Homework 7.8 Improper Integrals

Tashfeen Omran

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1 Improper Integral Problems and Solutions

1.1 Problem 1

Determine whether the integral $\int_1^\infty 3x^{-4}\,dx$ is convergent or divergent. If it is convergent, evaluate it.

Solution

This is a Type 1 improper integral because it has an infinite limit of integration.

$$\begin{aligned}
\int_1^\infty 3x^{-4}\,dx &= \lim_{t\to\infty}\int_1^t 3x^{-4}\,dx \\
&= \lim_{t\to\infty}\left[3\frac{x^{-3}}{-3}\right]_1^t \\
&= \lim_{t\to\infty}\left[-x^{-3}\right]_1^t = \lim_{t\to\infty}\left[-\frac{1}{x^3}\right]_1^t \\
&= \lim_{t\to\infty}\left(-\frac{1}{t^3}-\left(-\frac{1}{1^3}\right)\right) \\
&= \lim_{t\to\infty}\left(-\frac{1}{t^3}+1\right) \\
&= 0+1=1
\end{aligned}$$

Answer: Convergent. The integral evaluates to **1**.

1.2 Problem 2

Determine whether the integral $\int_{-\infty}^{-1}\frac{1}{\sqrt[3]{x}}\,dx$ is convergent or divergent. If it is convergent, evaluate it.

Solution

This is a Type 1 improper integral.

$$\begin{aligned}
\int_{-\infty}^{-1} x^{-1/3}\,dx &= \lim_{t\to-\infty}\int_t^{-1} x^{-1/3}\,dx \\
&= \lim_{t\to-\infty}\left[\frac{x^{2/3}}{2/3}\right]_t^{-1} = \lim_{t\to-\infty}\left[\frac{3}{2}x^{2/3}\right]_t^{-1} \\
&= \lim_{t\to-\infty}\left(\frac{3}{2}(-1)^{2/3}-\frac{3}{2}t^{2/3}\right) \\
&= \frac{3}{2}(1)-\lim_{t\to-\infty}\frac{3}{2}t^{2/3} \\
&= \frac{3}{2}-\infty=-\infty
\end{aligned}$$

As $t \to -\infty$, $t^{2/3}$ (which is $(\sqrt[3]{t})^2$) approaches $+\infty$. The limit does not exist. **Answer: DIVERGES**.

1.3 Problem 3

Determine whether the integral $\int_{-6}^{\infty} \frac{1}{x+7}\,dx$ is convergent or divergent.

Solution

This is a Type 1 improper integral.

$$\begin{aligned}
\int_{-6}^{\infty} \frac{1}{x+7}\,dx &= \lim_{t\to\infty} \int_{-6}^{t} \frac{1}{x+7}\,dx \\
&= \lim_{t\to\infty} [\ln|x+7|]_{-6}^{t} \\
&= \lim_{t\to\infty} (\ln|t+7| - \ln|-6+7|) \\
&= \lim_{t\to\infty} \ln(t+7) - \ln(1) \\
&= \infty - 0 = \infty
\end{aligned}$$

The limit does not exist. **Answer: DIVERGES**.

1.4 Problem 4

Determine whether the integral $\int_{8}^{\infty} \frac{1}{(x-7)^{3/2}}\,dx$ is convergent or divergent.

Solution

This is a Type 1 improper integral.

$$\begin{aligned}
\int_{8}^{\infty} (x-7)^{-3/2}\,dx &= \lim_{t\to\infty} \int_{8}^{t} (x-7)^{-3/2}\,dx \\
&= \lim_{t\to\infty} \left[\frac{(x-7)^{-1/2}}{-1/2}\right]_{8}^{t} = \lim_{t\to\infty} \left[\frac{-2}{\sqrt{x-7}}\right]_{8}^{t} \\
&= \lim_{t\to\infty} \left(\frac{-2}{\sqrt{t-7}} - \left(\frac{-2}{\sqrt{8-7}}\right)\right) \\
&= \lim_{t\to\infty} \left(\frac{-2}{\sqrt{t-7}} + 2\right) \\
&= 0 + 2 = 2
\end{aligned}$$

Answer: Convergent. The integral evaluates to **2**.

1.5 Problem 5

Determine whether the integral $\int_{0}^{\infty} \frac{1}{\sqrt{1+x}}\,dx$ is convergent or divergent.

Solution

This is a Type 1 improper integral.

$$\begin{aligned}
\int_0^\infty (1+x)^{-1/2}\,dx &= \lim_{t\to\infty}\int_0^t (1+x)^{-1/2}\,dx \\
&= \lim_{t\to\infty}\left[\frac{(1+x)^{1/2}}{1/2}\right]_0^t = \lim_{t\to\infty}\left[2\sqrt{1+x}\right]_0^t \\
&= \lim_{t\to\infty}\left(2\sqrt{1+t}-2\sqrt{1+0}\right) \\
&= \infty - 2 = \infty
\end{aligned}$$

The limit does not exist. **Answer: DIVERGES**.

1.6 Problem 6

Determine whether the integral $\int_{-\infty}^0 \frac{x}{(x^2+4)^2}\,dx$ is convergent or divergent.

Solution

This is a Type 1 improper integral. Use u-substitution: $u = x^2+4$, $du = 2x\,dx \implies x\,dx = du/2$.

$$\begin{aligned}
\int_{-\infty}^0 \frac{x}{(x^2+4)^2}\,dx &= \lim_{t\to-\infty}\int_t^0 \frac{x}{(x^2+4)^2}\,dx \\
&= \lim_{t\to-\infty}\int_{x=t}^{x=0}\frac{1}{u^2}\frac{du}{2} \\
&= \frac{1}{2}\lim_{t\to-\infty}\left[-\frac{1}{u}\right]_{x=t}^{x=0} = \frac{1}{2}\lim_{t\to-\infty}\left[-\frac{1}{x^2+4}\right]_t^0 \\
&= \frac{1}{2}\lim_{t\to-\infty}\left(-\frac{1}{0^2+4}-\left(-\frac{1}{t^2+4}\right)\right) \\
&= \frac{1}{2}\lim_{t\to-\infty}\left(-\frac{1}{4}+\frac{1}{t^2+4}\right) \\
&= \frac{1}{2}\left(-\frac{1}{4}+0\right) = -\frac{1}{8}
\end{aligned}$$

Answer: Convergent. The integral evaluates to **-1/8**.

1.7 Problem 7

Determine whether the integral $\int_1^\infty \frac{x^3+x+1}{x^5}\,dx$ is convergent or divergent.

Solution

First, simplify the integrand.

$$\frac{x^3+x+1}{x^5} = \frac{x^3}{x^5}+\frac{x}{x^5}+\frac{1}{x^5} = x^{-2}+x^{-4}+x^{-5}$$

Now evaluate the integral:

$$\begin{aligned}
\int_1^\infty (x^{-2}+x^{-4}+x^{-5})\,dx &= \lim_{t\to\infty}\int_1^t (x^{-2}+x^{-4}+x^{-5})\,dx \\
&= \lim_{t\to\infty}\left[-x^{-1}-\frac{x^{-3}}{3}-\frac{x^{-4}}{4}\right]_1^t \\
&= \lim_{t\to\infty}\left[-\frac{1}{x}-\frac{1}{3x^3}-\frac{1}{4x^4}\right]_1^t \\
&= \lim_{t\to\infty}\left(\left(-\frac{1}{t}-\frac{1}{3t^3}-\frac{1}{4t^4}\right)-\left(-1-\frac{1}{3}-\frac{1}{4}\right)\right) \\
&= (0-0-0)-\left(-\frac{12}{12}-\frac{4}{12}-\frac{3}{12}\right) = -\left(-\frac{19}{12}\right) = \frac{19}{12}
\end{aligned}$$

Answer: Convergent. The integral evaluates to **19/12**.

1.8 Problem 8

Determine whether the integral $\int_0^\infty \frac{e^x}{(8+e^x)^2}\,dx$ is convergent or divergent.

Solution

Use u-substitution: $u = 8+e^x$, $du = e^x\,dx$.

$$\begin{aligned}
\int_0^\infty \frac{e^x}{(8+e^x)^2}\,dx &= \lim_{t\to\infty}\int_0^t \frac{e^x}{(8+e^x)^2}\,dx \\
&= \lim_{t\to\infty}\left[-\frac{1}{8+e^x}\right]_0^t \\
&= \lim_{t\to\infty}\left(-\frac{1}{8+e^t}-\left(-\frac{1}{8+e^0}\right)\right) \\
&= \lim_{t\to\infty}\left(-\frac{1}{8+e^t}+\frac{1}{9}\right) \\
&= 0+\frac{1}{9} = \frac{1}{9}
\end{aligned}$$

Answer: Convergent. The integral evaluates to **1/9**.

1.9 Problem 9

Determine whether the integral $\int_{-\infty}^\infty 9xe^{-x^2}\,dx$ is convergent or divergent.

Solution

The integral is over $(-\infty,\infty)$, so we split it at an arbitrary point, like $x=0$.

$$\int_{-\infty}^\infty 9xe^{-x^2}\,dx = \int_{-\infty}^0 9xe^{-x^2}\,dx + \int_0^\infty 9xe^{-x^2}\,dx$$

Evaluate the second integral first. Use u-substitution: $u=-x^2, du=-2x\,dx \implies 9x\,dx = -\frac{9}{2}du$.

$$\begin{aligned}
\int_0^\infty 9xe^{-x^2}\,dx &= \lim_{t\to\infty}\int_0^t 9xe^{-x^2}\,dx \\
&= \lim_{t\to\infty}\left[-\frac{9}{2}e^{-x^2}\right]_0^t \\
&= \lim_{t\to\infty}\left(-\frac{9}{2}e^{-t^2}-\left(-\frac{9}{2}e^0\right)\right) = 0+\frac{9}{2} = \frac{9}{2}
\end{aligned}$$

Now evaluate the first integral:

$$\begin{aligned}
\int_{-\infty}^{0} 9xe^{-x^2}\,dx &= \lim_{s\to-\infty}\int_{s}^{0} 9xe^{-x^2}\,dx \\
&= \lim_{s\to-\infty}\left[-\frac{9}{2}e^{-x^2}\right]_{s}^{0} \\
&= \lim_{s\to-\infty}\left(-\frac{9}{2}e^{0}-\left(-\frac{9}{2}e^{-s^2}\right)\right) = -\frac{9}{2}-0=-\frac{9}{2}
\end{aligned}$$

Since both integrals converge, the original integral converges. The total value is $\frac{9}{2}+(-\frac{9}{2})=0$. **Answer:** Convergent. The integral evaluates to **0**. (Note: This could also be solved by recognizing $f(x)=9xe^{-x^2}$ is an odd function.)

1.10 Problem 10

Determine whether the integral $\int_{-\infty}^{\infty}\frac{x^5}{x^6+1}\,dx$ is convergent or divergent.

Solution

Split the integral at $x=0$. Let's evaluate the part from $[0,\infty)$. Use u-substitution: $u=x^6+1, du=6x^5\,dx \implies x^5\,dx=du/6$.

$$\begin{aligned}
\int_{0}^{\infty}\frac{x^5}{x^6+1}\,dx &= \lim_{t\to\infty}\int_{0}^{t}\frac{x^5}{x^6+1}\,dx \\
&= \lim_{t\to\infty}\left[\frac{1}{6}\ln|x^6+1|\right]_{0}^{t} \\
&= \frac{1}{6}\lim_{t\to\infty}\left(\ln(t^6+1)-\ln(1)\right)=\frac{1}{6}(\infty-0)=\infty
\end{aligned}$$

Since one part of the split integral diverges, the entire integral diverges. **Answer: DIVERGES**.

1.11 Problem 11

Determine whether the integral $\int_{0}^{\infty}4\sin^2(\alpha)\,d\alpha$ is convergent or divergent.

Solution

Use the half-angle identity: $\sin^2(\alpha)=\frac{1-\cos(2\alpha)}{2}$.

$$\begin{aligned}
\int_{0}^{\infty}4\sin^2(\alpha)\,d\alpha &= \int_{0}^{\infty}4\left(\frac{1-\cos(2\alpha)}{2}\right)\,d\alpha=\int_{0}^{\infty}(2-2\cos(2\alpha))\,d\alpha \\
&= \lim_{t\to\infty}\int_{0}^{t}(2-2\cos(2\alpha))\,d\alpha \\
&= \lim_{t\to\infty}\left[2\alpha-\sin(2\alpha)\right]_{0}^{t} \\
&= \lim_{t\to\infty}((2t-\sin(2t))-(0-0))
\end{aligned}$$

As $t\to\infty$, the $2t$ term goes to infinity. The term $\sin(2t)$ oscillates between -1 and 1, but it cannot stop the $2t$ term from growing infinitely large. The limit does not exist. **Answer: DIVERGES**.

1.12 Problem 12

(a) Evaluate the integral: $\int_{0}^{t}8\sin^2(\alpha)\,d\alpha$ (b) Determine whether $\int_{0}^{\infty}8\sin^2(\alpha)\,d\alpha$ is convergent or divergent.

Solution (a)

This is a standard definite integral. Using the half-angle identity:

$$\int_0^t 8\sin^2(\alpha)\,d\alpha = \int_0^t 8\left(\frac{1-\cos(2\alpha)}{2}\right)d\alpha = \int_0^t (4-4\cos(2\alpha))\,d\alpha$$
$$= [4\alpha - 2\sin(2\alpha)]_0^t$$
$$= (4t - 2\sin(2t)) - (0-0)$$

Answer (a): $4t - 2\sin(2t)$

Solution (b)

To determine if the improper integral converges, we take the limit of the result from part (a) as $t \to \infty$.

$$\lim_{t\to\infty}(4t - 2\sin(2t)) = \infty$$

The limit does not exist. **Answer (b): DIVERGES**.

1.13 Problem 13

Determine whether the integral $\int_0^\infty \sin(\theta)e^{\cos(\theta)}\,d\theta$ is convergent or divergent.

Solution

Use u-substitution: $u = \cos(\theta), du = -\sin(\theta)\,d\theta$.

$$\int_0^\infty \sin(\theta)e^{\cos(\theta)}\,d\theta = \lim_{t\to\infty}\int_0^t \sin(\theta)e^{\cos(\theta)}\,d\theta$$
$$= \lim_{t\to\infty}\left[-e^{\cos(\theta)}\right]_0^t$$
$$= \lim_{t\to\infty}\left(-e^{\cos(t)} - (-e^{\cos(0)})\right)$$
$$= \lim_{t\to\infty}\left(e - e^{\cos(t)}\right)$$

As $t \to \infty$, $\cos(t)$ oscillates between -1 and 1. Therefore, $e^{\cos(t)}$ oscillates between e^{-1} and e^1. The limit does not settle on a single value. **Answer: DIVERGES**.

1.14 Problem 14

Determine whether the integral $\int_1^\infty \frac{1}{x^2+x}\,dx$ is convergent or divergent.

Solution

Use partial fraction decomposition: $\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} \implies 1 = A(x+1) + Bx$. If $x = 0, A = 1$. If $x = -1, B = -1$. So, $\frac{1}{x^2+x} = \frac{1}{x} - \frac{1}{x+1}$.

$$\int_1^\infty \left(\frac{1}{x} - \frac{1}{x+1}\right)dx = \lim_{t\to\infty}\int_1^t \left(\frac{1}{x} - \frac{1}{x+1}\right)dx$$
$$= \lim_{t\to\infty}\left[\ln|x| - \ln|x+1|\right]_1^t$$
$$= \lim_{t\to\infty}\left[\ln\left|\frac{x}{x+1}\right|\right]_1^t$$
$$= \lim_{t\to\infty}\left(\ln\left(\frac{t}{t+1}\right) - \ln\left(\frac{1}{2}\right)\right)$$

As $t \to \infty$, $\frac{t}{t+1} \to 1$, so $\ln(\frac{t}{t+1}) \to \ln(1) = 0$. The result is $0 - \ln(1/2) = -(-\ln(2)) = \ln(2)$. **Answer:** Convergent. The integral evaluates to $\mathbf{ln(2)}$.

1.15 Problem 15

Determine whether the integral $\int_2^\infty \frac{dv}{v^2+2v-3}$ is convergent or divergent.

Solution

Use partial fraction decomposition: $\frac{1}{(v+3)(v-1)} = \frac{A}{v+3} + \frac{B}{v-1} \implies 1 = A(v-1) + B(v+3)$. If $v = 1, B = 1/4$. If $v = -3, A = -1/4$.

$$\begin{aligned}
\int_2^\infty \frac{1/4}{v-1} - \frac{1/4}{v+3}\,dv &= \frac{1}{4}\lim_{t\to\infty}\int_2^t \left(\frac{1}{v-1} - \frac{1}{v+3}\right) dv \\
&= \frac{1}{4}\lim_{t\to\infty}\left[\ln|v-1| - \ln|v+3|\right]_2^t \\
&= \frac{1}{4}\lim_{t\to\infty}\left[\ln\left|\frac{v-1}{v+3}\right|\right]_2^t \\
&= \frac{1}{4}\lim_{t\to\infty}\left(\ln\left(\frac{t-1}{t+3}\right) - \ln\left(\frac{1}{5}\right)\right) \\
&= \frac{1}{4}(\ln(1) - \ln(1/5)) = \frac{1}{4}(0 - (-\ln(5))) = \frac{\ln(5)}{4}
\end{aligned}$$

Answer: Convergent. The integral evaluates to $\mathbf{ln(5)/4}$.

1.16 Problem 16

Determine whether the integral $\int_{-\infty}^0 \frac{z}{z^4+81}\,dz$ is convergent or divergent.

Solution

Use u-substitution: $u = z^2, du = 2z\,dz \implies z\,dz = du/2$.

$$\begin{aligned}
\int_{-\infty}^0 \frac{z}{z^4+81}\,dz &= \lim_{t\to-\infty}\int_t^0 \frac{z}{(z^2)^2+81}\,dz \\
&= \lim_{t\to-\infty}\frac{1}{2}\int_{z=t}^{z=0}\frac{du}{u^2+81} \\
&= \frac{1}{2}\lim_{t\to-\infty}\left[\frac{1}{9}\arctan\left(\frac{u}{9}\right)\right]_{z=t}^{z=0} = \frac{1}{18}\lim_{t\to-\infty}\left[\arctan\left(\frac{z^2}{9}\right)\right]_t^0 \\
&= \frac{1}{18}\lim_{t\to-\infty}\left(\arctan(0) - \arctan\left(\frac{t^2}{9}\right)\right)
\end{aligned}$$

As $t \to -\infty, t^2 \to \infty$, so $\arctan(t^2/9) \to \pi/2$. The result is $\frac{1}{18}(0 - \frac{\pi}{2}) = -\frac{\pi}{36}$. **Answer:** Convergent. The integral evaluates to **-π/36**.

1.17 Problem 17

(a) Evaluate the integral: $\int_t^0 \frac{z}{z^4+36}\,dz$ (b) Determine whether $\int_{-\infty}^0 \frac{z}{z^4+36}\,dz$ is convergent or divergent.

Solution (a)

Similar to problem 16, use u-substitution $u = z^2, du = 2z\,dz$.

$$\begin{aligned}\int_t^0 \frac{z}{z^4+36}\,dz &= \frac{1}{2}\int_{z=t}^{z=0} \frac{du}{u^2+36} \\ &= \frac{1}{2}\left[\frac{1}{6}\arctan\left(\frac{u}{6}\right)\right]_{z=t}^{z=0} = \frac{1}{12}\left[\arctan\left(\frac{z^2}{6}\right)\right]_t^0 \\ &= \frac{1}{12}\left(\arctan(0) - \arctan\left(\frac{t^2}{6}\right)\right)\end{aligned}$$

Answer (a): $-\frac{1}{12}\arctan\left(\frac{t^2}{6}\right)$

Solution (b)

Take the limit of the result from part (a) as $t \to -\infty$.

$$\lim_{t\to-\infty}\left(-\frac{1}{12}\arctan\left(\frac{t^2}{6}\right)\right) = -\frac{1}{12}\left(\frac{\pi}{2}\right) = -\frac{\pi}{24}$$

Answer (b): Convergent. The integral evaluates to **-π/24**.

1.18 Problem 18

Determine whether the integral $\int_0^9 \frac{3}{\sqrt[3]{x-1}}\,dx$ is convergent or divergent.

Solution

This is a Type 2 improper integral because the function has an infinite discontinuity at $x = 1$, which is within the interval $[0, 9]$. We must split the integral at the point of discontinuity.

$$\int_0^9 \frac{3}{(x-1)^{1/3}}\,dx = \int_0^1 3(x-1)^{-1/3}\,dx + \int_1^9 3(x-1)^{-1/3}\,dx$$

Evaluate the first part:

$$\begin{aligned}\lim_{t\to1^-}\int_0^t 3(x-1)^{-1/3}\,dx &= \lim_{t\to1^-}\left[3\frac{(x-1)^{2/3}}{2/3}\right]_0^t = \lim_{t\to1^-}\left[\frac{9}{2}(x-1)^{2/3}\right]_0^t \\ &= \lim_{t\to1^-}\left(\frac{9}{2}(t-1)^{2/3} - \frac{9}{2}(-1)^{2/3}\right) \\ &= 0 - \frac{9}{2}(1) = -\frac{9}{2}\end{aligned}$$

This part converges. Now evaluate the second part:

$$\begin{aligned}\lim_{s\to1^+}\int_s^9 3(x-1)^{-1/3}\,dx &= \lim_{s\to1^+}\left[\frac{9}{2}(x-1)^{2/3}\right]_s^9 \\ &= \lim_{s\to1^+}\left(\frac{9}{2}(9-1)^{2/3} - \frac{9}{2}(s-1)^{2/3}\right) \\ &= \frac{9}{2}(8)^{2/3} - 0 = \frac{9}{2}(4) = 18\end{aligned}$$

This part also converges. Since both parts converge, the original integral converges. The total value is $-\frac{9}{2} + 18 = \frac{27}{2}$.
Answer: Convergent. The integral evaluates to **27/2**.

2 Analysis of Problems and Techniques

2.1 Types of Improper Integrals Encountered

1. **Type 1: Infinite Intervals**
   - Integrals over $[a, \infty)$: Problems 1, 3, 4, 5, 7, 8, 11, 12, 13, 14, 15.
   - Integrals over $(-\infty, b]$: Problems 2, 6, 16, 17.
   - Integrals over $(-\infty, \infty)$: Problems 9, 10. These must be split into two separate improper integrals, e.g., $\int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx$. The integral converges only if BOTH parts converge.
2. **Type 2: Infinite Discontinuity**
   - The integrand has a vertical asymptote at $x = c$ within the interval $[a, b]$. The integral must be split at the discontinuity: $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$. This was seen in Problem 18 at $x = 1$ on the interval $[0, 9]$.

2.2 Convergence and Divergence Rules (p-Integrals)

A key tool for quickly assessing convergence is the **p-integral test**.

- For Type 1 integrals: $\int_a^\infty \frac{1}{x^p}\,dx$ (where $a > 0$) **converges if** $p > 1$ and **diverges if** $p \leq 1$.
  - Problem 1: $p = 4 > 1$, converges.
  - Problem 4: Form is $1/u^{3/2}$, $p = 3/2 > 1$, converges.
  - Problem 2: $p = 1/3 \leq 1$ on an infinite interval, diverges.
  - Problem 5: Form is $1/u^{1/2}$, $p = 1/2 \leq 1$, diverges.
- For Type 2 integrals: $\int_0^a \frac{1}{x^p}\,dx$ (discontinuity at 0) **converges if** $p < 1$ and **diverges if** $p \geq 1$.
  - Problem 18: Form is $1/u^{1/3}$, $p = 1/3 < 1$, converges.

2.3 Techniques and Algebraic Manipulations Used

- **Limit Definition**: The fundamental technique for all problems was to rewrite the improper integral as a limit of a proper integral.
- **U-Substitution**: Used in Problems 6, 8, 9, 10, 13, 16, 17 to simplify the integrand before integration.
- **Partial Fraction Decomposition**: Necessary for integrating rational functions where the denominator is factorable. Used in Problems 14 and 15.
- **Trigonometric Identities**: The half-angle identity $\sin^2(\alpha) = (1 - \cos(2\alpha))/2$ was crucial for Problems 11 and 12.
- **Splitting Integrals**: Required for Type 1 integrals over $(-\infty, \infty)$ (Problems 9, 10) and for Type 2 integrals with an interior discontinuity (Problem 18).
- **Recognizing Odd/Even Functions**: In Problem 9, the integrand is an odd function integrated over a symmetric interval $(-\infty, \infty)$. If such an integral converges, its value must be 0. In Problem 10, the integrand was also odd, but it was shown to diverge. *Trick: You must still prove convergence of one half of the integral before concluding the value is 0.*
- **Simplifying the Integrand**: In Problem 7, dividing each term in the numerator by the denominator simplified the problem into a sum of p-integrals.

2.4 Essential Limits to Know

For evaluating improper integrals, knowledge of limits at infinity is critical.

- $\lim_{x\to\infty} \frac{1}{x^p} = 0$ for any $p > 0$.
- $\lim_{x\to\infty} e^x = \infty$ and $\lim_{x\to-\infty} e^x = 0$.
- $\lim_{x\to\infty} \ln(x) = \infty$ and $\lim_{x\to 0^+} \ln(x) = -\infty$.
- $\lim_{x\to\infty} \arctan(x) = \frac{\pi}{2}$ and $\lim_{x\to-\infty} \arctan(x) = -\frac{\pi}{2}$.
- Limits of oscillating functions like $\sin(x)$ or $\cos(x)$ as $x \to \infty$ do not exist.

2.5 Additional Tricks and Untested Concepts

The provided problems did not cover all aspects of improper integrals. Here are other important concepts:

- **The Comparison Test**: For an integrand that is difficult to integrate directly, you can compare it to a simpler function. If $f(x) \geq g(x) \geq 0$:
  - If $\int_a^\infty f(x)\,dx$ converges, then $\int_a^\infty g(x)\,dx$ also converges.
  - If $\int_a^\infty g(x)\,dx$ diverges, then $\int_a^\infty f(x)\,dx$ also diverges.
  - **Example**: To determine if $\int_1^\infty \frac{1}{x^2+5}\,dx$ converges, we can note that $\frac{1}{x^2+5} < \frac{1}{x^2}$. Since $\int_1^\infty \frac{1}{x^2}\,dx$ converges (p-integral with p=2), our integral must also converge.
- **The Limit Comparison Test**: If $f(x)$ and $g(x)$ are positive functions and $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$, where L is a finite, positive number, then $\int f(x)\,dx$ and $\int g(x)\,dx$ either both converge or both diverge.
  - **Example**: For $\int_1^\infty \frac{x}{x^3-2}\,dx$, we can compare it to $g(x) = \frac{x}{x^3} = \frac{1}{x^2}$. Since $\lim_{x\to\infty} \frac{x/(x^3-2)}{1/x^2} = 1$, and we know $\int_1^\infty \frac{1}{x^2}\,dx$ converges, our integral must also converge.
- **Checking the Domain**: As a trick, always check that the function is defined over the integration interval. A problem like $\int_0^2 \frac{1}{\sqrt{x-3}}\,dx$ is invalid because the integrand is not real for any value in the interval. The initial (incorrect) reading of Problem 18 as $1/\sqrt{x-1}$ would have made the integral from $[0,1]$ invalid in the real number system.