Practice Problems for Section 11.6: The Ratio and Root Tests

Generated by Gemini

November 5, 2025

Practice Problems

Use the Ratio Test, Root Test, or any other appropriate test to determine whether the following series are absolutely convergent, conditionally convergent, or divergent. Show all your work.

1. $\sum_{n=1}^{\infty} \frac{n^3}{5^n}$
2. $\sum_{n=1}^{\infty} \left(\frac{2n^2+1}{3n^2-4}\right)^n$
3. $\sum_{n=1}^{\infty} \frac{(-10)^n}{n!}$
4. $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n+1}}$
5. $\sum_{n=1}^{\infty} \frac{n^{10}+n^5}{n!}$
6. $\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{e^n}$
7. $\sum_{n=1}^{\infty} \left(1+\frac{3}{n}\right)^{n^2}$
8. $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$
9. $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$
10. $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ (Hint: Use Ratio Test and know the limit for e.)
11. $\sum_{n=1}^{\infty} \frac{3^n}{2^{n+1}}$
12. $\sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{5^n n!}$
13. $\sum_{n=1}^{\infty} \left(\frac{-2n}{3n+1}\right)^{5n}$
14. $\sum_{k=1}^{\infty} \frac{k!}{(k+2)!}$
15. $\sum_{n=1}^{\infty} \frac{n+5}{n^3+2n}$
16. $\sum_{n=1}^{\infty} n^2 \left(\frac{2}{3}\right)^n$
17. $\sum_{n=1}^{\infty} \left(\frac{n}{\ln(n+1)}\right)^n$
18. $\sum_{n=1}^{\infty} \frac{(3n)!}{100^n (n!)^3}$
19. A series $\sum a_n$ is tested with the Ratio Test, yielding $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \frac{e}{2}$. What can you conclude about the series?
20. $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^n$
21. $\sum_{n=1}^{\infty} \frac{n\cdot(-2)^n}{3^n}$
22. $\sum_{n=1}^{\infty} \frac{n!e^n}{n^n}$ (This is a more challenging version of Problem 10)

23. $\sum_{n=1}^{\infty} 2\left(1-\frac{1}{n}\right)^{n^2}$

24. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2+1}$ (Determine if it's absolutely or conditionally convergent).

25. $\sum_{n=1}^{\infty} \frac{2\cdot 4\cdot 6\cdots(2n)}{1\cdot 4\cdot 7\cdots(3n-2)}$

26. $\sum_{n=1}^{\infty} \frac{1000n^{100}}{n!}$

27. $\sum_{n=1}^{\infty} \frac{\arctan(n)}{n^2}$

28. $\sum_{n=1}^{\infty} (\arctan(n))^{-n}$

29. Without calculating the limit, predict whether $\sum \frac{n^{10}10^n}{n!}$ converges or diverges, and justify your prediction based on the hierarchy of growth.

30. $\sum_{n=1}^{\infty} \frac{n\pi^n}{4^n}$

Solutions to Practice Problems

1. $\sum \frac{n^3}{5^n}$. Use Ratio Test. $L = \lim_{n\to\infty} \left|\frac{(n+1)^3/5^{n+1}}{n^3/5^n}\right| = \lim_{n\to\infty} \frac{(n+1)^3}{n^3}\frac{5^n}{5^{n+1}} = \lim_{n\to\infty}(\frac{n+1}{n})^3\frac{1}{5} = 1^3 \cdot \frac{1}{5} = \frac{1}{5}$. Since $L < 1$, the series is **absolutely convergent**.

2. $\sum(\frac{2n^2+1}{3n^2-4})^n$. Use Root Test. $L = \lim_{n\to\infty}\left|\frac{2n^2+1}{3n^2-4}\right| = \lim_{n\to\infty}\frac{2+1/n^2}{3-4/n^2} = \frac{2}{3}$. Since $L < 1$, the series is **absolutely convergent**.

3. $\sum \frac{(-10)^n}{n!}$. Use Ratio Test. $L = \lim_{n\to\infty}\left|\frac{(-10)^{n+1}/(n+1)!}{(-10)^n/n!}\right| = \lim_{n\to\infty}\frac{10^{n+1}}{10^n}\frac{n!}{(n+1)!} = \lim_{n\to\infty} 10 \cdot \frac{1}{n+1} = 0$. Since $L < 1$, the series is **absolutely convergent**.

4. $\sum \frac{1}{n\sqrt{n+1}}$. Ratio test gives $L = 1$. Use Limit Comparison Test with $b_n = \frac{1}{n\sqrt{n}} = \frac{1}{n^{3/2}}$, a convergent p-series ($p = 3/2 > 1$). $\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{1/(n\sqrt{n+1})}{1/(n\sqrt{n})} = \lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n+1}} = 1$. Since the limit is finite and positive, the series **converges**.

5. $\sum \frac{n^{10}+n^5}{n!}$. Use Ratio Test. $L = \lim_{n\to\infty}\left|\frac{((n+1)^{10}+(n+1)^5)/(n+1)!}{(n^{10}+n^5)/n!}\right| = \lim_{n\to\infty}\frac{(n+1)^{10}(1+\dots)}{n^{10}(1+\dots)}\frac{1}{n+1} = 0$. Since $L < 1$, the series is **absolutely convergent**.

6. $\sum \frac{(-1)^n n^2}{e^n}$. Use Ratio Test. $L = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+1)^2/e^{n+1}}{(-1)^n n^2/e^n}\right| = \lim_{n\to\infty}\frac{(n+1)^2}{n^2}\frac{1}{e} = \frac{1}{e}$. Since $L < 1$, the series is **absolutely convergent**.

7. $\sum(1+\frac{3}{n})^{n^2}$. Use Root Test. $L = \lim_{n\to\infty}\left((1+\frac{3}{n})^{n^2}\right)^{1/n} = \lim_{n\to\infty}(1+\frac{3}{n})^n = e^3$. Since $L > 1$, the series **diverges**.

8. $\sum \frac{(n!)^2}{(2n)!}$. Use Ratio Test. $L = \lim_{n\to\infty}\left|\frac{((n+1)!)^2/(2(n+1))!}{(n!)^2/(2n)!}\right| = \lim_{n\to\infty}\frac{((n+1)n!)^2}{(n!)^2}\frac{(2n)!}{(2n+2)!} = \lim_{n\to\infty}(n+1)^2\frac{1}{(2n+2)(2n+1)} = \lim_{n\to\infty}\frac{n^2+2n+1}{4n^2+6n+2} = \frac{1}{4}$. Since $L < 1$, the series is **absolutely convergent**.

9. $\sum \frac{(-1)^n}{\ln n}$. Ratio test on $|\frac{(-1)^n}{\ln n}|$ gives $L = 1$. Use Alternating Series Test. $b_n = \frac{1}{\ln n}$. b_n is positive and decreasing, and $\lim_{n\to\infty} b_n = 0$. The series converges. However, $\sum \frac{1}{\ln n}$ diverges by comparison with $\sum \frac{1}{n}$. Thus, the series is **conditionally convergent**.

10. $\sum \frac{n^n}{n!}$. Use Ratio Test. $L = \lim_{n\to\infty}\left|\frac{(n+1)^{n+1}/(n+1)!}{n^n/n!}\right| = \lim_{n\to\infty}\frac{(n+1)^{n+1}}{n^n}\frac{n!}{(n+1)!} = \lim_{n\to\infty}\frac{(n+1)^n(n+1)}{n^n}\frac{1}{n+1} = \lim_{n\to\infty}(\frac{n+1}{n})^n = e$. Since $L > 1$, the series **diverges**.

11. $\sum \frac{3^n}{2^{n+1}}$. This is a geometric series: $\frac{1}{2}\sum(\frac{3}{2})^n$. Since the ratio $|r| = 3/2 \geq 1$, the series **diverges**.

12. $\sum \frac{1\cdot 3\cdots(2n-1)}{5^n n!}$. Use Ratio Test. $L = \lim_{n\to\infty}\left|\frac{1\cdot 3\cdots(2n-1)(2n+1)}{5^{n+1}(n+1)!}\cdot\frac{5^n n!}{1\cdot 3\cdots(2n-1)}\right| = \lim_{n\to\infty}\frac{2n+1}{5(n+1)} = \frac{2}{5}$. Since $L < 1$, the series is **absolutely convergent**.

13. $\sum(\frac{-2n}{3n+1})^{5n}$. Use Root Test. $L = \lim_{n\to\infty}\left|\left(\frac{-2n}{3n+1}\right)^5\right| = \left(\lim_{n\to\infty}\frac{2n}{3n+1}\right)^5 = (\frac{2}{3})^5 = \frac{32}{243}$. Since $L < 1$, the series is **absolutely convergent**.

14. $\sum \frac{k!}{(k+2)!}$. Simplify $a_k = \frac{k!}{(k+2)(k+1)k!} = \frac{1}{(k+1)(k+2)}$. This is a telescoping series or can be shown to converge by Limit Comparison to $\sum \frac{1}{k^2}$. The series **converges**.

15. $\sum \frac{n+5}{n^3+2n}$. Ratio test gives $L = 1$. Use Limit Comparison with $b_n = \frac{n}{n^3} = \frac{1}{n^2}$, a convergent p-series ($p = 2 > 1$). $\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{n+5}{n^3+2n}\cdot n^2 = \lim_{n\to\infty}\frac{n^3+5n^2}{n^3+2n} = 1$. The series **converges**.

16. $\sum n^2(\frac{2}{3})^n$. Use Ratio Test. $L = \lim_{n\to\infty}\left|\frac{(n+1)^2(2/3)^{n+1}}{n^2(2/3)^n}\right| = \lim_{n\to\infty}(\frac{n+1}{n})^2\frac{2}{3} = \frac{2}{3}$. Since $L < 1$, the series is **absolutely convergent**.

17. $\sum(\frac{n}{\ln(n+1)})^n$. Use Root Test. $L = \lim_{n\to\infty}\frac{n}{\ln(n+1)}$. Using L'Hôpital's Rule: $\lim_{n\to\infty}\frac{1}{1/(n+1)} = \lim_{n\to\infty} n+1 = \infty$. Since $L > 1$, the series **diverges**.

18. $\sum \frac{(3n)!}{100^n(n!)^3}$. Use Ratio Test. $L = \lim_{n\to\infty}\frac{(3n+3)!}{100^{n+1}((n+1)!)^3}\cdot\frac{100^n(n!)^3}{(3n)!} = \lim_{n\to\infty}\frac{(3n+3)(3n+2)(3n+1)}{100(n+1)^3} = \frac{27}{100}$. Since $L < 1$, the series is **absolutely convergent**.

19. Since $L = e/2 \approx 2.718/2 \approx 1.359 > 1$, the series **diverges** by the Ratio Test.

20. $\sum(\frac{n}{n+1})^n$. Root Test gives $L = 1$. Use Test for Divergence. $\lim_{n\to\infty} a_n = \lim_{n\to\infty}(\frac{n}{n+1})^n = \lim_{n\to\infty} \frac{1}{((n+1)/n)^n} = \frac{1}{(1+1/n)^n} = \frac{1}{e}$. Since the limit is not 0, the series **diverges**.

21. $\sum \frac{n\cdot(-2)^n}{3^n}$. Use Ratio Test. $L = \lim_{n\to\infty} \left|\frac{(n+1)(-2)^{n+1}/3^{n+1}}{n(-2)^n/3^n}\right| = \lim_{n\to\infty} \frac{n+1}{n}\frac{2}{3} = \frac{2}{3}$. Since $L < 1$, the series is **absolutely convergent**.

22. $\sum \frac{n!e^n}{n^n}$. Use Ratio Test. $L = \lim_{n\to\infty} \frac{(n+1)!e^{n+1}/(n+1)^{n+1}}{n!e^n/n^n} = \lim_{n\to\infty} \frac{(n+1)!}{n!}\frac{e^{n+1}}{e^n}\frac{n^n}{(n+1)^{n+1}} = \lim_{n\to\infty}(n+1)e\frac{n^n}{(n+1)^n(n+1)} = \lim_{n\to\infty} e(\frac{n}{n+1})^n = e \cdot \frac{1}{e} = 1$. The test is inconclusive. (Note: This is related to Stirling's approximation; the series diverges).

23. $\sum 2(1-\frac{1}{n})^{n^2}$. Use Root Test. $L = \lim_{n\to\infty}(2^{1/n})(1-\frac{1}{n})^n = 1 \cdot e^{-1} = \frac{1}{e}$. Since $L < 1$, the series **converges**.

24. $\sum \frac{(-1)^{n-1}}{n^2+1}$. Test for absolute convergence: $\sum \frac{1}{n^2+1}$ converges by limit comparison with $\sum \frac{1}{n^2}$. Since the series of absolute values converges, the original series is **absolutely convergent**.

25. $\sum \frac{2\cdot4\cdots(2n)}{1\cdot4\cdots(3n-2)}$. Use Ratio Test. $L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{2n+2}{3(n+1)-2} = \lim_{n\to\infty} \frac{2n+2}{3n+1} = \frac{2}{3}$. Since $L < 1$, the series is **absolutely convergent**.

26. $\sum \frac{1000n^{100}}{n!}$. Use Ratio Test. $L = \lim_{n\to\infty}\left|\frac{1000(n+1)^{100}/(n+1)!}{1000n^{100}/n!}\right| = \lim_{n\to\infty}(\frac{n+1}{n})^{100}\frac{1}{n+1} = 1 \cdot 0 = 0$. Since $L < 1$, the series is **absolutely convergent**.

27. $\sum \frac{\arctan(n)}{n^2}$. Ratio test is inconclusive. Use Limit Comparison with $b_n = \frac{1}{n^2}$. $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \arctan(n) = \frac{\pi}{2}$. Since the limit is finite and positive, and $\sum b_n$ converges, the series **converges**.

28. $\sum(\arctan(n))^{-n}$. Use Root Test. $L = \lim_{n\to\infty}\left|(\arctan(n))^{-1}\right| = \lim_{n\to\infty} \frac{1}{\arctan(n)} = \frac{1}{\pi/2} = \frac{2}{\pi}$. Since $L < 1$, the series is **absolutely convergent**.

29. Prediction: The series **converges**. Justification: According to the hierarchy of growth, factorial functions ($n!$) grow much faster than any exponential function (10^n) or any polynomial function (n^{10}). Since the dominant term $n!$ is in the denominator, the terms of the series will go to zero very rapidly, suggesting convergence.

30. $\sum \frac{n\pi^n}{4^n}$. This can be written as $\sum n(\frac{\pi}{4})^n$. Since $\pi/4 < 1$, we can use the Ratio Test. $L = \lim_{n\to\infty} \frac{(n+1)(\pi/4)^{n+1}}{n(\pi/4)^n} = \lim_{n\to\infty} \frac{n+1}{n}\frac{\pi}{4} = \frac{\pi}{4}$. Since $L < 1$, the series is **absolutely convergent**.

Concept Checklist and Problem Cross-Reference

- **C1: Ratio Test - Basic Application (L ¡ 1 or L ¿ 1)**
  - C1a: Polynomial / Exponential: Problems 1, 16, 30
  - C1b: Exponential / Exponential: Problem 11
  - C1c: Alternating Polynomial / Exponential: Problems 6, 21
- **C2: Ratio Test - Factorials**
  - C2a: Simple Factorial: Problem 14
  - C2b: Factorial with Polynomials: Problems 5, 26
  - C2c: Factorial with Exponentials: Problems 3, 22
  - C2d: Complex Factorial Expressions: Problems 8, 18
- **C3: Ratio Test - Product Series**
  - C3a: Products in numerator/denominator: Problems 12, 25
- **C4: Root Test - Basic Application (L ¡ 1 or L ¿ 1)**
  - C4a: Simple form $(f(n))^n$: Problems 2, 13, 28
  - C4b: Complex form leading to the limit 'e': Problems 7, 23
  - C4c: Form requiring L'Hôpital's Rule: Problem 17
- **C5: Inconclusive Case (L = 1) and Follow-up Tests**
  - C5a: Divergence by Test for Divergence: Problems 10, 20
  - C5b: Requires Limit Comparison Test: Problems 4, 15, 27
- **C6: Absolute vs. Conditional Convergence**
  - C6a: Requires Alternating Series Test after Ratio Test fails for absolute convergence: Problems 9, 24
- **C7: Conceptual Understanding**
  - C7a: State conclusion from a given limit L: Problem 19
  - C7b: Hierarchy of Growth prediction: Problem 29