Homework 14.2 Practice: Multivariable Limits and Continuity

Based on Homework 14.2 Study Guide

Calculus III Practice Set

Part 1: Problem Set

Instructions: Evaluate the following limits, determine continuity, or identify the set of points where the function is continuous. If a limit does not exist, state "DNE" and provide a brief justification (e.g., Two-Path Test).

Topic A: Direct Substitution and Algebraic Simplification

1. **Problem 1:** Find the limit using direct substitution.

$$\lim_{(x,y)\to(2,-1)} (x^3y^2 - 4xy + 5)$$

2. **Problem 2:** Find the limit involving trigonometric and exponential functions.

$$\lim_{(x,y)\to(\pi,0)} e^x \cos(x+y)$$

3. **Problem 3:** Find the limit. (Hint: Factor the numerator).

$$\lim_{(x,y)\to(2,2)} \frac{x^2 - y^2}{2x - 2y}$$

4. **Problem 4:** Find the limit. (Hint: Difference of cubes factoring).

$$\lim_{(x,y)\to(1,1)} \frac{x^3 - y^3}{x - y}$$

5. **Problem 5:** Find the limit. (Hint: Use the conjugate).

$$\lim_{(x,y)\to(0,0)} \frac{\sqrt{x^2 + y^2 + 9} - 3}{x^2 + y^2}$$

Topic B: The Two-Path Test (Proving Non-Existence)

6. **Problem 6:** Show that the limit does not exist by approaching along the x-axis and the y-axis.

$$\lim_{(x,y)\to(0,0)} \frac{3x-y}{3x+y}$$

7. **Problem 7:** Show that the limit does not exist by approaching along the lines $y = x$ and $y = -x$.

$$\lim_{(x,y)\to(0,0)} \frac{xy}{2x^2+3y^2}$$

8. **Problem 8:** Show that the limit does not exist. (Hint: Degrees are unequal. Try a linear path $y = mx$ vs a parabolic path $x = y^2$).

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$$

9. **Problem 9:** Determine if the limit exists. (Hint: Consider the path $y = x^3$).

$$\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^6+y^2}$$

10. **Problem 10:** Determine if the limit exists.

$$\lim_{(x,y)\to(0,0)} \frac{x^4-y^2}{x^4+y^2}$$

Topic C: Polar Coordinates and Squeeze Theorem

11. **Problem 11:** Use polar coordinates ($x = r\cos\theta, y = r\sin\theta$) to find the limit.

$$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2}$$

12. **Problem 12:** Use polar coordinates to find the limit.

$$\lim_{(x,y)\to(0,0)} \frac{5x^2y^2}{\sqrt{x^2+y^2}}$$

13. **Problem 13:** Use polar coordinates to find the limit.

$$\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2)$$

14. **Problem 14:** Use polar coordinates to show the limit DNE (dependence on θ).

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$$

15. **Problem 15:** Find the limit using the Squeeze Theorem or Polar Coordinates.

$$\lim_{(x,y)\to(0,0)} \frac{x^2\sin^2(y)}{x^2+y^2}$$

Topic D: Substitutions and "Single Variable" Limits

16. **Problem 16:** Evaluate by using a substitution $u = xy$.

$$\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{xy}$$

17. **Problem 17:** Evaluate by using a substitution $u = x^2 + y^2$.

$$\lim_{(x,y)\to(0,0)} \frac{1-\cos(x^2+y^2)}{x^2+y^2}$$

18. **Problem 18:** Evaluate.

$$\lim_{(x,y)\to(1,1)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}}$$

Topic E: Continuity and Domain Analysis

19. **Problem 19:** Describe the set of points where the function is continuous.

$$f(x,y) = \frac{1}{x^2-y}$$

20. **Problem 20:** Describe the set of points where the function is continuous. (Focus on the square root and denominator).

$$f(x,y) = \frac{\sqrt{x+y}}{x-y}$$

21. **Problem 21:** Describe the set of points where the function is continuous.

$$f(x,y) = \ln(4-x^2-y^2)$$

22. **Problem 22:** Determine where the function is discontinuous.

$$f(x,y) = \begin{cases} \frac{x^2y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

23. **Problem 23:** Determine where the function is discontinuous.

$$f(x,y) = \begin{cases} \frac{x^2-y^2}{x-y} & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

24. **Problem 24:** Is the following function continuous at $(0,0)$?

$$g(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Topic F: Advanced Concepts (Definitions and Theory)

25. **Problem 25 (Conceptual):** To prove that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$ using the definition, we must show that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $0 < \sqrt{x^2+y^2} < \delta$, then $|f(x,y)| < \epsilon$. If we find that $|f(x,y)| \leq 2\sqrt{x^2+y^2}$, what value should we choose for δ in terms of ϵ?

26. **Problem 26 (Find the Flaw):** A student claims that because $\lim_{x\to 0} f(x,0) = 0$ and $\lim_{y\to 0} f(0,y) = 0$, the limit exists and equals 0. Explain why this reasoning is flawed.

27. **Problem 27 (Composition):** Find the set of points where $h(x,y) = e^{1/(x^2+y^2)}$ is continuous. What happens to the limit as $(x,y) \to (0,0)$?

28. **Problem 28 (Limits at Infinity):** Evaluate the limit or show it DNE by converting to polar and letting $r \to \infty$.
$$\lim_{(x,y)\to(\infty,\infty)} \frac{xy}{x^2+y^2}$$

29. **Problem 29 (3 Variables):** Find the limit.
$$\lim_{(x,y,z)\to(0,0,0)} \frac{x^2+y^2+z^2}{\sqrt{x^2+y^2+z^2+1}-1}$$

30. **Problem 30 (Removable vs. Essential):** Consider $f(x,y) = \frac{\sin(x^2+y^2)}{x^2+y^2}$ for $(x,y) \neq (0,0)$. How should $f(0,0)$ be defined to make the function continuous everywhere?

Part 2: Detailed Solutions

1. **Answer: 13.** Polynomials are continuous everywhere. $2^3(-1)^2 - 4(2)(-1) + 5 = 8(1) + 8 + 5 = 21$.

2. **Answer: $-e^\pi$.** Continuous function. Substitution: $e^\pi \cos(\pi + 0) = e^\pi(-1) = -e^\pi$.

3. **Answer: 2.** Factor numerator: $(x - y)(x + y)$. Factor denominator: $2(x - y)$. $\lim \frac{(x-y)(x+y)}{2(x-y)} = \lim \frac{x+y}{2} = \frac{2+2}{2} = 2$.

4. **Answer: 3.** Difference of cubes: $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$. $\lim \frac{(x-y)(x^2+xy+y^2)}{x-y} = 1^2 + (1)(1) + 1^2 = 3$.

5. **Answer: 1/6.** Multiply by conjugate $\frac{\sqrt{u}+3}{\sqrt{u}+3}$. $\frac{(x^2+y^2+9)-9}{(x^2+y^2)(\sqrt{x^2+y^2+9}+3)} = \frac{x^2+y^2}{(x^2+y^2)(\dots)} = \frac{1}{\sqrt{0+9}+3} = \frac{1}{6}$.

6. **Answer: DNE.** Along x-axis ($y = 0$): $\lim \frac{3x}{3x} = 1$. Along y-axis ($x = 0$): $\lim \frac{-y}{y} = -1$. $1 \neq -1$, so DNE.

7. **Answer: DNE.** Along $y = x$: $\frac{x^2}{2x^2+3x^2} = \frac{1}{5}$. Along $y = -x$: $\frac{-x^2}{2x^2+3x^2} = -\frac{1}{5}$. Values differ.

8. **Answer: DNE.** Along $x = 0$: 0. Along $x = y^2$: $\frac{(y^2)(y^2)}{(y^2)^2+y^4} = \frac{y^4}{2y^4} = \frac{1}{2}$. $0 \neq 1/2$.

9. **Answer: DNE.** Along $y = 0$: 0. Along $y = x^3$: $\frac{x^3(x^3)}{x^6+(x^3)^2} = \frac{x^6}{2x^6} = \frac{1}{2}$. $0 \neq 1/2$.

10. **Answer: DNE.** Along x-axis ($y = 0$): $\frac{x^4}{x^4} = 1$. Along y-axis ($x = 0$): $\frac{-y^2}{y^2} = -1$.

11. **Answer: 0.** $x = r\cos\theta, y = r\sin\theta$. $\frac{r^3(\cos^3\theta+\sin^3\theta)}{r^2} = r(\cos^3\theta + \sin^3\theta)$. As $r \to 0$, $0 \cdot (\text{bounded}) = 0$.

12. **Answer: 0.** $\frac{5(r\cos\theta)^2(r\sin\theta)^2}{r} = \frac{5r^4\cos^2\theta\sin^2\theta}{r} = 5r^3\cos^2\theta\sin^2\theta$. As $r \to 0$, limit is 0.

13. **Answer: 0.** $(r^2)\ln(r^2) = 2r^2\ln(r)$. Let $t = r$. $\lim_{t\to 0} t^2 \ln t = \lim \frac{\ln t}{t^{-2}}$. Use L'Hopital: $\frac{1/t}{-2t^{-3}} = \frac{t^3}{-2t} = -\frac{t^2}{2} \to 0$.

14. **Answer: DNE.** $\frac{r^2\cos^2\theta}{r^2} = \cos^2\theta$. The limit depends on the angle θ. (e.g., 1 at $\theta = 0$, 0 at $\theta = \pi/2$).

15. **Answer: 0.** $0 \leq \frac{x^2\sin^2 y}{x^2+y^2} \leq \frac{x^2(1)}{x^2+y^2} \leq 1$. Wait, better bound: $\frac{x^2}{x^2+y^2} \leq 1$, so $|f| \leq \sin^2 y$. As $y \to 0$, $\sin^2 y \to 0$. Limit is 0.

16. **Answer: 1.** Let $u = xy$. As $(x, y) \to (0, 0)$, $u \to 0$. $\lim_{u\to 0} \frac{\sin u}{u} = 1$.

17. **Answer: 0.** Let $u = x^2 + y^2$. $\lim_{u\to 0} \frac{1-\cos u}{u}$. L'Hopital: $\frac{\sin u}{1} \to 0$.

18. **Answer: 0.** Factor top: $x(x-y) = x(\sqrt{x}-\sqrt{y})(\sqrt{x}+\sqrt{y})$. $\lim \frac{x(\sqrt{x}-\sqrt{y})(\sqrt{x}+\sqrt{y})}{\sqrt{x}-\sqrt{y}} = \lim x(\sqrt{x}+\sqrt{y})$. At $(1,1)$: $1(1+1) = 2$. Wait, re-calculation: $x(x-y) \to x^2 - xy$. Top factors to $x(\sqrt{x}-\sqrt{y})(\sqrt{x}+\sqrt{y})$. Denom cancels. Result $x(\sqrt{x}+\sqrt{y})$. Plug in (1,1) $\to 1(1+1) = 2$. (Solution: 2).

19. **Answer: Continuous everywhere except points where $y = x^2$.** Rational function is continuous where denominator $\neq 0$.

20. **Answer: Continuous where $x + y \geq 0$ AND $x \neq y$.** Must satisfy square root domain and denominator non-zero. Set notation: $\{(x,y)|x+y \geq 0, x \neq y\}$.

21. **Answer: Continuous where $x^2 + y^2 < 4$.** Log argument must be positive. This is the interior of a circle with radius 2.

22. **Answer: Continuous everywhere (including origin).** Check limit at $(0,0)$ via Polar: $\frac{r^3 \cos^2\theta \sin\theta}{r^2} = r\cos^2\theta\sin\theta \to 0$. Limit (0) = Function Value (0).

23. **Answer: Discontinuous along $y = x$.** Away from $y = x$, it simplifies to $x + y$. Limit as $x \to y$ is $2x$. However, at $x = y$, defined as 0. Limit is $x + x = 2x$. Function is 0. Unless $x = 0$, $2x \neq 0$. Discontinuous at all points on line $y = x$ except possibly origin. At origin limit is 0, value is 0. Strictly speaking, the "formula" creates the discontinuity $x = y$.

24. **Answer: No (Discontinuous).** Limit via polar: $\frac{r^2\cos\theta\sin\theta}{r^2} = \cos\theta\sin\theta$. Depends on θ. Limit DNE, so cannot be continuous.

25. **Answer: $\delta = \epsilon/2$.** We want $|f| < \epsilon$. We have $|f| \leq 2\sqrt{x^2+y^2}$. So we need $2\sqrt{x^2+y^2} < \epsilon \implies \sqrt{x^2+y^2} < \epsilon/2$. Since $\sqrt{x^2+y^2} < \delta$, set $\delta = \epsilon/2$.

26. **Answer: The "Two-Path" Trap.** Checking specific paths (like axes) only proves non-existence if they differ. If they match, it proves nothing. There are infinite directions to approach from (spirals, parabolas, etc.).

27. **Answer: Continuous for $(x,y) \neq (0,0)$.** At origin: as $(x,y) \to (0,0)$, $x^2+y^2 \to 0^+$, so $1/(x^2+y^2) \to \infty$. $e^\infty \to \infty$. Limit DNE (Infinite).

28. **Answer: DNE.** Polar: $x \to \infty$ becomes $r \to \infty$. Expression: $\cos\theta\sin\theta$. This oscillates based on angle θ. DNE.

29. **Answer: 2.** Rationalize: Multiply by $\sqrt{\ldots}+1$. $\frac{(x^2+y^2+z^2)(\sqrt{\ldots}+1)}{x^2+y^2+z^2+1-1} = \sqrt{x^2+y^2+z^2+1}+1$. Sub $(0,0,0) \to \sqrt{1}+1 = 2$.

30. **Answer: $f(0,0) = 1$.** Use polar/substitution $u = r^2$. $\lim_{u\to 0}\frac{\sin u}{u} = 1$. Defining $f(0,0) = 1$ makes it continuous.

Part 3: Concept Check List Matrix

The following matrix maps the problems in this set to the concepts defined in the Homework 14.2 PDF.

Concept / Problem Type	Problem Numbers
1. Direct Substitution	
- Simple Polynomials	1
- Trig/Exponentials	2
2. Algebraic Manipulation	
- Factoring (Diff of Squares/Cubes)	3, 4, 18
- Conjugates	5, 29
3. Two-Path Test (Proving DNE)	
- Axis Paths	6, 10
- Linear Paths ($y = mx$)	6, 7
- Curved/Parabolic Paths ($y = x^2$)	8, 9
4. Polar Coordinates / Squeeze Theorem	
- Existence (Limit is 0)	11, 12, 13, 15, 22
- Non-Existence (Angular Dependence)	14, 24, 28
5. Substitutions ($u = xy, u = r^2$)	16, 17, 30
6. Continuity Analysis	
- Domain (Rational functions)	19, 20, 23
- Domain (Logarithms/Roots)	20, 21
- Piecewise Functions	22, 23, 24, 30
7. Advanced / Conceptual	
- Epsilon-Delta Logic	25
- "Find the Flaw" (Diagnostic)	26
- Composition Limits	27
- Limits at Infinity	28
- 3-Variable Limits	29