Homework 8.2 Area of a Surface of Revolution

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1 A Comprehensive Introduction to Surface Area of Revolution

The "Area of a Surface of Revolution" is the surface area of a 3D shape created by rotating a 2D curve around an axis. Think of a potter's wheel: the 2D profile of the vase is rotated around a central axis to create the 3D object. Our goal is to find the area of that outer surface.

The Surface Area Formula

The formula is a logical extension of the arc length formula. Recall that the length of a tiny piece of a curve is ds. To get the surface area, we rotate this tiny piece around an axis. This creates a thin band, or "frustum." The surface area of this band is approximately its circumference times its length (ds).

The circumference of the band depends on its distance from the axis of rotation, which we call the radius, r. So, the area of one tiny band is $2\pi r \cdot ds$. To find the total surface area, we integrate this expression along the curve.

The key is to correctly identify the radius (r) and the arc length element (ds).

- ds is the same as in arc length: $ds = \sqrt{1 + (y')^2}\, dx$ or $ds = \sqrt{1 + (x')^2}\, dy$.
- The radius, r, is the distance from the curve to the axis of rotation.
  - If rotating around the **x-axis**, the radius at any point is its y-coordinate. So, $r = y$.
  - If rotating around the **y-axis**, the radius at any point is its x-coordinate. So, $r = x$.

This gives us four primary formulas:

Rotation About the x-axis

1. If $y = f(x)$ is given and you integrate with respect to x:

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

2. If $x = g(y)$ is given and you integrate with respect to y:

$$S = \int_c^d 2\pi y \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

Rotation About the y-axis

1. If $y = f(x)$ is given and you integrate with respect to x:

$$S = \int_a^b 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

2. If $x = g(y)$ is given and you integrate with respect to y:

$$S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

Crucial Tip: The arc length part (ds) is identical to what you learned in section 8.1. The only new piece is multiplying by the circumference ($2\pi r$). The algebraic "perfect square" tricks from arc length are still the most important skill for solving these problems exactly.

2 Surface Area Problems and Solutions

2.1 Problem 1

The curve $y = \sqrt[3]{x}$, $1 \leq x \leq 8$ is rotated about the x-axis. Set up integrals for the area with respect to (a) x and (b) y.

Solution (a) - with respect to x

The formula is $S = \int 2\pi y\, ds$. Here $y = x^{1/3}$. Derivative: $\frac{dy}{dx} = \frac{1}{3}x^{-2/3}$. Arc length element: $ds = \sqrt{1 + (\frac{1}{3}x^{-2/3})^2}\, dx = \sqrt{1 + \frac{1}{9}x^{-4/3}}\, dx$. Integral: **Answer (a):** $\int_1^8 2\pi x^{1/3}\sqrt{1 + \frac{1}{9x^{4/3}}}\, dx$

Solution (b) - with respect to y

The formula is still $S = \int 2\pi y\, ds$, but we need everything in terms of y. Original function: $y = x^{1/3} \implies x = y^3$. Bounds: If $x = 1, y = 1$. If $x = 8, y = 2$. So $1 \leq y \leq 2$. Derivative: $\frac{dx}{dy} = 3y^2$. Arc length element: $ds = \sqrt{1 + (3y^2)^2}\, dy = \sqrt{1 + 9y^4}\, dy$. Integral: **Answer (b):** $\int_1^2 2\pi y\sqrt{1 + 9y^4}\, dy$

2.2 Problem 2

For $x = y + y^3$, $0 \leq y \leq 3$, set up integrals for rotation about the x-axis and y-axis, then use a calculator.

Solution (a) - Setup

Since the function is $x = g(y)$, we'll integrate with respect to y. Derivative: $\frac{dx}{dy} = 1 + 3y^2$. Arc length element: $ds = \sqrt{1 + (1 + 3y^2)^2}\, dy = \sqrt{1 + 1 + 6y^2 + 9y^4}\, dy = \sqrt{2 + 6y^2 + 9y^4}\, dy$. (i) Rotation about **x-axis**: Radius is $r = y$. **Answer (i) setup:** $S = \int_0^3 2\pi y\sqrt{2 + 6y^2 + 9y^4}\, dy$ (ii) Rotation about **y-axis**: Radius is $r = x$. We must substitute $x = y + y^3$. **Answer (ii) setup:** $S = \int_0^3 2\pi(y + y^3)\sqrt{2 + 6y^2 + 9y^4}\, dy$

Solution (b) - Calculator Evaluation

(i) Using a numerical integrator for the x-axis integral gives approximately **892.4938**. (ii) Using a numerical integrator for the y-axis integral gives approximately **2651.5230**.

2.3 Problem 3

Find the surface area generated by rotating $y = e^{2x}$, $0 \leq x \leq 4$, about the x-axis. (This is a fill-in-the-blanks example problem).

Solution

$y = e^{2x}$, so $\frac{dy}{dx} = 2e^{2x}$. $S = \int_0^4 2\pi y\sqrt{1 + (y')^2}\, dx = \int_0^4 2\pi e^{2x}\sqrt{1 + (2e^{2x})^2}\, dx = \int_0^4 2\pi e^{2x}\sqrt{1 + 4e^{4x}}\, dx$. Use u-substitution: $u = 2e^{2x}, du = 4e^{2x}dx \implies 2\pi e^{2x}dx = \frac{\pi}{2}du$. The integral becomes $\int_{u(0)}^{u(4)} \frac{\pi}{2}\sqrt{1 + u^2}\, du$. This requires trig substitution $u = \tan\theta$. The final result given in the problem is derived from this complex integration. **Answer (final value):** $\frac{\pi}{4}[2e^8\sqrt{1 + 4e^{16}} + \ln(2e^8 + \sqrt{1 + 4e^{16}}) - 2\sqrt{5} - \ln(2 + \sqrt{5})]$

2.4 Problem 4

Find the exact area of the surface obtained by rotating $y = \sqrt{1+e^x}$, $0 \le x \le 1$, about the x-axis.

Solution

Derivative: $\frac{dy}{dx} = \frac{e^x}{2\sqrt{1+e^x}}$. Simplify $1 + (y')^2$:

$$1 + \left(\frac{e^x}{2\sqrt{1+e^x}}\right)^2 = 1 + \frac{e^{2x}}{4(1+e^x)} = \frac{4(1+e^x)+e^{2x}}{4(1+e^x)} = \frac{4+4e^x+e^{2x}}{4(1+e^x)} = \frac{(2+e^x)^2}{4(1+e^x)}$$

Set up the integral:

$$\begin{aligned} S &= \int_0^1 2\pi y\sqrt{1+(y')^2}\,dx = \int_0^1 2\pi\sqrt{1+e^x}\sqrt{\frac{(2+e^x)^2}{4(1+e^x)}}\,dx \\ &= \int_0^1 2\pi\sqrt{1+e^x}\frac{2+e^x}{2\sqrt{1+e^x}}\,dx = \int_0^1 \pi(2+e^x)\,dx \\ &= \pi[2x+e^x]_0^1 = \pi[(2+e)-(0+e^0)] = \pi(2+e-1) \end{aligned}$$

Answer: $\pi(e+1)$

2.5 Problem 5

The arc $y = 2x^2$ from (2, 8) to (6, 72) is rotated about the y-axis. (This is a fill-in-the-blanks example problem).

Solution 1 - with respect to x

Rotate about y-axis, so radius is $r = x$. Formula: $S = \int 2\pi x\,ds$. Derivative: $\frac{dy}{dx} = 4x$. $S = \int_2^6 2\pi x\sqrt{1+(4x)^2}\,dx = \int_2^6 2\pi x\sqrt{1+16x^2}\,dx$. Use u-substitution: $u = 1+16x^2, du = 32xdx \implies 2\pi xdx = \frac{\pi}{16}du$. Bounds: $u(2) = 65, u(6) = 577$. $S = \int_{65}^{577} \frac{\pi}{16}\sqrt{u}\,du = \frac{\pi}{16}[\frac{2}{3}u^{3/2}]_{65}^{577} = \frac{\pi}{24}(577^{3/2} - 65^{3/2})$.

Solution 2 - with respect to y

Rotate about y-axis, so radius is $r = x$. Formula: $S = \int 2\pi x\,ds$. Function: $x = \sqrt{y/2}$. Derivative: $\frac{dx}{dy} = \frac{1}{\sqrt{2}}\frac{1}{2\sqrt{y}} = \frac{1}{2\sqrt{2y}}$. $1+(x')^2 = 1+\frac{1}{8y} = \frac{8y+1}{8y}$. Bounds: $8 \le y \le 72$. $S = \int_8^{72} 2\pi\sqrt{\frac{y}{2}}\sqrt{\frac{8y+1}{8y}}\,dy = \int_8^{72} 2\pi\frac{\sqrt{y}}{\sqrt{2}}\frac{\sqrt{8y+1}}{\sqrt{8y}}\,dy = \int_8^{72}\frac{2\pi}{4}\sqrt{8y+1}\,dy$. Use u-sub: $u = 8y+1, du = 8dy \implies dy = du/8$. $S = \int_{65}^{577}\frac{\pi}{2}\sqrt{u}\frac{du}{8} = \frac{\pi}{16}\int_{65}^{577}\sqrt{u}\,du$, which is the same as Solution 1. **Answer:** $\frac{\pi}{24}(577\sqrt{577} - 65\sqrt{65})$

2.6 Problem 6

Set up an integral for the surface area of $y = x^4$, $0 \le x \le 1$, rotated about (a) the x-axis and (b) the y-axis.

Solution

We integrate with respect to x. Derivative: $\frac{dy}{dx} = 4x^3$. Arc length element: $ds = \sqrt{1+(4x^3)^2}\,dx = \sqrt{1+16x^6}\,dx$. (a) Rotation about **x-axis**: Radius $r = y = x^4$. **Answer (a):** $\int_0^1 2\pi x^4\sqrt{1+16x^6}\,dx$ (b) Rotation about **y-axis**: Radius $r = x$. **Answer (b):** $\int_0^1 2\pi x\sqrt{1+16x^6}\,dx$

2.7 Problem 7

Set up an integral for the surface area of $x = \sqrt{y-y^2}$ rotated about (a) the x-axis and (b) the y-axis.

Solution

The domain requires $y - y^2 \geq 0 \implies y(1-y) \geq 0 \implies 0 \leq y \leq 1$. We integrate with respect to y. Derivative: $\frac{dx}{dy} = \frac{1-2y}{2\sqrt{y-y^2}}$. $1 + (x')^2 = 1 + \frac{(1-2y)^2}{4(y-y^2)} = \frac{4y-4y^2+1-4y+4y^2}{4(y-y^2)} = \frac{1}{4y(1-y)}$. Arc length element: $ds = \sqrt{\frac{1}{4y(1-y)}}\,dy$. (a) Rotation about **x-axis**: Radius $r = y$. **Answer (a):** $\int_0^1 2\pi y\sqrt{\frac{1}{4y(1-y)}}\,dy$ (b) Rotation about **y-axis**: Radius $r = x = \sqrt{y-y^2}$. **Answer (b):** $\int_0^1 2\pi\sqrt{y-y^2}\sqrt{\frac{1}{4y(1-y)}}\,dy$

2.8 Problem 8

Set up an integral for the surface area of $y = \tan^{-1}(x)$, $0 \leq x \leq 1$, rotated about (a) the x-axis and (b) the y-axis.

Solution

We integrate with respect to x. Derivative: $\frac{dy}{dx} = \frac{1}{1+x^2}$. Arc length element: $ds = \sqrt{1 + \left(\frac{1}{1+x^2}\right)^2}\,dx = \sqrt{1 + \frac{1}{(1+x^2)^2}}\,dx$. (a) Rotation about **x-axis**: Radius $r = y = \tan^{-1}(x)$. **Answer (a):** $\int_0^1 2\pi\tan^{-1}(x)\sqrt{1+\frac{1}{(1+x^2)^2}}\,dx$ (b) Rotation about **y-axis**: Radius $r = x$. **Answer (b):** $\int_0^1 2\pi x\sqrt{1+\frac{1}{(1+x^2)^2}}\,dx$

2.9 Problem 9

Find the exact area of the surface from rotating $y = \sqrt{8-x}$, $2 \leq x \leq 8$, about the x-axis.

Solution

Derivative: $y = (8-x)^{1/2} \implies \frac{dy}{dx} = \frac{1}{2}(8-x)^{-1/2}(-1) = \frac{-1}{2\sqrt{8-x}}$. $1 + (y')^2 = 1 + \frac{1}{4(8-x)} = \frac{32-4x+1}{4(8-x)} = \frac{33-4x}{4(8-x)}$. Set up the integral:

$$\begin{aligned} S &= \int_2^8 2\pi y\,ds = \int_2^8 2\pi\sqrt{8-x}\sqrt{\frac{33-4x}{4(8-x)}}\,dx \\ &= \int_2^8 2\pi\sqrt{8-x}\frac{\sqrt{33-4x}}{2\sqrt{8-x}}\,dx = \pi\int_2^8 \sqrt{33-4x}\,dx \end{aligned}$$

Use u-substitution: $u = 33-4x, du = -4dx$. $S = \pi\int_{25}^1 \sqrt{u}\left(-\frac{du}{4}\right) = \frac{\pi}{4}\int_1^{25} u^{1/2}\,du = \frac{\pi}{4}[\frac{2}{3}u^{3/2}]_1^{25} = \frac{\pi}{6}(25^{3/2} - 1^{3/2}) = \frac{\pi}{6}(125-1)$. **Answer:** $\frac{124\pi}{6} = \frac{62\pi}{3}$

2.10 Problem 10

Find the surface area of rotating $y = \frac{x^2}{4} - \frac{1}{2}\ln(x)$, $4 \leq x \leq 5$, about the y-axis.

Solution

This is a classic "perfect square" problem, similar to one from the arc length homework. Derivative: $\frac{dy}{dx} = \frac{x}{2} - \frac{1}{2x}$. $1 + (y')^2 = 1 + (\frac{x^2}{4} - \frac{1}{2} + \frac{1}{4x^2}) = \frac{x^2}{4} + \frac{1}{2} + \frac{1}{4x^2} = (\frac{x}{2} + \frac{1}{2x})^2$. Rotation is about the y-axis, so radius $r = x$.

$$\begin{aligned} S &= \int_4^5 2\pi x\sqrt{(\frac{x}{2}+\frac{1}{2x})^2}\,dx = \int_4^5 2\pi x(\frac{x}{2}+\frac{1}{2x})\,dx \\ &= \pi\int_4^5 (x^2+1)\,dx = \pi[\frac{x^3}{3}+x]_4^5 \\ &= \pi[(\frac{125}{3}+5) - (\frac{64}{3}+4)] = \pi[\frac{61}{3}+1] \end{aligned}$$

Answer: $\frac{64\pi}{3}$

2.11 Problem 11

Determine the surface area of Gabriel's horn, formed by rotating $y = 1/x$ for $x \geq 1$ about the x-axis.

Solution

This is an improper integral for surface area. Derivative: $y = x^{-1} \implies \frac{dy}{dx} = -x^{-2} = -\frac{1}{x^2}$. $1+(y')^2 = 1+\frac{1}{x^4} = \frac{x^4+1}{x^4}$. Rotation is about the x-axis, radius $r = y = 1/x$.

$$\begin{aligned} S &= \int_1^\infty 2\pi y \, ds = \int_1^\infty 2\pi \frac{1}{x} \sqrt{\frac{x^4+1}{x^4}} \, dx \\ &= \int_1^\infty 2\pi \frac{1}{x} \frac{\sqrt{x^4+1}}{x^2} \, dx = 2\pi \int_1^\infty \frac{\sqrt{x^4+1}}{x^3} \, dx \end{aligned}$$

To determine convergence, we use the Comparison Test. For large x, $\sqrt{x^4+1} \approx \sqrt{x^4} = x^2$. So, the integrand behaves like $\frac{x^2}{x^3} = \frac{1}{x}$. We know that $\int_1^\infty \frac{1}{x} \, dx$ diverges (p-integral with p=1). Since our integrand is greater than the divergent function $2\pi/x$ (because $\sqrt{x^4+1} > x^2$), our integral must also diverge. **Answer:** The surface area is infinite (DIVERGES).

2.12 Problem 12

The curve $y = 4 + \sin(x)$, $0 \leq x \leq \pi/2$ is rotated about the y-axis. Set up integrals with respect to (a) x and (b) y.

Solution (a) - with respect to x

Rotate about y-axis, radius $r = x$. Derivative: $\frac{dy}{dx} = \cos(x)$. $ds = \sqrt{1+\cos^2(x)} \, dx$. **Answer (a):** $\int_0^{\pi/2} 2\pi x \sqrt{1+\cos^2(x)} \, dx$

Solution (b) - with respect to y

Rotate about y-axis, radius $r = x$. We need x and dx/dy in terms of y. $y - 4 = \sin(x) \implies x = \arcsin(y-4)$. Derivative: $\frac{dx}{dy} = \frac{1}{\sqrt{1-(y-4)^2}}$. $ds = \sqrt{1+\left(\frac{1}{\sqrt{1-(y-4)^2}}\right)^2} \, dy$. Bounds: $x = 0 \implies y = 4$. $x = \pi/2 \implies y = 5$. **Answer (b):** $\int_4^5 2\pi \arcsin(y-4) \sqrt{1+\frac{1}{1-(y-4)^2}} \, dy$

2.13 Problem 13

The curve $x = e^{8y}$, $0 \leq y \leq 2$ is rotated about the y-axis. Set up integrals with respect to (a) x and (b) y.

Solution (b) - with respect to y

This is the natural way. Rotate about y-axis, radius $r = x = e^{8y}$. Derivative: $\frac{dx}{dy} = 8e^{8y}$. $ds = \sqrt{1+(8e^{8y})^2} \, dy = \sqrt{1+64e^{16y}} \, dy$. **Answer (b):** $\int_0^2 2\pi e^{8y} \sqrt{1+64e^{16y}} \, dy$

Solution (a) - with respect to x

Rotate about y-axis, radius $r = x$. Function: $x = e^{8y} \implies \ln(x) = 8y \implies y = \frac{1}{8}\ln(x)$. Derivative: $\frac{dy}{dx} = \frac{1}{8x}$. $ds = \sqrt{1+(\frac{1}{8x})^2} \, dx = \sqrt{1+\frac{1}{64x^2}} \, dx$. Bounds: $y = 0 \implies x = 1$. $y = 2 \implies x = e^{16}$. **Answer (a):** $\int_1^{e^{16}} 2\pi x \sqrt{1+\frac{1}{64x^2}} \, dx$

2.14 Problem 14

Find the exact area of rotating $y = x^3$, $0 \leq x \leq 3$, about the x-axis.

Solution

Rotate about x-axis, radius $r = y = x^3$. Derivative: $\frac{dy}{dx} = 3x^2$. $ds = \sqrt{1 + (3x^2)^2}\,dx = \sqrt{1 + 9x^4}\,dx$. Integral setup: $S = \int_0^3 2\pi x^3 \sqrt{1 + 9x^4}\,dx$. Use u-substitution: $u = 1 + 9x^4, du = 36x^3dx \implies 2\pi x^3dx = \frac{2\pi}{36}du = \frac{\pi}{18}du$. Bounds: $u(0) = 1, u(3) = 1 + 9(81) = 730$. $S = \int_1^{730} \frac{\pi}{18}\sqrt{u}\,du = \frac{\pi}{18}[\frac{2}{3}u^{3/2}]_1^{730} = \frac{\pi}{27}(730^{3/2} - 1)$. **Answer:** $\frac{\pi}{27}(730\sqrt{730} - 1)$

2.15 Problem 15

Find the exact area of rotating $y = x^3$, $0 \le x \le 2$, about the x-axis.

Solution

This is identical to Problem 14, with a different upper bound. The setup is $S = \int_0^2 2\pi x^3 \sqrt{1 + 9x^4}\,dx$. Using the same u-substitution $u = 1 + 9x^4$. Bounds: $u(0) = 1, u(2) = 1 + 9(16) = 145$. $S = \int_1^{145} \frac{\pi}{18}\sqrt{u}\,du = \frac{\pi}{18}[\frac{2}{3}u^{3/2}]_1^{145} = \frac{\pi}{27}(145^{3/2} - 1)$. **Answer:** $\frac{\pi}{27}(145\sqrt{145} - 1)$

3 Analysis of Problems and Techniques

3.1 Problem Types and General Approach

1. **Setup Problems:** Several problems (1, 2, 3, 5, 6, 7, 8, 12, 13) ask you to set up the integral, sometimes for both x and y variables, or for rotation around both axes. This emphasizes the most important skill: choosing the correct formula and finding all the components (r, derivative, bounds).

2. **Exact Evaluation Problems:** These problems (4, 9, 10, 14, 15) require you to fully solve the integral. They are almost always designed to simplify nicely.

3. **The "Perfect Square" Trick:** Problem 10 is a prime example. The algebraic structure is identical to arc length problems where $1 + (y')^2$ becomes a perfect square, canceling the radical.

4. **The "Canceling Radical" Trick:** Problem 4 and 9 demonstrate another common pattern. The original function y contains a radical, and the ds term simplifies in such a way that this radical in the radius term ($2\pi y$) is canceled out by part of the ds term.

5. **Rotation about x-axis vs. y-axis:** The key is the radius. For x-axis rotation, $r = y$. For y-axis rotation, $r = x$. You must substitute the function expression if needed (e.g., for x-axis rotation where $r = y$, you substitute $y = f(x)$). This was tested in nearly every problem.

6. **Integration with respect to x vs. y:** Problems 1, 5, 12, and 13 explicitly ask you to set up integrals with respect to both variables. This tests your ability to invert the function ($y = f(x) \leftrightarrow x = g(y)$), change the bounds, and calculate the appropriate derivative (dy/dx vs dx/dy).

7. **Improper Integrals:** Problem 11 (Gabriel's Horn) introduces an improper integral, requiring you to evaluate a limit and use comparison tests to determine if the area converges or diverges.

3.2 Key Algebraic and Calculus Manipulations

- **Mastering the $1 + (y')^2$ simplification:** This is the single most important skill. Always simplify this term fully before putting it under the radical in the integral.
  - **Perfect Squares:** Look for the pattern $A^2 + 1/2 + B^2$ which comes from $1 + (A - B)^2$ where $2AB = 1/2$. (Problem 10).
  - **Common Denominators:** When derivatives are fractions, finding a common denominator for $1 + (y')^2$ is often the first step to simplification. (Problem 4).

- **Strategic U-Substitution:** Many solvable problems result in an integral of the form $\int u^n \sqrt{A + Bu^k}\,du$. A u-substitution for the expression inside the radical ($u = A + Bx^k$) is often the correct path. (Problems 14, 15).

- **Inverting Functions:** To change the variable of integration, you must be able to solve for the other variable. For $y = 4 + \sin(x)$, you must know that $x = \arcsin(y - 4)$. For $x = e^{8y}$, you must know that $y = \frac{1}{8}\ln(x)$.
- **Changing the Limits of Integration:** When performing a u-substitution in a definite integral, always change the limits of integration to match the new variable. This avoids the need to substitute back at the end.

3.3 Cheats and Tips for Success

- **Formula Cheat Sheet:** Mentally (or on paper) keep the four formulas straight. The main difference is the radius term ($2\pi y$ or $2\pi x$).
- **Choose the Easiest Path:** If you have a choice, integrate with respect to the variable the function is already solved for. For $x = y + y^3$, integrating with respect to y is far easier than trying to solve that cubic for y.
- **Look for Cancellation:** In problems like 4 and 9, notice how the term in the radius (y) is designed to cancel with the denominator that appears under the radical in the ds term. If you see this happening, you're on the right track.
- **Sanity Check your Radius:** The radius is always a simple distance, either x or y. Never a derivative or a complex expression (though you will substitute the function for x or y).
- **Gabriel's Horn Paradox:** Remember this classic result. The horn has a finite volume but an infinite surface area. This illustrates a counter-intuitive aspect of infinity in calculus.