Comprehensive Study Guide: Deriving the Black-Scholes-Merton Framework

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1 Conceptual Foundations: The Delta-Hedged Portfolio

The derivation of the Black-Scholes-Merton (BSM) model begins not with the formula itself, but with the construction of a specific portfolio designed to eliminate directional risk.

1.1 The Portfolio Construction

To mitigate directional risk, option traders combine options with a position in the underlying asset. We define a portfolio Π consisting of:

- A long position in a call option (C).
- A short position in the underlying stock.

The value of this portfolio at time t is given by:

$$\Pi_t = C - \Delta S_t$$

Where:

- C is the value of the call option.
- S_t is the price of the underlying at time t.
- Δ is the number of shares sold short. Crucially, in this context, Δ represents the hedge ratio (the partial derivative $\frac{\partial C}{\partial S}$).

1.2 Analyzing Portfolio Changes and Financing

Over a small time step, the change in the portfolio's value includes the change in asset prices and the cost of financing the position. The total change is expressed as:

$$\text{Change} = \underbrace{[C(S_{t+1}) - C(S_t)]}_{\text{Change in Option}} - \underbrace{\Delta(S_{t+1} - S_t)}_{\text{P/L from Short Stock}} - \underbrace{r(C - \Delta S_t)}_{\text{Financing Cost}}$$

Understanding the Financing Term: The term $-r(C-\Delta S_t)$ represents the interest paid or received on the net value of the portfolio.

- **Cost of Call ($-rC$):** We must borrow money to buy the option, incurring a debit of interest.
- **Interest on Short Stock ($+r\Delta S_t$):** When we short Δ shares, we receive cash proceeds equal to ΔS_t. This cash earns the risk-free rate r.

Thus, shorting the stock provides financing that offsets the cost of holding the long option.

2 The Mathematical Engine: Taylor Expansions and The Greeks

To solve for the option price, we must approximate how the option's value C changes as S and t change. We utilize a **Second-Order Taylor Expansion**.

2.1 The Taylor Expansion of Option Value

The change in the option value dC is approximated by expanding the function $C(S, t)$:

$$dC \approx \underbrace{\frac{\partial C}{\partial t}dt}_{\text{Time Decay}} + \underbrace{\frac{\partial C}{\partial S}dS}_{\text{Linear Price Move}} + \underbrace{\frac{1}{2}\frac{\partial^2 C}{\partial S^2}(dS)^2}_{\text{Curvature}}$$

Mapping this to the Greeks:

$$\Theta(\text{Theta}) = \frac{\partial C}{\partial t}$$
$$\Delta(\text{Delta}) = \frac{\partial C}{\partial S}$$
$$\Gamma(\text{Gamma}) = \frac{\partial^2 C}{\partial S^2}$$

Substituting these into the portfolio change equation, the linear delta terms (ΔdS) cancel out perfectly. This is the definition of being "Delta Neutral." We are left with:

$$\text{Portfolio Change} \approx \Theta dt + \frac{1}{2}\Gamma(dS)^2 - r(C - \Delta S)dt$$

2.2 Deep Dive: The Sign of Theta

A common confusion arises from the term $+\Theta$ in the equation, given that option holders typically lose money due to time decay.

Clarification: Why is Theta Positive in the Equation?

The symbol Θ represents the partial derivative $\frac{\partial C}{\partial t}$. For a long option position, the value decreases as time passes. Therefore, the mathematical value of the variable Θ is **negative** (e.g., $\Theta = -0.05$). The equation adds the variable Θ. Since Θ is a negative number, adding it results in a reduction of value. The text states "the option holder loses money," which is consistent with adding a negative parameter.

3 The Stochastic Leap: From Realized Returns to Volatility

The derivation relies on a critical substitution derived from **Ito's Lemma**. This is the bridge between randomness and deterministic pricing.

3.1 Realized Squared Change vs. Variance

In the Taylor expansion, we encounter the term $(dS)^2$, or $(S_{t+1} - S_t)^2$.

- **The Random Variable:** $(S_{t+1} - S_t)^2$ is the *realized* squared price change over a single time step. It is unknown until it happens.
- **The Parameter:** σ^2 (Variance) is the *expected* dispersion of returns over time.

3.2 The Fundamental Substitution

Under the assumption that stock prices follow a Geometric Brownian Motion, Ito's Lemma allows us to replace the random realized squared change with a deterministic expected value as the time step $dt \to 0$:

$$(S_{t+1} - S_t)^2 \approx \sigma^2 S^2 dt$$

Important Distinctions:

- σ^2 is the Variance.
- σ is the Standard Deviation (Volatility).
- The S^2 term exists because volatility acts on returns (percentages), so the dollar variance scales with the square of the stock price (S^2).

By substituting $\sigma^2 S^2 dt$ for $(dS)^2$, we eliminate the random component of the portfolio's change. The portfolio becomes risk-free.

4 Drift, Risk Neutrality, and No Arbitrage

4.1 The Irrelevance of Drift (μ)

A counter-intuitive result of the BSM derivation is that the expected return of the stock (drift) does not appear in the final equation.

- While a call option benefits from upward drift, the delta-hedged portfolio includes a short stock position that *loses* from upward drift.
- By holding the correct ratio (Δ), the effects of drift cancel out perfectly.
- Since drift is hedged away, the market offers no risk premium for it.

4.2 The Law of One Price

Because the delta-hedged portfolio has had its random risk terms (Γ) converted to deterministic terms (via Ito's Lemma) and its directional risk (Δ) eliminated, it is a **risk-free asset**.

By the **No Arbitrage** principle, a risk-free portfolio must earn exactly the risk-free rate r. If it earned more, arbitrageurs would borrow at r to buy the portfolio; if less, they would short the portfolio to lend at r.

$$\text{Change in Portfolio} = r \times (\text{Portfolio Value})dt$$

Setting the derived change equal to the risk-free return yields the Black-Scholes PDE.

5 Practical Application: The Reality of Hedging

In the theoretical derivation, we assume continuous rebalancing. In practice, traders face the "Trader's Dilemma."

5.1 Static vs. Dynamic Hedging

- **Static Hedging:** Setting a hedge and leaving it. This fails because Delta changes as Spot changes (Gamma).
- **Dynamic Hedging:** Continuously adjusting the hedge ratio.
- **Transaction Costs:** In reality, continuous hedging implies infinite transaction costs. Traders must balance the cost of rebalancing against the risk of being unhedged (Gamma risk).

5.2 Trading Volatility

A delta-hedged portfolio is not "safe" from all risks; it transforms directional risk into volatility risk.

- **Long Gamma/Vega:** The trader profits if realized volatility (magnitude of moves) exceeds the implied volatility paid for the option.
- **Short Theta:** The trader pays "rent" (time decay) every day for the privilege of holding the Gamma/Vega exposure.

5.3 Hedging the Greeks

- **Delta Neutral:** Hedged against small directional moves (via stock).
- **Gamma/Vega Neutral:** Requires buying/selling other options to offset curvature and volatility sensitivity.
- **Theta Neutral:** Generally impossible without exiting the position. Theta is the cost of doing business.