Quantitative Analyst Role for Exotic Derivative Desk

You are interviewing for a quantitative analyst position on the exotic derivatives desk at a leading investment bank. The desk deals with complex products where modeling the dynamics of volatility is crucial. Consider the following:

Background:

1. **Local Volatility Model:**

   The local volatility model assumes that the volatility of an asset is a deterministic function of the asset price S and time t, denoted as $\sigma_{\text{loc}}(S,t)$. The asset price follows:

   $$dS_t = rS_t dt + \sigma_{\text{loc}}(S_t,t)S_t dW_t$$

   where:

   - r is the constant risk-free interest rate,
   - W_t is a standard Brownian motion.

2. **Stochastic Volatility Model (Heston Model):**

   The Heston model introduces stochastic volatility by modeling the variance as a separate stochastic process:

   $$\begin{cases} dS_t = rS_t dt + \sqrt{v_t} S_t dW_t^S \\ dv_t = \kappa(\theta - v_t)dt + \xi\sqrt{v_t}dW_t^v \end{cases}$$

   with:

   - v_t being the instantaneous variance,
   - κ (mean-reversion rate), θ (long-term variance), and ξ (volatility of variance) are positive constants,
   - W_t^S and W_t^v are correlated Brownian motions with correlation coefficient ρ.

1. **Derive the Dupire Local Volatility Formula:**

   Starting from the risk-neutral pricing formula and using the Fokker-Planck (forward Kolmogorov) equation, derive the Dupire formula for the local volatility $\sigma_{\text{loc}}(S, t)$ in terms of the market-observed European call option prices $C_{\text{mkt}}(K, t)$.

2. **Explain Differences Between Local and Stochastic Volatility Models:**

   Discuss the theoretical and practical differences between local volatility models and stochastic volatility models, particularly in how they capture the implied volatility surface and their suitability for pricing and hedging exotic options.

3. **Derive the Characteristic Function of $\ln(S_T)$ in the Heston Model:**

   Under the Heston stochastic volatility model, derive the characteristic function $\phi(u) = E[e^{iu\ln(S_T)}]$ necessary for pricing European options using Fourier transform methods.

4. **Impact on Exotic Option Pricing:**

   Consider a barrier option and an Asian option. Analyze how the choice between a local volatility model and a stochastic volatility model affects the pricing and hedging of these exotic options.

5. **Design a Numerical Scheme for Pricing Under the Heston Model:**

   Propose a finite difference method to price a European barrier option under the Heston stochastic volatility model. Detail how you would set up the two-dimensional PDE, discretize it, handle boundary conditions, and ensure numerical stability and convergence.