Risk-Neutral Pricing: An Intuitive Approach

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October 2024

Abstract

This paper aims to provide a straightforward and intuitive introduction to Risk-Neutral Pricing for derivative securities. We begin by reviewing foundational concepts from Itô calculus, including Itô processes and martingales, along with the key theorems necessary for deriving the risk-neutral pricing formula. We then proceed to derive the formula, prioritizing intuition over mathematical rigor.

1 Towards an Arbitrage-Free Market

One of the main assumptions of the Black-Scholes model is that the market is arbitrage-free. We can define arbitrage as a trading strategy that begins with no money, has zero probability of losing money, and has a positive probability of making money [2]. In derivatives pricing, assuming a market with arbitrage leads to paradoxes, as money could effectively be created out of nothing. Although arbitrage opportunities can appear briefly in real markets, they are typically short-lived; as soon as participants identify these opportunities, trading activities quickly eliminate them.

For this reason, assuming the absence of arbitrage is crucial for determining fair prices of derivative securities. Additionally, the price of any asset is closely related to its associated risk, as investors generally demand higher returns for taking on greater risk. Since each investor has unique risk preferences, it is essential to establish a common pricing framework to determine a single price for a derivative. To achieve this, we adopt an equilibrium measure (often called the risk-neutral measure) in the market, aligning all investors to a standardized risk preference and thereby leading to a unique price for the derivative security.

The need for an arbitrage-free market and an equilibrium measure led to the development of the fundamental theorem of asset pricing, which states that the condition of no-arbitrage is equivalent to the existence of a unique risk-neutral measure (an equilibrium measure) that allows computing fair values for derivative securities. Under this risk-neutral probability measure, the price of every asset can be found by taking the present value of its expected payoff.

In this paper, we will develop the mathematical framework for determining the risk-neutral measure in the context of option pricing, adopting an intuitive and accessible approach.

2 Itô Processes and Martingales

The continuous-time option pricing models rely on the mathematical machinery of stochastic calculus. In our case, we will focus only on Itô processes and martingales, as these are the basic stochastic tools that are required to develop a risk-neutral measure for the market. We will begin with Itô processes.

An **Itô process** X_t is a type of stochastic process that can be expressed as:

$$dX_t = \mu(t, X_t)\,dt + \sigma(t, X_t)\,dW_t \tag{1}$$

where:

- $\mu(t, X_t)$ is the drift term, describing the deterministic rate of change,
- $\sigma(t, X_t)$ is the diffusion term describing the stochastic rate of change,
- W_t is a standard Wiener process (Brownian motion).

The key takeaway of (1) is that this type of process has two main components: the drift, the deterministic component, and the diffusion, the random component. As we will see later, stocks can be modeled using this process. A stock with positive constant drift has a tendency to rise. A special type of Itô process with no drift component is a martingale.

A **martingale** is a type of stochastic process where the future expected value of the process, given all current information, is equal to its present value. In other words, a martingale has the following property:

$$\mathbb{E}[X_{t+1} \mid X_t] = X_t \tag{2}$$

This means that, on average, a martingale neither tends to increase nor decrease over time. It has no drift in expectation. In the context of option pricing, a martingale is a stock that does not tend to rise or fall. If we eliminate the drift term in (1), we can deduce that W_t is a martingale. In other words, a standard Wiener process (or Brownian Motion) is a martingale.

3 Main Theorems of Risk-Neutral Pricing

Equipped with the basic tools of stochastic calculus, we are now ready to take on the main theorems that lead to the risk-neutral pricing formula for options. We will begin with the Fundamental Theorem of Asset Pricing.

Theorem 1. *The **Fundamental Theorem of Asset Pricing** states that:*

1. *A market is **arbitrage-free** if and only if there exists an equivalent risk-neutral measure $\tilde{\mathbb{P}}$ under which the discounted stock price processes are martingales.*
2. *A market is **complete** (meaning every asset has a price) if and only if this risk-neutral measure $\tilde{\mathbb{P}}$ is unique.*

In other words, for a financial market to be free of arbitrage opportunities, there must exist a probability measure $\tilde{\mathbb{P}}$ (which is unique) equivalent to the real-world probability measure $\mathbb{P}$, under which the discounted asset prices follow a martingale process.

The main idea behind this theorem is that if we want to eliminate arbitrage in the market, we need to have all the discounted stock price processes behaving as martingales (with no tendency to rise or fall). This is the equilibrium condition we have talked about previously, and with this, we accomplish the goal of harmonizing the distinct risk preferences of all the investors. To do this, we have no other choice but to re-weigh the probabilities of the discounted stock processes and eliminate the drift component. This is also known as a change of probability measure, and for this task, we will make use of the next theorem.

Theorem 2. *The **Radon-Nikodym derivative** Z describes how one measure $\mathbb{P}$ relates to another measure $\tilde{\mathbb{P}}$. It can be defined as:*

$$Z = \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} \tag{3}$$

In simpler terms, Z acts like a weighting factor that adjusts $\mathbb{P}$ to give us the measure $\tilde{\mathbb{P}}$. This derivative exists only if $\mathbb{P}$ and $\tilde{\mathbb{P}}$ are equivalent. This means that if an event has zero probability of occurring under $\mathbb{P}$, so it does under $\tilde{\mathbb{P}}$. Additionally, we need to have $\mathbb{E}(Z) = 1$ as:

$$\mathbb{E}(Z) = \int Z d\mathbb{P} = \int d\tilde{\mathbb{P}} = 1$$

In the next section, we will see that stock price processes follow a Geometric Brownian Motion (GBM), which is a special case of an Itô process. The GBM has a constant drift that we will have to shift by changing the probability measure as in (3), so the discounted price process behaves like a martingale, eliminating arbitrage.

The Girsanov theorem specifies exactly how to change the probability measure of a Geometric Brownian Motion, by using the Randon-Nykodym derivative.

Theorem 3. *The* ***Girsanov Theorem*** *states that if we have a standard Brownian motion W_t under a probability measure $\mathbb{P}$, and we define a new process by adding a certain drift θ_t:*

$$\tilde{W_t} = W_t + \int_0^t \theta_s \, ds \tag{4}$$

then under the new measure $\tilde{\mathbb{P}}$, the process $\tilde{W_t}$ behaves like a standard Brownian motion, thus $\tilde{W_t}$ is a martingale.

This is only possible if the Radon-Nikodym derivative related to this change of probability measure, which is expressed as:

$$Z_t = \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \exp\left(\int_0^t \theta_s \, dW_s - \frac{1}{2}\int_0^t \theta_s^2 \, ds\right) \tag{5}$$

is a martingale. In simpler terms, the Girsanov theorem shows how we can shift the drift of a stochastic process and still maintain the structure of a Brownian motion (a martingale structure, after all) under a new measure.

4 Risk-Neutral Pricing

After having reviewed the main theorems in the derivatives pricing theory, we are now ready to derive a risk-neutral pricing model for options. We will start with the Geometric Brownian Motion, which is the model for stock price processes and is expressed as:

$$dS_t = \mu \, S_t \, dt + \sigma \, S_t \, dW_t \tag{6}$$

This is an Itô process with drift. We want to derive the expression for a process

$$\tilde{S_t} = S_t \, e^{-rt}$$

that is a martingale under the risk-neutral probability measure $\tilde{\mathbb{P}}$. If we differentiate this expression, we have that:

$$d(S_t\, e^{-rt}) = dS_t\, e^{-rt} - S_t\, r\, e^{-rt}\, dt = e^{-rt}(\mu\, S_t\, dt + \sigma\, S_t\, dW_t - r\, S_t\, dt)$$

Therefore we get that:

$$d\tilde{S}_t = e^{-rt}\, S_t\, [(\mu - r)\, dt + \sigma\, dW_t] = \sigma\, \tilde{S}_t\, \left(\frac{\mu - r}{\sigma}\, dt + dW_t\right)$$

And if we set:

$$\theta = \frac{\mu - r}{\sigma}$$

we reach the following expression:

$$d\tilde{S}_t = \sigma\, \tilde{S}_t\, (\theta\, dt + dW_t) \tag{7}$$

This is not a martingale under $\mathbb{P}$ as we have a drift component. However, we can change the probability measure to $\tilde{\mathbb{P}}$ to accomplish this. If we set the following:

$$d\tilde{W}_t = \theta\, dt + dW_t \tag{8}$$

we can define a new process $\tilde{W}_t$ as the one in (4) so:

$$\tilde{W}_t = W_t + \int_0^t \theta\, ds$$

Which is indeed a martingale under $\tilde{\mathbb{P}}$ according to Girnasov's theorem, so under this new risk-neutral probability, we have that:

$$d\tilde{S}_t = \sigma\, \tilde{S}_t\, d\tilde{W}_t \tag{9}$$

This last expression has no drift. Consequently, we have accomplished the goal of defining a stock price process that is a martingale under $\tilde{\mathbb{P}}$. Putting back (8) into (6) we have that:

$$dS_t = r\, S_t\, dt + \sigma\, S_t\, d\tilde{W}_t$$

This means that under $\tilde{\mathbb{P}}$, stock price processes must have a constant drift r, the risk-free rate of the market, so if we discount these processes, we get a martingale. In other words, under the risk-neutral probability, all the assets grow at the same rate, the risk-free rate of the market, so investors are indifferent about risk. This is the equilibrium condition we have been looking for.

Lastly, for the Girsanov theorem to be true, we have to make sure that the Randon-Nykodim derivative of this change of measure, Z_t, is a martingale. According to (5), the expression for Z_t is:

$$Z = \exp(\theta\, W_t - \frac{1}{2}\, \theta^2\, t)$$

By expressing this in differential form [1], we can clearly see that there is no drift component thus Z_t is a martingale:

$$dZ_t = \theta Z_t\, dW_t$$

References

[1] Rutkowski Musiela. *Martingale methods in financial modelling.* Springer Finance, 2004.

[2] Steven Shreve. *Stochastic calculus for finance I: the binomial asset pricing model.* Springer Science & Business Media, 2005.

[3] Steven E Shreve et al. *Stochastic calculus for finance II: Continuous-time models*, volume 11. Springer, 2004.