Homework 7.5 Strategy for Integration

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1 Integration Problems and Solutions

1.1 Problem 1

(a) Evaluate the integral: $\int \frac{3x}{1+x^2}\,dx$

Solution

To evaluate the integral of (3x)/(1+x**2**), u-substitution is used. Let $u = 1 + x^2$, so $du = 2x\,dx$. The integral becomes:

$$\int \frac{3}{2u}\,du = \frac{3}{2}\int \frac{1}{u}\,du$$

This evaluates to: **Answer:** $\frac{3}{2}\ln|1 + x^2| + C$

(b) Evaluate the integral: $\int \frac{3}{1+x^2}\,dx$

Solution

The integral of 3/(1+x**2**) is a standard form. It evaluates to: **Answer:** $3\tan^{-1}(x) + C$

(c) Evaluate the integral: $\int \frac{3}{1-x^2}\,dx$

Solution

For the integral of 3/(1-x**2**), partial fraction decomposition is applied. The expression can be rewritten as:

$$\frac{3}{2}\left[\int \frac{1}{1-x}\,dx + \int \frac{1}{1+x}\,dx\right]$$

This integrates to $-(\frac{3}{2})\ln|1 - x| + (\frac{3}{2})\ln|1 + x| + C$, which can be simplified to: **Answer:** $\frac{3}{2}\ln\left|\frac{1+x}{1-x}\right| + C$

1.2 Problem 2

(a) Evaluate the integral: $\int 7x\sqrt{x^2 - 1}\,dx$

Solution

To solve the integral of 7x(x**2**-1), let $u = x^2 - 1$, which gives $du = 2x\,dx$. The integral becomes:

$$\frac{7}{2}\int \sqrt{u}\,du$$

This evaluates to: **Answer:** $\frac{7}{3}(x^2 - 1)^{3/2} + C$

(b) Evaluate the integral: $\int \frac{7}{x\sqrt{x^2-1}}\,dx$

Solution

The integral of 7/(x(x²-1)) is a standard form related to inverse trigonometric functions. The solution is: **Answer:** $7\sec^{-1}|x| + C$

(c) Evaluate the integral: $\int \frac{7\sqrt{x^2-1}}{x}\,dx$

Solution

For the integral of 7(x²-1)/x, a trigonometric substitution is appropriate. Let $x = \sec(\theta)$, so $dx = \sec(\theta)\tan(\theta)\,d\theta$. The integral simplifies to:

$$7\int \tan^2(\theta)\,d\theta = 7\int (\sec^2(\theta) - 1)\,d\theta$$

This integrates to $7(\tan(\theta) - \theta) + C$. Substituting back gives: **Answer:** $7(\sqrt{x^2-1} - \sec^{-1}(x)) + C$

1.3 Problem 3

(a) Evaluate the integral: $\int \frac{2\ln(x)}{x}\,dx$

Solution

To integrate 2ln(x)/x, use u-substitution with $u = \ln(x)$, so $du = (1/x)\,dx$. The integral becomes:

$$2\int u\,du$$

which results in: **Answer:** $(\ln(x))^2 + C$

(b) Evaluate the integral: $\int 2\ln(2x)\,dx$

Solution

The integral of 2ln(2x) requires integration by parts. Let $u = \ln(2x)$ and $dv = 2\,dx$. Then $du = (1/x)\,dx$ and $v = 2x$. The formula $\int u\,dv = uv - \int v\,du$ gives:

$$2x\ln(2x) - \int 2\,dx$$

which evaluates to: **Answer:** $2x\ln(2x) - 2x + C$

(c) Evaluate the integral: $\int 2x\ln|x|\,dx$

Solution

For the integral of 2xln—x—, integration by parts is also used. Let $u = \ln|x|$ and $dv = 2x\,dx$. Then $du = (1/x)\,dx$ and $v = x^2$. This leads to:

$$x^2\ln|x| - \int x\,dx$$

which results in: **Answer:** $x^2\ln|x| - \frac{x^2}{2} + C$

1.4 Problem 4

(a) Evaluate the integral: $\int 2\sin^2(x)\,dx$

Solution

To evaluate the integral of 2sin**²**(x), the half-angle identity $\sin^2(x) = (1 - \cos(2x))/2$ is used. The integral becomes:

$$\int (1 - \cos(2x))\, dx$$

which evaluates to: **Answer:** $x - \frac{1}{2}\sin(2x) + C$

(b) Evaluate the integral: $\int 2\sin^3(x)\, dx$

Solution

For the integral of 2sin**³**(x), rewrite $\sin^3(x)$ as $\sin(x)(1 - \cos^2(x))$. Using u-substitution with $u = \cos(x)$ and $du = -\sin(x)\, dx$, the integral becomes:

$$-2\int (1 - u^2)\, du$$

This evaluates to: **Answer:** $-2(\cos(x) - \frac{1}{3}\cos^3(x)) + C$

(c) Evaluate the integral: $\int 2\sin(2x)\, dx$

Solution

The integral of 2sin(2x) is a straightforward integration. It evaluates to: **Answer:** $-\cos(2x) + C$

1.5 Problem 5

Evaluate the integral: $\int \frac{\cos(x)}{3-\sin(x)}\, dx$

Solution

To solve the integral of cos(x)/(3-sin(x)), a u-substitution is performed. Let $u = 3 - \sin(x)$, so $du = -\cos(x)\, dx$. The integral becomes:

$$-\int \frac{1}{u}\, du$$

which evaluates to: **Answer:** $-\ln|3 - \sin(x)| + C$

1.6 Problem 6

Evaluate the integral: $\int \frac{x}{x^4+16}\, dx$

Solution

For the integral of x/(x+16), a substitution is made. Let $u = x^2$, so $du = 2x\, dx$. The integral becomes:

$$\frac{1}{2}\int \frac{1}{u^2+16}\, du$$

This is a standard arctangent form, and the result is: **Answer:** $\frac{1}{8}\tan^{-1}\left(\frac{x^2}{4}\right) + C$

1.7 Problem 7

Evaluate the integral: $\int 5t\sin(t)\cos(t)\, dt$

Solution

To evaluate the integral of 5t sin(t)cos(t), the double angle identity $\sin(2t) = 2\sin(t)\cos(t)$ is used. The integral becomes:

$$\frac{5}{2}\int t\sin(2t)\,dt$$

Integration by parts is then applied, with $u = t$ and $dv = \sin(2t)\,dt$. This yields $du = dt$ and $v = -(\frac{1}{2})\cos(2t)$. The final result is: **Answer:** $\frac{5}{2}\left[-\frac{t}{2}\cos(2t) + \frac{1}{4}\sin(2t)\right] + C$

1.8 Problem 8

Evaluate the integral: $\int \frac{2x-3}{x^2+3x}\,dx$

Solution

For the integral of (2x-3)/(x^2+3x), we can split the denominator by factoring: x(x+3). However, a u-substitution is more direct. Let u = x^2+3x, then du = (2x+3) dx. The integral becomes (2x-3)/(u) * (du/(2x+3)). The integral of (2x-3)/(x$^2 + 3x)is$: **Answer:** $\ln|x^2 + 3x| + C$

1.9 Problem 9

Evaluate the integral: $\int x\sec(x)\tan(x)\,dx$

Solution

To integrate x sec(x)tan(x), integration by parts is the appropriate method. Let $u = x$ and $dv = \sec(x)\tan(x)\,dx$. Then $du = dx$ and $v = \sec(x)$. This gives:

$$x\sec(x) - \int \sec(x)\,dx$$

The final answer is: **Answer:** $x\sec(x) - \ln|\sec(x) + \tan(x)| + C$

1.10 Problem 10

Evaluate the integral: $\int 9\theta\tan^2(\theta)\,d\theta$

Solution

To evaluate the integral of 9 tan^2(), use the identity $\tan^2(\theta) = \sec^2(\theta) - 1$. The integral becomes:

$$9\int \theta(\sec^2(\theta) - 1)\,d\theta = 9\left[\int \theta\sec^2(\theta)\,d\theta - \int \theta\,d\theta\right]$$

The first part requires integration by parts with $u = \theta$ and $dv = \sec^2(\theta)\,d\theta$, leading to $\theta\tan(\theta) - \int \tan(\theta)\,d\theta$. The final result is: **Answer:** $9\left[\theta\tan(\theta) + \ln|\cos(\theta)| - \frac{\theta^2}{2}\right] + C$

2 Problem Types and Techniques Used

- **U-Substitution:** This was a fundamental technique used in problems 1a, 2a, 3a, 4b, 5, 6, and 8.
- **Integration by Parts:** This method was essential for problems 3b, 3c, 7, 9, and 10.
- **Trigonometric Integrals:** Problems 4a, 4b, and 4c involved powers and multiples of trigonometric functions, requiring specific identities.
- **Trigonometric Substitution:** Problem 2c utilized this technique to simplify a radical expression.

- **Partial Fraction Decomposition:** Problem 1c was solved using this algebraic method.
- **Standard Integral Forms:** Problems 1b and 2b were recognizable as basic antiderivatives involving inverse trigonometric functions.

3 Algebraic Manipulations and Tricks

- **Trigonometric Identities:**
  - Half-angle identity: $\sin^2(x) = (1 - \cos(2x))/2$ (Problem 4a).
  - Pythagorean identity: $\sin^2(x) = 1 - \cos^2(x)$ (Problem 4b).
  - Double angle identity: $\sin(2t) = 2\sin(t)\cos(t)$ (Problem 7).
  - Identity for $\tan^2(\theta)$: $\tan^2(\theta) = \sec^2(\theta) - 1$ (Problem 10).
- **Completing the Square and Factoring:** While not explicitly used in the final solutions, these are often considered for rational functions and were part of the initial analysis for problems like 1c and 8.
- **Substitution before Integration by Parts:** In problem 7, a trigonometric identity was used to simplify the integrand before applying integration by parts.