

## 7.8: Improper Integrals - Problem Set

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### Part I: Problems

#### Problem 1

Determine whether the integral is convergent or divergent. If it is convergent, evaluate it.

$$\int_2^{\infty} \frac{5}{x^3} dx$$

#### Problem 2

Determine whether the integral is convergent or divergent. If it is convergent, evaluate it.

$$\int_1^{\infty} \frac{1}{\sqrt[4]{x}} dx$$

#### Problem 3

Determine whether the integral is convergent or divergent. If it is convergent, evaluate it.

$$\int_0^{\infty} e^{-2x} dx$$

#### Problem 4

Determine whether the integral is convergent or divergent. If it is convergent, evaluate it.

$$\int_e^{\infty} \frac{1}{x(\ln x)^2} dx$$

#### Problem 5

Determine whether the integral is convergent or divergent. If it is convergent, evaluate it.

$$\int_1^{\infty} \frac{x^2 + 2}{x^3} dx$$

#### Problem 6

Determine whether the integral is convergent or divergent. If it is convergent, evaluate it.

$$\int_{-\infty}^0 \frac{1}{(1-x)^{3/2}} dx$$

#### Problem 7

Determine whether the integral is convergent or divergent. If it is convergent, evaluate it.

$$\int_{-\infty}^{-1} \frac{1}{x^5} dx$$

**Problem 8**

Determine whether the integral is convergent or divergent. If it is convergent, evaluate it.

$$\int_{-\infty}^0 \frac{x}{(x^2 + 1)^2} dx$$

**Problem 9**

Determine whether the integral is convergent or divergent. If it is convergent, evaluate it.

$$\int_{-\infty}^{\infty} \frac{x}{1 + x^2} dx$$

**Problem 10**

Determine whether the integral is convergent or divergent. If it is convergent, evaluate it.

$$\int_{-\infty}^{\infty} \frac{1}{x^2 + 4} dx$$

**Problem 11**

Determine whether the integral is convergent or divergent. If it is convergent, evaluate it.

$$\int_{-\infty}^{\infty} x^2 e^{-x^3} dx$$

**Problem 12**

Determine whether the integral is convergent or divergent. If it is convergent, evaluate it.

$$\int_0^1 \frac{1}{\sqrt[3]{x}} dx$$

**Problem 13**

Determine whether the integral is convergent or divergent. If it is convergent, evaluate it.

$$\int_0^2 \frac{1}{(x - 2)^2} dx$$

**Problem 14**

Determine whether the integral is convergent or divergent. If it is convergent, evaluate it.

$$\int_0^3 \frac{1}{\sqrt{3 - x}} dx$$

**Problem 15**

Determine whether the integral is convergent or divergent. If it is convergent, evaluate it.

$$\int_{-1}^8 \frac{1}{\sqrt[3]{x}} dx$$

**Problem 16**

Determine whether the integral is convergent or divergent. If it is convergent, evaluate it.

$$\int_1^{\infty} \frac{1}{x^2 + x} dx$$

**Problem 17**

Determine whether the integral is convergent or divergent. If it is convergent, evaluate it.

$$\int_2^{\infty} \frac{4}{x^2 - 1} dx$$

**Problem 18**

Determine whether the integral is convergent or divergent. If it is convergent, evaluate it.

$$\int_0^{\infty} x e^{-x} dx$$

**Problem 19**

Determine whether the integral is convergent or divergent. If it is convergent, evaluate it.

$$\int_1^{\infty} \frac{\ln x}{x^2} dx$$

**Problem 20**

Determine whether the integral is convergent or divergent. If it is convergent, evaluate it.

$$\int_0^{\infty} \cos(x) dx$$

**Problem 21**

Determine whether the integral is convergent or divergent. If it is convergent, evaluate it.

$$\int_0^{\infty} 2 \cos^2(x) dx$$

**Problem 22**

Determine whether the integral is convergent or divergent. If it is convergent, evaluate it.

$$\int_1^{\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}} dx$$

**Problem 23**

Determine whether the integral is convergent or divergent. If it is convergent, evaluate it.

$$\int_{-\infty}^0 x e^x dx$$

**Problem 24**

Determine whether the integral is convergent or divergent. If it is convergent, evaluate it.

$$\int_0^1 \frac{1}{4y - 1} dy$$

**Problem 25**

Determine whether the integral is convergent or divergent. If it is convergent, evaluate it.

$$\int_1^{\infty} \frac{\arctan(x)}{x^2 + 1} dx$$

**Problem 26**

Determine whether the integral is convergent or divergent. If it is convergent, evaluate it.

$$\int_0^{\pi/2} \tan(x) \, dx$$

**Problem 27**

Determine whether the integral is convergent or divergent. If it is convergent, evaluate it.

$$\int_{-\infty}^{\infty} \frac{e^x}{1 + e^{2x}} \, dx$$

**Problem 28**

Determine whether the integral is convergent or divergent. If it is convergent, evaluate it.

$$\int_0^1 \ln(x) \, dx$$

**Problem 29**

Determine whether the integral is convergent or divergent. If it is convergent, evaluate it.

$$\int_1^{\infty} \frac{1}{x\sqrt{x^2 - 1}} \, dx$$

**Problem 30**

Determine whether the integral is convergent or divergent. If it is convergent, evaluate it.

$$\int_{-\infty}^1 \frac{1}{x^2 - 4x + 5} \, dx$$

## Part II: Detailed Solutions

### Solution 1

This is a Type 1 improper integral, which is a p-integral with  $p = 3 > 1$ , so it converges.

$$\begin{aligned}\int_2^\infty 5x^{-3} dx &= \lim_{t \rightarrow \infty} \int_2^t 5x^{-3} dx \\&= \lim_{t \rightarrow \infty} \left[ \frac{5x^{-2}}{-2} \right]_2^t = \lim_{t \rightarrow \infty} \left[ -\frac{5}{2x^2} \right]_2^t \\&= \lim_{t \rightarrow \infty} \left( -\frac{5}{2t^2} - \left( -\frac{5}{2(2)^2} \right) \right) \\&= 0 + \frac{5}{8} = \frac{5}{8}\end{aligned}$$

**Answer:** Convergent, value is  $5/8$ .

### Solution 2

This is a Type 1 improper integral, which is a p-integral with  $p = 1/4 \leq 1$ , so it diverges.

$$\begin{aligned}\int_1^\infty x^{-1/4} dx &= \lim_{t \rightarrow \infty} \int_1^t x^{-1/4} dx \\&= \lim_{t \rightarrow \infty} \left[ \frac{x^{3/4}}{3/4} \right]_1^t = \lim_{t \rightarrow \infty} \left[ \frac{4}{3} x^{3/4} \right]_1^t \\&= \lim_{t \rightarrow \infty} \left( \frac{4}{3} t^{3/4} - \frac{4}{3} (1)^{3/4} \right) \\&= \infty - \frac{4}{3} = \infty\end{aligned}$$

**Answer:** Diverges.

### Solution 3

This is a Type 1 improper integral.

$$\begin{aligned}\int_0^\infty e^{-2x} dx &= \lim_{t \rightarrow \infty} \int_0^t e^{-2x} dx \\&= \lim_{t \rightarrow \infty} \left[ -\frac{1}{2} e^{-2x} \right]_0^t \\&= \lim_{t \rightarrow \infty} \left( -\frac{1}{2} e^{-2t} - \left( -\frac{1}{2} e^0 \right) \right) \\&= 0 + \frac{1}{2} = \frac{1}{2}\end{aligned}$$

**Answer:** Convergent, value is  $1/2$ .

### Solution 4

This is a Type 1 improper integral. Use u-substitution with  $u = \ln x$ , so  $du = \frac{1}{x} dx$ . When  $x = e$ ,  $u = 1$ . When  $x \rightarrow \infty$ ,  $u \rightarrow \infty$ .

$$\begin{aligned}\int_e^\infty \frac{1}{x(\ln x)^2} dx &= \int_1^\infty \frac{1}{u^2} du \\&= \lim_{t \rightarrow \infty} \int_1^t u^{-2} du = \lim_{t \rightarrow \infty} [-u^{-1}]_1^t \\&= \lim_{t \rightarrow \infty} \left( -\frac{1}{t} - (-1) \right) = 0 + 1 = 1\end{aligned}$$

**Answer:** Convergent, value is 1.

### Solution 5

This is a Type 1 improper integral. First, simplify the integrand.

$$\begin{aligned}\int_1^\infty \left( \frac{x^2}{x^3} + \frac{2}{x^3} \right) dx &= \int_1^\infty \left( \frac{1}{x} + 2x^{-3} \right) dx \\&= \lim_{t \rightarrow \infty} \int_1^t \left( \frac{1}{x} + 2x^{-3} \right) dx \\&= \lim_{t \rightarrow \infty} [\ln|x| - x^{-2}]_1^t \\&= \lim_{t \rightarrow \infty} \left( (\ln t - \frac{1}{t^2}) - (\ln 1 - 1) \right) \\&= (\infty - 0) - (0 - 1) = \infty\end{aligned}$$

The integral diverges because the  $\int \frac{1}{x} dx$  part diverges ( $p = 1$ ). **Answer:** Diverges.

### Solution 6

This is a Type 1 improper integral.

$$\begin{aligned}\int_{-\infty}^0 (1-x)^{-3/2} dx &= \lim_{t \rightarrow -\infty} \int_t^0 (1-x)^{-3/2} dx \\&= \lim_{t \rightarrow -\infty} \left[ 2(1-x)^{-1/2} \right]_t^0 \\&= \lim_{t \rightarrow -\infty} \left( 2(1)^{-1/2} - 2(1-t)^{-1/2} \right) \\&= \lim_{t \rightarrow -\infty} \left( 2 - \frac{2}{\sqrt{1-t}} \right) \\&= 2 - 0 = 2\end{aligned}$$

**Answer:** Convergent, value is 2.

### Solution 7

This is a Type 1 improper integral. The p-integral with  $p = 5 > 1$  converges on  $[1, \infty)$ , and similarly converges on  $(-\infty, -1]$ .

$$\begin{aligned}\int_{-\infty}^{-1} x^{-5} dx &= \lim_{t \rightarrow -\infty} \int_t^{-1} x^{-5} dx \\&= \lim_{t \rightarrow -\infty} \left[ \frac{x^{-4}}{-4} \right]_t^{-1} \\&= \lim_{t \rightarrow -\infty} \left( \frac{(-1)^{-4}}{-4} - \frac{t^{-4}}{-4} \right) \\&= \lim_{t \rightarrow -\infty} \left( -\frac{1}{4} + \frac{1}{4t^4} \right) = -\frac{1}{4} + 0 = -\frac{1}{4}\end{aligned}$$

**Answer:** Convergent, value is  $-1/4$ .

### Solution 8

This is a Type 1 improper integral. Use u-substitution with  $u = x^2 + 1$ ,  $du = 2x dx$ . When  $x = 0$ ,  $u = 1$ . When  $x \rightarrow -\infty$ ,  $u \rightarrow \infty$ .

$$\begin{aligned}\int_{-\infty}^0 \frac{x}{(x^2+1)^2} dx &= \lim_{t \rightarrow -\infty} \int_t^0 \frac{x}{(x^2+1)^2} dx \\&= \int_{\infty}^1 \frac{1}{u^2} \frac{du}{2} = -\frac{1}{2} \int_1^{\infty} u^{-2} du \\&= -\frac{1}{2} \lim_{t \rightarrow \infty} [-u^{-1}]_1^t \\&= -\frac{1}{2} \lim_{t \rightarrow \infty} \left( -\frac{1}{t} - (-1) \right) = -\frac{1}{2}(0+1) = -\frac{1}{2}\end{aligned}$$

**Answer:** Convergent, value is  $-1/2$ .

### Solution 9

This is a Type 1 integral over  $(-\infty, \infty)$ . We split it at  $x = 0$ .

$$\int_{-\infty}^{\infty} \frac{x}{1+x^2} dx = \int_{-\infty}^0 \frac{x}{1+x^2} dx + \int_0^{\infty} \frac{x}{1+x^2} dx$$

Let's evaluate the second part. Use  $u = 1 + x^2$ ,  $du = 2x dx$ .

$$\begin{aligned}\int_0^{\infty} \frac{x}{1+x^2} dx &= \lim_{t \rightarrow \infty} \int_0^t \frac{x}{1+x^2} dx \\&= \lim_{t \rightarrow \infty} \left[ \frac{1}{2} \ln(1+x^2) \right]_0^t \\&= \frac{1}{2} \lim_{t \rightarrow \infty} (\ln(1+t^2) - \ln(1)) = \infty\end{aligned}$$

Since one part diverges, the whole integral diverges. Note: The integrand is an odd function, but for the integral to be 0, it must first converge. **Answer:** Diverges.

### Solution 10

This is a Type 1 integral over  $(-\infty, \infty)$ . Split at  $x = 0$ . The integrand is even.

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1}{x^2+4} dx &= 2 \int_0^{\infty} \frac{1}{x^2+4} dx \\2 \lim_{t \rightarrow \infty} \int_0^t \frac{1}{x^2+2^2} dx &= 2 \lim_{t \rightarrow \infty} \left[ \frac{1}{2} \arctan\left(\frac{x}{2}\right) \right]_0^t \\&= \lim_{t \rightarrow \infty} \left( \arctan\left(\frac{t}{2}\right) - \arctan(0) \right) \\&= \frac{\pi}{2} - 0 = \frac{\pi}{2}\end{aligned}$$

The original integral is  $2 \times (\pi/2) = \pi$ . **Answer:** Convergent, value is  $\pi$ .

### Solution 11

This is a Type 1 integral over  $(-\infty, \infty)$ . Split at  $x = 0$ .

$$\int_{-\infty}^0 x^2 e^{-x^3} dx + \int_0^{\infty} x^2 e^{-x^3} dx$$

Let's evaluate the second part. Use  $u = -x^3, du = -3x^2 dx$ .

$$\begin{aligned}\int_0^\infty x^2 e^{-x^3} dx &= \lim_{t \rightarrow \infty} \int_0^t x^2 e^{-x^3} dx \\ &= \lim_{t \rightarrow \infty} \left[ -\frac{1}{3} e^{-x^3} \right]_0^t \\ &= -\frac{1}{3} \lim_{t \rightarrow \infty} (e^{-t^3} - e^0) = -\frac{1}{3}(0 - 1) = \frac{1}{3}\end{aligned}$$

The first part diverges:

$$\begin{aligned}\int_{-\infty}^0 x^2 e^{-x^3} dx &= \lim_{t \rightarrow -\infty} \int_t^0 x^2 e^{-x^3} dx \\ &= \lim_{t \rightarrow -\infty} \left[ -\frac{1}{3} e^{-x^3} \right]_t^0 \\ &= -\frac{1}{3} \lim_{t \rightarrow -\infty} (e^0 - e^{-t^3}) = -\frac{1}{3}(1 - \infty) = \infty\end{aligned}$$

Since one part diverges, the whole integral diverges. **Answer:** Diverges.

## Solution 12

This is a Type 2 improper integral with a discontinuity at  $x = 0$ . It's a p-integral with  $p = 1/3 < 1$ , so it converges.

$$\begin{aligned}\int_0^1 x^{-1/3} dx &= \lim_{t \rightarrow 0^+} \int_t^1 x^{-1/3} dx \\ &= \lim_{t \rightarrow 0^+} \left[ \frac{3}{2} x^{2/3} \right]_t^1 \\ &= \lim_{t \rightarrow 0^+} \left( \frac{3}{2} (1)^{2/3} - \frac{3}{2} t^{2/3} \right) = \frac{3}{2} - 0 = \frac{3}{2}\end{aligned}$$

**Answer:** Convergent, value is  $3/2$ .

## Solution 13

This is a Type 2 improper integral with a discontinuity at  $x = 2$ . It's a p-integral with  $p = 2 > 1$ , so it diverges.

$$\begin{aligned}\int_0^2 (x-2)^{-2} dx &= \lim_{t \rightarrow 2^-} \int_0^t (x-2)^{-2} dx \\ &= \lim_{t \rightarrow 2^-} \left[ -(x-2)^{-1} \right]_0^t \\ &= \lim_{t \rightarrow 2^-} \left( -\frac{1}{t-2} - \left( -\frac{1}{-2} \right) \right) \\ &= -(-\infty) - \frac{1}{2} = \infty\end{aligned}$$

**Answer:** Diverges.

## Solution 14

This is a Type 2 improper integral with a discontinuity at  $x = 3$ .

$$\begin{aligned}\int_0^3 (3-x)^{-1/2} dx &= \lim_{t \rightarrow 3^-} \int_0^t (3-x)^{-1/2} dx \\ &= \lim_{t \rightarrow 3^-} \left[ -2(3-x)^{1/2} \right]_0^t \\ &= \lim_{t \rightarrow 3^-} \left( -2\sqrt{3-t} - (-2\sqrt{3}) \right) \\ &= 0 + 2\sqrt{3} = 2\sqrt{3}\end{aligned}$$



**Answer:** Convergent, value is  $2\sqrt{3}$ .

### Solution 15

This is a Type 2 improper integral with a discontinuity at  $x = 0$  inside the interval. We must split it.

$$\int_{-1}^8 x^{-1/3} dx = \int_{-1}^0 x^{-1/3} dx + \int_0^8 x^{-1/3} dx$$

First part:

$$\begin{aligned} \lim_{t \rightarrow 0^-} \int_{-1}^t x^{-1/3} dx &= \lim_{t \rightarrow 0^-} \left[ \frac{3}{2} x^{2/3} \right]_{-1}^t \\ &= \lim_{t \rightarrow 0^-} \left( \frac{3}{2} t^{2/3} - \frac{3}{2} (-1)^{2/3} \right) = 0 - \frac{3}{2} = -\frac{3}{2} \end{aligned}$$

Second part:

$$\begin{aligned} \lim_{t \rightarrow 0^+} \int_t^8 x^{-1/3} dx &= \lim_{t \rightarrow 0^+} \left[ \frac{3}{2} x^{2/3} \right]_t^8 \\ &= \lim_{t \rightarrow 0^+} \left( \frac{3}{2} (8)^{2/3} - \frac{3}{2} t^{2/3} \right) = \frac{3}{2} (4) - 0 = 6 \end{aligned}$$

Both parts converge, so the total is  $-\frac{3}{2} + 6 = \frac{9}{2}$ . **Answer:** Convergent, value is  $9/2$ .

### Solution 16

This is a Type 1 integral. Use partial fractions:  $\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$ .

$$\begin{aligned} \int_1^\infty \left( \frac{1}{x} - \frac{1}{x+1} \right) dx &= \lim_{t \rightarrow \infty} \int_1^t \left( \frac{1}{x} - \frac{1}{x+1} \right) dx \\ &= \lim_{t \rightarrow \infty} [\ln |x| - \ln |x+1|]_1^t \\ &= \lim_{t \rightarrow \infty} \left[ \ln \left| \frac{x}{x+1} \right| \right]_1^t \\ &= \lim_{t \rightarrow \infty} \left( \ln \left( \frac{t}{t+1} \right) - \ln \left( \frac{1}{2} \right) \right) \\ &= \ln(1) - \ln(1/2) = 0 - (-\ln 2) = \ln 2 \end{aligned}$$

**Answer:** Convergent, value is  $\ln 2$ .

### Solution 17

This is a Type 1 integral. Use partial fractions:  $\frac{4}{x^2-1} = \frac{2}{x-1} - \frac{2}{x+1}$ .

$$\begin{aligned} \int_2^\infty \left( \frac{2}{x-1} - \frac{2}{x+1} \right) dx &= \lim_{t \rightarrow \infty} [2 \ln |x-1| - 2 \ln |x+1|]_2^t \\ &= 2 \lim_{t \rightarrow \infty} \left[ \ln \left| \frac{x-1}{x+1} \right| \right]_2^t \\ &= 2 \lim_{t \rightarrow \infty} \left( \ln \left( \frac{t-1}{t+1} \right) - \ln \left( \frac{1}{3} \right) \right) \\ &= 2(\ln(1) - \ln(1/3)) = 2(0 - (-\ln 3)) = 2 \ln 3 \end{aligned}$$

**Answer:** Convergent, value is  $2 \ln 3$ .

### Solution 18

This is a Type 1 integral. Use integration by parts with  $u = x, dv = e^{-x}dx$ . Then  $du = dx, v = -e^{-x}$ .

$$\begin{aligned}\int_0^\infty x e^{-x} dx &= \lim_{t \rightarrow \infty} \int_0^t x e^{-x} dx \\&= \lim_{t \rightarrow \infty} \left( [-x e^{-x}]_0^t - \int_0^t -e^{-x} dx \right) \\&= \lim_{t \rightarrow \infty} \left( [-x e^{-x} - e^{-x}]_0^t \right) \\&= \lim_{t \rightarrow \infty} \left( \left( -\frac{t}{e^t} - \frac{1}{e^t} \right) - (0 - e^0) \right) \\&= (0 - 0) - (-1) = 1\end{aligned}$$

(Used L'Hôpital's Rule for  $\lim_{t \rightarrow \infty} t/e^t = 0$ ). **Answer:** Convergent, value is 1.

### Solution 19

This is a Type 1 integral. Use integration by parts with  $u = \ln x, dv = x^{-2}dx$ . Then  $du = 1/x dx, v = -x^{-1}$ .

$$\begin{aligned}\int_1^\infty \frac{\ln x}{x^2} dx &= \lim_{t \rightarrow \infty} \int_1^t (\ln x)(x^{-2}) dx \\&= \lim_{t \rightarrow \infty} \left( \left[ -\frac{\ln x}{x} \right]_1^t - \int_1^t -\frac{1}{x^2} dx \right) \\&= \lim_{t \rightarrow \infty} \left( \left[ -\frac{\ln x}{x} - \frac{1}{x} \right]_1^t \right) \\&= \lim_{t \rightarrow \infty} \left( \left( -\frac{\ln t}{t} - \frac{1}{t} \right) - \left( -\frac{\ln 1}{1} - \frac{1}{1} \right) \right) \\&= (0 - 0) - (0 - 1) = 1\end{aligned}$$

(Used L'Hôpital's Rule for  $\lim_{t \rightarrow \infty} \ln t/t = 0$ ). **Answer:** Convergent, value is 1.

### Solution 20

This is a Type 1 integral with an oscillating function.

$$\begin{aligned}\int_0^\infty \cos(x) dx &= \lim_{t \rightarrow \infty} \int_0^t \cos(x) dx \\&= \lim_{t \rightarrow \infty} [\sin(x)]_0^t \\&= \lim_{t \rightarrow \infty} (\sin(t) - \sin(0)) = \lim_{t \rightarrow \infty} \sin(t)\end{aligned}$$

The limit does not exist as  $\sin(t)$  oscillates between -1 and 1. **Answer:** Diverges.

### Solution 21

This is a Type 1 integral. Use the power-reducing identity  $\cos^2(x) = \frac{1+\cos(2x)}{2}$ .

$$\begin{aligned}\int_0^\infty 2 \left( \frac{1 + \cos(2x)}{2} \right) dx &= \int_0^\infty (1 + \cos(2x)) dx \\&= \lim_{t \rightarrow \infty} \int_0^t (1 + \cos(2x)) dx \\&= \lim_{t \rightarrow \infty} \left[ x + \frac{1}{2} \sin(2x) \right]_0^t \\&= \lim_{t \rightarrow \infty} \left( \left( t + \frac{1}{2} \sin(2t) \right) - 0 \right) = \infty\end{aligned}$$

The limit is infinite. **Answer:** Diverges.

### Solution 22

This is a Type 1 integral. Use u-substitution with  $u = -\sqrt{x}$ ,  $du = -\frac{1}{2\sqrt{x}}dx$ .

$$\begin{aligned}\int_1^\infty \frac{e^{-\sqrt{x}}}{\sqrt{x}} dx &= \lim_{t \rightarrow \infty} \int_1^t \frac{e^{-\sqrt{x}}}{\sqrt{x}} dx \\&= \lim_{t \rightarrow \infty} \left[ -2e^{-\sqrt{x}} \right]_1^t \\&= \lim_{t \rightarrow \infty} \left( -2e^{-\sqrt{t}} - (-2e^{-1}) \right) \\&= 0 + \frac{2}{e} = \frac{2}{e}\end{aligned}$$

**Answer:** Convergent, value is  $2/e$ .

### Solution 23

This is a Type 1 integral. It is the same integral as problem 18, but over a different interval. Use integration by parts with  $u = x$ ,  $dv = e^x dx$ .

$$\begin{aligned}\int_{-\infty}^0 xe^x dx &= \lim_{t \rightarrow -\infty} \int_t^0 xe^x dx \\&= \lim_{t \rightarrow -\infty} [xe^x - e^x]_t^0 \\&= \lim_{t \rightarrow -\infty} ((0 - e^0) - (te^t - e^t)) \\&= -1 - (0 - 0) = -1\end{aligned}$$

(Used L'Hôpital's Rule for  $\lim_{t \rightarrow -\infty} te^t = \lim_{t \rightarrow -\infty} t/e^{-t} = 0$ ). **Answer:** Convergent, value is  $-1$ .

### Solution 24

This is a Type 2 integral with a discontinuity at  $y = 1/4$ , which is inside  $[0, 1]$ . Must split.

$$\int_0^{1/4} \frac{1}{4y-1} dy + \int_{1/4}^1 \frac{1}{4y-1} dy$$

Let's evaluate the first part.

$$\begin{aligned}\lim_{t \rightarrow 1/4^-} \int_0^t \frac{1}{4y-1} dy &= \lim_{t \rightarrow 1/4^-} \left[ \frac{1}{4} \ln |4y-1| \right]_0^t \\&= \frac{1}{4} \lim_{t \rightarrow 1/4^-} (\ln |4t-1| - \ln |-1|) \\&= \frac{1}{4} (-\infty - 0) = -\infty\end{aligned}$$

Since one part diverges, the whole integral diverges. **Answer:** Diverges.

### Solution 25

This is a Type 1 integral. Use u-substitution with  $u = \arctan(x)$ ,  $du = \frac{1}{1+x^2}dx$ . When  $x = 1$ ,  $u = \pi/4$ . When  $x \rightarrow \infty$ ,  $u \rightarrow \pi/2$ .

$$\begin{aligned}\int_1^\infty \frac{\arctan(x)}{x^2+1} dx &= \int_{\pi/4}^{\pi/2} u du \\&= \left[ \frac{u^2}{2} \right]_{\pi/4}^{\pi/2} \\&= \frac{1}{2} \left( \left( \frac{\pi}{2} \right)^2 - \left( \frac{\pi}{4} \right)^2 \right) \\&= \frac{1}{2} \left( \frac{\pi^2}{4} - \frac{\pi^2}{16} \right) = \frac{1}{2} \left( \frac{3\pi^2}{16} \right) = \frac{3\pi^2}{32}\end{aligned}$$

**Answer:** Convergent, value is  $3\pi^2/32$ .

### Solution 26

This is a Type 2 integral since  $\tan(x)$  has a vertical asymptote at  $x = \pi/2$ .

$$\begin{aligned}\int_0^{\pi/2} \tan(x) dx &= \lim_{t \rightarrow \pi/2^-} \int_0^t \tan(x) dx \\&= \lim_{t \rightarrow \pi/2^-} [-\ln |\cos(x)|]_0^t \\&= \lim_{t \rightarrow \pi/2^-} (-\ln |\cos(t)| - (-\ln |\cos(0)|)) \\&= -(-\infty) + \ln(1) = \infty\end{aligned}$$

**Answer:** Diverges.

### Solution 27

This is a Type 1 integral over  $(-\infty, \infty)$ . Let  $u = e^x, du = e^x dx$ . When  $x \rightarrow -\infty, u \rightarrow 0$ . When  $x \rightarrow \infty, u \rightarrow \infty$ .

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{e^x}{1 + (e^x)^2} dx &= \int_0^{\infty} \frac{1}{1 + u^2} du \\&= \lim_{t \rightarrow \infty} \int_0^t \frac{1}{1 + u^2} du \\&= \lim_{t \rightarrow \infty} [\arctan(u)]_0^t \\&= \lim_{t \rightarrow \infty} (\arctan(t) - \arctan(0)) = \frac{\pi}{2} - 0 = \frac{\pi}{2}\end{aligned}$$

**Answer:** Convergent, value is  $\pi/2$ .

### Solution 28

This is a Type 2 integral with a discontinuity at  $x = 0$ . Use integration by parts with  $u = \ln x, dv = dx$ . Then  $du = 1/x dx, v = x$ .

$$\begin{aligned}\int_0^1 \ln(x) dx &= \lim_{t \rightarrow 0^+} \int_t^1 \ln(x) dx \\&= \lim_{t \rightarrow 0^+} \left( [x \ln x]_t^1 - \int_t^1 1 dx \right) \\&= \lim_{t \rightarrow 0^+} [x \ln x - x]_t^1 \\&= \lim_{t \rightarrow 0^+} ((1 \ln 1 - 1) - (t \ln t - t)) \\&= (0 - 1) - (0 - 0) = -1\end{aligned}$$

(Used L'Hôpital's Rule for  $\lim_{t \rightarrow 0^+} t \ln t = \lim_{t \rightarrow 0^+} \frac{\ln t}{1/t} = 0$ ). **Answer:** Convergent, value is  $-1$ .

### Solution 29

This is a Type 2 integral with a discontinuity at  $x = 1$ . The antiderivative of the integrand is  $\operatorname{arcsec}(x)$ .

$$\begin{aligned}\int_1^\infty \frac{1}{x\sqrt{x^2-1}} dx &= \text{We must split this integral, for example at } x = 2. \\ &= \int_1^2 \frac{1}{x\sqrt{x^2-1}} dx + \int_2^\infty \frac{1}{x\sqrt{x^2-1}} dx \\ \text{First part: } \lim_{t \rightarrow 1^+} \int_t^2 \frac{1}{x\sqrt{x^2-1}} dx &= \lim_{t \rightarrow 1^+} [\operatorname{arcsec}(x)]_t^2 \\ &= \operatorname{arcsec}(2) - \lim_{t \rightarrow 1^+} \operatorname{arcsec}(t) = \frac{\pi}{3} - 0 = \frac{\pi}{3} \\ \text{Second part: } \lim_{t \rightarrow \infty} \int_2^t \frac{1}{x\sqrt{x^2-1}} dx &= \lim_{t \rightarrow \infty} [\operatorname{arcsec}(x)]_2^t \\ &= \lim_{t \rightarrow \infty} \operatorname{arcsec}(t) - \operatorname{arcsec}(2) = \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}\end{aligned}$$

Both parts converge. Total value is  $\frac{\pi}{3} + \frac{\pi}{6} = \frac{\pi}{2}$ . **Answer:** Convergent, value is  $\pi/2$ .

### Solution 30

This is a Type 1 integral. Complete the square for the denominator:  $x^2 - 4x + 5 = (x^2 - 4x + 4) + 1 = (x - 2)^2 + 1$ .

$$\begin{aligned}\int_{-\infty}^1 \frac{1}{(x-2)^2+1} dx &= \lim_{t \rightarrow -\infty} \int_t^1 \frac{1}{(x-2)^2+1} dx \\ &= \lim_{t \rightarrow -\infty} [\arctan(x-2)]_t^1 \\ &= \lim_{t \rightarrow -\infty} (\arctan(1-2) - \arctan(t-2)) \\ &= \arctan(-1) - (-\pi/2) = -\frac{\pi}{4} + \frac{\pi}{2} = \frac{\pi}{4}\end{aligned}$$

**Answer:** Convergent, value is  $\pi/4$ .

## Concept and Problem Number Index

Here is a list of the concepts tested and the corresponding problem numbers.

- **Type 1 Integrals, upper limit  $\infty$ :** 1, 2, 3, 4, 5, 16, 17, 18, 19, 21, 22, 25
- **Type 1 Integrals, lower limit  $-\infty$ :** 6, 7, 8, 23, 30
- **Type 1 Integrals, on  $(-\infty, \infty)$ :** 9, 10, 11, 27
- **Type 2 Integrals, discontinuity at endpoint:** 12, 13, 14, 26, 28
- **Type 2 Integrals, discontinuity inside interval:** 15, 24
- **Mixed Type 1 and Type 2:** 29
- **p-Test (Direct or after substitution):**
  - Convergent ( $p > 1$ ): 1, 4, 7, 8
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- **u-Substitution:** 4, 6, 8, 11, 22, 25, 27, 30
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