Assignment 4

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Ans 1	From $$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$ One gets that $$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$ Giving $$xe^{-x} = x - x^2 + \frac{x^3}{2!} - \frac{x^4}{3!} + \frac{x^5}{4!} + \cdots$$ Thus, First three terms of Maclaurin series of xe^{-x} will be $x - x^2 + \frac{x^3}{2!}$
Ans 2	A Maclaurin series is a Taylor series expansion of a function about 0, $$f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \cdots$$ $f(x) = \frac{x}{1+x^2}$ $f(0) = 0$ $f'(x) = -\frac{x^2-1}{(x^2+1)^2}$ $f'(0) = 1$ $f''(x) = \frac{2x(x^2-3)}{(x^2+1)^3}$ $f''(0) = 0$ $f'''(0) = -6$ $f''''(0) = 0$ $f'''''(0) = -120$ Thus, First three terms of Maclaurin series of $\frac{x}{1+x^2}$ will be $x - x^3 + x^5$
Ans 3	A Maclaurin series is a Taylor series expansion of a function about 0, $$f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \cdots$$ $f(x) = cos(sinx)$ $f(0) = 1$ $f'(x) = -\cos(x).\sin(\sin(x))$ $f'(0) = 0$ $f''(x) = \sin(x).\sin(\sin(x)) - \cos^2(x).\cos(\sin(x))$ $f''(0) = -1$ $f'''(0) = 0$ $f''''(0) = 5$ Thus, First three terms of Maclaurin series of $\cos(\sin(x)$ will be $1 - \frac{x^2}{2} + \frac{5x^4}{24}$
Ans 4	A Maclaurin series is a Taylor series expansion of a function about 0, $$f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \cdots$$

$f(x) = sin(sinx)$
$f(0) = 0$
$f'(x) = \cos(x) . \cos(\sin(x))$
$f'(0) = 1$
$f''(x) = -\sin(x) . \cos(\sin(x)) - \cos^2(x). \sin(\sin(x))$
$f''(0) = 0$
$f'''(0) =$ -2
$f''''(0) = 0$
$f'''''(0) = 12$

Thus, First three terms of Maclaurin series of $\sin(\sin(x))$ will be $x - \frac{x^3}{3} + \frac{x^5}{10}$

Ans 5

A Taylor series expansion of a function about a=0.5 is given as,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \cdots$$

$f(x) = \sin(\pi x)$
$f(a) = 1$
$f'(x) = \pi\cos(\pi x)$
$f'(a) = 0$
$f''(x) = -\pi^2\sin(\pi x)$
$f''(a) = -\pi^2$
$f'''(a) = 0$
$f''''(a) = \pi^4$

Thus, First three terms of Taylor series of $\sin(\pi x)$ will be
$1 - \pi^2 \frac{(x-0.5)^2}{2} + \pi^4 \frac{(x-0.5)^4}{24}$

$Thus,\ sin(\frac{\pi}{2} + \frac{\pi}{10})$ = 0.9511

Ans 6

A) Limit of a function $f(x,y)$ as (x,y) approaches (x_o, y_o) is the value that $f(x,y)$ approaches to. It may or may not be equal to $f(x_o, y_o)$.

$$L = \lim_{(x,y)\to(x_o,y_o)} f(x,y)$$

Continuity of a function $f(x,y)$ at a point means that its graph does not have breaks, holes, jump at that point. And Just like a one variable function.

A function is said to be continuous if it satifies the following euqation.

$$f(x_o, y_o) = \lim_{(x,y)\to(x_o,y_o)} f(x,y)$$

B) Evaluate the following limits:

1. $\lim_{(x,y)\to(\infty,2)} \frac{xy+4}{x^2+2y^2}$

   As x approaches ∞ we can neglect y as it is insignificant in comparison to y.
   Thus New function becomes,
   $\lim_{x\to\infty} \frac{x}{x^2}$
   $\Rightarrow L = 0$
   $\Rightarrow L = \lim_{x\to\infty} \frac{1}{x}$

2. $lim_{(x,y)\to(0,2)} \frac{2xy+1}{x^2+y^2}$

   Limit of $lim_{(x,y)\to(0,2)} \frac{2xy+1}{x^2+y^2}$ along x = 0

$$\Rightarrow L = \lim_{(y\to 2)} \frac{1}{y^2}$$
$$\Rightarrow L = \frac{1}{4}$$
Limit of $lim_{(x,y)\to(0,2)} \frac{2xy+1}{x^2+y^2}$ along y = 2
$$\Rightarrow L = \lim_{(x\to 0)} \frac{4x+1}{x^2+4}$$
$$\Rightarrow L = \frac{1}{4}$$

Hence, $lim_{(x,y)\to(0,2)} \frac{2xy+1}{x^2+y^2} = \frac{1}{4}$

3. $\lim_{(x,y)\to(1,1)} \frac{x(y-2)}{y(x-2)}$

Limit of $\lim_{(x,y)\to(1,1)} \frac{x(y-2)}{y(x-2)}$ along x = 1
$$\Rightarrow L = \lim_{(y\to 1)} \frac{(y-2)}{-y}$$
$$\Rightarrow L = \frac{-1}{-1}$$
$$\Rightarrow L = 1$$
Limit of $\lim_{(x,y)\to(1,1)} \frac{x(y-2)}{y(x-2)}$ along y = 1
$$\Rightarrow L = \lim_{(x\to 1)} \frac{-x}{(x-2)}$$
$$\Rightarrow L = \frac{-1}{-1}$$
$$\Rightarrow L = 1$$

Hence, $\lim_{(x,y)\to(1,1)} \frac{x(y-2)}{y(x-2)} = 1$

C) Examine the continuity of $f(x,y) = \frac{xy}{x^2+y^2}$ at the origin
$$f(0,0) = \frac{0}{0+0} = \frac{0}{0}$$
Since the function gives a value $\frac{0}{0}$ at the origin, hence it is discontinous at the origin.

D) Euler's theorem for Mixed Partial Derivatives:

If $f(x,y)$ is a homogenous function of degree 'n' in x and y, then
$$x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2} = n(n-1)f$$

1) $f(x,y) = \ln(2x+3y)$
$$f = \ln(2x+3y)$$
$$2x + 3y = e^f$$
$$e^{f(kx,ky)} = k(2x+3y) = k\,e^f$$
Thus, e^f is a Homogenous function of degree 1.

By Euler's theorem we have $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$
$$xe^f\frac{\partial f}{\partial x} + ye^f\frac{\partial f}{\partial y} = e^f$$
$$\Rightarrow\ x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 1$$
$$x^2 f_{xx} + 2xyf_{xy} + y^2 f_{yy} = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\left(x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}\right)$$

	$= \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)(1) = 0$ Using Euler's method $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = 1(1-1)f$ $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = 0$ 2) $f(x,y) = x\sin y + y\sin x + xy$ $f = x\sin y + y\sin x + xy$ since $f(kx,ky) \neq k^n f(x,y)$ it is not a homogenous function. Hence Euler's theorem can't be applied to it.
Ans 7	$0 = xy + z^3x - 2yz$ $0 = y\frac{\partial y}{\partial x} + 3z^2x\frac{\partial z}{\partial x} + z^3 - 2yz$ At point (1,1,1) $0 = 1 + 3\frac{\partial z}{\partial x} + 1 - 2\frac{\partial z}{\partial x}$ $-2 = \frac{\partial z}{\partial x}$
Ans 8	