Homework 14.2: Multivariable Limits and Continuity

A Comprehensive Study Guide and Solution Set

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Contents

1 Part 1: Introduction, Context, and Prerequisites

1.1 Core Concepts

In single-variable calculus, a limit $\lim_{x\to a} f(x)$ exists if approaching a from the left yields the same value as approaching from the right. In multivariable calculus, specifically for functions of two variables $z = f(x, y)$, the domain is a plane, not a line.

To find $\lim_{(x,y)\to(a,b)} f(x, y)$, we must approach the point (a, b) from **infinite directions**. For the limit to exist, the function must approach the same value L along *every possible path* (lines, parabolas, spirals, etc.) passing through (a, b).

1.2 Intuition and Motivation

Imagine a hiking terrain described by a surface $z = f(x, y)$.

- **Limit Existence:** If you and your friends walk toward a specific map coordinate (a, b) from different directions (North, East, South-West, spiraling in), you should all meet at the same altitude L.
- **Continuity:** Not only must you meet at the same altitude, but there must also be actual ground beneath your feet at that point. If there is a hole in the terrain (a removable discontinuity) or a vertical cliff (an infinite discontinuity), the function is not continuous there.

1.3 Historical Context

The rigorous definition of limits (using $\epsilon - \delta$) was formalized by Augustin-Louis Cauchy and Karl Weierstrass in the 19th century. This development was motivated by the need to put calculus on a solid logical foundation. Early mathematicians struggled with "pathological" functions—surfaces that looked smooth but behaved chaotically at specific points. The rigorous theory of limits was necessary to distinguish between "well-behaved" smooth surfaces and those with subtle tears or singularities.

1.4 Key Formulas and Definitions

1. Definition of the Limit:

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

means that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta$, then $|f(x, y) - L| < \epsilon$.

2. Two-Path Test (To prove non-existence): If $f(x, y) \to L_1$ along path C_1 and $f(x, y) \to L_2$ along path C_2, and $L_1 \neq L_2$, then the limit **does not exist (DNE)**.

3. Continuity: A function f is continuous at (a, b) if:

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$$

1.5 Prerequisites

To master this topic, you must be proficient in:

- **Factoring Polynomials:** Difference of squares $(a^2 - b^2)$ and sum/difference of cubes.
- **L'Hôpital's Rule:** (Only valid after reducing a multivariable limit to a single variable limit).
- **Polar Coordinates:** Converting $x = r\cos\theta$, $y = r\sin\theta$, with $x^2 + y^2 = r^2$.
- **Domain Restrictions:** $\sqrt{A} \implies A \geq 0$, $\ln(A) \implies A > 0$, $\frac{1}{A} \implies A \neq 0$.

2 Part 2: Detailed Homework Solutions

Problem 1 (SCalcET9 14.2.005)

Find the limit:

$$\lim_{(x,y)\to(2,5)} (x^2y^3 - 7y^2)$$

Solution: This is a polynomial function. Polynomials are continuous everywhere. We can use **Direct Substitution**.

$$\begin{aligned} L &= (2)^2(5)^3 - 7(5)^2 \\ &= (4)(125) - 7(25) \\ &= 500 - 175 \\ &= 325 \end{aligned}$$

Final Answer: 325

Problem 2 (SCalcET9 14.2.008)

Find the limit:

$$\lim_{(x,y)\to(9,-3)} \frac{x^2y + xy^2}{x^2 - y^2}$$

Solution: First, attempt direct substitution to check for indeterminacy.

$$\text{Denominator: } x^2 - y^2 = (9)^2 - (-3)^2 = 81 - 9 = 72$$

Since the denominator is not zero ($72 \neq 0$), the function is continuous at this point. We can substitute directly.

$$\begin{aligned} \text{Numerator: } x^2y + xy^2 &= (9)^2(-3) + (9)(-3)^2 \\ &= 81(-3) + 9(9) \\ &= -243 + 81 \\ &= -162 \end{aligned}$$

$$\text{Limit} = \frac{-162}{72}$$

Simplify by dividing numerator and denominator by 18:

$$\frac{-162 \div 18}{72 \div 18} = \frac{-9}{4}$$

Alternative Algebraic Approach (Factoring):

$$\frac{xy(x+y)}{(x-y)(x+y)} = \frac{xy}{x-y}$$

Substitute $(9, -3)$:

$$\frac{9(-3)}{9-(-3)} = \frac{-27}{12} = -\frac{9}{4}$$

Final Answer: $-\frac{9}{4}$

Problem 3 (SCalcET9 14.2.009)

Find the limit:

$$\lim_{(x,y)\to(3\pi/2,\pi)} y\sin(x-y)$$

Solution: The function is a product of polynomial and trigonometric functions, which are continuous everywhere. Use Direct Substitution.

$$\begin{aligned} L &= \pi\sin\left(\frac{3\pi}{2}-\pi\right) \\ &= \pi\sin\left(\frac{\pi}{2}\right) \\ &= \pi(1) \\ &= \pi \end{aligned}$$

Final Answer: π

Problem 4 (SCalcET9 14.2.020)

Find the limit, if it exists:

$$\lim_{(x,y)\to(\pi,1/2)} 5e^{xy}\sin(xy)$$

Solution: Exponentials and sine functions are continuous on their domains. Use Direct Substitution.

$$\begin{aligned} L &= 5e^{(\pi)(1/2)}\sin((\pi)(1/2)) \\ &= 5e^{\pi/2}\sin\left(\frac{\pi}{2}\right) \\ &= 5e^{\pi/2}(1) \\ &= 5e^{\pi/2} \end{aligned}$$

Final Answer: $5e^{\pi/2}$

Problem 5 (SCalcET9 14.2.023)

Find the limit, if it exists:

$$\lim_{(x,y)\to(0,0)} \frac{xy^2\cos(y)}{5x^2+y^4}$$

Solution: Direct substitution yields 0/0. We must test different paths.

Path 1: Approach along the x-axis ($y=0$) Let $y=0$. As $x\to 0$:

$$\lim_{x\to 0}\frac{x(0)^2\cos(0)}{5x^2+0} = \lim_{x\to 0}\frac{0}{5x^2} = 0$$

Limit along this path is 0.

Path 2: Approach along the parabola $x=y^2$ Substitute $x=y^2$ into the function:

$$\begin{aligned} f(y^2,y) &= \frac{(y^2)(y^2)\cos(y)}{5(y^2)^2+y^4} \\ &= \frac{y^4\cos(y)}{5y^4+y^4} \\ &= \frac{y^4\cos(y)}{6y^4} \end{aligned}$$

Cancel y^4 (assuming $y \neq 0$):

$$\lim_{y \to 0} \frac{\cos(y)}{6} = \frac{1}{6}$$

Limit along this path is 1/6.

Conclusion: Since $0 \neq 1/6$, the limits along two different paths are different.

Final Answer: DNE

Problem 6 (SCalcET9 14.2.032.MI.SA)

Find the limit, if it exists:

$$\lim_{(x,y) \to (0,0)} \frac{xy}{\sqrt{x^2 + y^2}}$$

Solution: Direct substitution gives 0/0. The term $\sqrt{x^2 + y^2}$ suggests **Polar Coordinates**. Let $x = r\cos\theta$ and $y = r\sin\theta$. As $(x, y) \to (0, 0)$, $r \to 0^+$.

$$\begin{aligned} \lim_{r \to 0^+} \frac{(r\cos\theta)(r\sin\theta)}{\sqrt{r^2}} &= \lim_{r \to 0^+} \frac{r^2 \cos\theta \sin\theta}{r} \\ &= \lim_{r \to 0^+} r\cos\theta\sin\theta \end{aligned}$$

Since $-1 \leq \cos\theta\sin\theta \leq 1$, the term $\cos\theta\sin\theta$ is bounded. As $r \to 0$, $r \times (\text{bounded value}) \to 0$. The limit is 0 regardless of θ.

Final Answer: 0

Problem 7 (SCalcET9 14.2.039)

Analyze continuity:

$$f(x, y) = e^{1/(x-y)}$$

Graph the function and observe where it is discontinuous. Explanation using composition of functions.

Solution: The exponential function e^u is continuous everywhere. The discontinuity arises solely from the exponent $u = \frac{1}{x-y}$. A rational function is discontinuous where the denominator is zero.

$$x - y = 0 \implies y = x$$

Therefore, f is discontinuous along the line $y = x$.

Composition Explanation: Looking at $f(x, y) = e^{1/(x-y)}$ as a composition $(h \circ g)(x, y)$: Let $g(x, y) = \frac{1}{x-y}$. This is a rational function and is continuous everywhere except where $x - y = 0$. Let $h(t) = e^t$. Since $h(t)$ is continuous **everywhere** (on $\mathbb{R}$), the composition is continuous everywhere except at the discontinuity of the inner function g.

Fill-in-the-blanks: 1. f is discontinuous at: **The set of points** (x, y) **such that** $y = x$. 2. Since $h(t) = e^t$ is continuous **everywhere**, the composition is continuous everywhere except at the discontinuity above.

Final Answer: Discontinuous at $y = x$.

Problem 8 (SCalcET9 14.2.042)

Determine the set of points at which the function is continuous:

$$F(x, y) = \cos\left(\sqrt{1 + x - y}\right)$$

Solution: The cosine function is continuous everywhere. The square root function $\sqrt{u}$ is continuous for $u \geq 0$. Therefore, we require the inside of the square root to be non-negative:

$$1 + x - y \geq 0$$

Rearranging for y:

$$1 + x \geq y \quad \text{or} \quad y \leq x + 1$$

This describes the region on and below the line $y = x + 1$.

Final Answer: $\{(x, y) \mid y \leq x + 1\}$ (Select the last option in the screenshot).

Problem 9 (SCalcET9 14.2.046)

Determine the set of points at which the function is continuous:

$$G(x, y) = \ln(4 + x - y)$$

Solution: The natural logarithm $\ln(u)$ is continuous only when $u > 0$.

$$4 + x - y > 0$$

Rearranging for y:

$$4 + x > y \quad \text{or} \quad y < x + 4$$

This describes the region strictly below the line $y = x + 4$.

Final Answer: $\{(x, y) \mid y < x + 4\}$ (Select the first option in the screenshot).

Problem 10 (SCalcET9 14.2.049)

Determine the set of points at which the function is continuous:

$$f(x, y) = \begin{cases} \dfrac{x^2y^3}{8x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 1 & \text{if } (x, y) = (0, 0) \end{cases}$$

Solution: The function is rational away from $(0, 0)$ where the denominator is non-zero, so it is continuous for all $(x, y) \neq (0, 0)$. We must check continuity at $(0, 0)$. 1. Find the limit as $(x, y) \to (0, 0)$. Use Polar Coordinates.

$$x = r\cos\theta, \quad y = r\sin\theta$$

$$\lim_{r\to 0} \frac{(r^2\cos^2\theta)(r^3\sin^3\theta)}{8r^2\cos^2\theta + r^2\sin^2\theta} = \lim_{r\to 0} \frac{r^5\cos^2\theta\sin^3\theta}{r^2(8\cos^2\theta + \sin^2\theta)}$$

$$= \lim_{r\to 0} r^3 \left(\frac{\cos^2\theta\sin^3\theta}{8\cos^2\theta + \sin^2\theta}\right)$$

The denominator $8\cos^2\theta + \sin^2\theta \geq 1$ (it is never zero). The fraction is bounded. As $r \to 0$, $r^3 \to 0$. So, $\lim_{(x,y)\to(0,0)} f(x, y) = 0$.

2. Compare Limit to Defined Value. $f(0, 0)$ is defined as 1. Since Limit (0) $\neq$ Function Value (1), f is **not** continuous at $(0, 0)$.

Conclusion: The function is continuous everywhere except $(0, 0)$. This corresponds to the set where $(x, y) \neq (0, 0)$.

Final Answer: $\{(x, y) \mid (x, y) \neq (0, 0)\}$ (Select the third option).

3 Part 3: In-Depth Analysis of Problems and Techniques

3.1 A) Problem Types and General Approach

1. **Direct Substitution (Problems 1, 2, 3, 4):**
   - **Strategy:** Plug the numbers in immediately.
   - **Check:** Ensure the denominator is not zero and you don't take the square root/log of invalid numbers. If you get a number, you are done.
2. **The Two-Path Test (Problem 5):**
   - **Strategy:** Use this when substitution yields 0/0 and the degrees of numerator and denominator look somewhat "balanced" or specific paths simplify the expression.
   - **Common Paths:** Axis ($y = 0$ or $x = 0$), Linear ($y = mx$), Parabolic ($x = y^2$ or $y = x^2$).
3. **Polar Coordinates / Squeeze Theorem (Problem 6, 10):**
   - **Strategy:** Use this when you have terms like $x^2 + y^2$ in the denominator and direct sub yields 0/0.
   - **Method:** Swap $x \to r\cos\theta, y \to r\sin\theta$. If the limit depends only on r and vanishes as $r \to 0$ (independent of θ), the limit exists.
4. **Domain Continuity (Problems 7, 8, 9):**
   - **Strategy:** Identify the restriction function (denominator, sqrt, log). Set up the inequality (e.g., inside of log > 0) and solve for y.

3.2 B) Key Algebraic and Calculus Manipulations

- **The Parabolic Path Trick (Problem 5):** For $\frac{xy^2}{5x^2+y^4}$, notice that x^2 is degree 2 and y^4 is degree 4. To make them "compatible" (addable), we chose $x = y^2$, turning x^2 into $(y^2)^2 = y^4$. This allowed terms to combine in the denominator.
- **Polar Bounding (Problem 10):** When simplifying polar limits, we often isolate the "angular part" (functions of θ). If the denominator is strictly non-zero (e.g., $8\cos^2\theta + \sin^2\theta \geq 1$), the whole angular fraction is bounded. An r^n term multiplying a bounded term goes to 0.

4 Part 4: "Cheatsheet" and Tips for Success

Summary of Formulas

- **Polar Conversion:** $x = r\cos\theta, \quad y = r\sin\theta, \quad x^2 + y^2 = r^2$.
- **Continuous Functions:** Polynomials, sin, cos, e^x are continuous everywhere. Rationals, ln, $\sqrt{x}$ are continuous on their domains.

Tricks and Shortcuts

- **Degree Analysis Shortcut:** Look at $f(x, y) = \frac{P(x,y)}{Q(x,y)}$. Sum the exponents of terms.
  - If total degree of Numerator $>$ total degree of Denominator, Limit is likely 0 (Use Polar).
  - If Num Degree = Denom Degree, Limit likely DNE (Use Two-Path: $y = mx$).
  - If you see $x^2 + y^2$, immediately think **Polar**.

Common Pitfalls

- **The "Two Paths Match" Trap:** Finding that $\lim_{x\to 0} = 0$ and $\lim_{y\to 0} = 0$ **does NOT** prove the limit is 0. You must use Polar or Squeeze theorem to prove existence. You can only use paths to prove *non-existence*.
- **Algebraic Errors:** $(x+y)^2 \neq x^2 + y^2$.

5 Part 5: Conceptual Synthesis and The "Big Picture"

5.1 Thematic Connections

The core theme of Multivariable Limits is **Locality vs. Directionality**. In 1D, you only have "left" and "right". In 2D, a point $(0, 0)$ is surrounded by a 360-degree neighborhood. The behavior of the function must be consistent regardless of the angle of approach. This concept prepares you for **Directional Derivatives** (how slope changes based on direction) and **Gradients**.

5.2 Forward and Backward Links

- **Backward:** Relies heavily on Limits (Calc I) and Parametric/Polar equations (Calc II).
- **Forward:** Continuity is required for **Partial Differentiation**. If a function is not continuous, it cannot be differentiable. Furthermore, defining Multiple Integrals (Volume under a surface) requires the surface to be largely continuous.

6 Part 6: Real-World Application and Modeling

6.1 A) Concrete Scenarios (Finance & Economics)

1. **Options Pricing (Black-Scholes Model):** The Black-Scholes equation for pricing European options relies on variables like Stock Price (S), Time (t), and Volatility (σ). As time approaches expiration ($t \to T$), the value of the option must converge continuously to the Payoff Function $\max(S - K, 0)$. Analyzing this limit ensures the model is consistent with contract rules at expiration.
2. **Portfolio Correlation Singularities:** In Modern Portfolio Theory, risk is calculated using a correlation matrix of assets. If the correlation ρ between two assets approaches 1 (perfect correlation), the determinant of the covariance matrix approaches 0. This creates a singularity when trying to invert the matrix for optimization. Understanding the limit behavior as $\rho \to 1$ is crucial for building stable algorithmic trading models.

6.2 B) Model Problem Setup: High-Frequency Trading Continuity

Scenario: An algorithmic trading bot executes trades based on a liquidity function $L(v, t)$, where v is trade volume and t is time gap between trades.

$$L(v, t) = \frac{v \cdot t}{\sqrt{v^2 + t^2}}$$

Problem: The algorithm crashes if $L(v, t)$ is undefined or jumps discontinuously. **Task:** Determine if the liquidity score is stable as volume and time gap both vanish $(v, t) \to (0, 0)$. This is exactly Problem 6 from your homework. We proved the limit is 0, so the function can be defined as $L(0, 0) = 0$ to ensure stability.

7 Part 7: Common Variations and Untested Concepts

Your homework covered the basics, but omitted a few advanced standard curriculum concepts:

7.1 1. The Epsilon-Delta Proof

The homework relied on techniques to find answers, but not the rigorous proof. **Concept:** Proving $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{\sqrt{x^2+y^2}} = 0$ using definition. **Example:** We need to show $|\frac{3x^2y}{\sqrt{x^2+y^2}} - 0| < \epsilon$. Using $|y| \le \sqrt{x^2+y^2}$ and $x^2 \le x^2+y^2$:

$$\left|\frac{3x^2y}{\sqrt{x^2+y^2}}\right| \le \frac{3(x^2+y^2)\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} = 3(x^2+y^2)$$

We want $3(x^2+y^2) < \epsilon$, or $\sqrt{x^2+y^2} < \sqrt{\epsilon/3}$. Thus, choose $\delta = \sqrt{\epsilon/3}$.

7.2 2. Limits at Infinity

Concept: Limits where $x \to \infty$ or $y \to \infty$. **Example:** $\lim_{(x,y)\to(\infty,\infty)} \frac{xy}{x^2+y^2}$. Convert to polar: $r \to \infty$.

$$\frac{r^2\cos\theta\sin\theta}{r^2} = \cos\theta\sin\theta$$

As $r \to \infty$, the value oscillates between -0.5 and 0.5 depending on θ. The limit DNE.

8 Part 8: Advanced Diagnostic Testing: "Find the Flaw"

Problem 1: Path Independence? *Student Work:* Evaluate $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$. Path 1 ($x = 0$): Limit is 0. Path 2 ($y = x$): $\frac{x^3}{x^4+x^2} \approx \frac{x^3}{x^2} = x \to 0$. *Conclusion:* Since both paths are 0, the limit is 0. **Flaw:** Checking linear paths is insufficient. **Correction:** Use path $y = x^2$. Limit becomes $\frac{x^2(x^2)}{x^4+(x^2)^2} = \frac{x^4}{2x^4} = 1/2$. Limit DNE.

Problem 2: L'Hôpital's Rule Misuse *Student Work:* Evaluate $\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{xy}$. Apply L'Hôpital's Rule to top and bottom: $\lim \frac{\cos(xy)(y)}{y} = \cos(0) = 1$. **Flaw:** You cannot apply L'Hôpital's Rule directly to multivariable functions. **Correction:** Let $u = xy$. As $(x,y) \to (0,0)$, $u \to 0$. $\lim_{u\to 0} \frac{\sin u}{u} = 1$ via single-variable Calc I definition.

Problem 3: Direct Sub Algebra Error *Student Work:* $\lim_{(x,y)\to(1,1)} \frac{x-y}{x^2-y^2} = \frac{0}{0}$. Factor: $\frac{x-y}{(x-y)^2} = \frac{1}{x-y}$. Sub in (1,1) $\to 1/0 \to \infty$. **Flaw:** Factoring x^2-y^2 gives $(x-y)(x+y)$, not $(x-y)^2$. **Correction:** $\frac{x-y}{(x-y)(x+y)} = \frac{1}{x+y}$. Limit is 1/2.

Problem 4: Polar Bound Error *Student Work:* $\lim_{(x,y)\to(0,0)} \frac{2x}{x^2+y+1}$. Use Polar. $\frac{2r\cos\theta}{r^2+r\sin\theta+1}$. As $r \to 0$, num $\to 0$, den $\to 1$. Limit 0. **Flaw:** This solution is actually correct, but the *method* is overkill. **Correction:** This is a continuous function. Just use direct substitution. $\frac{0}{0+0+1} = 0$. Using polar unnecessarily complicates simple problems.

Problem 5: Continuity vs Limit *Student Work:* Function $f(x,y) = \frac{xy}{x^2+y^2}$ if $\neq (0,0)$, else 0. Limit DNE (proved by paths). *Conclusion:* The function is continuous because $f(0,0)$ is defined. **Flaw:** Existence of value $f(0,0)$ does not imply continuity. The limit must exist AND equal the value. **Correction:** Since Limit DNE, function is discontinuous at $(0,0)$.