sum_{n=2}^{\infty} n(n-1)x^{n-2}$. Radius remains $R = 1$.

Solution to Problem 19

Note that $\int \frac{1}{1+x} dx = \ln(1+x) + C$. The series for $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$.

$$\ln(1+x) = \int \left(\sum_{n=0}^{\infty} (-1)^n x^n \right) dx = C_0 + \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$$

At $x = 0$, $\ln(1) = 0$, so $C_0 = 0$.

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^k}{k}$$

Radius of convergence remains $R = 1$.

Solution to Problem 20

Note that $\int \frac{-1}{1-x} dx = \ln(1-x) + C$. The series for $\frac{-1}{1-x} = -\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} -x^n$.

$$\ln(1-x) = \int \left(\sum_{n=0}^{\infty} -x^n \right) dx = C_0 + \sum_{n=0}^{\infty} -\frac{x^{n+1}}{n+1}$$

At $x = 0$, $\ln(1) = 0$, so $C_0 = 0$.

$$\ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = -\sum_{k=1}^{\infty} \frac{x^k}{k}$$

Radius of convergence remains $R = 1$.

Solution to Problem 21

Note that $\int \frac{1}{3+x} dx = \ln(3+x) + C$. Series for $\frac{1}{3+x} = \frac{1}{3(1+x/3)} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{3^{n+1}}$.

$$\ln(3+x) = \int \left(\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{3^{n+1}} \right) dx = C_0 + \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{(n+1)3^{n+1}}$$

At $x = 0$, $\ln(3) = C_0$.

$$\ln(3+x) = \ln(3) + \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{(n+1)3^{n+1}} = \ln(3) + \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^k}{k \cdot 3^k}$$

Radius remains $R = 3$.

Solution to Problem 22

Note that $\int \frac{2x}{1+x^2} dx = \ln(1+x^2) + C$. First find series for $\frac{2x}{1+x^2}$. $\frac{2x}{1+x^2} = 2x \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} 2(-1)^n x^{2n+1}$.

$$\ln(1+x^2) = \int \left(\sum_{n=0}^{\infty} 2(-1)^n x^{2n+1} \right) dx = C_0 + \sum_{n=0}^{\infty} \frac{2(-1)^n x^{2n+2}}{2n+2} = C_0 + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{n+1}$$

At $x = 0$, $\ln(1) = 0$, so $C_0 = 0$.

$$\ln(1+x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{n+1} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^{2k}}{k}$$

Radius remains $R = 1$.

Solution to Problem 23

From Problem 19, $\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$.

$$x \ln(1+x) = x \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{n+1}}{n}$$

Solution to Problem 24

From Problem 22, $\ln(1+u) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} u^k}{k}$. Let $u = -x^2$. $\ln(1-x^2) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} (-x^2)^k}{k} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} (-1)^k x^{2k}}{k} = \sum_{k=1}^{\infty} \frac{x^{2k}}{k}$.

$$x^2 \ln(1-x^2) = x^2 \sum_{k=1}^{\infty} \frac{-x^{2k}}{k} = \sum_{k=1}^{\infty} \frac{-x^{2k+2}}{k}$$

Solution to Problem 25

Note that $\int \frac{1}{1+x^2} dx = \arctan(x) + C$. Series for $\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$.

$$\arctan(x) = \int \left(\sum_{n=0}^{\infty} (-1)^n x^{2n} \right) dx = C_0 + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

At $x = 0$, $\arctan(0) = 0$, so $C_0 = 0$.

$$\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

Radius of convergence remains $R = 1$.

Solution to Problem 26

Use the result from Problem 25, substituting x^3 for x .

$$\arctan(x^3) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^3)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+3}}{2n+1}$$

Solution to Problem 27

Use the result from Problem 25 and multiply by x .

$$x \arctan(x) = x \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{2n+1}$$

Solution to Problem 28

$$f(x) = \frac{x-1+3}{x-1} = 1 + \frac{3}{x-1} = 1 - \frac{3}{1-x} = 1 - 3 \sum_{n=0}^{\infty} x^n = 1 - \sum_{n=0}^{\infty} 3x^n$$

Solution to Problem 29

Using polynomial long division or synthetic division, $\frac{x^2}{x+3} = x-3 + \frac{9}{x+3}$. Series for $\frac{9}{x+3} = \frac{9}{3(1+x/3)} = 3 \sum_{n=0}^{\infty} (-x/3)^n = \sum_{n=0}^{\infty} \frac{3(-1)^n x^n}{3^n}$.

$$f(x) = x-3 + \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{3^{n-1}}$$

Solution to Problem 30

The integrand is $\frac{1}{1-(-x^5)} = \sum_{n=0}^{\infty} (-x^5)^n = \sum_{n=0}^{\infty} (-1)^n x^{5n}$.

$$\int \left(\sum_{n=0}^{\infty} (-1)^n x^{5n} \right) dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{5n+1}}{5n+1}$$

Radius of convergence is $R = 1$.

Solution to Problem 31

The integrand is $x \cdot \frac{1}{1-x^4} = x \sum_{n=0}^{\infty} (x^4)^n = \sum_{n=0}^{\infty} x^{4n+1}$.

$$\int \left(\sum_{n=0}^{\infty} x^{4n+1} \right) dx = C + \sum_{n=0}^{\infty} \frac{x^{4n+2}}{4n+2}$$

Radius of convergence is $R = 1$.

Solution to Problem 32

From Problem 20, $\ln(1-x) = -\sum_{k=1}^{\infty} \frac{x^k}{k}$.

$$\int \left(-\sum_{k=1}^{\infty} \frac{x^k}{k} \right) dx = C - \sum_{k=1}^{\infty} \frac{x^{k+1}}{k(k+1)}$$

Solution to Problem 33

From Problem 26, $\arctan(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{2n+1}$. Integrand is $x \arctan(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{2n+1}$.

$$\int \left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{2n+1} \right) dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+4}}{(2n+1)(4n+4)}$$

Solution to Problem 34

From Problem 25, $\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$

$$\begin{aligned} \int_0^{0.5} \left(x - \frac{x^3}{3} + \frac{x^5}{5} \right) dx &= \left[\frac{x^2}{2} - \frac{x^4}{12} + \frac{x^6}{30} \right]_0^{0.5} \\ &= \frac{(0.5)^2}{2} - \frac{(0.5)^4}{12} + \frac{(0.5)^6}{30} = \frac{0.25}{2} - \frac{0.0625}{12} + \frac{0.015625}{30} \approx 0.125 - 0.0052 + 0.0005 = 0.1203 \end{aligned}$$

Solution to Problem 35

The series $\sum_{n=1}^{\infty} nx^{n-1}$ is the derivative of $\sum_{n=0}^{\infty} x^n$. The sum of $\sum_{n=0}^{\infty} x^n$ is $\frac{1}{1-x}$. So, the sum of the series is $\frac{d}{dx} \left(\frac{1}{1-x} \right) = \frac{1}{(1-x)^2}$.

Solution to Problem 36

This is the Maclaurin series for e^x .

Solution to Problem 37

This is the Maclaurin series for $\cos(x)$.

Solution to Problem 38

This is the series for e^x with $x = 3/5$. So the sum is $e^{3/5}$.

$$\sum_{n=0}^{\infty} \frac{(3/5)^n}{n!} = e^{3/5}$$

Solution to Problem 39

This is the series for e^x with $x = -\ln(2)$. So the sum is $e^{-\ln(2)} = e^{\ln(2^{-1})} = 2^{-1} = 1/2$.

Solution to Problem 40

Let $S(x) = \sum_{n=2}^{\infty} \frac{x^n}{n}$. Then $S'(x) = \sum_{n=2}^{\infty} x^{n-1} = x + x^2 + x^3 + \dots$. This is a geometric series with $a = x, r = x$. The sum is $\frac{x}{1-x}$. So $S(x) = \int \frac{x}{1-x} dx = \int (-1 + \frac{1}{1-x}) dx = -x - \ln(1-x) + C$. Since $S(0) = 0$, we have $0 = -0 - \ln(1) + C \implies C = 0$. The sum is $-x - \ln(1-x)$.

Solution to Problem 41

$f(x) = (1+x)^{1/2}$. Here $k = 1/2$.

$$\begin{aligned}(1+x)^{1/2} &= 1 + \frac{1}{2}x + \frac{(1/2)(-1/2)}{2!}x^2 + \frac{(1/2)(-1/2)(-3/2)}{3!}x^3 + \dots \\ &= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \dots\end{aligned}$$

Radius is $R = 1$.

Solution to Problem 42

$f(x) = (1+x)^{-1/2}$. Here $k = -1/2$.

$$\begin{aligned}(1+x)^{-1/2} &= 1 - \frac{1}{2}x + \frac{(-1/2)(-3/2)}{2!}x^2 + \frac{(-1/2)(-3/2)(-5/2)}{3!}x^3 + \dots \\ &= 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \dots\end{aligned}$$

Radius is $R = 1$.

Solution to Problem 43

$f(x) = (1+x)^{1/3}$. Here $k = 1/3$.

$$\begin{aligned}(1+x)^{1/3} &= 1 + \frac{1}{3}x + \frac{(1/3)(-2/3)}{2!}x^2 + \frac{(1/3)(-2/3)(-5/3)}{3!}x^3 + \dots \\ &= 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3 - \dots\end{aligned}$$

Radius is $R = 1$.

Solution to Problem 44

$f(x) = (1-x^2)^{-1/2}$. Use result from Problem 42 with $x \rightarrow -x^2$.

$$= 1 - \frac{1}{2}(-x^2) + \frac{3}{8}(-x^2)^2 - \frac{5}{16}(-x^2)^3 + \dots = 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{5}{16}x^6 + \dots$$

Radius is $R = 1$.

Solution to Problem 45

$f(x) = (1+x)^{3/2}$. Here $k = 3/2$.

$$\begin{aligned}(1+x)^{3/2} &= 1 + \frac{3}{2}x + \frac{(3/2)(1/2)}{2!}x^2 + \frac{(3/2)(1/2)(-1/2)}{3!}x^3 + \dots \\ &= 1 + \frac{3}{2}x + \frac{3}{8}x^2 - \frac{1}{16}x^3 + \dots\end{aligned}$$

Solution to Problem 46

$$f(x) = \frac{1}{x} \ln(1+x) = \frac{1}{x} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{n-1}}{n}$$

Solution to Problem 47

Series for $\frac{1}{1+x^3} = \sum_{n=0}^{\infty} (-x^3)^n = \sum_{n=0}^{\infty} (-1)^n x^{3n}$.

$$f(x) = \frac{d}{dx} \left(\sum_{n=0}^{\infty} (-1)^n x^{3n} \right) = \sum_{n=1}^{\infty} (-1)^n (3n) x^{3n-1}$$

Solution to Problem 48

$f(x) = \frac{(x-1)(x+1)}{x-2} = (x+1)\frac{x-1}{x-2} = (x+1)\left(1 + \frac{1}{x-2}\right) = (x+1)\left(1 - \frac{1}{2-x}\right)$. $\frac{1}{2-x} = \sum_{n=0}^{\infty} \frac{x^n}{2^{n+1}}$. So $f(x) = (x+1)\left(1 - \sum_{n=0}^{\infty} \frac{x^n}{2^{n+1}}\right)$. This is more complex to write in a single sum. An easier way: $f(x) = \frac{x^2-1}{x-2} = x+2 + \frac{3}{x-2} = x+2 - \frac{3}{2-x} = x+2 - \sum_{n=0}^{\infty} \frac{3x^n}{2^{n+1}}$.

Solution to Problem 49

$f(x) = \ln(1+x) - \ln(1-x)$. Use series from Problems 19 and 20.

$$f(x) = \left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} \right) - \left(- \sum_{n=1}^{\infty} \frac{x^n}{n} \right) = \sum_{n=1}^{\infty} \frac{x^n}{n} ((-1)^{n-1} + 1)$$

If n is even, the term is 0. If n is odd, the term is $2x^n/n$.

$$f(x) = 2 \sum_{k=0}^{\infty} \frac{x^{2k+1}}{2k+1}$$

Solution to Problem 50

We need to differentiate $\frac{1}{4-x}$ twice. The second derivative is $\frac{2}{(4-x)^3}$. So $f(x) = \frac{1}{2} \frac{d^2}{dx^2} \left(\frac{1}{4-x} \right)$. Series for $\frac{1}{4-x} = \sum_{n=0}^{\infty} \frac{x^n}{4^{n+1}}$. First derivative: $\sum_{n=1}^{\infty} \frac{nx^{n-1}}{4^{n+1}}$. Second derivative: $\sum_{n=2}^{\infty} \frac{n(n-1)x^{n-2}}{4^{n+1}}$.

$$f(x) = \frac{1}{2} \sum_{n=2}^{\infty} \frac{n(n-1)x^{n-2}}{4^{n+1}}$$

Radius is $R = 4$.

Solution to Problem 51

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \dots. e^x - 1 = \sum_{n=1}^{\infty} \frac{x^n}{n!} \cdot \frac{e^x - 1}{x} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!}.$$

$$\int \left(\sum_{n=1}^{\infty} \frac{x^{n-1}}{n!} \right) dx = C + \sum_{n=1}^{\infty} \frac{x^n}{n \cdot n!}$$

Solution to Problem 52

The series for $\sin(u) = \sum_{n=0}^{\infty} \frac{(-1)^n u^{2n+1}}{(2n+1)!}$. Let $u = x^2$.

$$\sin(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!}$$

Solution to Problem 53

The series for $e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!}$. Let $u = -x^2$.

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$$

Solution to Problem 54

Using the result from Problem 53:

$$\int e^{-x^2} dx = \int \left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} \right) dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(2n+1)}$$

Solution to Problem 55

The series for $\cos(u) = \sum_{n=0}^{\infty} \frac{(-1)^n u^{2n}}{(2n)!}$. Let $u = \sqrt{x}$.

$$\cos(\sqrt{x}) = \sum_{n=0}^{\infty} \frac{(-1)^n (\sqrt{x})^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n)!}$$

Solution to Problem 56

$$f(x) = \frac{1-x^2+2x^2}{1-x^2} = \frac{1-x^2}{1-x^2} + \frac{2x^2}{1-x^2} = 1 + 2x^2 \sum_{n=0}^{\infty} (x^2)^n = 1 + \sum_{n=0}^{\infty} 2x^{2n+2}$$

Solution to Problem 57

$f(x) = (16-x)^{1/4} = 16^{1/4}(1-x/16)^{1/4} = 2(1-x/16)^{1/4}$. Use binomial series for $(1+u)^k$ with $u = -x/16, k = 1/4$.

$$\begin{aligned} & 2 \left[1 + \frac{1}{4} \left(-\frac{x}{16} \right) + \frac{(1/4)(-3/4)}{2!} \left(-\frac{x}{16} \right)^2 + \frac{(1/4)(-3/4)(-7/4)}{3!} \left(-\frac{x}{16} \right)^3 + \dots \right] \\ &= 2 \left[1 - \frac{x}{64} - \frac{3x^2}{8192} - \frac{7x^3}{786432} - \dots \right] = 2 - \frac{x}{32} - \frac{3x^2}{4096} - \frac{7x^3}{393216} - \dots \end{aligned}$$

Solution to Problem 58

Partial fractions: $\frac{5x-1}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}$. This gives $A = 3, B = 2$. $f(x) = \frac{3}{x-2} + \frac{2}{x+1} = \frac{-3}{2-x} + \frac{2}{1+x} = -\frac{3}{2} \frac{1}{1-x/2} + 2 \frac{1}{1-(-x)}$.

$$f(x) = -\frac{3}{2} \sum_{n=0}^{\infty} \left(\frac{x}{2} \right)^n + 2 \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} \left(-\frac{3}{2^{n+1}} + 2(-1)^n \right) x^n$$

Solution to Problem 59

Consider $f(x) = \sum_{n=1}^{\infty} nx^n = x \sum_{n=1}^{\infty} nx^{n-1} = x \frac{d}{dx} \left(\frac{1}{1-x} \right) = \frac{x}{(1-x)^2}$. The sum is $f(1/2) = \frac{1/2}{(1-1/2)^2} = \frac{1/2}{(1/2)^2} = 2$.

Solution to Problem 60

This is the series for $\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$ with $x = 1/3$. The sum is $\ln(1+1/3) = \ln(4/3)$.

Solution to Problem 61

Series for $\frac{1}{(1-u)^2} = \sum_{n=1}^{\infty} nu^{n-1}$. Let $u = 2x$. $\frac{1}{(1-2x)^2} = \sum_{n=1}^{\infty} n(2x)^{n-1} = \sum_{n=1}^{\infty} n2^{n-1}x^{n-1}$.

$$f(x) = x^3 \sum_{n=1}^{\infty} n2^{n-1}x^{n-1} = \sum_{n=1}^{\infty} n2^{n-1}x^{n+2}$$

Solution to Problem 62

$f(x) = \ln(4(1+x^2/4)) = \ln(4) + \ln(1+x^2/4)$. Use series for $\ln(1+u)$ with $u = x^2/4$.

$$f(x) = \ln(4) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x^2/4)^n}{n} = \ln(4) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^{2n}}{n \cdot 4^n}$$

Solution to Problem 63

This matches the series for $\sin(x) = x - x^3/3! + x^5/5! - \dots$ evaluated at $x = \pi/2$. The sum is $\sin(\pi/2) = 1$.

Solution to Problem 64

Series for $\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$. $\frac{\arctan(x)}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2n+1}$.

$$\int \frac{\arctan(x)}{x} dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)^2}$$

Solution to Problem 65

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots \quad x - \ln(1+x) = x - \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \dots \right) = \frac{x^2}{2} - \frac{x^3}{3} + \dots$$

$$\frac{x - \ln(1+x)}{x^2} = \frac{\frac{x^2}{2} - \frac{x^3}{3} + \dots}{x^2} = \frac{1}{2} - \frac{x}{3} + \dots$$

As $x \rightarrow 0$, the limit is $1/2$.

Concept and Problem Cross-Reference

Concept Checklist

Below is the list of concepts and problem types tested in this problem set, with the corresponding problem numbers for reference.

- **Basic Geometric Series Transformation** ($1/(a \pm bx)$)
 - Problems: 1, 2, 3, 4
- **Geometric Series with Powers of x** ($1/(a \pm bx^k)$)
 - Problems: 5, 6, 7
- **Multiplying a Series by a Polynomial** ($x^k \cdot f(x)$)
 - Problems: 8, 9, 10, 11, 12, 16, 23, 24, 27, 46, 61
- **Algebraic Pre-processing (Long Division, Partial Fractions, etc.)**
 - Problems: 28, 29, 48, 56, 58
- **Term-by-Term Differentiation (to find series for $1/(a + bx)^k$)**
 - Problems: 13, 14, 15, 17, 18, 47, 50
- **Term-by-Term Integration (to find series for $\ln(\dots)$ and $\arctan(\dots)$)**
 - Problems: 19, 20, 21, 22, 25, 26, 49, 62
- **Representing an Integral as a Power Series**
 - Problems: 30, 31, 32, 33, 51, 54, 64
- **The Binomial Series** ($(1+x)^k$)
 - Problems: 41, 42, 43, 44, 45, 57
- **Recognizing and Summing Known Maclaurin Series**
 - Problems: 35, 36, 37, 38, 39, 59, 60, 63
- **Substituting into Known Series** ($x \rightarrow x^k$, $x \rightarrow \sqrt{x}$)
 - Problems: 26, 44, 52, 53, 55
- **Multi-Step and Combination Problems**
 - Problems: 17, 24, 33, 40, 49, 50, 56, 58, 61
- **Applications (Approximation, Limits)**
 - Problems: 34, 65