Credit Risk

Lecture 1 – Introduction, reduced-form models and CDS

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Objectives of the lecture

Teaching objectives

At the end of this lecture, you will:

- Understand why credit risk is at the **basis of our economies**;
- Have a clear view on the credit risk modeling **challenges and outcomes**;
- Know the **basic concepts of credit risk**, that is, how to price a bond, what a spread is and how to extract it from the price of a bond, what are the Exposure At Default (EAD), Loss Given Default (LGD) and Probability of Default (PD);
- Know what **reduced-form models** are and how to calibrate them;
- Know what **Credit Default Swaps** are and how to price them.

2 - Credit risk and economics

Credit risk management is at the base of our economies

What is credit risk?

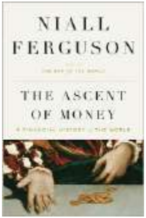
credo: I believe (latin)
resecare: To break (latin)

Credit risk – Definition

Credit risk is the risk of **default** on a debt, that may arise from a borrower failing to make required payments. [BCBS, 2000]

[Fergusson, 2008] is an interesting reference to tackle the subject from an historical perspective.

YouTube



Credit risk management is at the base of our economies

Why is there credit risk?

There is a **discrepancy of financial needs** among economic agents.
Some agents need money to fulfill their projects (firms, states, people, etc.) and other do not need an immediate access to their wealth.

$\downarrow$

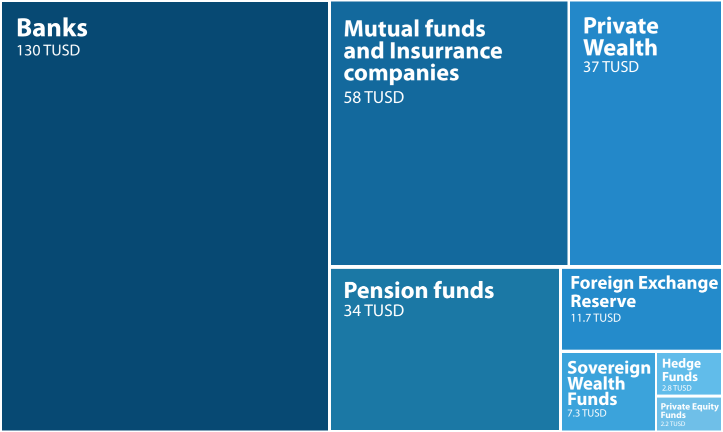
To fill this gap, lenders lend to borrowers, based on the **belief** that they will retrieve their money.

$\downarrow$

This belief – this trust – is at **the origin of credit risk**.

Credit risk management is at the base of our economies

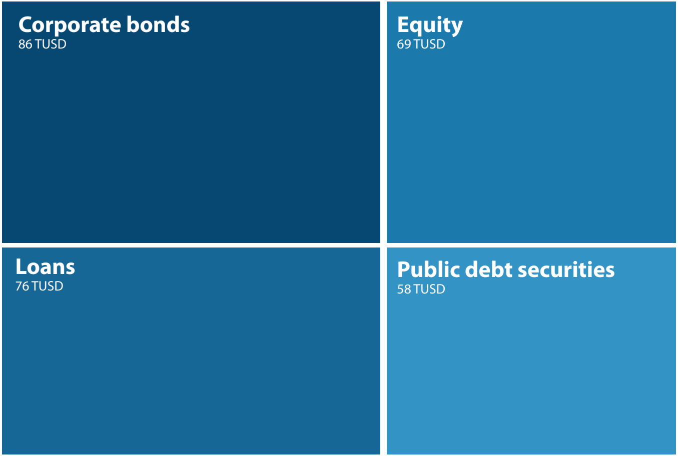
Who finance the economy?



Source: Aspects of Global Asset Allocation, IMF. and personal cross-checkings.

Credit risk management is at the base of our economies

Who borrow?



Source: The Random Walk, Mapping the world financial markets, 2014, DB research.

Credit risk management is at the base of our economies

In what banks differ from other lenders?

They have an **expertise** and a defined economic purpose as financial intermediaries.

- They have an expertise in **maturity transformation** (ALM[1] department);
- They have much more **information** on the economy and on their counterparties than any other agent;
- They know how to dissociate risks and underlying assets thanks to **derivative products**;
- They can deal with credit risk on a **macro level** (portfolio approach, dynamic management of assets, macro hedging strategy);
- They **create money** when allowing credits.

See several references at the end of the slides.

[1] Asset Liability Management.

Credit risk management is at the base of our economies

Regulatory requirements for banks

Banks are at the center of our economies

- Finance the economy;
- Provide way to transfer risks;
- At the basis of money creation.

Banks face numerous and complex risks

- Credit Risk (~80 %*);
- Market Risk (~10 %*);
- Liquidity Risk;
- Operational Risk (~10 %*).

* Computed as a percentage of Risk Weighted Assets (RWA, see Lecture 5).

It is thus a highly regulated sector

Banks must set capital apart to face an unlikely rise of these risks.

↓

Incentives to reduce balance sheet size

↓

Moving towards an *originate to distribute* model ? (vs *originate to hold*)

Conclusion

Credit risk management is at the base of our economies

- Credit risk is the risk that a **borrower fails to make required payments**;
- There is **no financing of the economy without credit risk**;
- **Banks finance** a big chunk of the economy and are thus prone to credit risk.

3 - Credit risk outcomes

The outcomes we will face in this class

Very different outcomes in comparison with market risk

	Market Risk	Credit Risk
Amount of data	A lot	Few
Liquidity of the assets	Liquid	Not liquid
Shape of the loss function	Symmetric	Asymmetric
Correlations	High	Low
Risk Management	Hedging	Diversifying
Backtesting	Possible	Impossible

The outcomes we will face in this class

The outcomes we will face in this class

Estimation of the prob. of default	**Point** in time	⚔	**Through** the Cycle
Time horizon	**One** Year	⚔	**Several** Years
Accountancy	**In** balance-sheet	⚔	**Off** balance-sheet
Number of counterparties	**Single-name** models	⚔	**Portfolio** models
Assets	**With** assets	⚔	**Without** assets
What to predict?	**Two-state** model	⚔	**Continuous** model
Purpose	**Regulatory** purpose	⚔	**Internal** purpose
Probability	**Risk Neutral** Probability	⚔	**Real World** Probability
Data	**Lack** of data	⚔	**Lot** of data

Let us take a closer look at the two latter.

Real World and Risk Neutral probability

Real World and Risk Neutral probability – Definitions

Let S_t be the variable equals to s, if the future state of the economy, in t, is s.

Real World Probability, $\mathbb{P}$

Probability that an event, s, occurs.

Risk Neutral Probability, $\mathbb{Q}$

Probability measure which weights the future state of the economy, s, according to **the price to be risk neutral to that specific state**, proportionally to the price to be risk neutral to all the future states of the economy.

This formalization was made by [Harrison and Kreps, 1979].

Real World and Risk Neutral probability

A simplified example to understand Risk Neutral Probability

A simplified example – Real World and Risk neutral probability

Definition

Let us say that a future state of the economy (in 3 years) will occur with a (Real World) probability of 10 %. Let us suppose that:

- to be sure (i.e. to be risk neutral) to have a cash flow of 1 EUR in 3 years (that is the price of a risk-free zero-coupon bond) costs 0.8 EUR;
- to be sure (i.e. to be risk neutral) to have a cash flow of 1 EUR, only if, the specific state s of the economy occurs in 3 years, costs 0.1 EUR.

We have that: $\mathbb{Q}(S_t = s) = \frac{0.1}{0.8} = 12.5\ \%$ even if $\mathbb{P}(S_t = s) = 10\ \%$.

In that case, the cost of protecting oneself against s is higher than suggested by the Real World probability.

Real World and Risk Neutral probability

Risk Neutral Default rates vs Real World Default rates

Rating (in %)	Real World Default rate	Risk Neutral Default rate
AAA	0.03	0.60
AA	0.06	0.73
A	0.18	1.15
BBB	0.44	2.13
BB	2.23	4.67
B	6.09	8.02
CCC	13.52	18.39

Risk Neutral Default rates are higher than Real World Default rates.

That could be because of:

- The **lack of liquidity** on the debt market;
- The **lack of information of investors** on the market;
- The **risk aversion of investors** on the market, etc.

Credit risk modeling and the challenge of data

Where to find data for credit risk modeling?

Data to estimate with real world probability		Data to estimate with risk neutral probability
Financial Reports		
Rating agencies	⟷	Financial markets prices (bonds, equity, CDS, etc.)
Banks knowledge		

Conclusion

Main credit risk modeling outcomes and challenges

- Contrary to market risk, credit risk faces the following **challenges**: there is few data, the market is illiquid, the loss functions are asymmetric, correlations are low and backtesting is hardly possible;
- Additionally, credit risk can be **envisioned in many different ways**: on several time spans, with a real-world or risk neutral approach, in a continuous or binary perspective, etc. making this risk particularly technical.

4 - Credit risk: The basics

Pricing of bonds – Continous version

The spread of a bond

Pricing a bond – Continuous version

Let us denote the continuous and constant coupon rate by c, the risky rate of the firm A by r^A, the maturity of the bond T, and assume the nominal, N, is equal to 1, **the price of a bond** is:

$$\bar{B}^A(0,T) = 1 + (c - r^A)\frac{1 - e^{-r^A T}}{r^A}$$

> The formula is a simple result consequent to the no-arbitrage assumption and the integral resolution of: $\bar{B}^A(0,T) = \int_0^T ce^{-r^A t}dt + Ne^{-r^A T}$

The spread of a bond – Continuous version

For a given risk-free rate, r, the spread of a bond of price $\bar{B}^A(0,T)$, is the value s^A so that:

$$\bar{B}^A(0,T) = 1 + (c - (r + \boldsymbol{s^A}))\frac{1 - e^{-(r+\boldsymbol{s^A})T}}{(r + \boldsymbol{s^A})}$$

Pricing of bonds

Pricing and implied survival probability

The implied survival probability – Computed with prices

Let τ be the time of default of firm A. Let $\bar{B}^A(t,T)$ be the price of a zero-coupon risky bond of firm A, at t, of maturity T, and nominal N. Let $B(t,T)$ be the price of a zero-coupon risk-free bond, at t, of maturity T, and nominal N.
The implied survival probability of firm A, in T, from t, is:

$$\mathbb{Q}\left(\tau > T \mid \tau > t\right) = \frac{\bar{B}^A(t,T)}{B(t,T)}$$

> It is a consequence of the no-arbitrage assumption.

The implied survival probability – Computed with constant continuous spreads

Let s^A be the spread of Firm A. The **implied survival probability** can be written:

$$\mathbb{Q}\left(\tau > T \mid \tau > t\right) = e^{-s^A(T-t)}$$

> $$\mathbb{Q}\left(T > \tau \mid \tau > t\right) = \frac{Ne^{-r(T-t)}}{Ne^{-(r+s^A)(T-t)}} = e^{-s^A(T-t)}$$

Tutorial | R Markdown

Three points of attention – Discrete version, recovery and risk-free rate

Continous vs Discrete (I/IV)

Pricing a bond – Discrete version

Let N be the nominal of a bond from firm A, $t_1, \dots, t_n = T$ the dates when the coupons C are paid, $r_1^A, \dots, r_n^A$ the respective risky rates and, T, its maturity. The **price of the bond** $\bar{B}^A(0,T)$, in 0, is:

$$\bar{B}^A(0,T) = \sum_{i=1}^{n} \frac{C}{(1+r_i^A)^{t_i}} + \frac{N}{(1+r_T^A)^T}$$

The spread of a bond – Discrete version

Let $\bar{B}^A(0,T)$ be the price of a bond and $r_1, \dots, r_n$ the risk-free rates in $t_1, \dots, t_n$. **The spread** of A is s^A so that:

$$\bar{B}^A(0,T) = \sum_{i=1}^{n} \frac{C}{(1+r_i+\boldsymbol{s^A})^{t_i}} + \frac{N}{(1+r_n+\boldsymbol{s^A})^T}$$

Three points of attention – Discrete version, recovery and risk-free rate

Continous vs Discrete (II/IV)

Bootstrap technique to compute implied survival probability – Discrete version

Suppose firm A has n bonds, $B_1^A, \ldots, B_n^A$, paying coupons, C, in $t_1, \ldots, t_n$ and of maturity, respectively $t_1, \ldots, t_n$. **Bootstrap** works the following way:

- With B_1^A, it is possible to compute r_1^A;
- With B_2^A, substracting its cash-flow in t_1 discounted by $1 + r_1^A$, it is possible to compute r_2^A;
- etc.

That way, one can compute iteratively $r_1^A, r_2^A, \ldots, r_n^A$ and thus deduce $s_1^A, s_2^A, \ldots, s_n^A$ using the risk-free rates.

Three points of attention – Discrete version, recovery and risk-free rate

Continous vs Discrete (III/IV)

Bond clean and dirty prices

Discrete payments implies discontinuity in bond valuation each time a coupon is paid : these discountinuous prices are called **dirty prices**. On the markets, the prices do not suffer such a problem as the so-called **clean prices** are quoted. The formula that links both is:

$$\text{Dirty price} = \text{Clean price} + \text{Accrued interests}$$

where $\text{Accrued interests} = \frac{\#\text{ days since last coupon}}{\#\text{ days between coupons}} \times \text{Coupon rate} \times \text{Nominal}$

Three points of attention – Discrete version, recovery and risk-free rate

Continous vs Discrete (IV/IV)

Adobe System Inc. bond valuation

Adobe Systems Inc. (NASDAQ: ADBE) has 600 MUSD worth of bond payable outstanding. The 1 000 USD par, 3.25 % semi-annual coupon bonds are due to mature on 1st February 2015. The coupon dates are 1st February and 1st August. They follow 30/360 day count convention and next coupon is due on 1st August 2013. Yvonne Barnet bought 1 000 such bonds from Charles Schwab on 20th July 2013. The market requires buyer to compensate seller for the accrued interest. How much Yvonne must pay Charles? Yvonne must pay the dirty price, but she only knows the clean price: 1036.10 USD.

- days between the transaction date and next coupon date = 11 = 10 days of July plus 1 day of August;
- days in the coupon period = 180 (since 30/360 day count convention is used).

Thus, the dirty price is: $1036.10 + \frac{1000 \times 3.25\%}{2} \times \frac{169}{180} = 1051.36$ USD

Three points of attention – Discrete version, recovery and risk-free rate

The importance of the recovery rate

The importance of the recovery rate

To simplify math formulas, the recovery rate – the extent to which principal and accrued interests on defaulted debt can be recovered, expressed as a percentage of face value – is often forgotten (or equivalently supposed equal to one). Nonetheless, in practice, **the recovery rate must be taken into account** when extracting the probability of default from a market price.

Newspapers

The implied probability of default taking into account recovery

Be Careful!

Let R be the recovery rate, the **implied probability of default taking into account recovery** is:

$$\mathrm{PD} = \frac{1 - \frac{\bar{B}(0,T)}{B(0,T)}}{1 - R}$$

This is a consequence of the no-arbitrage hypothesis.

Tutorial

Three points of attention – Discrete version, recovery and risk-free rate

What is the risk-free rate?

What is the risk-free rate?

The rates used as a **risk-free rates** evolved during the last decades:

- from the government rates, first;
- to the LIBOR rates, then;
- to the Overnight Indexed Swap (OIS) rate, now.

ICE Swap Rate

The general framework of credit risk modeling: PD, EAD and LGD

Definitions of PD and EAD

Probability of Default – PD

It is the **one-year default probability** of a counterparty.

Exposure At Default – EAD

Exposure At Default is the **loss** a bank would suffer if its counterparty defaults, and there would be no guarantee.

This value can either be **known** (loans for example) or **unknown** (lines of credit for instance).

The general framework of credit risk modeling: PD, EAD and LGD

Focus on the Loss Given Default (I/II)

Loss Given Default – LGD

Let R be the **recovery rate**, that is the percentage of the exposure recovered by the bank after the default, we have:

$$\text{LGD} = 1 - R$$

Loss Given Default and R depend on the underlying credit contract

Contracts	**R**
Bank loans	80.3 %
Senior secured bonds	63.5 %
Senior unsecured bonds	49.2 %
Senior subordinated bonds	29.4 %
Subordinated bonds	29.5 %
Junior subordinated	18.4 %

Source: Moody's statistics.

Moody's

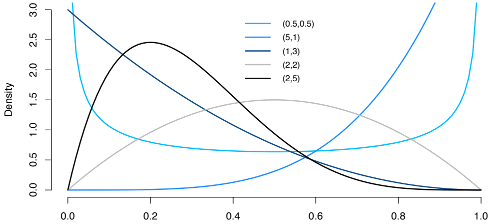
The general framework of credit risk modeling: PD, EAD and LGD

Focus on the Loss Given Default (II/II)

Modeling LGD with a beta distribution using a Maximum Likelihood estimator

Let $\alpha > 0$, $\beta > 0$. The density of a beta distribution is:

$$f(x;\alpha,\beta) = \begin{cases} \dfrac{x^{\alpha-1}(1-x)^{\beta-1}}{\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}\,\mathrm{d}u} & \text{for } x \in [0,1] \\ 0 & \text{else} \end{cases}$$



From data, it appears that the shape of LGD distributions is usually a **U-shaped curve**.

The general framework of credit risk modeling: PD, EAD and LGD

Independence of PD, EAD and LGD

The Expected Loss – EL

We define the Expected Loss as:

$$\begin{aligned}\mathrm{EL} &= \mathbb{E}(\mathrm{EAD} \times \mathbb{1}_{\{\tau<M\}} \times \mathrm{LGD}) \\ &\underbrace{=}_{\text{assump. } \perp} \mathbb{E}(\mathrm{EAD}) \times \underbrace{\mathbb{E}(\mathbb{1}_{\{\tau<M\}})}_{\mathrm{PD}} \times \mathbb{E}(\mathrm{LGD})\end{aligned}$$

Be Careful!

The independence of EAD, PD and LGD

Formula

There is no reason why one should assume **independence** between EAD, PD and LGD. Actually, some phenomena and papers proved the contrary:

- The phenomenon of **gambling for resurrection**;
- The study by [Frye, 2013] postulates that **LGD is a function of the probability of default**.

Quiz

Conclusion

The basics of credit risk

- Pricing of bonds, in a continuous or discrete framework, is based on the **no-arbitrage assumption**;
- **Coupons and recovery** are complexities that need to be taken into account when implying probabilities of default from bond prices through respectively accrued interests cleaning and bootstrapping;
- There are **three key components** of any credit expected losses estimator, the Exposure At Default, the Loss Given Default and the Probability of Default – their independence, while often assumed in models, is not always tenable;
- The **Probability of Default (PD)** is the major focus of this course.

5 - Reduced-form models

Reduced-form models

Definition

Reduced-form models

Reduced-form models consist in modeling the **conditional law of the random time of default**.

Conditional default probability

In a reduced-form model, **conditional default probability** is defined as:

$$\mathbb{Q}(\tau < t + \mathrm{d}t \mid \tau > t) = \lambda \mathrm{d}t$$

λ is the **default intensity** and corresponds to the **instantaneous forward default rate**. This variable is **exogenous** to the problem.

Survival function

The **survival function** is thus defined as:

$$S(t) = \mathbb{Q}(\tau > t) = \exp(-\lambda t)$$

Reduced-form models

What is default intensity?

Is the default intensity, λ, constant or stochastic?

It depends:

- **Constant**: Time homogeneous Poisson Process;
- **Deterministic**: Time deterministic inhomogeneous Poisson Process;
- **Stochastic**: Time-varying and stochastic Poisson Process as the Cox, Ingersoll, Ross (CIR) model.

Reduced-form models

Calibration of default intensity models

The implied survival probability

Let $B(0,t)$ be a **zero-coupon risk-free bond** and $\bar{B}(0,t)$ be a risky zero-coupon bond. We have:

$$\mathbb{Q}(\tau > t) = \frac{\bar{B}(0,t)}{B(0,t)}$$

This is a result based on the no-arbitrage assumption.

The implied survival probability – Application for calibration of intensity models

We deduce from the above formula the expression of the **default intensity**, λ:

$$\lambda = -\frac{\log\left(\frac{\bar{B}(0,t)}{B(0,t)}\right)}{t}$$

Be Careful!

We have seen that, $\mathbb{Q}(\tau > t) = e^{-\lambda t}$

Conclusion

Reduced-form models

- Reduced-form models are models based on an **exogenous variable**, called here, the default intensity;
- The **default intensity is often assumed constant** but can also be non-constant or even stochastic.

6 - Single-name derivatives

Single-name credit derivatives and Credit Default Swap (CDS)

What are CDS?

Credit Default Swap

Credit Default Swaps (CDS) are financial agreements that allows the **transfer of the credit risk** of a loan to another counterparty.

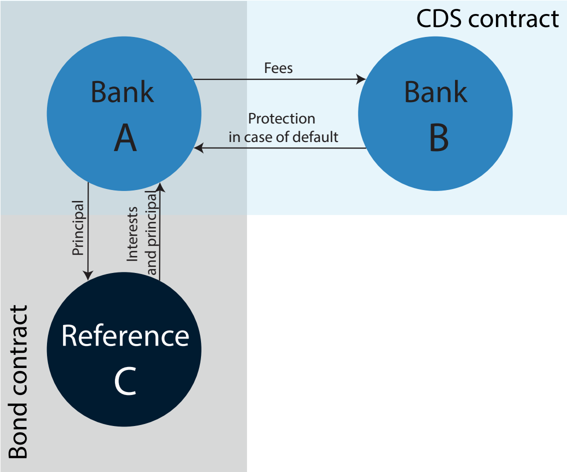
- Bank A holds a loan on corporate C on its balance-sheet and buys protection on the credit risk related to counterparty C;
- Bank B sells protection on corporate C and receives the payment of a premium.

Are CDS insurance contracts?

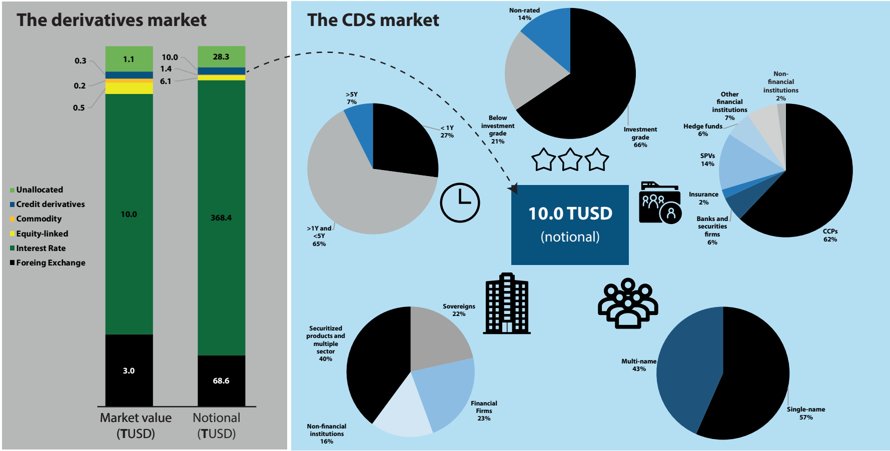
CDS are **neither insurance contracts, nor guarentees**.

Single-name credit derivatives and Credit Default Swap (CDS)

CDS cash flows



Statistics on the CDS market



Legal aspect of CDS

The International Swaps and Derivatives Association (ISDA)

ISDA defines **standards** for credit derivatives transactions:

- **1999**: definitions;
- **2003**: supplements;
- **2009**: big-bang protocol.

ISDA

Examples of important specifications in a CDS

- What is a CDS **credit event**?
  - bankruptcy;
  - failure to pay (coupon or nominal);
  - restructuring of a bond.
- **Settlement**:
  - Physical settlement: the buyer of the CDS gives the defaulted bonds to the seller, and receives the nominal (N);
  - Cash settlement: the buyer receives $N(1-R)$, and keeps the defaulted bonds.
- **Normal vs Digital CDS**:
  - Digital: the payoff is N;
  - Normal: the payoff is $(1-R)N$.

Single-name credit derivatives and Credit Default Swap (CDS)

Use of CDS

What CDS can be used for? Risk management and investment strategies.

- Risk mitigation;
- Management of credit lines;
- Bank's capital management;
- Balance sheet management;
- Leverage effect (FtD, CDO);
- Access to the market.

Who uses CDS?

- Banks;
- Corporates;
- Insurers and reinsurers;
- Asset managers;
- Hedge funds.

Reduced-form models applied to CDS pricing

Fixed leg of a CDS

Value of the fixed leg of a CDS

The value of the **fixed leg** of a CDS is:

$$\text{Fixed}(0, T) = s(0, T)\frac{1 - e^{-(r+\lambda)T}}{r + \lambda}$$

Fixed leg pays the reference spread at inception of the CDS up to the minimum of the default date (τ) and maturity T.
For a continuous paid spread:

$$\begin{aligned}
\text{Fixed}(0, T) &= \mathbb{E}^{\mathbb{Q}}\left(s(0, T)\int_0^{\tau \wedge T} e^{-rt}\mathrm{d}t\right) \\
&= s(0, T)\int_0^{T} e^{-(r+\lambda)t}\mathrm{d}t \\
&= s(0, T)\frac{1 - e^{-(r+\lambda)T}}{r + \lambda}
\end{aligned}$$

Reduced-form models applied to CDS pricing

Floating leg of a CDS

Value of the floating leg of a CDS

The value of the **floating leg** of a CDS is:

$$\text{Floating}(0, T) = (1 - R)\frac{\lambda}{\lambda + r}\left(1 - e^{-(\lambda+r)T}\right)$$

> The floating leg is paid when a default occurs: the protection seller pays the difference between the notional and the recovery.
>
> $$\begin{aligned}\text{Floating}(0, T) &= \mathbb{E}^{\mathbb{Q}}\left((1 - R)e^{-r\tau}\mathbb{1}_{\{\tau<T\}}\right) \\ &= (1 - R)\frac{\lambda}{\lambda + r}\left(1 - e^{-(\lambda+r)T}\right)\end{aligned}$$

Reduced-form models applied to CDS pricing

Spread of a CDS, its Mark-to-Market value and time sensitivity

The spread of a CDS

The **spread of a CDS** is:

$$s = \lambda(1 - R)$$

> At inception, the Net Present Value (NPV) for the protection seller is:
>
> $$\text{MtM}(0, T) = \text{Fixed}(0, T) - \text{Floating}(0, T)$$
>
> The fair spread sets the initial MtM at 0. We thus have $s = \lambda(1 - R)$.

MtM through the time and sensitivity of the CDS to the time

The sensitivity of the MtM is the **risky duration**, DV:

$$\text{DV}(t, T, \lambda) = \frac{1 - e^{-(r+\lambda)(T-t)}}{r + \lambda}$$

Tutorial

Reduced-form models applied to CDS pricing

CDS sensitivity

Present Value of a CDS through the time

Let s_0, be the spread in $t = 0$, and s, the spread today, in t.
The **Present Value of the protection seller** is:

$$PV(s_0, s_t) = \mathrm{DV}(0, t, \lambda)(s_0 - s_t)$$

- At inception, the Net Present Value (NPV) for the protection seller is: $\mathrm{MtM}(0, T) = 0$.
- Day 2: the market spread moves to s. The PV of the fixed leg does not change. The PV of the floating leg changes.
- If the spread increases, credit risk increases and then, the PV of the floating leg increases.
- A CDS issued day 2, would have equal floating and fixed legs, with fair spread equal to market spread s.
- The PV of the protection seller is then: $\mathrm{DV}(0, t, \lambda)(s_0 - s)$.

Reduced-form models applied to CDS pricing

CDS – Discrete vs continuous pricing

CDS – Discrete vs continuous pricing

Once again, these formulas suppose continuous interest rates payments and spread payments. These **formulas are nice proxies** to assess CDS spreads, but for precise pricing, **one must take into account each flow separately**.

Tutorial

Reduced-form models applied to CDS pricing

CDS spread vs Bond spread: the CDS basis

CDS spread and Bond spread

Generally, CDS spreads are larger than Bond spreads.

> Several reasons explain this phenomenon:
> - definition of credit event is different;
> - the protection buyer has no impact on the bond issuer through covenants.
>
> On the other hand, funds and insurers sell massively protection, making CDS spread tighten.

For more information on the subject, you can take a look at [Choudhry, 2006].

Default swaptions and other swaps

An introduction to default swaptions

Knock-out swaptions

Knock-out swaptions:

- Allow to sell or buy protection at maturity;
- Payoff is null if default occurs before maturity.

The pricing is done with models similar to Black-Scholes model.

Default swaptions and other swaps

Other swaps

Other swaps

- **Constant Maturity Credit Default Swaps (CMCDS)**: is a usual CDS except that the *fixed* leg of the contract is computed each semester with the new data. Its *pay-off* is often capped.
- **Loan-CDS (LCDS)**: they are linked to a specific loan, not on a specific name; thus, would the loan be reimbursed in advance, the CDS would be cancelled.
- **Forward CDS**: they are CDS that offer protection from a future date, T to a future data $T + M$.
- **Cancellable CDS**: they are CDS that are cancellable at a or several future dates;
- **Total Return Swap (TRS)**: they are financial contracts that transfer both the credit and the market risk of an asset against either a variable or a fixed rate (they are in fact like funded Interest Rate Swaps – IRS).

Tutorial

Conclusion

Reduced-form models

- CDS are contacts that **protect against default**;
- Their valuations reconcile the **no-arbitrage assumption and the reduced-form models**;
- CDS spreads **can differ from bond spreads**.

Quiz

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Credit Risk

Lecture 2 – Statistical tools for scoring and default modeling

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1 - Default and Ratings

The various conceptions of default

Is default a binary concept?

- For an **accountant**: it is possible to book losses even though the counterparty has made all its payments until now (cf. IAS 39 / IFRS 9, see Lecture 5);
- For the **regulator**, according to Basel Committee: "A default is considered to have occurred with regard to a particular obligor when one or more of the following events has taken place:
  - It is determined that the obligor is **unlikely to pay** its debt obligations (principal, interest, or fees) in full;
  - A **credit loss event** associated with any obligation of the obligor, such as a charge-off, specific provision, or distressed restructuring involving the forgiveness or postponement of principal, interest, or fees;
  - The obligor is past due **more than 90 days** on any credit obligation;
  - The obligor has **filed for bankruptcy** or similar protection from creditors."[1]
- For the market and rating agencies:
  - **bankruptcy**, liquidation or stop of activity;
  - **unpaid flow**;
  - **restructuring**.

[1] Basel Committee on Banking Supervision, The Internal Rating Based Approach

Ratings and rating agencies

What is the role of rating agencies?

Rating agencies

Rating agencies give grades to economic agents that reflect their **ability to reimburse** borrowed money, thanks to **qualitative and quantitative** criteria gathered by their analysts. These critiera can be:

- expected future cash flows;
- short term, long term liabilities;
- structure of the liabilities;
- countries of activity;
- competition in the market;
- quality of the management.

The most famous rating agencies are **Moody's**, **Fitch Ratings** and **Standard and Poors**.

Ratings and rating agencies

What does a rating scale look like?

Rating scale

The grades are the following (for S&P):

Investment Grades	**Speculative Grades**
AAA	BB+
AA+	BB
AA	BB-
AA-	B+
A+	B
A	B-
A-	CCC+
BBB+	CCC
BBB	CCC-
BBB-	CC
	C

Ratings and rating agencies

How ratings relate to default?

It appears from the historical data that there is a strong exponential relationship between the rating of a firm and its one-year probability of default. That is to say the probability that the firm will default within the next year.

We have: $\text{PD} = 6 \times 10^{-6} \times e^{0.64 * \text{Rating}}$.

Ratings of well-known firms

Firm	**S&P rating**
SG	A
BNP	A
Total	A
EDF	A-
Accor	BBB-

Transition matrix

How do ratings migrate?

Definition of Transition Matrix

In credit risk, **a transition matrix**, $\boldsymbol{M}_{t,t+1} = (m_{ij})_{ij}$, is a matrix where:

$$m_{ij} = \mathbb{P}(\text{Grade}_{t+1} = j \mid \text{Grade}_t = i)$$

Where i and j are the grades presented earlier.

S&P's transition matrix – From 1981 to 2013

(in %)	AAA	AA	A	BBB	BB	B	CCC	D	NR[2]
AAA	87.11	8.88	0.53	0.05	0.08	0.03	0.05	0	3.27
AA	0.55	86.39	8.26	0.56	0.06	0.07	0.02	0.02	4.07
A	0.03	1.87	87.34	5.48	0.35	0.14	0.02	0.07	4.7
BBB	0.01	0.12	3.59	85.22	3.82	0.59	0.13	0.21	6.31
BB	0.02	0.04	0.15	5.2	76.28	7.08	0.69	0.8	9.74
B	0	0.03	0.11	0.22	5.48	73.89	4.46	4.11	11.7
CCC	0	0	0.15	0.23	0.69	13.49	43.81	26.87	14.76

[2] "Non-rated" or "Unrated". Some "companies" may decide to stop being rated by a given rating agency.

Transition matrix

What properties should be expected from a transition matrix?

Several properties of transition matrices

Among the **properties of the transition matrices**, note that:

- Each row sums to 1;
- They are dominant;
- In the case of homogeneity we have that: $\boldsymbol{M}_{t,t+n} = \boldsymbol{M}^n_{t,t+1}$.

The generator for homogeneous Markov chains

The **generator for a Markov chain** $(\boldsymbol{M}_{t,t+n})_n$ is the matrix $\boldsymbol{Q}$ so that:

$$\forall (t, T), \quad \boldsymbol{M}_{t,T} = \exp\left((T-t)\boldsymbol{Q}\right) \qquad \text{with } \exp(\boldsymbol{A}) = \sum_{n \geq 0} \frac{\boldsymbol{A}^n}{n!}$$

Would such a matrix exist (see [Israel et al., 2001]), we have:

$$\boldsymbol{Q} = \sum_{n>0} (-1)^{n-1} \frac{(\boldsymbol{M}_{t,t+1} - \boldsymbol{I})^n}{n}$$

Transition matrix

How to estimate transition matrices?

Two techniques to estimate transition matrices

There are two techniques to estimate the generator of transition matrices:

- By **cohorts**: it consists in computing the average number of agents that change from rating i to j within one year, for all (i, j);
- By **durations**: it consists in looking for instantaneous probability, for an agent, of changing from rating i to j. The likelihood of changing of rating, from i to j, in t, is:

$$e^{\lambda_{ii} t} \lambda_{ij}$$

By maximizing the likelihood of these transitions, one can estimate $(\lambda_{ij})_{ij}$.

Transition matrix

Do the maths fit reality?

Transition matrices are not Markov matrices

The markovian assumption is in **contradiction** with phenomena described by the data.

For example, a firm which has recently experienced a downgrade to rating j, is more likely to experience another one, as opposed to a firm that has had rating j for a long time.

Quiz

Conclusion

Default and rating-based models

- Default can have **various definitions depending on the context**;
- Rating agencies provide **ratings that can be used to estimate the one-year probability of default of firms**;
- A transition matrix that allow to **describe ratings migrations**.

2 - Logistic regression

What do we want to predict?

Can we predict default?

- Will a firm/customer **default within a given period of time** (year, month, etc.)?

$$Y = \begin{cases} 1 & \text{if default within a given period,} \\ 0 & \text{otherwise.} \end{cases}$$

- Or better yet: What is the probability that a firm/customer will default within a given period of time, given the information we have?

$$p(X) = \mathbb{P}(Y = 1 | X)$$

- For a firm: X can be financial data, market data, country, activity sector, etc.
- For a customer: X can be age, job situation, salary, debt level, etc.

Theoretical framework of the logistic regression

The logistic regression's assumptions

Logistic regression's model

The **logistic regression model** can be defined the following ways:

$$p(X) = \mathbb{P}(Y = 1|X) = \frac{e^{\beta_0+\beta_1 X_1+\ldots+\beta_p X_p}}{1 + e^{\beta_0+\beta_1 X_1+\ldots+\beta_p X_p}}$$

or equivalently

$$\ln\left(\frac{p(X)}{1-p(X)}\right) = \beta_0 + \beta_1 X_1 + \ldots + \beta_p X_p \tag{1}$$

where $X = (X_1, \ldots, X_p)$

No closed-form solution

In practice, the vector of parameters $(\beta_0, \ldots, \beta_p)$ is estimated by maximizing the likelihood. Contrary to linear regression, there is **no closed-form solution**, which can therefore lead to different estimations depending on the algorithm/software chosen.

Theoretical framework of the logistic regression

How to interpret the coefficients? (I/II)

Logistic regression's coefficients can be interpreted through the concept of **odds ratio** using equation (1).

Coefficient interpretation of a logistic regression with one binary predictor (I/II)

We consider the following logistic modelling where the default ($Y = 1$) only depends on being a student ($X = 1$) or not ($X = 0$):

$$\ln\left(\frac{p(X)}{1-p(X)}\right) = \beta_0 + \beta_1 X$$

Then $\exp(\beta_0)$ is the odds ratio of a non-student being in the honor class (default):

$$\exp(\beta_0) = \frac{\mathbb{P}(Y=1|X=0)}{1-\mathbb{P}(Y=1|X=0)} = \frac{\mathbb{P}(\text{default}|\text{non-student})}{\mathbb{P}(\text{no default}|\text{non-student})}$$

$\exp(\beta_1)$ is the ratio of the odds for student to the odds for non-student:

$$\exp(\beta_1) = \frac{\mathbb{P}(Y=1|X=1)}{1-\mathbb{P}(Y=1|X=1)} \Big/ \frac{\mathbb{P}(Y=1|X=0)}{1-\mathbb{P}(Y=1|X=0)}$$

Theoretical framework of the logistic regression

How to interpret the coefficients? (II/II)

Coefficient interpretation of a logistic regression with one binary predictor (II/II)

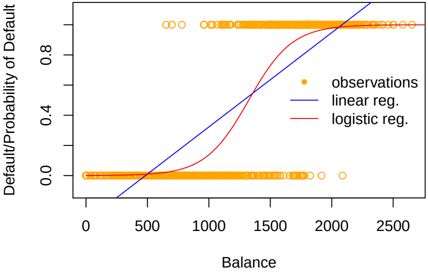
- We deduce from the previous slide that according to the model a non-student is $\exp(\beta_0)$ time(s) as likely to default as to not.
- Let's assume that $\beta_1 > 0$. The odds for a student is $\exp(\beta_1) - 1$ times higher than the odds for a non-student.

To learn more about logistic regression coefficients' interpretation:

Tutorial

Linear vs logistic regression

Why not use linear regression?



Source: The Default dataset from [Gareth et al., 2009]

The main reason for using logistic regression instead of linear regression is that the predictions are inside the $[0, 1]$ interval, making them easier to **interpret as probabilities**.

Conclusion

Why and why not use logistic regression to model default?

Pros:

- Can be **easily interpreted**:
  - reducing the risk of modelling errors;
  - making it easier to be audited (validation team, regulator, etc.).
- Provides an **estimation of the probability of default**;
- Can be converted into a **score card** to facilitate the use of the model.

Cons:

- Lacks prediction power (Logistic regression **cannot model non-linear relationships**)
- In practice the continuous predictors must be binned manually
- Requires variables selection/regularization (Lasso)

▶ R Markdown

3 - Tree-based algorithms

Classification tree

What is a classification tree?

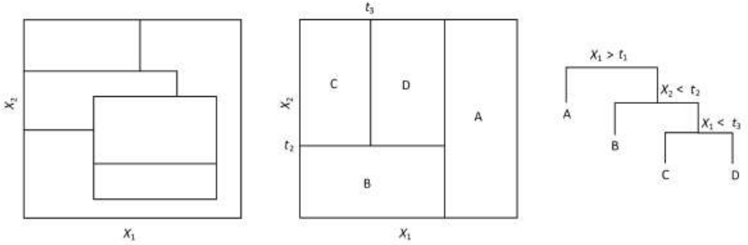
Definition of classification tree

A classification tree is a model with a tree-like structure. It contains **nodes and edges/branches**. There are two types of nodes:

- **Intermediate nodes**: An intermediate node is labeled by a single attribute, and the edges extending from the intermediate node are predicates on that attribute.
- **Leaf nodes**: A leaf node is labeled by the class label which contains the values for the prediction.

Classification tree

What can and cannot model a classification tree?



Source: Wikipedia.

Left: A partionning of predictors' space that cannot be obtained with trees.
Center & Right: **A partioning of the predictors' space** and its corresponding classification tree.

Classification tree

How to grow a classification tree?

Recursive binary splitting

- **Step 1**: We select the predictor X_k and the cutting point s in order to split the predictor space into the two regions $\{X, X_k < s\}$ and $\{X, X_k \geq s\}$ that gives the greatest decrease in the **criterion**[3] we want to minimize.
- **Step 2**: We repeat step 1 but instead of splitting the whole predictor space we split one of the two regions identified at step 1. The predictor space is now divided into three regions.
- **Step 3**: The regions are then **recursively split** until no split can decrease the criterion.

[3] For classifcation the Gini index is often used. The Gini index for the region m is $G_m = \sum_i \hat{p_{m,i}} \left(1 - \hat{p_{m,i}}\right)$, where $\hat{p_{m,i}}$ is the proportion of training observations in the m^{th} region that are from class i. The Gini index is close to zero when the $\hat{p_{m,i}}$ are close to zero or 1. It is therefore often referred as a **purity measure**.

Why and why not use classification trees to model default?

Pros and cons

Pros:

- Can be easily **interpreted and visualised**:
  - reducing the risk of modelling errors
  - making it easier to be audited (validation team, regulator, etc.)
- Provides non linear predictions

Cons:

- Does not provide a straightforward estimation of the probability of default
- Can exhibit **high variance**
- Error in the top split propagates all the way down to the leaves
- Requires regularization (pruning) to prevent overfitting
- Greedy algorithms do not guarantee getting the globally optimal decision tree
- Can be biased when the sample is unbalanced

Bootstrap aggregation or Bagging

What can be done to improve trees' predictive accuracy?

Trees are very unstable

Decision trees have **high variance**. Training two trees on two distinct halves of the training set can lead to very different outcomes.

Therefore, reducing the variance of trees is critical to increase their predictive accuracy.

Bootstrap aggregation or Bagging

The bagging training

The bagging algorithm

Let's assume we have a training set S of n observations.

- draw B samples of size m ($m \leq n$) from S with replacement. These B samples have duplicated observations.
- train B unpruned classification trees on the B samples drawn at the previous step. These trees exhibit low bias but high variance.
- For a new observation, the prediction will be the most common prediction among the B trees.

The last step of this procedure allows for **decreased variance** of the model compared to a single-tree approach. Moreover, the single-trees being unbiased the classifier consisting in aggregating these trees is also unbiased. Bagging can therefore lead to substantial gains in terms of **predictive accuracy**.

Bootstrap aggregation or Bagging

Can we do better?

The trees are correlated!

The **gain in variance** from aggregating the trees through bagging is **limited** since the the trees are still very correlated. Indeed, averaging over very correlated quantities does not lead to substantial variance reduction.

Let's assume that among the available predictors, one is particularly strong. It would then be used at the top split in many of the trained trees. The resulting trees would then look very similar. This problem can be overcome using **Random Forest**.

Random Forest

How to obtain less correlated trees?

To **circumvent the high correlation** among trees from the bagging procedure, the latter can be adjusted very simply.

Random Forest algorithm

Let's assume we have a training set S of n observations and p predictors.

- draw B samples of size m ($m \leq n$) from S with replacement.
- for each of these sample train a deep classification tree but at each split only consider a random fraction (usually of size $m \approx \sqrt{p}$) of the entire set of predictors.
- for a new observation, the prediction will be the most common prediction among the B trees.

Since the trees in a random forest are trained using a small and random **fraction of the predictors**, they happen to be much less correlated than those from a mere bagging procedure. The variance of random forest is therefore lower than that of tree bagging.

Random Forest

Advantages of Random Forests over basic trees

- RF exhibit much **less variance** than basic trees
- RF are less prone to overfitting
- RF provide variable importance measures

Lack of interpretability

While the predictive power is greatly enhanced with RF, the **interpretability** and representation capability of basic trees is **completely lost**!

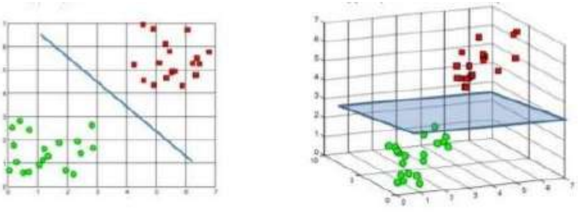
R Markdown

4 - SVM

Definition of hyperplane

Defintion of hyperplane

In a p-dimensional space, a hyperplane is a **flat affine subspace** of dimension $p - 1$

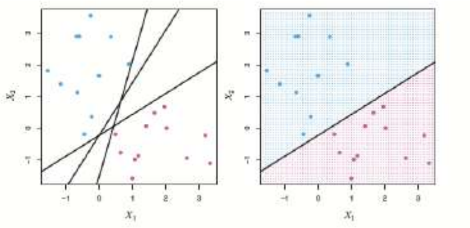


Source: KDAG Website

Left: In a 2-dimensional space a hyperplane is a line.
Right: In a 3-dimensional space a hyperplane is a plane.

Hyperplanes

Can we use a hyperplane to classify data?



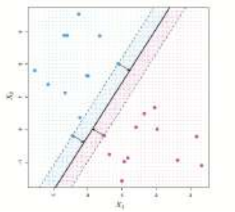
Source: [Gareth et al., 2009]

Which hyperplane to choose?

If the observations are separable, there exists an **infinity of separating hyperplanes** that could be used as classifiers. All these hyperplanes would lead to very different classifications.

The maximal margin classifier

Maximal margin classifier definition



Source: [Gareth et al., 2009]

The maximal margin classifier

- If the observations are separable, the maximum margin classifier is the separating hyperplane that is at **equal distance** from the two clouds and that maximizes this distance (margin).
- Using this hyperplane as classifier relies on the assumption that if the classification boundary (the separating hyperplan) is the **furthest from the two types of points** in the training set, it will also be the case in the test set.

The maximal margin classifier

Maximal margin classifier training

Maximal margin classifier optimization problem

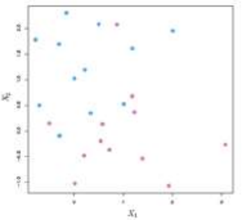
Let $y_i \in \{-1, 1\}$ denote the class of the i^{th} observation and x_{ik} denote the value of the k^{th} variable for the i^{th} observation ($\forall i \in \{1, \ldots, n\}$ and $\forall i \in \{1, \ldots, p\}$), where n denotes the number of training observations and p denotes the number of variables/predictors.

$$\begin{aligned}
&\underset{\beta_0, \ldots, \beta_p}{\text{maximize}} \quad M \\
&\text{subject to} \quad \sum_{k=1}^{n} \beta_k^2 = 1, \\
&y_i \left(\beta_0 + \beta_1 x_{i1} + \ldots + \beta_p x_{ip}\right) \geq M \quad \forall i \in \{1, \ldots, n\}
\end{aligned}$$

The first constraint has no meaning on its own, but the two constraints combined mean that the distance of each data point to the hyperplane is greater or equal than M, where M represents the margin.

The maximal margin classifier

Limits of the maximal margin classifier (I/II)



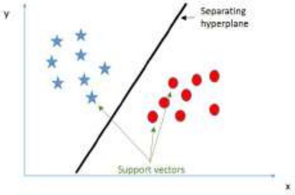
Source: [Gareth et al., 2009]

Data points are usually linearly inseparable!

In practice, observations are often **not separable**. In this case, the maximal margin classifier cannot be used.

The maximal margin classifier

Limits of the maximal margin classifier (II/II)



The maximal margin classifier is not robust

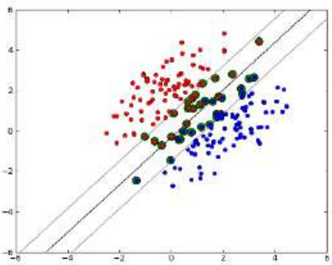
The hyperplane boundary only depends on the support vectors (the closest observations to the hyperplane). Hence, the maximal margin classifier is very sensitive to a small change in data, which also suggests that it is **likely to overfit** the training set.

Support Vector Classifier

SVC definition

Definition of the Support Vector Classifier

The **Support Vector Classifier** is an extension of the maximal margin classifier that allows training points to be on the wrong side of the margin or the hyperlane.



By allowing the hyperplane to **not perfectly separate** the observations from the two classes, the support vector classifier **overcomes the two main limits** of the maximal margin classifier.

Support Vector Classifier

SVC training

Let $y_i \in \{-1, 1\}$ denote the class of the i^{th} observation and x_{ik} denote the value of the k^{th} variable for the i^{th} observation ($\forall i \in \{1, \dots, n\}$ and $\forall i \in \{1, \dots, p\}$), where n denotes the number of training observations and p denotes the number of variables/predictors.

$$\underset{\beta_0,\dots,\beta_p,\epsilon_1,\dots,\epsilon_n}{\text{maximize}} \quad M$$

$$\text{subject to} \quad \sum_{k=1}^{n} \beta_k^2 = 1,$$

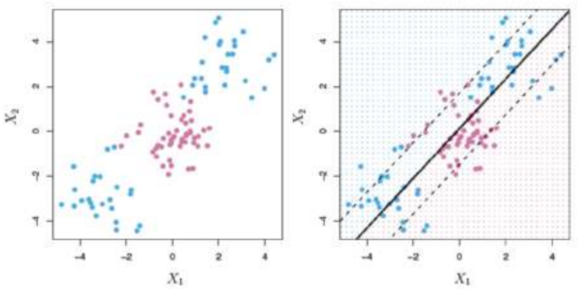
$$y_i\left(\beta_0 + \beta_1 x_{i1} + \ldots + \beta_p x_{ip}\right) \geq M\left(1 - \epsilon_i\right), \quad \forall i \in \{1, \dots, n\}$$

$$\epsilon_i \geq 0, \ \sum_{i=1}^{n} \epsilon_i \leq C$$

The hyperparameter C is kown as the *budget*. It represents the tolerance to the margin or hyperplane violation. If one sets $C = 0$ then $\epsilon_i = 0$, ($\forall i \in \{1, \dots, n\}$) and a maximum margin classifier is trained. Be careful, its definition may vary accross the different implementations of SVC.

Support Vector Classifier

SVC limitations



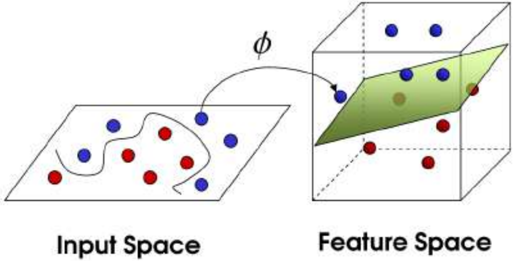
Source: [Gareth et al., 2009]

SVC is not suitable for non-linear boundary

When the data points cannot be separated by a linear boundary the SVC fails to provide a good classification. This problem can be overcome by using the **kernel trick**.

Support Vector Machine

SVM: Mapping the input space to an enlarged feature space



Source: stackoverflow website

Definition of Support Vector Machine

The Support Vector Machine (SVM) is an extension of the SVC consisting in mapping the inputs (data points) into **high-dimensional feature spaces** in which a linear boundary can be found. It results in a non-linear boundary in the input space.

Support Vector Machine

The idea behind SVM

The kernel trick

The support vector classifier optimization problem only requires computing **inner products** of the observations:

$$\langle x_i, x_j \rangle = \sum_{k=1}^{p} x_{ik} x_{jk}$$

Replacing the inner product by a **kernel function** such as:

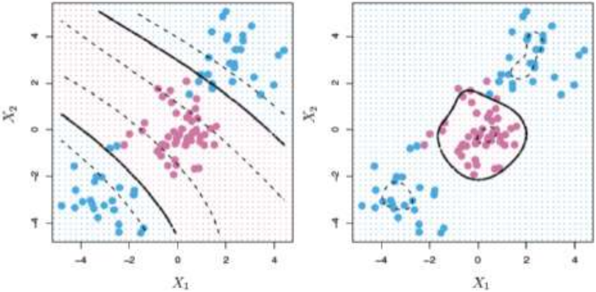
$$K(x_i, x_j) = \left(1 + \sum_{k=1}^{p} x_{ik} x_{jk}\right)^p \quad \text{(polynomial kernel)}$$

$$K(x_i, x_j) = \exp\left(-\gamma \sum_{k=1}^{p} (x_{ik} - x_{jk})^2\right) \quad \text{(radial kernel)}$$

is *equivalent* to mapping the input into a higher-dimensional feature space and performing a SVC in this space. However, it is **computationally much more efficient**.

Support Vector Machine

Polynomial and radial kernels



Source: [Gareth et al., 2009]

Left: SVM with a **polynomial** kernel
Right: SVM with a **radial** kernel

Depending on the kernel chosen to determine the separating hyperplane in the feature space, the decision boundaries in the input space may be very different.

Support Vector Machine

Pros and cons of using SVM to model default

Pros:

- Can capture **highly non-linear** patterns
- Can be very robust to small changes in data

Cons:

- Is **not easy to interpret**
- Does not provide a stragihtforward estimation of the probability of default
- Is prone to overfitting if C is set inadequately

R Markdown

Quiz

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Credit Risk

Lecture 3 – Credit Risk and Climate Risk

Loïc BRIN

École Nationale des Ponts et Chaussées
Département Ingénieurie Mathématique et Informatique (IMI) – Master II

Objectives of the lecture

Teaching objectives

At the end of this lecture, you will:

- get a first flavour on Ratings
- understand how basic models can be calibrated
- get introduced to credit and climate risk modeling

Table of Contents

The various conceptions of default

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Rating agencies

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Ratings and rating agencies

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Rating scale

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BBB+	CCC
BBB	CCC-
BBB-	CC
	C

Ratings and rating agencies

How ratings relate to default?

It appears from the historical data that there is a strong exponential relationship between the rating of a firm and its one-year probability of default. That is to say the probability that the firm will default within the next year.

We have: $\text{PD} = 6 \times 10^{-6} \times e^{0.64 * \text{Rating}}$.

Ratings of well-known firms

Firm	**S&P rating**
SG	A
BNP	A
Total	A
EDF	A-
Accor	BBB-

Transition matrix

How do ratings migrate?

Definition of Transition Matrix

In credit risk, **a transition matrix**, $\boldsymbol{M}_{t,t+1} = (m_{ij})_{ij}$, is a matrix where:

$$m_{ij} = \mathbb{P}(\text{Grade}_{t+1} = j \mid \text{Grade}_t = i)$$

Where i and j are the grades presented earlier.

S&P's transition matrix – From 1981 to 2013

(in %)	AAA	AA	A	BBB	BB	B	CCC	D	NR[2]
AAA	87.11	8.88	0.53	0.05	0.08	0.03	0.05	0	3.27
AA	0.55	86.39	8.26	0.56	0.06	0.07	0.02	0.02	4.07
A	0.03	1.87	87.34	5.48	0.35	0.14	0.02	0.07	4.7
BBB	0.01	0.12	3.59	85.22	3.82	0.59	0.13	0.21	6.31
BB	0.02	0.04	0.15	5.2	76.28	7.08	0.69	0.8	9.74
B	0	0.03	0.11	0.22	5.48	73.89	4.46	4.11	11.7
CCC	0	0	0.15	0.23	0.69	13.49	43.81	26.87	14.76

[2] "Non-rated" or "Unrated". Some "companies" may decide to stop being rated by a given rating agency.

Transition matrix

What properties should be expected from a transition matrix?

Several properties of transition matrices

Among the **properties of the transition matrices**, note that:

- Each row sums to 1;
- They are dominant;
- In the case of homogeneity we have that: $\boldsymbol{M}_{t,t+n} = \boldsymbol{M}_{t,t+1}^{n}$.

The generator for homogeneous Markov chains

The **generator for a Markov chain** $(\boldsymbol{M}_{t,t+n})_n$ is the matrix $\boldsymbol{Q}$ so that:

$$\forall (t, T), \quad \boldsymbol{M}_{t,T} = \exp\left((T-t)\boldsymbol{Q}\right) \qquad \text{with } \exp(\boldsymbol{A}) = \sum_{n \geq 0} \frac{\boldsymbol{A}^n}{n!}$$

Would such a matrix exist (see [Israel et al., 2001]), we have:

$$\boldsymbol{Q} = \sum_{n>0} (-1)^{n-1} \frac{(\boldsymbol{M}_{t,t+1} - \boldsymbol{I})^n}{n}$$

Transition matrix

How to estimate transition matrices?

Two techniques to estimate transition matrices

There are two techniques to estimate the generator of transition matrices:

- By **cohorts**: it consists in computing the average number of agents that change from rating i to j within one year, for all (i, j);
- By **durations**: it consists in looking for instantaneous probability, for an agent, of changing from rating i to j. The likelihood of changing of rating, from i to j, in t, is:

$$e^{\lambda_{ii} t} \lambda_{ij}$$

By maximizing the likelihood of these transitions, one can estimate $(\lambda_{ij})_{ij}$.

Transition matrix

Do the maths fit reality?

Transition matrices are not Markov matrices

The markovian assumption is in **contradiction** with phenomena described by the data.

For example, a firm which has recently experienced a downgrade to rating j, is more likely to experience another one, as opposed to a firm that has had rating j for a long time.

Quiz

Conclusion
Default and rating-based models

- Default can have **various definitions depending on the context**;
- Rating agencies provide **ratings that can be used to estimate the one-year probability of default of firms**;
- A transition matrix that allow to **describe ratings migrations**.

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What do we want to predict?

Can we predict default?

- Will a firm/customer **default within a given period of time** (year, month, etc.)?

$$Y = \begin{cases} 1 & \text{if default within a given period,} \\ 0 & \text{otherwise.} \end{cases}$$

- Or better yet: What is the probability that a firm/customer will default within a given period of time, given the information we have?

$$p(X) = \mathbb{P}(Y = 1|X)$$

- For a firm: X can be financial data, market data, country, activity sector, etc.
- For a customer: X can be age, job situation, salary, debt level, etc.

Theoretical framework of the logistic regression

The logistic regression's assumptions

Logistic regression's model

The **logistic regression model** can be defined the following ways:

$$p(X) = \mathbb{P}(Y = 1|X) = \frac{e^{\beta_0+\beta_1 X_1+\ldots+\beta_p X_p}}{1 + e^{\beta_0+\beta_1 X_1+\ldots+\beta_p X_p}}$$

or equivalently

$$\ln\left(\frac{p(X)}{1 - p(X)}\right) = \beta_0 + \beta_1 X_1 + \ldots + \beta_p X_p \tag{1}$$

where $X = (X_1, \ldots, X_p)$

No closed-form solution

In practice, the vector of parameters $(\beta_0, \ldots, \beta_p)$ is estimated by maximizing the likelihood. Contrary to linear regression, there is **no closed-form solution**, which can therefore lead to different estimations depending on the algorithm/software chosen.

Theoretical framework of the logistic regression

How to interpret the coefficients? (I/II)

Logistic regression's coefficients can be interpreted through the concept of **odds ratio** using equation (1).

Coefficient interpretation of a logistic regression with one binary predictor (I/II)

We consider the following logistic modelling where the default ($Y = 1$) only depends on being a student ($X = 1$) or not ($X = 0$):

$$\ln\left(\frac{p(X)}{1-p(X)}\right) = \beta_0 + \beta_1 X$$

Then $\exp(\beta_0)$ is the odds ratio of a non-student being in the honor class (default):

$$\exp(\beta_0) = \frac{\mathbb{P}(Y=1|X=0)}{1-\mathbb{P}(Y=1|X=0)} = \frac{\mathbb{P}(\text{default}|\text{non-student})}{\mathbb{P}(\text{no default}|\text{non-student})}$$

$\exp(\beta_1)$ is the ratio of the odds for student to the odds for non-student:

$$\exp(\beta_1) = \frac{\mathbb{P}(Y=1|X=1)}{1-\mathbb{P}(Y=1|X=1)} \Big/ \frac{\mathbb{P}(Y=1|X=0)}{1-\mathbb{P}(Y=1|X=0)}$$

Theoretical framework of the logistic regression

How to interpret the coefficients? (II/II)

Coefficient interpretation of a logistic regression with one binary predictor (II/II)

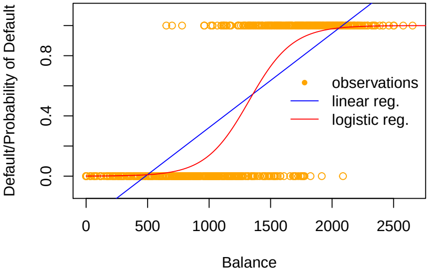
- We deduce from the previous slide that according to the model a non-student is $\exp(\beta_0)$ time(s) as likely to default as to not.
- Let's assume that $\beta_1 > 0$. The odds for a student is $\exp(\beta_1) - 1$ times higher than the odds for a non-student.

To learn more about logistic regression coefficients' interpretation:

▸ Tutorial

Linear vs logistic regression

Why not use linear regression?



Source: The Default dataset from [Gareth et al., 2009]

The main reason for using logistic regression instead of linear regression is that the predictions are inside the $[0, 1]$ interval, making them easier to **interpret as probabilities**.

Conclusion

Why and why not use logistic regression to model default?

Pros:

- Can be **easily interpreted**:
  - reducing the risk of modelling errors;
  - making it easier to be audited (validation team, regulator, etc.).
- Provides an **estimation of the probability of default**;
- Can be converted into a **score card** to facilitate the use of the model.

Cons:

- Lacks prediction power (Logistic regression **cannot model non-linear relationships**)
- In practice the continuous predictors must be binned manually
- Requires variables selection/regularization (Lasso)

▸ R Markdown

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The tragedy of the horizon

There is a growing international consensus that climate change is unequivocal . This change raises two issues and challenges of importance for the banking system and financial stability:

- Climate risk is a growing source of systemic risk. It will materialize under the forms of physical risk, liability risk and transition risk.
- Physical risks (e.g. flood, exceptional droughts - insurance natural events losses tripled since 90s) and
- transition risks (e.g. strengthening of carbon tax or environmental requirements such as petrol ban) may drastically affect banking portfolios and turn out to be a credible source of systemic risk and require therefore attention from central banks, in their financial stability mandate.
- In line with Paris Accords (2015), there is an urgent need to support green investment and favour the transition to a low carbon economy. Some proposals have been made through the introduction of a Green Supporting Factor (GSF) whose aim is -through a relief in the capital charge on 'green' investments- to accelerate the financing of a low carbon economy. The fierce debate surrounding this proposal confirms the need to envisage several mechanisms fostering the transition to a low carbon economy, taking the into account the objectives of central banks and banking supervisors as well as the society.

The tragedy of the horizon

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The tragedy of the horizon

The tragedy of the horizon

See Mark Carney: Speech at Lloyd's of London, 2015 (Breaking the tragedy of the horizon – climate change and financial stability) (see : https://www.youtube.com/watch?v=V5c-eqNxeSQ, you may start at 4:57)

- physical impacts of climate change will be felt over a long-term horizon, with massive costs and possible civilisational impacts on future generations vs
- the time horizon in which financial, economic and political players plan and act is much shorter

Implications for banks

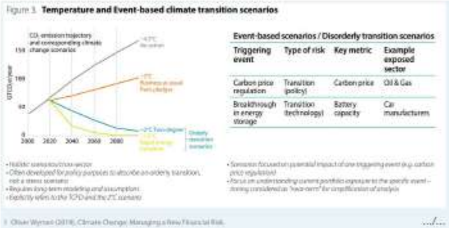
Which impact on real economy

- Extreme natural events
- Population Moves
- Black Swans/Green Swans
- Changes in consumption
- Reputation
- Political responses
- Stranded assets
- Changes in business models (hydrogen-tbd, no diesel as the most obvious today)

Climate risk (impact of carbon prices)and credit risk

Credit model

Link between CO2 emissions and global temperature increase



Source: Extract from "Overview of Environmental Risk Analysis by Financial Institutions", Technical Report from NGFS

Solution

- Carbon tax
- Impact on firms EBITDA (hence profitability)
- impact on firm credit risk

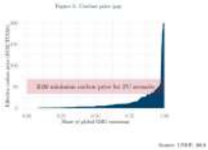
Link between Asset Value and EBITDA

The Asset Value of the firm is equal to the expected sum of discounted future cash flows (Modigliani-Miller)

$$Asset \equiv Cashflows(EBITDA - tbd/simplification)$$

Bouchet/Le Guenedal idea ("Credit Risk Sensitivity to Carbon Price", Working Paper 95-2019, Amundi, 2020):

- assessing the impact of carbon tax on EBITDA (through several scenarios)
  - collect all firms cash flows
  - collect all firms CO2 emissions and compute hypothetical carbon tax
  - deduce loss in cashflow from additional CO2 tax
- converting this impact in a proportional impact (w.r.t. cashflow level)
- apply same impact on Asset Value
- through Merton model, assess the impact on credit risk valuation (debt value, default probability)



Source: Extract from Bouchet/Le Guenedal

Computing probability of default

Observable variables are:

- Equity and Total Debt in current scenario/situation
- equity volatility
- assess average Debt Duration T

Then solve $(Assets$ and $\sigma_{Assets})$:

$$Equity = Assets\ \Phi(d_1) - Debt\ e^{-rT}\Phi(d_2) \tag{2}$$

$$Equity = \frac{\sigma_{Assets}}{\sigma_{Equity}}\Phi(d_1)\ Assets \tag{3}$$

recalling that d_1 and d_2 are deduced from $Assets$ and σ_{Assets} and observable variables :

$$d_1 = \frac{\ln\left(\frac{Assets}{Debt}\right) + (r + \frac{\sigma_V^2}{2}) \times T}{\sigma_V\sqrt{T}} \tag{4}$$

$$d_2 = d_1 - \sigma_V\sqrt{T} \tag{5}$$

Bouchet/Le Guenedal results/use

Extracts from Bouchet/Le Guenedal:

Figure 7: Scenario selection and global distribution of carbon price (all models and scenarios

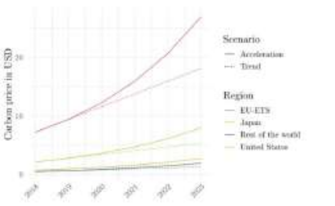


Figure 8: Medium-term impact on EBITDA

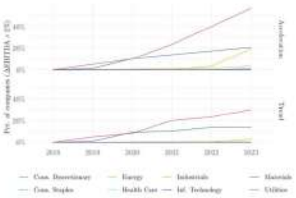
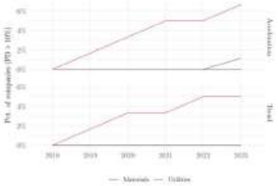


Figure 9: Medium-term impact on probabilities of default



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Credit Risk

Lecture 4 – Portfolio models and Asset Backed Securities (ABS)

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Objectives of the lecture

Teaching objectives

At the end of this lecture, you will:

- Know how to derive the **loss of a loan portfolio** under a series of assumptions;
- Better understand the **concept of dependence structure** through the theory of **copulas**;
- Be familiar with the **most common credit derivatives** such as CDO and CSO;
- Understand what are these credit derivatives used for and **how they are priced**.

1 - Vasicek Model

Vasicek Model – Framework

The Vasicek Model – Purpose and assumptions

Vasicek model's purpose

Vasicek model provides a way to assess **the loss distribution of a portfolio of defaultable assets**.

Assumptions of the infinite homogeneous Vasicek portfolio model

The Vasicek Model usually refers to the infinite homogeneous Vasicek portfolio model that supposes that:

- there is a **countable infinite** number of bonds (loans, mortgages, etc.);
- of **equal nominal**;
- **same maturity**;
- **same probability of default** at maturity (PD);
- and with the **same recovery rate** (R).

At individual bond level, Vasicek Model is a combination of Merton Model (asset return determines default or non-default) and reduced form model, with a fixed recovery rate.

Tutorial

The Vasicek Model – Framework

The Vasicek Model – Modeling the returns of the debtors

The Vasicek Model – Definition of the latent variable of return

We define a **latent variable of return**, for each asset as:

$$\forall i \in \mathbb{N}, \qquad R_i = \underbrace{\sqrt{\rho}}_{\substack{\text{corelation}\\ \text{factor}}} \underbrace{F}_{\substack{\text{systemic}\\ \text{factor}}} + \sqrt{1-\rho} \underbrace{e_i}_{\substack{\text{idiosyncratic}\\ \text{factor}}}$$

with $(e_i)_{i \in \mathbb{N}}$ and F are standardized, independent, normal variables, and thus $(R_i)_{i \in \mathbb{N}}$ are standardized and correlated, with correlation ρ.

Vasicek Model – Framework

The Vasicek Model – Definition of the default

Definition of the default in the Vasicek model

In the Vasicek model, the bond i **defaults** when:

$$\{R_i < s\}$$

that is when the **latent variable**, R_i, is smaller than s, the **latent threshold** (common for all bonds).

Economic interpretation of the Vasicek model

There is a latent variable for each counterparty in the studied portfolio whose behaviour is due to a **(unique) systemic factor** and a **idiosyncratic one**. The latent variable can be understood as some measure of the return of the counterparty, and the systemic factor as a measure of the economic soundness of the economy (GDP, Unemployment Rate).

- The smaller F, the harsher the economic environment and the smaller the latent return for **all** the counterparties;
- The smaller e_i, the smaller the return of the **ith** counterparty and the higher its probability of default.

Vasicek Model – Framework

The Vasicek Model – Definition of the default

The Vasicek Model – Link between the latent threshold and the probability of default

We can deduce the expression of the **common latent threshold of default**:
Given that:

$$\mathrm{PD} = \mathbb{P}(R_i < s) = \underbrace{\Phi}_{\substack{\text{Normal}\\ \text{cdf}}}(s)$$

We deduce that:

$$s = \Phi^{-1}(\mathrm{PD})$$

ρ and PD are not output of the Vasicek model

ρ and PD are input parameters of the model not outputs. The loss distribution of the portfolio is the output of the model.

Vasicek Model - Loss distribution

What is the distribution of the portofolio's loss?

The loss distribution of the infinite homogeneous Vasicek portfolio model

We thus have that for **the random variable of the losses of the porfolio**, expressed as a percentage, is:

$$
\begin{aligned}
L \mid F &= \frac{1-R}{N}\sum_{i=1}^{+\infty}\mathbb{1}_{\{R_i<s\}} \\
&= \frac{1-R}{N}\sum_{i=1}^{+\infty}\mathbb{1}_{\left\{e_i<\frac{\Phi^{-1}(PD)-\sqrt{\rho}F}{\sqrt{1-\rho}}\right\}} \\
&\underbrace{=}_{\substack{\text{Law of} \\ \text{large numbers}}} (1-R)\Phi\left(\frac{\Phi^{-1}(PD)-\sqrt{\rho}F}{\sqrt{1-\rho}}\right)
\end{aligned}
$$

Note that L is conditionned to the value of F, the stochastic systemic factor. R is the recovery rate for each loan.

▸ R Markdown

Conclusion

Vasicek Model

- Vasicek model assumes an **infinite homogeneous portfolios of correlated loans**;
- The loans are independent conditionnally to a **systemic factor**;
- Under assumptions **comparable to Merton's approach**, the distribution of the **portfolio loss** can be derived.

2 - Copulas

Correlation vs Dependence

Do correlation and dependence refer to the same concept?

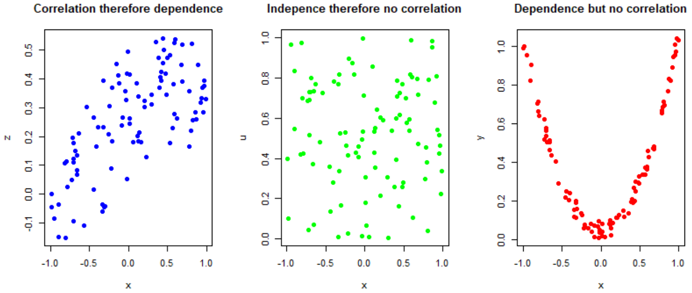
Correlation and dependence

Correlation $\neq$ Dependence

Dependence and correlation **do not refer to the same concept**. In fact, dependence is a much broader concept than correlation. Dependence structures can therefore be much **more complex** than correlation structures. **Analogy**: difference between mean and quantiles.

Correlation vs Dependence

Link between correlation and dependence



Correlation entails dependence not the other way around!

Copulas - definition and main results

Definition of a copula

Copula – Definition

A copula is (the cdf of) the joint distribution of random variables $U_1, \dots, U_d$, each of which being marginally uniformly distributed on $[0;1]$.

$$\forall (u_1, \dots, u_d) \in [0;1]^d, \quad C(u_1, \dots, u_d) = \mathbb{P}\left(U_1 \leq u_1, \dots, U_d \leq u_d\right)$$

It is therefore a function from $[0;1]^d$ to $[0;1]$.

Copulas allow to model the **dependence structure** of a random vector **regardless of its marginal behaviors**.

Copulas - Definition and main results

What is the link between a multivariate distribution and its copula?

Sklar theorem

Sklar's theorem asserts that from any continuous multivariate distribution F, a copula can be deduced with the following formula:

$$\forall (u_1, \ldots, u_d) \in [0;1]^d, \quad C(u_1, \ldots, u_d) = F(F_1^{-1}(u_1), \ldots, F_d^{-1}(u_d))$$

and equivalently

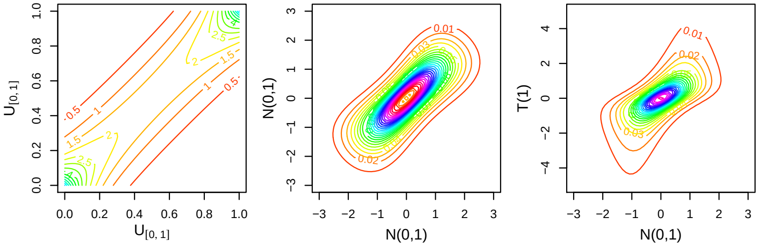
$$\forall (x_1, \ldots, x_d) \in \mathbb{R}^d, \quad F(x_1, \ldots, x_d) = C(F_1(x_1), \ldots, F_d(x_d))$$

where $F_1, \ldots, F_d$ are the cdf of the margins of F and $F_1^{-1}, \ldots, F_d^{-1}$ are their inverse distribution functions (or quantile functions).

By transforming the margins of a **continuous** multivariate distribution by their cumulative distribution functions respectively, we obtain the **copula characterizing the dependence structure (and the dependence structure only!)** of this multivariate distribution.

Copulas - Definition and main results

Can we build a multivariate distribution from a copula?



Densities of various bivariate distributions with different marginal distributions but the same dependence structure (Frank copula ($\theta = 7$)).

We can choose **any copula** (to characterize the dependence structure) **and any marginal distributions** (to characterize the marginal behaviors) **to build a multivariate distribution**.

Copulas - Definition and main results

Density function of a copula

We saw that $F(x_1, ..., x_d) = C(F_1(x_1), ..., F_d(x_d))$. If F is continuous, by derivating n times this expression, we can find the joint density, that is:

$$f(x_1, ..., x_d) = f_1(x_1) \times ... \times f_d(x_d) \times \frac{\partial^d C}{\partial x_1 ... \partial x_d}(F_1(x_1), ..., F_d(x_d))$$

With f the density of the joint distribution and $(f_1, ..., f_d)$ the ones of the marginal distributions.

Density of a copula

▸ Definition

We define the **density of a copula**, c:

$$c(u_1, ..., u_d) = \frac{\partial^d C}{\partial u_1 ... \partial u_d}(u_1, ..., u_d) = \frac{f\left(F_1^{-1}(u_1), ..., F_d^{-1}(u_d)\right)}{f_1\left(F_1^{-1}(u_1)\right) \times ... \times f_d\left(F_d^{-1}(u_d)\right)}$$

Estimation of a copula

The multivariate likelihood as a sum of likelihoods

We saw that:

$$f(x_1, ..., x_d) = c(F_1(x_1), ..., F_d(x_d)) \prod_{i=1}^{d} f_i(x_i)$$

where c, is the d-dimensional density of the copula C. In the following, we consider that we have n, d-dimensional observations: $(\mathbf{x_j^{(i)}})$. We can then deduce the likelihood L_C and the loglikelihood LL_C:

$$\begin{aligned} L_C &= \prod_{i=1}^{n} f(x_1^{(i)}), ..., x_d^{(i)})) \\ &= \prod_{i=1}^{n} \left[c(F_1(x_1^{(i)}), ..., F_d(x_d^{(i)})) \prod_{i=j}^{d} f_i(x_j^{(i)}) \right] \\ LL_C &= \sum_{i=1}^{n} \log \left(c(F_1(x_1^{(i)}), ..., F_d(x_d^{(i)})) \right) + \sum_{i=1}^{n} \sum_{i=j}^{d} \log \left(f_i(x_j^{(i)}) \right) \end{aligned}$$

The first term deals with **the dependence** when the second one deals with **the distributions of the margins**.

Estimation of a copula

How to fit a copula with the Maximum Likelihood Estimator (MLE)?

From now on, we will denote by θ the parameters of the copula and α_i the parameters of the ith marginal distribution.

They are **two techniques** to fit a copula:

- The Maximum Likelihood Estimator (**MLE**);
- The Inference Functions for Margins method (**IFM**).

The Maximum Likelihood Estimator to fit copulas

The **Maximum Likelihood Estimator** consists in estimating $(\theta, \alpha_1, ..., \alpha_n)$ by

$$(\theta^{MLE}, \alpha_1^{MLE}, ..., \alpha_n^{MLE})$$

with:

$$(\theta^{MLE}, \alpha_1^{MLE}, ..., \alpha_n^{MLE}) = \text{argmax}_{(\theta, \alpha_1, ..., \alpha_n)} L((\theta, \alpha_1, ..., \alpha_n))$$

Estimation of a copula

How to fit a copula with the Inference Functions for Margins (IFM) method?

The Inference Functions for Margins

The **Inference Functions for Margins (IFM)** consists in a two-step procedure:

1. Computing $\forall i \in [1; d], \quad \boldsymbol{\alpha_i}^{IMF} = \text{argmax}_{\alpha_i} L_i(\boldsymbol{\alpha_i})$
2. Computing $^{IMF} = \text{argmax}_{\boldsymbol{\theta}} L_C(, \hat{\boldsymbol{\alpha}}_{\boldsymbol{1}}{}^{IFM}, \ldots, \hat{\boldsymbol{\alpha}}_{\boldsymbol{n}}{}^{IFM})$

Estimation of a copula

Copulas – Difference between MLE and IFM

Difference between MLE and IFM

There is a slight but decisive **difference between the two methods** that confers to both methods different asymptotic properties:

The MLE estimator $(^{MLE}, \boldsymbol{\alpha}_{\mathbf{1}}^{MLE}, \dots, \boldsymbol{\alpha}_{\boldsymbol{n}}^{MLE})$ solves:

$$\left(\frac{\partial L}{\partial}, \frac{\partial L}{\partial \boldsymbol{\alpha}_{\mathbf{1}}}, \dots, \frac{\partial L}{\partial \boldsymbol{\alpha}_{\boldsymbol{n}}}\right) = 0$$

While the IFM one $(^{IFM}, \boldsymbol{\alpha}_{\mathbf{1}}^{IFM}, \dots, \boldsymbol{\alpha}_{\boldsymbol{n}}^{IMF})$ solves:

$$\left(\frac{\partial L}{\partial}, \frac{\partial L_1}{\partial \boldsymbol{\alpha}_{\mathbf{1}}}, \dots, \frac{\partial L_n}{\partial \boldsymbol{\alpha}_{\boldsymbol{n}}}\right) = 0$$

[Joe et al., 1996] shows that MLE and IFM estimation procedures are equivalent in a very particular case: the one where the copula and the margins are Gaussian.

Well-know copulas

The Gaussian copula – Definition

The Gaussian copula

As Gaussian univariate and multivariate cumulative distributions are continuous, applying Sklar's theorem, we can define **the unique Gaussian copula**:

$\forall \mathbf{u} \in [0;1]^d$,

$$
\begin{aligned}
C_{\mathbf{R}}^{\mathcal{N}}(u_1, \ldots, u_d) &= \Phi_{\mathbf{R}}(u_1, \ldots, u_d) \\
&= \int_{-\infty}^{\Phi^{-1}(u_1)} \cdots \int_{-\infty}^{\Phi^{-1}(u_d)} \frac{1}{(2\pi)^{\frac{d}{2}} \mid \mathbf{R} \mid^{\frac{1}{2}}} \exp\left(-\frac{1}{2}\mathbf{x}^{\star}\mathbf{R}^{-1}\mathbf{x}\right) \mathrm{d}\mathbf{x}
\end{aligned}
$$

Well-know copulas

The Gaussian copula – Density of the copula

Density of the Gaussian copula

Using the above definition of the density of a copula, we can deduce **the density of the Gaussian copula** with correlation matrix $\mathbf{R}$:

$$\forall \mathbf{u} \in [0;1]^d, \quad c_{\mathbf{R}}^{\mathcal{N}}(u_1, \ldots, u_d) = \frac{1}{\sqrt{|\mathbf{R}|}} \exp\left(-\frac{1}{2} \begin{pmatrix} \Phi^{-1}(u_1) \\ \vdots \\ \Phi^{-1}(u_d) \end{pmatrix}^{\star} \cdot \left(\mathbf{R}^{-1} - \mathbf{I}_d\right) \cdot \begin{pmatrix} \Phi^{-1}(u_1) \\ \vdots \\ \Phi^{-1}(u_d) \end{pmatrix} \right)$$

Well-know copulas

The Gaussian copula – Simulation

It often happens that modeling involves complex univariate and multivariate variables so that there is no close formula to compute the risk metric: in such a case, one must use Monte Carlo techniques and thus simulate the copula.

How to simulate a Gaussian copula?

In order to **simulate a Gaussian copula** $C_{\mathbf{R}}^{\mathcal{N}}$, one must apply this two-step procedure:

1. First, one must **simulate a normal reduced centered vector** with correlation matrix R, $\boldsymbol{X} = (X_1, X_2, ..., X_d)$;
2. Second, one must **compose each variable of the vector by the inverse cumulative distribution function of a univariate centered and reduced Gaussian distribution**, $(\Phi(X_1), ..., \Phi(X_d))$.

And its goes the same way for any other copula deduced from a multivariate distribution applying Sklar's theorem.

Well-know copulas

Other well-known copulas

Other well-known copulas

There are other well-known copulas:

- **Other copulas deduced from multivariate distributions** applying Sklar's theorem: Student copulas, grouped t-copulas, individual t-copulas, etc.;
- **the so-called Archimedean copulas**, that can be written as:

$$C(u_1, \ldots, u_d; \theta) = \psi^{-1}\left(\psi(u_1; \theta) + \cdots + \psi(u_d; \theta); \theta\right)$$

where $\psi \colon [0,1] \times \Theta \to [0, \infty)$ is a continuous, strictly decreasing and convex function such that $\psi(1; \theta) = 0$, called the **generator of the Archimedean copula**.

▸ Tutorial ▸ R Markdown ▸ Quiz

Portfolio models and copulas

Link between the Vasicek model and copulas

Vasicek model and Gaussian copula

The Vasicek model is a copula-based model. Indeed, the dependence structure between the default times is based on a Gaussian copula.

> The formalization of such a point was made in [Burtschell et al., 2008].

▸ R Markdown

Extension of the Vasicek model based on other copulas

A lot of models can be deduced from this finding:

- for a more extreme dependence structure, one can use a **Student copula** to link default times;
- for an assymetric dependence structure of the default times, one can use the **Gumbel copula**;
- etc.

Conclusion
Copulas

- **Correlation** is only **one aspect of the concept of dependence**;
- Copulas are a tool to model **complex structures of dependence**;
- Copulas allow to **apprehend dependence structures separately from the marginal behavior**;
- The **estimation** of copulas remains a **complex issue**;
- Going **beyond the Gaussian** copulas is crucial to **model the reality more accurately**.

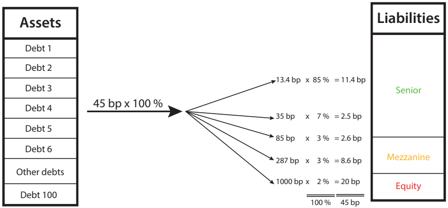
3 - CDO and CSO

Collateralized Debt Obligation (CDO)

CDO capital structure

Collateralized Debt Obligation – Capital structure

- A **SPV (Special Vehicule Purpose)** issues several tranches of debts to buy assets (debt instruments);
- The **tranches are rated** by rating agencies (Fitch, Moody's, S&P);
- The **tranches offer different risk / return ratios**:
  - Losses impact first the junior tranches;
  - Principal cash-flows are redirected to senior tranche first.

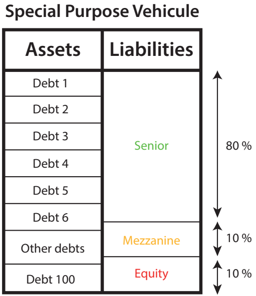


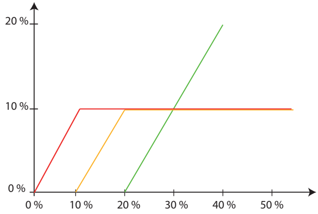
Collateralized Debt Obligation (CDO)

Option theory and description of CDO

Let L be the percentage of losses:

- If L is smaller than 10%: losses only affect equity;
- If L is between 10% and 20% : losses affect equity and mezzanine;
- If L is larger than 20%: losses affect all the tranches.





So tranching is a **non-linear** operation.

Collateralized Debt Obligation (CDO)

CDO's economic purposes (I/III)

Balance sheet CDO	Arbitrage CDO
▶ Refinancing of a portfolio (Private investors, ECB, etc); ▶ A bank wants to transfer the risk of its loan portfolio; ▶ Balance-sheet reduction; ▶ Regulatory and economic capital optimization; ▶ Increase ROE and RAROC; ▶ Close a business line.	▶ An asset manager wants to build a corporate portfolio; ▶ Funding through the issuance of debt securities and equity; ▶ That generates management and structuration fees; ▶ Increases Asset under Management (AuM); ▶ And offers diversification to the clients.

Simple, Transparent and Standardised (STS) Securitisations

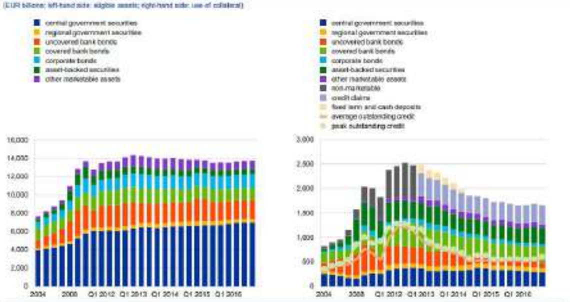
Collateralized Debt Obligation (CDO)

CDO's economic purposes (III/III)

CDOs/Securitized products offer refinancing possibilities.

Figure 3

Eligible assets and use of collateral

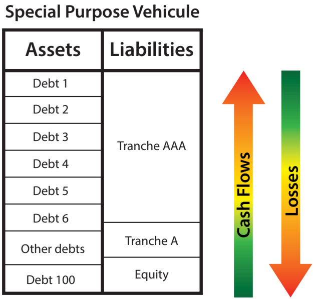


Source: ECB.
Notes: collateral used is reported after valuation and haircuts in averages of end of month data over each time period shown.
Since 2013 Q1, the category "Non-marketable assets" is split into two categories: "Fixed term and cash deposits" and "Credit claims".
Last observation: 2016 Q4.

Collateralized Debt Obligation (CDO)

CDO's economic purposes (III/III)

CDO intends to offer the optimal spread/rating duo for every investor.



- **The senior tranch is generally rated AAA;**
- **One or several mezzanine tranches are rated AAA to B;**
- **The equity tranche is generally not rated.**

For more details on the subject, you can take a look at [Bluhm and Christian, 2003].

Collateralized Debt Obligation (CDO)

The concept of credit enhancement

Credit enhancement

There are several way to improve the credit profile of an ABS:

- **Excess spread**: the received rate is higher than the served one;
- **Overcollateralization**: the face value of the underlying loan portfolio is larger than the security it backs;
- **Monolines and wrapped securities**: CDS on the underlying assets are bought to monolines to cover part of the losses.

Collateralized Debt Obligation (CDO)

Pricing of CDO (I/III)

Expected loss on tranche $[A; D]$

The **expected loss** at time t on tranche $[A; D]$, EL_t, is a simple function of the loss on the underlying portfolio at time t:

$$\mathrm{EL}_t = \mathbb{E}\left((L(t) - A)^+ - (L(t) - D)^+\right)$$

The loss distribution function

For a granular homogeneous credit portfolio, the loss at time t depends on the systemic factor F and the default time cdf H at time t, and **the loss distribution function** expression is:

$$L(t, F) = (1 - R)\Phi\left(\frac{\Phi^{-1}(\mathrm{H(t)}) - \sqrt{\rho}F}{\sqrt{1-\rho}}\right)$$

Collateralized Debt Obligation (CDO)

Pricing of CDO (II/III)

Floating leg market value

The **floating leg market value** of the CDO tranche $[A; D]$ is:

$$\mathrm{JV}^{[A;D]}(0; T) = \int_0^T e^{-rt} \mathrm{dEL}_t = e^{-rT}\mathrm{EL}_T + r\int_0^T e^{-rt}\mathrm{EL}_t \mathrm{d}t$$

Fix leg market value

The **fix leg market value** of the CDO tranche $[A; D]$ is:

$$\begin{aligned}
\mathrm{JF}^{[A;D]}(0; T) &= s^{[A;D]} \int_0^T e^{-rt}(D - A - \mathrm{EL}_t)\mathrm{d}t \\
&= s^{[A;D]}\mathrm{DV}^{[A;D]}(0; T) \\
&= s^{[A;D]}\left(\left(\frac{D-A}{r}\right)(1 - e^{-rT}) - \frac{1}{r}(\mathrm{JV}^{[A;D]}(0; T) - e^{-rT}\mathrm{EL}_T)\right)
\end{aligned}$$

Collateralized Debt Obligation (CDO)

Pricing of CDO (III/III)

As for CDS, we can use the no arbitrage assumption to calculate the spread of the studied CDO tranche.

Spread of the tranche of a CDO

Thus, the **spread of the CDO tranche** $[A; D]$ is:

$$s^{[A;D]} = \frac{JF^{[A;D]}(0, T)}{DV^{[A;D]}(0, T)}$$

▸ R Markdown

Collateralized Synthetic Obligation (CSO)
CSO vs CDO

Cash	Synthetic
Large AAA size	Mezzanine AAA+ large super senior
High funding cost	Low funding cost
Limited invested universe	Very large investment universe
Transfer of the assets	Risk transfer only
High management fees	Low management fees
10-15 % high yield	100% investment grade
Average rating BBB-	Average rating A
Low leverage (equity 10 %)	High leverage (2 / 3 %)

Collateralized Synthetic Obligation (CSO)

CSO and CDS indices

Credit Index

iTraxx is a Credit Index used in Europe and Asia with 125 references (the equivalent in the US is **CDX**). It has the following caracteritics:

- Spreads are usually from 10 bp to 120 bp with an average around 35 bp;
- Spread volatility is around 2 bp a day;
- Listed tranches are $[0\%, 3\%]$, $[3\%, 6\%]$, $[6\%, 9\%]$, $[9\%, 12\%]$, $[12\%, 22\%]$;
- Maturities are of 3, 5, 7, 10 years, rolled every 6 months;

iTraxx indice

Implied correlation and base correlation

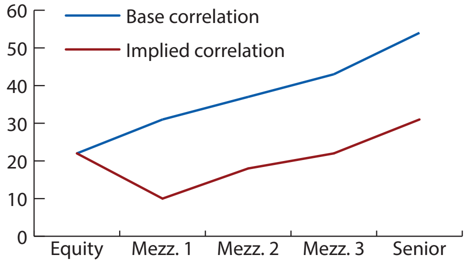
Implied correlation and base correlation

Implied correlation of tranche $[A; D]$

The **implied correlation** of $[A; D]$, knowing the spread of the tranche $s_{A,D}$, is the correlation required in the Vasicek model to price the CSO of tranche $[A; D]$, $s^{[A,D]}$.

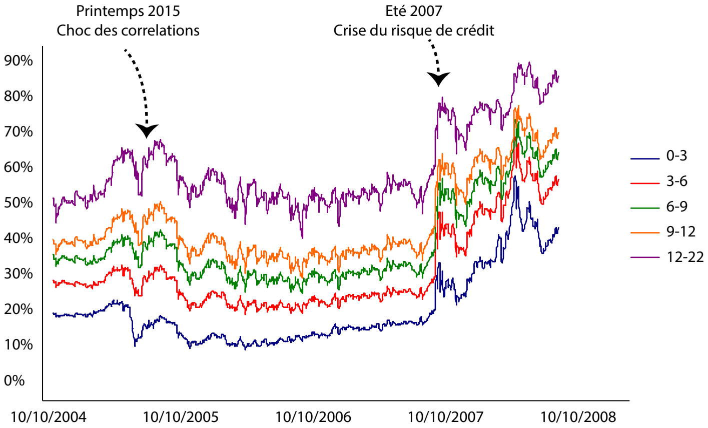
Base correlation

The **base correlation** in K is the implied correlation of $[0; K]$.



Implied correlation and base correlation

Base correlations dynamics



Implied correlation and base correlation

Implied correlation and base correlation – Bijectivity with CDO tranche prices

Bijective relationship between base correlation and CDO tranches spreads

There is a **bijective relationship** between the base correlation and the spread of a CDO tranche :

$$s^{[A;D]} = \frac{JV^{[0;D]}(\rho^{[0;D]}) - JV^{[0;A]}(\rho^{[0;A]})}{DV^{[0;D]}(\rho^{[0;D]}) - DV^{[0;A]}(\rho^{[0;A]})}$$

As option traders usually quote prices with implicit volatilities, CDO traders quote their prices using base correlations.

Implied correlation and base correlation

Implied correlation and base correlation – Interpretation

What do implied and base correlations tell us?

[D'Amato et al., 2005] presents several possible explanations for the correlation smile:

- there is a **segmentation among investors** across tranches;
- the used **models are inefficient**.

Implied correlation and base correlation

Implied correlation and base correlation – Limits of the Vasicek model to price CDO tranches

Wall Street's Math Wizards Forgot a Few Variables

By STEVE LOHR SEPT. 12, 2009

IN the aftermath of the great meltdown of 2008, Wall Street's quants have been cast as the financial engineers of profit-driven innovation run amok. They, after all, invented the exotic securities that proved so troublesome.

▸ The New York Times

Where models the reason of the subprime crisis?

The method used to price CDO tranches has been proved wrong:

- they are too many **homogeneity assumptions** (for correlation, default, maturity, nominal, etc.);
- the **dependence structure** in the model is not extreme enough.

There are a lot of other reasons (quality of the data – Garbage In Garbage Out logic among others) why the subprime crisis happened, most of them will be presented during the Subprimes Crisis Case Study (Lecture 7).

Delta hedging

Delta hedging

A trader wants to buy a protection on a mezzanine tranche;

- He hedges the market value fluctuations of his book by selling protection on individual CDS names;
- Trader's book value is:

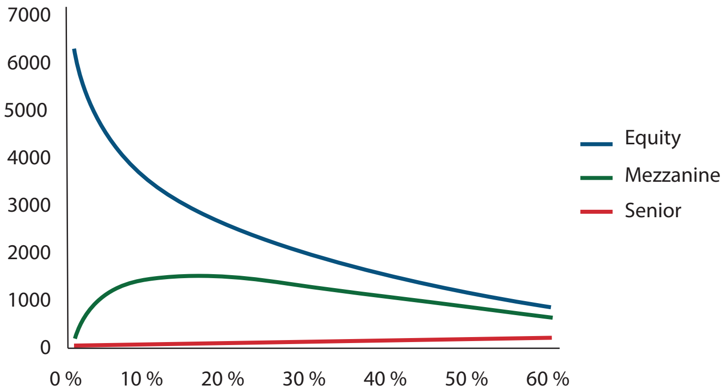
$$P(t) = V_{Tr}(t) + \sum_i \Delta_i V_{CDS_i}(t)$$

Thus, the **hedge ratio** is:

$$\frac{\partial P(t)}{\partial s_j} = 0 \Rightarrow \Delta_j = \frac{\partial V_{Tr}(t)}{DV_j \partial s_j}$$

Hedging single tranche exposure

Pricing sensitivy to correlation



4 - Other products

First-To-Default products – Definition

First-To-Default products – Definition

First-To-Default product

FtD products are similar to CDS contracts except that:

- They are based on a pool of **10 names maximum**;
- The protection buyer pays a **constant spread up to the first default** on the reference basket (if it occurs before maturity);
- When (and if) the first default occurs the protection buyer **delivers the defaulted bond and receives par**.

Would the underlying assets perfectly dependent, the FtD would be equivalent to a single-name CDS.

First-To-Default purpose and arbitrage bounds

First-To-Default purpose and arbitrage bounds

First-To-Default purpose

- They FtD is **riskier than the most risky reference** entity of the basket;
- Buying FtD protection is **cheaper** than buying the protection of each reference name in the basket.

Arbitrage bound of FtD products

Let $(s_1, ..., s_d)$ be the spreads of the underlying names, we have that the **the spread of the FtD, s_{FtD} arbitrage bounds** are:

$$\max(s_1, ..., s_d) \leq s_{FtD} \leq \sum_{i=1}^{d} s_i$$

It is a consequence of the no arbitrage asumption.

Rule of thumb for FtD pricing

$$s_{FtD} \approx \frac{2}{3} \sum_{i=1}^{d} s_i$$

Other synthetic products

Other synthetic products (I/II)

Other syntetic products

- **CDO squared**
  - Synthetic CDO on mezzanine synthetic single tranches;
  - More leverage;
  - Caution to systemic risk and overlaps.
- **Leveraged super senior**
  - Super senior tranche leveraged 6-10 times;
  - AAA rating, spread = 60 pb instead of 15 pb;
  - More credit enhancement compared to mezzanine AAA.
- **Combo notes**
  - Combination of A mezzanine and equity;
  - Principal rated A- by the rating agencies.

Other synthetic products

Other synthetic products (II/II)

Other syntetic products

- **EDS: Equity Default Swap**
  - An "equity event" replaces the usual "credit event";
  - The floating leg of the swap pays a cash-flow when the underlying stock hit the threshold of 30% of its value at inception;
  - Need of equity-credit model.
- **CEO: Collateralized Equity Obligation**
  - For example a CDO of EDS or of private equity;
  - In some cases, the maturity of the assets is an issue (ex: private equity).
- **CFO: Collateralized Fund Obligation**
  - CDO collateralized by shares of funds or hedge funds

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Credit Risk

Lecture 5 – Risk modeling and bank steering

Loïc BRIN

École Nationale des Ponts et Chaussées
Département Ingénieurie Mathématique et Informatique (IMI) – Master II

1. Credit risk models to fulfill regulatory requirements and prevent the bank from failure

2. Reevaluating the profitability of activities taking credit risk into account

3. Lessons from the field

Objectives of the lecture

Teaching objectives

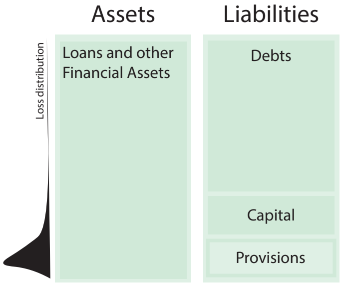
At the end of this lecture, you will:

- Understand the role and differences of **provisions and capital requirements** for banks;
- Know the **model-origin of the credit risk capital requirements** in details;
- Have a clear view of the **different profitability indicators** of business lines, their limits and how to appreciate them with regards to other indicators.
- get familiar through a practical example (thanks to our friend Denis Alexandre)

Table of Contents

Provisioning/Impairement and capital

In order to protect the economy, banks must set aside provisions and capital to be resilient in case of turmoil. The guidelines for credit risk impairement are defined within IFRS 9 framework (implemented in 2018, January 1st). The level of provisioning depends on the deterioration of the quality of the underlying loan and shall cover the expected loss (1 year time horzion or full maturity).



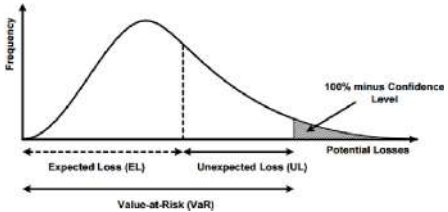
▸ Société Générale ▸ BNPP ▸ CASA ▸ JPM ▸ GS ▸ MS

Expected and Unexpected

The loss distribution of a bank is often split in two parts: the expected loss, from 0 to the average loss, and the unexpected loss above the average up to a percentile

Expected Loss and Unexpected Loss

- **Expected Loss (EL)**: the average loss, that is the normal cost of doing business covered by provisioning and pricing policies;
- **Unexpected Loss (UL)**: potential unexpected loss for wich capital should be held.



Provisions

Provisions are required by accounting rules and are designed to cover expected losses (the ECL: Expected Credit Loss)

- **IASB**: International Accounting Standards Board;
- A liability on **incurred losses** (on **Non Performing Loans**, NPL);
- That is **estimated**;
- and **adjusted** when closing the case.

IFRS9 – New provisioning rules

- A **non procyclical** rule for provisioning;
- A **forward looking**, **Point-In-Time**, provisions on all the loans (**IFRS 9**).

Website

Basel Accords 1 (1988)

Capital must be set aside to cover unexpected losses since Basel Accords 1 in 1988

Basel Accords 1 (1988)

- Basel Committee on Banking Supervision (BCBS) – 27 countries;
- **Cook ratio** and **standard** and simple computation of regulatory capital:
  - on the **banking book**:

$$\text{Regulatory Capital} = 8\% \times \text{Weights by counterparty type} \times \text{EAD}$$

  - on the **trading book** (since 1996) too.

Weight depends on the **nature** of the counterparty: Sovereign OECD : 0 %; Bank OECD: 20 %; Mortgages: 50 %; Other: 100 %.

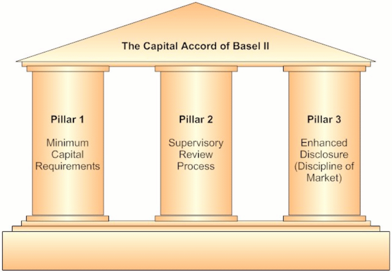
Basel Accords 2 (2006)

Basel Accords 2 were designed to increase the readability for a supervisor of banks activities and risks (i.e. better alignment between risk and capital levels)

Basel Accords 2 (2006)

- **Pillar 1**: Minimal Capital Requirements: capital for credit + market + operational risk;
- **Pillar 2**: Supervisory Review Process;
- **Pillar 3**: Enhanced Disclosure.

Basel 2 is amended after the 2007-2008 financial crisis, enlarging the scope of covered risk (e.g. market risk, credit and market risks).

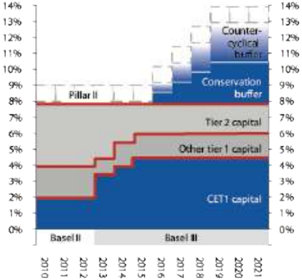


Basel Accords 3 (2014-2019)

Basel Accords 3 is more refined as for capital definitions and is more macro-prudential as it requires liquidity and funding requirements and treats differently systemic banks

Basel Accords 3 (2014-2019)

- **Liquidity Ratios**: Liquidity Coverage Ration (**LCR**) and Net Stable Funding Ratio (**NSFR**);
- **Leverage Ratio** (3 %);
- New definitions of **capital**.



Website

The Standard Approach for Credit risk

The regulator lets banks several option to fulfill capital requirements: from standard, model-free, approach, to more complex internal models approaches

Rating	$>$ **AA-**	$>$ **A-**	$>$ **BBB-**	$>$ **BB-**	$>$ **B-**	$<$ **B-**	**NR**
Sovereign	0%	20%	50%	100%	100%	150%	100%
Banks	20%	50%	100%	100%	100%	150%	100%
Corporates	20%	20%	100%	100%	150%	150%	100%

Weighted Assets

$$\text{RWA} = \text{Weights} \times \text{EAD}$$

Not just based on the **nature** of the relation but on its **grade** too.

The Internal Rating Based – IRB

Credit RWA, even with more advanced approaches, require the use of the regulator model, as the bank only has parameters PD, LGD, and EAD at its hands

The Internal Rating-Based Approach

- IRB **Fondation**: modeling of PD only;
- IRB **Advanced**: modeling of PD, LGD and EAD.

$$\mathrm{RWA} = f(\mathrm{PD}, \mathrm{LGD}, \mathrm{EAD})$$

IRB Approach – The formula

The IRB Approach is an estimation of the unexpected loss on a loan based on a modified version of the Vasicek model

IRB Approach – The formula

Using the IRB Approach, the **Risk-Weighted Assets formula** is:

$$RWA = LGD \times \left(\Phi \left(\frac{\Phi^{-1}(PD) + \sqrt{\rho}\Phi^{-1}(0.999)}{\sqrt{1-\rho}} \right) - PD \right) \times MA \times SF \times MCR \times EAD$$

- The Maturity Adjustment, $MA = \frac{1+(M-2.5)\times b}{1-1.5\times b}$ with $b = (0.11852 - 0.05478 \times \log(PD))^2$;
- The Scaling Factor, $SF = 1.06$;
- The Minimal Capital Requirements, $MCR = 12.5$.

Notebook

IRB Approach – The correlation parameter

The correlation parameter

The **correlation parameter** depends on the type of the countractor:

Type	Value for ρ
Large Corporates Institutions	$0.24\text{-}0.12 \times \frac{1-e^{-50\times PD}}{1-e^{-50}}$
SME with turn over <5 MEUR	$0.20\text{-}0.12 \times \frac{1-e^{-50\times PD}}{1-e^{-50}}$
SME	$0.24\text{-}0.12 \times \frac{1-e^{-50\times PD}}{1-e^{-50}} - 0.04 \times \left(1 - \frac{T-5}{45}\right)$
Residential Mortgages	0.15
Revolving	0.04
Other retail exposure	$0.16\text{-}0.13 \times \frac{1-e^{-35\times PD}}{1-e^{-35}}$

During the tutorial, we will interpret this formula and understand in details its link with the Vasicek model.

Website Tutorial

Economic Capital and stress testing

As regulatory capital is arbitrary, banks also calculate economic capital to have their own vision of their resilience

Economic Capital

- Regulatory capital does not take into account **correlation risk**, **concentration risk** and has other limits;
- Thus, banks have their **own internal models to steer their activity**.

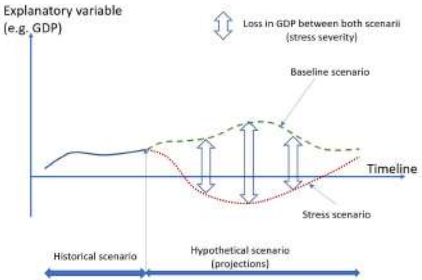
They thus compute a so called **economic capital**. Economic capital is however -as a adverse risk measure- replaced by stress testing that gets a better buy-in.

Economic Capital and stress testing

The objective of stress testing is to forecast the financial and solvability trajectory of the bank along two scenarios: a baseline scenario and a stress scenario. We offen refer to 'global stress tests'. There are complete by specific stress tests that focus on a reduced number of risk factors or reduced perimeter (e.g. impact of oil prices decrease on corporate portfolio)

Stress testing

- Global stress test allow to ensure that the minimal capital requirement and/or financial KPI are fullfi lled during a **remote crisis scenario**. There are also used by regulators (e.g. EBA)
- **The correlation between the different risks lies within the description of the scenario**, enabling the aggregation of risks.



Conclusion

Credit risk models to fulfill regulatory requirements and prevent banks failures

- **Provisions** are **accounting requirements** dedicated to cover **expected losses** of the bank;
- **Capital requirements** are buffers designed to prevent banks from failure in case of **unexpected losses**;
- **Basel Accords** are at the origin of the current capital requirements framework and is based, for the most advanced approach, on the **Vasicek model**.

Quiz

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Return On Equity – ROE

The ROE is a basic profitability indicator, that fail to grasp specificities of the banking sector

Return On Equity – ROE

The **Return On Equity** of a Business Line is:

$$\text{ROE} = \frac{\text{Net Income of the BL}}{\text{Regulatory Capital allocated to the BL}}$$

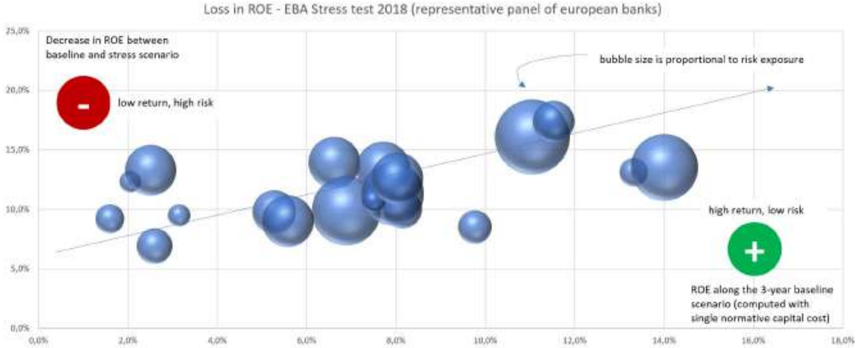
ROE's limits

- **In the numerator**: does **not** take into account the **risk**;
- **In the denominator**: does not take into account the **diversification effect** of the BL and suffers from **arbitrary and discretionary regulations choices**.

As risks are central in the banking sector, they must be taken into account when measuring return.

Return On Equity – ROE

Comparing the ROE sensitivity to stress scenario: the example of EBA stress tests



So EBA Website (2018 Stress tests results) and own simplified computations

Risk Adjusted Return On Capital – RAROC

As RAROC accounts for risks it is more appropriate in the banking sector

Risk Adjusted Return On Capital – RAROC

The **Risk Adjusted Return On Capital** of a Business Line is:

$$\text{RAROC} = \frac{\text{Net Income of the BL} - \text{Average loss of the BL}}{\text{Economic Capital allocated to the BL}}$$

RAROC's limits

What should it be compared to?

Hurdle rate and Weighted Average Cost of Capital – WACC

Profitability indicators can be compared to WACC as it is the hurdle rate of a business

Weighted Average Cost of Capital – WACC

The **Weighted Average Cost of Capital** is:

$$\text{WACC} = (r + k_1)T_1 + (r + k_2)T_2 + (r + k_d)D$$

where, T_1/r_1, T_2/r_2 and D/r_D is the proportion / the cost (spread) of Tier 1 capital, Tier 2 Capital and debt in the liabilities of the bank and r the risk free rate.

Hurdle rate and WACC

The WACC is the **hurdle rate** of bank activities, that is, the minimum return necessary to be profitable.

Economic Value Added – EVA

EVA offers an opportunity to measure profitability in absolute

Economic Value Added – EVA

The **Economic Value Added** for a bank is:

$$\text{EVA} = \text{Net Income of the bank} - \text{Average loss of the bank} - \text{WACC} \times \text{Liabilities}$$

Economic Value Added 2 – EVA 2

The **Economic Value Added 2** for a bank is:

$$\text{EVA 2} = \text{Net Income of the bank} - \text{Average loss of the bank} - k \times \text{Economic Capital}$$

where k is the cost of capital, that is: $k = (r + k_1)T_1 + (r + k_2)T_2$.

Risk Adjusted Return On Risk Adjusted Capital – RARORAC

These profitability measures require to be tractable to know the cost of capital of the bank: CAPM is an option to estimate that cost

Risk Adjusted Return On Risk Adjusted Capital – RARORAC

The **Risk Adjusted Return On Risk Adjusted Capital** for a bank is:

$$\text{RARORAC} = \text{RAROC} - k$$

Risk Adjusted Return On Risk Adjusted Capital 2 – RARORAC 2

The **Risk Adjusted Return On Risk Adjusted Capital 2** for a bank is:

$$\text{RARORAC 2} = \text{RAROC} - k \times \frac{\text{Allocated Economic Capital}}{\text{Used Economic Capital}}$$

▸ Notebook ▸ Tutorial

How to make from the theoretical Cost of capital k, a practical tool?

- How to estimate k, the cost of capital?
- Can we use a unique k for all the business lines?

How to estimate the cost of capital?

Cost of capital of the bank – Using the CAPM

CAPM and cost of capital

The **Capital Asset Pricing Model** [Sharpe, 1964], states that the shareholder's expected return for the firm i ($k = \mathbb{E}(r_i)$) is equal to the risk free rate, plus a market premium multiplied by a factor, β_i:

$$k = \mathbb{E}(r_i) = r_f + \beta_i \times \underbrace{(\mathbb{E}(r_M) - r_f)}_{\text{Market premium}}$$

where $\beta_i = \rho_{i,M} \frac{\sigma_i}{\sigma_M}$, $\rho_{i,M}$ being the correlation between the share of the firm i and the market (M), σ_i and σ_M being the volatility of the share of i and the market, r_f being the risk free rate.

How to estimate the cost of capital using CAPM?

Estimated β on the markets by [Matten, 1996]

Universal Bank	0.97
Investment Banks	1.16
Asset Management	1.21
Retail Bank	1.09
Banking Sector	**1.11**

The cost of capital for banks seems to have declined over the last decades

According to [King, 2009], the cost of capital for banks has declined due (i) the decrease in risk-free rates over this period, and (ii) a decline in the sensitivity of bank stock returns to market risk (the CAPM beta) in all countries except Japan.

How to estimate the cost of capital ?

Gordon-Shapiro formula is an alternative to measure the cost of capital

Gordon-Shapiro formula and cost of capital

The **Gordon-Shapiro equation**, [Gordon, 1959], states that the market capitalization of a firm, P, is equal to the sum of the expected future dividends:

$$P = \sum_{t=1}^{\infty} \frac{D_t}{(1+k)^t} = \sum_{t=1}^{\infty} \frac{D_1(1+g)^t}{(1+k)^t} = \frac{D_1}{k-g}$$

where D_t is the expected dividend distributed in year t by the firm, k is the cost of capital (the discount rate adequate given the risk born by the shareholder) and g is the expected annual dividend growth rate.

Limits of Gordon-Shapiro method?

The estimation is of k, the cost of capital, using the Gordon-Shapiro equation is often too **volatile** as the equity market is, and thus the market capitalization of firms are.

Conclusion

Reevaluating activities rentability taking credit risk into account

- The basic **Return On Equity (ROE)** profitability indicator **lacks to grasp credit risk** in its appreciation of value creation;
- **RAROC and RARORAC** thus offer a more **refined** profitability appreciation;
- These profitability indicators can be compared to the **cost of capital** and the **Weighted Average Cost of Capital (WACC)** which values can be extracted from markets valuations.
- Alternatively, ROE needs to be compared between baseline and stress scenarios.

Quiz

Conclusion

Risk modeling and Bank Steering

- Given their sizes and their risks, **banks are required to set aside provisions and capital** to face expected and unexpected losses;
- **These potential losses and these requirements** must be taken into account to assess **the profitability of a business lines**.

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Bonus

Lessons from the field

In this short video, Denis Alexandre proposes to share with you some practical example on market risk management on credit risk instrument

- Presentation of the skew, as the spread difference between ITRAXX and the sum of its individual components
- How to handle with a market request (link with stress tests)
- Video available here: https://youtu.be/bRzZh9utIfQ

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Proofs of the lecture

Proofs of the lecture

- Proof 1

Proof – Proof 1

Credit Risk

Lecture 6 – Counterparty risk

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Département Ingénieurie Mathématique et Informatique (IMI) – Master II

Objectives of the lecture

Teaching objectives

At the end of this lecture, you will:

- understand what **counterparty risk** is, why it is a very exotic risk and how it is mitigated;
- know how **counterparty risk is measured** in a risk management, regulatory and pricing perspective;
- what **other valuation adjustments (XVA)** are studied to go beyond counterparty risk in evaluating the value of OTC derivatives, taking into account contextual information.

1 - Counterparty risk is a complex risk

Introduction to the derivatives market

Derivatives can be treated on a listed market or over-the-counter (OTC)

Derivatives – Definition

A derivative is a financial security with a value that is reliant upon or derived from an **underlying asset** or group of assets.

Where to buy derivatives?

Derivatives can be exchanged on an **listed market** or **Over The Counter (OTC)**.

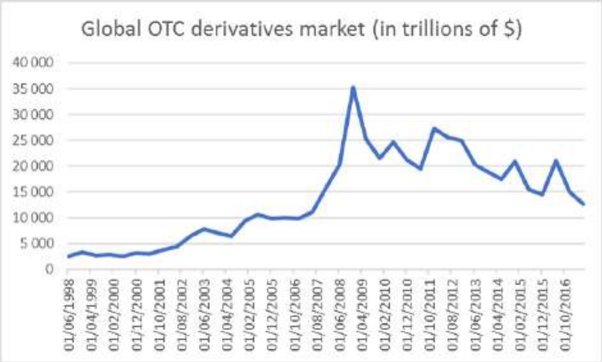
OTC derivatives – Pros and Cons

OTC derivatives are efficient tools to transfer financial risks between market participants. As a by product of such a transfer:

- they **create credit risk** between the counterparties;
- they **increase connectedness** within the financial system.

Introduction to the derivatives market

The derivatives market is a large market were interest rate swaps are predominant



Source: www.bis.org

Counterparty risk definition, specificities and mitigators

Counterparty risk is mostly the risk of default of a counterparty on the contract of a derivative

Counterparty risk

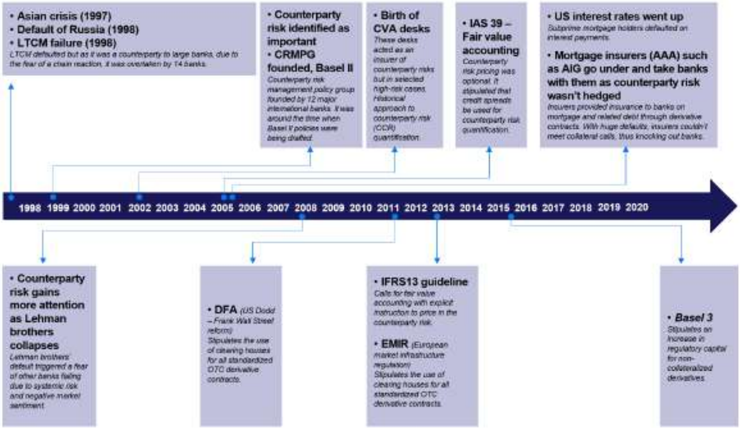
The counterparty risk is defined as the risk the counterparty to a transaction could default **before the final settlement of the transaction's cash flows**. An economic loss would occur if the transactions or portfolio of transactions with the counterparty has a positive economic value at the time of default.
[BCBS, 2006]

Counterparty risk is generated by the derivatives market

As derivatives' contracts generate cash flows between two counterparties, based on one or several underlying assets, **they generate counterparty risk**.

Counterparty risk definition, specificities and mitigators

Counterparty risk has materialized during the last 20 years and yielded to new accounting, capital and collateralization regulations



Counterparty risk definition, specificities and mitigators

Counterparty risk is more complex than credit risk as it is bilateral, fluctuating and dependent on many risk factors

Counterparty risk – A bilateral risk

Unlike a firm's exposure to credit risk through loan, where the exposure to credit risk is unilateral and only the lending bank faces the risk of loss, the counterparty credit risk creates a bilateral risk of loss: **the market value of the transaction can be positive or negative to either counterparty of the transaction**.
[BCBS, 2006]

Counterparty risk – A fluctuating exposure

The core value at risk in case of default is the market value of the derivative product (if it has a positive economic value). **This exposure is not constant, nor deterministic**, as it depends on market movements. Thus, counterparty risk is subject to a fluctuating exposure.

A direct inheritance of the underlying derivatives

The market value of the derivative generating counterparty risk is dependent on the underlying asset the derivative is based on. Counterparty risks, as they depend on the exposure to derivatives, **inherit of all the underlying risk factors**.

Counterparty risk definition, specificities and mitigators

Counterparty risk on market activities is thus more complex than credit risk on the lending business

Difference between the lending business and the derivatives business:

- **Loans**: exposure at any future date is the outstanding balance, which is certain (without considering prepayments);
- **Derivatives**: exposure at any future date is determined by the market value and the date is uncertain.

Counterparty risk can be:

- **Unilateral**: one party (the investor) is considered default-free and only the exposure to the counterparty matters;
- **Bilateral**: both parties are considered risky and face exposures depending on the value of the positions they hold against each other.

Counterparty risk definition, specificities and mitigators

Counterparty risk was a reason for the subprime crisis

Counterparty risk – A systemic risk

- Derivatives create credit risk **between the counterparties**;
- Derivatives increase the **connectedness** of the financial system.

Counterparty risk during the 2008 financial crisis

The 2008 financial crisis showed that counterparty-related losses (e.g. changes is credit spreads of the counterparties and changes in the market prices that drive the underlying derivative exposures) have been **much larger than default losses**.

Counterparty risk definition, specificities and mitigators

Counteprary risk is thus driven by the market value of the derivatives, the counterparty credit spread and the correlation between all these parameters

Drivers of counterparty risk

Counterparty risk is affected by several complex risk drivers:

- the Over the Counter (OTC) contract's **market value** risk drivers;
- the **counterparty credit spread**;
- the **correlation between the underlying and default of the counterparty**.

Counterparty risk definition, specificities and mitigators

For one counterparty, netting consists in aggregating all the transactions instead of considering each deal separately

Netting

In presence of multiple trades with a counterparty, netting agreements allow, in the event of a default of one of the counterparties, to **aggregate the transactions before settlings claims**.

Netting – Math formalization

- In the absence of netting, the exposure is:

$$E(t) = \sum_i E_i(t) = \sum_i V^+_{i(t)}$$

- A **netting agreement** is a legally binding contract between two counterparties based on which, in the event of default, the exposure results in:

$$E(t) = \left(\sum_i V_i(t) \right)^+$$

Counterparty risk definition, specificities and mitigators

For one counterparty, netting consist in aggregating all the transactions instead of considering each deal separately

Netting – Example

Two counterparties, a bank B and the counterparty C, such that:

- C holds a currency option written by B with a mark-to-market value of 50;
- B has an IRS with C, having a marked to market value in favor B of 80.

Exposures:

- The exposure of bank B to the counterparty C is 80;
- The exposure of the counterparty C to the bank B is 50;
- The exposure of the bank B to the counterparty C, with netting, is 30.

Counterparty risk definition, specificities and mitigators

Collateral posting is another mitigator as assets are segregated to protect both counterparties in case of a default

Collateral – Definition

Consider the bank B and the counterparty C. Let $C(t)$ be the (cash) **collateral amount posted** by C to B, at time t. B has no exposure to the contract up to the collateral amount, while its losses are reduced by the collateral amount whenever the exposure exceeds it. The collateralized exposure $EC(t)$ is defined as:

$$EC(t) = (E(t) - C(t))^+$$

Credit Support Annexes – CSA

A **Credit Support Annex (CSA)** provides credit protection by **setting forth the rules** governing the mutual posting of collateral. CSAs are used in documenting collateral arrangements between two parties that trade privately negotiated (OTC) derivative securities.

Counterparty risk definition, specificities and mitigators

Collateral posting can be very different from one counterparty to another even if the heterogeneity is often based to the same parameters

Posting threshold (H)

The **posting threshold** $H > 0$ is the **threshold** below which, no collateral is posted.

Margin period (δ)

The **marginal period is the interval** at which margin is monitored and called for:

$$C(t) = [E(t - \delta) - H]$$

Minimum Transfer Amount - MTA

The Minimum Transfer Amount is the amount below which **no margin transfer** is made:

$$C(t) = [E(t - \delta) - H] + \mathbb{1}_{\{E(t-\delta)-H>MTA\}}$$

Downgrade triggers

Downgrade triggers are triggers used to ensure more collateral to be posted, if the counterparty is **downgraded below** a certain level.

Counterparty risk definition, specificities and mitigators

The regulation pushed for more collateralization on the OTC market

American and European legislations pushed for more collateralization

Two major legislations made compulsory collateralization for financial institutions:

- **Dodd-Frank** Act in the United States;
- **European Market Infrastructure Regulation (EMIR)** in Europe.

Collateralization can be done through CCP or not

Collateralization can be made with:

- **CCP**: Central Clearing Houses (CCP) take charge of the margin calls;
- **Without CCP**: the two counterparties use accounts and call margins without any CCP.

Counterparty risk mitigations

The collateralization can be performed through CCP or not and require or not Initial Margins

Central Clearing Houses

A CCP becomes the counterparty to the buyer and the seller and guarantees the terms of a trade even if one party defaults on the agreement. The CCP collects enough money from each buyer and seller for covering potential losses incurred by not following through on an agreement, resulting in the entity replacing the trade at the current market price. Monetary requirements are based on each traders exposures and open obligations. Two counterparties can also collateralize using regular banking accounts and call marginal amounts to adjust the collateralization.

Initial Margin

More than requiring collateralization for financial institutions, Dodd-Frank Act and EMIR requires to complete the classical margin calls (Variation Margin – VM) with Initial Margin (IM), that comes to **cover the settlement risk** in case of default.

SIMM

The **Standard Initial Margin Model** (SIMM) is a common methodology to help market participants calculate initial margin on non-cleared derivatives under the framework developed by the Basel Committee on Banking Supervision and the International Organization of Securities Commissions.

ISDA

Wrong Way Risks and Right Way Risks

Correlations between the different risk factors can lead to cumulative risks called Wrong Way Risks

Wrong Way Risk

WWR is defined as the risk that occurs when "exposure to a counterparty is **adversely correlated** with the credit quality of that counterparty". It arises when default risk and credit exposure increase together.

- Specific WWR arises due to counterparty **specific factors**: a rating downgrade, poor earnings or litigation.
- General WWR occurs when the trade position is affected by **macroeconomic factors**: interest rates, inflation, political tension in a particuliar region, etc.

Wrong Way Risks and Right Way Risks

Wrong Way Risks happened during the subprime crisis

Monoline insurers (e.g. Ambac and MBIA)

During the subprime crisis, the **monolines** specialized in **guaranteeing Mortgage Backed Securities (MBS)**, saw their creditworthiness deteriorate and found themselves unable to pay all of the insurance claims. Almost all exposure mitigation from monoline insurance fell short due to the guarantor's increased probability of default under exactly the same conditions when the insurance was most needed.

Collateralized Loan

Bank A enters into a **collateralized loan** with bank B (the counterparty). The collateral that Bank B provides to A can be of different nature: bonds issued by Bank B (specific WWR) bonds issued by a different issuer belonging to a similar industry, or the same country or geographical region (general WWR).

Wrong Way Risks and Right Way Risks

These correlation effects can also mitigate the risk and are then called Right Way Risks

Right Way Risks

Right Way Risk is **the opposite to Wrong Way Risk**: it is the effect observed when **the exposure decreases as the default probability increases**, i.e., when there is a negative dependency between the two. The size of credit risk decreases as the counterparty approaches a potential default. RWR occurs when a company enters into transactions to partially hedge an existing exposure.

Examples of Right Way Risks

- An airline usually protects itself against a rise in fuel prices by entering into long oil derivatives contracts;
- A company would normally issue calls and not puts on its stocks. WWR and RWR are together referred to as **DWR (Directional Way Risk)**.

Wrong Way Risks and Right Way Risks

As regulatory do not require to model these correlation, they introduce an α multiplier to be conservative on that correlation risk

Regulatory Treatment of WWR

Basel II deals with WWR using the so called α **multiplier** to increase the exposure, in the version of the model where exposure and counterparty creditworthiness are assumed to be independent.
The effect is to increase exposure by the α multiplier that is set to 1.4 (or can banks can use their own model but there is a floor at 1.2).

Regulatory treatment of WWR

- if a bank uses its own model, at minimum, the exposure has to be **20 % higher** than given by the model where default and exposure are independent ;
- if a bank does not have its own model for WWR, it has to be **40 % higher**.

Estimates of α reported by banks range from 1.07 to 1.10.

Conclusion – Counterparty risk is a complex risk

Counterparty risk is a complex risk because of its dependency to potentially many underlying assets, the heterogeneity of the mitigators and the correlations that can amplify this risk

- Counterparty risk is **complex** as it is bilateral, fluctuating and dependent to all the risk factors of the underlying OTC derivatives;
- **Mitigators** are numerous and not that homogeneous on the market, thus adding more complexity in the measurement of the risk;
- Eventually, **correlations** between the different risk factors can amplify the risk (Wrong Way Risk) or decrease it (Right Way Risk) adding second order effects to the model.

2 - Counterparty risk metrics

Basic counterparty metrics, EEPE and KCCR

The two purposes of counterparty risk metrics is to manage risk and take it into account in the pricing

Measuring counterparty risk can fulfill two purposes:

- **counterparty risk management**: for **internal purpose** and for **regulatory capital requirements**, following Basel II;
- **counterparty risk from a pricing point of view**: **Credit Value Adjustment (CVA)** is computed during the pricing to account for possible default of the counterparty.

Basic counterparty metrics, EEPE and KCCR

PE, EE, EPE are three basic counterparty risk metrics

Let us denote $V(t)$ the market value of a derivative or of several derivatives if a netting agreement is in place, at time t. Counterparty exposure is equal to $E(t) = V(t)^+ = \max(0, V(t))$. It is also known a **Potential Future Exposure (PFE)**.

Peak Exposure (PE) at level α

$$\mathrm{PE}_\alpha(t) = \inf\{X(t) \,|\, \mathbb{P}[\mathrm{E}(t) \geq X(t)] \leq 1-\alpha\}$$

Expected Exposure (EE)

$$\mathrm{EE}(t) = \mathbb{E}^{\mathbb{Q}}(E(t))$$

Effective Positive Exposure (EPE)

$$\mathrm{EPE}(t) = \frac{1}{t}\int_0^t \mathrm{EE}(s)\mathrm{d}s$$

Basic counterparty metrics, EEPE and KCCR

EEPE is a basic counterparty risk metric that has been retained by the regulator to calculate KCCR (Credit RWA)

Expected Effective Positive Exposure (EEPE)

$$\text{EEPE}(t) = \frac{1}{t}\int_0^t \max_{h<s}(\text{EE}(h))\,\mathrm{d}s$$

KCCR (Capital for Counterparty Credit Risk)

Uncollateralized OTC derivatives generate Credit RWA as regular loans. Nonetheless, as their Exposure At Default (EAD) is fluctuating the regulator asks banks to use for $\text{EAD} = \alpha \times \text{EEPE}$ in the IRBA approach (see Lecture 5), with EEPE calculated for t equal to one year, and $\alpha > 1$ in order **to account for correlations in a conservative way**.

Basic counterparty metrics, EEPE and KCCR

Let us look at a fictitious example of a KCCR calculus

KCCR on a fictitious swap

A bank strikes a deal with Total. They sold a 8-years EUR/USD currency swap. To calculate the KCCR in an IRBA approach, the bank must use the regulator formula and plug into it: PD, EAD and LGD. We assume that for this bank, $\alpha = 1.5$.

- LGD: we can assume LGD is equal to 35%;
- PD: the internal rating and model of the bank (Lecture 2 on Statistical models) give us a value of 0.63 %;
- Diffusing the risk parameters (EUR, USD, risk-free rates), we can see that the Expected Effective Positive Exposure over one year is 1 MEUR.

Thus: $KCCR = f_{\text{regulator}}(0.18, 35\%, 1.5 \times 1MEUR)$

Counterparty and Debt Value Adjustment – CVA and DVA

Counterparty Value Adjustment (CVA) accounts for counterparty risk the evaluation of a derivative value and is an accounting requirement that brings volatility in the balance sheet of the banks

Counterparty Value Adjustment – CVA – Definition

CVA is defined as:

- the **difference between the risk-free value and the risk value** of one or more trades or, alternatively,
- the expected loss arising from a **future counterparty default**.

CVA – An accounting rule

CVA is first and foremost a **provision** that is compulsory since IFRS13 was published.

CVA – A source of P&L fluctuations

IFRS13 requires this provision to be risk neutral (future exposures should be risk neutral and PD must be implied from the CDS market), which make these reserves **fluctuating and volatile**. Given the amount at stakes and to avoid P&L jumps, banks have created **CVA desks** which purpose are to hedge theses provisions fluctuations by buying CDS and futures which values will evolve in the opposite way of CVA in case of market changes.

Counterparty and Debt Value Adjustment – CVA and DVA

Because this provision is volatile, banks hedge these and make the client pay for these hedging costs

CVA – An hedging cost

Of course, **these hedging strategies have a cost** (CDS premium in particular). As the provision are risk neutral, their value is equal to the cost to hedge them.

CVA – A component of the price

The market practice is to **make the client pay for this/these hedging strategy/provision moves**. When the deal is complex, the additive CVA component of the price is calculated by the CVA desk.

CVA equations

$$\textbf{CVA reserves change} = \textbf{Cost of hedging them} = \textbf{Fee charged to the customers}$$

Counterparty and Debt Value Adjustment – CVA and DVA

CVA are concentrated on corporate and fixed income departments

CVA are generally concentrated on Fixed Income and Corporate departments

As financial institutions must collateralized (DFA and EMIR), the vast majority of the CVA generated in a bank comes from **corporate customers**.
Additionally, as most of the time corporate customers buy fixed income derivatives (e.g. currency swaps, rates swaps), **Fixed Income departments** in banks concentrate the vast majority of the CVA.

Counterparty and Debt Value Adjustment – CVA and DVA

Counterparty Value Adjustment (CVA) – Unilateral CVA

Riskless price of an OTC deal does not take into account counterparty risk

$$P_{\text{riskless}} = \mathbb{E}^{\mathcal{Q}}\left[\int_0^T CF_t e^{-rt} dt\right]$$

Replacement cost

$$L_{\tau_C} = (1 - R)\mathbb{1}_{\{\tau_C \leq T\}} E(\tau_C)$$

Risky price

$$P_{\text{risky}} = \mathbb{E}^{\mathcal{Q}}\left[\int_0^T (CF_t - L_{\tau_C} \cdot \mathbb{1}_{\{\tau_c = t\}}) e^{-rt} dt\right]$$

Counterparty and Debt Value Adjustment – CVA and DVA

Counterparty Value Adjustment (CVA) is the component to add to the riskless price to take into account counterparty risk

Counterparty Value Adjustment – CVA

CVA is equal to the **market value of the expected loss**:

$$CVA = \mathbb{E}^{\mathbb{Q}}(L_{\tau_C} e^{-r\tau_C}) = (1-R)\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{\tau_C \leq T\}} E(\tau_C) e^{-r\tau_C}\right]$$

Counterparty Value Adjustment – CVA – Case of independence between τ_C and $E(\tau_C)$

In the case of independence between the default and the exposure:

$$CVA = (1-R)\int_0^T \mathbb{E}^{\mathbb{Q}}\left[E(t)\right] dQ_C(t)$$

where $Q_C(t) = 1 - e^{\frac{s_t}{1-R}t}$

The latter expression can be discretized:

$$CVA \approx (1-R)\sum_{i=0}^{n-1} \frac{\mathbb{E}^{\mathbb{Q}}(E(t_i)) + (E(t_{i+1}))}{2} \times \left(e^{-\frac{s_{t_i}}{1-R}t_i} - e^{-\frac{s_{t_{i+1}}}{1-R}t_{i+1}}\right)$$

Counterparty and Debt Value Adjustment – CVA and DVA

As counterparty risk is bilateral, so is CVA through DVA (Debt Valuation Adjustment)

Bilateral CVA

Bilateral CVA is expressed as:

$$CVA_{\text{bilateral}} = CVA_{unilateral} - DVA$$

where DVA is the Debt Value Adjustment, that is, my own CVA.

DVA and CVA – Expressions

Let denote B and C the two counterparties in the transactions, **DVA and CVA** are expressed as:

$$CVA_{\text{unilateral}} = (1 - R)\mathbb{E}^{\mathbb{Q}}(\mathbb{1}_{\{\tau_C \leq \min(T;\tau_B)\}} E(\tau_C) e^{-r\tau_C})$$

$$DVA = (1 - R)\mathbb{E}^{\mathbb{Q}}(\mathbb{1}_{\{\tau_B \leq \min(T;\tau_C)\}} E(\tau_B) e^{-r\tau_B})$$

Counterparty and Debt Value Adjustment – CVA and DVA

Let us take a look at the exposure on a CDS and how to infer a CVA proxy on a CDS

Exposure on a CDS protection contract

Let us consider the exposure on a CDS contract:

$$V(t) = (s_t - s_0) \cdot DV(t, T)$$

We assume that the spread process is normal:

$$s_t = s_0 + \sigma W_t$$

$$\begin{aligned} EE(t) &= \mathbb{E}^{\mathbb{Q}}(V(t)^+) \\ &\approx \mathbb{E}^{\mathbb{Q}}((s_t - s_0)^+) DV(t, T) \\ &\propto \sigma\sqrt{t} DV(t, T) \\ &\propto \sigma\sqrt{t}(e^{-(r+\lambda)t} - e^{-(r+\lambda)^T}) \end{aligned}$$

Counterparty and Debt Value Adjustment – CVA and DVA

CVA and DVA induce a lot of computational complexities

There are not that many CDS

There are **not that many CDS quoted on the market**. According to [Gregory, 2015], only 1600 CDS are traded.

Use of proxies' rules and of indices

As IFRS13 requires to use market estimation of the PD (risk neutral), banks use either **credit indices** as proxies (e.g. iTraxx, CDX) or **CDS of similar counterparties**.
The CVA Desk strategy is aligned with the CDS used to estimate the CVA.

Supercomplex calculations

CVA often require the use of a Monte Carlo in a Monte Carlo, as calculating the exposure in a future scenario requires to use derivatives pricers, which might be itself a Monte Carlo Pricer. We talk about **Nested Monte-Carlo**. For that reason, calculations of CVA is complex and can limit the ability of the CVA Desk to have a complete measurement of all its sensibilities to all the risk parameters it is sensible to.

Counterparty and Debt Value Adjustment – CVA and DVA

Additionally to CVA and DVA, the regulator asks for a capital buffer to face potential highly fluctuating CVA and DVA reserves

Definition of KCVA

The **KCVA is a VaR on CVA** used to calculate a capital regulatory requirement, additional to KCCR, and which economic purpose is to have a capital buffer in case of high fluctuant CVA reserves.

Different scopes

Paradoxically in Europe, **KCVA only applies to Financial Institutions** (even if most of the CVA is generated by corporates customers as seen before).

Role of the CVA desk on KCVA

Additionaly to its pricing role and hedging role, the CVA desk often buys CDS to **reduce the KCVA** VaR and thus the capital requirement.

Counterparty and Debt Value Adjustment – CVA and DVA

Many elements make CVA, DVA and KCVA models quite risky in terms of model risk

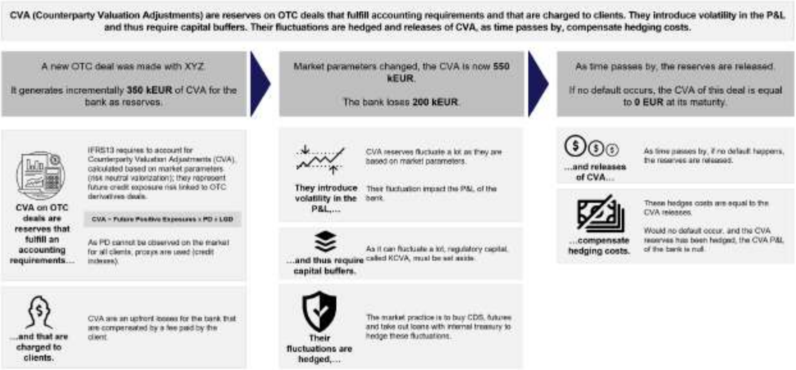
Complex models that entail a high risk model

Several aspects of CVA models make them risky in a risk model perspective:

- they are used for **several purpose**: accounting, regulatory requirements, pricing;
- they are based on **hardly observable** input parameters: use of proxies for credit, illiquid underlying risk factors;
- they are **prospective**: they require nested simulations (nested Monte Carlo);
- they must be **risk neutral**: martingality hypothesis to consider;
- they must take into account **many complexities**: netting, CSA aggreements;
- they are **transversal and global**: they use data from all the market activities systems, all the pricers;
- they are used on **different granularities**: at the bank-level for accounting requirements, at the counterparty level for pricing.

Counterparty and Debt Value Adjustment – CVA and DVA

Let us sum all of this up in one infography



Other market credit risks that are not counterparty risks

Incremental Risk Charge (IRC) is a capital requirement (Market RWA) generated by credit risk on the underlying asset of a derivative, not by counterparty risk

Defintion of IRC

The IRC is a risk metrics that **captures risk due to adverse rating migration** on vanilla credit securities such as bonds and CDS on corporate and sovereign within the trading book. It is a component of Market RWA.

Other market credit risks that are not counterparty risks

Incremental Risk Charge (IRC) is a VaR but which calculation can differ a lot from one bank to another

Example of a simple IRC modeling

The IRC can be computed as follows:

- Simulate random rating migration using **transition matrices** and risk drivers;
- For each simulation derive a **spread variation** from the rating migration;
- For each spread variation derive **a proxy of the P&L** on the portfolio;
- Compute **a 99.9% quantile** of the portfolio PL distribution at a one year horizon.

Many IRC implementations!

There is **not a unique definition** of the IRC.

Other market credit risks that are not counterparty risks

Comprehensive Risk Measure (CRM) is a capital requirement (Market RWA) generated by credit risk on the underlying asset of a derivative, not by counterparty risk

Definition of CRM

- The CRM is a risk metrics that, as the IRC, captures the risks due to **adverse rating migration** on credit securities. It applies to credit correlation portfolio (CDO, CLO, CBO, etc.) within the trading book.
- The CRM also **captures risks due to credit spread variations, recovery rates and base correlations**.
- CRM is also a component of the market RWA of a bank.

Conclusion – Counterparty risk metrics

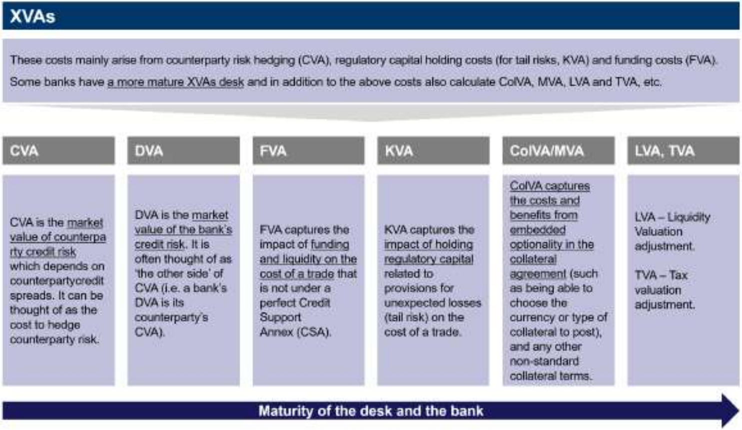
Counterparty risk is exotic and complex and is measured through metrics dedicated to pricing and risk management

- Counterparty risk is taken into account **through KCCR**, as EAD is substituted by α times EEPE;
- Counterparty risk is taken into account in the **provision called CVA and in the regulatory requirement KCVA**;
- Credit Risk impacts market activities too through the credit derivatives and give rise to capital requirement through **IRC and CRM**.

3 - Other Valuation Adjustments (XVA)

Other Valuation Adjustments

There other valuation adjustements that could be taken into account



Complex to take them all into account

All XVA are not easy to compute / charge to customer / likely to be provisioned

- A bank **cannot always compute super complex valuation adjustments**: as they would cost a lot because of their computational costs;
- A bank **cannot always reserve what it wants**: reserving, means reducing the P&L and thus require the approval of external auditors;
- A bank **cannot always charge to customer what it wants**: the banl needs to take into account market practices.

Thus, these complex XVAs are studied essentially to **appreciate a more refined vision of profitability** when they appear to be an important component of the price.

Conclusion – Introduction to other XVA

Over The Counter Derivatives sometimes need to take into account other effects than counterparty risk

- **Other effects** such as collateral type, initial margin, scarce resources requirements **can impact the profitability of a deal**;
- For that reason, banks, and in particular XVA desks, work on **new metrics to have a more complete vision of profitability**;
- These refined metrics are nonetheless **complex to price**.

Conclusion

Counterparty risk is exotic and complex and is measured through metrics dedicated to pricing and risk management

- **Counterparty risk is complex and exotic** as it is fluctuating, bilateral, dependent on many factors among which correlations;
- Counterparty metrics are used to comply with **accounting and regulatory requirements** and also to determine **prices of OTC deals**;
- Other XVA are currently studied to take into account in profitability's metrics **more complex effects**.

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