Optimal Variance Swap Replication

Peter B Lee, AlphaVols[1], SAAM[2]

December 7, 2024

Abstract

We make a simple observation that variance swap replication formula is an integral of smooth function over strike space from zero to infinity and hence can be well approximated by quadrature methods. Our main insight of the paper is that not only can quadrature methods be used for efficient pricing of fair variance strike, the weights of quadrature summation can be interpreted as weights for strip of vanilla options that replicate the variance swap itself. For SPX, we demonstrate it requires around ten vanilla options to closely approximate variance swap fair value and requires even less number to nearly fully hedge variance swap risk exposure. We conclude by demonstrating its effectiveness on August 5, 2024 when carry unwind lead to unprecedented volatility shock. Finally, our methodology can be easily applied to other European style derivative contracts such as gamma swaps and conditional variance swaps.

[1] peter@alphavols.com – Peter Lee is Founder and CEO of AlphaVols
[2] peter@saamfund.com – Peter Lee is Founding Principal of Sage Arin Asset Management

Introduction

We make a simple observation that variance swap replication formula is an integral of smooth function over strike space from zero to infinity and hence can be well approximated by quadrature methods. Our main insight of the paper is that not only can quadrature methods be used for efficient pricing of fair variance strike, the weights of quadrature summation can be interpreted as weights for strip of vanilla options that replicate the variance swap itself. For SPX, we demonstrate it requires around ten vanilla options to closely approximate variance swap fair value and requires even less number to nearly fully hedge variance swap risk exposure. We conclude by demonstrating its effectiveness on August 5, 2024 when carry unwind lead to unprecedented volatility shock. Finally, our methodology can be easily applied to other European style derivative contracts such as gamma swaps and conditional variance swaps.

Variance Swap

For ease of discussion, we take risk-free rate and dividend to be zero and work with continuous time. Using Taylor expansion for the log function, one has up to 2nd order in $\frac{dS}{S}$

$$\frac{1}{2}\left(\frac{dS}{S}\right)^2 = \frac{dS}{S} - d\,ln\,S \qquad (1)$$

Payoff of a variance swap contract with vega notional v and strike K_v is given as

$$P = \frac{v}{2K_v}\left(\frac{1}{T}\int_0^T \left(\frac{dS_t}{S_t}\right)^2 - K_v^2\right) \qquad (2)$$

Plugging equation (1) into equation (2), the payoff can be expressed as

$$\begin{aligned} P &= \frac{v}{2K_v}\left(\frac{2}{T}\int_0^T \left(-dln\,S_t + \frac{dS_t}{S_t}\right) - K_v^2\right) \\ &= \frac{v}{2K_v}\left(-\frac{2}{T}ln\,\frac{S_T}{S_0} + 2\int_0^T \frac{dS_t}{S_t} - K_v^2\right) \qquad (3) \end{aligned}$$

The first two terms can be viewed as replication strategy for variance swap by entering in to European style log payoff contract and continuously delta hedging to maintain fixed notional quantity of underlying throughout the life of the trade. As usual, we take expectation value in risk-neutral measure to get fair value of variance swap contract

$$V = \frac{v}{2K_v}\left(-\frac{2}{T}\left(E_0[ln\ \frac{S_T}{S_0}]\right) - K_v^2\right) \tag{4}$$

Any European payoff can be decomposed using calls as puts by recalling that

$$\frac{\partial \max\,(S-K,0)}{\partial K} \equiv \frac{\partial (S-K)^+}{\partial K} = -H(S-K)$$

$$\frac{\partial^2 (S-K)^+}{\partial K^2} = \delta(S-K)$$

where *H* is Heaviside step function and δ is the Dirac delta function. Similarly, we have

$$\frac{\partial^2 (K-S)^+}{\partial K^2} = \delta(S-K)$$

Therefore, we have

$$f(S) = \int_0^\infty \delta(S-K) f(K) dK = \int_0^\infty \frac{\partial^2 (S-K)^+}{\partial K^2} f(K) dK$$

Integrating above equation by parts twice yields the well-known result

$$f(S) = \int_0^\infty (S-K)^+ \frac{\partial^2 f(K)}{\partial K^2} dK + \frac{\partial (S-K)^+}{\partial K} f(K)\Big|_0^\infty - (S-K)^+ \frac{\partial f(K)}{\partial K}\Big|_0^\infty$$

$$E[f(S)] = \int_0^\infty call(S,K) \frac{\partial^2 f(K)}{\partial K^2} dK - f(0) + \frac{\partial f(K=0)}{\partial K} S$$

And similarly for puts. Applying above results to the log payoff contract and splitting calls and puts at arbitrary strike point a and taking consideration of boundary conditions leads to

$$E_0[ln\ \frac{S_T}{S_0}] = -\int_0^a put(S_0,K) \frac{1}{K^2} dK - \int_a^\infty call(S_0,K) \frac{1}{K^2} dK + \frac{S_0 - a}{a} \tag{5}$$

As usual for computational convenience (in addition out-of-money options are more liquid), we set $a = S_0$ in equation (5) to derive fair value expression in terms of vanilla options

$$V = \frac{v}{2K_v}\left(\frac{2}{T}\left(\int_0^{S_0} put(S_0,K) \frac{1}{K^2} dK + \int_{S_0}^\infty call(S_0,K) \frac{1}{K^2} dK\right) - K_v^2\right) \tag{6}$$

Finally, setting $V = 0$ and solving for strike gives equation for fair variance strike

$$K_v^2 = \frac{2}{T}\left(\int_0^{S_0} put(S_0, K)\frac{1}{K^2}dK + \int_{S_0}^{\infty} call(S_0, K)\frac{1}{K^2}dK\right) \quad (7)$$

In figure 1, we plot the integrand as of 4pm EST on December 6, 2024 for Jan 17, 2025 PM expiry.

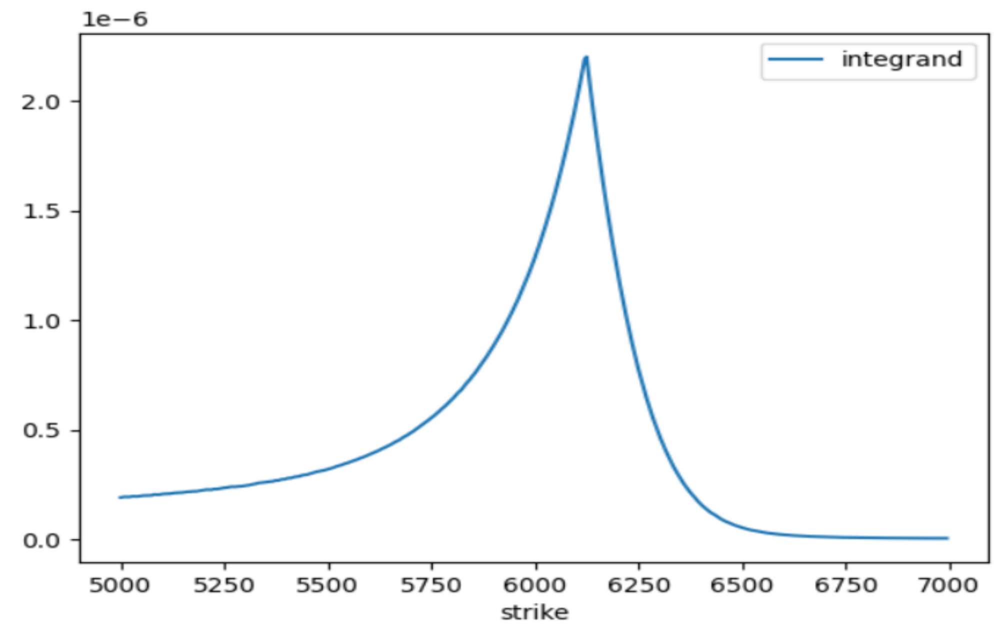

Fig 1. Jan 17, 2025 SPXW variance swap integrand as of Dec 6, 2024 close

Replication Procedure

For exact replication of variance swaps, one needs to trade all available strikes in the market using a close approximation to the integral formula which in theory requires infinite number of strikes. Usually, rectangle or trapezoidal and even Simpson rule discretization methods are used which are part of the Newton-Cotes method [2]. Other authors have tried to solve for vanilla option weights by minimizing error between the strip of replicating vanilla options to the log contract [3]. However, motivated by the smoothness of the function and empirical studies which show that volatility surface is driven mostly by only three factors (level, skew, smile) [1], we propose a procedure to replicate the integral using more effective quadrature methods such as the Gaussian-Quadrature. The Gaussian-Quadrature method can not only be used to efficiently price variance swaps, but the weights of the Legendre polynomial can be interpreted as number of options required to replicate it. For puts one has

$$K_{put}^2 = \frac{2}{T}\int_{K_{min}}^{S_0} put(S_0, K)\frac{1}{K^2}dK \cong \frac{S_0 - K_{min}}{T}\sum_{i=1}^{n} w_i \frac{put(S_0, K_i)}{K_i^2} \tag{8}$$

where

$$K_i = \frac{1}{2}\big(S_0 + K_{min} + \xi_i(S_0 - K_{min})\big) \tag{9}$$

$$w_i = \frac{2}{(1-\xi_i^2)[P_n'(\xi_i)]^2} \tag{10}$$

and ξ_i are i^{th} roots of the Legendre polynomials, $P_n(x)$, with n being the number of put options used. For calls, we make a change of variable, $x = \frac{1}{K}$, such that the domain of integration is finite for faster numerical convergence.

$$K_{call}^2 = \frac{2}{T}\int_{S_0}^{K_{max}} call(S_0, K)\frac{1}{K^2}dK = -\frac{2}{T}\int_{\frac{1}{S_0}}^{\frac{1}{K_{max}}} call(S_0, K(x))dx \tag{11}$$

Similarly applying the Gaussian-Qudrature method, we have for calls using m number of strikes

$$K_{call}^2 = \frac{2}{T}\int_{S_0}^{K_{max}} call(S_0, K)\frac{1}{K^2}dK = \frac{2}{T}\int_{\frac{1}{K_{max}}}^{\frac{1}{S_0}} call(S_0, K(x))dx \tag{12}$$

$$\cong \frac{\frac{1}{S_0} - \frac{1}{K_{max}}}{T}\sum_{i=1}^{m} w_i call(S_0, K(x_i))$$

where

$$x_i = \frac{1}{2}\left(\frac{1}{S_0} + \frac{1}{K_{max}} + \xi_i\left(\frac{1}{S_0} - \frac{1}{K_{max}}\right)\right) \tag{13}$$

The option weights to replicate variance swaps is just $\frac{v}{2K}\frac{S_0 - K_{min}}{TK_i^2}w_i$ for puts and $\frac{v}{2K}\frac{\frac{1}{S_0} - \frac{1}{K_{max}}}{T}w_i$ for calls. Finally, fair value of variance swap using this procedure yields

$$K_v^2 = \frac{S_0 - K_{min}}{T}\sum_{i=1}^{n} w_i \frac{put(S_0, K_i)}{K_i^2} + \frac{\frac{1}{S_0} - \frac{1}{K_{max}}}{T}\sum_{j=1}^{m} w_j call(S_0, K(x_j)) \tag{14}$$

where

$$K_i^{put} = \frac{1}{2}\left(S_0 + K_{min} + \xi_i(S_0 - K_{min})\right) \qquad \text{for } i = 1 \; to \; n \qquad (15)$$

$$K_j^{call} = \frac{2}{\frac{1}{S_0} + \frac{1}{K_{max}} + \xi_j\left(\frac{1}{S_0} - \frac{1}{K_{max}}\right)} \qquad \text{for } j = 1 \; to \; m \qquad (16)$$

Using the results from Gaussian quadrature - Wikipedia, we have following roots and weights

Number of points, n	Points, x_i		Weights, w_i	
1	0		2	
2	$\pm\frac{1}{\sqrt{3}}$	±0.57735...	1	
3	0		$\frac{8}{9}$	0.888889...
	$\pm\sqrt{\frac{3}{5}}$	±0.774597...	$\frac{5}{9}$	0.555556...
4	$\pm\sqrt{\frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}}$	±0.339981...	$\frac{18+\sqrt{30}}{36}$	0.652145...
	$\pm\sqrt{\frac{3}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}}$	±0.861136...	$\frac{18-\sqrt{30}}{36}$	0.347855...
5	0		$\frac{128}{225}$	0.568889...
	$\pm\frac{1}{3}\sqrt{5 - 2\sqrt{\frac{10}{7}}}$	±0.538469...	$\frac{322+13\sqrt{70}}{900}$	0.478629...
	$\pm\frac{1}{3}\sqrt{5 + 2\sqrt{\frac{10}{7}}}$	±0.90618...	$\frac{322-13\sqrt{70}}{900}$	0.236927...

Below we summarize main results of this paper

Table 1. Variance Swap Replicating Vanilla Option Portfolio

Replication Procedure	Puts ($i = 1\ to\ n$)	Calls ($j = 1\ to\ m$)
Number of options	$\frac{v}{2K_v K_i^2 T}(S_0 - K_{min})w_i$	$\frac{v}{2K_v T}\left(\frac{1}{S_0} - \frac{1}{K_{max}}\right)w_j$
Strikes to use	$\frac{1}{2}\left(S_0 + K_{min} + \xi_i(S_0 - K_{min})\right)$	$\frac{2}{\frac{1}{S_0} + \frac{1}{K_{max}} + \xi_j\left(\frac{1}{S_0} - \frac{1}{K_{max}}\right)}$

Observation

Below, we show how the ten-point (five puts and five calls) replication method discussed above compares to integrating over all strikes in the market over 500 to 8000 strikes for SPX from 1/1/2024 to 12/6/2024 to get the fair value one-month variance swap strike

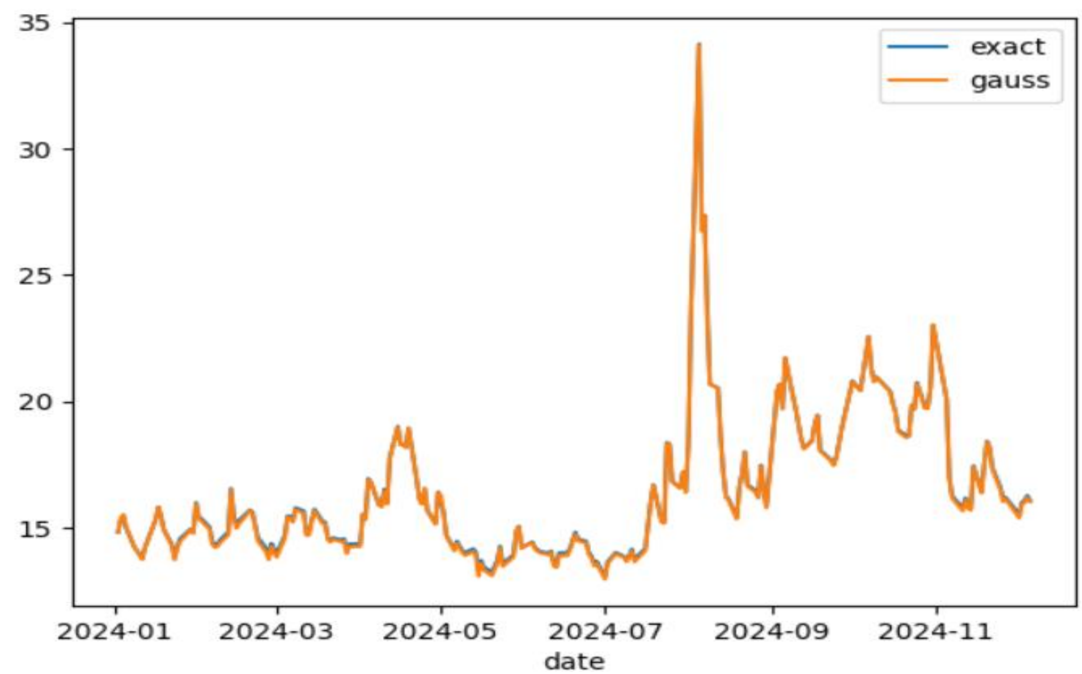

Fig 2. One-month SPXW variance swap vs 10-point approximation

We demonstrate robustness of our methodology by seeing how the 10-point Gaussian Quadrature replication portfolio performed on August 5th, 2024 for SPX variance swap with expiry October 18th, 2024 entered as of August 2nd, 2024. For $1mm vega and variance strike of 22 strike with $K_{min} = 1000$ and $K_{max} = 10000$, Oct 18th expiry variance swap using the quadrature method had following weights (risks are calculated using AlphaVols risk tool) on August 2nd.

port	tkr	tkr_type	prod_type	expiry	strike	type	qty	mult	fit_time	spot	vol	mark	bs_delta	bs_vega	premium_$	intrinsic_$	delta_$	gamma_$1%	vega_1v
grand total											nan	265.52	3.60	25.05	12,469,343	0	-80,323,738	25,183,129	1,341,256
spx	spxw	index	option	10/18/2024 16:00	1200	p	7600	100	08/02 16:00	5341.44	100.95	0.21	-0.03	0.03	156,290	0	-1,166,144	92,098	20,002
spx	spxw	index	option	10/18/2024 16:00	2000	p	5500	100	08/02 16:00	5341.44	77.28	0.97	-0.16	0.13	535,882	0	-4,722,212	425,972	70,820
spx	spxw	index	option	10/18/2024 16:00	3200	p	2600	100	08/02 16:00	5341.44	52.37	5.60	-1.16	0.75	1,456,047	0	-16,114,977	1,734,554	195,417
spx	spxw	index	option	10/18/2024 16:00	4375	p	1160	100	08/02 16:00	5341.44	31.98	25.50	-6.89	3.29	2,957,976	0	-42,704,060	5,541,688	381,289
spx	spxw	index	option	10/18/2024 16:00	5190	p	410	100	08/02 16:00	5341.44	20.54	114.40	-32.66	8.93	4,690,436	0	-71,516,347	8,288,101	366,312
spx	spxw	index	option	10/18/2024 16:00	5510	c	215	100	08/02 16:00	5341.44	16.56	113.45	40.44	9.60	2,439,166	0	46,438,471	5,791,575	206,369
spx	spxw	index	option	10/18/2024 16:00	6035	c	433	100	08/02 16:00	5341.44	13.29	4.65	3.63	1.97	201,293	0	8,406,657	2,989,735	85,500
spx	spxw	index	option	10/18/2024 16:00	7000	c	515	100	08/02 16:00	5341.44	19.83	0.37	0.26	0.20	19,311	0	723,170	242,929	10,363
spx	spxw	index	option	10/18/2024 16:00	8350	c	435	100	08/02 16:00	5341.44	30.17	0.23	0.11	0.09	9,853	0	262,023	62,416	4,052
spx	spxw	index	option	10/18/2024 16:00	9600	c	215	100	08/02 16:00	5341.44	37.42	0.14	0.06	0.05	3,089	0	69,681	14,061	1,132
sub total											nan	16.41	3.60	25.05	12,469,343	0	-80,323,738	25,183,129	1,341,256

Fig 3. $1m vega 10/18/2024 variance swap replication using 10-point approximation

port	tkr	tkr_type	prod_type	expiry	strike	type	qty	mult	fit_time	spot	vol	mark	bs_delta	bs_vega	premium_$	intrinsic_$	delta_$	gamma_$1%	vega_1v
grand total											nan	394.82	-27.93	26.92	26,207,352	18,612	-192,539,897	25,918,677	1,923,992
spx	spxw	index	option	10/18/2024 16:00	1200	p	7600	100	08/05 16:00	5189.55	113.74	0.90	-0.10	0.08	683,846	0	-4,083,285	262,298	62,997
spx	spxw	index	option	10/18/2024 16:00	2000	p	5500	100	08/05 16:00	5189.55	89.47	4.20	-0.54	0.37	2,309,682	0	-15,535,344	1,082,605	204,526
spx	spxw	index	option	10/18/2024 16:00	3200	p	2600	100	08/05 16:00	5189.55	60.69	17.45	-2.84	1.55	4,536,967	0	-38,323,449	3,145,533	403,129
spx	spxw	index	option	10/18/2024 16:00	4375	p	1160	100	08/05 16:00	5189.55	38.52	67.70	-13.50	5.18	7,853,135	0	-81,246,205	7,383,216	600,488
spx	spxw	index	option	10/18/2024 16:00	5190	p	410	100	08/05 16:00	5189.55	24.90	213.25	-44.67	9.43	8,743,256	18,612	-95,053,698	7,353,756	386,573
spx	spxw	index	option	10/18/2024 16:00	5510	c	215	100	08/05 16:00	5189.55	19.72	86.10	30.22	8.32	1,851,141	0	33,718,215	4,294,898	178,853
spx	spxw	index	option	10/18/2024 16:00	6035	c	433	100	08/05 16:00	5189.55	16.17	4.43	3.04	1.64	191,665	0	6,835,578	2,081,362	71,072
spx	spxw	index	option	10/18/2024 16:00	7000	c	515	100	08/05 16:00	5189.55	22.91	0.55	0.34	0.24	28,337	0	910,125	260,337	12,594
spx	spxw	index	option	10/18/2024 16:00	8350	c	435	100	08/05 16:00	5189.55	31.81	0.18	0.09	0.07	7,884	0	204,106	47,403	3,184
spx	spxw	index	option	10/18/2024 16:00	9600	c	215	100	08/05 16:00	5189.55	37.55	0.07	0.03	0.03	1,439	0	34,061	7,268	576
sub total											nan	34.48	-27.93	26.92	26,207,352	18,612	-192,539,897	25,918,677	1,923,992

Fig 4. 10-point approximation risk snapshot as of 8/5/2024 market close

On August 5th SPX dropped 3%, hence the total PnL in $m of replication portfolio was

$$-0.03 \times 80+(26.2-12.5) = -2.4+13.7=11.3$$

Integrating over strip of SPX options with strikes from 1000 to 10000, one gets variance swap fair value of around 23.5 on 8/2/2024 and 33.2 on 8/5/2024, hence the variance swap PnL would've been

$$\frac{1}{2\times 22}(33.2^2 - 23.5^2) = 12.5$$

Even in such extreme scenario, our methodology closely mirrored full variance swap exposure. More importantly, vega of the portfolio increased by 43% almost exactly in line with increase in volatility from 23.5 to 33.2 since variance swap vega scales linearly with vol as can be seen in figure 4 below. Lastly, we plot $gamma of portfolio in figure 3 vs both vol and spot shock to illustrate that $gamma is stable as should be for a variance swap replicating portfolio. Lastly, we note that call options with highest strikes barely contribute to overall risk and for practical hedging purposes, one requires only a handful of options (six to eight) while for fair value calculation, one needs around ten.

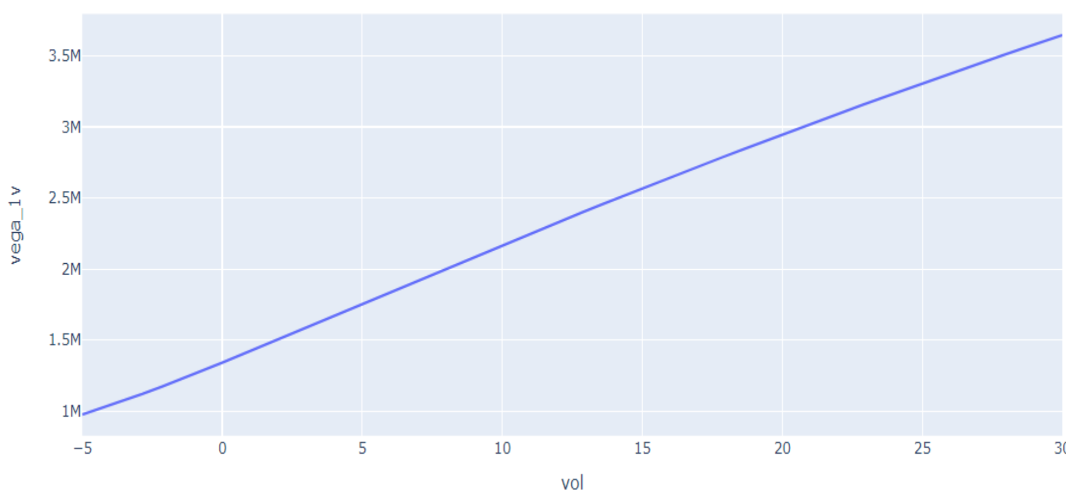

Fig 5. 10-point approximation vega profile vs vol shock from -5 to +30

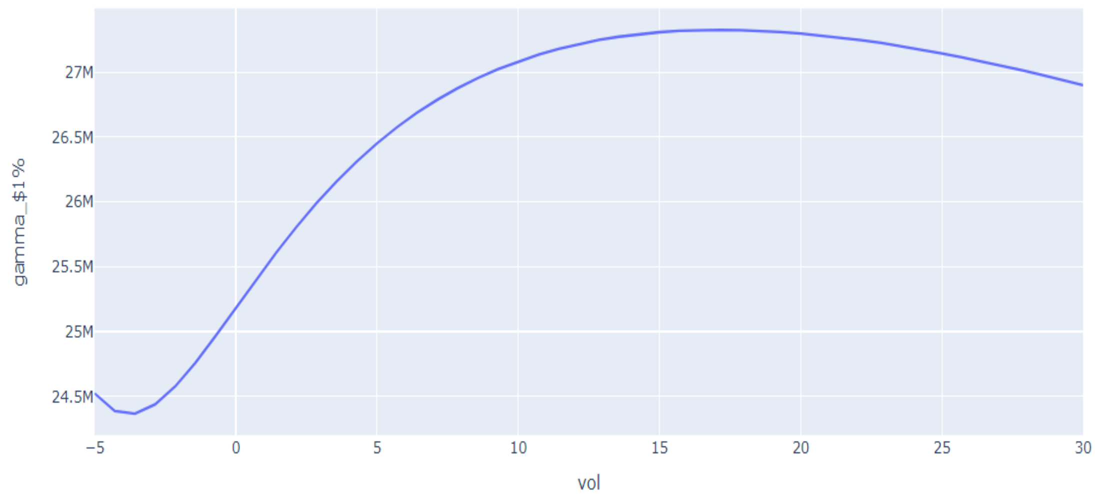

Fig 6. 10-point approximation $gamma profile vs vol shock from -5 to +30

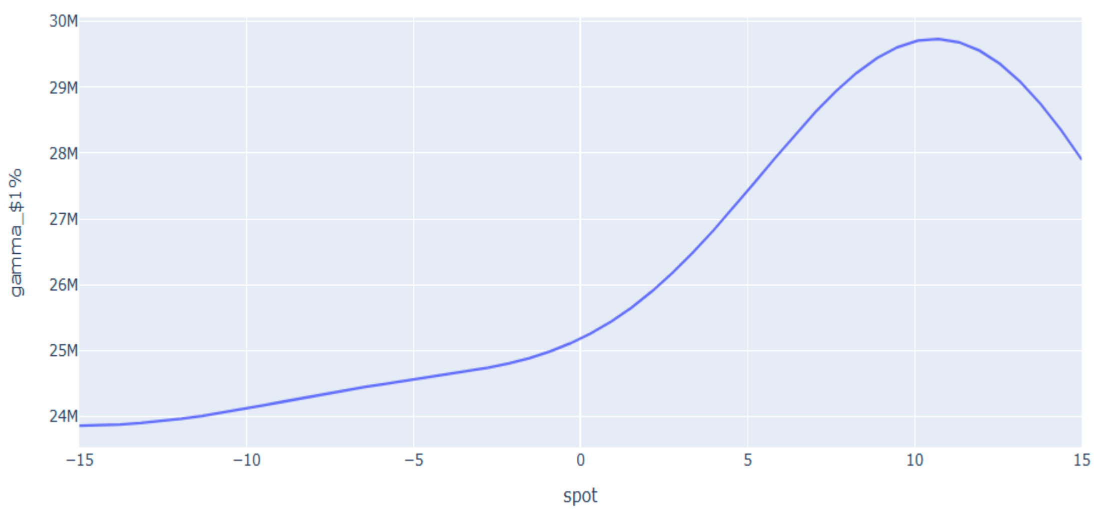

Fig 7. 10-point approximation $gamma profile vs spot shock from -15% to +15%

Reference

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