Calculus II - Test 2 Review Solutions

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1. Evaluate the convergence and divergence of the following integrals

a. **Integral:** $\displaystyle\int_2^{\infty} \frac{2}{x^2 - x}\, dx$

Explanation: This is an improper integral because the upper limit of integration is infinite. We first use partial fraction decomposition on the integrand.

$$\begin{aligned}
\frac{2}{x^2 - x} = \frac{2}{x(x-1)} &= \frac{A}{x} + \frac{B}{x-1} \\
2 &= A(x-1) + Bx \\
\text{Let } x = 0 : 2 &= A(-1) \implies A = -2 \\
\text{Let } x = 1 : 2 &= B(1) \implies B = 2 \\
\text{So, } \frac{2}{x(x-1)} &= \frac{2}{x-1} - \frac{2}{x}
\end{aligned}$$

Now we evaluate the integral as a limit:

$$\begin{aligned}
\int_2^{\infty} \left(\frac{2}{x-1} - \frac{2}{x}\right) dx &= \lim_{b\to\infty} \int_2^{b} \left(\frac{2}{x-1} - \frac{2}{x}\right) dx \\
&= \lim_{b\to\infty} \left[2\ln|x-1| - 2\ln|x|\right]_2^b \\
&= \lim_{b\to\infty} \left[2\ln\left|\frac{x-1}{x}\right|\right]_2^b \\
&= \lim_{b\to\infty} \left(2\ln\left|\frac{b-1}{b}\right| - 2\ln\left|\frac{2-1}{2}\right|\right) \\
&= 2\ln(1) - 2\ln\left(\frac{1}{2}\right) = 0 - 2(-\ln 2) = 2\ln 2
\end{aligned}$$

Result: The integral **converges** to $2\ln 2$.

b. **Integral:** $\displaystyle\int_0^{1} \frac{x+1}{\sqrt{x^2 + 2x}}\, dx$

Explanation: This integral is improper because the integrand has a discontinuity at $x = 0$. We use a u-substitution. Let $u = x^2 + 2x$, so $du = (2x+2)dx = 2(x+1)dx$.

$$\begin{aligned}
\int \frac{x+1}{\sqrt{x^2+2x}}\, dx &= \int \frac{1}{\sqrt{u}} \cdot \frac{du}{2} = \frac{1}{2}\int u^{-1/2}\, du \\
&= \frac{1}{2} \cdot 2u^{1/2} = \sqrt{u} = \sqrt{x^2+2x}
\end{aligned}$$

Now evaluate the limit:

$$\begin{aligned}\int_0^1 \frac{x+1}{\sqrt{x^2+2x}}\,dx &= \lim_{a\to 0^+}\int_a^1 \frac{x+1}{\sqrt{x^2+2x}}\,dx \\ &= \lim_{a\to 0^+}\left[\sqrt{x^2+2x}\right]_a^1 \\ &= \lim_{a\to 0^+}\left(\sqrt{1^2+2(1)}-\sqrt{a^2+2a}\right) \\ &= \sqrt{3}-0=\sqrt{3}\end{aligned}$$

Result: The integral **converges** to $\sqrt{3}$.

c. **Integral:** $\displaystyle\int_0^{\pi/2} \tan\theta\,d\theta$

Explanation: This is improper because $\tan\theta$ has a vertical asymptote at $\theta=\pi/2$.

$$\begin{aligned}\int_0^{\pi/2} \tan\theta\,d\theta &= \lim_{b\to(\pi/2)^-}\int_0^b \tan\theta\,d\theta \\ &= \lim_{b\to(\pi/2)^-}\left[-\ln|\cos\theta|\right]_0^b \\ &= \lim_{b\to(\pi/2)^-}\left(-\ln|\cos b|-(-\ln|\cos 0|)\right) \\ &= \lim_{b\to(\pi/2)^-}\left(-\ln|\cos b|\right)+\ln(1)\end{aligned}$$

As $b\to(\pi/2)^-$, $\cos b\to 0^+$, so $-\ln|\cos b|\to\infty$. **Result:** The integral **diverges**.

d. **Integral:** $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

Explanation: Improper due to a discontinuity at $x=1$. This is a standard integral form.

$$\begin{aligned}\int_0^1 \frac{dx}{\sqrt{1-x^2}} &= \lim_{b\to 1^-}\int_0^b \frac{dx}{\sqrt{1-x^2}} \\ &= \lim_{b\to 1^-}\left[\arcsin(x)\right]_0^b \\ &= \lim_{b\to 1^-}\left(\arcsin(b)-\arcsin(0)\right) \\ &= \arcsin(1)-0=\frac{\pi}{2}\end{aligned}$$

Result: The integral **converges** to $\frac{\pi}{2}$.

e. **Integral:** $\displaystyle\int_{-\infty}^{\infty} \frac{2x}{(x^2+1)^2}\,dx$

Explanation: This is improper because of the infinite limits. We split it at $x=0$.

$$\int_{-\infty}^{\infty} \frac{2x}{(x^2+1)^2}\,dx = \int_{-\infty}^{0} \frac{2x}{(x^2+1)^2}\,dx + \int_{0}^{\infty} \frac{2x}{(x^2+1)^2}\,dx$$

Let $u=x^2+1$, $du=2x\,dx$. The antiderivative is $\int \frac{1}{u^2}\,du=-\frac{1}{u}=-\frac{1}{x^2+1}$.

$$\text{Part 1: } \lim_{a\to-\infty}\left[-\frac{1}{x^2+1}\right]_a^0 = \left(-\frac{1}{0^2+1}\right)-\lim_{a\to-\infty}\left(-\frac{1}{a^2+1}\right) = -1-0=-1$$

$$\text{Part 2: } \lim_{b\to\infty}\left[-\frac{1}{x^2+1}\right]_0^b = \lim_{b\to\infty}\left(-\frac{1}{b^2+1}\right)-\left(-\frac{1}{0^2+1}\right) = 0-(-1)=1$$

The total value is $-1+1=0$. **Result:** The integral **converges** to 0.

2. Find the length of the curves

Arc Length Formula: $L = \int_a^b \sqrt{1 + (y')^2}\, dx$

a. **Curve:** $y = \frac{1}{3}(x^2+2)^{3/2}$ from $x = 0$ to $x = 3$

$$\begin{aligned}
y' &= \frac{1}{3} \cdot \frac{3}{2}(x^2+2)^{1/2} \cdot 2x = x\sqrt{x^2+2} \\
1 + (y')^2 &= 1 + (x\sqrt{x^2+2})^2 = 1 + x^2(x^2+2) \\
&= 1 + x^4 + 2x^2 = (x^2+1)^2 \\
L &= \int_0^3 \sqrt{(x^2+1)^2}\, dx = \int_0^3 (x^2+1)\, dx \\
&= \left[\frac{x^3}{3} + x\right]_0^3 = \left(\frac{27}{3} + 3\right) - 0 = 9 + 3 = 12
\end{aligned}$$

Length: 12.

b. **Curve:** $y = \ln(x) - \frac{x^2}{8}$ from $x = 1$ to $x = 2$

$$\begin{aligned}
y' &= \frac{1}{x} - \frac{2x}{8} = \frac{1}{x} - \frac{x}{4} \\
1 + (y')^2 &= 1 + \left(\frac{1}{x} - \frac{x}{4}\right)^2 = 1 + \left(\frac{1}{x^2} - \frac{1}{2} + \frac{x^2}{16}\right) \\
&= \frac{1}{x^2} + \frac{1}{2} + \frac{x^2}{16} = \left(\frac{1}{x} + \frac{x}{4}\right)^2 \\
L &= \int_1^2 \sqrt{\left(\frac{1}{x} + \frac{x}{4}\right)^2}\, dx = \int_1^2 \left(\frac{1}{x} + \frac{x}{4}\right)\, dx \\
&= \left[\ln|x| + \frac{x^2}{8}\right]_1^2 = \left(\ln 2 + \frac{4}{8}\right) - \left(\ln 1 + \frac{1}{8}\right) = \ln 2 + \frac{1}{2} - \frac{1}{8} = \ln 2 + \frac{3}{8}
\end{aligned}$$

Length: $\ln 2 + \frac{3}{8}$.

c. **Curve:** $y = \ln(\sec x)$ from $x = 0$ to $x = \pi/4$

$$\begin{aligned}
y' &= \frac{1}{\sec x} \cdot (\sec x \tan x) = \tan x \\
1 + (y')^2 &= 1 + \tan^2 x = \sec^2 x \\
L &= \int_0^{\pi/4} \sqrt{\sec^2 x}\, dx = \int_0^{\pi/4} \sec x\, dx \\
&= [\ln|\sec x + \tan x|]_0^{\pi/4} \\
&= \ln|\sec(\pi/4) + \tan(\pi/4)| - \ln|\sec(0) + \tan(0)| \\
&= \ln|\sqrt{2} + 1| - \ln|1 + 0| = \ln(\sqrt{2} + 1)
\end{aligned}$$

Length: $\ln(\sqrt{2} + 1)$.

3. Calculate the surface area generated by revolving the curves about the indicated axis

Surface Area Formula (x-axis): $S = \displaystyle\int_a^b 2\pi y\sqrt{1+(y')^2}\,dx$

a. **Curve:** $y = \sqrt{2x - x^2}$, $1 \le x \le 2$; about the x-axis

$$\begin{aligned}
y' &= \frac{1}{2\sqrt{2x-x^2}} \cdot (2-2x) = \frac{1-x}{\sqrt{2x-x^2}} \\
1+(y')^2 &= 1 + \frac{(1-x)^2}{2x-x^2} = \frac{2x-x^2+(1-2x+x^2)}{2x-x^2} = \frac{1}{2x-x^2} \\
S &= \int_1^2 2\pi y\sqrt{1+(y')^2}\,dx \\
&= \int_1^2 2\pi\sqrt{2x-x^2}\sqrt{\frac{1}{2x-x^2}}\,dx \\
&= \int_1^2 2\pi\,dx = [2\pi x]_1^2 = 2\pi(2) - 2\pi(1) = 2\pi
\end{aligned}$$

Surface Area: 2π.

b. **Curve:** $y = \sqrt{1+e^x}$, $0 \le x \le 1$; about the x-axis

$$\begin{aligned}
y' &= \frac{e^x}{2\sqrt{1+e^x}} \\
1+(y')^2 &= 1 + \frac{e^{2x}}{4(1+e^x)} = \frac{4(1+e^x)+e^{2x}}{4(1+e^x)} = \frac{4+4e^x+e^{2x}}{4(1+e^x)} = \frac{(e^x+2)^2}{4(1+e^x)} \\
S &= \int_0^1 2\pi\sqrt{1+e^x}\sqrt{\frac{(e^x+2)^2}{4(1+e^x)}}\,dx \\
&= \int_0^1 2\pi\sqrt{1+e^x}\frac{e^x+2}{2\sqrt{1+e^x}}\,dx \\
&= \pi\int_0^1 (e^x+2)\,dx = \pi\left[e^x+2x\right]_0^1 \\
&= \pi\big((e^1+2)-(e^0+0)\big) = \pi(e+2-1) = \pi(e+1)
\end{aligned}$$

Surface Area: $\pi(e+1)$.

4. Find and graph the Cartesian equation

a. **Equations:** $x = 4\cos t, y = 2\sin t, 0 \le t \le 2\pi$

Eliminate the parameter: From $x = 4\cos t \implies \cos t = x/4$. From $y = 2\sin t \implies \sin t = y/2$. Using the identity $\cos^2 t + \sin^2 t = 1$:

$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{2}\right)^2 = 1 \implies \frac{x^2}{16} + \frac{y^2}{4} = 1$$

Path: This is an ellipse centered at $(0,0)$ with a horizontal major axis of length 8 and a vertical minor axis of length 4.

Direction: At $t = 0$, $(x,y) = (4,0)$. At $t = \pi/2$, $(x,y) = (0,2)$. At $t = \pi$, $(x,y) = (-4,0)$. The particle travels **counter-clockwise** starting from $(4,0)$.

b. **Equations:** $x = \sin t, y = \cos 2t, -\pi/2 \le t \le \pi/2$

Eliminate the parameter: Using the identity $\cos 2t = 1 - 2\sin^2 t$:

$$y = 1 - 2x^2$$

Path: This is a parabola opening downwards with its vertex at $(0,1)$.

Direction: At $t = -\pi/2$, $(x,y) = (-1,-1)$. At $t = 0$, $(x,y) = (0,1)$. At $t = \pi/2$, $(x,y) = (1,-1)$. The particle moves from $(-1,-1)$ to $(1,-1)$ along the parabola.

c. **Equations:** $x = 1 + \sin t, y = \cos t - 2, 0 \le t \le \pi$

Eliminate the parameter: From the equations, $\sin t = x - 1$ and $\cos t = y + 2$. Using $\sin^2 t + \cos^2 t = 1$:

$$(x-1)^2 + (y+2)^2 = 1$$

Path: This is a circle of radius 1 centered at $(1,-2)$.

Direction: At $t = 0$, $(x,y) = (1,-1)$. At $t = \pi/2$, $(x,y) = (2,-2)$. At $t = \pi$, $(x,y) = (1,-3)$. The particle traces the **top semi-circle** from right to left, starting at $(1,-1)$.

5. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ as a function of t

Formulas: $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ and $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(dy/dx)}{dx/dt}$

a. **Equations:** $x = t - t^2, y = t - t^3$

$$\frac{dx}{dt} = 1 - 2t \qquad \frac{dy}{dt} = 1 - 3t^2$$

$$\frac{dy}{dx} = \frac{1 - 3t^2}{1 - 2t}$$

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{-6t(1-2t) - (1-3t^2)(-2)}{(1-2t)^2} = \frac{-6t + 12t^2 + 2 - 6t^2}{(1-2t)^2} = \frac{6t^2 - 6t + 2}{(1-2t)^2}$$

$$\frac{d^2y}{dx^2} = \frac{(6t^2 - 6t + 2)/(1-2t)^2}{1 - 2t} = \frac{2(3t^2 - 3t + 1)}{(1-2t)^3}$$

b. **Equations:** $x = \frac{1}{t+1}, y = \frac{t}{t-1}$

$$\frac{dx}{dt} = -(t+1)^{-2} = \frac{-1}{(t+1)^2}$$

$$\frac{dy}{dt} = \frac{1(t-1) - t(1)}{(t-1)^2} = \frac{-1}{(t-1)^2}$$

$$\frac{dy}{dx} = \frac{-1/(t-1)^2}{-1/(t+1)^2} = \frac{(t+1)^2}{(t-1)^2}$$

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{2(t+1)(t-1)^2 - (t+1)^2(2(t-1))}{(t-1)^4}$$

$$= \frac{2(t+1)(t-1)[(t-1) - (t+1)]}{(t-1)^4} = \frac{2(t+1)(-2)}{(t-1)^3} = \frac{-4(t+1)}{(t-1)^3}$$

$$\frac{d^2y}{dx^2} = \frac{-4(t+1)/(t-1)^3}{-1/(t+1)^2} = \frac{4(t+1)^3}{(t-1)^3}$$

6. Find an equation of the line tangent to the curve

a. **Curve:** $x = \sec t, y = \tan t$ at $t = \pi/4$

- **Point:** $x(\pi/4) = \sec(\pi/4) = \sqrt{2}$, $y(\pi/4) = \tan(\pi/4) = 1$. Point is $(\sqrt{2}, 1)$.
- **Slope:** $\frac{dx}{dt} = \sec t \tan t$, $\frac{dy}{dt} = \sec^2 t$. $\frac{dy}{dx} = \frac{\sec^2 t}{\sec t \tan t} = \frac{\sec t}{\tan t} = \csc t$. At $t = \pi/4$, slope $m = \csc(\pi/4) = \sqrt{2}$.
- **Equation:** $y - 1 = \sqrt{2}(x - \sqrt{2}) \implies y - 1 = \sqrt{2}x - 2 \implies y = \sqrt{2}x - 1$.

b. **Curve:** $x = t - \sin t, y = 1 - \cos t$ at $t = \pi/3$

- **Point:** $x(\pi/3) = \frac{\pi}{3} - \sin(\pi/3) = \frac{\pi}{3} - \frac{\sqrt{3}}{2}$. $y(\pi/3) = 1 - \cos(\pi/3) = 1 - \frac{1}{2} = \frac{1}{2}$. Point is $(\frac{\pi}{3} - \frac{\sqrt{3}}{2}, \frac{1}{2})$.
- **Slope:** $\frac{dx}{dt} = 1 - \cos t$, $\frac{dy}{dt} = \sin t$. $\frac{dy}{dx} = \frac{\sin t}{1-\cos t}$. At $t = \pi/3$, slope $m = \frac{\sin(\pi/3)}{1-\cos(\pi/3)} = \frac{\sqrt{3}/2}{1-1/2} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$.
- **Equation:** $y - \frac{1}{2} = \sqrt{3}\left(x - \left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)\right)$.

7. Find the lengths of the curves

Parametric Arc Length Formula: $L = \int_{t_1}^{t_2} \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\, dt$

a. **Curve:** $x = \frac{t^2}{2}, y = \frac{1}{3}(2t+1)^{3/2}, 0 \le t \le 4$

$$\frac{dx}{dt} = t \qquad \frac{dy}{dt} = \frac{1}{3} \cdot \frac{3}{2}(2t+1)^{1/2} \cdot 2 = \sqrt{2t+1}$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = t^2 + (\sqrt{2t+1})^2 = t^2 + 2t + 1 = (t+1)^2$$

$$L = \int_0^4 \sqrt{(t+1)^2}\, dt = \int_0^4 (t+1)\, dt$$

$$= \left[\frac{t^2}{2} + t\right]_0^4 = \left(\frac{16}{2} + 4\right) - 0 = 8 + 4 = 12$$

Length: 12.

b. **Curve:** $x = t^3, y = \frac{3t^2}{2}, 0 \le t \le \sqrt{3}$

$$\frac{dx}{dt} = 3t^2 \qquad \frac{dy}{dt} = 3t$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (3t^2)^2 + (3t)^2 = 9t^4 + 9t^2 = 9t^2(t^2+1)$$

$$L = \int_0^{\sqrt{3}} \sqrt{9t^2(t^2+1)}\, dt = \int_0^{\sqrt{3}} 3t\sqrt{t^2+1}\, dt$$

Let $u = t^2 + 1$, $du = 2t\, dt \implies \frac{3}{2}du = 3t\, dt$. When $t = 0, u = 1$. When $t = \sqrt{3}, u = 4$.

$$L = \int_1^4 \frac{3}{2}\sqrt{u}\, du = \frac{3}{2}\int_1^4 u^{1/2}\, du = \frac{3}{2}\left[\frac{2}{3}u^{3/2}\right]_1^4$$

$$= \left[u^{3/2}\right]_1^4 = 4^{3/2} - 1^{3/2} = 8 - 1 = 7$$

Length: 7.

8. For what values of t does the curve have a vertical tangent?

Equations: $x = t^3 - t^2 - 1, y = t^4 + 2t^2 - 8t$

Explanation: A vertical tangent occurs when the slope $\frac{dy}{dx}$ is undefined. This happens when $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$.

1. Find when $\frac{dx}{dt} = 0$:
$$\frac{dx}{dt} = 3t^2 - 2t = t(3t - 2)$$
Setting $\frac{dx}{dt} = 0$ gives $t = 0$ and $t = 2/3$.

2. Check if $\frac{dy}{dt} \neq 0$ at these values:
$$\frac{dy}{dt} = 4t^3 + 4t - 8$$
   - At $t = 0$: $\frac{dy}{dt} = 4(0)^3 + 4(0) - 8 = -8 \neq 0$.
   - At $t = 2/3$: $\frac{dy}{dt} = 4(2/3)^3 + 4(2/3) - 8 = 4(8/27) + 8/3 - 8 = 32/27 + 72/27 - 216/27 = -112/27 \neq 0$.

Result: A vertical tangent exists at both $t = 0$ and $t = 2/3$.

9. For what value of t is the particle at rest?

Equations: $x = t^3 - 3t^2, y = 2t^3 - 3t^2 - 12t$

Explanation: A particle is at rest when its velocity is zero. This means both velocity components, $\frac{dx}{dt}$ and $\frac{dy}{dt}$, must be zero simultaneously.

1. Set $\frac{dx}{dt} = 0$:
$$\frac{dx}{dt} = 3t^2 - 6t = 3t(t - 2)$$
The solutions are $t = 0$ and $t = 2$.

2. Set $\frac{dy}{dt} = 0$:
$$\frac{dy}{dt} = 6t^2 - 6t - 12 = 6(t^2 - t - 2) = 6(t - 2)(t + 1)$$
The solutions are $t = 2$ and $t = -1$.

3. Find the common solution: The only value of t that makes both derivatives zero is $t = 2$.

Result: The particle is at rest when $t = 2$.