Black-Scholes Option Pricing

Black and Scholes [1] use an arbitrage argument to derive a formula for option pricing.

Notation

s	Stock price
c	Call price
x	Exercise price
r	Risk-free rate of return
μ	Stock return risk premium
σ	Stock return standard deviation
τ	Time to expiration
t	Time

Random Walk

The risk-free asset has the constant return

$$r\,\mathrm{d}t\,.$$

The stock price follows a random walk, with constant mean and variance:

$$\frac{\mathrm{d}s}{s} = (r+\mu)\,\mathrm{d}t + \sigma\,\mathrm{d}z.$$

The stock pays no dividend, so this expression is the return on the stock.

Call Price

Given the model and its parameters, it seems natural that the call price is some function of the stock price and the time to expiration,

$$c(s,\tau).$$

Of course, at the expiration date, the call value is known:

$$c(s,0) = \max[s-x,0]. \tag{1}$$

We solve for $c(s,\tau)$.

Hedge Ratio

Definition 1 (Hedge Ratio) *The hedge ratio is*

$$h := \frac{\partial c}{\partial s}.$$

(In finance, “to hedge” means to take action to reduce or to eliminate risk.)

Risk-Free Portfolio

If the stock price determines the call price, then one can form a risk-free portfolio from the stock and the call.

For example, suppose that the hedge ratio $h = 1/2$. This value means that a one dollar increase in the stock price raises the call price by one-half dollar.

Then buying one share of stock and selling two calls achieves a risk-free portfolio: any increase in the stock price is offset by an equal decline in the value of the two calls.

By Itô's formula,

$$\mathrm{d}c = c_s\,\mathrm{d}s + \frac{1}{2}c_{ss}\,(\mathrm{d}s)^2 - c_\tau\,\mathrm{d}t$$

(as time passes, the time to expiration shrinks, so $\mathrm{d}\tau/\mathrm{d}t = -1$).

The change in the value of the portfolio is

$$
\begin{aligned}
\mathrm{d}s - \frac{1}{h}\mathrm{d}c &= \mathrm{d}s - \frac{1}{c_s}\mathrm{d}c \\
&= \mathrm{d}s - \frac{1}{c_s}\left[c_s\,\mathrm{d}s + \frac{1}{2}c_{ss}\,(\mathrm{d}s)^2 - c_\tau\,\mathrm{d}t\right] \\
&= s\left[(r+\mu)\,\mathrm{d}t + \sigma\,\mathrm{d}z\right] - \frac{1}{c_s}\Big(c_s\left\{s\left[(r+\mu)\,\mathrm{d}t + \sigma\,\mathrm{d}z\right]\right\} \\
&\qquad + \frac{1}{2}c_{ss}\left\{s\left[(r+\mu)\,\mathrm{d}t + \sigma\,\mathrm{d}z\right]\right\}^2 - c_\tau\,\mathrm{d}t\Big) \\
&= -\frac{1}{c_s}\left(\frac{1}{2}c_{ss}s^2\sigma^2 - c_\tau\right)\mathrm{d}t,
\end{aligned}
$$

which is risk-free.

Arbitrage

Since the portfolio is risk-free, to rule out an arbitrage opportunity its return must be the risk-free return. The cost of the portfolio is

$$s - \frac{1}{h}c,$$

so

$$\left(s - \frac{c}{c_s}\right) r\,\mathrm{d}t = -\frac{1}{c_s}\left(\frac{1}{2}c_{ss}s^2\sigma^2 - c_\tau\right)\mathrm{d}t.$$

Black-Scholes Partial Differential Equation

Rearranging gives the following.

Definition 2 (Black-Scholes Partial Differential Equation)

$$c_\tau + rc - rsc_s - \frac{1}{2} c_{ss} s^2 \sigma^2 = 0.$$

As it is not profitable to exercise the option prior to the expiration date, the boundary condition (1) applies, and using it one solves this partial differential equation. The equation is a transformation of the heat equation in physics and has a unique solution.

Black-Scholes Formula

Solution 3

$$
\begin{aligned}
c(s,\tau) = {} & s\mathrm{N}\left[\frac{\ln(s/x) + \left(r + \sigma^2/2\right)\tau}{\sigma\sqrt{\tau}}\right] \\
& - x\mathrm{e}^{-r\tau}\mathrm{N}\left[\frac{\ln(s/x) + \left(r - \sigma^2/2\right)\tau}{\sigma\sqrt{\tau}}\right].
\end{aligned}
$$

Here $\mathrm{N}(v)$ is the cumulative unit normal, the probability that the value is less than or equal to v.

Hedge Ratio

The hedge ratio is not constant but instead changes as time passes, following a stochastic process. To maintain a risk-free portfolio of the stock and the call thus requires a continuous realignment of the portfolio.

Comparative Statics

An arbitrage argument shows that the call price rises as the time to expiration increases and that the call price rises as the exercise price falls. Hence Black-Scholes formula must satisfy this condition, and one can indeed verify this property.

The Stock Price and the Call Price

Using the solution (3), it is possible to show that an increase in the stock price raises the call price.

This property is taken for granted in options markets. However it is not a consequence just of arbitrage, if the stochastic process for the stock price is unrestricted.

For example, consider an out-of-the-money call such that a higher current stock price is paired with an expectation that the future stock price will be less. Then a higher stock price now might lower the call value.

The Black-Scholes model precludes this possibility by assuming that the stock price follows a random walk. Then a higher current stock price implies a higher expected future stock price.

Variance and the Call Price

One can verify that increasing the variance raises the call price.

For an out-of-the-money option, this result is intuitive. Higher variance increases the chance that profitable exercise will happen.

For an in-the-money option, the result remains valid. Higher variance increases the chance that the option will expire unexercised. But in the other direction, higher variance also increases the chance of a large profit. It turns out that the second effect dominates, so the call price rises.

Risk Premium

Perhaps surprisingly, the risk premium on the stock has *no effect* on the call price: this parameter does not appear in the Black-Scholes partial differential equation.

A higher mean return does imply a greater chance of profitable arbitrage, so the expected profit from arbitrage rises.

However a high mean return also implies that these profits should be discounted at a higher rate. The Black-Scholes partial differential equation implies that this discount effect must offset *exactly* the higher expected profit.

Simple Calculation of the Black-Scholes Formula

That the risk premium has no effect on the call price allows a simple calculation of the Black-Scholes formula: set the risk premium to zero. Apply the basic model of asset-market equilibrium, in which each asset has the same expected rate of return (the market interest rate—the risk-free rate of return). This rate-of-return condition is equivalent to the present-value condition. Consequently the call price must be the expected value of the option at expiration, discounted at the risk-free rate of return.

The Black-Scholes partial differential equation implies that this same formula applies even if the risk premium is not zero.

Risk-Free Rate of Return

An increase in the risk-free rate of return lowers the call price.

Implied Standard Deviation

A test of the Black-Scholes formula is via the *implied standard deviation*.

Consider a real option selling at a particular price. Using the Black-Scholes formula, calculate what standard deviation is needed to yield this price.

The test is to compare this implied standard deviation to the sample standard deviation of the stock-price changes. In fact the correspondence is good, and thus the Black-Scholes model fits the data very well.

References

[1] Fischer Black and Myron Scholes. The pricing of options and corporate liabilities. *Journal of Political Economy*, 81(3):637–654, May/June 1973. HB1J7.