Homework 11.1 Sequences: Additional Practice Problems

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Problems

1. List the first five terms of the sequence $a_n = \frac{(-1)^n(n+1)}{n^2}$.
2. List the first five terms of the sequence given by $a_1 = 2$ and $a_{n+1} = \frac{a_n}{a_n+1}$.
3. List the first five terms of the sequence $\{\cos(n\pi)\}_{n=1}^{\infty}$.
4. List the first five terms of the sequence $a_n = \frac{(n+2)!}{n!}$.
5. Find a formula for the general term a_n of the sequence, assuming the pattern continues: $\{\frac{2}{3}, \frac{3}{5}, \frac{4}{7}, \frac{5}{9}, \dots\}$.
6. Find a formula for the general term a_n of the sequence, assuming the pattern continues: $\{-5, 10, -15, 20, -25, \dots\}$.
7. Find a formula for the general term a_n of the sequence, assuming the pattern continues: $\{1, -\frac{1}{4}, \frac{1}{9}, -\frac{1}{16}, \frac{1}{25}, \dots\}$.
8. Find a formula for the general term a_n of the sequence, which is an arithmetic sequence: $\{11, 7, 3, -1, -5, \dots\}$.
9. Determine if the sequence $a_n = \frac{5n^2-3n+1}{2n^2+n}$ converges or diverges. If it converges, find the limit.
10. Determine if the sequence $a_n = \frac{\sqrt{4n^2+n}}{3n-1}$ converges or diverges. If it converges, find the limit.
11. Determine if the sequence $a_n = \frac{n^3}{n^2+8}$ converges or diverges. If it converges, find the limit.
12. Determine if the sequence $a_n = n - \sqrt{n^2 - 4n}$ converges or diverges. If it converges, find the limit.
13. Determine if the sequence $a_n = \frac{8^n+1}{8^n-1}$ converges or diverges. If it converges, find the limit.
14. Determine if the sequence $a_n = \frac{\ln(n^2)}{n}$ converges or diverges. If it converges, find the limit.
15. Determine if the sequence $a_n = n^2e^{-n}$ converges or diverges. If it converges, find the limit.
16. Determine if the sequence $a_n = (1 + \frac{2}{n})^n$ converges or diverges. If it converges, find the limit. (Hint: Consider the limit of $\ln(a_n)$).
17. Determine if the sequence $a_n = \frac{\arctan(n)}{n}$ converges or diverges. If it converges, find the limit.
18. Determine if the sequence $a_n = n\sin(\frac{1}{n})$ converges or diverges. If it converges, find the limit.
19. Determine if the sequence $a_n = \frac{\sin(n)}{n^2}$ converges or diverges. If it converges, find the limit.
20. Determine if the sequence $a_n = \frac{(-1)^n}{n!}$ converges or diverges. If it converges, find the limit.
21. Determine if the sequence $a_n = \frac{5^n}{n!}$ converges or diverges. If it converges, find the limit. (Hint: For $n > 5$, $\frac{5^n}{n!} = \frac{5^5}{5!} \cdot \frac{5}{6} \cdot \frac{5}{7} \cdots \frac{5}{n}$).
22. Determine if the sequence $a_n = \frac{100^n}{n!}$ converges or diverges. If it converges, find the limit.
23. Determine if the sequence $a_n = \frac{(\ln n)^2}{\sqrt{n}}$ converges or diverges. If it converges, find the limit.
24. Determine if the sequence with terms $\{0.9, 0.99, 0.999, 0.9999, \dots\}$ converges or diverges. If it converges, find the limit.

25. Determine if the sequence $a_n = (\frac{\pi}{e})^n$ converges or diverges. If it converges, find the limit.
26. Consider the sequence defined by $a_1 = 1$ and $a_{n+1} = \sqrt{6 + a_n}$. Show that the sequence is increasing and bounded above by 3. Then find its limit.
27. Determine if the sequence $a_n = \frac{n}{n+1}$ is monotonic and bounded.
28. Determine if the sequence $a_n = n + \frac{1}{n}$ is monotonic and bounded.
29. A sequence is defined by $a_1 = 4$ and $a_{n+1} = \frac{1}{2}(a_n + \frac{9}{a_n})$. Assuming the sequence converges, find its limit.
30. **Problem:** Determine if $a_n = (\frac{n-1}{n})^n$ converges. **Flawed Solution:** The base is $\frac{n-1}{n} = 1 - \frac{1}{n}$. As $n \to \infty$, the base goes to 1. Since 1 raised to any power is 1, the limit is 1. **Your Task:** Identify the flaw, explain the error, and provide the correct solution.
31. **Problem:** Find the limit of the sequence $a_n = \frac{n^2}{n^3+1}$. **Flawed Solution:** This is an indeterminate form $\frac{\infty}{\infty}$, so we use L'Hôpital's Rule. $\lim_{n\to\infty} \frac{n^2}{n^3+1} \stackrel{L'H}{=} \lim_{n\to\infty} \frac{2n}{3n^2} \stackrel{L'H}{=} \lim_{n\to\infty} \frac{2}{6n} = \infty$. The sequence diverges. **Your Task:** Identify the flaw, explain the error, and provide the correct solution.
32. **Problem:** Determine if $a_n = \frac{(-1)^n n}{n+1}$ converges. **Flawed Solution:** Let's consider the absolute value: $|a_n| = \frac{n}{n+1}$. The limit is $\lim_{n\to\infty} \frac{n}{n+1} = 1$. Since the limit of the absolute value is 1, the sequence converges to 1. **Your Task:** Identify the flaw, explain the error, and provide the correct solution.
33. A patient receives a 200 mg dose of a drug at the same time every day. Between doses, 25% of the drug in the body is eliminated. Let Q_n be the quantity of the drug in the body immediately after the n-th dose. Write a recursive formula for Q_n. Assuming it converges, find the long-term quantity of the drug in the body (the limit).
34. Does every bounded sequence converge? Explain your reasoning and provide an example if necessary.

Solutions

1. $a_1 = \frac{-2}{1} = -2$, $a_2 = \frac{3}{4}$, $a_3 = \frac{-4}{9}$, $a_4 = \frac{5}{16}$, $a_5 = \frac{-6}{25}$.

2. $a_1 = 2$, $a_2 = \frac{2}{3}$, $a_3 = \frac{2/3}{2/3+1} = \frac{2/3}{5/3} = \frac{2}{5}$, $a_4 = \frac{2/5}{2/5+1} = \frac{2/5}{7/5} = \frac{2}{7}$, $a_5 = \frac{2/7}{2/7+1} = \frac{2/7}{9/7} = \frac{2}{9}$.

3. $\cos(\pi) = -1$, $\cos(2\pi) = 1$, $\cos(3\pi) = -1$, $\cos(4\pi) = 1$, $\cos(5\pi) = -1$. The terms are $\{-1, 1, -1, 1, -1, \dots\}$.

4. $a_1 = \frac{3!}{1!} = 6$, $a_2 = \frac{4!}{2!} = 12$, $a_3 = \frac{5!}{3!} = 20$, $a_4 = \frac{6!}{4!} = 30$, $a_5 = \frac{7!}{5!} = 42$. (Note: $a_n = (n+2)(n+1)$).

5. Numerator is $n+1$. Denominator is an arithmetic sequence $3, 5, 7, \dots$ which is $2n+1$. So, $a_n = \frac{n+1}{2n+1}$.

6. The terms are alternating multiples of 5. $a_n = (-1)^n(5n)$.

7. Alternating signs and denominators are perfect squares. $a_n = (-1)^{n+1}\frac{1}{n^2}$.

8. Common difference is $d = 7 - 11 = -4$. $a_n = a_1 + (n-1)d = 11 + (n-1)(-4) = 11 - 4n + 4 = 15 - 4n$.

9. Divide numerator and denominator by n^2: $\lim_{n\to\infty} \frac{5-3/n+1/n^2}{2+1/n} = \frac{5-0+0}{2+0} = \frac{5}{2}$. Converges to $5/2$.

10. Divide by n (which is $\sqrt{n^2}$ inside the radical): $\lim_{n\to\infty} \frac{\sqrt{4+1/n}}{3-1/n} = \frac{\sqrt{4+0}}{3-0} = \frac{2}{3}$. Converges to $2/3$.

11. The degree of the numerator is greater than the degree of the denominator. The limit is ∞. The sequence diverges.

12. Multiply by the conjugate: $\lim_{n\to\infty}(n - \sqrt{n^2-4n})\frac{n+\sqrt{n^2-4n}}{n+\sqrt{n^2-4n}} = \lim_{n\to\infty} \frac{n^2-(n^2-4n)}{n+\sqrt{n^2-4n}} = \lim_{n\to\infty} \frac{4n}{n+\sqrt{n^2(1-4/n)}} = \lim_{n\to\infty} \frac{4n}{n+n\sqrt{1-4/n}} = \lim_{n\to\infty} \frac{4}{1+\sqrt{1-4/n}} = \frac{4}{1+\sqrt{1}} = 2$. Converges to 2.

13. Divide by the fastest-growing term, 8^n: $\lim_{n\to\infty} \frac{1+1/8^n}{1-1/8^n} = \frac{1+0}{1-0} = 1$. Converges to 1.

14. Use L'Hôpital's Rule on $f(x) = \frac{2\ln(x)}{x}$. $\lim_{x\to\infty} \frac{2\ln(x)}{x} \overset{L'H}{=} \lim_{x\to\infty} \frac{2/x}{1} = 0$. Converges to 0.

15. Use L'Hôpital's Rule on $f(x) = \frac{x^2}{e^x}$. $\lim_{x\to\infty} \frac{x^2}{e^x} \overset{L'H}{=} \lim_{x\to\infty} \frac{2x}{e^x} \overset{L'H}{=} \lim_{x\to\infty} \frac{2}{e^x} = 0$. Converges to 0.

16. Let $L = \lim_{n\to\infty}(1+\frac{2}{n})^n$. Then $\ln L = \lim_{n\to\infty} n\ln(1+\frac{2}{n}) = \lim_{n\to\infty} \frac{\ln(1+2/n)}{1/n}$. Using L'Hôpital's Rule for $x \to \infty$: $\lim_{x\to\infty} \frac{\frac{1}{1+2/x}(-2/x^2)}{-1/x^2} = \lim_{x\to\infty} \frac{2}{1+2/x} = 2$. So $\ln L = 2$, which means $L = e^2$. Converges to e^2.

17. As $n \to \infty$, $\arctan(n) \to \frac{\pi}{2}$ and the denominator $n \to \infty$. A constant divided by infinity is 0. Converges to 0.

18. Rewrite as $\lim_{n\to\infty} \frac{\sin(1/n)}{1/n}$. Let $x = 1/n$. As $n \to \infty$, $x \to 0$. The limit becomes $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$. Converges to 1.

19. We know $-1 \le \sin(n) \le 1$. Thus, $-\frac{1}{n^2} \le \frac{\sin(n)}{n^2} \le \frac{1}{n^2}$. Since $\lim_{n\to\infty} -\frac{1}{n^2} = 0$ and $\lim_{n\to\infty} \frac{1}{n^2} = 0$, by the Squeeze Theorem, the sequence converges to 0.

20. We know $-1 \le (-1)^n \le 1$. Thus, $-\frac{1}{n!} \le \frac{(-1)^n}{n!} \le \frac{1}{n!}$. Since $\lim_{n\to\infty} \frac{1}{n!} = 0$, by the Squeeze Theorem, the sequence converges to 0.

21. For $n > 5$, we have $0 \le a_n = \frac{5^n}{n!} = \frac{5^5}{5!}\left(\frac{5}{6}\right)\left(\frac{5}{7}\right)\cdots\left(\frac{5}{n}\right) \le \frac{5^5}{5!}\left(\frac{5}{6}\right)^{n-5}$. The right side is a geometric sequence with $|r| = 5/6 < 1$, so it converges to 0. By the Squeeze Theorem, the sequence converges to 0.

22. For $n > 100$, $a_n = \frac{100^n}{n!} = \frac{100^{100}}{100!} \cdot \frac{100}{101} \cdots \frac{100}{n} \to 0$. Converges to 0 as factorials grow faster than exponentials.

23. Polynomials (even roots) grow faster than logarithms. The limit is 0. Using L'Hôpital's twice on $f(x) = (\ln x)^2/\sqrt{x}$ confirms this. Converges to 0.

24. The terms are $1-0.1, 1-0.01, 1-0.001, \dots$. The sequence can be written as $a_n = 1-10^{-n}$. As $n \to \infty$, $10^{-n} \to 0$, so the limit is 1. Converges to 1.

25. This is a geometric sequence with ratio $r = \pi/e$. Since $\pi \approx 3.14159$ and $e \approx 2.71828$, $r > 1$. The sequence diverges to ∞.

26. **Monotonic:** $a_1 = 1, a_2 = \sqrt{7} \approx 2.64$. Assume $a_k > a_{k-1}$. Then $a_k + 6 > a_{k-1} + 6$, so $\sqrt{a_k+6} > \sqrt{a_{k-1}+6}$, which means $a_{k+1} > a_k$. Sequence is increasing. **Bounded:** $a_1 < 3$. Assume $a_k < 3$. Then $a_k + 6 < 9$, so $a_{k+1} = \sqrt{a_k+6} < \sqrt{9} = 3$. Bounded above by 3. Since it is monotonic and bounded, it converges. Let $L = \lim a_n$. Then $L = \sqrt{6+L} \implies L^2 = 6+L \implies L^2 - L - 6 = 0 \implies (L-3)(L+2) = 0$. Since terms are positive, $L = 3$.

27. $a_{n+1} - a_n = \frac{n+1}{n+2} - \frac{n}{n+1} = \frac{(n+1)^2 - n(n+2)}{(n+2)(n+1)} = \frac{n^2+2n+1-n^2-2n}{(n+2)(n+1)} = \frac{1}{(n+2)(n+1)} > 0$. It is increasing (monotonic). It is bounded below by $a_1 = 1/2$ and above by its limit, 1.

28. $a_1 = 2, a_2 = 2.5, a_3 \approx 3.33$. It appears to be increasing. $f(x) = x + 1/x \implies f'(x) = 1 - 1/x^2 > 0$ for $x > 1$. So it's increasing. It is monotonic. It is bounded below by $a_1 = 2$, but it is not bounded above since $\lim_{n\to\infty} n + 1/n = \infty$.

29. Let L be the limit. Then $L = \frac{1}{2}(L + \frac{9}{L}) \implies 2L = L + \frac{9}{L} \implies L = \frac{9}{L} \implies L^2 = 9$. Since all terms are positive, the limit must be positive. $L = 3$.

30. **Flaw:** The expression is of the indeterminate form 1^∞, which is not necessarily 1. **Explanation:** The base approaches 1 while the exponent approaches infinity. Their interaction determines the limit. **Correction:** This is a famous limit. Let $L = \lim_{n\to\infty}(1-\frac{1}{n})^n$. Use the fact that $\lim_{n\to\infty}(1+\frac{x}{n})^n = e^x$. Here $x = -1$, so the limit is $e^{-1} = \frac{1}{e}$. Converges to $1/e$.

31. **Flaw:** The final step in the calculation is wrong. **Explanation:** $\lim_{n\to\infty} \frac{2}{6n}$ is of the form "constant over infinity", which is 0, not ∞. **Correction:** The limit is 0. Alternatively, dividing by the highest power in the denominator (n^3) gives $\lim_{n\to\infty} \frac{1/n}{1+1/n^3} = \frac{0}{1+0} = 0$. Converges to 0.

32. **Flaw:** The Theorem for Absolute Values only states that if $\lim |a_n| = 0$, then $\lim a_n = 0$. It does not apply if the limit of the absolute value is non-zero. **Explanation:** The sequence terms are approximately $\{-1, 1, -1, 1, \dots\}$ for large n. The sequence oscillates and does not approach a single value. **Correction:** The sequence oscillates between values close to -1 and 1. It does not converge. The sequence diverges.

33. After a dose, the amount remaining from the previous day is $0.75Q_{n-1}$. A new 200 mg dose is added. So, $Q_n = 0.75Q_{n-1} + 200$. Let L be the limit. Then $L = 0.75L + 200 \implies 0.25L = 200 \implies L = 800$. The long-term quantity is 800 mg.

34. No. A bounded sequence does not necessarily converge. For a sequence to be guaranteed to converge, it must be both bounded and monotonic (Monotonic Sequence Theorem). **Example:** The sequence $a_n = (-1)^n$ is bounded (between -1 and 1) but it diverges because it oscillates and never settles on a single limit.

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