Integration Techniques Revision

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1 Integration Problems and Solutions

1.1 Problem 1

Evaluate the integral: $\int \ln(\cos(x)) \tan(x)\, dx$

Solution

This integral is solved using a clever u-substitution. Let $u = \ln(\cos(x))$. Then $du = \frac{1}{\cos(x)}(-\sin(x))\, dx = -\tan(x)\, dx$. Substituting these into the integral gives:

$$\int u\,(-du) = -\int u\, du = -\frac{u^2}{2} + C$$

Substituting back for u: **Answer:** $-\frac{(\ln(\cos(x)))^2}{2} + C$

1.2 Problem 2

Evaluate the integral: $\int e^x \sin(x)\, dx$

Solution

This requires "looping" integration by parts. Let $I = \int e^x \sin(x)\, dx$. First IBP ($u = \sin(x), dv = e^x dx$): $I = e^x \sin(x) - \int e^x \cos(x)\, dx$. Second IBP ($u = \cos(x), dv = e^x dx$): $\int e^x \cos(x)\, dx = e^x \cos(x) + \int e^x \sin(x)\, dx = e^x \cos(x) + I$. Substitute back: $I = e^x \sin(x) - (e^x \cos(x) + I) \implies 2I = e^x(\sin(x) - \cos(x))$. **Answer:** $\frac{e^x(\sin(x)-\cos(x))}{2} + C$

1.3 Problem 3

Evaluate the integral: $\int \arctan(x)\, dx$

Solution

This integral requires the "Stealth dx" IBP trick. Let $u = \arctan(x)$ and $dv = dx$. Then $du = \frac{1}{1+x^2}\, dx$ and $v = x$. The integral becomes: $x \arctan(x) - \int \frac{x}{1+x^2}\, dx$. The second integral is a u-substitution with $w = 1 + x^2$, giving $\frac{1}{2}\ln(1 + x^2)$. **Answer:** $x \arctan(x) - \frac{1}{2}\ln(1 + x^2) + C$

1.4 Problem 4

Evaluate the integral: $\int \ln(x)\, dx$

Solution

This uses the "Stealth dx" IBP trick. Let $u = \ln(x)$ and $dv = dx$. Then $du = \frac{1}{x}\, dx$ and $v = x$. The integral becomes: $x \ln(x) - \int x \cdot \frac{1}{x}\, dx = x \ln(x) - \int 1\, dx$. **Answer:** $x \ln(x) - x + C$

1.5 Problem 5

Evaluate the integral: $\int \ln(x^2+4)\,dx$

Solution

Use IBP with $u = \ln(x^2+4), dv = dx$. This gives $x\ln(x^2+4) - \int \frac{2x^2}{x^2+4}\,dx$. The second integral requires algebraic manipulation:

$$\int \frac{2x^2}{x^2+4}\,dx = 2\int \frac{x^2+4-4}{x^2+4}\,dx = 2\int \left(1 - \frac{4}{x^2+4}\right)dx = 2(x - 2\arctan(\frac{x}{2}))$$

Answer: $x\ln(x^2+4) - 2x + 4\arctan\left(\frac{x}{2}\right) + C$

1.6 Problem 6

Evaluate the integral: $\int \ln(x^2+5)\,dx$

Solution

Use IBP with $u = \ln(x^2+5), dv = dx$. This gives $x\ln(x^2+5) - \int \frac{2x^2}{x^2+5}\,dx$. The second integral requires algebraic manipulation:

$$2\int \frac{x^2+5-5}{x^2+5}\,dx = 2\int \left(1 - \frac{5}{x^2+5}\right)dx = 2\left(x - \frac{5}{\sqrt{5}}\arctan\left(\frac{x}{\sqrt{5}}\right)\right)$$

Answer: $x\ln(x^2+5) - 2x + 2\sqrt{5}\arctan\left(\frac{x}{\sqrt{5}}\right) + C$

1.7 Problem 7

Evaluate the integral: $\int \frac{1}{x^2+6x+13}\,dx$

Solution

This integral requires completing the square in the denominator. $x^2+6x+13 = (x^2+6x+9)+4 = (x+3)^2+2^2$. The integral becomes $\int \frac{1}{(x+3)^2+2^2}\,dx$. This is the standard arctangent form with $u = x+3$ and $a = 2$. **Answer:** $\frac{1}{2}\arctan\left(\frac{x+3}{2}\right) + C$

1.8 Problem 8

Evaluate the integral: $\int x\arctan(x)\,dx$

Solution

Use IBP with $u = \arctan(x), dv = xdx$. This gives $\frac{x^2}{2}\arctan(x) - \frac{1}{2}\int \frac{x^2}{1+x^2}\,dx$. The second integral requires algebraic manipulation:

$$\int \frac{x^2}{1+x^2}\,dx = \int \frac{1+x^2-1}{1+x^2}\,dx = \int \left(1 - \frac{1}{1+x^2}\right)dx = x - \arctan(x)$$

Answer: $\frac{x^2}{2}\arctan(x) - \frac{1}{2}(x - \arctan(x)) + C$

1.9 Problem 9

Evaluate the integral: $\int_0^{\pi} e^{\cos(t)}\sin(2t)\,dt$

Solution

Use identity $\sin(2t) = 2\sin(t)\cos(t)$. Let $u = \cos(t)$, so $du = -\sin(t)dt$. Limits: $t = 0 \to u = 1$, $t = \pi \to u = -1$. The integral becomes $\int_1^{-1} e^u \cdot 2u \cdot (-du) = 2\int_{-1}^1 ue^u\, du$. Using IBP gives $2[ue^u - e^u]_{-1}^1 = 2[(e-e)-(-e^{-1}-e^{-1})] = 2(2/e)$. **Answer:** $\frac{4}{e}$

1.10 Problem 10

Evaluate the integral: $\int_{\pi/6}^{\pi/2} \sin(2x)\ln(\sin(x))\, dx$

Solution

Use identity $\sin(2x) = 2\sin(x)\cos(x)$. Let $u = \sin(x)$, so $du = \cos(x)dx$. Limits: $x = \pi/6 \to u = 1/2$, $x = \pi/2 \to u = 1$. The integral becomes $\int_{1/2}^1 2u\ln(u)\, du$. Using IBP gives $2[\frac{u^2}{2}\ln(u) - \frac{u^2}{4}]_{1/2}^1 = [\frac{1}{2}\ln(1) - \frac{1}{4}] - [\frac{(1/4)}{2}\ln(1/2) - \frac{1/4}{4}]$. **Answer:** $\frac{\ln(2)}{4} - \frac{3}{8}$

1.11 Problem 11

Evaluate the integral: $\int_0^{\pi/2} e^{\cos(x)}\sin(2x)\, dx$

Solution

Use identity $\sin(2x) = 2\sin(x)\cos(x)$. Let $u = \cos(x)$, so $du = -\sin(x)dx$. Limits: $x = 0 \to u = 1$, $x = \pi/2 \to u = 0$. The integral becomes $\int_1^0 e^u \cdot 2u \cdot (-du) = 2\int_0^1 ue^u\, du$. Using IBP gives $2[ue^u - e^u]_0^1 = 2[(e-e)-(0-e^0)] = 2(1)$. **Answer:** 2

1.12 Problem 12

Evaluate the integral: $\int_{\pi/3}^{\pi/2} \sin(2x)\cos(\cos(x))\, dx$

Solution

Use identity $\sin(2x) = 2\sin(x)\cos(x)$. Let $u = \cos(x)$, so $du = -\sin(x)dx$. Limits: $x = \pi/3 \to u = 1/2$, $x = \pi/2 \to u = 0$. The integral becomes $\int_{1/2}^0 2u\cos(u)(-du) = 2\int_0^{1/2} u\cos(u)\, du$. Using IBP gives $2[u\sin(u) + \cos(u)]_0^{1/2} = 2[(\frac{1}{2}\sin(\frac{1}{2}) + \cos(\frac{1}{2})) - (0 + \cos(0))]$. **Answer:** $\sin(\frac{1}{2}) + 2\cos(\frac{1}{2}) - 2$

1.13 Problem 13

Evaluate the integral: $\int_0^{\pi/2} \sin(2x)e^{\sin(x)}\, dx$

Solution

Use identity $\sin(2x) = 2\sin(x)\cos(x)$. Let $u = \sin(x)$, so $du = \cos(x)dx$. Limits: $x = 0 \to u = 0$, $x = \pi/2 \to u = 1$. The integral becomes $\int_0^1 2ue^u\, du$. Using IBP gives $2[ue^u - e^u]_0^1 = 2[(e-e)-(0-e^0)] = 2(1)$. **Answer:** 2

1.14 Problem 14

Evaluate the integral: $\int_0^{\pi} \sin(2x)\cos(\cos(x))\, dx$

Solution

Use identity $\sin(2x) = 2\sin(x)\cos(x)$. Let $u = \cos(x)$, so $du = -\sin(x)dx$. Limits: $x = 0 \to u = 1$, $x = \pi \to u = -1$. The integral becomes $\int_1^{-1} 2u\cos(u)(-du) = 2\int_{-1}^1 u\cos(u)\, du$. Using IBP gives $2[u\sin(u) + \cos(u)]_{-1}^1 = 2[(\sin(1) + \cos(1)) - (-\sin(-1) + \cos(-1))]$. Since sine is odd and cosine is even, this simplifies to $2[(\sin(1)+\cos(1)) - (\sin(1)+\cos(1))]$. **Answer:** 0

1.15 Problem 15

Evaluate the definite integral: $\int_0^{\sqrt{\ln(2)}} xe^{x^2}\cos(e^{x^2})\,dx$

Solution

This integral can be solved with a u-substitution. Let $u = e^{x^2}$. Then $du = 2xe^{x^2}\,dx$, which means $xe^{x^2}\,dx = \frac{1}{2}du$. We must also change the bounds of integration. When $x = 0$, $u = e^{0^2} = 1$. When $x = \sqrt{\ln(2)}$, $u = e^{(\sqrt{\ln(2)})^2} = e^{\ln(2)} = 2$. The integral becomes $\int_1^2 \cos(u) \cdot \frac{1}{2}du = \frac{1}{2}\int_1^2 \cos(u)\,du$. Integrating gives $\frac{1}{2}[\sin(u)]_1^2$. **Answer:** $\frac{1}{2}(\sin(2) - \sin(1))$

1.16 Problem 16

Evaluate the integral: $\int \frac{x}{1+\sqrt{x+4}}\,dx$

Solution

This requires a "clever" u-substitution to rationalize the expression. Let $u = \sqrt{x+4}$. Then $u^2 = x+4$, so $x = u^2 - 4$. The differential is $dx = 2u\,du$. Substitute everything into the integral: $\int \frac{u^2-4}{1+u}(2u)\,du = \int \frac{2u^3-8u}{u+1}\,du$. Using polynomial long division, we get $\int \left(2u^2 - 2u - 6 + \frac{6}{u+1}\right)du$. Integrating term by term yields $\frac{2}{3}u^3 - u^2 - 6u + 6\ln|u+1| + C$. Finally, substitute back $u = \sqrt{x+4}$. **Answer:** $\frac{2}{3}(x+4)^{3/2} - (x+4) - 6\sqrt{x+4} + 6\ln|\sqrt{x+4}+1| + C$

1.17 Problem 17

Evaluate the integral: $\int \sin^2(x)\cos^4(x)\,dx$

Solution

This integral requires power-reducing trigonometric identities. Rewrite the integrand: $\int(\sin(x)\cos(x))^2\cos^2(x)\,dx = \int\left(\frac{1}{2}\sin(2x)\right)^2\left(\frac{1+\cos(2x)}{2}\right)dx$. This simplifies to $\frac{1}{8}\int\sin^2(2x)(1+\cos(2x))\,dx = \frac{1}{8}\int\sin^2(2x)\,dx + \frac{1}{8}\int\sin^2(2x)\cos(2x)\,dx$. For the first integral, use the identity $\sin^2(\theta) = \frac{1-\cos(2\theta)}{2}$: $\frac{1}{8}\int\frac{1-\cos(4x)}{2}\,dx = \frac{1}{16}\left(x - \frac{1}{4}\sin(4x)\right)$. For the second integral, let $u = \sin(2x)$, $du = 2\cos(2x)dx$: $\frac{1}{16}\int u^2\,du = \frac{1}{16}\frac{u^3}{3} = \frac{1}{48}\sin^3(2x)$. Combining the parts gives the final answer. **Answer:** $\frac{1}{16}x - \frac{1}{64}\sin(4x) + \frac{1}{48}\sin^3(2x) + C$

1.18 Problem 18

Evaluate the integral: $\int \frac{2x+3}{x^2-4x+20}\,dx$

Solution

First, complete the square in the denominator: $x^2 - 4x + 20 = (x^2 - 4x + 4) + 16 = (x-2)^2 + 16$. Next, split the numerator to create the derivative of the denominator's quadratic part, which is $2x - 4$. $2x + 3 = (2x-4) + 7$. The integral becomes $\int\frac{2x-4}{x^2-4x+20}\,dx + \int\frac{7}{(x-2)^2+16}\,dx$. The first integral is a logarithm: $\ln(x^2 - 4x + 20)$. The second integral is an arctangent form: $7\int\frac{1}{(x-2)^2+4^2}\,dx = \frac{7}{4}\arctan\left(\frac{x-2}{4}\right)$. **Answer:** $\ln(x^2 - 4x + 20) + \frac{7}{4}\arctan\left(\frac{x-2}{4}\right) + C$

1.19 Problem 19

Evaluate the integral: $\int x\sqrt{x+1}\,dx$

Solution

Use the substitution $u = x + 1$. This implies $x = u - 1$ and $dx = du$. The integral transforms to $\int(u-1)\sqrt{u}\,du = \int(u^{3/2} - u^{1/2})\,du$. Integrating term by term: $\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + C$. Substitute back for $u = x + 1$. **Answer:** $\frac{2}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2} + C$

1.20 Problem 20

Evaluate the integral: $\int \frac{1}{e^x+e^{-x}}\,dx$

Solution

Multiply the numerator and denominator by e^x to simplify the expression. $\int \frac{e^x}{(e^x)(e^x+e^{-x})}\,dx = \int \frac{e^x}{e^{2x}+1}\,dx$. Now, perform a u-substitution. Let $u = e^x$, so $du = e^x\,dx$. The integral becomes $\int \frac{1}{u^2+1}\,du$, which is the standard form for arctangent. **Answer:** $\arctan(e^x) + C$

1.21 Problem 21

Evaluate the integral: $\int \frac{1}{\sqrt{x}+\sqrt[3]{x}}\,dx$

Solution

To handle the different roots, substitute using the least common multiple of the denominators of the fractional exponents (2 and 3), which is 6. Let $u = x^{1/6}$. Then $x = u^6$ and $dx = 6u^5\,du$. Also, $\sqrt{x} = u^3$ and $\sqrt[3]{x} = u^2$. The integral becomes $\int \frac{1}{u^3+u^2}(6u^5)\,du = \int \frac{6u^5}{u^2(u+1)}\,du = 6\int \frac{u^3}{u+1}\,du$. By polynomial long division, this is $6\int \left(u^2 - u + 1 - \frac{1}{u+1}\right)du$. Integrating yields $6\left(\frac{u^3}{3} - \frac{u^2}{2} + u - \ln|u+1|\right) + C$. Substitute back $u = x^{1/6}$. **Answer:** $2\sqrt{x} - 3\sqrt[3]{x} + 6\sqrt[6]{x} - 6\ln(x^{1/6}+1) + C$

1.22 Problem 22

Evaluate the integral: $\int \frac{x^3}{(x^2+1)^2}\,dx$

Solution

Rewrite the integrand as $\int \frac{x^2 \cdot x}{(x^2+1)^2}\,dx$. Let $u = x^2 + 1$. Then $du = 2x\,dx$, so $x\,dx = \frac{1}{2}du$. Also, $x^2 = u - 1$. Substituting gives $\int \frac{u-1}{u^2} \cdot \frac{1}{2}du = \frac{1}{2}\int \left(\frac{1}{u} - \frac{1}{u^2}\right)du$. Integrating gives $\frac{1}{2}\left(\ln|u| + \frac{1}{u}\right) + C$. Substitute back $u = x^2 + 1$. **Answer:** $\frac{1}{2}\ln(x^2+1) + \frac{1}{2(x^2+1)} + C$

1.23 Problem 23

Evaluate the definite integral: $\int_0^7 \frac{x}{\sqrt{x+1}}\,dx$

Solution

Let $u = x + 1$. Then $x = u - 1$ and $dx = du$. Change the bounds: when $x = 0$, $u = 1$; when $x = 7$, $u = 8$. The integral becomes $\int_1^8 \frac{u-1}{\sqrt{u}}\,du = \int_1^8 (u^{1/2} - u^{-1/2})\,du$. The antiderivative is $\left[\frac{2}{3}u^{3/2} - 2u^{1/2}\right]_1^8$. Evaluate at the bounds: $\left(\frac{2}{3}8^{3/2} - 2\cdot 8^{1/2}\right) - \left(\frac{2}{3} - 2\right) = \left(\frac{32\sqrt{2}}{3} - 4\sqrt{2}\right) - \left(-\frac{4}{3}\right)$. **Answer:** $\frac{20\sqrt{2}+4}{3}$

1.24 Problem 24

Evaluate the integral: $\int \frac{e^x}{9+e^{2x}}\,dx$

Solution

This integral is in a form that leads to arctangent. Rewrite as $\int \frac{e^x}{3^2+(e^x)^2}\,dx$. Let $u = e^x$, so $du = e^x\,dx$. The integral becomes $\int \frac{1}{3^2+u^2}\,du$. **Answer:** $\frac{1}{3}\arctan\left(\frac{e^x}{3}\right) + C$

1.25 Problem 25

Evaluate the integral: $\int \frac{e^{2x}}{\sqrt{1-e^{4x}}}\,dx$

Solution

This integral is in a form that leads to arcsine. Rewrite as $\int \frac{e^{2x}}{\sqrt{1^2-(e^{2x})^2}}\,dx$. Let $u = e^{2x}$, so $du = 2e^{2x}\,dx$, which means $e^{2x}\,dx = \frac{1}{2}du$. The integral becomes $\frac{1}{2}\int \frac{1}{\sqrt{1-u^2}}\,du$. **Answer:** $\frac{1}{2}\arcsin(e^{2x}) + C$

1.26 Problem 26

Evaluate the integral: $\int \frac{e^x+4}{e^{2x}+2e^x+5}\,dx$

Solution

Split the integral: $\int \frac{e^x}{e^{2x}+2e^x+5}\,dx + \int \frac{4}{e^{2x}+2e^x+5}\,dx$. Complete the square on the denominator: $(e^x+1)^2+4$. The first integral $I_1 = \int \frac{e^x}{(e^x+1)^2+4}\,dx$. Let $u = e^x+1, du = e^x dx$. This becomes $\int \frac{du}{u^2+4} = \frac{1}{2}\arctan(\frac{u}{2}) = \frac{1}{2}\arctan(\frac{e^x+1}{2})$. The second integral $I_2 = 4\int \frac{dx}{(e^x+1)^2+4}$. Let $u = e^x, dx = du/u$. This is $4\int \frac{du}{u((u+1)^2+4)} = 4\int \frac{du}{u(u^2+2u+5)}$. Use partial fractions: $\frac{4}{u(u^2+2u+5)} = \frac{4}{5u} - \frac{4u+8}{5(u^2+2u+5)}$. This integrates to $\frac{4}{5}\ln|u| - \frac{2}{5}\ln(u^2+2u+5) - \frac{2}{5}\arctan(\frac{u+1}{2})$. Substituting back $u = e^x$ and combining with I_1 gives the final answer. **Answer:** $\frac{1}{10}\arctan\left(\frac{e^x+1}{2}\right) - \frac{2}{5}\ln(e^{2x}+2e^x+5) + \frac{4}{5}x + C$

1.27 Problem 27

Evaluate the integral: $\int \frac{e^x}{e^x\sqrt{4e^{2x}-1}}\,dx$

Solution

First, cancel the e^x terms: $\int \frac{1}{\sqrt{4e^{2x}-1}}\,dx$. To transform this into a standard form, let $u = 2e^x$. Then $du = 2e^x dx = u\,dx$, so $dx = \frac{du}{u}$. The integral becomes $\int \frac{1}{\sqrt{u^2-1}} \cdot \frac{du}{u} = \int \frac{1}{u\sqrt{u^2-1}}\,du$. This is the standard form for the arcsecant function. **Answer:** $\operatorname{arcsec}(2e^x) + C$

1.28 Problem 28

Evaluate the definite integral: $\int_0^{\ln(2)} \frac{1}{\sqrt{4-e^{2x}}}\,dx$

Solution

Let $u = e^x$. Then $dx = du/u$. The bounds become $u = 1$ and $u = 2$. The integral is $\int_1^2 \frac{1}{\sqrt{4-u^2}}\frac{du}{u} = \int_1^2 \frac{du}{u\sqrt{4-u^2}}$. Use a trigonometric substitution. Let $u = 2\sin\theta$, so $du = 2\cos\theta d\theta$. The integral becomes $\int \frac{2\cos\theta d\theta}{2\sin\theta\sqrt{4-4\sin^2\theta}} = \int \frac{\cos\theta d\theta}{2\sin\theta\cos\theta} = \frac{1}{2}\int \csc\theta d\theta$. The antiderivative is $-\frac{1}{2}\ln|\csc\theta + \cot\theta|$. Convert back to u: $\sin\theta = u/2$, $\csc\theta = 2/u$, $\cot\theta = \sqrt{4-u^2}/u$. Antiderivative in terms of u: $-\frac{1}{2}\ln\left|\frac{2+\sqrt{4-u^2}}{u}\right|$. Evaluate from $u = 1$ to $u = 2$: $\left[-\frac{1}{2}\ln\left|\frac{2+\sqrt{4-u^2}}{u}\right|\right]_1^2 = -\frac{1}{2}(\ln(1) - \ln(2+\sqrt{3}))$. **Answer:** $\frac{1}{2}\ln(2+\sqrt{3})$

1.29 Problem 29

Find the area of the region enclosed by the curves: $y = 5x - x^2$ and $y = x$.

Solution

First, find the points of intersection by setting the equations equal to each other: $5x - x^2 = x \implies 4x - x^2 = 0 \implies x(4-x) = 0$. The intersection points are at $x = 0$ and $x = 4$. These are the limits of integration. In the interval $[0, 4]$, the curve $y = 5x - x^2$ is above the line $y = x$. The area A is given by the integral of the upper curve minus the lower curve:

$$A = \int_0^4 ((5x - x^2) - x)\,dx = \int_0^4 (4x - x^2)\,dx$$

Now, evaluate the integral:

$$A = \left[2x^2 - \frac{x^3}{3}\right]_0^4 = \left(2(4)^2 - \frac{4^3}{3}\right) - (0) = 32 - \frac{64}{3} = \frac{96-64}{3} = \frac{32}{3}$$

Answer: $\frac{32}{3}$

1.30 Problem 30

Find the area of the region enclosed by the curves $y = e^x$, $y = x^6$, from $x = 0$ to $x = 1$.

Solution

The limits of integration are given as $x = 0$ and $x = 1$. In this interval, $y = e^x$ is the upper curve and $y = x^6$ is the lower curve. Set up the integral for the area A:

$$A = \int_0^1 (e^x - x^6)\, dx$$

Evaluate the integral:

$$A = \left[e^x - \frac{x^7}{7}\right]_0^1 = \left(e^1 - \frac{1^7}{7}\right) - \left(e^0 - \frac{0^7}{7}\right) = \left(e - \frac{1}{7}\right) - (1 - 0) = e - \frac{8}{7}$$

Answer: $e - \frac{8}{7}$

1.31 Problem 31

Find the area of the region enclosed by the curves $x = y^2$, $x = y^2 - 5$, from $y = -1$ to $y = 1$.

Solution

The region is bounded by functions of y, so we integrate with respect to y. The limits are given as $y = -1$ and $y = 1$. The right boundary is $x_{\text{right}} = y^2$ and the left boundary is $x_{\text{left}} = y^2 - 5$. Set up the integral for the area A:

$$A = \int_{-1}^1 (x_{\text{right}} - x_{\text{left}})\, dy = \int_{-1}^1 (y^2 - (y^2 - 5))\, dy = \int_{-1}^1 5\, dy$$

Evaluate the integral:

$$A = [5y]_{-1}^1 = 5(1) - 5(-1) = 5 + 5 = 10$$

Answer: 10

1.32 Problem 32

Find the area of the region enclosed by the curves $x = 2y - y^2$ and $x = y^2 - 4y$.

Solution

First, find the points of intersection: $2y - y^2 = y^2 - 4y \implies 2y^2 - 6y = 0 \implies 2y(y-3) = 0$. The intersection points are at $y = 0$ and $y = 3$. In the interval $[0, 3]$, the curve $x = 2y - y^2$ is the right boundary. Set up the integral with respect to y:

$$A = \int_0^3 ((2y - y^2) - (y^2 - 4y))\, dy = \int_0^3 (6y - 2y^2)\, dy$$

Evaluate the integral:

$$A = \left[3y^2 - \frac{2y^3}{3}\right]_0^3 = \left(3(3)^2 - \frac{2(3)^3}{3}\right) - (0) = 27 - 18 = 9$$

Answer: 9

1.33 Problem 33

Find the area of the region enclosed by the curves $y = x^3 - 15x$ and $y = x$.

Solution

Find intersections: $x^3 - 15x = x \implies x^3 - 16x = 0 \implies x(x-4)(x+4) = 0$. Intersections are at $x = -4, 0, 4$. This defines two regions. Region 1 (from -4 to 0): The curve $y = x^3 - 15x$ is above $y = x$.

$$A_1 = \int_{-4}^{0} (x^3 - 16x)\, dx = \left[\frac{x^4}{4} - 8x^2\right]_{-4}^{0} = 0 - (64 - 128) = 64$$

Region 2 (from 0 to 4): The line $y = x$ is above $y = x^3 - 15x$.

$$A_2 = \int_{0}^{4} (16x - x^3)\, dx = \left[8x^2 - \frac{x^4}{4}\right]_{0}^{4} = (128 - 64) - 0 = 64$$

Total Area: $A = A_1 + A_2 = 64 + 64 = 128$. **Answer:** 128

1.34 Problem 34

Find the area of the region enclosed by the curves $y = x^2$, $y = \frac{2}{3}x + \frac{16}{3}$, and $y = 8 - 2x$.

Solution

The region must be split into two parts. The intersection points are at $x = -2, 1, 2$. Region 1 (from -2 to 1): The upper boundary is $y = \frac{2}{3}x + \frac{16}{3}$ and the lower is $y = x^2$.

$$A_1 = \int_{-2}^{1} \left(\frac{2}{3}x + \frac{16}{3} - x^2\right) dx = \left[\frac{x^2}{3} + \frac{16x}{3} - \frac{x^3}{3}\right]_{-2}^{1} = \left(\frac{16}{3}\right) - \left(-\frac{20}{3}\right) = 12$$

Region 2 (from 1 to 2): The upper boundary is $y = 8 - 2x$ and the lower is $y = x^2$.

$$A_2 = \int_{1}^{2} (8 - 2x - x^2)\, dx = \left[8x - x^2 - \frac{x^3}{3}\right]_{1}^{2} = \left(12 - \frac{8}{3}\right) - \left(7 - \frac{1}{3}\right) = \frac{8}{3}$$

Total Area: $A = A_1 + A_2 = 12 + \frac{8}{3} = \frac{44}{3}$. **Answer:** $\frac{44}{3}$

1.35 Problem 35

Set up an integral representing the area A of the region enclosed by the curves $x = y^4$ and $x = 2 - y^2$.

Solution

Find intersections: $y^4 = 2 - y^2 \implies y^4 + y^2 - 2 = 0 \implies (y^2 + 2)(y^2 - 1) = 0$. The real solutions are $y = \pm 1$, which are the limits of integration. In the interval $[-1, 1]$, the curve $x = 2 - y^2$ is the right boundary. The integral for the area A is:

$$A = \int_{-1}^{1} ((2 - y^2) - y^4)\, dy = \int_{-1}^{1} (2 - y^2 - y^4)\, dy$$

Answer: $A = \int_{-1}^{1} (2 - y^2 - y^4)\, dy$

1.36 Problem 36

Find the area of the region enclosed by the curves $y = 3 + x^3$, $y = 5 - x$, for $x = -1$ to $x = 0$.

Solution

The limits are given as $x = -1$ and $x = 0$. In this interval, the line $y = 5 - x$ is the upper boundary. Set up the integral for the area A:

$$A = \int_{-1}^{0} ((5 - x) - (3 + x^3))\, dx = \int_{-1}^{0} (2 - x - x^3)\, dx$$

Evaluate the integral:

$$A = \left[2x - \frac{x^2}{2} - \frac{x^4}{4}\right]_{-1}^{0} = (0) - \left(-2 - \frac{1}{2} - \frac{1}{4}\right) = \frac{11}{4}$$

Answer: $\frac{11}{4}$

1.37 Problem 37

Find the area of the region enclosed by the curves $y = 4\cos(x)$, $y = 4e^x$, and $x = \frac{\pi}{2}$.

Solution

The curves intersect when $4\cos(x) = 4e^x$, which occurs at $x = 0$. The limits of integration are $[0, \pi/2]$. In this interval, $y = 4e^x$ is the upper boundary.

$$A = \int_{0}^{\pi/2} (4e^x - 4\cos(x))\, dx = 4\int_{0}^{\pi/2} (e^x - \cos(x))\, dx$$

$$A = 4\left[e^x - \sin(x)\right]_0^{\pi/2} = 4\left((e^{\pi/2} - \sin(\pi/2)) - (e^0 - \sin(0))\right) = 4(e^{\pi/2} - 1 - 1)$$

Answer: $4e^{\pi/2} - 8$

1.38 Problem 38

Find the area of the region enclosed by the curves $y = x^2 - 4x$ and $y = 4x$.

Solution

Find intersections: $x^2 - 4x = 4x \implies x^2 - 8x = 0 \implies x(x - 8) = 0$. The limits are $x = 0$ and $x = 8$. In the interval $[0, 8]$, the line $y = 4x$ is the upper boundary.

$$A = \int_{0}^{8} (4x - (x^2 - 4x))\, dx = \int_{0}^{8} (8x - x^2)\, dx$$

$$A = \left[4x^2 - \frac{x^3}{3}\right]_0^8 = \left(4(8)^2 - \frac{8^3}{3}\right) - 0 = 256 - \frac{512}{3} = \frac{768 - 512}{3} = \frac{256}{3}$$

Answer: $\frac{256}{3}$

1.39 Problem 39

Find the area of the region enclosed by the curves $x = 4 - y^2$ and $x = y^2 - 4$.

Solution

Find intersections: $4 - y^2 = y^2 - 4 \implies 8 = 2y^2 \implies y^2 = 4$. The limits are $y = -2$ and $y = 2$. The right boundary is $x = 4 - y^2$.

$$A = \int_{-2}^{2} ((4 - y^2) - (y^2 - 4))\, dy = \int_{-2}^{2} (8 - 2y^2)\, dy$$

$$A = \left[8y - \frac{2y^3}{3}\right]_{-2}^{2} = \left(16 - \frac{16}{3}\right) - \left(-16 + \frac{16}{3}\right) = 32 - \frac{32}{3} = \frac{64}{3}$$

Answer: $\frac{64}{3}$

1.40 Problem 40

Find the area of the region enclosed by the curves $2x + y^2 = 8$ and $x = y$.

Solution

Solve for x: $x = 4 - \frac{1}{2}y^2$. Find intersections: $y = 4 - \frac{1}{2}y^2 \implies y^2 + 2y - 8 = 0 \implies (y+4)(y-2) = 0$. The limits are $y = -4$ and $y = 2$. The right boundary is $x = 4 - \frac{1}{2}y^2$.

$$A = \int_{-4}^{2} \left(\left(4 - \frac{1}{2}y^2\right) - y \right) dy$$

$$A = \left[4y - \frac{y^2}{2} - \frac{y^3}{6}\right]_{-4}^{2} = \left(8 - 2 - \frac{8}{6}\right) - \left(-16 - 8 + \frac{64}{6}\right) = \left(6 - \frac{4}{3}\right) - \left(-24 + \frac{32}{3}\right) = \frac{14}{3} - \left(-\frac{40}{3}\right) = \frac{54}{3}$$

Answer: 18

1.41 Problem 41

Find the area of the region enclosed by the curves $x = 8y^2$ and $x = 28 + y^2$.

Solution

Find intersections: $8y^2 = 28 + y^2 \implies 7y^2 = 28 \implies y^2 = 4$. The limits are $y = -2$ and $y = 2$. The right boundary is $x = 28 + y^2$.

$$A = \int_{-2}^{2} ((28 + y^2) - 8y^2)\, dy = \int_{-2}^{2} (28 - 7y^2)\, dy$$

$$A = \left[28y - \frac{7y^3}{3}\right]_{-2}^{2} = \left(56 - \frac{56}{3}\right) - \left(-56 + \frac{56}{3}\right) = 112 - \frac{112}{3} = \frac{224}{3}$$

Answer: $\frac{224}{3}$

1.42 Problem 42

Find the area of the region enclosed by the curves $x = y^2 - 5$ and $x = e^y$, from $y = -1$ to $y = 1$.

Solution

The limits are given as $y = -1$ and $y = 1$. The right boundary is $x = e^y$.

$$A = \int_{-1}^{1} (e^y - (y^2 - 5))\, dy = \int_{-1}^{1} (e^y - y^2 + 5)\, dy$$

$$A = \left[e^y - \frac{y^3}{3} + 5y\right]_{-1}^{1} = \left(e - \frac{1}{3} + 5\right) - \left(e^{-1} + \frac{1}{3} - 5\right) = e - e^{-1} - \frac{2}{3} + 10 = e - \frac{1}{e} + \frac{28}{3}$$

Answer: $e - \frac{1}{e} + \frac{28}{3}$

1.43 Problem 43

Find the area of the region enclosed by the curves $y = \sqrt{x}$ and $y = \frac{1}{5}x$, for $0 \le x \le 36$.

Solution

Find intersection: $\sqrt{x} = \frac{1}{5}x \implies x = \frac{1}{25}x^2 \implies x(x-25) = 0$. Intersection is at $x = 25$. The region is split. Region 1 ($0 \le x \le 25$): $y = \sqrt{x}$ is upper. $A_1 = \int_0^{25}(\sqrt{x} - \frac{1}{5}x)\,dx = [\frac{2}{3}x^{3/2} - \frac{x^2}{10}]_0^{25} = \frac{250}{3} - \frac{625}{10} = \frac{125}{6}$. Region 2 ($25 \le x \le 36$): $y = \frac{1}{5}x$ is upper. $A_2 = \int_{25}^{36}(\frac{1}{5}x - \sqrt{x})\,dx = [\frac{x^2}{10} - \frac{2}{3}x^{3/2}]_{25}^{36} = (\frac{1296}{10} - \frac{2}{3}(216)) - (\frac{625}{10} - \frac{250}{3}) = (129.6 - 144) - (62.5 - \frac{250}{3}) = -14.4 - (-\frac{125}{6}) = \frac{193}{30}$. Total Area: $A = \frac{125}{6} + \frac{193}{30} = \frac{625+193}{30} = \frac{818}{30} = \frac{409}{15}$. **Answer:** $\frac{409}{15}$

1.44 Problem 44

Find the area of the region enclosed by the curves $y = \cos(x)$ and $y = 2 - \cos(x)$, for $0 \le x \le 2\pi$.

Solution

In the interval $[0, 2\pi]$, the curve $y = 2 - \cos(x)$ is always above $y = \cos(x)$. They touch at $x = 0$ and $x = 2\pi$.

$$A = \int_0^{2\pi} ((2 - \cos(x)) - \cos(x))\,dx = \int_0^{2\pi} (2 - 2\cos(x))\,dx$$

$$A = [2x - 2\sin(x)]_0^{2\pi} = (4\pi - 2\sin(2\pi)) - (0 - 2\sin(0)) = 4\pi$$

Answer: 4π

1.45 Problem 45

Find the area of the region enclosed by the curves $y = \cos(x)$ and $y = \sin(2x)$, for $0 \le x \le \frac{\pi}{2}$.

Solution

Find intersection: $\cos(x) = \sin(2x) = 2\sin(x)\cos(x) \implies \cos(x)(1 - 2\sin(x)) = 0$. Intersections at $x = \pi/6$ and $x = \pi/2$. Region 1 ($0 \le x \le \pi/6$): $y = \cos(x)$ is upper. $A_1 = \int_0^{\pi/6}(\cos(x) - \sin(2x))\,dx = [\sin(x) + \frac{1}{2}\cos(2x)]_0^{\pi/6} = (\frac{1}{2} + \frac{1}{4}) - (\frac{1}{2}) = \frac{1}{4}$. Region 2 ($\pi/6 \le x \le \pi/2$): $y = \sin(2x)$ is upper. $A_2 = \int_{\pi/6}^{\pi/2}(\sin(2x) - \cos(x))\,dx = [-\frac{1}{2}\cos(2x) - \sin(x)]_{\pi/6}^{\pi/2} = (\frac{1}{2} - 1) - (-\frac{1}{4} - \frac{1}{2}) = -\frac{1}{2} - (-\frac{3}{4}) = \frac{1}{4}$. Total Area: $A = A_1 + A_2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$. **Answer:** $\frac{1}{2}$

1.46 Problem 46

Evaluate the integral. (Remember the constant of integration.)

$$\int 2\sin^2(x)\cos^3(x)\,dx$$

Solution

We have an odd power of cosine, so we save one cosine factor and convert the rest to sines using $\cos^2(x) = 1 - \sin^2(x)$.

$$\int 2\sin^2(x)\cos^2(x)\cos(x)\,dx = \int 2\sin^2(x)(1 - \sin^2(x))\cos(x)\,dx$$

Let $u = \sin(x)$, so $du = \cos(x)\,dx$. The integral becomes:

$$\int 2u^2(1 - u^2)\,du = \int (2u^2 - 2u^4)\,du$$

$$= \frac{2}{3}u^3 - \frac{2}{5}u^5 + C$$

Substituting back $u = \sin(x)$:

$$= \frac{2}{3}\sin^3(x) - \frac{2}{5}\sin^5(x) + C$$

Answer: $\frac{2}{3}\sin^3(x) - \frac{2}{5}\sin^5(x) + C$

1.47 Problem 47

Evaluate the integral. (Remember the constant of integration.)

$$\int \sin^3(y)\cos^4(y)\,dy$$

Solution

We have an odd power of sine, so we save one sine factor and convert the rest to cosines using $\sin^2(y) = 1 - \cos^2(y)$.

$$\int \sin^2(y)\cos^4(y)\sin(y)\,dy = \int (1-\cos^2(y))\cos^4(y)\sin(y)\,dy$$

Let $u = \cos(y)$, so $du = -\sin(y)\,dy$. The integral becomes:

$$\int (1-u^2)u^4(-du) = -\int (u^4 - u^6)\,du$$

$$= -\left(\frac{u^5}{5} - \frac{u^7}{7}\right) + C = \frac{1}{7}u^7 - \frac{1}{5}u^5 + C$$

Substituting back $u = \cos(y)$:

$$= \frac{1}{7}\cos^7(y) - \frac{1}{5}\cos^5(y) + C$$

Answer: $\frac{1}{7}\cos^7(y) - \frac{1}{5}\cos^5(y) + C$

1.48 Problem 48

Evaluate the integral.

$$\int_0^{\pi/2} \cos^{13}(x)\sin^5(x)\,dx$$

Solution

The power of sine is odd. We save a $\sin(x)$ factor and convert the rest.

$$\int_0^{\pi/2} \cos^{13}(x)\sin^4(x)\sin(x)\,dx = \int_0^{\pi/2} \cos^{13}(x)(1-\cos^2(x))^2\sin(x)\,dx$$

Let $u = \cos(x)$, so $du = -\sin(x)\,dx$. The bounds change: $x = 0 \implies u = 1$, and $x = \pi/2 \implies u = 0$.

$$\int_1^0 u^{13}(1-u^2)^2(-du) = \int_0^1 u^{13}(1 - 2u^2 + u^4)\,du$$

$$= \int_0^1 (u^{13} - 2u^{15} + u^{17})\,du = \left[\frac{u^{14}}{14} - \frac{2u^{16}}{16} + \frac{u^{18}}{18}\right]_0^1$$

$$= \left(\frac{1}{14} - \frac{1}{8} + \frac{1}{18}\right) - (0) = \frac{36 - 63 + 28}{504} = \frac{1}{504}$$

Answer: $\frac{1}{504}$

1.49 Problem 49

Evaluate the integral.

$$\int_0^{\pi/2} 9\sin^2(x)\cos^2(x)\,dx$$

Solution

We use the identity $\sin(x)\cos(x) = \frac{1}{2}\sin(2x)$, and then $\sin^2(\theta) = \frac{1-\cos(2\theta)}{2}$.

$$\int_0^{\pi/2} 9(\sin(x)\cos(x))^2\,dx = \int_0^{\pi/2} 9\left(\frac{1}{2}\sin(2x)\right)^2 dx = \frac{9}{4}\int_0^{\pi/2} \sin^2(2x)\,dx$$

$$= \frac{9}{4}\int_0^{\pi/2} \frac{1-\cos(4x)}{2}\,dx = \frac{9}{8}\int_0^{\pi/2} (1-\cos(4x))\,dx$$

$$= \frac{9}{8}\left[x - \frac{1}{4}\sin(4x)\right]_0^{\pi/2} = \frac{9}{8}\left[(\frac{\pi}{2} - \frac{1}{4}\sin(2\pi)) - (0 - \frac{1}{4}\sin(0))\right]$$

$$= \frac{9}{8}\left(\frac{\pi}{2}\right) = \frac{9\pi}{16}$$

Answer: $\frac{9\pi}{16}$

1.50 Problem 50

Evaluate the integral.

$$\int_0^{\pi/2} 5\cos^2(\theta)\,d\theta$$

Solution

Using the half-angle identity $\cos^2(\theta) = \frac{1+\cos(2\theta)}{2}$.

$$\int_0^{\pi/2} 5\left(\frac{1+\cos(2\theta)}{2}\right) d\theta = \frac{5}{2}\int_0^{\pi/2} (1+\cos(2\theta))\,d\theta$$

$$= \frac{5}{2}\left[\theta + \frac{1}{2}\sin(2\theta)\right]_0^{\pi/2} = \frac{5}{2}\left[(\frac{\pi}{2} + \frac{1}{2}\sin(\pi)) - (0 + \frac{1}{2}\sin(0))\right]$$

$$= \frac{5}{2}\left(\frac{\pi}{2}\right) = \frac{5\pi}{4}$$

Answer: $\frac{5\pi}{4}$

1.51 Problem 51

Evaluate the integral.

$$\int \sqrt{\cos(\theta)}\sin^3(\theta)\,d\theta$$

Solution

The power of sine is odd. We save a $\sin(\theta)$ factor.

$$\int \sqrt{\cos(\theta)}\sin^2(\theta)\sin(\theta)\,d\theta = \int \sqrt{\cos(\theta)}(1-\cos^2(\theta))\sin(\theta)\,d\theta$$

Let $u = \cos(\theta)$, so $du = -\sin(\theta)\,d\theta$.

$$\int \sqrt{u}(1-u^2)(-du) = -\int (u^{1/2} - u^{5/2})\,du$$

$$= -\left(\frac{u^{3/2}}{3/2} - \frac{u^{7/2}}{7/2}\right) + C = -\frac{2}{3}u^{3/2} + \frac{2}{7}u^{7/2} + C$$

$$= \frac{2}{7}\cos^{7/2}(\theta) - \frac{2}{3}\cos^{3/2}(\theta) + C$$

Answer: $\frac{2}{7}\cos^{7/2}(\theta) - \frac{2}{3}\cos^{3/2}(\theta) + C$

1.52 Problem 52

Evaluate the integral.

$$\int \sin(3x)\sec^5(3x)\,dx$$

Solution

Rewrite $\sec(3x)$ as $1/\cos(3x)$.

$$\int \sin(3x)\frac{1}{\cos^5(3x)}\,dx = \int \frac{\sin(3x)}{\cos^5(3x)}\,dx$$

Let $u = \cos(3x)$, so $du = -3\sin(3x)\,dx$, which means $\sin(3x)dx = -du/3$.

$$\int \frac{1}{u^5}\left(-\frac{du}{3}\right) = -\frac{1}{3}\int u^{-5}\,du$$

$$= -\frac{1}{3}\frac{u^{-4}}{-4} + C = \frac{1}{12}u^{-4} + C = \frac{1}{12\cos^4(3x)} + C$$

$$= \frac{1}{12}\sec^4(3x) + C$$

Answer: $\frac{1}{12}\sec^4(3x) + C$

1.53 Problem 53

Evaluate the integral.

$$\int 4\tan(x)\sec^3(x)\,dx$$

Solution

We can rewrite the integrand to isolate a $\sec(x)\tan(x)$ factor.

$$4\int \sec^2(x)(\sec(x)\tan(x))\,dx$$

Let $u = \sec(x)$, so $du = \sec(x)\tan(x)\,dx$.

$$4\int u^2\,du = 4\left(\frac{u^3}{3}\right) + C = \frac{4}{3}u^3 + C$$

Substituting back $u = \sec(x)$:

$$= \frac{4}{3}\sec^3(x) + C$$

Answer: $\frac{4}{3}\sec^3(x) + C$

1.54 Problem 54

Evaluate the integral.

$$\int 5\tan^2(x)\,dx$$

Solution

Using the identity $\tan^2(x) = \sec^2(x) - 1$.

$$5\int (\sec^2(x) - 1)\,dx = 5(\int \sec^2(x)\,dx - \int 1\,dx)$$

$$= 5(\tan(x) - x) + C = 5\tan(x) - 5x + C$$

Answer: $5\tan(x) - 5x + C$

1.55 Problem 55

Evaluate the integral.

$$\int 11\tan^4(x)\sec^6(x)\,dx$$

Solution

The power of secant is even. We save a $\sec^2(x)$ factor and convert the rest to tangents using $\sec^2(x) = 1 + \tan^2(x)$.

$$11\int \tan^4(x)\sec^4(x)\sec^2(x)\,dx = 11\int \tan^4(x)(1 + \tan^2(x))^2\sec^2(x)\,dx$$

Let $u = \tan(x)$, so $du = \sec^2(x)\,dx$.

$$11\int u^4(1+u^2)^2\,du = 11\int u^4(1 + 2u^2 + u^4)\,du$$

$$= 11\int (u^4 + 2u^6 + u^8)\,du = 11\left(\frac{u^5}{5} + \frac{2u^7}{7} + \frac{u^9}{9}\right) + C$$

$$= \frac{11}{5}\tan^5(x) + \frac{22}{7}\tan^7(x) + \frac{11}{9}\tan^9(x) + C$$

Answer: $\frac{11}{9}\tan^9(x) + \frac{22}{7}\tan^7(x) + \frac{11}{5}\tan^5(x) + C$

1.56 Problem 56

Evaluate the integral.

$$\int \tan^3(x)\sec(x)\,dx$$

Solution

The power of tangent is odd. We save a $\sec(x)\tan(x)$ factor and convert the rest to secants.

$$\int \tan^2(x)(\sec(x)\tan(x))\,dx = \int (\sec^2(x)-1)(\sec(x)\tan(x))\,dx$$

Let $u = \sec(x)$, so $du = \sec(x)\tan(x)\,dx$.

$$\int (u^2-1)\,du = \frac{u^3}{3} - u + C$$

$$= \frac{1}{3}\sec^3(x) - \sec(x) + C$$

Answer: $\frac{1}{3}\sec^3(x) - \sec(x) + C$

1.57 Problem 57

Evaluate the integral.

$$\int \tan^3(x)\sec^6(x)\,dx$$

Solution

The power of tangent is odd, so we can let $u = \sec(x)$.

$$\int \tan^2(x)\sec^5(x)(\sec(x)\tan(x))\,dx = \int (\sec^2(x)-1)\sec^5(x)(\sec(x)\tan(x))\,dx$$

Let $u = \sec(x)$, so $du = \sec(x)\tan(x)\,dx$.

$$\int (u^2-1)u^5\,du = \int (u^7 - u^5)\,du$$

$$= \frac{u^8}{8} - \frac{u^6}{6} + C = \frac{1}{8}\sec^8(x) - \frac{1}{6}\sec^6(x) + C$$

Answer: $\frac{1}{8}\sec^8(x) - \frac{1}{6}\sec^6(x) + C$

1.58 Problem 58

Evaluate the integral.

$$\int_0^{\pi/6} \tan^4(t)\,dt$$

Solution

We reduce the power of tangent using $\tan^2(t) = \sec^2(t) - 1$.

$$\int_0^{\pi/6} \tan^2(t)(\sec^2(t)-1)\,dt = \int_0^{\pi/6} \tan^2(t)\sec^2(t)\,dt - \int_0^{\pi/6} \tan^2(t)\,dt$$

The first integral is $\int u^2 du$ with $u = \tan(t)$. The second integral becomes $\int(\sec^2(t)-1)dt$.

$$\left[\frac{1}{3}\tan^3(t)\right]_0^{\pi/6} - \int_0^{\pi/6} (\sec^2(t)-1)\,dt$$

$$= \left[\frac{1}{3}\tan^3(t)\right]_0^{\pi/6} - [\tan(t) - t]_0^{\pi/6} = \left[\frac{1}{3}\tan^3(t) - \tan(t) + t\right]_0^{\pi/6}$$

Since $\tan(\pi/6) = 1/\sqrt{3}$:

$$= \left(\frac{1}{3}\left(\frac{1}{\sqrt{3}}\right)^3 - \frac{1}{\sqrt{3}} + \frac{\pi}{6}\right) - (0) = \frac{1}{9\sqrt{3}} - \frac{1}{\sqrt{3}} + \frac{\pi}{6}$$

$$= \frac{1-9}{9\sqrt{3}} + \frac{\pi}{6} = -\frac{8}{9\sqrt{3}} + \frac{\pi}{6} = \frac{\pi}{6} - \frac{8\sqrt{3}}{27}$$

Answer: $\frac{\pi}{6} - \frac{8\sqrt{3}}{27}$

1.59 Problem 59

Evaluate the integral.

$$\int \tan^5(x)\,dx$$

Solution

We reduce the power of tangent.

$$\int \tan^3(x)\tan^2(x)\,dx = \int \tan^3(x)(\sec^2(x) - 1)\,dx$$

$$= \int \tan^3(x)\sec^2(x)\,dx - \int \tan^3(x)\,dx$$

The first integral is $\frac{1}{4}\tan^4(x)$. For the second integral:

$$\int \tan^3(x)\,dx = \int \tan(x)(\sec^2(x) - 1)\,dx = \int \tan(x)\sec^2(x)\,dx - \int \tan(x)\,dx$$

$$= \frac{1}{2}\tan^2(x) - \ln|\sec(x)|$$

Combining the results:

$$\frac{1}{4}\tan^4(x) - \left(\frac{1}{2}\tan^2(x) - \ln|\sec(x)|\right) + C$$

Answer: $\frac{1}{4}\tan^4(x) - \frac{1}{2}\tan^2(x) + \ln|\sec(x)| + C$

1.60 Problem 60

Evaluate the integral.

$$\int \frac{\tan(x)\sec^2(x)}{\cos(x)}\,dx$$

Solution

Since $1/\cos(x) = \sec(x)$, the integral becomes:

$$\int \tan(x)\sec^3(x)\,dx = \int \sec^2(x)(\sec(x)\tan(x))\,dx$$

Let $u = \sec(x)$, so $du = \sec(x)\tan(x)\,dx$.

$$\int u^2\,du = \frac{u^3}{3} + C$$

$$= \frac{1}{3}\sec^3(x) + C$$

Answer: $\frac{1}{3}\sec^3(x) + C$

1.61 Problem 61

Evaluate the integral.

$$\int_{\pi/6}^{\pi/2} 5\cot^2(x)\,dx$$

Solution

Using the identity $\cot^2(x) = \csc^2(x) - 1$.

$$5\int_{\pi/6}^{\pi/2} (\csc^2(x) - 1)\,dx = 5\left[-\cot(x) - x\right]_{\pi/6}^{\pi/2}$$

$$= 5\left[(-\cot(\frac{\pi}{2}) - \frac{\pi}{2}) - (-\cot(\frac{\pi}{6}) - \frac{\pi}{6})\right]$$

Since $\cot(\pi/2) = 0$ and $\cot(\pi/6) = \sqrt{3}$:

$$= 5\left[(0 - \frac{\pi}{2}) - (-\sqrt{3} - \frac{\pi}{6})\right] = 5\left[-\frac{\pi}{2} + \sqrt{3} + \frac{\pi}{6}\right]$$

$$= 5\left[\sqrt{3} - \frac{3\pi}{6} + \frac{\pi}{6}\right] = 5\left(\sqrt{3} - \frac{2\pi}{6}\right) = 5\left(\sqrt{3} - \frac{\pi}{3}\right)$$

Answer: $5\left(\sqrt{3} - \frac{\pi}{3}\right)$

1.62 Problem 62

Evaluate the integral.

$$\int \sin(8x)\cos(5x)\,dx$$

Solution

We use the product-to-sum identity $\sin(A)\cos(B) = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$.

$$\int \frac{1}{2}[\sin(8x+5x) + \sin(8x-5x)]\,dx = \frac{1}{2}\int (\sin(13x) + \sin(3x))\,dx$$

$$= \frac{1}{2}\left(-\frac{1}{13}\cos(13x) - \frac{1}{3}\cos(3x)\right) + C$$

$$= -\frac{1}{26}\cos(13x) - \frac{1}{6}\cos(3x) + C$$

Answer: $-\frac{1}{26}\cos(13x) - \frac{1}{6}\cos(3x) + C$

1.63 Problem 63

Evaluate the integral.

$$\int 5\tan^2(x)\sec(x)\,dx$$

Solution

Using the identity $\tan^2(x) = \sec^2(x) - 1$.

$$5\int (\sec^2(x) - 1)\sec(x)\,dx = 5\int (\sec^3(x) - \sec(x))\,dx$$

We use the standard integrals for $\sec^3(x)$ and $\sec(x)$.

$$\int \sec(x)\,dx = \ln|\sec(x) + \tan(x)| + C$$

$$\int \sec^3(x)\,dx = \frac{1}{2}(\sec(x)\tan(x) + \ln|\sec(x) + \tan(x)|) + C$$

So, the integral is:

$$5\left[\frac{1}{2}(\sec(x)\tan(x) + \ln|\sec(x) + \tan(x)|) - \ln|\sec(x) + \tan(x)|\right] + C$$

$$= 5\left[\frac{1}{2}\sec(x)\tan(x) - \frac{1}{2}\ln|\sec(x) + \tan(x)|\right] + C$$

Answer: $\frac{5}{2}(\sec(x)\tan(x) - \ln|\sec(x) + \tan(x)|) + C$

2 Summary of Rules, Formulas, and Tricks

This document covers a wide range of integration techniques, from basic substitutions to multi-stage problems requiring a sequence of tricks.

Category 1: U-Substitution

- **"Clever" First-Step Substitution:** Recognizing a non-obvious substitution that dramatically simplifies the integral from the start (e.g., Problem 1).
- **Standard Second-Step Substitution:** A more routine substitution that becomes necessary *after* an initial step like IBP has been applied.

Category 2: Integration by Parts (IBP)

- **The "Stealth dx" Trick:** Creating a product by setting $dv = dx$ to integrate single functions like $\ln(x)$ or $\arctan(x)$ (e.g., Problems 3, 4, 5, 6).
- **Looping / Circular Integration:** Applying IBP twice to problems like $\int e^x \sin(x)dx$ to solve for the original integral algebraically (e.g., Problem 2).
- **Standard IBP:** The direct application of the formula, often used as a key step in a larger problem (e.g., Problem 8 and the definite integral sequence).

Category 3: Algebraic & Trigonometric Manipulation

- **Trigonometric Identities:** Using identities like $\sin(2x) = 2\sin(x)\cos(x)$ to unlock the problem (e.g., Problems 9-14).
- **The "Add and Subtract" Trick:** A shortcut for polynomial long division to simplify rational functions where the numerator's degree is equal to the denominator's (e.g., Problems 5, 6, 8).
- **Completing the Square:** Rewriting a quadratic denominator to fit the standard arctangent form (e.g., Problem 7).

Category 4: Multi-Stage "Grand Challenge" Problems

- **The "Trig Identity → U-Sub → IBP" Sequence:** A powerful pattern for solving complex definite integrals (e.g., Problems 9-14).
- **The "IBP → Algebraic Trick → Standard Form" Sequence:** A common pattern for integrating logarithmic and inverse trig functions (e.g., Problems 5, 6, 8).

Category 5: Definite Integral Skills

- **Changing Limits of Integration:** A mandatory step when performing u-substitution on a definite integral to avoid having to substitute back.
- **Flipping Limits of Integration:** Using the property $\int_a^b f(x)dx = -\int_b^a f(x)dx$ to cancel negative signs and simplify calculations.
- **The F(b) - F(a) Evaluation:** The final, crucial arithmetic step of the Fundamental Theorem of Calculus.
- **Understanding Function Properties (Even/Odd):** Using symmetry, like $\cos(-x) = \cos(x)$, to simplify the final evaluation (e.g., Problem 14).

Untouched Tricks & Problem Types

The following are integration strategies that were identified but not yet practiced.

- **The DI Method (Tabular Method):** A shortcut for repeated IBP.
- **Reduction Formulas:** Using IBP to create a formula relating an integral to a simpler version of itself.
- **Partial Fraction Decomposition:** The main algebraic technique for integrating more complex rational functions.
- **The "King Property" Connection:** A specific trick for definite integrals.
- **Feynman's Trick (Differentiation Under the Integral Sign):** A powerful method where you introduce a parameter into the integral, differentiate with respect to that parameter, and then integrate back.
- **Weierstrass Substitution (Tangent Half-Angle Substitution):** A substitution that can convert any rational function of trigonometric functions into an algebraic rational function, which can then be solved using partial fractions.
- **The "King Property" Connection:** A specific trick for definite integrals.
- **Contour Integration and the Residue Theorem:**These are techniques from complex analysis that can be used to solve very difficult real-valued definite integrals.
- **Meijer G-Function:** A highly generalized function that can represent most elementary and many special functions. There are rules for finding the antiderivative of a Meijer G-function, making it a powerful, though complex, symbolic integration method.