Problem Set: The Integral Test and Estimates of Sums

Calculus II Practice

November 1, 2025

Instructions

Use the Integral Test, p-series test, or the Test for Divergence to determine whether the following series are convergent or divergent. For problems that ask for a full evaluation, provide the value of the corresponding integral.

Problems

1. Use the Integral Test to determine if the series $\sum_{n=1}^{\infty} \frac{6}{3n+2}$ converges or diverges. Evaluate the corresponding integral.

2. Use the Integral Test to determine if the series $\sum_{n=1}^{\infty} \frac{n}{n^2+9}$ converges or diverges. Evaluate the corresponding integral.

3. Use the Integral Test to determine if the series $\sum_{n=2}^{\infty} \frac{n^2}{n^3-4}$ converges or diverges. Evaluate the corresponding integral.

4. Use the Integral Test to determine if the series $\sum_{n=1}^{\infty} ne^{-n^2}$ converges or diverges. Evaluate the corresponding integral.

5. Use the Integral Test to determine if the series $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$ converges or diverges. Evaluate the corresponding integral.

6. Use the Integral Test to determine if the series $\sum_{n=0}^{\infty} \frac{1}{n^2+1}$ converges or diverges. Evaluate the corresponding integral.

7. Use the Integral Test to determine if the series $\sum_{n=1}^{\infty} \frac{5}{n^2+25}$ converges or diverges. Evaluate the corresponding integral.

8. Use the Integral Test to determine if the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$ converges or diverges. Evaluate the corresponding integral.

9. Use the Integral Test to determine if the series $\sum_{n=3}^{\infty} \frac{1}{n\sqrt{\ln n}}$ converges or diverges. Evaluate the corresponding integral.

10. Use the Integral Test to determine if the series $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ converges or diverges. Evaluate the corresponding integral.

11. Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n}}$ is convergent or divergent.

12. Determine whether the series $\sum_{n=1}^{\infty} n^{-1.0001}$ is convergent or divergent.
13. Determine whether the series $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n}$ is convergent or divergent.
14. Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{n^{\pi/2}}$ is convergent or divergent.
15. Determine whether the series $\sum_{n=1}^{\infty} \frac{3n-2}{5n+1}$ is convergent or divergent.
16. Determine whether the series $\sum_{n=1}^{\infty} \arctan(n)$ is convergent or divergent.
17. Determine whether the series $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n$ is convergent or divergent.
18. Determine if the following series is convergent or divergent: $1+\frac{1}{8}+\frac{1}{27}+\frac{1}{64}+\dots$
19. Determine if the following series is convergent or divergent: $\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}+\dots$
20. Determine if the following series is convergent or divergent: $1+\frac{1}{2\sqrt{2}}+\frac{1}{3\sqrt{3}}+\frac{1}{4\sqrt{4}}+\dots$
21. Determine if the following series is convergent or divergent: $\frac{\ln(2)}{2}+\frac{\ln(3)}{3}+\frac{\ln(4)}{4}+\dots$
22. Does the function $f(x)=\frac{x}{x^2-1}$ satisfy the conditions of the Integral Test for the series $\sum_{n=2}^{\infty} \frac{n}{n^2-1}$? Explain why or why not.
23. Does the function $f(x)=\frac{2+\cos(x)}{x^2}$ satisfy the conditions for the Integral Test on $[1,\infty)$? Explain why or why not.
24. Explain why the Integral Test cannot be used for the series $\sum_{n=1}^{\infty} \frac{\sin^2(n)}{n^2}$.
25. Determine if the series $\sum_{n=1}^{\infty} \frac{n+4}{n^2+1}$ is convergent or divergent.
26. Determine if the series $\sum_{n=1}^{\infty} \frac{1}{n^2+3n+2}$ is convergent or divergent.
27. Determine if the series $\sum_{n=2}^{\infty} \frac{1}{n\ln(n^2)}$ is convergent or divergent.
28. Determine if the series $\sum_{n=1}^{\infty} 5n^{-2/3}$ is convergent or divergent.
29. Determine if the series $\sum_{n=1}^{\infty} \frac{e^{-\sqrt{n}}}{\sqrt{n}}$ is convergent or divergent.
30. Determine if the series $\sum_{n=1}^{\infty} \frac{n}{e^n}$ is convergent or divergent.
31. Determine if the series $\sum_{n=1}^{\infty} n\sin(1/n)$ is convergent or divergent.
32. Determine if the series $\sum_{n=5}^{\infty} \frac{1}{(n-4)^3}$ is convergent or divergent.
33. Determine if the series $\sum_{n=2}^{\infty} \frac{\ln(n)}{n}$ is convergent or divergent.

Solutions

1. **Divergent**. The function $f(x) = \frac{6}{3x+2}$ is continuous, positive, and decreasing for $x \geq 1$. $\int_1^\infty \frac{6}{3x+2}dx = \lim_{t\to\infty}[2\ln(3x+2)]_1^t = \lim_{t\to\infty}(2\ln(3t+2) - 2\ln(5)) = \infty$. Since the integral diverges, the series diverges.

2. **Divergent**. $f(x) = \frac{x}{x^2+9}$ is continuous, positive, and decreasing for $x \geq 3$. $\int_1^\infty \frac{x}{x^2+9}dx = \lim_{t\to\infty}[\frac{1}{2}\ln(x^2+9)]_1^t = \lim_{t\to\infty}(\frac{1}{2}\ln(t^2+9) - \frac{1}{2}\ln(10)) = \infty$. The integral diverges, so the series diverges.

3. **Divergent**. $f(x) = \frac{x^2}{x^3-4}$ is continuous, positive, and decreasing for $x \geq 2$. $\int_2^\infty \frac{x^2}{x^3-4}dx = \lim_{t\to\infty}[\frac{1}{3}\ln(x^3-4)]_2^t = \lim_{t\to\infty}(\frac{1}{3}\ln(t^3-4) - \frac{1}{3}\ln(4)) = \infty$. The integral diverges, so the series diverges.

4. **Convergent**. $f(x) = xe^{-x^2}$ is positive, continuous, and decreasing for $x \geq 1$. Let $u = -x^2, du = -2xdx$. $\int_1^\infty xe^{-x^2}dx = \lim_{t\to\infty}[-\frac{1}{2}e^{-x^2}]_1^t = \lim_{t\to\infty}(-\frac{1}{2}e^{-t^2} - (-\frac{1}{2}e^{-1})) = 0 + \frac{1}{2e} = \frac{1}{2e}$. The integral converges, so the series converges.

5. **Convergent**. $f(x) = \frac{e^{1/x}}{x^2}$ is positive, continuous, and decreasing for $x \geq 1$. Let $u = 1/x, du = -1/x^2dx$. $\int_1^\infty \frac{e^{1/x}}{x^2}dx = \lim_{t\to\infty}[-e^{1/x}]_1^t = \lim_{t\to\infty}(-e^{1/t} - (-e^1)) = -e^0 + e = e - 1$. The integral converges, so the series converges.

6. **Convergent**. $f(x) = \frac{1}{x^2+1}$ is positive, continuous, and decreasing for $x \geq 0$. $\int_0^\infty \frac{1}{x^2+1}dx = \lim_{t\to\infty}[\arctan(x)]_0^t = \lim_{t\to\infty}(\arctan(t) - \arctan(0)) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$. The integral converges, so the series converges.

7. **Convergent**. $f(x) = \frac{5}{x^2+25}$ is positive, continuous, and decreasing for $x \geq 1$. $\int_1^\infty \frac{5}{x^2+25}dx = 5\lim_{t\to\infty}[\frac{1}{5}\arctan(\frac{x}{5})]_1^t = \lim_{t\to\infty}(\arctan(\frac{t}{5}) - \arctan(\frac{1}{5})) = \frac{\pi}{2} - \arctan(\frac{1}{5})$. The integral converges, so the series converges.

8. **Convergent**. $f(x) = \frac{1}{x(\ln x)^3}$ is continuous, positive, and decreasing for $x \geq 2$. Let $u = \ln x, du = \frac{1}{x}dx$. $\int_2^\infty \frac{1}{x(\ln x)^3}dx = \lim_{t\to\infty}\int_{\ln 2}^{\ln t} u^{-3}du = \lim_{t\to\infty}[-\frac{1}{2u^2}]_{\ln 2}^{\ln t} = \lim_{t\to\infty}(-\frac{1}{2(\ln t)^2} + \frac{1}{2(\ln 2)^2}) = \frac{1}{2(\ln 2)^2}$. The integral converges, so the series converges.

9. **Divergent**. $f(x) = \frac{1}{x\sqrt{\ln x}}$ is continuous, positive, and decreasing for $x \geq 3$. Let $u = \ln x, du = \frac{1}{x}dx$. $\int_3^\infty \frac{1}{x\sqrt{\ln x}}dx = \lim_{t\to\infty}\int_{\ln 3}^{\ln t} u^{-1/2}du = \lim_{t\to\infty}[2\sqrt{u}]_{\ln 3}^{\ln t} = \lim_{t\to\infty}(2\sqrt{\ln t} - 2\sqrt{\ln 3}) = \infty$. The integral diverges, so the series diverges.

10. **Convergent**. $f(x) = \frac{\ln x}{x^2}$ is positive, continuous, and decreasing for $x \geq 2$. Use Integration by Parts: $u = \ln x, dv = x^{-2}dx$. $\int_1^\infty \frac{\ln x}{x^2}dx = \lim_{t\to\infty}[-\frac{\ln x}{x} - \frac{1}{x}]_1^t = \lim_{t\to\infty}(-\frac{\ln t}{t} - \frac{1}{t}) - (0 - 1) = 0 - 0 + 1 = 1$. (Note: $\lim_{t\to\infty}\frac{\ln t}{t} = 0$ by L'Hôpital's Rule). The integral converges, so the series converges.

11. **Divergent**. This is a p-series $\sum \frac{1}{n^p}$ with $p = 1/5$. Since $p \leq 1$, the series diverges.

12. **Convergent**. This is a p-series with $p = 1.0001$. Since $p > 1$, the series converges.

13. **Divergent**. The series is $\sum \frac{n^{1/2}}{n} = \sum \frac{1}{n^{1/2}}$. This is a p-series with $p = 1/2$. Since $p \leq 1$, the series diverges.

14. **Convergent**. This is a p-series with $p = \pi/2 \approx 1.57$. Since $p > 1$, the series converges.

15. **Divergent**. Use the Test for Divergence: $\lim_{n\to\infty} \frac{3n-2}{5n+1} = \frac{3}{5} \neq 0$. The series diverges.

16. **Divergent**. Use the Test for Divergence: $\lim_{n\to\infty} \arctan(n) = \frac{\pi}{2} \neq 0$. The series diverges.

17. **Divergent**. Use the Test for Divergence: $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e \neq 0$. The series diverges.

18. **Convergent**. The series can be written as $\sum_{n=1}^{\infty} \frac{1}{n^3}$. This is a p-series with $p = 3$. Since $p > 1$, the series converges.

19. **Divergent**. The series is $\sum_{n=1}^{\infty} \frac{1}{2n+3}$. Let $f(x) = \frac{1}{2x+3}$. The integral $\int_1^{\infty} \frac{1}{2x+3} dx = \lim_{t\to\infty} [\frac{1}{2}\ln(2x+3)]_1^t = \infty$. The series diverges. (Can also use Limit Comparison Test with $\sum 1/n$).

20. **Convergent**. The series can be written as $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$. This is a p-series with $p = 3/2$. Since $p > 1$, the series converges.

21. **Divergent**. The series is $\sum_{n=2}^{\infty} \frac{\ln(n)}{n}$. The function $f(x) = \frac{\ln x}{x}$ is positive, continuous, and decreasing for $x \geq 3$. $\int_2^{\infty} \frac{\ln x}{x} dx = \lim_{t\to\infty} [\frac{1}{2}(\ln x)^2]_2^t = \infty$. The series diverges.

22. **Yes**. For $x \geq 2$:
    - **Positive:** For $x \geq 2$, $x > 0$ and $x^2 - 1 > 0$, so $f(x)$ is positive.
    - **Continuous:** $f(x)$ is a rational function, continuous wherever the denominator is not zero ($x \neq \pm 1$), so it is continuous on $[2, \infty)$.
    - **Decreasing:** $f'(x) = \frac{(x^2-1)(1)-x(2x)}{(x^2-1)^2} = \frac{-x^2-1}{(x^2-1)^2}$. Since the numerator is always negative and the denominator is always positive for $x \geq 2$, $f'(x) < 0$, so $f(x)$ is decreasing.

23. **Yes**. For $x \geq 1$:
    - **Positive:** Since $-1 \leq \cos(x) \leq 1$, the numerator $2 + \cos(x)$ is always between 1 and 3. The denominator x^2 is positive. So $f(x)$ is positive.
    - **Continuous:** The numerator and denominator are continuous, and the denominator is never zero on $[1, \infty)$, so $f(x)$ is continuous.
    - **Decreasing:** $f'(x) = \frac{-x\sin(x)-4-2\cos(x)}{x^3}$. For large x, the numerator is dominated by the -4 term, making $f'(x)$ negative. The function is eventually decreasing.

24. The Integral Test requires the function $f(x)$ to be **decreasing**. The function $f(x) = \frac{\sin^2(x)}{x^2}$ is not decreasing on $[1, \infty)$ because the $\sin^2(x)$ term oscillates between 0 and 1, causing the function to have many local maxima and minima.

25. **Divergent**. Use the Limit Comparison Test with the harmonic series $\sum \frac{1}{n}$. $\lim_{n\to\infty} \frac{(n+4)/(n^2+1)}{1/n} = \lim_{n\to\infty} \frac{n(n+4)}{n^2+1} = 1$. Since the limit is a finite positive number and $\sum \frac{1}{n}$ diverges, the series diverges. The Integral test could also be used.

26. **Convergent**. The function $f(x) = \frac{1}{x^2+3x+2}$ is positive, continuous, and decreasing for $x \geq 1$. This can be compared to $\sum \frac{1}{n^2}$, which converges. Using the Integral Test, $\int_1^\infty \frac{1}{(x+1)(x+2)}dx$ can be solved with partial fractions and is convergent.

27. **Divergent**. The series is $\sum_{n=2}^\infty \frac{1}{2n\ln(n)}$. This is a constant multiple (1/2) of the series $\sum_{n=2}^\infty \frac{1}{n\ln(n)}$, which diverges by the Integral Test (p-series test for logarithms with p=1).

28. **Divergent**. This is $5\sum n^{-2/3}$. It is a constant multiple of a p-series with $p = 2/3$. Since $p \leq 1$, the series diverges.

29. **Convergent**. Use the Integral Test with $f(x) = \frac{e^{-\sqrt{x}}}{\sqrt{x}}$. Let $u = -\sqrt{x}$, $du = -\frac{1}{2\sqrt{x}}dx$. $\int_1^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}}dx = \lim_{t\to\infty}[-2e^{-\sqrt{x}}]_1^t = \lim_{t\to\infty}(-2e^{-\sqrt{t}} + 2e^{-1}) = \frac{2}{e}$. The integral converges, so the series converges.

30. **Convergent**. Use the Integral Test. $f(x) = xe^{-x}$ is positive, continuous, and decreasing for $x \geq 1$. Integrate by parts: $\int_1^\infty xe^{-x}dx = \lim_{t\to\infty}[-xe^{-x} - e^{-x}]_1^t = (0-0) - (-e^{-1} - e^{-1}) = \frac{2}{e}$. The integral converges, so the series converges.

31. **Divergent**. Use the Test for Divergence. $\lim_{n\to\infty} n\sin(1/n) = \lim_{n\to\infty} \frac{\sin(1/n)}{1/n}$. Let $x = 1/n$. As $n \to \infty$, $x \to 0$. The limit becomes $\lim_{x\to 0} \frac{\sin(x)}{x} = 1 \neq 0$. The series diverges.

32. **Convergent**. This is a shifted p-series. Let $k = n - 4$. When $n = 5, k = 1$. The series is $\sum_{k=1}^\infty \frac{1}{k^3}$. This is a p-series with $p = 3$. Since $p > 1$, it converges.

33. **Divergent**. Use the Integral Test with $f(x) = \frac{\ln x}{x}$. Let $u = \ln x, du = \frac{1}{x}dx$. $\int_2^\infty \frac{\ln x}{x}dx = \lim_{t\to\infty}[\frac{1}{2}(\ln x)^2]_2^t = \infty$. The integral diverges, so the series diverges.

Concept Checklist and Problem Mapping

This checklist outlines the key concepts tested in this problem set. The numbers refer to the problems that primarily test each concept.

- **C1: The Integral Test Conditions:** Verifying if a function is continuous, positive, and decreasing.
  - Problems: 22, 23, 24
- **C2: Applying the Integral Test for Convergence/Divergence:**
  - **C2a: p-Series:** Directly applying the p-series test ($p > 1$ converges, $p \leq 1$ diverges).
    - Problems: 11, 12, 13, 14, 18, 20, 28, 32
  - **C2b: Logarithmic Functions:** Series of the form $\sum \frac{1}{n(\ln n)^p}$.
    - Problems: 8, 9, 21, 27, 33
  - **C2c: Exponential Functions:** Series involving exponential terms.
    - Problems: 4, 5, 29, 30
  - **C2d: Rational Functions:** Series where the corresponding integral is of a rational function.
    - Problems: 1, 2, 3, 25, 26
  - **C2e: Inverse Trig Functions:** Series where the integral leads to an arctan function.
    - Problems: 6, 7
- **C3: Prerequisite Skill - Evaluating Improper Integrals:**
  - **C3a, C3b:** Basic Power and Logarithmic Rules are implicit in many problems.
  - **C3c: U-Substitution:** Required for integral evaluation.
    - Problems: 1, 2, 3, 4, 5, 8, 9, 29, 33
  - **C3d: Integration by Parts:** Required for more complex integrals.
    - Problems: 10, 30
  - **C3e: Inverse Trig Integrals:** Integrals of the form $\int \frac{1}{x^2+a^2} dx$.
    - Problems: 6, 7
- **C4: Test for Divergence:** Applying the test where $\lim_{n\to\infty} a_n \neq 0$.
  - Problems: 15, 16, 17, 31
- **C5: Pattern Recognition:** Deducing the general term a_n from the first few terms of a series.
  - Problems: 18, 19, 20, 21