Stochastic Differential Equations

Let's start with the basics first.

ODE ⇒ Ordinary Differential Equation

It is a differential equation which depends on only one independent variable and its derivatives.

For example: we have a simple ODE like

$$\frac{dy}{dt} = 2y$$

This equation says that rate of change of y with respect to time is equal to 2 times y.

In other words if y is growing, it will continue to grow faster & faster over time.

RANDOMNESS

Now let's introduce the concept of randomness. In some situations, we may not be able to predict how a quantity will change over time, because there are random forces at play that affects its behaviour.

For example, lets say we are trying to model the movement of a particle in a fluid, but there are random fluctuations in the velocity of the fluid that affect the particles trajectory.

To model this kind of situation we need to use a Stochastic Differential equation (SDE), which is a differential equation in which one or more of the terms is a stochastic process. In other words, it is a classical differential equation which is perturbed by a random noise.

For example -

$$dx = a(x,t)dt + b(x,t)\,dW$$

This equation says that the change in x over an infinitesimal time period dt is equal to a deterministic component, $a(x,t)\,dt$ plus a stochastic component, $b(x,t)\,dW$.

The deterministic component represents the expected behaviour of x based on some set of rules, while the stochastic component represent the random fluctuation in x due to some source of randomness.

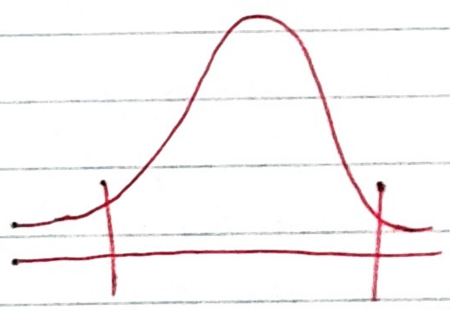
$dW \Rightarrow$ In this equation, dW represents Weiner process which is a type of random process that is often used to model Brownian motion.

This process is characterized by its mean, which is zero and its variance which is equal to time interval dt.

mean $(\mu) = 0$, variance $(\sigma^2) = dt$

Implementation of SDE in Finance

Example of SDE in Finance :
Imagine you are trying to predict the price of a stock.
We will have historical data to estimate the average daily return of the stock, and assume that the daily return are normally distributed around this average with some standard deviation (σ).



Returns normally distributed

However, you also know that the stock price can be influenced by unpredictable news events or other random factors. So, your predictions will never be 100% accurate.

Now, let's see how SDE will work here.

SDE describes the evolution of a system over time, where the evolution is influenced by random or stochastic forces.

In this case of the stock price we might use the following SDE.

$$dS = \mu S\,dt + \sigma S\,dW$$

here, S represents the stock price
μ = average daily returns
σ = standard deviation of daily returns
dW = Wiener process, mathematical model for Brownian motion

SDE says that the change in the stock price over a small increment of time is equal to the product of the current stock price, average daily returns (μ) and the time increment (dt), plus the product of the current stock price (S), the standard deviation of daily returns ~~current stock price~~ (σ) and random increment (dW).

To understand this equation intutively, think of it as saying that the stock price can either increase or decrease based on the average daily return but this change is also subject to random fluctuations.

The size of these fluctuations is determined by the standard deviation of the daily return and the Wiener process captures the randomness of these fluctuations.

How SDE can be used to predict the stock price?

This involves finding a stochastic process that satisfies the SDE, given some initial condition (i.e., current stock price)

One way to do this is to use Ito's lemma which is a formula for calculating the derivative of a stochastic process.

Using Ito's Lemma, we can show that the solution to the SDE is

$$S(t) = S(0) \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t\right) + \sigma \, dW(t)$$

$S(t)$ = stock price at time t

$S(0)$ = inintial stock price

$W(t)$ is Wiener process

The solution says that the stock price follows a log normal distribution where the expected return is given by mean (μ) and standard deviation is given by σ.

So, using this simple example of SDE we can make prediction about the future stock price by simulating the Wiener process and plugging into the solution formula.

For example: We might simulate 1000 future stock price path over the next year, and calculate the expected return and standard deviation of each path. This information can be used to make investment decision or hedge against risk.