Practice Set: Functions of Several Variables

Based on Homework 14.1 Study Guide

Comprehensive Review & Problem Set

Concept Check List

Before beginning the problems, review the following concepts derived from the study guide. These are the key areas tested in this problem set.

1. **Function Evaluation:** Substituting scalars and expressions into $f(x, y)$.
2. **Domain - Rational Functions:** Denominators $\neq 0$.
3. **Domain - Radical Functions:** Arguments of even roots ≥ 0.
4. **Domain - Logarithmic Functions:** Arguments of logarithms > 0.
5. **Domain - Visualizing Regions:** Identifying lines, circles, parabolas, and ellipses as boundaries; determining solid vs. dashed lines.
6. **Domain - 3 Variables:** Extending domain rules to $f(x, y, z)$.
7. **Range Analysis:** Determining possible output values (using properties of squares, exponentials, and roots).
8. **Level Curves (Contour Maps):** Finding equations for $z = k$ and identifying shapes (Circles, Ellipses, Hyperbolas, Lines).
9. **Topographic Intuition:** Relating contour spacing to steepness (Gradient intuition).
10. **Traces & Cross-Sections:** Fixing one variable to analyze 2D curves.
11. **Surface Identification:** Matching equations to Quadric Surfaces (Paraboloids, Cones, Spheres, Ellipsoids) and Cylinders.
12. **Applications:** Interpreting tabular data and word problems.

Problem Set

Part 1: Evaluation

1. Given $f(x,y) = \frac{x^3+2y}{x-y}$, evaluate $f(2,1)$.
2. Given $g(x,y) = x^2e^{xy}$, find $g(2,-0.5)$.
3. Given $h(x,y) = 3x^2 - y$, find and simplify the expression for $h(x+t,y)$.
4. Given $f(x,y) = \sqrt{x^2+y^2}$, find the value of $f(t,t)$ assuming $t > 0$.

Part 2: Domain and Range

Determine the domain of the following functions. Describe the region geometrically (e.g., "inside the circle," "above the line"). For sketching, note whether boundaries are solid or dashed.
5. $f(x,y) = \frac{x+y}{x^2+y^2-16}$
6. $f(x,y) = \sqrt{2x-y}$
7. $f(x,y) = \ln(y-x^2)$
8. $f(x,y) = \frac{\sqrt{x}}{y^2-1}$
9. $f(x,y) = \sqrt[4]{xy}$
10. $f(x,y) = \ln(16-x^2-y^2)$
11. $f(x,y) = \frac{1}{\sqrt{x+y-2}}$
12. $f(x,y) = e^{\sqrt{1-x^2}}$
13. $f(x,y,z) = \ln(z-x^2-y^2)$
14. $g(x,y,z) = \sqrt{36-4x^2-9y^2-z^2}$
15. **(Range)** Find the range of $z = 5 - \sqrt{x^2+y^2}$.
16. **(Range)** Find the range of $z = e^{-(x^2+y^2)}$.

Part 3: Level Curves and Contour Maps

For the following functions, describe the geometric shape of the level curves $f(x,y) = k$.
17. $z = x^2 + 4y^2$
18. $z = y - x^2$
19. $z = \sqrt{x^2+y^2}$
20. $z = xy$
21. $z = \frac{y}{x}$
22. Consider the function $f(x,y) = \sqrt{100-x^2-y^2}$.

(a) Determine the domain and range.

(b) Sketch the level curves for $k = 0, 6, 8$.

(c) As k increases from 0 to 10, do the level curves get closer together or farther apart? What does this imply about the shape of the surface?

23. Which of the following functions corresponds to a contour map consisting of parallel, equally spaced straight lines?

(A) $z = x^2 + y$

(B) $z = 3x - 2y + 5$

(C) $z = xy$

(D) $z = \sin(x)$

Part 4: Surface Identification and Traces

24. Identify the surface defined by $z = x^2$. Explain why the variable y is missing and what that implies about the graph.

25. Match the equation $x^2 + y^2 + z^2 = 25$ to its surface type. Describe its traces in the planes $z = 0$, $z = 3$, and $z = 5$.

26. Consider the surface $z = x^2 - y^2$.

(a) What is the shape of the trace when $z = 0$?

(b) What is the shape of the trace when $x = 0$?

(c) What is the common name for this surface?

27. Match the equation to the description:

- **A:** $z = \sqrt{x^2 + y^2}$
- **B:** $z = x^2 + y^2$

Description 1: Level curves are equally spaced circles. *Description 2:* Level curves are circles that get closer together as z increases.

28. Identify the surface $z = \sin(y)$. Describe the cross-sections for a fixed value of x.

Part 5: Applications and Interpretation

29. (Table Interpretation) The "Heat Index" $I(T, h)$ is a function of temperature T (in °F) and humidity h (%).

$T \setminus h$	40	50	60
80	80	81	82
85	86	88	90
90	91	95	100

(a) Evaluate $I(90, 50)$ and interpret its meaning.

(b) If $T = 85$ is held constant, estimate the rate of change of the Heat Index as humidity increases from 50 to 60.

30. (Conceptual) In a financial model, the cost C is given by $C(x, y) = 1000 + 5x + 2y$, where x is labor hours and y is material units. Describe the shape of the level curves of this cost function. What does moving along a level curve represent in economic terms?

Answer Key and Solutions

Part 1: Evaluation

1. Answer: 10. Substitute $x = 2, y = 1$: $\frac{2^3+2(1)}{2-1} = \frac{8+2}{1} = 10$.

2. Answer: $4e^{-1}$ **or** $4/e$**.** Substitute $x = 2, y = -0.5$: $(2)^2e^{2(-0.5)} = 4e^{-1}$.

3. Answer: $3x^2 + 6xt + 3t^2 - y$**.** Substitute $x \to (x + t)$: $3(x + t)^2 - y = 3(x^2 + 2xt + t^2) - y$.

4. Answer: $t\sqrt{2}$**.** $f(t, t) = \sqrt{t^2 + t^2} = \sqrt{2t^2} = |t|\sqrt{2}$. Since $t > 0$, result is $t\sqrt{2}$.

Part 2: Domain and Range

5. Domain: All real numbers except points on the circle $x^2 + y^2 = 16$. *Reason:* Denominator $\neq 0$.

6. Domain: $y \leq 2x$. Region on or below the line $y = 2x$. *Reason:* Radicand $\geq 0 \implies 2x - y \geq 0$. Solid boundary.

7. Domain: $y > x^2$. Region strictly above the parabola $y = x^2$. *Reason:* Log argument > 0. Dashed boundary.

8. Domain: $x \geq 0$ AND $y \neq 1, y \neq -1$. *Reason:* Square root $x \geq 0$ (right half-plane) minus the two horizontal lines $y = \pm 1$.

9. Domain: Quadrants I and III (including axes). *Reason:* Even root requires $xy \geq 0$. This happens if both $x, y \geq 0$ OR both $x, y \leq 0$.

10. Domain: Interior of the circle $x^2 + y^2 < 16$. *Reason:* $16 - x^2 - y^2 > 0 \implies x^2 + y^2 < 16$. Dashed boundary radius 4.

11. Domain: $y > -x + 2$. Region strictly above the line $y = -x + 2$. *Reason:* Denom $\neq 0$ AND root $\geq 0 \implies x + y - 2 > 0$.

12. Domain: $-1 \leq x \leq 1$. (Infinite strip in y-direction). *Reason:* Root requires $1 - x^2 \geq 0 \implies x^2 \leq 1$. y can be anything.

13. Domain: Inside the paraboloid $z > x^2+y^2$. *Reason:* Log argument $z-(x^2+y^2) > 0$.

14. Domain: Interior and surface of the ellipsoid $\frac{x^2}{9} + \frac{y^2}{4} + \frac{z^2}{36} \leq 1$. *Reason:* Radicand ≥ 0.

15. Range: $(-\infty, 5]$. *Reason:* $\sqrt{\dots} \geq 0$. Max value is $5 - 0 = 5$. No lower bound.

16. Range: $(0, 1]$. *Reason:* $x^2 + y^2 \geq 0$, so $-(x^2 + y^2) \leq 0$. $e^{\text{negative}} \in (0, 1]$.

Part 3: Level Curves

17. Ellipses. Equation $\frac{x^2}{k} + \frac{y^2}{k/4} = 1$ (for $k > 0$). **18. Parabolas.** $y = x^2 + k$. **19. Circles.** $x^2 + y^2 = k^2$ (Concentrically spaced). **20. Hyperbolas.** $y = k/x$. **21. Lines.** $y = kx$ (Lines passing through origin). **22. (a)** Domain: Circle $r = 10$. Range: $[0, 10]$. **(b)** $k = 0 \to r = 10$; $k = 6 \to r = 8$; $k = 8 \to r = 6$. **(c)** Closer together. The surface (hemisphere) gets steeper near the base ($z = 0$). **23. B.** This is a plane. Linear functions of x and y yield parallel linear contours.

Part 4: Surface Identification

24. Parabolic Cylinder. y is missing, meaning the curve $z = x^2$ is extended infinitely along the y-axis (like a trough). **25. Sphere.** Center $(0, 0, 0)$ radius 5. $z = 0$: Circle radius 5. $z = 3$: Circle radius 4. $z = 5$: Point $(0, 0)$. **26. (a)** $x = \pm y$ (Intersecting lines). **(b)** $z = -y^2$ (Parabola opening down). **(c)** Hyperbolic Paraboloid (Saddle). **27. A matches Desc 1** (Cone - linear slope). **B matches Desc 2** (Paraboloid - increasing slope). **28. Sine Cylinder.** Wave shape extending along the x-axis. Fixed x yields $z = \sin(y)$.

Part 5: Applications

29. (a) 95. At 90°F and 50% humidity, it feels like 95°F. **(b)** Change is $90 - 88 = 2$ degrees over 10% humidity. Rate ≈ 0.2 deg/%. **30.** Lines ($5x + 2y = C - 1000$). Moving along a level curve means keeping total cost constant while trading labor for materials.

Concept Check List Mapping

This table maps the problems to the concepts listed at the start of the document.

Concept / Problem Type	**Relevant Question #**
1. Function Evaluation	1, 2, 3, 4, 29(a)
2. Domain - Rational	5, 11
3. Domain - Radical	6, 8, 9, 11, 12, 14
4. Domain - Logarithmic	7, 10, 13
5. Domain - Visualizing Regions	5, 6, 7, 10, 11, 12
6. Domain - 3 Variables	13, 14
7. Range Analysis	15, 16, 22(a)
8. Level Curves - Identification	17, 18, 19, 20, 21, 23
9. Topographic Intuition (Spacing)	22(c), 27
10. Traces & Cross-Sections	26, 28
11. Surface Identification	24, 25, 26, 27, 28
12. Applications/Interpretation	29, 30