

Calculus II - Test 2 Review Solutions

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1. Evaluate the convergence and divergence of the following integrals

a. **Integral:** $\int_2^\infty \frac{2}{x^2 - x} dx$

Explanation: This is an improper integral because the upper limit of integration is infinite. We first use partial fraction decomposition on the integrand.

$$\begin{aligned}\frac{2}{x^2 - x} &= \frac{2}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1} \\ 2 &= A(x-1) + Bx \\ \text{Let } x = 0 : 2 &= A(-1) \implies A = -2 \\ \text{Let } x = 1 : 2 &= B(1) \implies B = 2 \\ \text{So, } \frac{2}{x(x-1)} &= \frac{2}{x-1} - \frac{2}{x}\end{aligned}$$

Now we evaluate the integral as a limit:

$$\begin{aligned}\int_2^\infty \left(\frac{2}{x-1} - \frac{2}{x} \right) dx &= \lim_{b \rightarrow \infty} \int_2^b \left(\frac{2}{x-1} - \frac{2}{x} \right) dx \\ &= \lim_{b \rightarrow \infty} [2 \ln|x-1| - 2 \ln|x|]_2^b \\ &= \lim_{b \rightarrow \infty} \left[2 \ln \left| \frac{x-1}{x} \right| \right]_2^b \\ &= \lim_{b \rightarrow \infty} \left(2 \ln \left| \frac{b-1}{b} \right| - 2 \ln \left| \frac{2-1}{2} \right| \right) \\ &= 2 \ln(1) - 2 \ln \left(\frac{1}{2} \right) = 0 - 2(-\ln 2) = 2 \ln 2\end{aligned}$$

Result: The integral **converges** to $2 \ln 2$.

b. **Integral:** $\int_0^1 \frac{x+1}{\sqrt{x^2+2x}} dx$

Explanation: This integral is improper because the integrand has a discontinuity at $x = 0$. We use a u-substitution. Let $u = x^2 + 2x$, so $du = (2x+2)dx = 2(x+1)dx$.

$$\begin{aligned}\int \frac{x+1}{\sqrt{x^2+2x}} dx &= \int \frac{1}{\sqrt{u}} \cdot \frac{du}{2} = \frac{1}{2} \int u^{-1/2} du \\ &= \frac{1}{2} \cdot 2u^{1/2} = \sqrt{u} = \sqrt{x^2+2x}\end{aligned}$$

Now evaluate the limit:

$$\begin{aligned}
 \int_0^1 \frac{x+1}{\sqrt{x^2+2x}} dx &= \lim_{a \rightarrow 0^+} \int_a^1 \frac{x+1}{\sqrt{x^2+2x}} dx \\
 &= \lim_{a \rightarrow 0^+} \left[\sqrt{x^2+2x} \right]_a^1 \\
 &= \lim_{a \rightarrow 0^+} \left(\sqrt{1^2+2(1)} - \sqrt{a^2+2a} \right) \\
 &= \sqrt{3} - 0 = \sqrt{3}
 \end{aligned}$$

Result: The integral **converges** to $\sqrt{3}$.

c. **Integral:** $\int_0^{\pi/2} \tan \theta d\theta$

Explanation: This is improper because $\tan \theta$ has a vertical asymptote at $\theta = \pi/2$.

$$\begin{aligned}
 \int_0^{\pi/2} \tan \theta d\theta &= \lim_{b \rightarrow (\pi/2)^-} \int_0^b \tan \theta d\theta \\
 &= \lim_{b \rightarrow (\pi/2)^-} [-\ln |\cos \theta|]_0^b \\
 &= \lim_{b \rightarrow (\pi/2)^-} (-\ln |\cos b| - (-\ln |\cos 0|)) \\
 &= \lim_{b \rightarrow (\pi/2)^-} (-\ln |\cos b|) + \ln(1)
 \end{aligned}$$

As $b \rightarrow (\pi/2)^-$, $\cos b \rightarrow 0^+$, so $-\ln |\cos b| \rightarrow \infty$. **Result:** The integral **diverges**.

d. **Integral:** $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

Explanation: Improper due to a discontinuity at $x = 1$. This is a standard integral form.

$$\begin{aligned}
 \int_0^1 \frac{dx}{\sqrt{1-x^2}} &= \lim_{b \rightarrow 1^-} \int_0^b \frac{dx}{\sqrt{1-x^2}} \\
 &= \lim_{b \rightarrow 1^-} [\arcsin(x)]_0^b \\
 &= \lim_{b \rightarrow 1^-} (\arcsin(b) - \arcsin(0)) \\
 &= \arcsin(1) - 0 = \frac{\pi}{2}
 \end{aligned}$$

Result: The integral **converges** to $\frac{\pi}{2}$.

e. **Integral:** $\int_{-\infty}^{\infty} \frac{2x}{(x^2+1)^2} dx$

Explanation: This is improper because of the infinite limits. We split it at $x = 0$.

$$\int_{-\infty}^{\infty} \frac{2x}{(x^2+1)^2} dx = \int_{-\infty}^0 \frac{2x}{(x^2+1)^2} dx + \int_0^{\infty} \frac{2x}{(x^2+1)^2} dx$$

Let $u = x^2 + 1$, $du = 2x dx$. The antiderivative is $\int \frac{1}{u^2} du = -\frac{1}{u} = -\frac{1}{x^2+1}$.

$$\text{Part 1: } \lim_{a \rightarrow -\infty} \left[-\frac{1}{x^2+1} \right]_a^0 = \left(-\frac{1}{0^2+1} \right) - \lim_{a \rightarrow -\infty} \left(-\frac{1}{a^2+1} \right) = -1 - 0 = -1$$

$$\text{Part 2: } \lim_{b \rightarrow \infty} \left[-\frac{1}{x^2+1} \right]_0^b = \lim_{b \rightarrow \infty} \left(-\frac{1}{b^2+1} \right) - \left(-\frac{1}{0^2+1} \right) = 0 - (-1) = 1$$

The total value is $-1 + 1 = 0$. **Result:** The integral **converges** to 0.

2. Find the length of the curves

Arc Length Formula: $L = \int_a^b \sqrt{1 + (y')^2} dx$

a. **Curve:** $y = \frac{1}{3}(x^2 + 2)^{3/2}$ from $x = 0$ to $x = 3$

$$\begin{aligned} y' &= \frac{1}{3} \cdot \frac{3}{2}(x^2 + 2)^{1/2} \cdot 2x = x\sqrt{x^2 + 2} \\ 1 + (y')^2 &= 1 + (x\sqrt{x^2 + 2})^2 = 1 + x^2(x^2 + 2) \\ &= 1 + x^4 + 2x^2 = (x^2 + 1)^2 \\ L &= \int_0^3 \sqrt{(x^2 + 1)^2} dx = \int_0^3 (x^2 + 1) dx \\ &= \left[\frac{x^3}{3} + x \right]_0^3 = \left(\frac{27}{3} + 3 \right) - 0 = 9 + 3 = 12 \end{aligned}$$

Length: 12.

b. **Curve:** $y = \ln(x) - \frac{x^2}{8}$ from $x = 1$ to $x = 2$

$$\begin{aligned} y' &= \frac{1}{x} - \frac{2x}{8} = \frac{1}{x} - \frac{x}{4} \\ 1 + (y')^2 &= 1 + \left(\frac{1}{x} - \frac{x}{4} \right)^2 = 1 + \left(\frac{1}{x^2} - \frac{1}{2} + \frac{x^2}{16} \right) \\ &= \frac{1}{x^2} + \frac{1}{2} + \frac{x^2}{16} = \left(\frac{1}{x} + \frac{x}{4} \right)^2 \\ L &= \int_1^2 \sqrt{\left(\frac{1}{x} + \frac{x}{4} \right)^2} dx = \int_1^2 \left(\frac{1}{x} + \frac{x}{4} \right) dx \\ &= \left[\ln|x| + \frac{x^2}{8} \right]_1^2 = \left(\ln 2 + \frac{4}{8} \right) - \left(\ln 1 + \frac{1}{8} \right) = \ln 2 + \frac{1}{2} - \frac{1}{8} = \ln 2 + \frac{3}{8} \end{aligned}$$

Length: $\ln 2 + \frac{3}{8}$.

c. **Curve:** $y = \ln(\sec x)$ from $x = 0$ to $x = \pi/4$

$$\begin{aligned} y' &= \frac{1}{\sec x} \cdot (\sec x \tan x) = \tan x \\ 1 + (y')^2 &= 1 + \tan^2 x = \sec^2 x \\ L &= \int_0^{\pi/4} \sqrt{\sec^2 x} dx = \int_0^{\pi/4} \sec x dx \\ &= [\ln |\sec x + \tan x|]_0^{\pi/4} \\ &= \ln |\sec(\pi/4) + \tan(\pi/4)| - \ln |\sec(0) + \tan(0)| \\ &= \ln |\sqrt{2} + 1| - \ln |1 + 0| = \ln(\sqrt{2} + 1) \end{aligned}$$

Length: $\ln(\sqrt{2} + 1)$.

3. Calculate the surface area generated by revolving the curves about the indicated axis

Surface Area Formula (x -axis): $S = \int_a^b 2\pi y \sqrt{1 + (y')^2} dx$

a. **Curve:** $y = \sqrt{2x - x^2}$, $1 \leq x \leq 2$; about the x -axis

$$\begin{aligned} y' &= \frac{1}{2\sqrt{2x-x^2}} \cdot (2-2x) = \frac{1-x}{\sqrt{2x-x^2}} \\ 1 + (y')^2 &= 1 + \frac{(1-x)^2}{2x-x^2} = \frac{2x-x^2+(1-2x+x^2)}{2x-x^2} = \frac{1}{2x-x^2} \\ S &= \int_1^2 2\pi y \sqrt{1 + (y')^2} dx \\ &= \int_1^2 2\pi \sqrt{2x-x^2} \sqrt{\frac{1}{2x-x^2}} dx \\ &= \int_1^2 2\pi dx = [2\pi x]_1^2 = 2\pi(2) - 2\pi(1) = 2\pi \end{aligned}$$

Surface Area: 2π .

b. **Curve:** $y = \sqrt{1+e^x}$, $0 \leq x \leq 1$; about the x -axis

$$\begin{aligned} y' &= \frac{e^x}{2\sqrt{1+e^x}} \\ 1 + (y')^2 &= 1 + \frac{e^{2x}}{4(1+e^x)} = \frac{4(1+e^x) + e^{2x}}{4(1+e^x)} = \frac{4+4e^x+e^{2x}}{4(1+e^x)} = \frac{(e^x+2)^2}{4(1+e^x)} \\ S &= \int_0^1 2\pi \sqrt{1+e^x} \sqrt{\frac{(e^x+2)^2}{4(1+e^x)}} dx \\ &= \int_0^1 2\pi \sqrt{1+e^x} \frac{e^x+2}{2\sqrt{1+e^x}} dx \\ &= \pi \int_0^1 (e^x+2) dx = \pi [e^x+2x]_0^1 \\ &= \pi ((e^1+2)-(e^0+0)) = \pi(e+2-1) = \pi(e+1) \end{aligned}$$

Surface Area: $\pi(e+1)$.

4. Find and graph the Cartesian equation

- a. **Equations:** $x = 4 \cos t, y = 2 \sin t, 0 \leq t \leq 2\pi$

Eliminate the parameter: From $x = 4 \cos t \implies \cos t = x/4$. From $y = 2 \sin t \implies \sin t = y/2$. Using the identity $\cos^2 t + \sin^2 t = 1$:

$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{2}\right)^2 = 1 \implies \frac{x^2}{16} + \frac{y^2}{4} = 1$$

Path: This is an ellipse centered at $(0, 0)$ with a horizontal major axis of length 8 and a vertical minor axis of length 4.

Direction: At $t = 0$, $(x, y) = (4, 0)$. At $t = \pi/2$, $(x, y) = (0, 2)$. At $t = \pi$, $(x, y) = (-4, 0)$. The particle travels **counter-clockwise** starting from $(4, 0)$.

- b. **Equations:** $x = \sin t, y = \cos 2t, -\pi/2 \leq t \leq \pi/2$

Eliminate the parameter: Using the identity $\cos 2t = 1 - 2 \sin^2 t$:

$$y = 1 - 2x^2$$

Path: This is a parabola opening downwards with its vertex at $(0, 1)$.

Direction: At $t = -\pi/2$, $(x, y) = (-1, -1)$. At $t = 0$, $(x, y) = (0, 1)$. At $t = \pi/2$, $(x, y) = (1, -1)$. The particle moves from $(-1, -1)$ to $(1, -1)$ along the parabola.

- c. **Equations:** $x = 1 + \sin t, y = \cos t - 2, 0 \leq t \leq \pi$

Eliminate the parameter: From the equations, $\sin t = x - 1$ and $\cos t = y + 2$. Using $\sin^2 t + \cos^2 t = 1$:

$$(x - 1)^2 + (y + 2)^2 = 1$$

Path: This is a circle of radius 1 centered at $(1, -2)$.

Direction: At $t = 0$, $(x, y) = (1, -1)$. At $t = \pi/2$, $(x, y) = (2, -2)$. At $t = \pi$, $(x, y) = (1, -3)$. The particle traces the **top semi-circle** from right to left, starting at $(1, -1)$.

5. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ as a function of t

Formulas: $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ and $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(dy/dx)}{dx/dt}$

a. Equations: $x = t - t^2$, $y = t - t^3$

$$\begin{aligned}\frac{dx}{dt} &= 1 - 2t & \frac{dy}{dt} &= 1 - 3t^2 \\ \frac{dy}{dx} &= \frac{1 - 3t^2}{1 - 2t} \\ \frac{d}{dt} \left(\frac{dy}{dx} \right) &= \frac{-6t(1 - 2t) - (1 - 3t^2)(-2)}{(1 - 2t)^2} = \frac{-6t + 12t^2 + 2 - 6t^2}{(1 - 2t)^2} = \frac{6t^2 - 6t + 2}{(1 - 2t)^2} \\ \frac{d^2y}{dx^2} &= \frac{(6t^2 - 6t + 2)/(1 - 2t)^2}{1 - 2t} = \frac{2(3t^2 - 3t + 1)}{(1 - 2t)^3}\end{aligned}$$

b. Equations: $x = \frac{1}{t+1}$, $y = \frac{t}{t-1}$

$$\begin{aligned}\frac{dx}{dt} &= -(t + 1)^{-2} = \frac{-1}{(t + 1)^2} \\ \frac{dy}{dt} &= \frac{1(t - 1) - t(1)}{(t - 1)^2} = \frac{-1}{(t - 1)^2} \\ \frac{dy}{dx} &= \frac{-1/(t - 1)^2}{-1/(t + 1)^2} = \frac{(t + 1)^2}{(t - 1)^2} \\ \frac{d}{dt} \left(\frac{dy}{dx} \right) &= \frac{2(t + 1)(t - 1)^2 - (t + 1)^2(2(t - 1))}{(t - 1)^4} \\ &= \frac{2(t + 1)(t - 1)[(t - 1) - (t + 1)]}{(t - 1)^4} = \frac{2(t + 1)(-2)}{(t - 1)^3} = \frac{-4(t + 1)}{(t - 1)^3} \\ \frac{d^2y}{dx^2} &= \frac{-4(t + 1)/(t - 1)^3}{-1/(t + 1)^2} = \frac{4(t + 1)^3}{(t - 1)^3}\end{aligned}$$

6. Find an equation of the line tangent to the curve

a. **Curve:** $x = \sec t, y = \tan t$ at $t = \pi/4$

- **Point:** $x(\pi/4) = \sec(\pi/4) = \sqrt{2}, y(\pi/4) = \tan(\pi/4) = 1$. Point is $(\sqrt{2}, 1)$.
- **Slope:** $\frac{dx}{dt} = \sec t \tan t, \frac{dy}{dt} = \sec^2 t$. $\frac{dy}{dx} = \frac{\sec^2 t}{\sec t \tan t} = \frac{\sec t}{\tan t} = \csc t$. At $t = \pi/4$, slope $m = \csc(\pi/4) = \sqrt{2}$.
- **Equation:** $y - 1 = \sqrt{2}(x - \sqrt{2}) \implies y - 1 = \sqrt{2}x - 2 \implies y = \sqrt{2}x - 1$.

b. **Curve:** $x = t - \sin t, y = 1 - \cos t$ at $t = \pi/3$

- **Point:** $x(\pi/3) = \frac{\pi}{3} - \sin(\pi/3) = \frac{\pi}{3} - \frac{\sqrt{3}}{2}$. $y(\pi/3) = 1 - \cos(\pi/3) = 1 - \frac{1}{2} = \frac{1}{2}$. Point is $(\frac{\pi}{3} - \frac{\sqrt{3}}{2}, \frac{1}{2})$.
- **Slope:** $\frac{dx}{dt} = 1 - \cos t, \frac{dy}{dt} = \sin t$. $\frac{dy}{dx} = \frac{\sin t}{1 - \cos t}$. At $t = \pi/3$, slope $m = \frac{\sin(\pi/3)}{1 - \cos(\pi/3)} = \frac{\sqrt{3}/2}{1 - 1/2} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$.
- **Equation:** $y - \frac{1}{2} = \sqrt{3} \left(x - \left(\frac{\pi}{3} - \frac{\sqrt{3}}{2} \right) \right)$.

7. Find the lengths of the curves

Parametric Arc Length Formula: $L = \int_{t_1}^{t_2} \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} dt$

a. **Curve:** $x = \frac{t^2}{2}, y = \frac{1}{3}(2t+1)^{3/2}, 0 \leq t \leq 4$

$$\begin{aligned}\frac{dx}{dt} &= t & \frac{dy}{dt} &= \frac{1}{3} \cdot \frac{3}{2}(2t+1)^{1/2} \cdot 2 = \sqrt{2t+1} \\ \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 &= t^2 + (\sqrt{2t+1})^2 = t^2 + 2t + 1 = (t+1)^2 \\ L &= \int_0^4 \sqrt{(t+1)^2} dt = \int_0^4 (t+1) dt \\ &= \left[\frac{t^2}{2} + t \right]_0^4 = \left(\frac{16}{2} + 4 \right) - 0 = 8 + 4 = 12\end{aligned}$$

Length: 12.

b. **Curve:** $x = t^3, y = \frac{3t^2}{2}, 0 \leq t \leq \sqrt{3}$

$$\begin{aligned}\frac{dx}{dt} &= 3t^2 & \frac{dy}{dt} &= 3t \\ \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 &= (3t^2)^2 + (3t)^2 = 9t^4 + 9t^2 = 9t^2(t^2 + 1) \\ L &= \int_0^{\sqrt{3}} \sqrt{9t^2(t^2 + 1)} dt = \int_0^{\sqrt{3}} 3t\sqrt{t^2 + 1} dt\end{aligned}$$

Let $u = t^2 + 1, du = 2t dt \implies \frac{3}{2}du = 3t dt$. When $t = 0, u = 1$. When $t = \sqrt{3}, u = 4$.

$$\begin{aligned}L &= \int_1^4 \frac{3}{2}\sqrt{u} du = \frac{3}{2} \int_1^4 u^{1/2} du = \frac{3}{2} \left[\frac{2}{3}u^{3/2} \right]_1^4 \\ &= \left[u^{3/2} \right]_1^4 = 4^{3/2} - 1^{3/2} = 8 - 1 = 7\end{aligned}$$

Length: 7.

8. For what values of t does the curve have a vertical tangent?

Equations: $x = t^3 - t^2 - 1$, $y = t^4 + 2t^2 - 8t$

Explanation: A vertical tangent occurs when the slope $\frac{dy}{dx}$ is undefined. This happens when $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$.

1. Find when $\frac{dx}{dt} = 0$:

$$\frac{dx}{dt} = 3t^2 - 2t = t(3t - 2)$$

Setting $\frac{dx}{dt} = 0$ gives $t = 0$ and $t = 2/3$.

2. Check if $\frac{dy}{dt} \neq 0$ at these values:

$$\frac{dy}{dt} = 4t^3 + 4t - 8$$

- At $t = 0$: $\frac{dy}{dt} = 4(0)^3 + 4(0) - 8 = -8 \neq 0$.
- At $t = 2/3$: $\frac{dy}{dt} = 4(2/3)^3 + 4(2/3) - 8 = 4(8/27) + 8/3 - 8 = 32/27 + 72/27 - 216/27 = -112/27 \neq 0$.

Result: A vertical tangent exists at both $t = 0$ and $t = 2/3$.

9. For what value of t is the particle at rest?

Equations: $x = t^3 - 3t^2$, $y = 2t^3 - 3t^2 - 12t$

Explanation: A particle is at rest when its velocity is zero. This means both velocity components, $\frac{dx}{dt}$ and $\frac{dy}{dt}$, must be zero simultaneously.

1. Set $\frac{dx}{dt} = 0$:

$$\frac{dx}{dt} = 3t^2 - 6t = 3t(t - 2)$$

The solutions are $t = 0$ and $t = 2$.

2. Set $\frac{dy}{dt} = 0$:

$$\frac{dy}{dt} = 6t^2 - 6t - 12 = 6(t^2 - t - 2) = 6(t - 2)(t + 1)$$

The solutions are $t = 2$ and $t = -1$.

3. Find the common solution: The only value of t that makes both derivatives zero is $t = 2$.

Result: The particle is at rest when $t = 2$.