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Topic: Part 3: Arriving at the Black-Scholes-Merton Formula and Its Practical Implementation

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What we will learn:

- Deriving the Black-Scholes Partial Differential Equation (PDE).
- The Black-Scholes Formula
- Python Implementation

Derivation of the Black-Scholes Partial Differential Equation (PDE)

The **Black-Scholes PDE** is the cornerstone of the Black-Scholes-Merton model, describing how the price of an option evolves over time under certain assumptions. It is derived by constructing a **delta-hedged portfolio** to eliminate risk, allowing the portfolio to grow at the risk-free rate r.

Here's a detailed step-by-step derivation:

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1. The Option Price as a Function

Let the option price V depend on:

- The price of the underlying asset, S_t.
- Time to expiration, t.

Using **Itô's Lemma**, the change in V is:

$$dV = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \left(\sigma^2 S^2\right) dt$$

Where:

- $\frac{\partial V}{\partial t} dt$: Change in the option value due to the passage of time.
- $\frac{\partial V}{\partial S} dS$: Change in the option value due to changes in the underlying asset price.
- $\frac{1}{2} \frac{\partial^2 V}{\partial S^2} \left(\sigma^2 S^2\right) dt$: Captures the effect of volatility on the option price.

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2. Constructing the Delta-Hedged Portfolio

Construct a portfolio Π consisting of:

1. A **long position** in the option $+V$.
2. A **short position** in the underlying asset $-\Delta S$, where $\Delta=\frac{\partial V}{\partial S}$ (the delta of the option).

The portfolio value is:

$$\Pi=V-\Delta S$$

The change in the portfolio value is:

$$d\Pi=dV-\Delta dS$$

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3. Substitute for dV and dS

Substitute the expressions for dV (from Itô's Lemma) and dS (from Geometric Brownian Motion) into the equation for $d\Pi$.

From GBM:

$$dS=\mu S dt+\sigma S dW$$

Substituting dV and dS:

$$d\Pi=\left(\frac{\partial V}{\partial t}dt+\frac{\partial V}{\partial S}dS+\frac{1}{2}\frac{\partial^2 V}{\partial S^2}\left(\sigma^2 S^2\right)dt\right)-\Delta\left(\mu S dt+\sigma S dW\right)$$

Simplify:

$$d\Pi=\left(\frac{\partial V}{\partial t}+\frac{1}{2}\frac{\partial^2 V}{\partial S^2}\left(\sigma^2 S^2\right)+\frac{\partial V}{\partial S}\mu S-\Delta\mu S\right)dt-\Delta\sigma S dW$$

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4. Eliminating Risk

To make the portfolio risk-free, the stochastic term $\Delta\sigma S dW$ must cancel out. This is achieved by setting:

$$\Delta=\frac{\partial V}{\partial S}$$

The change in the portfolio then becomes:

$$d\Pi=\left(\frac{\partial V}{\partial t}+\frac{1}{2}\frac{\partial^2 V}{\partial S^2}\left(\sigma^2 S^2\right)+\mu S\frac{\partial V}{\partial S}-\mu S\frac{\partial V}{\partial S}\right)dt$$

Simplify further:

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \left(\sigma^2 S^2 \right) \right) dt$$

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5. Applying the No-Arbitrage Condition

Since the portfolio is risk-free, it must grow at the risk-free rate r. Thus:

$$d\Pi = r\Pi\, dt$$

Substitute (\Pi = V - \Delta S) into the equation:

$$r\left(V - \Delta S\right) dt = \left(\frac{\partial V}{\partial t} + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \left(\sigma^2 S^2 \right) \right) dt$$

Substitute (\Delta = \frac{\partial V}{\partial S}):

$$r\left(V - S \frac{\partial V}{\partial S}\right) dt = \frac{\partial V}{\partial t} dt + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \left(\sigma^2 S^2 \right) dt$$

Divide through by (dt):

$$rV - rS \frac{\partial V}{\partial S} = \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}$$

Rearrange terms to obtain the **Black-Scholes PDE**:

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0$$

Key Insights

- The Black-Scholes PDE ensures that the option price evolves consistently with the no-arbitrage principle.
- The equation incorporates:
  - Time decay $\frac{\partial V}{\partial t}$.
  - The impact of volatility $\frac{\partial^2 V}{\partial S^2}$.
  - The sensitivity of the option to the underlying price $\frac{\partial V}{\partial S}$.
  - The risk-free rate r.

The Black-Scholes PDE governs the behavior of the option price over time. Solving this equation under specific boundary conditions (e.g., payoff at expiration) yields the celebrated **Black-Scholes formula** for European call and put options. In the next section, we'll solve this PDE and implement the formula in Python.

Solving the Black-Scholes PDE and Deriving the Black-Scholes Formula

The **Black-Scholes Partial Differential Equation (PDE)**, derived in the previous section, governs how the price of an option evolves over time. To solve this PDE, we apply specific **boundary conditions**, which reflect the payoff of the option at expiration. Let's outline the steps to solve the PDE and derive the famous **Black-Scholes formula**.

Boundary Conditions

The solution to the Black-Scholes PDE depends on the option type (call or put) and its payoff at expiration:

1. **European Call Option**: At expiration (T):

   $$C(S,T) = max(S-K,0)$$

   This means the call option is worth the difference between the asset price S and the strike price K, or zero if the option is out-of-the-money.

2. **European Put Option**: At expiration T:

   $$P(S,T) = max(K-S,0)$$

   This means the put option is worth the difference between the strike price K and the asset price S, or zero if the option is out-of-the-money.

Transforming the PDE

To solve the Black-Scholes PDE, a series of transformations simplifies the problem:

1. **Substitute the Variable**:
   Change the variables from S and t to dimensionless variables x and τ, where:

   $$x = \ln\left(\frac{S}{K}\right) \text{and} \, \tau = T - t$$

2. **Reduce to the Heat Equation**:
   Using these substitutions and other transformations, the Black-Scholes PDE is reduced to a **heat equation**, a well-known equation in mathematical physics:

$$\frac{\partial u}{\partial \tau}=\frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial x^2}$$

   Solving this equation gives us the option price in terms of $u(x,\tau)$, which is then transformed back to $V(S,t)$.

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2. The Black-Scholes Formula

The **Black-Scholes formula** provides the theoretical value of European call and put options under a set of simplifying assumptions. It is the solution to the **Black-Scholes Partial Differential Equation (PDE)** derived earlier. Here's a breakdown of the formula, its components, and the intuition behind each term.

The Black-Scholes Formula

For a European Call Option:

$$C(S,t)=S\,N(d_1)-K\,e^{-r(T-t)}N(d_2)$$

For a European Put Option:

$$P(S,t)=K\,e^{-r(T-t)}N(-d_2)-S\,N(-d_1)$$

Terms Explained

1. S: Current price of the underlying asset.

2. K: Strike price of the option (price at which the option can be exercised).

3. $T-t$: Time to expiration (in years).

4. r: Risk-free interest rate (continuously compounded).

5. σ: Volatility of the underlying asset (annualized standard deviation of returns).

6. $N(d)$: The cumulative distribution function (CDF) of the standard normal distribution.
   - $N(d)$ represents the probability that a standard normal variable is less than d. For example, $N(1.0) \approx 0.8413$, indicating an 84.13% probability.

Key Components

1. Defining d_1 and d_2:

The Black-Scholes formula involves two key variables, d_1 and d_2, which are calculated as:

$$d_1 = \frac{\ln(S/K) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

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2. Intuition Behind d_1 and d_2:

- d_1:
  Measures the **"moneyness"** of the option, adjusted for volatility and time to expiration. It represents the likelihood (under the risk-neutral measure) that the option will be profitable if exercised at a future date.
- d_2:
  Represents the **probability that the option will expire in the money** under the risk-neutral measure. It adjusts d_1 by subtracting the uncertainty (volatility over time).

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3. The Role of $N(d_1)$ and $N(d_2)$:

- $N(d_1)$:
  The probability (in a risk-neutral world) that the option will be exercised, considering both volatility and time.
- $N(d_2)$:
  The probability that the option will expire in the money.

Formula Intuition: Breaking It Down

For a Call Option:

$$C(S,t)=S\,N(d_1)-K\,e^{-r(T-t)}N(d_2)$$

1. $S\,N(d_1)$**:**
   Represents the **current value of the option** if it were exercised immediately. It's the product of the asset price and the probability that the option will be exercised.

2. $K\,e^{-r(T-t)}N(d_2)$**:**
   Represents the **discounted present value** of paying the strike price K at expiration. It accounts for the time value of money and the probability of expiring in the money.

The formula essentially calculates:

- The potential gain if the option is exercised, minus

- The cost of exercising it, adjusted for time and probability.

For a Put Option:

$$P(S,t)=K\,e^{-r(T-t)}N(-d_2)-S\,N(-d_1)$$

1. $K\,e^{-r(T-t)}N(-d_2)$**:**
   The present value of receiving the strike price at expiration, adjusted for the probability of expiring in the money.

2. $S\,N(-d_1)$**:**
   The current value of the underlying asset, weighted by the probability that the option will not be exercised.

Interpretation of the Formula

1. **Call Options:** The value of a call option depends on:
   - How far the current price is from the strike price $(S-K)$.
   - The time remaining until expiration.
   - The volatility of the underlying asset.

   Intuitively, a call option becomes more valuable as:
   - The asset price increases.
   - Time to expiration increases.
   - Volatility increases.
2. **Put Options:** The value of a put option increases as:
   - The asset price decreases.

- Time to expiration increases.
- Volatility increases.

Practical Uses

The Black-Scholes formula is widely used in:

1. **Option Pricing**:
   Determining whether options are fairly priced in the market.

2. **Risk Management**:
   Calculating the Greeks (e.g., delta, gamma) to hedge portfolios effectively.

3. **Trading Strategies**:
   Exploiting mispriced options or constructing arbitrage strategies.

Key Assumptions

While the formula is groundbreaking, it is based on certain assumptions:

1. The underlying asset follows Geometric Brownian Motion.
2. Volatility (σ) is constant.
3. The risk-free interest rate (r) is constant.
4. The market is frictionless (no transaction costs or taxes).
5. The option can only be exercised at expiration (European-style).

Next Steps

Now that we've understood the formula and its components, the next step is to implement it in Python.

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3. Python Implementation

Here's how to implement the Black-Scholes formula in Python:

```python
import numpy as np
from scipy.stats import norm

def black_scholes(S, K, T, r, sigma, option_type="call"):
    """
    Black-Scholes formula for European options.
```

```
    Parameters:
    S : float : Current price of the underlying asset
    K : float : Strike price
    T : float : Time to expiration (in years)
    r : float : Risk-free interest rate
    sigma : float : Volatility of the underlying asset
    option_type : str : "call" for call option, "put" for put option

    Returns:
    float : Option price
    """
    d1 = (np.log(S / K) + (r + 0.5 * sigma**2) * T) / (sigma *
np.sqrt(T))
    d2 = d1 - sigma * np.sqrt(T)

    if option_type == "call":
        return S * norm.cdf(d1) - K * np.exp(-r * T) * norm.cdf(d2)
    elif option_type == "put":
        return K * np.exp(-r * T) * norm.cdf(-d2) - S * norm.cdf(-d1)
    else:
        raise ValueError("Invalid option type. Use 'call' or 'put'.")

# Example usage
S = 100  # Current stock price
K = 110  # Strike price
T = 1    # Time to maturity (1 year)
r = 0.05  # Risk-free rate (5%)
sigma = 0.2  # Volatility (20%)

call_price = black_scholes(S, K, T, r, sigma, option_type="call")
put_price = black_scholes(S, K, T, r, sigma, option_type="put")

print(f"Call Option Price: {call_price:.2f}")
print(f"Put Option Price: {put_price:.2f}")
```

```
Call Option Price: 6.04
Put Option Price: 10.68
```

Practical Implications

- **For Traders**: The Black-Scholes formula helps assess whether options are mispriced in the market, enabling arbitrage opportunities.

- **For Risk Managers**: The formula and its derivatives (the Greeks) guide hedging strategies.