1. Elementary operations are row and column operations on a matrix which allow:
   1. scaling of rows/ columns,
   2. addition of rows/ columns

Example: let matrix A:

$$A = \begin{bmatrix} a & b & c \\ c & d & e \\ e & f & g \end{bmatrix}$$

applying $R_1 \to R_1 + R_2$
gives:

$$A = \begin{bmatrix} a+c & b+d & c+e \\ c & d & e \\ e & f & g \end{bmatrix}$$

and applying $R_1 \to 2R_1$ on the original matrix gives:

$$A = \begin{bmatrix} 2a & 2b & 2c \\ c & d & e \\ e & f & g \end{bmatrix}$$

2. The rank of matrix is defined as the no. of linearly independent rows or columns.

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 0 & 4 \\ 1 & 1 & 1 \end{bmatrix}$$

is a 3×3 matrix with rank 2 as it has exactly 2 linearly independent rows (when reduced to the simplest form)

---

3. Given:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

Applying $R_3 \to R_3 - R_2$ and $R_2 \to R_2 - R_1$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

Applying $R_1 \to R_1 - R_2$ and rearranging we get

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Applying $R_1 \to R_1 - R_2$ and rearranging we get

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

which can be reduced to

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Which has 3 linearly independent rows/columns and so has rank 3.

---

4. A matrix is said to be in normal form when it can be represented as

$$A=\begin{bmatrix} I_r & 0 \end{bmatrix}, A=\begin{bmatrix} I_r \\ 0 \end{bmatrix}, A=\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, A=\begin{bmatrix} I_r \end{bmatrix}$$

Where I_r is the identity matrix of order r and 0 is the null matrix of any order. A matrix A is said to be in Echelon form when: 1. Every row that has all entries 0 is below every row with non-zero entries 2. First non zero entry is 1 3. tuber of zeros before first non-zero entry of a row is greater than the previous row.

---

5. a. Given matrix

$$A=\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$$

Apply $R_3 \to R_3-(R_1+R_2)$ to get:

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Apply $R_2 \rightarrow R_2 - 2R_1$ and divide by -3 , and apply $C_2 \rightarrow C_2/2$ to get:

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & -\frac{2}{3} \\ 3 & 2 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Apply $C_1 \rightarrow C_1 - C_2$ to get:

$$A = \begin{bmatrix} 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & -\frac{2}{3} \\ 2 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Simplifying further and rearranging, we get

$$A = \begin{bmatrix} 1 & 1/2 & 1/2 & 3/2 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Which is in Echelon form, and we can now see that the rank of given matrix is 3.

---

b. Given matrix

$$A = \begin{bmatrix} -1 & 1 & -1 & 1 \\ 1 & -1 & 2 & -1 \\ 3 & 1 & 0 & 1 \end{bmatrix}$$

in this matrix, columns 2 and 4 are identical.

$$A = \begin{bmatrix} -1 & 1 & -1 & 0 \\ 1 & -1 & 2 & 0 \\ 3 & 1 & 0 & 0 \end{bmatrix}$$

Apply $C_1 \to C_1 + C_2$ and $R_2 \to R_2 + R_3$

$$A = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & 0 \\ 4 & -1 & -5 & 0 \end{bmatrix}$$

Apply $C_3 \to C_3 - (C_2 - C_1)$ and divide C_1 by 4

$$A = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & -1 & 1 & 0 \end{bmatrix}$$

Apply $C_1 \to C_1 + C_2$ and $R_2 \to R_2/2 + R_1$, followed by $R_3 \to R_3 + R_1$

$$A = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

There are 2 linearly independent rows and columns to the rank of the matrix is 2.

---

c.

$$A = \begin{bmatrix} 1 & 4 & 8 \\ 2 & 10 & 22 \\ 0 & 4 & 12 \end{bmatrix}$$

Apply $R_2 \to R_2/2$ and $R_2 \to R_2 - R_3$ to get

$$A = \begin{bmatrix} 1 & 4 & 8 \\ 1 & 1 & -1 \\ 0 & 4 & 12 \end{bmatrix}$$

Apply $R_1 \to R_1 - R_3$ and $R_3 \to R_3/4$

$$A = \begin{bmatrix} 1 & 0 & -4 \\ 1 & 1 & -1 \\ 0 & 1 & 3 \end{bmatrix}$$

Apply $R_3 \to R_3 + (R_1 - R_2)$

$$A = \begin{bmatrix} 1 & 0 & -4 \\ 1 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

simplify to get

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

Which is a matrix in echelon form with rank 2.

---

6. a.

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 3 & 2 & 2 \\ 2 & 4 & 3 & 4 \\ 3 & 7 & 4 & 6 \end{bmatrix}$$

apply
$R_2 \to R_2 - R_1,\ R_3 \to R_3 - 2R_1,\ R_4 \to R_4 - (3R_1 +$

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

apply $R_4 \to R_4 + R_3$ and reduce and rearrange the last column

$$A = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is a matrix in echelon form with rank 3.

---

b.

$$A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$$

apply $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - R_1$

$$A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 1 & 1 & 3 \end{bmatrix}$$

taking negative of second row, subtracting from third:

$$A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is a matrix in echelon form and the rank is 2

---

c.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 2 \\ 2 & 1 & 3 \end{bmatrix}$$

apply $R_2 \to R_2 + R_1$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix}$$

apply $R_3 \to R_3 - 2R_1$

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & -3 & 1 \end{bmatrix}$$

make first no zero elements 1:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3/2 \\ 0 & 1 & -1/3 \end{bmatrix}$$

apply $R_3 \to R_3 - R_2$ and appropriately scale the last row

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3/2 \\ 0 & 0 & 1 \end{bmatrix}$$

is a matrix in echelon form with rank 3

---

7. find α s.t. rank of

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3/2 \\ 0 & 1 & -1/3 \end{bmatrix}$$

   is 3

(done in notebook, copy from pdf later)

---

8. reduce to echelon form:

$$A = \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 11 & -5 & 3 \\ 2 & -5 & 3 & 1 \\ 4 & 1 & 1 & 5 \end{bmatrix}$$

   apply $R_3 \rightarrow R_3 - 2R_1$

$$A = \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 11 & -5 & 3 \\ 0 & -11 & 5 & -3 \\ 4 & 1 & 1 & 5 \end{bmatrix}$$

subtract row 1 from 2 and rearrange:

$$A = \begin{bmatrix} 1 & 3 & -1 & 2 \\ 4 & 1 & 1 & 5 \\ 0 & 11 & -5 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is in echelon form and has rank 3

---

9. $$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 4 & 1 & 2 & 1 \\ 3 & -1 & 1 & 2 \\ 1 & 2 & 0 & 1 \end{bmatrix}$$

apply $R_2 \to R_2 - 4R1$ and $R_3 \to R_3 - 3R_1$ and $R_4 \to R_4 - R_1$

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 6 & -11 \\ 0 & -7 & -2 & -7 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

apply $R_3 \to R_3 - R_2$

$$A=\begin{bmatrix}1 & 2 & -1 & 3\\ 0 & -7 & 6 & 4\\ 0 & 0 & -8 & 4\\ 0 & 0 & 1 & -2\end{bmatrix}$$

apply $C_4 \rightarrow C_4 + 2C_3$

$$A=\begin{bmatrix}1 & 2 & -1 & 1\\ 0 & -7 & 6 & 16\\ 0 & 0 & -8 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}$$

which can be reduced to:

$$A=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

is a matrix of rank 3 in normal form.

---

10. Consistent system of linear equations has one or more solutions, and an inconsistent system has no solution.

A linear equation of the form $AX = B$ is Non-Homogeneous if B is not a 0 vector, and it is called Homogeneous if B is a null vector.

For unique solution: if $\rho(A) = \rho([A|B]) = n = r$ The system has a unique solution.

if $\rho(A) < \rho([A|B])$ the system is inconsistent or has no solutions.

if $m < n$ then there will be infinite solutions.
