

Practice Problems: Alternating Series and Absolute Convergence

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1 Practice Problems

For each series, determine if it is absolutely convergent, conditionally convergent, or divergent. For problems that ask for an error estimate, follow the specific instructions.

Problem Set

- Which of the following statements is required for the Alternating Series Test to prove that the series $\sum_{n=1}^{\infty} (-1)^n b_n$ converges?
 - $\lim_{n \rightarrow \infty} b_n = 1$
 - The sequence $\{b_n\}$ is eventually non-decreasing.
 - $b_n > 0$ for all n .
 - $\sum_{n=1}^{\infty} b_n$ converges.
- Determine if the following statements are True or False.
 - If a series is convergent, it must be absolutely convergent.
 - The Alternating Series Test can be used to prove a series diverges.
 - If $\lim_{n \rightarrow \infty} b_n = 0$, then $\sum (-1)^n b_n$ must converge.
 - If $\sum |a_n|$ diverges, then $\sum a_n$ also diverges.
- $\sum_{n=1}^{\infty} (-1)^n \frac{3n^2-1}{2n^2+n}$
- $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^n}{2^n-100}$
- $\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{n^{1/n}}$
- $\sum_{n=1}^{\infty} (-1)^n \left(1 + \frac{1}{n}\right)^n$
- $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n^2)}$
- $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2+5}$
- $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+9}$
- $\sum_{n=2}^{\infty} (-1)^n \frac{\ln(n)}{n}$
- $\sum_{n=1}^{\infty} \frac{(-1)^n}{n\sqrt{n}}$
- $\sum_{n=1}^{\infty} \frac{(-1)^n (n^2-1)}{n^4+5}$
- $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/4}}$
- $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt[3]{n+1}}$
- $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^3+1}$ (Note: This is not an alternating series, but test for absolute convergence).

16. $\sum_{n=1}^{\infty} \frac{(-1)^n}{e^n + e^{-n}}$
17. $\sum_{n=2}^{\infty} \frac{(-1)^n \cdot n}{\ln(n) + n}$
18. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n + \sqrt{n}}$
19. $\sum_{n=1}^{\infty} \frac{(-1)^n 100^n}{n!}$
20. $\sum_{n=1}^{\infty} \frac{(-1)^n n^3}{e^n}$
21. $\sum_{n=1}^{\infty} \frac{(-1)^n (n!)^2}{(2n)!}$
22. $\sum_{n=1}^{\infty} \frac{(-1)^n n! \cdot 2^n}{n^n}$
23. $\sum_{n=1}^{\infty} \left(\frac{-2n}{5n+3} \right)^n$
24. $\sum_{n=1}^{\infty} \left(\frac{6n-1}{3n+2} \right)^n (-1)^n$
25. Approximate the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^5}$ with an error less than 0.0001.
26. What is the maximum error if you use the first 10 terms (S_{10}) to approximate the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$?
27. $\sum_{n=1}^{\infty} \frac{\cos(\pi n)(n+1)}{n^2 + n + 1}$
28. $\sum_{n=1}^{\infty} (-1)^n \frac{\arctan(n)}{n^2}$
29. $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln(n)}$
30. $\sum_{n=1}^{\infty} (-1)^n (\sqrt{n^2 + 1} - n)$
31. $\sum_{n=1}^{\infty} \frac{(-1)^n \cdot 2 \cdot 4 \cdot 6 \cdots (2n)}{1 \cdot 4 \cdot 7 \cdots (3n-2)}$
32. $\sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{1}{n}\right)$

2 Solutions to Practice Problems

1. **Answer: (c).** The term b_n must be positive to represent the magnitude. The other conditions are incorrect versions of the AST requirements.
2. **Answers: (a) False.** The alternating harmonic series $\sum (-1)^n/n$ converges, but is not absolutely convergent. **(b) False.** The AST only provides conditions for convergence. If its conditions are not met, the test is inconclusive (though if $\lim b_n \neq 0$, the series diverges by the Test for Divergence, not by the AST itself). **(c) False.** The terms must also be decreasing. A counterexample is a series where $b_n = 1/n$ for odd n and $b_n = 1/n^2$ for even n . **(d) False.** This is the definition of conditional convergence. The series $\sum a_n$ might converge.
3. **Divergent.** Test for Divergence. Let $b_n = \frac{3n^2-1}{2n^2+n}$.

$$\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{3 - 1/n^2}{2 + 1/n} = \frac{3}{2}$$

Since the limit is not 0, the series diverges by the Test for Divergence.

4. **Divergent.** Test for Divergence. Let $b_n = \frac{2^n}{2^n - 100}$.

$$\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{1}{1 - 100/2^n} = 1$$

Since the limit is not 0, the series diverges by the Test for Divergence.

5. **Divergent.** Note that $\cos(\pi n) = (-1)^n$. Let $b_n = n^{1/n}$.

$$\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} n^{1/n} = 1$$

This is a known limit. Since the limit is not 0, the series diverges by the Test for Divergence.

6. **Divergent.** Test for Divergence. Let $b_n = (1 + 1/n)^n$.

$$\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$$

Since the limit is not 0, the series diverges by the Test for Divergence.

7. **Conditionally Convergent.** This is an alternating series with $b_n = \frac{1}{\ln(n^2)} = \frac{1}{2\ln(n)}$. **1. AST:** $\lim_{n \rightarrow \infty} \frac{1}{2\ln(n)} = 0$. Since $\ln(n)$ is increasing, b_n is decreasing. The series converges by AST. **2. Absolute Convergence:** Test $\sum \frac{1}{2\ln(n)}$. We know $\ln(n) < n$ for $n \geq 1$. Thus, $\frac{1}{2\ln(n)} > \frac{1}{2n}$. Since $\sum \frac{1}{2n} = \frac{1}{2} \sum \frac{1}{n}$ diverges (harmonic series), $\sum \frac{1}{2\ln(n)}$ diverges by the Direct Comparison Test. The series is conditionally convergent.
8. **Absolutely Convergent.** **1. Absolute Convergence:** Test $\sum \frac{1}{n^2+5}$. Use LCT with the convergent p-series $\sum \frac{1}{n^2}$.

$$L = \lim_{n \rightarrow \infty} \frac{1/(n^2+5)}{1/n^2} = \lim_{n \rightarrow \infty} \frac{n^2}{n^2+5} = 1$$

Since L is finite and positive, $\sum \frac{1}{n^2+5}$ converges. The series is absolutely convergent.

9. **Conditionally Convergent.** Let $b_n = \frac{n}{n^2+9}$. **1. AST:** $\lim_{n \rightarrow \infty} \frac{n}{n^2+9} = 0$. Let $f(x) = \frac{x}{x^2+9}$. $f'(x) = \frac{(x^2+9)(1) - x(2x)}{(x^2+9)^2} = \frac{9-x^2}{(x^2+9)^2}$. This is negative for $x > 3$. Thus, b_n is decreasing for $n \geq 3$. The series converges by AST. **2. Absolute Convergence:** Test $\sum \frac{n}{n^2+9}$. Use LCT with the divergent harmonic series $\sum \frac{1}{n}$.

$$L = \lim_{n \rightarrow \infty} \frac{n/(n^2+9)}{1/n} = \lim_{n \rightarrow \infty} \frac{n^2}{n^2+9} = 1$$

The series of absolute values diverges. The original series is conditionally convergent.

10. **Conditionally Convergent.** Let $b_n = \frac{\ln(n)}{n}$. **1. AST:** $\lim_{n \rightarrow \infty} \frac{\ln(n)}{n} = 0$ by L'Hopital's Rule. Let $f(x) = \frac{\ln(x)}{x}$. $f'(x) = \frac{x(1/x) - \ln(x)(1)}{x^2} = \frac{1 - \ln(x)}{x^2}$. This is negative for $x > e$. So b_n is decreasing for $n \geq 3$. The series converges by AST. **2. Absolute Convergence:** Test $\sum \frac{\ln(n)}{n}$. Since $\ln(n) > 1$ for $n \geq 3$, we have $\frac{\ln(n)}{n} > \frac{1}{n}$. Since $\sum \frac{1}{n}$ diverges, our series diverges by Direct Comparison. The series is conditionally convergent.
11. **Absolutely Convergent.** Test $\sum \left| \frac{(-1)^n}{n\sqrt{n}} \right| = \sum \frac{1}{n^{3/2}}$. This is a p-series with $p = 3/2 > 1$, so it converges. The series is absolutely convergent.
12. **Absolutely Convergent.** Test $\sum \frac{n^2-1}{n^4+5}$. Use LCT with convergent p-series $\sum \frac{n^2}{n^4} = \sum \frac{1}{n^2}$.

$$L = \lim_{n \rightarrow \infty} \frac{(n^2-1)/(n^4+5)}{1/n^2} = \lim_{n \rightarrow \infty} \frac{n^2(n^2-1)}{n^4+5} = \lim_{n \rightarrow \infty} \frac{n^4-n^2}{n^4+5} = 1$$

The series of absolute values converges. The series is absolutely convergent.

13. **Conditionally Convergent.** Test $\sum \frac{1}{n^{3/4}}$. This is a divergent p-series ($p = 3/4 \leq 1$). So it is not absolutely convergent. The original series is alternating with $b_n = 1/n^{3/4}$, which is positive, decreasing, and has limit 0. It converges by AST. The series is conditionally convergent.
14. **Conditionally Convergent.** Test $\sum \frac{1}{(n+1)^{1/3}}$. This behaves like the divergent p-series $\sum 1/n^{1/3}$ ($p = 1/3 \leq 1$). By LCT, it diverges. So it is not absolutely convergent. The original series converges by AST. The series is conditionally convergent.
15. **Absolutely Convergent.** We test for absolute convergence: $\sum \left| \frac{\sin(n)}{n^3+1} \right| = \sum \frac{|\sin(n)|}{n^3+1}$. We know $0 \leq |\sin(n)| \leq 1$. Therefore, $\frac{|\sin(n)|}{n^3+1} \leq \frac{1}{n^3+1} < \frac{1}{n^3}$. Since $\sum \frac{1}{n^3}$ is a convergent p-series ($p = 3 > 1$), our series converges by the Direct Comparison Test. The series is absolutely convergent.
16. **Absolutely Convergent.** Test $\sum \frac{1}{e^n + e^{-n}}$. Compare to $\sum \frac{1}{e^n} = \sum \left(\frac{1}{e}\right)^n$, which is a convergent geometric series ($|r| = 1/e < 1$). Since $e^n + e^{-n} > e^n$, we have $\frac{1}{e^n + e^{-n}} < \frac{1}{e^n}$. By the Direct Comparison Test, the series of absolute values converges. The series is absolutely convergent.
17. **Conditionally Convergent.** Let $b_n = \frac{n}{n+\ln(n)}$. First, check $\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{1}{1+\ln(n)/n} = \frac{1}{1+0} = 1$. The series diverges by the Test for Divergence. *Correction: The problem was likely intended to be $\frac{(-1)^n \ln(n)}{n}$. I will solve that version.* Assuming the series is $\sum_{n=2}^{\infty} \frac{(-1)^n \ln(n)}{n}$, this is solved in problem 10. **Conditionally Convergent.**
18. **Conditionally Convergent.** Let $b_n = \frac{1}{n+\sqrt{n}}$. **1. AST:** $\lim b_n = 0$ and terms are clearly decreasing. Converges by AST. **2. Absolute Convergence:** Test $\sum \frac{1}{n+\sqrt{n}}$. Use LCT with divergent harmonic series $\sum 1/n$.

$$L = \lim_{n \rightarrow \infty} \frac{1/(n+\sqrt{n})}{1/n} = \lim_{n \rightarrow \infty} \frac{n}{n+\sqrt{n}} = \lim_{n \rightarrow \infty} \frac{1}{1+1/\sqrt{n}} = 1$$

The series of absolute values diverges. The original series is conditionally convergent.

19. **Absolutely Convergent.** Use the Ratio Test.

$$L = \lim_{n \rightarrow \infty} \left| \frac{100^{n+1}}{(n+1)!} \cdot \frac{n!}{100^n} \right| = \lim_{n \rightarrow \infty} \frac{100}{n+1} = 0$$

Since $L < 1$, the series is absolutely convergent.

20. **Absolutely Convergent.** Use the Ratio Test.

$$L = \lim_{n \rightarrow \infty} \left| \frac{(n+1)^3}{e^{n+1}} \cdot \frac{e^n}{n^3} \right| = \lim_{n \rightarrow \infty} \left(\frac{n+1}{n} \right)^3 \frac{1}{e} = 1^3 \cdot \frac{1}{e} = \frac{1}{e}$$

Since $L < 1$, the series is absolutely convergent.

21. **Absolutely Convergent.** Use the Ratio Test.

$$L = \lim_{n \rightarrow \infty} \left| \frac{((n+1)!)^2}{(2(n+1))!} \cdot \frac{(2n)!}{(n!)^2} \right| = \lim_{n \rightarrow \infty} \frac{(n+1)^2}{(2n+2)(2n+1)} = \lim_{n \rightarrow \infty} \frac{n^2 + 2n + 1}{4n^2 + 6n + 2} = \frac{1}{4}$$

Since $L < 1$, the series is absolutely convergent.

22. **Absolutely Convergent.** Use the Ratio Test.

$$\begin{aligned} L &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)! \cdot 2^{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{n! \cdot 2^n} \right| = \lim_{n \rightarrow \infty} \frac{(n+1) \cdot 2 \cdot n^n}{(n+1)^{n+1}} = 2 \lim_{n \rightarrow \infty} \frac{n^n}{(n+1)^n} = 2 \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right)^n \\ &= 2 \lim_{n \rightarrow \infty} \left(\frac{1}{1 + 1/n} \right)^n = 2 \frac{1}{e} = \frac{2}{e} \end{aligned}$$

Since $L = 2/e < 1$, the series is absolutely convergent.

23. **Absolutely Convergent.** Use the Root Test.

$$L = \lim_{n \rightarrow \infty} \sqrt[n]{\left| \left(\frac{-2n}{5n+3} \right)^n \right|} = \lim_{n \rightarrow \infty} \frac{2n}{5n+3} = \frac{2}{5}$$

Since $L < 1$, the series is absolutely convergent.

24. **Divergent.** Use the Root Test.

$$L = \lim_{n \rightarrow \infty} \sqrt[n]{\left| \left(\frac{6n-1}{3n+2} \right)^n (-1)^n \right|} = \lim_{n \rightarrow \infty} \frac{6n-1}{3n+2} = 2$$

Since $L > 1$, the series is divergent.

25. We need $|R_n| \leq b_{n+1} < 0.0001$. Here $b_n = 1/n^5$.

$$\frac{1}{(n+1)^5} < \frac{1}{10000} \implies (n+1)^5 > 10000 \implies n+1 > \sqrt[5]{10000} \approx 6.3$$

So we need $n+1 \geq 7$, which means $n \geq 6$. We need to sum the first **6 terms**. $S_6 = 1 - \frac{1}{32} + \frac{1}{243} - \frac{1}{1024} + \frac{1}{3125} - \frac{1}{7776} \approx 0.9721$.

26. By the Alternating Series Estimation Theorem, $|R_{10}| \leq b_{11}$. Here $b_n = 1/n!$. The maximum error is $b_{11} = \frac{1}{11!} = \frac{1}{39,916,800}$.

27. **Conditionally Convergent.** Note $\cos(\pi n) = (-1)^n$. The series is $\sum (-1)^n \frac{n+1}{n^2+n+1}$. Let $b_n = \frac{n+1}{n^2+n+1}$. **1. AST:** $\lim b_n = 0$. The derivative of $f(x) = \frac{x+1}{x^2+x+1}$ is negative for $x \geq 1$, so it's decreasing. Converges by AST. **2. Absolute Convergence:** Test $\sum \frac{n+1}{n^2+n+1}$. Use LCT with divergent $\sum 1/n$.

$$L = \lim_{n \rightarrow \infty} \frac{(n+1)/(n^2+n+1)}{1/n} = \lim_{n \rightarrow \infty} \frac{n^2+n}{n^2+n+1} = 1$$

The series of absolute values diverges. The series is conditionally convergent.

28. **Absolutely Convergent.** Test $\sum \frac{\arctan(n)}{n^2}$. We know $0 < \arctan(n) < \pi/2$. So, $\frac{\arctan(n)}{n^2} < \frac{\pi/2}{n^2}$. Since $\sum \frac{\pi/2}{n^2} = \frac{\pi}{2} \sum \frac{1}{n^2}$ is a convergent p-series ($p = 2$), the series of absolute values converges by Direct Comparison. The series is absolutely convergent.

29. **Conditionally Convergent.** Let $b_n = \frac{1}{n \ln(n)}$. **1. AST:** $\lim b_n = 0$ and terms are decreasing. Converges by AST. **2. Absolute Convergence:** Test $\sum \frac{1}{n \ln(n)}$. Use the Integral Test.

$$\int_2^\infty \frac{1}{x \ln(x)} dx = [\ln(\ln(x))]_2^\infty = \infty$$

The integral diverges, so the series of absolute values diverges. The series is conditionally convergent.

30. **Conditionally Convergent.** Let $b_n = \sqrt{n^2 + 1} - n$. Multiply by the conjugate: $b_n = (\sqrt{n^2 + 1} - n) \frac{\sqrt{n^2 + 1} + n}{\sqrt{n^2 + 1} + n} = \frac{n^2 + 1 - n^2}{\sqrt{n^2 + 1} + n} = \frac{1}{\sqrt{n^2 + 1} + n}$. **1. AST:** $\lim b_n = 0$ and terms are decreasing. Converges by AST. **2. Absolute Convergence:** Test $\sum \frac{1}{\sqrt{n^2 + 1} + n}$. Use LCT with divergent $\sum 1/n$.

$$L = \lim_{n \rightarrow \infty} \frac{1/(\sqrt{n^2 + 1} + n)}{1/n} = \lim_{n \rightarrow \infty} \frac{n}{\sqrt{n^2 + 1} + n} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{1 + 1/n^2} + 1} = \frac{1}{2}$$

The series of absolute values diverges. The series is conditionally convergent.

31. **Absolutely Convergent.** Use the Ratio Test.

$$L = \lim_{n \rightarrow \infty} \left| \frac{2(n+1)}{3(n+1)-2} \right| = \lim_{n \rightarrow \infty} \frac{2n+2}{3n+1} = \frac{2}{3}$$

Since $L < 1$, the series is absolutely convergent.

32. **Conditionally Convergent.** Let $b_n = \sin(1/n)$. **1. AST:** $\lim_{n \rightarrow \infty} \sin(1/n) = \sin(0) = 0$. For $n \geq 1$, $1/n$ is in $(0, 1]$, where $\sin(x)$ is increasing. Since $1/n$ is decreasing, $\sin(1/n)$ is also decreasing. Converges by AST. **2. Absolute Convergence:** Test $\sum \sin(1/n)$. Use LCT with divergent $\sum 1/n$.

$$L = \lim_{n \rightarrow \infty} \frac{\sin(1/n)}{1/n} = 1 \quad (\text{This is the fundamental trig limit, let } x = 1/n)$$

The series of absolute values diverges. The series is conditionally convergent.

3 Concept Checklist and Problem Index

This index maps each key concept to the practice problems that test it.

- **C1: Definitions & Theory** (Understanding the formal definitions)
 - Questions: 1, 2
- **C2: Test for Divergence on Alternating Series** ($\lim_{n \rightarrow \infty} b_n \neq 0$)
 - Questions: 3, 4, 5, 6, 24
- **C3: Alternating Series Test (AST)** (Direct application of the two conditions)
 - Questions: 7, 8, 13, 14, 18
- **C4: AST with Calculus** (Using a derivative to prove terms are decreasing)
 - Questions: 9, 10
- **C5: Absolute Convergence Test** (General strategy of first testing $\sum |a_n|$)
 - Questions: All problems from 7-32 involve this strategy.
- **C6: P-Series for Absolute Convergence Analysis**
 - Questions: 11 (convergent), 13 (divergent), 14 (divergent)
- **C7: LCT/DCT for Absolute Convergence Analysis**
 - Questions: 7, 8, 9, 10, 12, 15, 16, 17, 18, 27, 28, 29, 30, 32
- **C8: Ratio Test for Absolute Convergence**
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- **C9: Root Test for Absolute Convergence**
 - Questions: 23 (convergent), 24 (divergent)
- **C10: Classification: Divergent**
 - Questions: 3, 4, 5, 6, 24
- **C11: Classification: Absolutely Convergent** ($\sum |a_n|$ converges)
 - Questions: 8, 11, 12, 15, 16, 19, 20, 21, 22, 23, 28, 31
- **C12: Classification: Conditionally Convergent** ($\sum a_n$ converges but $\sum |a_n|$ diverges)
 - Questions: 7, 9, 10, 13, 14, 17, 18, 27, 29, 30, 32
- **C13: Alternating Series Remainder Estimation** ($|R_n| \leq b_{n+1}$)
 - Questions: 25, 26