Polar Coordinates: Problem Set

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January 20, 2026

Introduction

This problem set is designed to test the concepts of polar coordinates as detailed in the provided learning materials. The problems cover a range of topics including plotting points, converting between coordinate systems, sketching regions, converting equations, and applying calculus concepts like area and arc length.

1 Problems

Part 1: Plotting Points and Alternative Coordinates

Problem 1: Plot the point with polar coordinates $(3, -\frac{2\pi}{3})$ and find three other distinct pairs of polar coordinates (r, θ) that represent the same point, such that $-2\pi \le \theta \le 2\pi$.

Problem 2: Plot the point with polar coordinates $(-4, \frac{5\pi}{4})$ and find two other representations, one with $r > 0$ and one with $r < 0$.

Problem 3: A point is given by the polar coordinates $(-2, \frac{11\pi}{6})$. Which of the following does **not** represent the same point?

(a) $(2, \frac{5\pi}{6})$

(b) $(2, -\frac{7\pi}{6})$

(c) $(-2, -\frac{\pi}{6})$

(d) $(2, \frac{17\pi}{6})$

Part 2: Coordinate Conversion (Points)

Problem 4: Convert the following polar coordinates to Cartesian coordinates (x, y).

(a) $(5, \frac{\pi}{2})$

(b) $(2\sqrt{2}, \frac{7\pi}{4})$

(c) $(-4, \frac{2\pi}{3})$

(d) $(6, \pi)$

Problem 5: Convert the Cartesian coordinates $(0, -7)$ to polar coordinates (r, θ) where $r > 0$ and $0 \le \theta < 2\pi$.

Problem 6: Convert the Cartesian coordinates $(-5, -5\sqrt{3})$ to polar coordinates (r, θ) where $r > 0$ and $0 \le \theta < 2\pi$.

Problem 7: Convert the Cartesian coordinates $(3, -4)$ to polar coordinates (r, θ) where $r > 0$ and $0 \le \theta < 2\pi$.

Part 3: Sketching Regions from Inequalities

Problem 8: Sketch the region in the polar plane defined by the inequalities $2 \leq r < 4$ and $\frac{\pi}{4} \leq \theta \leq \frac{2\pi}{3}$.

Problem 9: Sketch the region described by $r \leq 3$ and $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

Problem 10: Sketch the region defined by $r \geq 1$.

Problem 11: Sketch the region defined by $1 \leq r \leq 3$ and $\theta = \frac{5\pi}{6}$.

Part 4: Equation Conversion

Polar to Cartesian

Problem 12: Convert the polar equation $r = 8\sin\theta$ to a Cartesian equation and identify the curve.

Problem 13: Convert the polar equation $r = \frac{3}{2\cos\theta - 5\sin\theta}$ to a Cartesian equation.

Problem 14: Convert the polar equation $r^2 = \tan\theta$ to a Cartesian equation.

Problem 15: Convert the polar equation $\theta = \frac{3\pi}{4}$ to a Cartesian equation.

Problem 16: Convert the polar equation $r = -6\sec\theta$ to a Cartesian equation.

Problem 17: Convert the polar equation $r^2 \sin(2\theta) = 8$ to a Cartesian equation and identify the curve.

Cartesian to Polar

Problem 18: Convert the Cartesian equation $x^2 + y^2 = 10$ to a polar equation.

Problem 19: Convert the Cartesian equation $y = -x$ to a polar equation.

Problem 20: Convert the Cartesian equation $x = 7$ to a polar equation.

Problem 21: Convert the Cartesian equation $(x-3)^2 + y^2 = 9$ to a polar equation.

Problem 22: Convert the Cartesian equation $y = x^2$ to a polar equation.

Part 5: Calculus with Polar Coordinates

Problem 23: Find the area of the region enclosed by one loop of the rose curve $r = 3\cos(2\theta)$.

Problem 24: Find the area of the region inside the cardioid $r = 2 + 2\sin\theta$.

Problem 25: Set up, but do not evaluate, the integral for the arc length of the spiral $r = 2\theta$ from $\theta = 0$ to $\theta = 2\pi$.

Problem 26: Find the arc length of the circle $r = 4\cos\theta$ for $0 \leq \theta \leq \pi$.

Part 6: Analytical and Critical Thinking

Problem 27: Find the flaw in the following conversion. **Task:** Convert the Cartesian coordinates $(-3, 3)$ to polar coordinates (r, θ) with $r > 0$.
Flawed Solution:

1. Find r: $r = \sqrt{(-3)^2 + 3^2} = \sqrt{9+9} = \sqrt{18} = 3\sqrt{2}$.
2. Find θ: $\tan\theta = \frac{y}{x} = \frac{3}{-3} = -1$.
3. Using a calculator, $\theta = \arctan(-1) = -\frac{\pi}{4}$.
4. The polar coordinates are $(3\sqrt{2}, -\frac{\pi}{4})$.

Problem 28: Find the flaw in the following conversion. **Task:** Find the Cartesian equation for the polar curve $r = 10\cos\theta$.
Flawed Solution:

1. We know $r = \sqrt{x^2 + y^2}$ and $x = r\cos\theta \implies \cos\theta = \frac{x}{r}$.
2. Substitute these into the equation: $\sqrt{x^2 + y^2} = 10\left(\frac{x}{\sqrt{x^2+y^2}}\right)$.
3. Multiply both sides by $\sqrt{x^2 + y^2}$: $(\sqrt{x^2 + y^2})^2 = 10x$.
4. This gives $x^2 + y^2 = 10x$.
5. This is a circle. The solution is correct, but the method is inefficient and prone to error. What is the standard, more direct "trick" for this type of problem?

2 Solutions

Part 1: Solutions

Solution 1: To plot $(3, -\frac{2\pi}{3})$, rotate clockwise by $\frac{2\pi}{3}$ (or $120°$) and move 3 units out. This is in Quadrant III.
Other representations:

- Add 2π: $(3, -\frac{2\pi}{3} + 2\pi) = (3, \frac{4\pi}{3})$.
- Negative r, add π: $(-3, -\frac{2\pi}{3} + \pi) = (-3, \frac{\pi}{3})$.
- Negative r, subtract π: $(-3, -\frac{2\pi}{3} - \pi) = (-3, -\frac{5\pi}{3})$.

Solution 2: To plot $(-4, \frac{5\pi}{4})$, face the direction $\frac{5\pi}{4}$ ($225°$, Quadrant III) and move 4 units backward, which lands you in Quadrant I. This is the same point as $(4, \frac{5\pi}{4} - \pi) = (4, \frac{\pi}{4})$.

- With $r > 0$: $(4, \frac{\pi}{4})$.
- With $r < 0$: Find a coterminal angle for $\frac{5\pi}{4}$ by subtracting 2π. $\frac{5\pi}{4} - 2\pi = -\frac{3\pi}{4}$. So, $(-4, -\frac{3\pi}{4})$ is another representation.

Solution 3: The point $(-2, \frac{11\pi}{6})$ is in Quadrant II. Let's check the options.

- (a) $(2, \frac{5\pi}{6})$: Quadrant II. Angle is $\frac{11\pi}{6} - \pi = \frac{5\pi}{6}$. This is the same point.
- (b) $(2, -\frac{7\pi}{6})$: The angle $-\frac{7\pi}{6}$ is coterminal with $\frac{5\pi}{6}$. This is the same point.
- (c) $(-2, -\frac{\pi}{6})$: The angle $-\frac{\pi}{6}$ is coterminal with $\frac{11\pi}{6}$. This is the same point.
- (d) $(2, \frac{17\pi}{6})$: The angle $\frac{17\pi}{6} = \frac{5\pi}{6} + 2\pi$. So this is the point $(2, \frac{5\pi}{6})$. The original point is $(-2, \frac{11\pi}{6})$ which is equivalent to $(2, \frac{5\pi}{6})$. This is the same point. Let's re-evaluate. The point $(-2, 11\pi/6)$ means face $11\pi/6$ (Q IV) and move 2 units backwards into Q II. This point is equivalent to $(2, 11\pi/6 - \pi) = (2, 5\pi/6)$. (a) $(2, 5\pi/6)$ is correct. (b) $2, -7\pi/6$ is coterminal with $5\pi/6$. Correct. (c) $-2, -\pi/6$ is coterminal with $-2, 11\pi/6$. Correct. (d) $(2, 17\pi/6)$ is coterminal with $(2, 5\pi/6)$. Correct. There seems to be a mistake in the problem statement as written. Let's change option (d) to be incorrect. For example, let's change it to $(2, \frac{\pi}{6})$. The point $(2, \frac{\pi}{6})$ is in Quadrant I, while our point is in Quadrant II. Thus, $(2, \frac{\pi}{6})$ would be the answer. **Correction:** Assume option (d) was intended to be incorrect. The point $(2, \frac{17\pi}{6})$ is equivalent to $(2, \frac{5\pi}{6})$, which is correct. The problem as written has no incorrect option. Let's assume the intended incorrect answer was, for example, $(2, \frac{7\pi}{6})$. This point is in QIII and would be wrong.

Part 2: Solutions

Solution 4:

(a) $x = 5\cos(\frac{\pi}{2}) = 5(0) = 0$. $y = 5\sin(\frac{\pi}{2}) = 5(1) = 5$. Result: $(0, 5)$.

(b) $x = 2\sqrt{2}\cos(\frac{7\pi}{4}) = 2\sqrt{2}(\frac{\sqrt{2}}{2}) = 2$. $y = 2\sqrt{2}\sin(\frac{7\pi}{4}) = 2\sqrt{2}(-\frac{\sqrt{2}}{2}) = -2$. Result: $(2, -2)$.

(c) $x = -4\cos(\frac{2\pi}{3}) = -4(-\frac{1}{2}) = 2$. $y = -4\sin(\frac{2\pi}{3}) = -4(\frac{\sqrt{3}}{2}) = -2\sqrt{3}$. Result: $(2, -2\sqrt{3})$.

(d) $x = 6\cos(\pi) = 6(-1) = -6$. $y = 6\sin(\pi) = 6(0) = 0$. Result: $(-6, 0)$.

Solution 5: The point $(0, -7)$ is on the negative y-axis. $r = \sqrt{0^2 + (-7)^2} = 7$. The angle is $\theta = \frac{3\pi}{2}$. Result: $(7, \frac{3\pi}{2})$.

Solution 6: The point $(-5, -5\sqrt{3})$ is in Quadrant III. $r = \sqrt{(-5)^2 + (-5\sqrt{3})^2} = \sqrt{25 + 75} = \sqrt{100} = 10$. $\tan\theta = \frac{-5\sqrt{3}}{-5} = \sqrt{3}$. The reference angle is $\frac{\pi}{3}$. In Quadrant III, $\theta = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$. Result: $(10, \frac{4\pi}{3})$.

Solution 7: The point $(3, -4)$ is in Quadrant IV. $r = \sqrt{3^2 + (-4)^2} = \sqrt{9+16} = \sqrt{25} = 5$. $\tan\theta = \frac{-4}{3}$. $\theta = \arctan(-\frac{4}{3}) \approx -0.927$ radians. To get an angle in $[0, 2\pi)$, we add 2π: $\theta \approx -0.927 + 2\pi \approx 5.356$ radians. Result: $(5, \arctan(-\frac{4}{3}) + 2\pi)$.

Part 3: Solutions

Solution 8: This is a sector of an annulus (a washer shape). The inner radius is 2, the outer radius is 4 (not inclusive). The region is between the angles $45°$ and $120°$.

Solution 9: This describes a filled-in semicircle of radius 3 in the right half-plane (including the y-axis).

Solution 10: This describes the entire plane excluding the disk of radius 1 centered at the origin.

Solution 11: This is not a region, but a line segment. The angle is fixed at $150°$, and r ranges from 1 to 3. It's a line segment of length 2.

Part 4: Solutions

Polar to Cartesian

Solution 12: $r = 8\sin\theta$. Multiply by r: $r^2 = 8r\sin\theta$. Substitute: $x^2 + y^2 = 8y$. Complete the square: $x^2 + y^2 - 8y = 0 \implies x^2 + (y^2 - 8y + 16) = 16 \implies x^2 + (y-4)^2 = 16$. This is a circle centered at $(0, 4)$ with radius 4.

Solution 13: $r(2\cos\theta - 5\sin\theta) = 3$. Distribute r: $2r\cos\theta - 5r\sin\theta = 3$. Substitute: $2x - 5y = 3$. This is a line.

Solution 14: $r^2 = \tan\theta \implies x^2 + y^2 = \frac{y}{x}$. We can write this as $x(x^2 + y^2) = y$.

Solution 15: $\theta = \frac{3\pi}{4}$. Take the tangent of both sides: $\tan\theta = \tan(\frac{3\pi}{4})$. Substitute $\tan\theta = y/x$: $\frac{y}{x} = -1 \implies y = -x$. This is a line through the origin.

Solution 16: $r = -6\sec\theta \implies r = \frac{-6}{\cos\theta} \implies r\cos\theta = -6$. Substitute: $x = -6$. This is a vertical line.

Solution 17: $r^2\sin(2\theta) = 8$. Use the identity $\sin(2\theta) = 2\sin\theta\cos\theta$: $r^2(2\sin\theta\cos\theta) = 8$. Rearrange: $2(r\sin\theta)(r\cos\theta) = 8$. Substitute: $2yx = 8 \implies yx = 4$. This is a hyperbola.

Cartesian to Polar

Solution 18: $x^2 + y^2 = 10$. Substitute $r^2 = x^2 + y^2$: $r^2 = 10$. So, $r = \sqrt{10}$.

Solution 19: $y = -x$. Divide by x: $\frac{y}{x} = -1$. Substitute $\tan\theta = y/x$: $\tan\theta = -1$. So, $\theta = \frac{3\pi}{4}$ (or $\frac{7\pi}{4}$).

Solution 20: $x = 7$. Substitute $x = r\cos\theta$: $r\cos\theta = 7$. So, $r = \frac{7}{\cos\theta} = 7\sec\theta$.

Solution 21: $(x-3)^2 + y^2 = 9$. Expand: $x^2 - 6x + 9 + y^2 = 9$. Simplify: $x^2 + y^2 - 6x = 0$. Substitute $x^2 + y^2 = r^2$ and $x = r\cos\theta$: $r^2 - 6r\cos\theta = 0$. Factor out r: $r(r - 6\cos\theta) = 0$. This gives $r = 0$ (the pole) or $r = 6\cos\theta$.

Solution 22: $y = x^2$. Substitute $y = r\sin\theta$ and $x = r\cos\theta$: $r\sin\theta = (r\cos\theta)^2 = r^2\cos^2\theta$. Assuming $r \neq 0$, divide by r: $\sin\theta = r\cos^2\theta$. Solve for r: $r = \frac{\sin\theta}{\cos^2\theta} = \tan\theta\sec\theta$.

Part 5: Solutions

Solution 23: The curve $r = 3\cos(2\theta)$ is a four-petaled rose. One loop is traced as 2θ goes from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, which means θ goes from $-\frac{\pi}{4}$ to $\frac{\pi}{4}$. Area $A = \frac{1}{2}\int_{-\pi/4}^{\pi/4} r^2 d\theta = \frac{1}{2}\int_{-\pi/4}^{\pi/4}(3\cos(2\theta))^2 d\theta = \frac{9}{2}\int_{-\pi/4}^{\pi/4}\cos^2(2\theta)d\theta$. Use identity $\cos^2(x) = \frac{1+\cos(2x)}{2}$: $A = \frac{9}{2}\int_{-\pi/4}^{\pi/4}\frac{1+\cos(4\theta)}{2}d\theta = \frac{9}{4}\left[\theta + \frac{1}{4}\sin(4\theta)\right]_{-\pi/4}^{\pi/4}$. $A = \frac{9}{4}\left[(\frac{\pi}{4} + \frac{1}{4}\sin(\pi)) - (-\frac{\pi}{4} + \frac{1}{4}\sin(-\pi))\right] = \frac{9}{4}(\frac{\pi}{4} - (-\frac{\pi}{4})) = \frac{9}{4}(\frac{\pi}{2}) = \frac{9\pi}{8}$.

Solution 24: The cardioid $r = 2 + 2\sin\theta$ is traced once from $\theta = 0$ to $\theta = 2\pi$. Area $A = \frac{1}{2}\int_0^{2\pi}(2 + 2\sin\theta)^2 d\theta = \frac{1}{2}\int_0^{2\pi} 4(1 + \sin\theta)^2 d\theta = 2\int_0^{2\pi}(1 + 2\sin\theta + \sin^2\theta)d\theta$. Use $\sin^2\theta = \frac{1-\cos(2\theta)}{2}$: $A = 2\int_0^{2\pi}(1 + 2\sin\theta + \frac{1-\cos(2\theta)}{2})d\theta = 2\int_0^{2\pi}(\frac{3}{2} + 2\sin\theta - \frac{1}{2}\cos(2\theta))d\theta$. $A = 2\left[\frac{3}{2}\theta - 2\cos\theta - \frac{1}{4}\sin(2\theta)\right]_0^{2\pi} = 2[(\frac{3}{2}(2\pi) - 2\cos(2\pi) - 0) - (0 - 2\cos(0) - 0)] = 2[(3\pi - 2) - (-2)] = 6\pi$.

Solution 25: $r = 2\theta$, so $\frac{dr}{d\theta} = 2$. Arc Length $L = \int_0^{2\pi}\sqrt{r^2 + (\frac{dr}{d\theta})^2}d\theta = \int_0^{2\pi}\sqrt{(2\theta)^2 + 2^2}d\theta = \int_0^{2\pi}\sqrt{4\theta^2 + 4}d\theta = 2\int_0^{2\pi}\sqrt{\theta^2 + 1}d\theta$.

Solution 26: The curve $r = 4\cos\theta$ is a circle of diameter 4 centered at $(2, 0)$. The arc length should be the circumference, $\pi d = 4\pi$. Let's verify with the formula. $r = 4\cos\theta$, $\frac{dr}{d\theta} = -4\sin\theta$. $L = \int_0^{\pi}\sqrt{(4\cos\theta)^2 + (-4\sin\theta)^2}d\theta = \int_0^{\pi}\sqrt{16\cos^2\theta + 16\sin^2\theta}d\theta$. $L = \int_0^{\pi}\sqrt{16(\cos^2\theta + \sin^2\theta)}d\theta = \int_0^{\pi}\sqrt{16}d\theta = \int_0^{\pi} 4d\theta = [4\theta]_0^{\pi} = 4\pi$.

Part 6: Solutions

Solution 27: The flaw is in step 3 and 4. The "Quadrant Trap". The point $(-3, 3)$ is in Quadrant II. The angle given by $\arctan(-1) = -\frac{\pi}{4}$ is in Quadrant IV. To find the correct angle in Quadrant II that has a tangent of -1, we should use the reference angle $\frac{\pi}{4}$ and calculate $\theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$. The correct polar coordinates are $(3\sqrt{2}, \frac{3\pi}{4})$.

Solution 28: The flaw is not in the correctness, but in the method. The standard "trick" or more direct method is to multiply the entire equation by r at the very beginning. Starting with $r = 10\cos\theta$, multiplying by r immediately gives $r^2 = 10r\cos\theta$. This allows for a direct substitution of $r^2 = x^2 + y^2$ and $r\cos\theta = x$, leading to $x^2+y^2 = 10x$ in one step. This avoids working with square roots and fractions and is the standard manipulation for these types of equations.

3 Concept Checklist

This checklist maps the problems to the key concepts they are designed to test.

- **Fundamentals of Polar Coordinates**
  - Plotting points in the polar plane (including negative r): **1, 2**
  - Finding multiple representations for a single point: **1, 2, 3**
- **Coordinate Conversion (Points)**
  - Converting Polar coordinates to Cartesian: **4**
  - Converting Cartesian coordinates to Polar: **5, 6, 7**
  - Correctly determining the quadrant for θ (The "Quadrant Trap"): **6, 27**
- **Sketching Regions in Polar Coordinates**
  - Sketching regions defined by inequalities on r (disks, annuli): **8, 10**
  - Sketching regions defined by inequalities on θ (wedges): **8, 9**
  - Sketching regions defined by combined inequalities: **8, 9, 11**
- **Equation Conversion (Curves)**
  - **Polar to Cartesian**
    - $r = b\sin\theta$ or $r = a\cos\theta$ (Circles not at origin): **12**
    - Lines not through origin: **13**
    - General polar equations to Cartesian: **14**
    - $\theta = k$ (Lines through origin): **15**
    - $r = a\sec\theta$ or $r = b\csc\theta$ (Vertical/Horizontal Lines): **16**
    - Equations with double-angle identities: **17**
  - **Cartesian to Polar**
    - $x^2 + y^2 = k^2$ (Circles at origin): **18**
    - $y = mx$ (Lines through origin): **19**
    - $x = a$ or $y = b$ (Vertical/Horizontal Lines): **20**
    - Circles not centered at origin: **21**
    - General Cartesian equations: **22**
- **Calculus with Polar Coordinates**
  - Calculating the area of a polar region: **23, 24**
  - Calculating the arc length of a polar curve: **25, 26**
- **Analytical and Critical Thinking**
  - Identifying flaws in incorrect solutions ("Find the Flaw"): **27, 28**