

# Calculus II - Test 2 Review Solutions

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## 1. Evaluate the convergence and divergence of the following integrals

a. **Integral:**  $\int_2^{\infty} \frac{2}{x^2 - x} dx$

*Explanation:* This is an improper integral because the upper limit of integration is infinite. We first use partial fraction decomposition on the integrand.

$$\begin{aligned}\frac{2}{x^2 - x} &= \frac{2}{x(x - 1)} = \frac{A}{x} + \frac{B}{x - 1} \\ 2 &= A(x - 1) + Bx\end{aligned}$$

$$\text{Let } x = 0 : 2 = A(-1) \implies A = -2$$

$$\text{Let } x = 1 : 2 = B(1) \implies B = 2$$

$$\text{So, } \frac{2}{x(x - 1)} = \frac{2}{x - 1} - \frac{2}{x}$$

Now we evaluate the integral as a limit:

$$\begin{aligned}\int_2^{\infty} \left( \frac{2}{x - 1} - \frac{2}{x} \right) dx &= \lim_{b \rightarrow \infty} \int_2^b \left( \frac{2}{x - 1} - \frac{2}{x} \right) dx \\ &= \lim_{b \rightarrow \infty} [2 \ln |x - 1| - 2 \ln |x|]_2^b \\ &= \lim_{b \rightarrow \infty} \left[ 2 \ln \left| \frac{x - 1}{x} \right| \right]_2^b \\ &= \lim_{b \rightarrow \infty} \left( 2 \ln \left| \frac{b - 1}{b} \right| - 2 \ln \left| \frac{2 - 1}{2} \right| \right) \\ &= 2 \ln(1) - 2 \ln \left( \frac{1}{2} \right) = 0 - 2(-\ln 2) = 2 \ln 2\end{aligned}$$

**Result:** The integral **converges** to  $2 \ln 2$ .

b. **Integral:**  $\int_0^1 \frac{x + 1}{\sqrt{x^2 + 2x}} dx$

*Explanation:* This integral is improper because the integrand has a discontinuity at  $x = 0$ . We use a u-substitution. Let  $u = x^2 + 2x$ , so  $du = (2x + 2)dx = 2(x + 1)dx$ .

$$\begin{aligned}\int \frac{x + 1}{\sqrt{x^2 + 2x}} dx &= \int \frac{1}{\sqrt{u}} \cdot \frac{du}{2} = \frac{1}{2} \int u^{-1/2} du \\ &= \frac{1}{2} \cdot 2u^{1/2} = \sqrt{u} = \sqrt{x^2 + 2x}\end{aligned}$$

Now evaluate the limit:

$$\begin{aligned}
 \int_0^1 \frac{x+1}{\sqrt{x^2+2x}} dx &= \lim_{a \rightarrow 0^+} \int_a^1 \frac{x+1}{\sqrt{x^2+2x}} dx \\
 &= \lim_{a \rightarrow 0^+} \left[ \sqrt{x^2+2x} \right]_a^1 \\
 &= \lim_{a \rightarrow 0^+} \left( \sqrt{1^2+2(1)} - \sqrt{a^2+2a} \right) \\
 &= \sqrt{3} - 0 = \sqrt{3}
 \end{aligned}$$

**Result:** The integral **converges** to  $\sqrt{3}$ .

c. **Integral:**  $\int_0^{\pi/2} \tan \theta d\theta$

*Explanation:* This is improper because  $\tan \theta$  has a vertical asymptote at  $\theta = \pi/2$ .

$$\begin{aligned}
 \int_0^{\pi/2} \tan \theta d\theta &= \lim_{b \rightarrow (\pi/2)^-} \int_0^b \tan \theta d\theta \\
 &= \lim_{b \rightarrow (\pi/2)^-} [-\ln |\cos \theta|]_0^b \\
 &= \lim_{b \rightarrow (\pi/2)^-} (-\ln |\cos b| - (-\ln |\cos 0|)) \\
 &= \lim_{b \rightarrow (\pi/2)^-} (-\ln |\cos b|) + \ln(1)
 \end{aligned}$$

As  $b \rightarrow (\pi/2)^-$ ,  $\cos b \rightarrow 0^+$ , so  $-\ln |\cos b| \rightarrow \infty$ . **Result:** The integral **diverges**.

d. **Integral:**  $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

*Explanation:* Improper due to a discontinuity at  $x = 1$ . This is a standard integral form.

$$\begin{aligned}
 \int_0^1 \frac{dx}{\sqrt{1-x^2}} &= \lim_{b \rightarrow 1^-} \int_0^b \frac{dx}{\sqrt{1-x^2}} \\
 &= \lim_{b \rightarrow 1^-} [\arcsin(x)]_0^b \\
 &= \lim_{b \rightarrow 1^-} (\arcsin(b) - \arcsin(0)) \\
 &= \arcsin(1) - 0 = \frac{\pi}{2}
 \end{aligned}$$

**Result:** The integral **converges** to  $\frac{\pi}{2}$ .

e. **Integral:**  $\int_{-\infty}^{\infty} \frac{2x}{(x^2+1)^2} dx$

*Explanation:* This is improper because of the infinite limits. We split it at  $x = 0$ .

$$\int_{-\infty}^{\infty} \frac{2x}{(x^2+1)^2} dx = \int_{-\infty}^0 \frac{2x}{(x^2+1)^2} dx + \int_0^{\infty} \frac{2x}{(x^2+1)^2} dx$$

Let  $u = x^2 + 1$ ,  $du = 2x dx$ . The antiderivative is  $\int \frac{1}{u^2} du = -\frac{1}{u} = -\frac{1}{x^2+1}$ .

Part 1:  $\lim_{a \rightarrow -\infty} \left[ -\frac{1}{x^2+1} \right]_a^0 = \left( -\frac{1}{0^2+1} \right) - \lim_{a \rightarrow -\infty} \left( -\frac{1}{a^2+1} \right) = -1 - 0 = -1$

Part 2:  $\lim_{b \rightarrow \infty} \left[ -\frac{1}{x^2+1} \right]_0^b = \lim_{b \rightarrow \infty} \left( -\frac{1}{b^2+1} \right) - \left( -\frac{1}{0^2+1} \right) = 0 - (-1) = 1$

The total value is  $-1 + 1 = 0$ . **Result:** The integral **converges** to 0.

## 2. Find the length of the curves

Arc Length Formula:  $L = \int_a^b \sqrt{1 + (y')^2} dx$

a. **Curve:**  $y = \frac{1}{3}(x^2 + 2)^{3/2}$  from  $x = 0$  to  $x = 3$

$$\begin{aligned}y' &= \frac{1}{3} \cdot \frac{3}{2}(x^2 + 2)^{1/2} \cdot 2x = x\sqrt{x^2 + 2} \\1 + (y')^2 &= 1 + (x\sqrt{x^2 + 2})^2 = 1 + x^2(x^2 + 2) \\&= 1 + x^4 + 2x^2 = (x^2 + 1)^2 \\L &= \int_0^3 \sqrt{(x^2 + 1)^2} dx = \int_0^3 (x^2 + 1) dx \\&= \left[ \frac{x^3}{3} + x \right]_0^3 = \left( \frac{27}{3} + 3 \right) - 0 = 9 + 3 = 12\end{aligned}$$

**Length:** 12.

b. **Curve:**  $y = \ln(x) - \frac{x^2}{8}$  from  $x = 1$  to  $x = 2$

$$\begin{aligned}y' &= \frac{1}{x} - \frac{2x}{8} = \frac{1}{x} - \frac{x}{4} \\1 + (y')^2 &= 1 + \left( \frac{1}{x} - \frac{x}{4} \right)^2 = 1 + \left( \frac{1}{x^2} - \frac{1}{2} + \frac{x^2}{16} \right) \\&= \frac{1}{x^2} + \frac{1}{2} + \frac{x^2}{16} = \left( \frac{1}{x} + \frac{x}{4} \right)^2 \\L &= \int_1^2 \sqrt{\left( \frac{1}{x} + \frac{x}{4} \right)^2} dx = \int_1^2 \left( \frac{1}{x} + \frac{x}{4} \right) dx \\&= \left[ \ln|x| + \frac{x^2}{8} \right]_1^2 = \left( \ln 2 + \frac{4}{8} \right) - \left( \ln 1 + \frac{1}{8} \right) = \ln 2 + \frac{1}{2} - \frac{1}{8} = \ln 2 + \frac{3}{8}\end{aligned}$$

**Length:**  $\ln 2 + \frac{3}{8}$ .

c. **Curve:**  $y = \ln(\sec x)$  from  $x = 0$  to  $x = \pi/4$

$$\begin{aligned}y' &= \frac{1}{\sec x} \cdot (\sec x \tan x) = \tan x \\1 + (y')^2 &= 1 + \tan^2 x = \sec^2 x \\L &= \int_0^{\pi/4} \sqrt{\sec^2 x} dx = \int_0^{\pi/4} \sec x dx \\&= [\ln|\sec x + \tan x|]_0^{\pi/4} \\&= \ln|\sec(\pi/4) + \tan(\pi/4)| - \ln|\sec(0) + \tan(0)| \\&= \ln|\sqrt{2} + 1| - \ln|1 + 0| = \ln(\sqrt{2} + 1)\end{aligned}$$

**Length:**  $\ln(\sqrt{2} + 1)$ .

### 3. Calculate the surface area generated by revolving the curves about the indicated axis

Surface Area Formula (x-axis):  $S = \int_a^b 2\pi y \sqrt{1 + (y')^2} dx$

a. **Curve:**  $y = \sqrt{2x - x^2}$ ,  $1 \leq x \leq 2$ ; about the x-axis

$$\begin{aligned}y' &= \frac{1}{2\sqrt{2x - x^2}} \cdot (2 - 2x) = \frac{1 - x}{\sqrt{2x - x^2}} \\1 + (y')^2 &= 1 + \frac{(1 - x)^2}{2x - x^2} = \frac{2x - x^2 + (1 - 2x + x^2)}{2x - x^2} = \frac{1}{2x - x^2} \\S &= \int_1^2 2\pi y \sqrt{1 + (y')^2} dx \\&= \int_1^2 2\pi \sqrt{2x - x^2} \sqrt{\frac{1}{2x - x^2}} dx \\&= \int_1^2 2\pi dx = [2\pi x]_1^2 = 2\pi(2) - 2\pi(1) = 2\pi\end{aligned}$$

**Surface Area:**  $2\pi$ .

b. **Curve:**  $y = \sqrt{1 + e^x}$ ,  $0 \leq x \leq 1$ ; about the x-axis

$$\begin{aligned}y' &= \frac{e^x}{2\sqrt{1 + e^x}} \\1 + (y')^2 &= 1 + \frac{e^{2x}}{4(1 + e^x)} = \frac{4(1 + e^x) + e^{2x}}{4(1 + e^x)} = \frac{4 + 4e^x + e^{2x}}{4(1 + e^x)} = \frac{(e^x + 2)^2}{4(1 + e^x)} \\S &= \int_0^1 2\pi \sqrt{1 + e^x} \sqrt{\frac{(e^x + 2)^2}{4(1 + e^x)}} dx \\&= \int_0^1 2\pi \sqrt{1 + e^x} \frac{e^x + 2}{2\sqrt{1 + e^x}} dx \\&= \pi \int_0^1 (e^x + 2) dx = \pi [e^x + 2x]_0^1 \\&= \pi ((e^1 + 2) - (e^0 + 0)) = \pi(e + 2 - 1) = \pi(e + 1)\end{aligned}$$

**Surface Area:**  $\pi(e + 1)$ .

#### 4. Find and graph the Cartesian equation

- a. **Equations:**  $x = 4 \cos t, y = 2 \sin t, 0 \leq t \leq 2\pi$

*Eliminate the parameter:* From  $x = 4 \cos t \implies \cos t = x/4$ . From  $y = 2 \sin t \implies \sin t = y/2$ . Using the identity  $\cos^2 t + \sin^2 t = 1$ :

$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{2}\right)^2 = 1 \implies \frac{x^2}{16} + \frac{y^2}{4} = 1$$

**Path:** This is an ellipse centered at  $(0, 0)$  with a horizontal major axis of length 8 and a vertical minor axis of length 4.

**Direction:** At  $t = 0, (x, y) = (4, 0)$ . At  $t = \pi/2, (x, y) = (0, 2)$ . At  $t = \pi, (x, y) = (-4, 0)$ . The particle travels **counter-clockwise** starting from  $(4, 0)$ .

- b. **Equations:**  $x = \sin t, y = \cos 2t, -\pi/2 \leq t \leq \pi/2$

*Eliminate the parameter:* Using the identity  $\cos 2t = 1 - 2 \sin^2 t$ :

$$y = 1 - 2x^2$$

**Path:** This is a parabola opening downwards with its vertex at  $(0, 1)$ .

**Direction:** At  $t = -\pi/2, (x, y) = (-1, -1)$ . At  $t = 0, (x, y) = (0, 1)$ . At  $t = \pi/2, (x, y) = (1, -1)$ . The particle moves from  $(-1, -1)$  to  $(1, -1)$  along the parabola.

- c. **Equations:**  $x = 1 + \sin t, y = \cos t - 2, 0 \leq t \leq \pi$

*Eliminate the parameter:* From the equations,  $\sin t = x - 1$  and  $\cos t = y + 2$ . Using  $\sin^2 t + \cos^2 t = 1$ :

$$(x - 1)^2 + (y + 2)^2 = 1$$

**Path:** This is a circle of radius 1 centered at  $(1, -2)$ .

**Direction:** At  $t = 0, (x, y) = (1, -1)$ . At  $t = \pi/2, (x, y) = (2, -2)$ . At  $t = \pi, (x, y) = (1, -3)$ . The particle traces the **top semi-circle** from right to left, starting at  $(1, -1)$ .

**5. Find  $\frac{dy}{dx}$  and  $\frac{d^2y}{dx^2}$  as a function of  $t$**

Formulas:  $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$  and  $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(dy/dx)}{dx/dt}$

a. **Equations:**  $x = t - t^2, y = t - t^3$

$$\begin{aligned}\frac{dx}{dt} &= 1 - 2t & \frac{dy}{dt} &= 1 - 3t^2 \\ \frac{dy}{dx} &= \frac{1 - 3t^2}{1 - 2t} \\ \frac{d}{dt} \left( \frac{dy}{dx} \right) &= \frac{-6t(1 - 2t) - (1 - 3t^2)(-2)}{(1 - 2t)^2} = \frac{-6t + 12t^2 + 2 - 6t^2}{(1 - 2t)^2} = \frac{6t^2 - 6t + 2}{(1 - 2t)^2} \\ \frac{d^2y}{dx^2} &= \frac{(6t^2 - 6t + 2)/(1 - 2t)^2}{1 - 2t} = \frac{2(3t^2 - 3t + 1)}{(1 - 2t)^3}\end{aligned}$$

b. **Equations:**  $x = \frac{1}{t+1}, y = \frac{t}{t-1}$

$$\begin{aligned}\frac{dx}{dt} &= -(t+1)^{-2} = \frac{-1}{(t+1)^2} \\ \frac{dy}{dt} &= \frac{1(t-1) - t(1)}{(t-1)^2} = \frac{-1}{(t-1)^2} \\ \frac{dy}{dx} &= \frac{-1/(t-1)^2}{-1/(t+1)^2} = \frac{(t+1)^2}{(t-1)^2} \\ \frac{d}{dt} \left( \frac{dy}{dx} \right) &= \frac{2(t+1)(t-1)^2 - (t+1)^2(2(t-1))}{(t-1)^4} \\ &= \frac{2(t+1)(t-1)[(t-1) - (t+1)]}{(t-1)^4} = \frac{2(t+1)(-2)}{(t-1)^3} = \frac{-4(t+1)}{(t-1)^3} \\ \frac{d^2y}{dx^2} &= \frac{-4(t+1)/(t-1)^3}{-1/(t+1)^2} = \frac{4(t+1)^3}{(t-1)^3}\end{aligned}$$

## 6. Find an equation of the line tangent to the curve

a. **Curve:**  $x = \sec t, y = \tan t$  at  $t = \pi/4$

- **Point:**  $x(\pi/4) = \sec(\pi/4) = \sqrt{2}, y(\pi/4) = \tan(\pi/4) = 1$ . Point is  $(\sqrt{2}, 1)$ .
- **Slope:**  $\frac{dx}{dt} = \sec t \tan t, \frac{dy}{dt} = \sec^2 t. \frac{dy}{dx} = \frac{\sec^2 t}{\sec t \tan t} = \frac{\sec t}{\tan t} = \csc t$ . At  $t = \pi/4$ , slope  $m = \csc(\pi/4) = \sqrt{2}$ .
- **Equation:**  $y - 1 = \sqrt{2}(x - \sqrt{2}) \implies y - 1 = \sqrt{2}x - 2 \implies y = \sqrt{2}x - 1$ .

b. **Curve:**  $x = t - \sin t, y = 1 - \cos t$  at  $t = \pi/3$

- **Point:**  $x(\pi/3) = \frac{\pi}{3} - \sin(\pi/3) = \frac{\pi}{3} - \frac{\sqrt{3}}{2}$ .  $y(\pi/3) = 1 - \cos(\pi/3) = 1 - \frac{1}{2} = \frac{1}{2}$ . Point is  $(\frac{\pi}{3} - \frac{\sqrt{3}}{2}, \frac{1}{2})$ .
- **Slope:**  $\frac{dx}{dt} = 1 - \cos t, \frac{dy}{dt} = \sin t. \frac{dy}{dx} = \frac{\sin t}{1 - \cos t}$ . At  $t = \pi/3$ , slope  $m = \frac{\sin(\pi/3)}{1 - \cos(\pi/3)} = \frac{\sqrt{3}/2}{1 - 1/2} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$ .
- **Equation:**  $y - \frac{1}{2} = \sqrt{3} \left( x - \left( \frac{\pi}{3} - \frac{\sqrt{3}}{2} \right) \right)$ .

## 7. Find the lengths of the curves

Parametric Arc Length Formula:  $L = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$

a. **Curve:**  $x = \frac{t^2}{2}, y = \frac{1}{3}(2t+1)^{3/2}, 0 \leq t \leq 4$

$$\begin{aligned}\frac{dx}{dt} &= t & \frac{dy}{dt} &= \frac{1}{3} \cdot \frac{3}{2}(2t+1)^{1/2} \cdot 2 = \sqrt{2t+1} \\ \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 &= t^2 + (\sqrt{2t+1})^2 = t^2 + 2t + 1 = (t+1)^2 \\ L &= \int_0^4 \sqrt{(t+1)^2} dt = \int_0^4 (t+1) dt \\ &= \left[\frac{t^2}{2} + t\right]_0^4 = \left(\frac{16}{2} + 4\right) - 0 = 8 + 4 = 12\end{aligned}$$

**Length:** 12.

b. **Curve:**  $x = t^3, y = \frac{3t^2}{2}, 0 \leq t \leq \sqrt{3}$

$$\begin{aligned}\frac{dx}{dt} &= 3t^2 & \frac{dy}{dt} &= 3t \\ \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 &= (3t^2)^2 + (3t)^2 = 9t^4 + 9t^2 = 9t^2(t^2 + 1) \\ L &= \int_0^{\sqrt{3}} \sqrt{9t^2(t^2 + 1)} dt = \int_0^{\sqrt{3}} 3t\sqrt{t^2 + 1} dt\end{aligned}$$

Let  $u = t^2 + 1, du = 2t dt \implies \frac{3}{2}du = 3t dt$ . When  $t = 0, u = 1$ . When  $t = \sqrt{3}, u = 4$ .

$$\begin{aligned}L &= \int_1^4 \frac{3}{2}\sqrt{u} du = \frac{3}{2} \int_1^4 u^{1/2} du = \frac{3}{2} \left[\frac{2}{3}u^{3/2}\right]_1^4 \\ &= \left[u^{3/2}\right]_1^4 = 4^{3/2} - 1^{3/2} = 8 - 1 = 7\end{aligned}$$

**Length:** 7.



## 8. For what values of $t$ does the curve have a vertical tangent?

**Equations:**  $x = t^3 - t^2 - 1, y = t^4 + 2t^2 - 8t$

*Explanation:* A vertical tangent occurs when the slope  $\frac{dy}{dx}$  is undefined. This happens when  $\frac{dx}{dt} = 0$  and  $\frac{dy}{dt} \neq 0$ .

1. Find when  $\frac{dx}{dt} = 0$ :

$$\frac{dx}{dt} = 3t^2 - 2t = t(3t - 2)$$

Setting  $\frac{dx}{dt} = 0$  gives  $t = 0$  and  $t = 2/3$ .

2. Check if  $\frac{dy}{dt} \neq 0$  at these values:

$$\frac{dy}{dt} = 4t^3 + 4t - 8$$

- At  $t = 0$ :  $\frac{dy}{dt} = 4(0)^3 + 4(0) - 8 = -8 \neq 0$ .
- At  $t = 2/3$ :  $\frac{dy}{dt} = 4(2/3)^3 + 4(2/3) - 8 = 4(8/27) + 8/3 - 8 = 32/27 + 72/27 - 216/27 = -112/27 \neq 0$ .

**Result:** A vertical tangent exists at both  $t = 0$  and  $t = 2/3$ .

## 9. For what value of $t$ is the particle at rest?

**Equations:**  $x = t^3 - 3t^2, y = 2t^3 - 3t^2 - 12t$

*Explanation:* A particle is at rest when its velocity is zero. This means both velocity components,  $\frac{dx}{dt}$  and  $\frac{dy}{dt}$ , must be zero simultaneously.

1. Set  $\frac{dx}{dt} = 0$ :

$$\frac{dx}{dt} = 3t^2 - 6t = 3t(t - 2)$$

The solutions are  $t = 0$  and  $t = 2$ .

2. Set  $\frac{dy}{dt} = 0$ :

$$\frac{dy}{dt} = 6t^2 - 6t - 12 = 6(t^2 - t - 2) = 6(t - 2)(t + 1)$$

The solutions are  $t = 2$  and  $t = -1$ .

3. Find the common solution: The only value of  $t$  that makes both derivatives zero is  $t = 2$ .

**Result:** The particle is at rest when  $t = 2$ .