Correlation between Spot Rates and Two-Factor Models

Kshitij Anand

01-12-2024

Building the Background

- One-factor and two-factor short-rate models are widely discussed in interest rate modeling.
- In one-factor models, the short rate is represented as a single stochastic process.
- Two-factor models, on the other hand, are represented using two correlated stochastic processes. The short-rate equation can also include a deterministic function of time.
- This document aims to understand the motivation for using two-factor models.

Building the Context

The Vasicek model provides the zero-coupon bond (ZCB) price as:

$$P(t, T) = A(t, T) \exp(-B(t, T) r_t).$$

The continuously compounded spot rate is then defined as:

$$R(t, T) = -\frac{\ln P(t, T)}{T - t}.$$

Substituting the expression for $P(t, T)$, we get:

$$R(t, T) = -\frac{\ln A(t, T)}{T - t} + \frac{B(t, T) r_t}{T - t}.$$

Spot Rate Expression

The continuously compounded spot rate can be expressed as:

$$R(t, T) = a(t, T) + b(t, T)r_t,$$

where:

$$a(t, T) = -\frac{\ln A(t, T)}{T - t}, \quad b(t, T) = \frac{B(t, T)}{T - t}.$$

Here, r_t represents the short rate at time t.

Spot Rates for Two Maturities

In a one-factor short-rate model, the spot rate is expressed as:

$$R(t, T) = a(t, T) + b(t, T)r_t,$$

where:

- $a(t, T) = -\frac{\ln A(t,T)}{T-t}$: A deterministic function of time.
- $b(t, T) = \frac{B(t,T)}{T-t}$: A deterministic function of time.
- r_t: The stochastic short rate at time t.

For two maturities T_1 and T_2, the spot rates are:

$$R(t, T_1) = a(t, T_1) + b(t, T_1)r_t, \quad R(t, T_2) = a(t, T_2) + b(t, T_2)r_t.$$

Correlation Between Spot Rates

The correlation between spot rates is:

$$\mathrm{Corr}(R(t, T_1), R(t, T_2)).$$

Substituting the expressions for $R(t, T_1)$ and $R(t, T_2)$:

$$\mathrm{Corr}(R(t, T_1), R(t, T_2)) = \mathrm{Corr}(b(t, T_1)r_t, b(t, T_2)r_t).$$

Since r_t is the common stochastic term, the correlation simplifies to:

$$\mathrm{Corr}(R(t, T_1), R(t, T_2)) = 1.$$

Insights from One-Factor Models

- In one-factor models, the spot rates for any maturities are perfectly correlated.
- For example, $R(0, 6)$ and $R(0, 8)$ are perfectly correlated.
- This implies that any perturbation in the interest rate curve affects all parts of the curve equally.
- However, this behavior does not match real-world observations.

Two-Factor Model Formulation

In a two-factor model (e.g., G2), the short rate r_t is expressed as:

$$r_t = x_t + y_t,$$

where:

$$dx_t = k_x(\theta_x - x_t)dt + \sigma_x dW_1(t), \quad dy_t = k_y(\theta_y - y_t)dt + \sigma_y dW_2(t).$$

Here, $dW_1(t)$ and $dW_2(t)$ are correlated Brownian motions with:

$$\mathrm{Corr}(dW_1, dW_2) = \rho.$$

Flexible Correlation Structure

The bond price in a two-factor model remains affine:

$$P(t,T) = A(t,T)\exp(-B_x(t,T)x_t - B_y(t,T)y_t).$$

The correlation between spot rates becomes:

$$\mathrm{Corr}(R(t,T_1), R(t,T_2)) = \mathrm{Corr}(b_x(T_1)x_t + b_y(T_1)y_t, b_x(T_2)x_t + b_y(T_2)y_t)$$

This depends on the correlation ρ between x_t and y_t, allowing for more realistic term structure behavior.

Advantages of Two-Factor Models

- Popular models include:
  - Two-Additive-Factor Gaussian Model (G2++)
  - Two-Additive-Factor Extended CIR Model (CIR2++)
  - Two-Factor Hull-White Model
- Two-factor models offer:
  - Greater flexibility due to more parameters.
  - Realistic representation of term structure movements.
  - Closed-form solutions for interest rate derivatives.