LINEAR ALGEBRA - II

Topic Learning Objectives:
Upon Completion of this unit, students will be able to:

- Study the orthogonal and orthonormal properties of vectors.
- Use Gram-Schmidt process to factorize a given matrix as a product of an orthogonal matrix(Q) and an upper triangular invertible matrix(R).
- Diagonalize symmetric matrices using eigenvalues and eigenvectors.
- Decompose a given matrix into product of an orthogonal matrix(U), a diagonal matrix (Σ) and an orthogonal matrix(V^T).

Introduction:

This section deals with the study of orthogonal and orthonormal vectors which forms the basis for the construction of an orthogonal basis for a vector space. The Gram-Schmidt process is applied to construct an orthogonal basis for the column space of a given matrix and further to decompose a given matrix to the form A = QR, where Q has orthonormal column vectors and R is an upper triangular invertible matrix with positive entries along the diagonal. This section also deals with finding the Eigen values and Eigen vectors of a square matrix, which is applied to diagonalize a square matrix as D = $P^{-1}AP$. Further the singular value decomposition is studied wherein, a given matrix is resolved as a product of an orthogonal matrix (U), a diagonal matrix(Σ) and an orthogonal matrix (V^T).

Orthogonal Vectors:

Two vectors u and v in $\mathbb{R}^n$ are orthogonal to each other if $u.v = 0$.
$u = (1, 2)$ and $v = (6, -3)$ are orthogonal in $\mathbb{R}^2$ as $u.v = (1, 2).(6, -3) = 0$.

Orthogonal Sets:

A set of vectors $\{u_1, u_2, ..., u_p\}$ in $\mathbb{R}^n$ is said to be an orthogonal set if each pair of distinct vectors from the set is orthogonal, that is, if $u_i.u_j = 0$ whenever $i \neq j$.
ex. $\{u_1, u_2, u_3\}$ such that $u_1 = (3, 1, 1), u_2 = (-1, 2, 1), u_3 = (-\frac{1}{2}, -2, \frac{7}{2})$.
$u_1.u_2 = (3, 1, 1).(-1, 2, 1) = -3 + 2 + 1 = 0$ $u_1.u_3 = (3, 1, 1).(-\frac{1}{2}, -2, \frac{7}{2}) = -\frac{3}{2} - 2 + \frac{7}{2} = 0$
$u_2.u_3 = (-1, 2, 1).(-\frac{1}{2}, -2, \frac{7}{2}) = \frac{1}{2} - 4 + \frac{7}{2} = 0$.
Each pair of distinct vectors is orthogonal and so $\{u_1, u_2, u_3\}$ is an orthogonal set.

Orthonormal Sets:

A set $\{u_1, u_2, ..., u_p\}$ is an orthonormal set if it is an orthogonal set of unit vectors.
$\{e_1, e_2, ..., e_n\}$, the standard basis for $\mathbb{R}^n$, is an orthonormal set.
Any non-empty subset of $\{e_1, e_2, ..., e_n\}$ is orthonormal.

Orthogonal Basis:

An orthogonal basis for a subspace W of $\mathbb{R}^n$ is a basis for W that is also an orthogonal set.
ex. $S = \{u_1, u_2, u_3\}, u_1 = (3, 1, 1), u_2 = (-1, 2, 1), u_3 = (-\frac{1}{2}, -2, \frac{7}{2})$ is an orthogonal basis for $\mathbb{R}^3$ as (i) S is an orthogonal set and (ii) S forms a basis of $\mathbb{R}^3$.

$$\begin{vmatrix} 3 & 1 & 1 \\ -1 & 2 & 1 \\ -\frac{1}{2} & -2 & \frac{7}{2} \end{vmatrix} = 3(7 + 2) - 1(-\frac{7}{2} + \frac{1}{2}) + 1(2 + 1) = 27 + 3 + 3 = 33 \neq 0$$

Orthonormal Basis:

An orthonormal basis for a subspace W of $\mathbb{R}^n$ is a basis for W that is also an orthonormal set.

Example:

1. Show that $\{v_1, v_2, v_3\}$ is an orthonormal basis of $\mathbb{R}^3$, where
$v_1 = (\frac{3}{\sqrt{11}}, \frac{1}{\sqrt{11}}, \frac{1}{\sqrt{11}}), v_2 = (-\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}), v_3 = (-\frac{1}{\sqrt{66}}, -\frac{4}{\sqrt{66}}, \frac{7}{\sqrt{66}})$.

Solution:

$v_1.v_2 = -\frac{3}{\sqrt{66}} + \frac{2}{\sqrt{66}} + \frac{1}{\sqrt{66}} = 0$, $v_1.v_3 = -\frac{3}{\sqrt{726}} - \frac{4}{\sqrt{726}} + \frac{7}{\sqrt{726}} = 0$, $v_2.v_3 = \frac{1}{\sqrt{396}} - \frac{8}{\sqrt{396}} + \frac{7}{\sqrt{396}} = 0$
Thus $\{v_1, v_2, v_3\}$ is a orthogonal set.
$v_1.v_1 = \frac{9}{11} + \frac{1}{11} + \frac{1}{11} = 1$, $v_2.v_2 = \frac{2}{6} + \frac{4}{6} + \frac{1}{6} = 1$, $v_3.v_3 = \frac{1}{66} + \frac{16}{66} + \frac{49}{66} = 1$ which shows that v_1, v_2, v_3 are unit vectors.
Thus $\{v_1, v_2, v_3\}$ is an orthonormal set.
Since the set is linearly independent, its three vectors form a basis for $\mathbb{R}^3$.

Orthogonal Matrix:

A square matrix A with real entries and satisfying the condition $A^{-1} = A^T$ is called an orthogonal matrix.
ex. Let $P = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ Then $P^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ and $P^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$
clearly $P^{-1} = P^T$
$\therefore P$ is an orthogonal matrix.

ex. The matrix $A = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix}$ is orthogonal,

since $A^T A = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
The row vector of A, namely $(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}), (\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3})$ and $(\frac{2}{3}, \frac{2}{3}, \frac{1}{3})$ are orthonormal.
So are the column vectors of A.

Note:

Suppose that A is an $n \times n$ matrix with real entries. Then
(a) A is orthogonal iff the row vectors of A form an orthonormal basis of $\mathbb{R}^n$.
(b) A is orthogonal iff the column vectors of A form an orthonormal basis of $\mathbb{R}^n$.

Orthogonal Projections:

Given a non-zero vector $\overrightarrow{u}$ in $\mathbb{R}^n$, consider the problem of decomposing a vector $\overrightarrow{y}$ in $\mathbb{R}^n$ into the sum of two vectors, one a multiple of $\overrightarrow{u}$ and the other orthogonal to $\overrightarrow{u}$. We wish to write $\overrightarrow{y} = \hat{y} + \overrightarrow{z} - (1)$, where $\hat{y} = \alpha\overrightarrow{u}$, for some scalar α and $\overrightarrow{z}$ is some vector orthogonal to $\overrightarrow{u}$.
Given any scalar α, let $\overrightarrow{z} = \overrightarrow{y} - \alpha\overrightarrow{u}$, so that (1) is satisfied.
Then $\overrightarrow{y} - \hat{y}$ is orthogonal to $\overrightarrow{u}$ iff $0 = (\overrightarrow{y} - \alpha\overrightarrow{u}).\overrightarrow{u} = \overrightarrow{y}.\overrightarrow{u} - (\alpha\overrightarrow{u}).\overrightarrow{u} = \overrightarrow{y}.\overrightarrow{u} - \alpha(\overrightarrow{u}.\overrightarrow{u})$
That is, (1) is satisfied with $\overrightarrow{z}$ orthogonal to $\overrightarrow{u}$ iff $\alpha = \dfrac{\overrightarrow{y}.\overrightarrow{u}}{\overrightarrow{u}.\overrightarrow{u}}$ and $\hat{y} = \dfrac{\overrightarrow{y}.\overrightarrow{u}}{\overrightarrow{u}.\overrightarrow{u}}\overrightarrow{u}$.
The vector $\hat{y}$ is called the orthogonal projection of $\overrightarrow{y}$ onto $\overrightarrow{u}$, and the vector $\overrightarrow{z}$ is called the component of $\overrightarrow{y}$ orthogonal to $\overrightarrow{u}$.

ex. Let $\overrightarrow{y} = (7, 6)$ and $\overrightarrow{u} = (4, 2)$.
The orthogonal projection of $\overrightarrow{y}$ onto $\overrightarrow{u}$ is given by,

$$\hat{y} = \frac{\overrightarrow{y}.\overrightarrow{u}}{\overrightarrow{u}.\overrightarrow{u}}\overrightarrow{u} = \frac{40}{20}\overrightarrow{u} = 2\overrightarrow{u} = 2(4, 2) = (8, 4)$$

Note:
The orthogonal projection of $\overrightarrow{y}$ onto a space W spanned by orthogonal vectors $\{u_1, u_2\}$ is given by $\hat{y} = \dfrac{\overrightarrow{y}.\overrightarrow{u_1}}{\overrightarrow{u_1}.\overrightarrow{u_1}}\overrightarrow{u_1} + \dfrac{\overrightarrow{y}.\overrightarrow{u_2}}{\overrightarrow{u_2}.\overrightarrow{u_2}}\overrightarrow{u_2}$
The distance from a point $\overrightarrow{y}$ in $\mathbb{R}^n$ to a subspace W is defined as the distance from $\overrightarrow{y}$ to the nearest point in W.
ex. The distance from $\overrightarrow{y}$ to $W = \text{Span}\{u_1, u_2\}$, where $\overrightarrow{y} = (-1, -5, 10), u_1 = (5, -2, 1), u_2 = (1, 2, -1)$. is given by
$\hat{y} = \dfrac{(-1, -5, 10).(5, -2, 1)}{(5, -2, 1).(5, -2, 1)}(5, -2, 1) + \dfrac{(-1, -5, 10).(1, 2, -1)}{(1, 2, -1).(1, 2, -1)}(1, 2, -1) = (-1, -8, 4)$
$\overrightarrow{y} - \hat{y} = (-1, -5, 10) - (-1, -8, 4) = (0, 3, 6)$
The distance from $\overrightarrow{y}$ to W is $\sqrt{0 + 3^2 + 6^2} = \sqrt{45} = 3\sqrt{5}$.

Exercise:
1. Determine which set of vectors are orthogonal.
(i) $u_1 = (-1, 4, -3), u_2 = (5, 2, 1), u_3 = (3, -4, -7)$,
(ii) $u_1 = (5, -4, 0, 3), u_2 = (-4, 1, -3, 8), u_3 = (3, 3, 5, -1)$.
2.Show that $\{(2, -3), (6, 4)\}$ forms an orthogonal basis of $\mathbb{R}^2$.
3.Show that $\{(1, 0, 1), (-1, 4, 1), (2, 1, -2)\}$ forms an orthogonal basis of $\mathbb{R}^3$.
4. Show that the matrix $U = \begin{bmatrix} \frac{3}{\sqrt{11}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{66}} \\ \frac{1}{\sqrt{11}} & \frac{2}{\sqrt{6}} & -\frac{4}{\sqrt{66}} \\ \frac{1}{\sqrt{11}} & \frac{1}{\sqrt{6}} & -\frac{7}{\sqrt{66}} \end{bmatrix}$ is an orthogonal matrix.
5. Find the orthogonal projection of $y = (2, 6)$ onto $u = (7, 1)$.
6. Let $u_1 = (2, 5, -1), u_2 = (-2, 1, 1)$ and $y = (1, 2, 3)$. $W = \text{Span}\{u_1, u_2\}$. Find the orthogonal projection of y onto $W = \text{Span}\{u_1, u_2\}$.

Answers:
1. u_1, u_2 and u_2, u_3.
2. u_1, u_2 , u_1, u_3.
5. $(14/5, 2/5)$
6. $(-2/5, 2, 1/5)$

Gram-Schmidt Orthogonalization
The Gram-Schmidt process is a simple algorithm for producing an orthogonal or orthonormal basis for any non-zero subspace of $\mathbb{R}^n$.
The construction converts a skewed set of axes into a perpendicular set.

Gram-Schmidt process
Given a basis $\{x_1, x_2, ..., x_p\}$ for a subspace W of $\mathbb{R}^n$
define, $v_1 = x_1$
$v_2 = x_2 - \dfrac{x_2.v_1}{v_1.v_1}v_1$
$v_3 = x_3 - \dfrac{x_3.v_1}{v_1.v_1}v_1 - \dfrac{x_3.v_2}{v_2.v_2}v_2$
.
.
$v_p = v_p - \dfrac{x_p.v_1}{v_1.v_1}v_1 - \dfrac{x_p.v_2}{v_2.v_2}v_2... - \dfrac{x_p.v_{p-1}}{v_{p-1}.v_{p-1}}v_{p-1}$
Then $\{v_1, v_2, ..., v_p\}$ is an orthogonal basis for W.
In addition $\text{Span}\{v_1, v_2, ..., v_p\} = \text{Span}\{x_1, x_2, ..., x_k\}$ for $1 \leq k \leq p$.

Examples:

1. Let $W = \text{Span}\{x_1, x_2\}$ where $x_1 = (3,6,0)$ and $x_2 = (1,2,2)$. Construct an orthogonal basis $\{v_1, v_2\}$ for W.

Solution:

Let $v_1 = x_1$ and $v_2 = x_2 - \dfrac{x_2.x_1}{x_1.x_1}x_1 = (1,2,2) - \dfrac{(1,2,2).(3,6,0)}{(3,6,0).(3,6,0)}(3,6,0) = (0,0,2)$.

Then $\{v_1, v_2\}$ is an orthogonal set of non-zero vectors in W. Since $\dim W = 2$, the set $\{v_1, v_2\}$ is a basis in W.

2. Let $W = \text{Span}\{v_1, v_2, v_3\}$, where $v_1 = (0,1,2), v_2 = (1,1,2), v_3 = (1,0,1)$. Construct an orthogonal basis $\{u_1, u_2, u_3\}$ for W.

Solution:

Set $u_1 = v_1$ and $u_2 = v_2 - \dfrac{v_2.u_1}{u_1.u_1}u_1 = (1,1,2) - \dfrac{(1,1,2).(0,1,2)}{(0,1,2).(0,1,2)}(0,1,2) = (1,0,0)$

and $u_3 = v_3 - \dfrac{v_3.u_1}{u_1.u_1}u_1 - \dfrac{v_3.u_2}{u_2,u_2}u_2$

$= (1,0,1) - \dfrac{(1,0,1).(0,1,2)}{(0,1,2).(0,1,2)}(0,1,2) - \dfrac{(1,0,1).(1,0,0)}{(1,0,0).(1,0,0)}(1,0,0)$

$= (1,0,1) - \dfrac{2}{5}(0,1,2) - (1,0,0) = (0, -\dfrac{2}{5}, \dfrac{1}{5})$.

QR Factorization:

If A is an $m \times n$ matrix with linearly independent columns, then A can be factored as $A = QR$, where Q is an $m \times n$ matrix whose columns form an orthonormal basis for col A and R is an $n \times n$ upper triangular invertible matrix with positive entries on its diagonal.

Examples:

1. Find a QR factorization of $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Solution:

<u>Construction an orthonormal basis for Col A</u>

The columns of A are the vectors $\{x_1, x_2, x_3\}$

Let $v_1 = x_1 = (1,1,1,1)$

$v_2 = x_2 - \dfrac{x_2.v_1}{v_1.v_1}v_1 = (0,1,1,1) - \dfrac{(0,1,1,1).(1,1,1,1)}{(1,1,1,1).(1,1,1,1)}(1,1,1,1)$

$= (0,1,1,1) - \dfrac{3}{4}(1,1,1,1) = (-\dfrac{3}{4}, \dfrac{1}{4}, \dfrac{1}{4}, \dfrac{1}{4})$

$v_3 = x_3 - \dfrac{x_3.v_1}{v_1.v_1}v_1 - \dfrac{x_3.v_2}{v_2.v_2}v_2$

$= (0,0,1,1) - \dfrac{(0,0,1,1).(1,1,1,1)}{(1,1,1,1).(1,1,1,1)}(1,1,1,1) - \dfrac{(0,0,1,1).(-\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})}{(-\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}).(-\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})}(-\dfrac{3}{4}, \dfrac{1}{4}, \dfrac{1}{4}, \dfrac{1}{4})$

$= (0,0,1,1) - \dfrac{2}{4}(1,1,1,1) - \dfrac{2}{3}(-\dfrac{3}{4}, \dfrac{1}{4}, \dfrac{1}{4}, \dfrac{1}{4}) = (0, -\dfrac{2}{3}, \dfrac{1}{3}, \dfrac{2}{3})$

$\therefore \{v_1, v_2, v_3\}$ forms an orthogonal basis of Col A .

$\{(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}), (-\frac{3}{\sqrt{12}}, \frac{1}{\sqrt{12}}, \frac{1}{\sqrt{12}}, \frac{1}{\sqrt{12}}), (0, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}})\}$ forms an orthonormal basis of Col A.

$$\therefore Q = \begin{bmatrix} 1/2 & -3/\sqrt{12} & 0 \\ 1/2 & 1/\sqrt{12} & -2/\sqrt{6} \\ 1/2 & 1/\sqrt{12} & 1/\sqrt{6} \\ 1/2 & 1/\sqrt{12} & 1/\sqrt{6} \end{bmatrix}$$

To construct an upper triangular invertible matrix

We have $A = QR \implies Q^T A = Q^T QR \implies Q^T A = IR \implies Q^T A = R$ i.e., $R = Q^T A$.

$$\therefore R = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ -3/\sqrt{12} & 1/\sqrt{12} & 1/\sqrt{12} & 1/\sqrt{12} \\ 0 & -2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3/2 & 1 \\ 0 & 3/\sqrt{12} & 2/\sqrt{12} \\ 0 & 0 & 2/\sqrt{6} \end{bmatrix}.$$

2. Find a QR factorization of $A = \begin{bmatrix} 1 & 2 & 5 \\ -1 & 1 & -4 \\ -1 & 4 & -3 \\ 1 & -4 & 7 \\ 1 & 2 & 1 \end{bmatrix}$

Solution:
$\{x_1, x_2, x_3\}$ are the columns of the matrix A.
Let $v_1 = x_1 = (1, -1, -1, 1, 1)$

$v_2 = x_2 - \dfrac{x_2.v_1}{v_1.v_1} v_1 = (2, 1, 4, -4, 2) - \dfrac{(2, 1, 4, -4, 2).(1, -1, -1, 1, 1)}{(1, -1, -1, 1, 1).(1, -1, -1, 1, 1)}(1, -1, -1, 1, 1)$

$= (2, 1, 4, -4, 2) - \dfrac{-5}{5}(1, -1, -1, 1, 1) = (3, 0, 3, -3, 3)$

$v_3 = x_3 - \dfrac{x_3.v_1}{v_1.v_1} v_1 - \dfrac{x_3.v_2}{v_2.v_2} v_2$

$= (5, -4, -3, 7, 1) - \frac{(5,-4,-3,7,1).(1,-1,-1,1,1)}{(1,-1,-1,1,1).(1,-1,-1,1,1)}(1, -1, -1, 1, 1) - \frac{(5,-4,-3,7,1).(3,0,3,-3,3)}{(3,0,3,-3,3).(3,0,3,-3,3)}(3, 0, 3, -3, 3)$

$= (5, -4, -3, 7, 1) - \dfrac{20}{5}(1, -1, -1, 1, 1) - \dfrac{-12}{36}(3, 0, 3, -3, 3) = (2, 0, 2, 2, -2).$

$\therefore \{(1, -1, -1, 1, 1), (3, 0, 3, -3, 3), (2, 0, 2, 2, -2)\}$ forms an orthogonal basis of Col A.
$\{(1/\sqrt{5}, -1/\sqrt{5}, -1/\sqrt{5}, 1/\sqrt{5}, 1/\sqrt{5}), (1/2, 0, 1/2, -1/2, 1/2), (1/2, 0, 1/2, 1/2, -1/2)\}$
forms an orthonormal basis of Col A.

$$\therefore Q = \begin{bmatrix} 1/\sqrt{5} & 1/2 & 1/2 \\ -1/\sqrt{5} & 0 & 0 \\ -1/\sqrt{5} & 1/2 & 1/2 \\ 1/\sqrt{5} & -1/2 & 1/2 \\ 1/\sqrt{5} & 1/2 & -1/2 \end{bmatrix}$$

$$R = Q^T A = \begin{bmatrix} 1/\sqrt{5} & -1/\sqrt{5} & -1/\sqrt{5} & 1/\sqrt{5} & 1/\sqrt{5} \\ 1/2 & 0 & 1/2 & -1/2 & 1/2 \\ 1/2 & 0 & 1/2 & 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 5 \\ -1 & 1 & -4 \\ -1 & 4 & -3 \\ 1 & -4 & 7 \\ 1 & 2 & 1 \end{bmatrix}$$

$$\therefore R = \begin{bmatrix} \sqrt{5} & -\sqrt{5} & 4\sqrt{5} \\ 0 & 6 & -2 \\ 0 & 0 & 4 \end{bmatrix}$$

3. Find the orthogonal basis for the column space of the matrix $\begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{bmatrix}$

Solution:
The columns of A are the vectors $\{x_1, x_2, x_3\}$
where $x_1 = (3,1,-1,3), x_2 = (-5,1,5,-7), x_3 = (1,1,-2,8)$.
Let $v_1 = (3,1,-1,3)$
$v_2 = x_2 - \dfrac{x_2.v_1}{v_1.v_1}v_1 = (-5,1,5,-7) - \frac{(-5,1,5,-7).(3,1,-1,3)}{(3,1,-1,3).(3,1,-1,3)}(3,1,-1,3)$
$= (-5,1,5,-7) - \frac{-40}{20}(3,1,-1,3) = (1,3,3,-1)$
$v_3 = \dfrac{x_3.v_1}{v_1.v_1}v_1 - \dfrac{x_3.v_2}{v_2.v_2}v_2 = (1,1,-2,8) - \frac{(1,1,-2,8).(3,1,-1,3)}{(3,1,-1,3).(3,1,-1,3)}(3,1,-1,3) - \frac{(1,1,-2,8).(1,3,3,-1)}{(1,3,3,-1).(1,3,3,-1)}(1,3,3,-1)$
$= (1,1,-2,8) - \frac{30}{20}(3,1,-1,3) - \frac{-10}{20}(1,3,3,-1) = (-3,1,1,3)$
$\{(3,1,-1,3),(1,3,3,-1),(-3,1,1,3)\}$ is an orthogonal basis for the column space of the given matrix.

4. Find the orthogonal basis for the column space of the matrix $\begin{bmatrix} -1 & 6 & 6 \\ 3 & -8 & 3 \\ 1 & -2 & 6 \\ 1 & -4 & -3 \end{bmatrix}$

Solution:
The columns of A are the vectors $\{x_1, x_2, x_3\}$
where $x_1 = (-1,3,1,1), x_2 = (6,-8,-2,-4), x_3 = (6,3,6,-3)$.
Let $v_1 = (-1,3,1,1)$
$v_2 = x_2 - \dfrac{x_2.v_1}{v_1.v_1}v_1 = (6,-8,-2,-4) - \frac{(6,-8,-2,-4).(-1,3,1,1)}{(-1,3,1,1).(-1,3,1,1)}(-1,3,1,1)$
$= (6,-8,-2,-4) - \frac{-36}{12}(-1,3,1,1) = (3,1,1,-1)$
$v_3 = \dfrac{x_3.v_1}{v_1.v_1}v_1 - \dfrac{x_3.v_2}{v_2.v_2}v_2 = (6,3,6,-3) - \frac{(6,3,6,-3).(-1,3,1,1)}{(-1,3,1,1).(-1,3,1,1)}(-1,3,1,1) - \frac{(6,3,6,-3).(3,1,1,-1)}{(3,1,1,-1).(3,1,1,-1)}(3,1,1,-1)$
$= (6,3,6,-3) - \frac{6}{12}(-1,3,1,1) - \frac{30}{12}(3,1,1,-1) = (-1,-1,3,-1)$
$\{(-1,3,1,1),(3,1,1,-1),(-1,-1,3,-1)\}$ is an orthogonal basis for the column space of the given matrix.

Exercise:
1. Let $W = \text{Span}\{v_1, v_2\}$, where $v_1 = (1,1)$ and $v_2 = (2,-1)$. Construct an orthogonal basis $\{u_1, u_2\}$ for W.
2. Find the orthonormal basis of the subspace spanned by the vectors $u_1 = (1,-4,0,1), u_2 = (7,-7,-4,1)$
3. Find the QR factorization of the matrix $A = \begin{bmatrix} 1 & 3 & 5 \\ -1 & -3 & 1 \\ 0 & 2 & 3 \\ 1 & 5 & 2 \\ 1 & 5 & 8 \end{bmatrix}$

Answer:
1. $v_1 = (1,1), v_2 = (\dfrac{3}{2}, -\dfrac{3}{2})$
2. $v_1 = (1,-4,0,1), v_2 = (5,1,-4,-1)$

Eigen Values and Eigen Vectors:

If A is a square matrix of order n, we can find the matrix $A - \lambda I$, where I is the n^{th} order unit matrix. The determinant of this matrix equated to zero, i.e,

$$|A - \lambda I| = \begin{bmatrix} a_{11}-\lambda & a_{12} & . & . & . & a_{1n} \\ a_{21} & a_{22}-\lambda & . & . & . & a_{2n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ a_{n1} & a_{n2} & . & . & . & a_{nn}-\lambda \end{bmatrix}$$

is called the characteristic equation of A.

On expanding the determinant, the characteristic equation takes the form

$(-1)^n\lambda^n + k_1\lambda^{n-1} + k_2\lambda^{n-2} + ... + k_n = 0$,

where k^s are expressible in terms of the elements a_{ij}. The roots of this equation are called the characteristic roots or latent roots or eigen-values of the matrix A.

If $x = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix}$ and $A = \begin{bmatrix} a_{11} & a_{12} & . & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & . & a_{2n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ a_{n1} & a_{n2} & . & . & . & a_{nn} \end{bmatrix}$,

then the linear transformation $y = Ax$ - (1) carries the column vector x into the column vector y by means of the square matrix A.

In practice it is often required to find such vectors which transform into themselves or to a scalar multiple of themselves.

Let x be such a vector which transforms into λx by means of the transformation (1).

Then, $\lambda x = Ax$ or $Ax - \lambda Ix = 0$ or $[A - \lambda I]x = 0$- (2)

The matrix equation represents n homogeneous linear equations,

$(a_{11} - \lambda)x_1 + a_{12}x_2 + ... + a_{1n}x_n = 0$

$a_{21}x_1 + (a_{22} - \lambda)x_2 + ... + a_{2n}x_n = 0$

......

$a_{n1}x_1 + a_{n2}x_2 + ... + (a_{nn} - \lambda)x_n = 0$ -(3)

which will have a non-trivial solution only if the coefficient matrix is singular.

i.e, if $|A - \lambda I| = 0$

This is called the characteristic equation of the transformation and is same as the characteristic equation of the matrix A.

It has $n$roots and corresponding to each root, the equation (2)(or equation (3)) will have a non-zero solution, $x = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix}$, which is known as the eigen vector or latent vector.

Observation 1:

Corresponding to n distinct eigen values, we get n independent eigen vectors. But when two or more eigen values are equal, it may or may not be possible to get linearly independent eigen vectors corresponding to the repeated roots.

Observation 2:

If x_i is a solution for a eigen value λ_i then it follows from (2) that cx_i is also a solution, where c ia an arbitrary constant. Thus the eigen vector corresponding to an eigen value is not unique, but may be any one of the vectors cx_i.

Examples:

1. Find the Eigen Values and Eigen vectors of the matrix $A = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$

Solution:

$|A - \lambda I| = 0 \implies \begin{vmatrix} \frac{1}{2} - \lambda & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} - \lambda \end{vmatrix} \implies \lambda^2 - \lambda = 0 \implies \lambda = 0, \lambda = 1.$

with $\lambda = 1$, $(A - \lambda I)x = 0 \implies \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies -x_1 + x_2 = 0 \implies x_2 = x_1$

Letting $x_1 = 1 \implies x_2 = 1 \therefore x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

with $\lambda = 0$, $(A - \lambda I)x = 0 \implies \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies x_1 + x_2 = 0 \implies x_2 = -x_1$

Letting $x_1 = 1 \implies x_2 = -1 \therefore x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

2. Find the Eigen Values and Eigen vectors of the matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Solution:

$|A - \lambda I| = 0 \implies \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} \implies \lambda^2 + 1 = 0 \implies \lambda = +i, \lambda = -i.$

with $\lambda = i$, $(A - \lambda I)x = 0 \implies \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies -ix_1 - x_2 = 0 \implies x_2 = -ix_1$

Letting $x_1 = 1 \implies x_2 = -i \therefore x = \begin{bmatrix} 1 \\ -i \end{bmatrix}$

with $\lambda = -i$, $(A - \lambda I)x = 0 \implies \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies ix_1 - x_2 = 0 \implies x_2 = ix_1$

Letting $x_1 = 1 \implies x_2 = i \therefore x = \begin{bmatrix} 1 \\ i \end{bmatrix}$

3. Find the Eigen Values and Eigen vectors of the matrix $A = \begin{bmatrix} 3 & 4 & 2 \\ 3 & 5 & 4 \\ 0 & 1 & 2 \end{bmatrix}$.

Solution:

$|A - \lambda I| = 0 \implies \begin{vmatrix} 3-\lambda & 4 & 2 \\ 3 & 5-\lambda & 4 \\ 0 & 1 & 2-\lambda \end{vmatrix} = 0 \implies \lambda^3 - 10\lambda^2 + 15\lambda = 0$

$\implies \lambda = 5 + \sqrt{10},\ 5 - \sqrt{10},\ 0$

with $\lambda = 5 + \sqrt{10}$ $|A - \lambda I| = 0 \implies \begin{bmatrix} -2-\sqrt{10} & 4 & 2 \\ 3 & -\sqrt{10} & 4 \\ 0 & 1 & -3-\sqrt{10} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\implies \dfrac{x_1}{3\sqrt{10}+6} = \dfrac{-x_2}{-9-3\sqrt{10}} = \dfrac{x_3}{3} \implies \dfrac{x_1}{2+\sqrt{10}} = \dfrac{x_2}{3+\sqrt{10}} = \dfrac{x_3}{1}$

$\therefore x = \begin{bmatrix} 2+\sqrt{10} \\ 3+\sqrt{10} \\ 1 \end{bmatrix}$

with $\lambda = 5 - \sqrt{10}$, $|A - \lambda I| = 0 \implies \begin{bmatrix} -2+\sqrt{10} & 4 & 2 \\ 3 & \sqrt{10} & 4 \\ 0 & 1 & -3+\sqrt{10} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\implies \dfrac{x_1}{-3\sqrt{10}+6} = \dfrac{-x_2}{-9+3\sqrt{10}} = \dfrac{x_3}{3} \implies \dfrac{x_1}{2-\sqrt{10}} = \dfrac{x_2}{3-\sqrt{10}} = \dfrac{x_3}{1}$

$\therefore x = \begin{bmatrix} 2-\sqrt{10} \\ 3-\sqrt{10} \\ 1 \end{bmatrix}$

with $\lambda = 0$, $|A - \lambda I| = 0 \implies \begin{bmatrix} 3 & 4 & 2 \\ 3 & 5 & 4 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\implies \frac{x_1}{6} = \frac{-x_2}{6} = \frac{x_3}{3} \implies \frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{1}$

$\therefore x = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

4. Find the Eigen Values and Eigen vectors of the matrix $A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$

Solution:

$|A - \lambda I| = 0 \implies \begin{vmatrix} 1-\lambda & 3 & 3 \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{vmatrix} = 0 \implies \lambda^3 + 3\lambda^2 - 4 = 0$

$\implies \lambda = 1, -2, -2$

with $\lambda = 1$, $|A - \lambda I| = 0 \implies \begin{bmatrix} 0 & 3 & 3 \\ -3 & -6 & -3 \\ 3 & 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\implies \frac{x_1}{9} = \frac{-x_2}{9} = \frac{x_3}{9} \implies \frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{1} \therefore x = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

with $\lambda = -2$, $|A - \lambda I| = 0 \implies \begin{bmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$3x_1 + 3x_2 + 3x_3 = 0 \implies x_1 = -x_2 - x_3$

Letting $x_2 = k_1, x_3 = k_2 \implies x_1 = -k_1 - k_2 \therefore x = \begin{bmatrix} -k_1 - k_2 \\ k_1 \\ k_2 \end{bmatrix}$

or $x = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, x = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ are the linearly independent eigen vectors corresponding to $\lambda = -2$.

Diagonalization of a Matrix:

Suppose the n by n matrix A has n linearly independent eigen vectors. If these eigen vectors are the columns of a matrix P, then $P^{-1}AP$ is a diagonal matrix D. The eigen values of A are on the diagonal of D

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & & & & & \\ & \lambda_2 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & \lambda_n \end{bmatrix}$$

Note:

1. Any matrix with distinct eigen values can be diagonalized.
2. The diagonalization matrix P is not unique.
3. Not all matrices posses n linearly independent eigen vectors, so not all matrices are diagonalizable.
4. Diagonalizability of A depends on enough eigen vectors.
5. Diagonalizability can fail only if there are repeated eigen values.
6. The eigen values of A^k are $\lambda_1^k, \lambda_2^k, ..., \lambda_n^k$ and each eigen vector of A is still an eigen vector of A^k.

$[D^k = D.D....D(k\,\text{times}) = (P^{-1}AP)(P^{-1}AP)...(P^{-1}AP) = P^{-1}A^kP]$.

Problems:

1. Diagonalize the matrix $A = \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix}$.

Solution:

$|A - \lambda I| = 0 \implies \begin{vmatrix} 7-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = 0 \implies \lambda^2 - 11\lambda + 24 = 0 \implies (\lambda - 3)(\lambda - 8) = 0 \implies \lambda = 3, \lambda = 8.$

With $\lambda = 3, (A - 3I)x = 0 \implies \begin{vmatrix} 4 & 2 \\ 2 & 1 \end{vmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies 2x_1 + x_2 = 0.$

Letting $x_1 = 1 \implies x_2 = -2$. Hence $\mathrm{x}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

With $\lambda = 8, (A - 8I)x = 0 \implies \begin{vmatrix} -1 & 2 \\ 2 & -4 \end{vmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies -x_1 + 2x_2 = 0.$

Letting $x_2 = 1 \implies x_1 = 2$. Hence $\mathrm{x}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\implies P = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}, D = \begin{bmatrix} 3 & 0 \\ 0 & 8 \end{bmatrix}, P^{-1} = \begin{bmatrix} 1/5 & -2/5 \\ 2/5 & 1/5 \end{bmatrix}$

2. Diagonalize the matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}$.

Solution:

$|A - \lambda I| = 0 \implies \begin{vmatrix} 1-\lambda & 0 \\ 0 & -3-\lambda \end{vmatrix} = 0 \implies (\lambda - 1)(\lambda + 3) = 0 \implies \lambda = -3, \lambda = 1.$

With $\lambda = -3, (A + 3I)x = 0 \implies \begin{vmatrix} 4 & 0 \\ 0 & 0 \end{vmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies 4x_1 = 0 \implies x_1 = 0$. Let $x_2 = 1$.

Hence $\mathrm{x}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

With $\lambda = 1, (A - I)x = 0 \implies \begin{vmatrix} 0 & 0 \\ 0 & -4 \end{vmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies 4x_2 = 0 \implies x_2 = 0$. Let $x_1 = 1$.

Hence $\mathrm{x}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\implies P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, D = \begin{bmatrix} -3 & 0 \\ 0 & 1 \end{bmatrix}, P^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$

3. Diagonalize the matrix $A = \begin{bmatrix} 6 & -2 & -1 \\ -2 & 6 & -1 \\ -1 & -1 & 5 \end{bmatrix}$

Soln: $|A - \lambda I| = 0 \implies \begin{vmatrix} 6-\lambda & -2 & -1 \\ -2 & 6-\lambda & -1 \\ -1 & -1 & 5-\lambda \end{vmatrix} = 0 \implies \lambda^3 - 17\lambda^2 + 90\lambda - 144 = 0$

$\implies \lambda = 3,\ 6,\ 8$

with $\lambda = 3$ $|A - \lambda I| = 0 \implies \begin{bmatrix} 3 & -2 & -1 \\ -2 & 3 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\implies \frac{x_1}{5} = \frac{-x_2}{-5} = \frac{x_3}{5} \implies \frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1} \therefore x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

with $\lambda = 6$ $|A - \lambda I| = 0 \implies \begin{bmatrix} 0 & -2 & -1 \\ -2 & 0 & -1 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\implies \frac{x_1}{-1} = \frac{-x_2}{1} = \frac{x_3}{2} \implies \frac{x_1}{-1} = \frac{x_2}{-1} = \frac{x_3}{2} \therefore x = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$

with $\lambda = 8$ $|A - \lambda I| = 0 \implies \begin{bmatrix} -2 & -2 & -1 \\ -2 & -2 & -1 \\ -1 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\implies \frac{x_1}{5} = \frac{-x_2}{5} = \frac{x_3}{0} \implies \frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{2} \therefore x = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

Hence $P = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & -1 \\ 1 & 2 & 0 \end{bmatrix}$, $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 8 \end{bmatrix}$, $P^{-1} = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ -1/6 & -1/6 & 1/3 \\ 1/2 & -1/2 & 0 \end{bmatrix}$.

4. Diagonalize the matrix $A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$

Soln: $|A - \lambda I| = 0 \implies \begin{vmatrix} 3-\lambda & -2 & 4 \\ -2 & 6-\lambda & 2 \\ 4 & 2 & 3-\lambda \end{vmatrix} = 0 \implies \lambda^3 - 12\lambda^2 - 21\lambda + 98 = 0$

$\implies \lambda = -2,\ 7,\ 7$

with $\begin{matrix} \lambda = -2, \\ |A - \lambda I| = 0 \end{matrix} \implies \begin{bmatrix} 5 & -2 & 4 \\ -2 & 8 & 2 \\ 4 & 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies \frac{x_1}{36} = \frac{-x_2}{-18} = \frac{x_3}{-36} \therefore \mathrm{x}_1 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$

with $\lambda = 7$, $|A - \lambda I| = 0 \implies \begin{bmatrix} -4 & -2 & 4 \\ -2 & -1 & 2 \\ 4 & 2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

As the second and third row are dependent on the first row, we get only one equation in three unknowns. i.e., $-4x_1 - 2x_2 + 4x_3 = 0$. Letting x_1 and x_3 as arbitrary implies $x_2 = -2x_1 + 2x_3$. With $x_1 = 1, x_3 = 2$ we get $x_2 = 2$. With $x_1 = 2, x_3 = 1$ we get $x_2 = -2$.

$\therefore \mathrm{x}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \therefore \mathrm{x}_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

Exercise:

1. Diagonalize the matrices (i) $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$, (ii) $\begin{bmatrix} 1 & 5 \\ 5 & 1 \end{bmatrix}$.

2. Diagonalize the matrices (i) $\begin{bmatrix} -2 & -36 & 0 \\ -36 & -23 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, (ii) $\begin{bmatrix} 7 & -4 & 4 \\ -4 & 5 & 0 \\ 4 & 0 & 9 \end{bmatrix}$.

Singular Value Decomposition:

Any $m \times n$ matrix A can be factored into $A = U\Sigma V^T =$ (orthogonal)(diagonal)(orthogonal). The columns of $U(m \,\text{by}\, m)$ are eigen vectors of AA^T, and the columns of $V(n \,\text{by}\, n)$ are eigen vectors of A^TA. The r singular values on the diagonal of $\Sigma(m \,\text{by}\, n)$ are the square roots of the non-zero eigen values of both AA^T and A^TA.

Note:

The diagonal(but rectangular) matrix Σ has eigen values from A^TA. These positive entries(also called sigma) will be $\sigma_1, \sigma_2, ..., \sigma_r$,such that $\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_r > 0$. They are the singular values of A.
When A multiplies a column v_j of V, it produces σ_j times a column of $U(A = U\Sigma V^T \implies AV = U\Sigma)$.

Examples:

1. Decompose $A = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$ as $U\Sigma V^T$, where U and V are orthogonal matrices.

Solution:

$AA^T = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} \begin{bmatrix} -1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix}$

$|AA^T - \lambda I| = 0 \implies \begin{vmatrix} 1-\lambda & -2 & -2 \\ -2 & 4-\lambda & 4 \\ -2 & 4 & 4-\lambda \end{vmatrix} = 0$

$\implies \lambda^3 - 9\lambda^2 = 0 \implies \lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 9$

with $\lambda = 9, [AA^T - \lambda I]x = 0 \implies$

$\begin{bmatrix} -8 & -2 & -2 \\ -2 & -5 & 4 \\ -2 & 4 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies -8x_1 - 2x_2 - 2x_3 = 0, -18x_2 + 18x_3 = 0$

$\implies x_1 = -(1/2)x_3, x_2 = x_3 \implies x = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$

with $\lambda = 0, [AA^T - \lambda I]x = 0 \implies$

$\begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies x_1 = 2x_2 + 2x_3 \implies x = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$ and $x = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$

Hence $U = \begin{bmatrix} -1/3 & 2/3 & 2/3 \\ 2/3 & -1/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \end{bmatrix}$

$A^TA = \begin{bmatrix} -1 & 2 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 9 \end{bmatrix}$

$|A^TA - \lambda I| = 0 \implies |9-\lambda| = 0 \implies \lambda = 9$

Then $A^TA - \lambda I)x = 0 \implies \begin{bmatrix} 0 \end{bmatrix} \begin{bmatrix} x_1 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$

Let $x_1 = 1 \therefore x = \begin{bmatrix} 1 \end{bmatrix}$

Hence $V = \begin{bmatrix} 1 \end{bmatrix}$ or $V^T = \begin{bmatrix} 1 \end{bmatrix}$

9 is an eigen value of both AA^T and A^TA.

And rank of $A = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$ is $r = 1$.

$\therefore \Sigma$ has only $\sigma_1 = \sqrt{9} = 3$. $\therefore \Sigma = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$

$\therefore$ the SVD of $A = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -1/3 & 2/3 & 2/3 \\ 2/3 & -1/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix}$

2. Obtain the SVD of $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

Solution:

$AA^T = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

$|AA^T - \lambda I| = 0, \begin{vmatrix} 2-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = 0 \implies \lambda^2 - 3\lambda + 1 = 0 \implies \lambda_1 = \dfrac{3-\sqrt{5}}{2}, \lambda_2 = \dfrac{3+\sqrt{5}}{2}$

with$\lambda = \dfrac{3-\sqrt{5}}{2}$
$(AA^T - \lambda I)x = 0$
$\implies$ $\begin{bmatrix} \dfrac{1+\sqrt{5}}{2} & 1 \\ 1 & \dfrac{-1+\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \dfrac{1+\sqrt{5}}{2} x_1 + x_2 = 0.$

Letting $x_1 = -1$, then $x_2 = \dfrac{1+\sqrt{5}}{2} \therefore x = \begin{bmatrix} -1 \\ \dfrac{1+\sqrt{5}}{2} \end{bmatrix} = \begin{bmatrix} -1 \\ \alpha \end{bmatrix}$, where $\alpha = \dfrac{1+\sqrt{5}}{2}$.

with $\lambda = \dfrac{3+\sqrt{5}}{2}$
$(AA^T - \lambda I)x = 0$
$\implies$ $\begin{bmatrix} \dfrac{1-\sqrt{5}}{2} & 1 \\ 1 & \dfrac{-1-\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \dfrac{1-\sqrt{5}}{2} x_1 + x_2 = 0.$

Letting $x_1 = -1$, then $x_2 = \dfrac{1-\sqrt{5}}{2} \therefore x = \begin{bmatrix} -1 \\ \dfrac{1-\sqrt{5}}{2} \end{bmatrix} = \begin{bmatrix} -1 \\ \beta \end{bmatrix}$, where $\beta = \dfrac{1-\sqrt{5}}{2}$.

Hence $U = \begin{bmatrix} \dfrac{-1}{\sqrt{1+\alpha^2}} & \dfrac{-1}{\sqrt{1+\beta^2}} \\ \dfrac{\alpha}{\sqrt{1+\alpha^2}} & \dfrac{\beta}{\sqrt{1+\beta^2}} \end{bmatrix}$

As $A^TA = AA^T \quad V^T = \begin{bmatrix} \dfrac{-1}{\sqrt{1+\alpha^2}} & \dfrac{\alpha}{\sqrt{1+\alpha^2}} \\ \dfrac{-1}{\sqrt{1+\beta^2}} & \dfrac{\beta}{\sqrt{1+\beta^2}} \end{bmatrix}$ and $\Sigma = \begin{bmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{bmatrix}$.

3. Obtain the SVD of $A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

Solution:

$AA^T = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

$|AA^T - \lambda I| = 0 \implies \begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = 0 \implies \lambda^2 - 4\lambda + 3 = 0 \implies \lambda_1 = 1, \lambda_2 = 3$

with $\lambda = 3$ $(AA^T - \lambda I)x = 0 \implies \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies x_1 + x_2 = 0 \implies x_1 = -x_2$

Letting $x_2 = 1 \implies x_1 = -1 \therefore x = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

with $\lambda = 1$ $(AA^T - \lambda I)x = 0 \implies \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies x_1 - x_2 = 0 \implies x_1 = x_2$

Letting $x_2 = 1 \implies x_1 = 1 \therefore x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$A^TA = \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$|A^TA - \lambda I| = 0 \implies \begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix} = 0 \implies \lambda^3 - 4\lambda^2 + 3\lambda = 0 \implies \lambda_1 = 0, \lambda_2 = 1, \lambda_3 = 3$

with $\lambda = 0$ $(A^TA - \lambda I)x = 0 \implies \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $\begin{matrix} x_1 - x_2 = 0 \\ x_2 - x_3 = 0 \end{matrix}$, $\begin{matrix} x_1 = x_2 \\ x_2 = x_3 \end{matrix}$ $\therefore x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

with$\lambda = 1$ $(A^TA - \lambda I)x = 0 \implies \begin{bmatrix} 0 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $\begin{matrix} -x_1 + x_2 \\ -x_3 = 0 \\ x_2 = 0 \end{matrix}$, $\begin{matrix} x_1 = -x_3 \\ x_2 = 0 \end{matrix}$ $\therefore x = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

with$\lambda = 3$ $(A^TA - \lambda I)x = 0 \implies \begin{bmatrix} -2 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $\begin{matrix} -2x_1 - x_2 = 0 \\ x_2 + 2x_3 = 0 \end{matrix}$, $\begin{matrix} 2x_1 = x_2 \\ x_2 = -2x_3 \end{matrix}$ $\therefore x = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

Hence $U = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ $\Sigma = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ $V = \begin{bmatrix} 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ -2/\sqrt{6} & 0 & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}$

Exercise:

1. Find the SVD of

(i) $\begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix}$, (ii) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}$, (iii) $\begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$