Calculus with Polar Coordinates Problem Set

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Problem Set

Area of a Simple Polar Region

Problem 1

Find the area of the region enclosed by the cardioid $r = 3 - 3\cos(\theta)$.

Solution: The cardioid is traced once from $\theta = 0$ to $\theta = 2\pi$.

$$\begin{aligned}
A &= \frac{1}{2}\int_0^{2\pi} (3 - 3\cos(\theta))^2 \, d\theta \\
&= \frac{9}{2}\int_0^{2\pi} (1 - 2\cos(\theta) + \cos^2(\theta)) \, d\theta \\
&= \frac{9}{2}\int_0^{2\pi} \left(1 - 2\cos(\theta) + \frac{1+\cos(2\theta)}{2}\right) d\theta \\
&= \frac{9}{2}\int_0^{2\pi} \left(\frac{3}{2} - 2\cos(\theta) + \frac{1}{2}\cos(2\theta)\right) d\theta \\
&= \frac{9}{2}\left[\frac{3}{2}\theta - 2\sin(\theta) + \frac{1}{4}\sin(2\theta)\right]_0^{2\pi} \\
&= \frac{9}{2}\left(\frac{3}{2}(2\pi) - 0 + 0\right) - 0 = \frac{27\pi}{2}
\end{aligned}$$

Problem 2

Find the area of the region bounded by the circle $r = 5\sin(\theta)$.

Solution: The circle is traced once from $\theta = 0$ to $\theta = \pi$.

$$\begin{aligned}
A &= \frac{1}{2}\int_0^{\pi} (5\sin(\theta))^2 \, d\theta = \frac{25}{2}\int_0^{\pi} \sin^2(\theta) \, d\theta \\
&= \frac{25}{2}\int_0^{\pi} \frac{1-\cos(2\theta)}{2} \, d\theta \\
&= \frac{25}{4}\left[\theta - \frac{1}{2}\sin(2\theta)\right]_0^{\pi} = \frac{25}{4}(\pi - 0) = \frac{25\pi}{4}
\end{aligned}$$

Problem 3

Find the area of the region that lies in the first quadrant and is bounded by the curve $r = 2\theta$.

Solution: The first quadrant corresponds to $0 \le \theta \le \pi/2$.

$$\begin{aligned}
A &= \frac{1}{2}\int_0^{\pi/2} (2\theta)^2 \, d\theta = \frac{1}{2}\int_0^{\pi/2} 4\theta^2 \, d\theta \\
&= 2\left[\frac{\theta^3}{3}\right]_0^{\pi/2} = 2\left(\frac{(\pi/2)^3}{3}\right) = \frac{2}{3}\frac{\pi^3}{8} = \frac{\pi^3}{12}
\end{aligned}$$

Area of a Single Loop or Petal

Problem 4

Find the area of one petal of the rose curve $r = 4\cos(3\theta)$.

Solution: Find bounds for one loop by setting $r = 0$. $4\cos(3\theta) = 0 \implies 3\theta = \pm\pi/2, \pm3\pi/2, \ldots$. The first loop is traced for 3θ from $-\pi/2$ to $\pi/2$, so θ from $-\pi/6$ to $\pi/6$.

$$\begin{aligned}
A &= \frac{1}{2}\int_{-\pi/6}^{\pi/6} (4\cos(3\theta))^2\, d\theta = 8\int_{-\pi/6}^{\pi/6} \cos^2(3\theta)\, d\theta \\
&= 8\int_{-\pi/6}^{\pi/6} \frac{1+\cos(6\theta)}{2}\, d\theta = 4\left[\theta + \frac{1}{6}\sin(6\theta)\right]_{-\pi/6}^{\pi/6} \\
&= 4\left[\left(\frac{\pi}{6} + \frac{1}{6}\sin(\pi)\right) - \left(-\frac{\pi}{6} + \frac{1}{6}\sin(-\pi)\right)\right] = 4\left(\frac{2\pi}{6}\right) = \frac{4\pi}{3}
\end{aligned}$$

Problem 5

Find the area enclosed by one loop of the lemniscate $r^2 = 9\cos(2\theta)$.

Solution: One loop is traced when $\cos(2\theta) \geq 0$. This occurs for 2θ between $-\pi/2$ and $\pi/2$, so θ between $-\pi/4$ and $\pi/4$.

$$\begin{aligned}
A &= \frac{1}{2}\int_{-\pi/4}^{\pi/4} r^2\, d\theta = \frac{1}{2}\int_{-\pi/4}^{\pi/4} 9\cos(2\theta)\, d\theta \\
&= \frac{9}{2}\left[\frac{1}{2}\sin(2\theta)\right]_{-\pi/4}^{\pi/4} \\
&= \frac{9}{4}\left[\sin(\pi/2) - \sin(-\pi/2)\right] = \frac{9}{4}(1-(-1)) = \frac{9}{2}
\end{aligned}$$

Problem 6

Find the area of the inner loop of the limaçon $r = 1 + 2\sin(\theta)$.

Solution: The inner loop is traced when $r < 0$. First find $r = 0$: $1 + 2\sin(\theta) = 0 \implies \sin(\theta) = -1/2$. This occurs at $\theta = 7\pi/6$ and $\theta = 11\pi/6$. The inner loop is traced between these angles.

$$\begin{aligned}
A &= \frac{1}{2}\int_{7\pi/6}^{11\pi/6} (1+2\sin(\theta))^2\, d\theta \\
&= \frac{1}{2}\int_{7\pi/6}^{11\pi/6} (1 + 4\sin(\theta) + 4\sin^2(\theta))\, d\theta \\
&= \frac{1}{2}\int_{7\pi/6}^{11\pi/6} \left(1 + 4\sin(\theta) + 4\frac{1-\cos(2\theta)}{2}\right) d\theta \\
&= \frac{1}{2}\int_{7\pi/6}^{11\pi/6} (3 + 4\sin(\theta) - 2\cos(2\theta))\, d\theta \\
&= \frac{1}{2}[3\theta - 4\cos(\theta) - \sin(2\theta)]_{7\pi/6}^{11\pi/6} \\
&= \frac{1}{2}\left[\left(\frac{33\pi}{6} - 4\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right) - \left(\frac{21\pi}{6} - 4(-\frac{\sqrt{3}}{2}) - (-\frac{\sqrt{3}}{2})\right)\right] \\
&= \frac{1}{2}\left[\frac{12\pi}{6} - 2\sqrt{3} - \frac{\sqrt{3}}{2} - 2\sqrt{3} - \frac{\sqrt{3}}{2}\right] = \frac{1}{2}[2\pi - 5\sqrt{3}] = \pi - \frac{5\sqrt{3}}{2}
\end{aligned}$$

Note: Area must be positive. Let's recheck the calculation. $A = \frac{1}{2}[(3\frac{11\pi}{6} - 2\sqrt{3} - (-\frac{\sqrt{3}}{2})) - (3\frac{7\pi}{6} - (-2\sqrt{3}) - (\frac{\sqrt{3}}{2}))] = \frac{1}{2}[\frac{11\pi}{2} - \frac{3\sqrt{3}}{2} - \frac{7\pi}{2} - \frac{3\sqrt{3}}{2}] = \frac{1}{2}[2\pi - 3\sqrt{3}] = \pi - \frac{3\sqrt{3}}{2}$.

Area from a Graph

Problem 7

The graph of $r = 2 + 2\cos(\theta)$ is a cardioid. Find the area of the region above the polar axis.

Solution: The region above the polar axis is traced from $\theta = 0$ to $\theta = \pi$.

$$\begin{aligned}
A &= \frac{1}{2}\int_0^\pi (2 + 2\cos(\theta))^2\, d\theta = 2\int_0^\pi (1 + 2\cos(\theta) + \cos^2(\theta))\, d\theta \\
&= 2\int_0^\pi \left(\frac{3}{2} + 2\cos(\theta) + \frac{1}{2}\cos(2\theta)\right) d\theta \\
&= 2\left[\frac{3}{2}\theta + 2\sin(\theta) + \frac{1}{4}\sin(2\theta)\right]_0^\pi = 2\left(\frac{3\pi}{2}\right) = 3\pi
\end{aligned}$$

Problem 8

Find the area of the shaded region for $r = 4 + 3\sin(\theta)$, which is the right half of the limaçon.

Solution: The right half is traced from $\theta = -\pi/2$ to $\theta = \pi/2$.

$$\begin{aligned}
A &= \frac{1}{2}\int_{-\pi/2}^{\pi/2} (4 + 3\sin(\theta))^2\, d\theta \\
&= \frac{1}{2}\int_{-\pi/2}^{\pi/2} (16 + 24\sin(\theta) + 9\sin^2(\theta))\, d\theta \\
&= \frac{1}{2}\int_{-\pi/2}^{\pi/2} \left(16 + 24\sin(\theta) + \frac{9}{2}(1 - \cos(2\theta))\right) d\theta \\
&= \frac{1}{2}\left[\frac{41}{2}\theta - 24\cos(\theta) - \frac{9}{4}\sin(2\theta)\right]_{-\pi/2}^{\pi/2} \\
&= \frac{1}{2}\left[\left(\frac{41\pi}{4}\right) - \left(-\frac{41\pi}{4}\right)\right] = \frac{41\pi}{4}
\end{aligned}$$

Area Between Two Curves

Problem 9

Find the area of the region that lies inside the circle $r = 3\sin(\theta)$ and outside the cardioid $r = 1 + \sin(\theta)$.

Solution: Find intersection points: $3\sin(\theta) = 1 + \sin(\theta) \implies 2\sin(\theta) = 1 \implies \sin(\theta) = 1/2$. Intersections at $\theta = \pi/6$ and $\theta = 5\pi/6$. In this interval, $3\sin(\theta) > 1 + \sin(\theta)$, so it's the outer curve.

$$\begin{aligned}
A &= \frac{1}{2}\int_{\pi/6}^{5\pi/6} \left[(3\sin(\theta))^2 - (1+\sin(\theta))^2\right]\, d\theta \\
&= \frac{1}{2}\int_{\pi/6}^{5\pi/6} [9\sin^2(\theta) - (1 + 2\sin(\theta) + \sin^2(\theta))]\, d\theta \\
&= \frac{1}{2}\int_{\pi/6}^{5\pi/6} (8\sin^2(\theta) - 2\sin(\theta) - 1)\, d\theta \\
&= \frac{1}{2}\int_{\pi/6}^{5\pi/6} (4(1-\cos(2\theta)) - 2\sin(\theta) - 1)\, d\theta \\
&= \frac{1}{2}\int_{\pi/6}^{5\pi/6} (3 - 4\cos(2\theta) - 2\sin(\theta))\, d\theta \\
&= \frac{1}{2}[3\theta - 2\sin(2\theta) + 2\cos(\theta)]_{\pi/6}^{5\pi/6} \\
&= \frac{1}{2}\left[\left(\frac{15\pi}{6} - 2(-\frac{\sqrt{3}}{2}) + 2(-\frac{\sqrt{3}}{2})\right) - \left(\frac{3\pi}{6} - 2(\frac{\sqrt{3}}{2}) + 2(\frac{\sqrt{3}}{2})\right)\right] \\
&= \frac{1}{2}\left[(\frac{5\pi}{2}) - (\frac{\pi}{2})\right] = \pi
\end{aligned}$$

Problem 10

Find the area of the region common to the two circles $r = \cos(\theta)$ and $r = \sin(\theta)$.

Solution: Intersection: $\cos(\theta) = \sin(\theta) \implies \tan(\theta) = 1 \implies \theta = \pi/4$. The area is the sum of two parts. By symmetry, we can find the area of the region bounded by $r = \sin(\theta)$ from 0 to $\pi/4$ and double it.

$$A = 2 \times \frac{1}{2}\int_0^{\pi/4} (\sin(\theta))^2\, d\theta + 2 \times \frac{1}{2}\int_{\pi/4}^{\pi/2} (\cos(\theta))^2\, d\theta$$

By symmetry, we can just calculate one and add:

$$\begin{aligned}
A &= \int_0^{\pi/4} \sin^2(\theta)\, d\theta + \int_{\pi/4}^{\pi/2} \cos^2(\theta)\, d\theta \\
&= \int_0^{\pi/4} \frac{1-\cos(2\theta)}{2}\, d\theta + \int_{\pi/4}^{\pi/2} \frac{1+\cos(2\theta)}{2}\, d\theta \\
&= \frac{1}{2}[\theta - \frac{1}{2}\sin(2\theta)]_0^{\pi/4} + \frac{1}{2}[\theta + \frac{1}{2}\sin(2\theta)]_{\pi/4}^{\pi/2} \\
&= \frac{1}{2}(\frac{\pi}{4} - \frac{1}{2}) + \frac{1}{2}[(\frac{\pi}{2}) - (\frac{\pi}{4} + \frac{1}{2})] = \frac{\pi}{8} - \frac{1}{4} + \frac{\pi}{8} - \frac{1}{4} = \frac{\pi}{4} - \frac{1}{2}
\end{aligned}$$

Problem 11

Find the area of the region inside the cardioid $r = 2 + 2\cos(\theta)$ and outside the circle $r = 3$.

Solution: Intersection: $2+2\cos(\theta)=3 \implies \cos(\theta)=1/2 \implies \theta=\pm\pi/3$.

$$
\begin{aligned}
A &= \frac{1}{2}\int_{-\pi/3}^{\pi/3}[(2+2\cos\theta)^2-3^2]d\theta \\
&= 2\cdot\frac{1}{2}\int_0^{\pi/3}[4(1+2\cos\theta+\cos^2\theta)-9]d\theta \\
&= \int_0^{\pi/3}[4+8\cos\theta+4(\frac{1+\cos(2\theta)}{2})-9]d\theta \\
&= \int_0^{\pi/3}[-3+8\cos\theta+2\cos(2\theta)]d\theta \\
&= [-3\theta+8\sin\theta+\sin(2\theta)]_0^{\pi/3} \\
&= -\pi+8(\frac{\sqrt{3}}{2})+\frac{\sqrt{3}}{2}=\frac{9\sqrt{3}}{2}-\pi
\end{aligned}
$$

Finding Intersection Points

Problem 12

Find all points of intersection of the curves $r=1+\cos(\theta)$ and $r=1-\cos(\theta)$.

Solution: Set equations equal: $1+\cos(\theta)=1-\cos(\theta) \implies 2\cos(\theta)=0 \implies \theta=\pi/2, 3\pi/2$. At $\theta=\pi/2$, $r=1$. At $\theta=3\pi/2$, $r=1$. Points are $(1,\pi/2)$ and $(1,3\pi/2)$. Check pole: For $r=1+\cos(\theta)$, $r=0$ when $\theta=\pi$. For $r=1-\cos(\theta)$, $r=0$ when $\theta=0$. The curves pass through the pole at different angles, so the pole is an intersection point. Points: $(1,\pi/2)$, $(1,3\pi/2)$, and the pole $(0,\theta)$.

Problem 13

Find all points of intersection of $r^2=\sin(\theta)$ and $r=\cos(\theta)$.

Solution: Substitute r: $\cos^2(\theta)=\sin(\theta) \implies 1-\sin^2(\theta)=\sin(\theta)$. Let $u=\sin(\theta)$: $u^2+u-1=0$. $u=\frac{-1\pm\sqrt{1-4(1)(-1)}}{2}=\frac{-1\pm\sqrt{5}}{2}$. Since $|\sin\theta|\le 1$, we must have $\sin(\theta)=\frac{\sqrt{5}-1}{2}$. Let $\alpha=\arcsin(\frac{\sqrt{5}-1}{2})$. The solutions are $\theta=\alpha$ and $\theta=\pi-\alpha$. For $\theta=\alpha$, $r=\cos(\alpha)=\sqrt{1-\sin^2\alpha}=\sqrt{1-(\frac{\sqrt{5}-1}{2})^2}$. For $\theta=\pi-\alpha$, $r=\cos(\pi-\alpha)=-\cos(\alpha)$. Check pole: $r=\cos\theta=0$ at $\theta=\pi/2$. $r^2=\sin\theta=0$ at $\theta=0,\pi$. No common pole intersection. Points: $(\cos(\alpha),\alpha)$ and $(-\cos(\alpha),\pi-\alpha)$, where $\alpha=\arcsin(\frac{\sqrt{5}-1}{2})$.

Problem 14

Find all points of intersection of $r=2$ and $r=4\sin(2\theta)$.

Solution: $2=4\sin(2\theta) \implies \sin(2\theta)=1/2$. $2\theta=\pi/6, 5\pi/6, 13\pi/6, 17\pi/6, \dots$ $\theta=\pi/12, 5\pi/12, 13\pi/12, 17\pi/12$. The radius is $r=2$ for all these angles. Points: $(2,\pi/12), (2,5\pi/12), (2,13\pi/12), (2,17\pi/12)$.

Arc Length

Problem 15

Find the exact length of the cardioid $r=1+\sin(\theta)$.

Solution: $r' = \cos(\theta)$. The curve is traced from 0 to 2π.

$$\begin{aligned}
L &= \int_0^{2\pi} \sqrt{(1+\sin\theta)^2 + (\cos\theta)^2}\, d\theta \\
&= \int_0^{2\pi} \sqrt{1 + 2\sin\theta + \sin^2\theta + \cos^2\theta}\, d\theta \\
&= \int_0^{2\pi} \sqrt{2 + 2\sin\theta}\, d\theta \\
&= \sqrt{2}\int_0^{2\pi} \sqrt{1+\sin\theta}\, d\theta \\
&= \sqrt{2}\int_0^{2\pi} \sqrt{1+\cos(\theta - \pi/2)}\, d\theta \\
&= \sqrt{2}\int_0^{2\pi} \sqrt{2\cos^2(\frac{\theta}{2} - \frac{\pi}{4})}\, d\theta \\
&= 2\int_0^{2\pi} |\cos(\frac{\theta}{2} - \frac{\pi}{4})|\, d\theta
\end{aligned}$$

The argument of cosine is positive from $\theta = -\pi/2$ to $3\pi/2$. By symmetry, we can integrate from $-\pi/2$ to $3\pi/2$: $L = 2[-\sin(\frac{\theta}{2} - \frac{\pi}{4}) \times 2]_{3\pi/2}^{2\pi} + 2[2\sin(\frac{\theta}{2} - \frac{\pi}{4})]_0^{3\pi/2}$. A simpler path is to use symmetry $L = 2\int_{-\pi/2}^{\pi/2} \sqrt{2+2\sin\theta} d\theta$. The classic solution to this integral is 8.

Problem 16

Find the exact length of the curve $r = \theta^2$ for $0 \le \theta \le \sqrt{5}$.

Solution: $r' = 2\theta$.

$$\begin{aligned}
L &= \int_0^{\sqrt{5}} \sqrt{(\theta^2)^2 + (2\theta)^2}\, d\theta = \int_0^{\sqrt{5}} \sqrt{\theta^4 + 4\theta^2}\, d\theta \\
&= \int_0^{\sqrt{5}} \theta\sqrt{\theta^2+4}\, d\theta
\end{aligned}$$

Let $u = \theta^2 + 4$, $du = 2\theta d\theta$. When $\theta = 0, u = 4$. When $\theta = \sqrt{5}, u = 9$.

$$\begin{aligned}
L &= \frac{1}{2}\int_4^9 \sqrt{u}\, du = \frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]_4^9 \\
&= \frac{1}{3}(9^{3/2} - 4^{3/2}) = \frac{1}{3}(27-8) = \frac{19}{3}
\end{aligned}$$

Problem 17

Find the arc length of the circle $r = 6\cos(\theta)$.

Solution: The circle is traced from 0 to π. $r' = -6\sin(\theta)$.

$$\begin{aligned}
L &= \int_0^{\pi} \sqrt{(6\cos\theta)^2 + (-6\sin\theta)^2}\, d\theta \\
&= \int_0^{\pi} \sqrt{36\cos^2\theta + 36\sin^2\theta}\, d\theta \\
&= \int_0^{\pi} \sqrt{36}\, d\theta = \int_0^{\pi} 6\, d\theta = [6\theta]_0^{\pi} = 6\pi
\end{aligned}$$

This matches the circumference $C = \pi d = \pi(6)$.

Slope of a Tangent Line

Problem 18

Find the slope of the tangent line to the curve $r = 1/\theta$ at $\theta = \pi$.

Solution: $r = 1/\theta$, $r' = -1/\theta^2$. At $\theta = \pi$, $r = 1/\pi$ and $r' = -1/\pi^2$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta} \\ &= \frac{(-1/\pi^2)\sin\pi + (1/\pi)\cos\pi}{(-1/\pi^2)\cos\pi - (1/\pi)\sin\pi} \\ &= \frac{0 + (1/\pi)(-1)}{(-1/\pi^2)(-1) - 0} = \frac{-1/\pi}{1/\pi^2} = -\pi\end{aligned}$$

Problem 19

Find the slope of the tangent line to $r = 2 - \sin(\theta)$ at $\theta = \pi/3$.

Solution: $r' = -\cos(\theta)$. At $\theta = \pi/3$: $r = 2 - \sin(\pi/3) = 2 - \sqrt{3}/2$. $r' = -\cos(\pi/3) = -1/2$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{(-1/2)(\sqrt{3}/2) + (2 - \sqrt{3}/2)(1/2)}{(-1/2)(1/2) - (2 - \sqrt{3}/2)(\sqrt{3}/2)} \\ &= \frac{-\sqrt{3}/4 + 1 - \sqrt{3}/4}{-1/4 - (2\sqrt{3}/2 - 3/4)} = \frac{1 - \sqrt{3}/2}{-1/4 - \sqrt{3} + 3/4} \\ &= \frac{1 - \sqrt{3}/2}{1/2 - \sqrt{3}} = \frac{(2 - \sqrt{3})/2}{(1 - 2\sqrt{3})/2} = \frac{2 - \sqrt{3}}{1 - 2\sqrt{3}}\end{aligned}$$

Problem 20

Find the slope of the tangent to the four-leaved rose $r = \cos(2\theta)$ at $\theta = \pi/4$.

Solution: $r' = -2\sin(2\theta)$. At $\theta = \pi/4$, $r = \cos(\pi/2) = 0$, $r' = -2\sin(\pi/2) = -2$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{(-2)\sin(\pi/4) + (0)\cos(\pi/4)}{(-2)\cos(\pi/4) - (0)\sin(\pi/4)} \\ &= \frac{-2(\sqrt{2}/2)}{-2(\sqrt{2}/2)} = 1\end{aligned}$$

Horizontal and Vertical Tangents

Problem 21

Find the points on the cardioid $r = 1 - \cos(\theta)$ where the tangent line is horizontal or vertical for $0 \le \theta < 2\pi$.

Solution: $r' = \sin\theta$. Numerator: $dy/d\theta = r'\sin\theta + r\cos\theta = \sin^2\theta + (1 - \cos\theta)\cos\theta = \sin^2\theta + \cos\theta - \cos^2\theta = 0$. $(1 - \cos^2\theta) + \cos\theta - \cos^2\theta = 0 \implies -2\cos^2\theta + \cos\theta + 1 = 0$. $2\cos^2\theta - \cos\theta - 1 = 0 \implies (2\cos\theta + 1)(\cos\theta - 1) = 0$. $\cos\theta = -1/2$ or $\cos\theta = 1$. $\theta = 2\pi/3, 4\pi/3$ or $\theta = 0$. Denominator: $dx/d\theta = r'\cos\theta - r\sin\theta = \sin\theta\cos\theta - (1 - \cos\theta)\sin\theta = \sin\theta(\cos\theta - 1 + \cos\theta) = \sin\theta(2\cos\theta - 1) = 0$. $\sin\theta = 0$ or $\cos\theta = 1/2$. $\theta = 0, \pi$ or $\theta = \pi/3, 5\pi/3$. At $\theta = 0$, both are zero (cusp). Horizontal tangents at $\theta = 2\pi/3, 4\pi/3$. Points: $(3/2, 2\pi/3), (3/2, 4\pi/3)$. Vertical tangents at $\theta = \pi, \pi/3, 5\pi/3$. Points: $(2, \pi), (1/2, \pi/3), (1/2, 5\pi/3)$.

Problem 22

Find the points on the circle $r = 4\sin(\theta)$ where the tangent line is horizontal or vertical.

Solution: $r' = 4\cos\theta$. $dy/d\theta = 4\cos\theta\sin\theta + 4\sin\theta\cos\theta = 8\sin\theta\cos\theta = 4\sin(2\theta) = 0 \implies 2\theta = 0, \pi, 2\pi, 3\pi \implies \theta = 0, \pi/2, \pi, 3\pi/2$. $dx/d\theta = 4\cos^2\theta - 4\sin^2\theta = 4\cos(2\theta) = 0 \implies 2\theta = \pi/2, 3\pi/2, 5\pi/2, 7\pi/2 \implies \theta = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$. Horizontal tangents: $\theta = 0, \pi$ (pole) and $\theta = \pi/2$ (point $(4, \pi/2)$). Vertical tangents: $\theta = \pi/4, 3\pi/4$. Points: $(4\sin(\pi/4), \pi/4) = (2\sqrt{2}, \pi/4)$ and $(2\sqrt{2}, 3\pi/4)$.

Mixed and Advanced Problems

Problem 23

Find the area enclosed by the outer loop of the limaçon $r = 2 + \sqrt{2}\cos(\theta)$.

Solution: This limaçon has no inner loop since $|2/\sqrt{2}| = \sqrt{2} > 1$. The entire curve is traced from 0 to 2π.

$$\begin{aligned} A &= \frac{1}{2}\int_0^{2\pi} (2 + \sqrt{2}\cos\theta)^2 d\theta = \frac{1}{2}\int_0^{2\pi} (4 + 4\sqrt{2}\cos\theta + 2\cos^2\theta) d\theta \\ &= \frac{1}{2}\int_0^{2\pi} (4 + 4\sqrt{2}\cos\theta + 1 + \cos(2\theta)) d\theta \\ &= \frac{1}{2}[5\theta + 4\sqrt{2}\sin\theta + \frac{1}{2}\sin(2\theta)]_0^{2\pi} = \frac{1}{2}(10\pi) = 5\pi \end{aligned}$$

Problem 24

A region is bounded by $r = e^{\theta/2}$ for $0 \le \theta \le \pi$. Find its area.

Solution:

$$A = \frac{1}{2}\int_0^{\pi} (e^{\theta/2})^2 d\theta = \frac{1}{2}\int_0^{\pi} e^{\theta} d\theta = \frac{1}{2}[e^{\theta}]_0^{\pi} = \frac{1}{2}(e^{\pi} - 1)$$

Problem 25

Find the arc length of the spiral $r = e^{\theta}$ for $0 \le \theta \le 2\pi$.

Solution: $r' = e^{\theta}$.

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{(e^{\theta})^2 + (e^{\theta})^2} d\theta = \int_0^{2\pi} \sqrt{2e^{2\theta}} d\theta \\ &= \int_0^{2\pi} \sqrt{2} e^{\theta} d\theta = \sqrt{2}[e^{\theta}]_0^{2\pi} = \sqrt{2}(e^{2\pi} - 1) \end{aligned}$$

Problem 26

Find the area of the region inside $r = 4$ and to the right of the line $x = 2$ (in Cartesian coordinates).

Solution: The line is $r\cos\theta = 2$, so $r = 2\sec\theta$. Intersection: $4 = 2\sec\theta \implies \cos\theta = 1/2 \implies \theta = \pm\pi/3$.

$$\begin{aligned} A &= \frac{1}{2}\int_{-\pi/3}^{\pi/3} (4^2 - (2\sec\theta)^2) d\theta = \int_0^{\pi/3} (16 - 4\sec^2\theta) d\theta \\ &= [16\theta - 4\tan\theta]_0^{\pi/3} = 16(\frac{\pi}{3}) - 4\tan(\frac{\pi}{3}) = \frac{16\pi}{3} - 4\sqrt{3} \end{aligned}$$

Problem 27

Find the area between the loops of the limaçon $r = 2 + 4\cos(\theta)$.

Solution: The entire area is $A_{total} = \frac{1}{2}\int_0^{2\pi}(2+4\cos\theta)^2 d\theta$. The inner loop bounds are where $r = 0$, $\cos\theta = -1/2 \implies \theta = 2\pi/3, 4\pi/3$. $A_{inner} = \frac{1}{2}\int_{2\pi/3}^{4\pi/3}(2+4\cos\theta)^2 d\theta$. The area between is $A_{total} - 2A_{inner}$? No, it's $A_{outer} - A_{inner}$. The outer loop is traced over $[0, 2\pi]$ excluding the inner loop interval. $A_{total} = \frac{1}{2}\int_0^{2\pi}(4+16\cos\theta+16\cos^2\theta)d\theta = \frac{1}{2}\int_0^{2\pi}(4+16\cos\theta+8(1+\cos 2\theta))d\theta = \frac{1}{2}[12\theta+16\sin\theta+4\sin 2\theta]_0^{2\pi} = 12\pi$. $A_{inner} = \frac{1}{2}\int_{2\pi/3}^{4\pi/3}(12+16\cos\theta+8\cos 2\theta)d\theta = \frac{1}{2}[12\theta+16\sin\theta+4\sin 2\theta]_{2\pi/3}^{4\pi/3} = 4\pi - 6\sqrt{3}$. Area between loops is $A_{total} - 2A_{inner}$ is not correct. It's the area of the big loop minus the area of the small loop. The area of the big loop is the total area calculated over $[0, 2\pi]$ minus the area of the inner loop, which gets counted twice. A simpler way is to find the total area and subtract the inner loop area. Wait, the area formula ‘1/2 r^2‘$canbenegative.Theareaoftheouterloopis1\frac{}{2\int_{-2\pi/3}^{2\pi/3}(2+4\cos\theta)^2 d\theta=8\pi+6\sqrt{3}}$. Area between loops $= A_{outer} - A_{inner} = (8\pi+6\sqrt{3}) - (4\pi - 6\sqrt{3}) = 4\pi + 12\sqrt{3}$.

Problem 28

Find the area shared by the cardioids $r = 2(1+\cos\theta)$ and $r = 2(1-\cos\theta)$.

Solution: Intersections: $1+\cos\theta = 1-\cos\theta \implies \cos\theta = 0 \implies \theta = \pi/2, 3\pi/2$. Also the pole. By symmetry, we can find the area in the first quadrant and multiply by 4. Or the top half and multiply by 2. Area of $r = 2(1-\cos\theta)$ from 0 to $\pi/2$ plus area of $r = 2(1+\cos\theta)$ from $\pi/2$ to π, then double it. By symmetry, it's $4 \times \frac{1}{2}\int_0^{\pi/2}(2(1-\cos\theta))^2 d\theta$.

$$A = 2\int_0^{\pi/2} 4(1-2\cos\theta+\cos^2\theta)d\theta = 8\int_0^{\pi/2}(1-2\cos\theta+\frac{1+\cos 2\theta}{2})d\theta$$
$$= 8[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin 2\theta]_0^{\pi/2} = 8(\frac{3\pi}{4} - 2) = 6\pi - 16$$

Problem 29

Find all values of θ for which the tangent line to $r = 3+\cos(4\theta)$ is perpendicular to the polar axis.

Solution: Perpendicular to polar axis means vertical. We need $dx/d\theta = 0$. $r' = -4\sin(4\theta)$. $dx/d\theta = r'\cos\theta - r\sin\theta = -4\sin(4\theta)\cos\theta - (3+\cos(4\theta))\sin\theta = 0$. This equation is difficult to solve analytically and typically requires numerical methods. Let's choose a simpler problem. **Replacement Problem 29:** Find the area of the region inside $r^2 = 6\cos(2\theta)$ and outside the circle $r = \sqrt{3}$.

Solution: Intersection: $3 = 6\cos(2\theta) \implies \cos(2\theta) = 1/2$. $2\theta = \pm\pi/3 \implies \theta = \pm\pi/6$.

$$A = \frac{1}{2}\int_{-\pi/6}^{\pi/6}[6\cos(2\theta) - (\sqrt{3})^2]d\theta = \int_0^{\pi/6}(6\cos(2\theta) - 3)d\theta$$
$$= [3\sin(2\theta) - 3\theta]_0^{\pi/6} = 3\sin(\pi/3) - 3(\pi/6) = 3(\frac{\sqrt{3}}{2}) - \frac{\pi}{2} = \frac{3\sqrt{3}-\pi}{2}$$

Problem 30

The equation $r = 4\sin\theta\cos^2\theta$ describes a "bifolium". Find the total area enclosed.

Solution: The curve is defined for $\sin\theta \geq 0$, so $0 \leq \theta \leq \pi$. The curve is traced once.

$$\begin{aligned}
A &= \frac{1}{2}\int_0^\pi (4\sin\theta\cos^2\theta)^2 d\theta = 8\int_0^\pi \sin^2\theta\cos^4\theta d\theta \\
&= 8\int_0^\pi (\frac{1-\cos 2\theta}{2})(\frac{1+\cos 2\theta}{2})^2 d\theta \\
&= \int_0^\pi (1-\cos 2\theta)(1+2\cos 2\theta+\cos^2 2\theta)d\theta \\
&= \int_0^\pi (1+\cos 2\theta-\cos^2 2\theta-\cos^3 2\theta)d\theta
\end{aligned}$$

This is getting complex. Let's use Wallis' formula after substitution.

$$A = 8\int_0^\pi \sin^2\theta(1-\sin^2\theta)^2 d\theta = 8\int_0^\pi (\sin^2\theta - 2\sin^4$$

Using $\int_0^\pi \sin^{2n}\theta d\theta = \frac{(2n-1)!!}{(2n)!!}\pi$: $\int_0^\pi \sin^2\theta d\theta = \frac{1}{2}\pi$ $\int_0^\pi \sin^4\theta d\theta = \frac{3\cdot 1}{4\cdot 2}\pi = \frac{3}{8}\pi$ $\int_0^\pi \sin^6\theta d\theta = \frac{5\cdot 3\cdot 1}{6\cdot 4\cdot 2}\pi = \frac{5}{16}\pi$
$A = 8[\frac{\pi}{2} - 2(\frac{3\pi}{8}) + \frac{5\pi}{16}] = 8[\frac{8\pi-12\pi+5\pi}{16}] = 8[\frac{\pi}{16}] = \frac{\pi}{2}$.

Concept Checklist and Problem Cross-Reference

I. Fundamental Concepts & Formulas

- **Area of a Simple Polar Region:** Problems 1, 2, 3, 7, 23, 24.
- **Area of a Single Loop/Petal:** Problems 4, 5.
- **Area of a Specified Sector from a Graph:** Problems 7, 8.
- **Area Between Two Polar Curves:** Problems 9, 10, 11, 26, 27, 28, 29.
- **Arc Length of a Polar Curve:** Problems 15, 16, 17, 25.
- **Slope of a Tangent Line:** Problems 18, 19, 20.
- **Horizontal and Vertical Tangents:** Problems 21, 22.
- **Finding All Intersection Points:** Problems 12, 13, 14.
- **Area of the Inner Loop of a Limaçon:** Problem 6. (Also used in 27).

II. Curve Types

- **Circles:** Problems 2, 10, 11, 17, 22, 26, 29.
- **Cardioids:** Problems 1, 7, 9, 12, 15, 21, 28.
- **Limaçons (with inner loop):** Problems 6, 27.
- **Limaçons (without inner loop):** Problems 8, 19, 23.
- **Roses:** Problems 4, 14, 20.
- **Lemniscates:** Problems 5, 13, 29.
- **Spirals:** Problems 3, 16, 24, 25.

III. Key Techniques & Manipulations

- **Trigonometric Power-Reducing Formulas:** Problems 1, 2, 4, 6, 7, 8, 9, 10, 11, 23, 27, 28, 30.
- **Squaring Binomials:** Problems 1, 6, 7, 8, 9, 11, 15, 23, 27, 28.
- **Solving Trigonometric Equations:** Problems 4, 5, 6, 9, 10, 11, 12, 13, 14, 21, 22, 26, 27, 29.
- **Double-Angle Identities:** Used implicitly in power-reduction and explicitly in some solutions.
- **Simplifying Radicals for Arc Length:** Problems 15, 16, 17, 25. (Problem 15 is a classic example involving absolute value).
- **U-Substitution in Integration:** Problem 16.
- **Using Symmetry:** Problems 4, 5, 7, 10, 11, 15, 28.