The T-Forward Measure

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Building the context

- Change of measure is highly useful in derivatives pricing.
- Each *Numeraire* is represented as a stochastic process and has an associated probability measure.
- The choice of numeraire depends on the specific derivatives we are pricing.
- The Zero-Coupon Bond is one such numeraire that can be used effectively in pricing.

Bank Account as Numeraire

- **Bank Account as Numeraire:**
- In many financial models, we use a *numeraire* to measure the value of financial assets. A common choice for the numeraire is a *bank account* or risk-free asset, which is a hypothetical account that earns a constant risk-free interest rate.
- Let $B(t)$ represent the value of the bank account at time t given B(0) = 1, which is assumed to follow the equation:

  $$B(t) = e^{\int_0^t r(u)\,du}$$

  where $r(u)$ is the instantaneous risk-free rate of interest at time u.
- The bank account $B(t)$ grows exponentially over time, as it accumulates interest at a rate $r(u)$, compounding continuously.

Bank Account as Numeraire

- **Deriving the Ratio $\frac{B(T)}{B(t)}$:**
- To calculate the ratio of the value of the bank account at two different times T and t, we consider the following:

$$\frac{B(T)}{B(t)} = \frac{e^{\int_0^T r(u)\,du}}{e^{\int_0^t r(u)\,du}}$$

- Using the properties of exponents, this simplifies to:

$$\frac{B(T)}{B(t)} = e^{\int_0^T r(u)\,du - \int_0^t r(u)\,du}$$

- Thus, we have the final expression:

$$\frac{B(T)}{B(t)} = e^{\int_t^T r(u)\,du}$$

ZCB as a numeraire

- We select a zero-coupon bond whose maturity T matches that of the derivative.
- Naturally, $P(T, T) = 1$, and $P(t, T)$ denotes the price of the zero-coupon bond at time t.
- The probability measure associated with the ZCB maturing at time T is known as the *T-forward measure*.

How is T-Forward measure helpful?

- Consider an interest rate derivative with payoff at time T denoted by $V(T)$.
- Note that it is not assumed that V depends solely on T.
- Pricing this payoff involves calculating the following expectation under risk-neutral measure B:

$$\text{Price of derivative at time } t : \mathbb{E}^{\mathbb{Q}_B}\left[e^{-\int_t^T r(u)\,du} \cdot V(T) \middle| \mathcal{F}_t \right]$$

Why can't we directly evaluate V(t) from risk-neutral expectation?

- In the case of interest rate derivatives, $e^{-\int_t^T r(u)\,du}$ cannot be taken out of the expectation because interest rates are stochastic and IRD are closely related with the dynamics of interest rates.
- This makes it challenging to calculate $V(t)$ directly under the risk-neutral measure.
- However, we can evaluate the same expression through a measure change.
- Note that $V(t)$ will remain the same regardless of the measure chosen, as measure change is simply a tool to simplify the calculation.

Change of measure

- **Change of Measure: How do we do it?**
- Recall that for any random variable X, the following relationship holds:

$$\mathbb{E}_{\mathbb{Q}_B}[X] = \mathbb{E}_{\mathbb{Q}_T}\left[X \cdot \frac{d\mathbb{Q}_B}{d\mathbb{Q}_T}\right]$$

- Here, $\frac{d\mathbb{Q}_B}{d\mathbb{Q}_T}$ represents the Radon-Nikodym derivative.
- $\mathbb{Q}_B$ is the risk-neutral measure, while $\mathbb{Q}_T$ is the T-forward measure.
- Additionally, the Radon-Nikodym derivative can be expressed as:

$$\frac{d\mathbb{Q}_B}{d\mathbb{Q}_T} = \frac{B(T)}{B(t)} \cdot \frac{P(t,T)}{P(T,T)}$$

  where $B(t)$ and $B(T)$ represent the values of the Bank Account at times t and T, and $P(t,T)$ denotes the price of a zero-coupon bond at time t maturing at time T.

Step-1

- **Changing to the T-Forward Measure:**
- Let $V(T)$ denote the payoff of an interest rate derivative at time T.
- We aim to calculate the price of the derivative at time t, represented as $V(t)$.
- Initially, in the risk-neutral measure $\mathbb{Q}_B$, this is given by:

$$V(t) = \mathbb{E}^{\mathbb{Q}_B}\left[e^{-\int_t^T r(u)\,du} \cdot V(T)\middle|\mathcal{F}_t\right]$$

where $e^{-\int_t^T r(u)\,du}$ accounts for discounting under the stochastic interest rate $r(u)$.

Step-2

- To simplify this expectation, we change to the T-forward measure $\mathbb{Q}_T$.
- By the measure change formula, we have:

$$\mathbb{E}^{\mathbb{Q}_B}[X] = \mathbb{E}^{\mathbb{Q}_T}\left[X \cdot \frac{d\mathbb{Q}_B}{d\mathbb{Q}_T}\right]$$

  for any random variable X.
- Using this, we can rewrite $V(t)$ as:

$$V(t) = \mathbb{E}^{\mathbb{Q}_T}\left[e^{-\int_t^T r(u)\,du} \cdot V(T) \cdot \frac{d\mathbb{Q}_B}{d\mathbb{Q}_T}\middle|\mathcal{F}_t\right]$$

Step-3

- The Radon-Nikodym derivative $\frac{d\mathbb{Q}_B}{d\mathbb{Q}_T}$ is given by:

$$\frac{d\mathbb{Q}_B}{d\mathbb{Q}_T} = \frac{B(T)}{B(t)} \cdot \frac{P(t,T)}{P(T,T)}$$

  where $B(t)$ and $B(T)$ are the values of the chosen numeraire at times t and T, and $P(t,T)$ is the price of a zero-coupon bond at time t maturing at T, and $P(T,T) = 1$.
- Given that $\frac{B(T)}{B(t)} = e^{\int_t^T r(u)\,du}$, we can simplify $\frac{d\mathbb{Q}_B}{d\mathbb{Q}_T}$ as:

$$\frac{d\mathbb{Q}_B}{d\mathbb{Q}_T} = e^{\int_t^T r(u)\,du} \cdot P(t,T)$$

Step-4

- Substituting this into the expression for $V(t)$, we get:

$$V(t) = \mathbb{E}^{\mathbb{Q}_T}\left[e^{-\int_t^T r(u)\,du} \cdot V(T) \cdot e^{\int_t^T r(u)\,du} \cdot P(t,T) \middle| \mathcal{F}_t\right]$$

- Since $e^{-\int_t^T r(u)\,du} \cdot e^{\int_t^T r(u)\,du} = 1$ and P(t, T) is known at time t, we can simplify further:

$$V(t) = P(t,T) \cdot \mathbb{E}^{\mathbb{Q}_T}\left[V(T) \middle| \mathcal{F}_t\right]$$

- The final expression for $V(t)$ is:

$$V(t) = P(t,T) \cdot \mathbb{E}^{\mathbb{Q}_T}\left[V(T) \middle| \mathcal{F}_t\right]$$

- This expectation can be easily computed.

Example

- Let us take a European call option with maturity T, strike K and written on a unit-principal zero-coupon bond with maturity S>T.
- The price V(t) for this derivative in risk-neutral measure will be:

$$V(t) = E^{Q^B}\left(e^{-\int_t^T r(u)\,du}(P(T,S)-K)^+ \middle| \mathcal{F}_t\right).$$

- Changing the measure to T-forward measure:

$$V(t) = P(t,T)\cdot E^{Q^T}\left(P(T,S)-K)^+ \middle| \mathcal{F}_t\right).$$

- If $P(T,S)$ follows a lognormal distribution conditional on $\mathcal{F}_t$ under the T-forward measure Q^T, this expectation can be computed similarly to a call option on a stock in the Black-Scholes framework. In this setup, we treat $P(T,S)$ as the "underlying asset" with strike K, and a suitable volatility is used corresponding to $P(T,S)$ under Q^T. Thus, the expectation simplifies to a Black-like formula.