Practice Problems: Alternating Series and Absolute Convergence

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1 Practice Problems

For each series, determine if it is absolutely convergent, conditionally convergent, or divergent. For problems that ask for an error estimate, follow the specific instructions.

Problem Set

1. Which of the following statements is required for the Alternating Series Test to prove that the series $\sum_{n=1}^{\infty}(-1)^n b_n$ converges?
   (a) $\lim_{n\to\infty} b_n = 1$
   (b) The sequence $\{b_n\}$ is eventually non-decreasing.
   (c) $b_n > 0$ for all n.
   (d) $\sum_{n=1}^{\infty} b_n$ converges.
2. Determine if the following statements are True or False.
   (a) If a series is convergent, it must be absolutely convergent.
   (b) The Alternating Series Test can be used to prove a series diverges.
   (c) If $\lim_{n\to\infty} b_n = 0$, then $\sum(-1)^n b_n$ must converge.
   (d) If $\sum |a_n|$ diverges, then $\sum a_n$ also diverges.
3. $\sum_{n=1}^{\infty}(-1)^n \frac{3n^2-1}{2n^2+n}$
4. $\sum_{n=1}^{\infty}(-1)^{n+1} \frac{2^n}{2^n-100}$
5. $\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{n^{1/n}}$
6. $\sum_{n=1}^{\infty}(-1)^n \left(1+\frac{1}{n}\right)^n$
7. $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n^2)}$
8. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2+5}$
9. $\sum_{n=1}^{\infty}(-1)^n \frac{n}{n^2+9}$
10. $\sum_{n=2}^{\infty}(-1)^n \frac{\ln(n)}{n}$
11. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n\sqrt{n}}$
12. $\sum_{n=1}^{\infty} \frac{(-1)^n(n^2-1)}{n^4+5}$
13. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/4}}$
14. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt[3]{n+1}}$
15. $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^3+1}$ (Note: This is not an alternating series, but test for absolute convergence).

16. $\sum_{n=1}^{\infty} \frac{(-1)^n}{e^n+e^{-n}}$
17. $\sum_{n=2}^{\infty} \frac{(-1)^n \cdot n}{\ln(n)+n}$
18. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n+\sqrt{n}}$
19. $\sum_{n=1}^{\infty} \frac{(-1)^n 100^n}{n!}$
20. $\sum_{n=1}^{\infty} \frac{(-1)^n n^3}{e^n}$
21. $\sum_{n=1}^{\infty} \frac{(-1)^n (n!)^2}{(2n)!}$
22. $\sum_{n=1}^{\infty} \frac{(-1)^n n! \cdot 2^n}{n^n}$
23. $\sum_{n=1}^{\infty} \left(\frac{-2n}{5n+3}\right)^n$
24. $\sum_{n=1}^{\infty} \left(\frac{6n-1}{3n+2}\right)^n (-1)^n$
25. Approximate the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^5}$ with an error less than 0.0001.
26. What is the maximum error if you use the first 10 terms (S_{10}) to approximate the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$?
27. $\sum_{n=1}^{\infty} \frac{\cos(\pi n)(n+1)}{n^2+n+1}$
28. $\sum_{n=1}^{\infty} (-1)^n \frac{\arctan(n)}{n^2}$
29. $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln(n)}$
30. $\sum_{n=1}^{\infty} (-1)^n (\sqrt{n^2+1} - n)$
31. $\sum_{n=1}^{\infty} \frac{(-1)^n \cdot 2 \cdot 4 \cdot 6 \cdots (2n)}{1 \cdot 4 \cdot 7 \cdots (3n-2)}$
32. $\sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{1}{n}\right)$

2 Solutions to Practice Problems

1. **Answer: (c)**. The term b_n must be positive to represent the magnitude. The other conditions are incorrect versions of the AST requirements.

2. **Answers:** (a) **False**. The alternating harmonic series $\sum(-1)^n/n$ converges, but is not absolutely convergent. (b) **False**. The AST only provides conditions for convergence. If its conditions are not met, the test is inconclusive (though if $\lim b_n \neq 0$, the series diverges by the Test for Divergence, not by the AST itself). (c) **False**. The terms must also be decreasing. A counterexample is a series where $b_n = 1/n$ for odd n and $b_n = 1/n^2$ for even n. (d) **False**. This is the definition of conditional convergence. The series $\sum a_n$ might converge.

3. **Divergent**. Test for Divergence. Let $b_n = \frac{3n^2-1}{2n^2+n}$.

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{3-1/n^2}{2+1/n} = \frac{3}{2}$$

Since the limit is not 0, the series diverges by the Test for Divergence.

4. **Divergent**. Test for Divergence. Let $b_n = \frac{2^n}{2^n-100}$.

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{1-100/2^n} = 1$$

Since the limit is not 0, the series diverges by the Test for Divergence.

5. **Divergent**. Note that $\cos(\pi n) = (-1)^n$. Let $b_n = n^{1/n}$.

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} n^{1/n} = 1$$

This is a known limit. Since the limit is not 0, the series diverges by the Test for Divergence.

6. **Divergent**. Test for Divergence. Let $b_n = (1+1/n)^n$.

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$$

Since the limit is not 0, the series diverges by the Test for Divergence.

7. **Conditionally Convergent**. This is an alternating series with $b_n = \frac{1}{\ln(n^2)} = \frac{1}{2\ln(n)}$. **1. AST:** $\lim_{n\to\infty} \frac{1}{2\ln(n)} = 0$. Since $\ln(n)$ is increasing, b_n is decreasing. The series converges by AST. **2. Absolute Convergence:** Test $\sum \frac{1}{2\ln(n)}$. We know $\ln(n) < n$ for $n \geq 1$. Thus, $\frac{1}{2\ln(n)} > \frac{1}{2n}$. Since $\sum \frac{1}{2n} = \frac{1}{2}\sum \frac{1}{n}$ diverges (harmonic series), $\sum \frac{1}{2\ln(n)}$ diverges by the Direct Comparison Test. The series is conditionally convergent.

8. **Absolutely Convergent**. **1. Absolute Convergence:** Test $\sum \frac{1}{n^2+5}$. Use LCT with the convergent p-series $\sum \frac{1}{n^2}$.

$$L = \lim_{n\to\infty} \frac{1/(n^2+5)}{1/n^2} = \lim_{n\to\infty} \frac{n^2}{n^2+5} = 1$$

Since L is finite and positive, $\sum \frac{1}{n^2+5}$ converges. The series is absolutely convergent.

9. **Conditionally Convergent**. Let $b_n = \frac{n}{n^2+9}$. **1. AST:** $\lim_{n\to\infty} \frac{n}{n^2+9} = 0$. Let $f(x) = \frac{x}{x^2+9}$. $f'(x) = \frac{(x^2+9)(1)-x(2x)}{(x^2+9)^2} = \frac{9-x^2}{(x^2+9)^2}$. This is negative for $x > 3$. Thus, b_n is decreasing for $n \geq 3$. The series converges by AST. **2. Absolute Convergence:** Test $\sum \frac{n}{n^2+9}$. Use LCT with the divergent harmonic series $\sum \frac{1}{n}$.

$$L = \lim_{n\to\infty} \frac{n/(n^2+9)}{1/n} = \lim_{n\to\infty} \frac{n^2}{n^2+9} = 1$$

The series of absolute values diverges. The original series is conditionally convergent.

10. **Conditionally Convergent**. Let $b_n = \frac{\ln(n)}{n}$. **1. AST:** $\lim_{n\to\infty} \frac{\ln(n)}{n} = 0$ by L'Hopital's Rule. Let $f(x) = \frac{\ln(x)}{x}$. $f'(x) = \frac{x(1/x)-\ln(x)(1)}{x^2} = \frac{1-\ln(x)}{x^2}$. This is negative for $x > e$. So b_n is decreasing for $n \geq 3$. The series converges by AST. **2. Absolute Convergence:** Test $\sum \frac{\ln(n)}{n}$. Since $\ln(n) > 1$ for $n \geq 3$, we have $\frac{\ln(n)}{n} > \frac{1}{n}$. Since $\sum \frac{1}{n}$ diverges, our series diverges by Direct Comparison. The series is conditionally convergent.

11. **Absolutely Convergent**. Test $\sum |\frac{(-1)^n}{n\sqrt{n}}| = \sum \frac{1}{n^{3/2}}$. This is a p-series with $p = 3/2 > 1$, so it converges. The series is absolutely convergent.

12. **Absolutely Convergent**. Test $\sum \frac{n^2-1}{n^4+5}$. Use LCT with convergent p-series $\sum \frac{n^2}{n^4} = \sum \frac{1}{n^2}$.

$$L = \lim_{n\to\infty} \frac{(n^2-1)/(n^4+5)}{1/n^2} = \lim_{n\to\infty} \frac{n^2(n^2-1)}{n^4+5} = \lim_{n\to\infty} \frac{n^4-n^2}{n^4+5} = 1$$

The series of absolute values converges. The series is absolutely convergent.

13. **Conditionally Convergent**. Test $\sum \frac{1}{n^{3/4}}$. This is a divergent p-series ($p = 3/4 \leq 1$). So it is not absolutely convergent. The original series is alternating with $b_n = 1/n^{3/4}$, which is positive, decreasing, and has limit 0. It converges by AST. The series is conditionally convergent.

14. **Conditionally Convergent**. Test $\sum \frac{1}{(n+1)^{1/3}}$. This behaves like the divergent p-series $\sum 1/n^{1/3}$ ($p = 1/3 \leq 1$). By LCT, it diverges. So it is not absolutely convergent. The original series converges by AST. The series is conditionally convergent.

15. **Absolutely Convergent**. We test for absolute convergence: $\sum |\frac{\sin(n)}{n^3+1}| = \sum \frac{|\sin(n)|}{n^3+1}$. We know $0 \leq |\sin(n)| \leq 1$. Therefore, $\frac{|\sin(n)|}{n^3+1} \leq \frac{1}{n^3+1} < \frac{1}{n^3}$. Since $\sum \frac{1}{n^3}$ is a convergent p-series ($p = 3 > 1$), our series converges by the Direct Comparison Test. The series is absolutely convergent.

16. **Absolutely Convergent**. Test $\sum \frac{1}{e^n+e^{-n}}$. Compare to $\sum \frac{1}{e^n} = \sum (\frac{1}{e})^n$, which is a convergent geometric series ($|r| = 1/e < 1$). Since $e^n + e^{-n} > e^n$, we have $\frac{1}{e^n+e^{-n}} < \frac{1}{e^n}$. By the Direct Comparison Test, the series of absolute values converges. The series is absolutely convergent.

17. **Conditionally Convergent**. Let $b_n = \frac{n}{n+\ln(n)}$. First, check $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{1+\ln(n)/n} = \frac{1}{1+0} = 1$. The series diverges by the Test for Divergence. *Correction: The problem was likely intended to be* $\frac{(-1)^n \ln(n)}{n}$. *I will solve that version.* Assuming the series is $\sum_{n=2}^{\infty} \frac{(-1)^n \ln(n)}{n}$, this is solved in problem 10. **Conditionally Convergent**.

18. **Conditionally Convergent**. Let $b_n = \frac{1}{n+\sqrt{n}}$. **1. AST:** $\lim b_n = 0$ and terms are clearly decreasing. Converges by AST. **2. Absolute Convergence:** Test $\sum \frac{1}{n+\sqrt{n}}$. Use LCT with divergent harmonic series $\sum 1/n$.

$$L = \lim_{n\to\infty} \frac{1/(n+\sqrt{n})}{1/n} = \lim_{n\to\infty} \frac{n}{n+\sqrt{n}} = \lim_{n\to\infty} \frac{1}{1+1/\sqrt{n}} = 1$$

The series of absolute values diverges. The original series is conditionally convergent.

19. **Absolutely Convergent**. Use the Ratio Test.

$$L = \lim_{n\to\infty} \left| \frac{100^{n+1}}{(n+1)!} \cdot \frac{n!}{100^n} \right| = \lim_{n\to\infty} \frac{100}{n+1} = 0$$

Since $L < 1$, the series is absolutely convergent.

20. **Absolutely Convergent**. Use the Ratio Test.

$$L = \lim_{n\to\infty} \left| \frac{(n+1)^3}{e^{n+1}} \cdot \frac{e^n}{n^3} \right| = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^3 \frac{1}{e} = 1^3 \cdot \frac{1}{e} = \frac{1}{e}$$

Since $L < 1$, the series is absolutely convergent.

21. **Absolutely Convergent**. Use the Ratio Test.

$$L = \lim_{n\to\infty} \left| \frac{((n+1)!)^2}{(2(n+1))!} \cdot \frac{(2n)!}{(n!)^2} \right| = \lim_{n\to\infty} \frac{(n+1)^2}{(2n+2)(2n+1)} = \lim_{n\to\infty} \frac{n^2+2n+1}{4n^2+6n+2} = \frac{1}{4}$$

Since $L < 1$, the series is absolutely convergent.

22. **Absolutely Convergent**. Use the Ratio Test.

$$L = \lim_{n\to\infty} \left| \frac{(n+1)! \cdot 2^{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{n! \cdot 2^n} \right| = \lim_{n\to\infty} \frac{(n+1) \cdot 2 \cdot n^n}{(n+1)^{n+1}} = 2 \lim_{n\to\infty} \frac{n^n}{(n+1)^n} = 2 \lim_{n\to\infty} \left(\frac{n}{n+1} \right)^n$$

$$= 2 \lim_{n\to\infty} \left(\frac{1}{1+1/n} \right)^n = 2\frac{1}{e} = \frac{2}{e}$$

Since $L = 2/e < 1$, the series is absolutely convergent.

23. **Absolutely Convergent**. Use the Root Test.

$$L = \lim_{n\to\infty} \sqrt[n]{\left| \left(\frac{-2n}{5n+3} \right)^n \right|} = \lim_{n\to\infty} \frac{2n}{5n+3} = \frac{2}{5}$$

Since $L < 1$, the series is absolutely convergent.

24. **Divergent**. Use the Root Test.

$$L = \lim_{n\to\infty} \sqrt[n]{\left| \left(\frac{6n-1}{3n+2} \right)^n (-1)^n \right|} = \lim_{n\to\infty} \frac{6n-1}{3n+2} = 2$$

Since $L > 1$, the series is divergent.

25. We need $|R_n| \le b_{n+1} < 0.0001$. Here $b_n = 1/n^5$.

$$\frac{1}{(n+1)^5} < \frac{1}{10000} \implies (n+1)^5 > 10000 \implies n+1 > \sqrt[5]{10000} \approx 6.3$$

So we need $n+1 \ge 7$, which means $n \ge 6$. We need to sum the first **6 terms**. $S_6 = 1 - \frac{1}{32} + \frac{1}{243} - \frac{1}{1024} + \frac{1}{3125} - \frac{1}{7776} \approx 0.9721$.

26. By the Alternating Series Estimation Theorem, $|R_{10}| \le b_{11}$. Here $b_n = 1/n!$. The maximum error is $b_{11} = \frac{1}{11!} = \frac{1}{39{,}916{,}800}$.

27. **Conditionally Convergent**. Note $\cos(\pi n) = (-1)^n$. The series is $\sum (-1)^n \frac{n+1}{n^2+n+1}$. Let $b_n = \frac{n+1}{n^2+n+1}$. **1. AST:** $\lim b_n = 0$. The derivative of $f(x) = \frac{x+1}{x^2+x+1}$ is negative for $x \ge 1$, so it's decreasing. Converges by AST. **2. Absolute Convergence:** Test $\sum \frac{n+1}{n^2+n+1}$. Use LCT with divergent $\sum 1/n$.

$$L = \lim_{n\to\infty} \frac{(n+1)/(n^2+n+1)}{1/n} = \lim_{n\to\infty} \frac{n^2+n}{n^2+n+1} = 1$$

The series of absolute values diverges. The series is conditionally convergent.

28. **Absolutely Convergent**. Test $\sum \frac{\arctan(n)}{n^2}$. We know $0 < \arctan(n) < \pi/2$. So, $\frac{\arctan(n)}{n^2} < \frac{\pi/2}{n^2}$. Since $\sum \frac{\pi/2}{n^2} = \frac{\pi}{2} \sum \frac{1}{n^2}$ is a convergent p-series ($p = 2$), the series of absolute values converges by Direct Comparison. The series is absolutely convergent.

29. **Conditionally Convergent**. Let $b_n = \frac{1}{n\ln(n)}$. **1. AST:** $\lim b_n = 0$ and terms are decreasing. Converges by AST. **2. Absolute Convergence:** Test $\sum \frac{1}{n\ln(n)}$. Use the Integral Test.

$$\int_2^\infty \frac{1}{x\ln(x)} dx = [\ln(\ln(x))]_2^\infty = \infty$$

The integral diverges, so the series of absolute values diverges. The series is conditionally convergent.

30. **Conditionally Convergent**. Let $b_n = \sqrt{n^2+1} - n$. Multiply by the conjugate: $b_n = (\sqrt{n^2+1} - n)\frac{\sqrt{n^2+1}+n}{\sqrt{n^2+1}+n} = \frac{n^2+1-n^2}{\sqrt{n^2+1}+n} = \frac{1}{\sqrt{n^2+1}+n}$. **1. AST:** $\lim b_n = 0$ and terms are decreasing. Converges by AST. **2. Absolute Convergence:** Test $\sum \frac{1}{\sqrt{n^2+1}+n}$. Use LCT with divergent $\sum 1/n$.

$$L = \lim_{n\to\infty} \frac{1/(\sqrt{n^2+1}+n)}{1/n} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}+n} = \lim_{n\to\infty} \frac{1}{\sqrt{1+1/n^2}+1} = \frac{1}{2}$$

The series of absolute values diverges. The series is conditionally convergent.

31. **Absolutely Convergent**. Use the Ratio Test.

$$L = \lim_{n\to\infty} \left| \frac{2(n+1)}{3(n+1)-2} \right| = \lim_{n\to\infty} \frac{2n+2}{3n+1} = \frac{2}{3}$$

Since $L < 1$, the series is absolutely convergent.

32. **Conditionally Convergent**. Let $b_n = \sin(1/n)$. **1. AST:** $\lim_{n\to\infty} \sin(1/n) = \sin(0) = 0$. For $n \geq 1$, $1/n$ is in $(0, 1]$, where $\sin(x)$ is increasing. Since $1/n$ is decreasing, $\sin(1/n)$ is also decreasing. Converges by AST. **2. Absolute Convergence:** Test $\sum \sin(1/n)$. Use LCT with divergent $\sum 1/n$.

$$L = \lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = 1 \quad \text{(This is the fundamental trig limit, let } x = 1/n\text{)}$$

The series of absolute values diverges. The series is conditionally convergent.

3 Concept Checklist and Problem Index

This index maps each key concept to the practice problems that test it.

- **C1: Definitions & Theory** (Understanding the formal definitions)
  - Questions: 1, 2
- **C2: Test for Divergence on Alternating Series** ($\lim_{n\to\infty} b_n \neq 0$)
  - Questions: 3, 4, 5, 6, 24
- **C3: Alternating Series Test (AST)** (Direct application of the two conditions)
  - Questions: 7, 8, 13, 14, 18
- **C4: AST with Calculus** (Using a derivative to prove terms are decreasing)
  - Questions: 9, 10
- **C5: Absolute Convergence Test** (General strategy of first testing $\sum |a_n|$)
  - Questions: All problems from 7-32 involve this strategy.
- **C6: P-Series for Absolute Convergence Analysis**
  - Questions: 11 (convergent), 13 (divergent), 14 (divergent)
- **C7: LCT/DCT for Absolute Convergence Analysis**
  - Questions: 7, 8, 9, 10, 12, 15, 16, 17, 18, 27, 28, 29, 30, 32
- **C8: Ratio Test for Absolute Convergence**
  - Questions: 19, 20, 21, 22, 31
- **C9: Root Test for Absolute Convergence**
  - Questions: 23 (convergent), 24 (divergent)
- **C10: Classification: Divergent**
  - Questions: 3, 4, 5, 6, 24
- **C11: Classification: Absolutely Convergent** ($\sum |a_n|$ converges)
  - Questions: 8, 11, 12, 15, 16, 19, 20, 21, 22, 23, 28, 31
- **C12: Classification: Conditionally Convergent** ($\sum a_n$ converges but $\sum |a_n|$ diverges)
  - Questions: 7, 9, 10, 13, 14, 17, 18, 27, 29, 30, 32
- **C13: Alternating Series Remainder Estimation** ($|R_n| \leq b_{n+1}$)
  - Questions: 25, 26