

# Homework 7.8 Improper Integrals

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## 1 Improper Integral Problems and Solutions

### 1.1 Problem 1

Determine whether the integral  $\int_1^\infty 3x^{-4} dx$  is convergent or divergent. If it is convergent, evaluate it.

#### Solution

This is a Type 1 improper integral because it has an infinite limit of integration.

$$\begin{aligned}\int_1^\infty 3x^{-4} dx &= \lim_{t \rightarrow \infty} \int_1^t 3x^{-4} dx \\&= \lim_{t \rightarrow \infty} \left[ 3 \frac{x^{-3}}{-3} \right]_1^t \\&= \lim_{t \rightarrow \infty} [-x^{-3}]_1^t = \lim_{t \rightarrow \infty} \left[ -\frac{1}{x^3} \right]_1^t \\&= \lim_{t \rightarrow \infty} \left( -\frac{1}{t^3} - \left( -\frac{1}{1^3} \right) \right) \\&= \lim_{t \rightarrow \infty} \left( -\frac{1}{t^3} + 1 \right) \\&= 0 + 1 = 1\end{aligned}$$

**Answer:** Convergent. The integral evaluates to **1**.

### 1.2 Problem 2

Determine whether the integral  $\int_{-\infty}^{-1} \frac{1}{\sqrt[3]{x}} dx$  is convergent or divergent. If it is convergent, evaluate it.

#### Solution

This is a Type 1 improper integral.

$$\begin{aligned}\int_{-\infty}^{-1} x^{-1/3} dx &= \lim_{t \rightarrow -\infty} \int_t^{-1} x^{-1/3} dx \\&= \lim_{t \rightarrow -\infty} \left[ \frac{x^{2/3}}{2/3} \right]_t^{-1} = \lim_{t \rightarrow -\infty} \left[ \frac{3}{2} x^{2/3} \right]_t^{-1} \\&= \lim_{t \rightarrow -\infty} \left( \frac{3}{2} (-1)^{2/3} - \frac{3}{2} t^{2/3} \right) \\&= \frac{3}{2} (1) - \lim_{t \rightarrow -\infty} \frac{3}{2} t^{2/3} \\&= \frac{3}{2} - \infty = -\infty\end{aligned}$$

As  $t \rightarrow -\infty$ ,  $t^{2/3}$  (which is  $(\sqrt[3]{t})^2$ ) approaches  $+\infty$ . The limit does not exist. **Answer: DIVERGES.**

### 1.3 Problem 3

Determine whether the integral  $\int_{-6}^{\infty} \frac{1}{x+7} dx$  is convergent or divergent.

#### Solution

This is a Type 1 improper integral.

$$\begin{aligned}\int_{-6}^{\infty} \frac{1}{x+7} dx &= \lim_{t \rightarrow \infty} \int_{-6}^t \frac{1}{x+7} dx \\ &= \lim_{t \rightarrow \infty} [\ln |x+7|]_{-6}^t \\ &= \lim_{t \rightarrow \infty} (\ln |t+7| - \ln |-6+7|) \\ &= \lim_{t \rightarrow \infty} \ln(t+7) - \ln(1) \\ &= \infty - 0 = \infty\end{aligned}$$

The limit does not exist. **Answer: DIVERGES.**

### 1.4 Problem 4

Determine whether the integral  $\int_8^{\infty} \frac{1}{(x-7)^{3/2}} dx$  is convergent or divergent.

#### Solution

This is a Type 1 improper integral.

$$\begin{aligned}\int_8^{\infty} (x-7)^{-3/2} dx &= \lim_{t \rightarrow \infty} \int_8^t (x-7)^{-3/2} dx \\ &= \lim_{t \rightarrow \infty} \left[ \frac{(x-7)^{-1/2}}{-1/2} \right]_8^t = \lim_{t \rightarrow \infty} \left[ \frac{-2}{\sqrt{x-7}} \right]_8^t \\ &= \lim_{t \rightarrow \infty} \left( \frac{-2}{\sqrt{t-7}} - \left( \frac{-2}{\sqrt{8-7}} \right) \right) \\ &= \lim_{t \rightarrow \infty} \left( \frac{-2}{\sqrt{t-7}} + 2 \right) \\ &= 0 + 2 = 2\end{aligned}$$

**Answer:** Convergent. The integral evaluates to **2**.

### 1.5 Problem 5

Determine whether the integral  $\int_0^{\infty} \frac{1}{\sqrt{1+x}} dx$  is convergent or divergent.

### Solution

This is a Type 1 improper integral.

$$\begin{aligned}\int_0^\infty (1+x)^{-1/2} dx &= \lim_{t \rightarrow \infty} \int_0^t (1+x)^{-1/2} dx \\&= \lim_{t \rightarrow \infty} \left[ \frac{(1+x)^{1/2}}{1/2} \right]_0^t = \lim_{t \rightarrow \infty} [2\sqrt{1+x}]_0^t \\&= \lim_{t \rightarrow \infty} (2\sqrt{1+t} - 2\sqrt{1+0}) \\&= \infty - 2 = \infty\end{aligned}$$

The limit does not exist. **Answer: DIVERGES.**

### 1.6 Problem 6

Determine whether the integral  $\int_{-\infty}^0 \frac{x}{(x^2+4)^2} dx$  is convergent or divergent.

### Solution

This is a Type 1 improper integral. Use u-substitution:  $u = x^2 + 4$ ,  $du = 2x dx \implies x dx = du/2$ .

$$\begin{aligned}\int_{-\infty}^0 \frac{x}{(x^2+4)^2} dx &= \lim_{t \rightarrow -\infty} \int_t^0 \frac{x}{(x^2+4)^2} dx \\&= \lim_{t \rightarrow -\infty} \int_{x=t}^{x=0} \frac{1}{u^2} \frac{du}{2} \\&= \frac{1}{2} \lim_{t \rightarrow -\infty} \left[ -\frac{1}{u} \right]_{x=t}^{x=0} = \frac{1}{2} \lim_{t \rightarrow -\infty} \left[ -\frac{1}{x^2+4} \right]_t^0 \\&= \frac{1}{2} \lim_{t \rightarrow -\infty} \left( -\frac{1}{0^2+4} - \left( -\frac{1}{t^2+4} \right) \right) \\&= \frac{1}{2} \lim_{t \rightarrow -\infty} \left( -\frac{1}{4} + \frac{1}{t^2+4} \right) \\&= \frac{1}{2} \left( -\frac{1}{4} + 0 \right) = -\frac{1}{8}\end{aligned}$$

**Answer:** Convergent. The integral evaluates to **-1/8**.

### 1.7 Problem 7

Determine whether the integral  $\int_1^\infty \frac{x^3+x+1}{x^5} dx$  is convergent or divergent.

### Solution

First, simplify the integrand.

$$\frac{x^3+x+1}{x^5} = \frac{x^3}{x^5} + \frac{x}{x^5} + \frac{1}{x^5} = x^{-2} + x^{-4} + x^{-5}$$

Now evaluate the integral:

$$\begin{aligned}
 \int_1^{\infty} (x^{-2} + x^{-4} + x^{-5}) dx &= \lim_{t \rightarrow \infty} \int_1^t (x^{-2} + x^{-4} + x^{-5}) dx \\
 &= \lim_{t \rightarrow \infty} \left[ -x^{-1} - \frac{x^{-3}}{3} - \frac{x^{-4}}{4} \right]_1^t \\
 &= \lim_{t \rightarrow \infty} \left[ -\frac{1}{x} - \frac{1}{3x^3} - \frac{1}{4x^4} \right]_1^t \\
 &= \lim_{t \rightarrow \infty} \left( \left( -\frac{1}{t} - \frac{1}{3t^3} - \frac{1}{4t^4} \right) - \left( -1 - \frac{1}{3} - \frac{1}{4} \right) \right) \\
 &= (0 - 0 - 0) - \left( -\frac{12}{12} - \frac{4}{12} - \frac{3}{12} \right) = - \left( -\frac{19}{12} \right) = \frac{19}{12}
 \end{aligned}$$

**Answer:** Convergent. The integral evaluates to **19/12**.

## 1.8 Problem 8

Determine whether the integral  $\int_0^{\infty} \frac{e^x}{(8+e^x)^2} dx$  is convergent or divergent.

**Solution**

Use u-substitution:  $u = 8 + e^x$ ,  $du = e^x dx$ .

$$\begin{aligned}
 \int_0^{\infty} \frac{e^x}{(8+e^x)^2} dx &= \lim_{t \rightarrow \infty} \int_0^t \frac{e^x}{(8+e^x)^2} dx \\
 &= \lim_{t \rightarrow \infty} \left[ -\frac{1}{8+e^x} \right]_0^t \\
 &= \lim_{t \rightarrow \infty} \left( -\frac{1}{8+e^t} - \left( -\frac{1}{8+e^0} \right) \right) \\
 &= \lim_{t \rightarrow \infty} \left( -\frac{1}{8+e^t} + \frac{1}{9} \right) \\
 &= 0 + \frac{1}{9} = \frac{1}{9}
 \end{aligned}$$

**Answer:** Convergent. The integral evaluates to **1/9**.

## 1.9 Problem 9

Determine whether the integral  $\int_{-\infty}^{\infty} 9xe^{-x^2} dx$  is convergent or divergent.

**Solution**

The integral is over  $(-\infty, \infty)$ , so we split it at an arbitrary point, like  $x = 0$ .

$$\int_{-\infty}^{\infty} 9xe^{-x^2} dx = \int_{-\infty}^0 9xe^{-x^2} dx + \int_0^{\infty} 9xe^{-x^2} dx$$

Evaluate the second integral first. Use u-substitution:  $u = -x^2$ ,  $du = -2x dx \implies 9x dx = -\frac{9}{2} du$ .

$$\begin{aligned}
 \int_0^{\infty} 9xe^{-x^2} dx &= \lim_{t \rightarrow \infty} \int_0^t 9xe^{-x^2} dx \\
 &= \lim_{t \rightarrow \infty} \left[ -\frac{9}{2} e^{-x^2} \right]_0^t \\
 &= \lim_{t \rightarrow \infty} \left( -\frac{9}{2} e^{-t^2} - \left( -\frac{9}{2} e^0 \right) \right) = 0 + \frac{9}{2} = \frac{9}{2}
 \end{aligned}$$

Now evaluate the first integral:

$$\begin{aligned}\int_{-\infty}^0 9xe^{-x^2} dx &= \lim_{s \rightarrow -\infty} \int_s^0 9xe^{-x^2} dx \\ &= \lim_{s \rightarrow -\infty} \left[ -\frac{9}{2} e^{-x^2} \right]_s^0 \\ &= \lim_{s \rightarrow -\infty} \left( -\frac{9}{2} e^0 - \left( -\frac{9}{2} e^{-s^2} \right) \right) = -\frac{9}{2} - 0 = -\frac{9}{2}\end{aligned}$$

Since both integrals converge, the original integral converges. The total value is  $\frac{9}{2} + (-\frac{9}{2}) = 0$ . **Answer:** Convergent. The integral evaluates to **0**. (Note: This could also be solved by recognizing  $f(x) = 9xe^{-x^2}$  is an odd function.)

### 1.10 Problem 10

Determine whether the integral  $\int_{-\infty}^{\infty} \frac{x^5}{x^6+1} dx$  is convergent or divergent.

#### Solution

Split the integral at  $x = 0$ . Let's evaluate the part from  $[0, \infty)$ . Use u-substitution:  $u = x^6 + 1, du = 6x^5 dx \implies x^5 dx = du/6$ .

$$\begin{aligned}\int_0^{\infty} \frac{x^5}{x^6+1} dx &= \lim_{t \rightarrow \infty} \int_0^t \frac{x^5}{x^6+1} dx \\ &= \lim_{t \rightarrow \infty} \left[ \frac{1}{6} \ln |x^6+1| \right]_0^t \\ &= \frac{1}{6} \lim_{t \rightarrow \infty} (\ln(t^6+1) - \ln(1)) = \frac{1}{6}(\infty - 0) = \infty\end{aligned}$$

Since one part of the split integral diverges, the entire integral diverges. **Answer: DIVERGES.**

### 1.11 Problem 11

Determine whether the integral  $\int_0^{\infty} 4 \sin^2(\alpha) d\alpha$  is convergent or divergent.

#### Solution

Use the half-angle identity:  $\sin^2(\alpha) = \frac{1 - \cos(2\alpha)}{2}$ .

$$\begin{aligned}\int_0^{\infty} 4 \sin^2(\alpha) d\alpha &= \int_0^{\infty} 4 \left( \frac{1 - \cos(2\alpha)}{2} \right) d\alpha = \int_0^{\infty} (2 - 2 \cos(2\alpha)) d\alpha \\ &= \lim_{t \rightarrow \infty} \int_0^t (2 - 2 \cos(2\alpha)) d\alpha \\ &= \lim_{t \rightarrow \infty} [2\alpha - \sin(2\alpha)]_0^t \\ &= \lim_{t \rightarrow \infty} ((2t - \sin(2t)) - (0 - 0))\end{aligned}$$

As  $t \rightarrow \infty$ , the  $2t$  term goes to infinity. The term  $\sin(2t)$  oscillates between -1 and 1, but it cannot stop the  $2t$  term from growing infinitely large. The limit does not exist. **Answer: DIVERGES.**

### 1.12 Problem 12

(a) Evaluate the integral:  $\int_0^t 8 \sin^2(\alpha) d\alpha$  (b) Determine whether  $\int_0^{\infty} 8 \sin^2(\alpha) d\alpha$  is convergent or divergent.

**Solution (a)**

This is a standard definite integral. Using the half-angle identity:

$$\begin{aligned}\int_0^t 8 \sin^2(\alpha) d\alpha &= \int_0^t 8 \left( \frac{1 - \cos(2\alpha)}{2} \right) d\alpha = \int_0^t (4 - 4 \cos(2\alpha)) d\alpha \\ &= [4\alpha - 2 \sin(2\alpha)]_0^t \\ &= (4t - 2 \sin(2t)) - (0 - 0)\end{aligned}$$

**Answer (a):**  $4t - 2 \sin(2t)$

**Solution (b)**

To determine if the improper integral converges, we take the limit of the result from part (a) as  $t \rightarrow \infty$ .

$$\lim_{t \rightarrow \infty} (4t - 2 \sin(2t)) = \infty$$

The limit does not exist. **Answer (b): DIVERGES.**

**1.13 Problem 13**

Determine whether the integral  $\int_0^\infty \sin(\theta) e^{\cos(\theta)} d\theta$  is convergent or divergent.

**Solution**

Use u-substitution:  $u = \cos(\theta)$ ,  $du = -\sin(\theta) d\theta$ .

$$\begin{aligned}\int_0^\infty \sin(\theta) e^{\cos(\theta)} d\theta &= \lim_{t \rightarrow \infty} \int_0^t \sin(\theta) e^{\cos(\theta)} d\theta \\ &= \lim_{t \rightarrow \infty} \left[ -e^{\cos(\theta)} \right]_0^t \\ &= \lim_{t \rightarrow \infty} \left( -e^{\cos(t)} - (-e^{\cos(0)}) \right) \\ &= \lim_{t \rightarrow \infty} \left( e - e^{\cos(t)} \right)\end{aligned}$$

As  $t \rightarrow \infty$ ,  $\cos(t)$  oscillates between -1 and 1. Therefore,  $e^{\cos(t)}$  oscillates between  $e^{-1}$  and  $e^1$ . The limit does not settle on a single value. **Answer: DIVERGES.**

**1.14 Problem 14**

Determine whether the integral  $\int_1^\infty \frac{1}{x^2+x} dx$  is convergent or divergent.

**Solution**

Use partial fraction decomposition:  $\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} \implies 1 = A(x+1) + Bx$ . If  $x = 0$ ,  $A = 1$ . If  $x = -1$ ,  $B = -1$ . So,  $\frac{1}{x^2+x} = \frac{1}{x} - \frac{1}{x+1}$ .

$$\begin{aligned}\int_1^\infty \left( \frac{1}{x} - \frac{1}{x+1} \right) dx &= \lim_{t \rightarrow \infty} \int_1^t \left( \frac{1}{x} - \frac{1}{x+1} \right) dx \\ &= \lim_{t \rightarrow \infty} [\ln|x| - \ln|x+1|]_1^t \\ &= \lim_{t \rightarrow \infty} \left[ \ln \left| \frac{x}{x+1} \right| \right]_1^t \\ &= \lim_{t \rightarrow \infty} \left( \ln \left( \frac{t}{t+1} \right) - \ln \left( \frac{1}{2} \right) \right)\end{aligned}$$

As  $t \rightarrow \infty$ ,  $\frac{t}{t+1} \rightarrow 1$ , so  $\ln(\frac{t}{t+1}) \rightarrow \ln(1) = 0$ . The result is  $0 - \ln(1/2) = -(-\ln(2)) = \ln(2)$ . **Answer:** Convergent. The integral evaluates to **ln(2)**.

### 1.15 Problem 15

Determine whether the integral  $\int_2^\infty \frac{dv}{v^2+2v-3}$  is convergent or divergent.

#### Solution

Use partial fraction decomposition:  $\frac{1}{(v+3)(v-1)} = \frac{A}{v+3} + \frac{B}{v-1} \implies 1 = A(v-1) + B(v+3)$ . If  $v = 1$ ,  $B = 1/4$ . If  $v = -3$ ,  $A = -1/4$ .

$$\begin{aligned} \int_2^\infty \frac{1/4}{v-1} - \frac{1/4}{v+3} dv &= \frac{1}{4} \lim_{t \rightarrow \infty} \int_2^t \left( \frac{1}{v-1} - \frac{1}{v+3} \right) dv \\ &= \frac{1}{4} \lim_{t \rightarrow \infty} [\ln|v-1| - \ln|v+3|]_2^t \\ &= \frac{1}{4} \lim_{t \rightarrow \infty} \left[ \ln \left| \frac{v-1}{v+3} \right| \right]_2^t \\ &= \frac{1}{4} \lim_{t \rightarrow \infty} \left( \ln \left( \frac{t-1}{t+3} \right) - \ln \left( \frac{1}{5} \right) \right) \\ &= \frac{1}{4} (\ln(1) - \ln(1/5)) = \frac{1}{4} (0 - (-\ln(5))) = \frac{\ln(5)}{4} \end{aligned}$$

**Answer:** Convergent. The integral evaluates to **ln(5)/4**.

### 1.16 Problem 16

Determine whether the integral  $\int_{-\infty}^0 \frac{z}{z^4+81} dz$  is convergent or divergent.

#### Solution

Use u-substitution:  $u = z^2$ ,  $du = 2z dz \implies z dz = du/2$ .

$$\begin{aligned} \int_{-\infty}^0 \frac{z}{z^4+81} dz &= \lim_{t \rightarrow -\infty} \int_t^0 \frac{z}{(z^2)^2+81} dz \\ &= \lim_{t \rightarrow -\infty} \frac{1}{2} \int_{z=t}^{z=0} \frac{du}{u^2+81} \\ &= \frac{1}{2} \lim_{t \rightarrow -\infty} \left[ \frac{1}{9} \arctan \left( \frac{u}{9} \right) \right]_{z=t}^{z=0} = \frac{1}{18} \lim_{t \rightarrow -\infty} \left[ \arctan \left( \frac{z^2}{9} \right) \right]_t^0 \\ &= \frac{1}{18} \lim_{t \rightarrow -\infty} \left( \arctan(0) - \arctan \left( \frac{t^2}{9} \right) \right) \end{aligned}$$

As  $t \rightarrow -\infty$ ,  $t^2 \rightarrow \infty$ , so  $\arctan(t^2/9) \rightarrow \pi/2$ . The result is  $\frac{1}{18}(0 - \frac{\pi}{2}) = -\frac{\pi}{36}$ . **Answer:** Convergent. The integral evaluates to **-π/36**.

### 1.17 Problem 17

(a) Evaluate the integral:  $\int_t^0 \frac{z}{z^4+36} dz$  (b) Determine whether  $\int_{-\infty}^0 \frac{z}{z^4+36} dz$  is convergent or divergent.

**Solution (a)**

Similar to problem 16, use u-substitution  $u = z^2, du = 2z dz$ .

$$\begin{aligned}\int_t^0 \frac{z}{z^4 + 36} dz &= \frac{1}{2} \int_{z=t}^{z=0} \frac{du}{u^2 + 36} \\ &= \frac{1}{2} \left[ \frac{1}{6} \arctan\left(\frac{u}{6}\right) \right]_{z=t}^{z=0} = \frac{1}{12} \left[ \arctan\left(\frac{z^2}{6}\right) \right]_t^0 \\ &= \frac{1}{12} \left( \arctan(0) - \arctan\left(\frac{t^2}{6}\right) \right)\end{aligned}$$

**Answer (a):**  $-\frac{1}{12} \arctan\left(\frac{t^2}{6}\right)$

**Solution (b)**

Take the limit of the result from part (a) as  $t \rightarrow -\infty$ .

$$\lim_{t \rightarrow -\infty} \left( -\frac{1}{12} \arctan\left(\frac{t^2}{6}\right) \right) = -\frac{1}{12} \left( \frac{\pi}{2} \right) = -\frac{\pi}{24}$$

**Answer (b):** Convergent. The integral evaluates to  $-\pi/24$ .

**1.18 Problem 18**

Determine whether the integral  $\int_0^9 \frac{3}{\sqrt[3]{x-1}} dx$  is convergent or divergent.

**Solution**

This is a Type 2 improper integral because the function has an infinite discontinuity at  $x = 1$ , which is within the interval  $[0, 9]$ . We must split the integral at the point of discontinuity.

$$\int_0^9 \frac{3}{(x-1)^{1/3}} dx = \int_0^1 3(x-1)^{-1/3} dx + \int_1^9 3(x-1)^{-1/3} dx$$

Evaluate the first part:

$$\begin{aligned}\lim_{t \rightarrow 1^-} \int_0^t 3(x-1)^{-1/3} dx &= \lim_{t \rightarrow 1^-} \left[ 3 \frac{(x-1)^{2/3}}{2/3} \right]_0^t = \lim_{t \rightarrow 1^-} \left[ \frac{9}{2} (x-1)^{2/3} \right]_0^t \\ &= \lim_{t \rightarrow 1^-} \left( \frac{9}{2} (t-1)^{2/3} - \frac{9}{2} (-1)^{2/3} \right) \\ &= 0 - \frac{9}{2} (1) = -\frac{9}{2}\end{aligned}$$

This part converges. Now evaluate the second part:

$$\begin{aligned}\lim_{s \rightarrow 1^+} \int_s^9 3(x-1)^{-1/3} dx &= \lim_{s \rightarrow 1^+} \left[ \frac{9}{2} (x-1)^{2/3} \right]_s^9 \\ &= \lim_{s \rightarrow 1^+} \left( \frac{9}{2} (9-1)^{2/3} - \frac{9}{2} (s-1)^{2/3} \right) \\ &= \frac{9}{2} (8)^{2/3} - 0 = \frac{9}{2} (4) = 18\end{aligned}$$

This part also converges. Since both parts converge, the original integral converges. The total value is  $-\frac{9}{2} + 18 = \frac{27}{2}$ .

**Answer:** Convergent. The integral evaluates to **27/2**.



## 2 Analysis of Problems and Techniques

### 2.1 Types of Improper Integrals Encountered

#### 1. Type 1: Infinite Intervals

- Integrals over  $[a, \infty)$ : Problems 1, 3, 4, 5, 7, 8, 11, 12, 13, 14, 15.
- Integrals over  $(-\infty, b]$ : Problems 2, 6, 16, 17.
- Integrals over  $(-\infty, \infty)$ : Problems 9, 10. These must be split into two separate improper integrals, e.g.,  $\int_{-\infty}^c f(x) dx + \int_c^{\infty} f(x) dx$ . The integral converges only if BOTH parts converge.

#### 2. Type 2: Infinite Discontinuity

- The integrand has a vertical asymptote at  $x = c$  within the interval  $[a, b]$ . The integral must be split at the discontinuity:  $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$ . This was seen in Problem 18 at  $x = 1$  on the interval  $[0, 9]$ .

### 2.2 Convergence and Divergence Rules (p-Integrals)

A key tool for quickly assessing convergence is the **p-integral test**.

- For Type 1 integrals:  $\int_a^{\infty} \frac{1}{x^p} dx$  (where  $a > 0$ ) **converges if**  $p > 1$  and **diverges if**  $p \leq 1$ .
  - Problem 1:  $p = 4 > 1$ , converges.
  - Problem 4: Form is  $1/u^{3/2}$ ,  $p = 3/2 > 1$ , converges.
  - Problem 2:  $p = 1/3 \leq 1$  on an infinite interval, diverges.
  - Problem 5: Form is  $1/u^{1/2}$ ,  $p = 1/2 \leq 1$ , diverges.
- For Type 2 integrals:  $\int_0^a \frac{1}{x^p} dx$  (discontinuity at 0) **converges if**  $p < 1$  and **diverges if**  $p \geq 1$ .
  - Problem 18: Form is  $1/u^{1/3}$ ,  $p = 1/3 < 1$ , converges.

### 2.3 Techniques and Algebraic Manipulations Used

- **Limit Definition:** The fundamental technique for all problems was to rewrite the improper integral as a limit of a proper integral.
- **U-Substitution:** Used in Problems 6, 8, 9, 10, 13, 16, 17 to simplify the integrand before integration.
- **Partial Fraction Decomposition:** Necessary for integrating rational functions where the denominator is factorable. Used in Problems 14 and 15.
- **Trigonometric Identities:** The half-angle identity  $\sin^2(\alpha) = (1 - \cos(2\alpha))/2$  was crucial for Problems 11 and 12.
- **Splitting Integrals:** Required for Type 1 integrals over  $(-\infty, \infty)$  (Problems 9, 10) and for Type 2 integrals with an interior discontinuity (Problem 18).
- **Recognizing Odd/Even Functions:** In Problem 9, the integrand is an odd function integrated over a symmetric interval  $(-\infty, \infty)$ . If such an integral converges, its value must be 0. In Problem 10, the integrand was also odd, but it was shown to diverge. *Trick: You must still prove convergence of one half of the integral before concluding the value is 0.*
- **Simplifying the Integrand:** In Problem 7, dividing each term in the numerator by the denominator simplified the problem into a sum of p-integrals.

## 2.4 Essential Limits to Know

For evaluating improper integrals, knowledge of limits at infinity is critical.

- $\lim_{x \rightarrow \infty} \frac{1}{x^p} = 0$  for any  $p > 0$ .
- $\lim_{x \rightarrow \infty} e^x = \infty$  and  $\lim_{x \rightarrow -\infty} e^x = 0$ .
- $\lim_{x \rightarrow \infty} \ln(x) = \infty$  and  $\lim_{x \rightarrow 0^+} \ln(x) = -\infty$ .
- $\lim_{x \rightarrow \infty} \arctan(x) = \frac{\pi}{2}$  and  $\lim_{x \rightarrow -\infty} \arctan(x) = -\frac{\pi}{2}$ .
- Limits of oscillating functions like  $\sin(x)$  or  $\cos(x)$  as  $x \rightarrow \infty$  do not exist.

## 2.5 Additional Tricks and Untested Concepts

The provided problems did not cover all aspects of improper integrals. Here are other important concepts:

- **The Comparison Test:** For an integrand that is difficult to integrate directly, you can compare it to a simpler function. If  $f(x) \geq g(x) \geq 0$ :
  - If  $\int_a^\infty f(x) dx$  converges, then  $\int_a^\infty g(x) dx$  also converges.
  - If  $\int_a^\infty g(x) dx$  diverges, then  $\int_a^\infty f(x) dx$  also diverges.
  - **Example:** To determine if  $\int_1^\infty \frac{1}{x^2+5} dx$  converges, we can note that  $\frac{1}{x^2+5} < \frac{1}{x^2}$ . Since  $\int_1^\infty \frac{1}{x^2} dx$  converges (p-integral with  $p=2$ ), our integral must also converge.
- **The Limit Comparison Test:** If  $f(x)$  and  $g(x)$  are positive functions and  $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = L$ , where  $L$  is a finite, positive number, then  $\int f(x) dx$  and  $\int g(x) dx$  either both converge or both diverge.
  - **Example:** For  $\int_1^\infty \frac{x}{x^3-2} dx$ , we can compare it to  $g(x) = \frac{x}{x^3} = \frac{1}{x^2}$ . Since  $\lim_{x \rightarrow \infty} \frac{x/(x^3-2)}{1/x^2} = 1$ , and we know  $\int_1^\infty \frac{1}{x^2} dx$  converges, our integral must also converge.
- **Checking the Domain:** As a trick, always check that the function is defined over the integration interval. A problem like  $\int_0^2 \frac{1}{\sqrt{x-3}} dx$  is invalid because the integrand is not real for any value in the interval. The initial (incorrect) reading of Problem 18 as  $1/\sqrt{x-1}$  would have made the integral from  $[0, 1]$  invalid in the real number system.