Module-1

PAGE NO.
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Linear Programming Problem: Slack and Surplus variables.

Standard form

sol^n of LPP → Basic sol^n

Basic feasible sol^n

Degenerate and non-degenerate sol^n

Optimal sol^n.

Sol^n by (1) Simplex method

(2) Big-M-method.

9. Suppose that a manufacturer produce 2 types of products, say A and B. Assume that both these products are to be processed in two different machines, say N_1 and N_2. The following table is the data.

	Profit/Item	Process time in machine M_1	Process time in machine M_2
Profit A x_1	₹ 3	2 hours	4 hours
Profit B x_2	₹ 4	3 hours	2 hours
Time available per week		48 hours	48 hours

Formulate an LPP and solve it.

→ Let x_1 be the no. of product A to be produced per week and x_2 be the no. of product B to be produced per week

The objective funcⁿ is to maximize the profit, $Z = 3x_1 + 4x_2$

The constraints are:-

$$2x_1 + 3x_2 \leq 48$$

$$4x_1 + 2x_2 \leq 48$$

∵ the no. of products is non-ve, +ve

$x_1, x_2 \geq 0$

To find x_1 and x_2 so as to

Max $Z = 3x_1 + 4x_2$

Subject to the constraints

$2x_1 + 3x_2 \le 48$

$4x_1 + 2x_2 \le 48$

$x_1, x_2 \ge 0$.

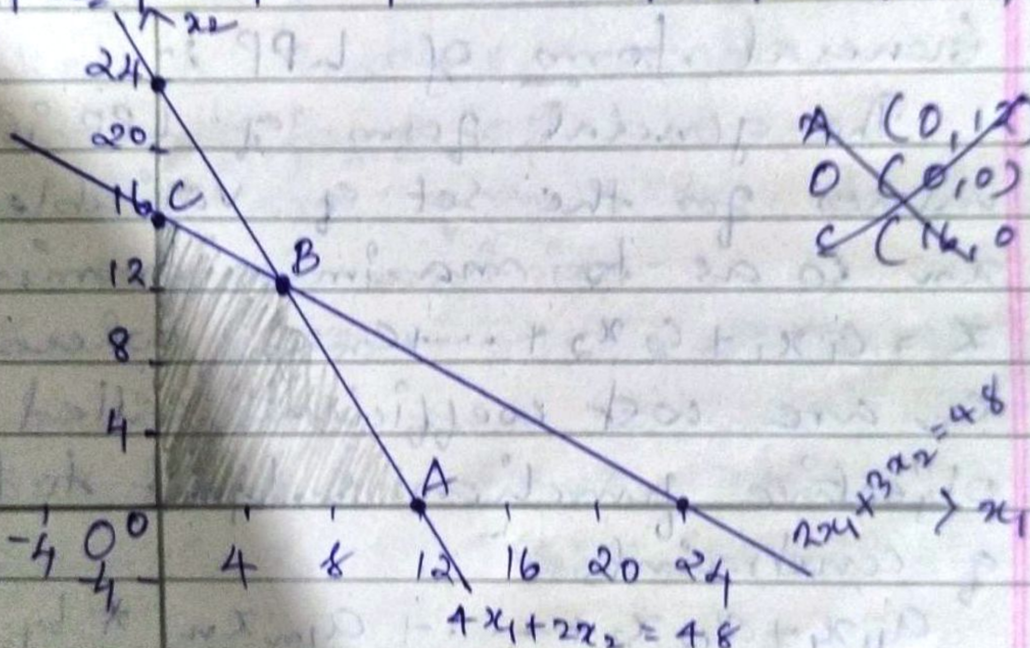
Graphical method :-

$2x_1 + 3x_2 = 48$

x_1	0	24
x_2	16	0

$4x_1 + 2x_2 = 48$

x_1	0	12
x_2	24	0



~~A (0,12)~~ ~~O (0,0)~~ ~~C (16,0)~~

A (12,0)
C (0,16)
O (0,0)
B (6,12)

$(2x_1 + 3x_2 = 48) \times 2$

$4x_1 + 6x_2 = 96$ —

$4x_1 + 2x_2 = 48$

$4x_2 = 48$

$x_2 = \underline{\underline{12}}.$

$x_2 = 12$

$\Rightarrow 2x_1 + 3 \times 12 = 48$

$2x_1 + 36 = 48$

$2x_1 = 12$

$x_1 = \underline{\underline{6}}.$

- A (12, 0) in Z.

$Z = 3x_1 + 4x_2 = 3 \times 12 = 36$

- B (6, 12) in Z

$Z = 3 \times 6 + 4 \times 12 = 18 + 48 = 66$

- C (0, 16) in Z

$Z = 3 \times 0 + 4 \times 16 = 64$

- O (0, 0) in Z

$Z = 3 \times 0 + 4 \times 0 = 0$

$\therefore$ Max $Z = 66$ at $x_1 = 6$ & $x_2 = 12$

=> General form of LPP :-

The general form of LPP is to find values for the set of variables $x_1, x_2 \ldots x_n$ so as to maximise or minimise $Z = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$ where $c_1, c_2, \ldots c_n$ are cost coefficients called the objective function subject to the set of constraints

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n * b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n * b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n * b_m$$

where * means $\leq$, $\geq$ or $=$ and

$x_1, x_2 \ldots x_n \geq 0$

Imp

⇒ Slack & Surplus Variables :-

If a constraint of an LPP is of the form $a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \le b_i$, then the nonnegative variable x_{n+i} such that $a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n + x_{n+i} = b_i$ is called a slack variable.

If a constraint of an LPP is of the form $a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \geq b_i$, then the nonnegative variable x_{n+i} such that $a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n - x_{n+i} = b_i$ is called a surplus variable.

22/11/2021

⇒ Standard form of LPP :-

The std form of an LPP is to find $x_1, x_2 \ldots x_n$ so as to maximize

$Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$ subject to the constraints :

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

where $x_i \geq 0 \;\forall\; i = 1, 2, 3 \ldots, n$ and
$b_j \geq 0 \;\forall\; j = 1, 2, 3 \ldots m$

In matrix notation :-

To find X such that

Max $z = C \cdot X$
Subjected to the constraints
$$AX = B$$
where $X, B \geq 0$.

- $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$; $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$

$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$; $C = [c_1, c_2, \ldots, c_n]$

Q. Convert the given problem to Standard form.

1. Minimize $z = x_1 + 2x_2 - 4x_3$
Subjected to
$$2x_1 + x_2 + 3x_3 \leq 16$$
$$x_1 + x_2 + x_3 = 8$$
$$x_1 - 2x_2 + x_3 \leq 7 \quad -x_1 + 2x_2 - x_3 \geq -7$$
$$x_1 + x_3 \geq 2$$
$$x_1, x_2, x_3 \geq 0.$$

should ≥ 0 +ve

Min change to max by multiplying a minus

To make constraints into equality add a surplus variable x_4

→ Standard form is
$$\text{Max } z = -(\text{Min } z)$$
$$= -(x_1 + 2x_2 - 4x_3)$$
$$= -x_1 - 2x_2 + 4x_3 + 0x_4 + 0x_5 + 0x_6.$$

subjected to constraints

$$2x_1 + x_2 + 3x_3 + x_4 = 16$$
$$x_1 + x_2 + x_3 = 8$$
$$x_1 - 2x_2 + x_3 + x_5 = 7$$
$$x_1 + x_3 - x_6 = 2.$$
$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0.$$

2. Maximise $z = 2x_1 + x_2 + 4x_3$

subjected to

$$-2x_1 + 4x_2 \leq 4$$
$$x_1 + 2x_2 + x_3 \geq 5$$
$$2x_1 + 3x_3 \leq 2$$
$$x_1, x_2, x_3 \geq 0.$$

→ Standard form is

Max $z = 2x_1 + x_2 + 4x_3 + 0x_4 + 0x_5 + 0x_6.$

subjected to

$$-2x_1 + 4x_2 + x_4 = 4$$
$$x_1 + 2x_2 + x_3 - x_5 = 5$$
$$2x_1 + 3x_3 + x_6 = 2$$
$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0.$$

⇒ Note:-

A variable is said to be unrestricted in signs, that means it can be expressed as the difference of two nonnegative variables.

? Convert the given problems to std form.

1. Min. $z = 2x_1 + 3x_2$

subjected to constraints

$x_1 + x_2 \le 4$

$2x_1 - x_2 \ge 3$

$3x_1 + 4x_2 = 10$

where $x_1 \ge 0$ and x_2 is unrestricted in sign.

x_2

$x_2' - x_2''$

$x_2', x_2'' \ge 0$.

→ Min $z = 2x_1 + 3x_2' - 3x_2''$

subjected to

$x_1 + x_2' - x_2'' \le 4$

$2x_1 - x_2' + x_2'' \ge 3$

$3x_1 + 4x_2' - 4x_2'' = 10$

$x_1, x_2', x_2'' \ge 0$.

Standard form is :-

Max $z = -2x_1 - 3x_2' + 3x_2'' + 0x_3 + 0x_4$

subjected to

$x_1 + x_2' - x_2'' + x_3 = 4$

$2x_1 - x_2' + x_2'' - x_4 = 3$

$3x_1 + 4x_2' - 4x_2'' = 10$

$x_1, x_2', x_2'', x_3, x_4 \ge 0$

2. Max $z = 2x_1 + x_2 + 4x_3$

subject to

$2x_1 - 4x_2 \geq -4$

$-x_1 - 2x_2 - x_3 \leq -5$

$-2x_1 - 3x_3 \geq -2$

$x_1, x_2 \geq 0$ & x_3 is unrestricted.

$\rightarrow$ $x_3 = x_3' - x_3''$; $x_3', x_3'' \geq 0$.

Max $z = 2x_1 + x_2 + 4x_3' - 4x_3''$

subject to

$2x_1 - 4x_2 \geq -4$

$-x_1 - 2x_2 - x_3' + x_3'' \leq -5$

$-2x_1 - 3x_3' + 3x_3'' \geq -2$

$x_1, x_2, x_3', x_3'' \geq 0$.

Std form is :-

Max $z = 2x_1 + x_2 + 4x_3' - 4x_3'' + 0x_4 + 0x_5 + 0x_6$

subject to

$-2x_1 + 4x_2 + x_4 = 4$

$x_1 + 2x_2 + x_3' - x_3'' - x_5 = 5$

$2x_1 + 3x_3' - 3x_3'' + x_6 = 2$

$x_1, x_2, x_3', x_3'', x_4, x_5, x_6 \geq 0$.

Imp $\Rightarrow$ Different Types of Sol^{ns} of LPP :-

Consider the std LPP,

Max $Z = CX$

Subject to $AX = B$, where $X, B \geq 0$ and A is an $m \times n$ matrix with $n > m$. ~~and~~ ie; It contains m eqns and n variables.

Suppose some $(n-m)$ variables are set equal to zero. If the resulting s/m of m eqns in m unknowns has a unique solⁿ. This solⁿ is called a basic solⁿ. Total no. of basic solⁿˢ is ${}^nC_m$.

A basic solⁿ which satisfies the non-ve restrictions is called a basic feasible solⁿ.

A basic feasible solⁿ which maximise the objective funcⁿ in a std LPP is called an Optimal solⁿ or Optimal basic feasible solⁿ.

A basic solⁿ is called degenerate if one or more of the basic variables equals zero. If all the basic variables are +ve, then the solⁿ is called non-degenerate solⁿ.

23/11/2021

Q. Find all basic solⁿˢ of $2x_1 + x_2 + 4x_3 = 11$ and $3x_1 + x_2 + 5x_3 = 14$. Which of them are feasible and which are degenerate?

→ No. of variables, $n = 3$
No. of constraints, $m = 2$
∴ Total no. of basic sol^{ns} = ${}^nC_m$
$= {}^3C_2 = 3$

Now, $n - m = 3 - 2 = 1$
Assign any one variable as zero.
case 1 : when x_1 is zero :-

$$\Rightarrow \quad x_2 + 4x_3 = 11$$
$$\underline{x_2 + 5x_3 = 14}$$
$$-x_3 = -3 \Rightarrow x_3 = 3$$

$x_2 + 4x_3 = 11 \Rightarrow x_2 + 4 \times 3 = 11$
$\Rightarrow x_2 + 12 = 11 \Rightarrow x_2 = -1$

∴ Sol^n is $(0, -1, 3)^T \Rightarrow X_1 = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$

Case 2: when x_2 is zero :-

$$\Rightarrow 2x_1 + 4x_3 = 11 \quad \times 3$$
$$\underline{3x_1 + 5x_3 = 14} \quad \times 2$$
$$\Rightarrow 6x_1 + 12x_3 = 33$$
$$\underline{6x_1 + 10x_3 = 28}$$
$$2x_3 = 5 \Rightarrow x_3 = \frac{5}{2} = 2.5$$

$2x_1 + 4x_3 = 11 \Rightarrow 2x_1 + \cancel{4}^{2} \times \frac{5}{\cancel{2}} = 11$
$\Rightarrow 2x_1 + 10 = 11$
$\Rightarrow 2x_1 = 1 \Rightarrow x_1 = 1/2$

$\therefore$ Sol^n $\Rightarrow$ $X_2 = \begin{bmatrix} 1/2 \\ 0 \\ 5/2 \end{bmatrix}$

Case 3: when x_3 is zero:-

$\Rightarrow 2x_1 + x_2 = 11$

$\underline{3x_1 + x_2 = 14}$

$-x_1 = -3 \Rightarrow x_1 = 3$ //

$2x_1 + x_2 = 11 \Rightarrow 2 \times 3 + x_2 = 11$

$\Rightarrow 6 + x_2 = 11$

$\Rightarrow x_2 = 5$ //

$\therefore$ Sol^n $\Rightarrow$ $X_3 = \begin{bmatrix} 3 \\ 5 \\ 0 \end{bmatrix}$ //

Here X_1, X_2, X_3 are basic sol^{ns} of which X_2 and X_3 are basic feasible sol^{ns}.

Here X_2 and X_3 are non-degenerate sol^{ns}.

Q. Find all the basic sol^{ns} and basic feasible sol^{ns} for

$2x_1 + 6x_2 + 2x_3 + x_4 = 3$

$6x_1 + 4x_2 + 4x_3 + 6x_4 = 2$

$\rightarrow$ No. of variables, $n = 4$

No. of constraints, $m = 2$

Total no. of basic sol^{ns} $= {}^nC_m = {}^4C_2 = 6$ //

Now, $n-m = 4-2 = 2$

$\Rightarrow$ Assign 2 variables to zero.

Case 1: when $x_1 = x_2 = 0$.

$\Rightarrow 2x_3 + x_4 = 3 \quad \times 2$

$4x_3 + 6x_4 = 2$

$4x_3 + 2x_4 = 6 \quad -$

$4x_3 + 6x_4 = 2$

$-4x_4 = 4 \Rightarrow x_4 = -1$

$2x_3 + x_4 = 3 \Rightarrow 2x_3 - 1 = 3$

$\Rightarrow 2x_3 = 4 \Rightarrow x_3 = 2$

Sol^n, $X_1 = \begin{bmatrix} 0 \\ 0 \\ 2 \\ -1 \end{bmatrix}$ - not feasible

Case 2: when $x_1 = x_3 = 0$.

$6x_2 + x_4 = 3 \quad \times 6$

$4x_2 + 6x_4 = 2$

$36x_2 + 6x_4 = 18 \quad -$

$4x_2 + 6x_4 = 2$

$32x_2 = 16 \Rightarrow x_2 = \frac{1}{2}$

$6x_2 + x_4 = 3 \Rightarrow 6 \times \frac{1}{2} + x_4 = 3$

$\Rightarrow 3 + x_4 = 3 \Rightarrow x_4 = 0$

Sol^n, $X_2 = \begin{bmatrix} 0 \\ 1/2 \\ 0 \\ 0 \end{bmatrix}$ is a degenerate basic feasible sol^n.

Case 3: $x_1 = x_4 = 0$

$6x_2 + 2x_3 = 3 \quad \times 2$

$4x_2 + 4x_3 = 2$

$12x_2 + 4x_3 = 6 \quad -$

$4x_2 + 4x_3 = 2$

$8x_2 = 4 \Rightarrow x_2 = 1/2$ //

$6x_2 + 2x_3 = 3 \Rightarrow 6 \times \frac{1}{2} + 2x_3 = 3$

$\Rightarrow 3 + 2x_3 = 3 \Rightarrow 2x_3 = 0$

$\Rightarrow x_3 = 0$ //

Sol^n:- $X_3 = \begin{bmatrix} 0 \\ 1/2 \\ 0 \\ 0 \end{bmatrix}$ // is a degenerate basic feasible sol^n.

Case 4: $x_2 = x_3 = 0$.

$2x_1 + x_4 = 3 \quad \times 6$.

$6x_1 + 6x_4 = 2$

$12x_1 + 6x_4 = 18$

$6x_1 + 6x_4 = 2$

$6x_1 = 16 \Rightarrow x_1 = 16/6 = 8/3$ //

$2 \times \frac{8}{3} + x_4 = 3 \Rightarrow \frac{16}{3} + x_4 = 3$

$\Rightarrow x_4 = 3 - \frac{16}{3} = \frac{9-16}{3} = -7/3$

Sol^n:- $X_4 = \begin{bmatrix} 8/3 \\ 0 \\ 0 \\ -7/3 \end{bmatrix}$ // is a basic sol^n.

Case 5 :- $x_2 = x_4 = 0$.

$2x_1 + 2x_3 = 3 \quad \times 2$

$\cancel{4}x_1 + 4x_3 = 2$

$4x_1 + 4x_3 = 6$

$6x_1 + 4x_3 = 2$

$-2x_1 = 4 \Rightarrow x_1 = -2$

$2x_1 + 2x_3 = 3 \Rightarrow 2 \times -2 + 2x_3 = 3$

$\Rightarrow -4 + 2x_3 = 3$

$\Rightarrow 2x_3 = 7 \Rightarrow x_3 = 7/2$

Sol^n :- $X_5 = \begin{bmatrix} -2 \\ 0 \\ 7/2 \\ 0 \end{bmatrix}$ is a basic sol^n :-

Case 6 :- when $x_3 = x_4 = 0$:

$2x_1 + 6x_2 = 3 \quad \times 3$

$6x_1 + 4x_2 = 2$

$6x_1 + 18x_2 = 9$

$6x_1 + 4x_2 = 2$

$14x_2 = 7 \Rightarrow x_2 = 1/2$

$2x_1 + 6x_2 = 3 \Rightarrow 2 \times x_1 + \overset{3}{\cancel{6}} \times \frac{1}{\cancel{2}} = 3$

$\Rightarrow 2x_1 + 3 = 3$

$\Rightarrow 2x_1 = 0 \Rightarrow x_1 = 0$.

$\therefore$ Sol^n :- $X_6 = \begin{bmatrix} 0 \\ 1/2 \\ 0 \\ 0 \end{bmatrix}$ is a degenerate basic feasible sol^n.

⇒ Simplex Method :-

Q. Solve the following LPP using simplex method.

Minimize $Z = x_1 - 3x_2 + 2x_3$

Subjected to the constraints

$3x_1 - x_2 + 2x_3 \le 7$

$-2x_1 + 4x_2 \le 12$

$-4x_1 + 3x_2 + 8x_3 \le 10$

$x_1, x_2, x_3 \ge 0$.

Sol^n → Step 1 :- Convert given problem to std form.

Max $Z = -(\text{Min } Z)$

Max $Z = -x_1 + 3x_2 - 2x_3 + 0x_4 + 0x_5 + 0x_6$.

Subject to $\quad$ Max $Z = CX$

$$\left.\begin{array}{l} 3x_1 - x_2 + 2x_3 + x_4 = 7 \\ -2x_1 + 4x_2 + x_5 = 12 \\ -4x_1 + 3x_2 + 8x_3 + x_6 = 10 \\ x_1, x_2, x_3, x_4, x_5, x_6 \ge 0 \end{array}\right\} AX = B.$$

→ Step 2 : Obtain an initial basic feasible sol^n.

Here no. of variables = 6

no. of eqns = 3

Then $n - m = 6 - 3 = 3$

Assume 3 variables to zero.

put $x_1 = x_2 = x_3 = 0$.

$\therefore x_4 = 7$

$x_5 = 12$

$x_6 = 10$

x_4, x_5, x_6 are slack variables.

$z_j = \sum C_B x_j$

→ Step 3: Simplex table:-

Table-1. key column

C_j		-1	3	-2	0	0	0		
C_B	X_B	x_1	x_2	x_3	x_4	x_5	x_6	b	θ
0	$x_4 = 7$	3	-1	2	1	0	0	7	$\frac{7}{-1} = -7$
0	$x_5 = 12$	-2	4	0	0	1	0	12	$\frac{12}{4} = 3$
0	$x_6 = 10$	-4	3	8	0	0	1	10	$\frac{10}{3} = 3.33$
	$z_j - C_j$	1	-3	2	0	0	0		

key row

key element / pivot element

← min +ve value of θ

most -ve $z_j - C_j$

* If all $C_j z_j \le 0$, soln is optimal.

Table-2. key col

$R_1 \to R_1 + R_2$

$R_3 \to R_3 - 3R_2$

C_j		-1	3	-2	0	0	0	
C_B	X_B	x_1	x_2	x_3	x_4	x_5	x_6	θ
0	$x_4 = 10$	$5/2$	0	2	1	$1/4$	0	$\frac{10}{5/2} = 4$
3	$x_2 = 3$	$-1/2$	1	0	0	$1/4$	0	$\frac{3}{-1/2} = -6$
0	$x_6 = 1$	$-5/2$	0	8	0	$-3/4$	1	$\frac{1}{-5/2} = \frac{-2}{5}$
	$z_j - C_j$	$-1/2$	0	2	0	$3/4$	0	

key row

÷ by 4

← min +ve

most -ve value

All $z_j - C_j \ge 0$ is the condition.

$-\frac{2}{5}+\frac{3}{5}=\frac{1}{5}$ $\frac{2}{5}+2=$

$R_2 \to R_2+\frac{1}{2}R_1$

$R_2 \to R_3+\frac{5}{2}R_1$

$R_3 \to$

PAGE NO.

DATE:

Table 3.

C_B	c_j / x_B	x_1 (-1)	x_2 (3)	x_3 (-2)	x_4 (0)	x_5 (0)	x_6 (0)
-1	$x_1=4$	1	0	4/5	2/5	1/10	0
3	$x_2=5$	0	1	2/5	1/5	3/10	0
0	$x_6=11$	0	0	10	1	-1/2	1
	z_j-c_j	0	0	12/5	1/5	4/5	0

Since all $z_j - c_j \geq 0$, soln is optimal.

$x_1 = 4$ $\quad x_3 = 0$

$x_2 = 5$ $\quad x_4 = 0$

$x_6 = 11$ $\quad x_5 = 0$

$\therefore z = -4+(3\times5)-(2\times0)+(0\times0)+(0\times5)+(0\times11)$

$\therefore$ Max $z = -4+15 = \underline{\underline{11}}$

9. Use simplex method to solve

Max $z = 2x_1 + 3x_2$

subjected to

$2x_1 + x_2 \leq 6$

$x_1 + 2x_2 \leq 8$

$x_1 - x_2 \leq 1$

$x_1 \leq 2$

$x_1, x_2 \geq 0$.

→ Step 1: Convert to std form:

Max $z = 2x_1 + 3x_2 + 0x_3 + 0x_4 + 0x_5 + 0x_6$

Max $= \frac{38}{3}$

$x_1 = 4/3$

$x_2 = 10/3$

$$2x_1 + x_2 + x_3 = 6$$
$$x_1 + 2x_2 + x_4 = 8$$
$$x_1 - x_2 + x_5 = 1$$
$$x_1 + x_6 = 2$$
$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0.$$

→ step 2: Obtain initial basic feasible sol^n.

$m = 6.$

$m = 4$

$n - m = 6 - 4 = 2$

put $x_1 = x_2 = 0$.

$\therefore x_6 = 2$; $x_5 = 1$; $x_4 = 8$; $x_3 = 6$

→ Step 3:- Simplex table:-

Table: 1

	C_j	2 (key col)	3	0	0	0	0	
C_B	X_B	X_1	X_2	X_3	X_4	X_5	X_6	θ
0	$X_3 = 6$	2	1	1	0	0	0	3
0	$X_4 = 8$	1	2	0	1	0	0	8
key row 0	$X_5 = 1$	1	-1	0	0	1	0	1 → min +ve value
0	$X_6 = 2$	1	0	0	0	0	1	2
	$Z_j - C_j$	-2	-3	0	0	0	0	

most -ve value

$R_2 \to R_2 - R_3$

$R_4 \to R_4 - R_3$

$R_1 \to R_1 - 2R_3$

Table 2:-

	C_j	2	3	0	0	0	0	
C_B	X_B	x_1	x_2	x_3	x_4	x_5	x_6	θ
0	$x_3=4$	0	3	1	0	-2	0	4/3
0	$x_4=7$	0	3	0	1	-1	0	7/3
2	$x_1=1$	1	-1	0	0	1	0	-1
0	$x_6=1$	0	1	0	0	-1	1	1
	z_j-c_j	0	-5	0	0	2	0	

$R_3 \to R_3 + R_4$
$R_2 \to R_2 + 3R_4$
$R_1 \to R_1 + 3R_4$

Table 3 :-

	C_j	2	3	0	0	0	0	
C_B	X_B	x_1	x_2	x_3	x_4	x_5	x_6	θ
0	$x_3=1$	0	0	1	0	1	-3	1
0	$x_4=4$	0	0	0	1	2	-3	2
2	$x_1=2$	1	0	0	0	0	1	∞
3	$x_2=1$	0	1	0	0	-1	1	-1
	z_j-c_j	0	0	0	0	-3	5	

$R_2 \to R_2 - 2R_1$
$R_4 \to R_4 + R_1$

Table 4:-

	C_j	2	3	0	0	0	0	
C_B	X_B	x_1	x_2	x_3	x_4	x_5	x_6	θ
0	$x_5=1$	0	0	1	0	1	-3	-1/3
0	$x_4=2$	0	0	-2	1	0	3	2/3
2	$x_1=2$	1	0	0	0	0	1	2
3	$x_2=2$	0	1	1	0	0	-2	-1
	z_j-c_j	0	0	0	0	0	-4	

$R_2 \to \frac{R_2}{3}$

$R_1 \to R_1 + 3R_2$

$R_3 \to R_3 - R_2$

$R_4 \to R_4 + 2R_2$

Table 5:-

C_B	C_j	2	3	0	0	0	0
	X_B	x_1	x_2	x_3	x_4	x_5	x_6
0	$x_5 = 3$	0	0	-5	1	1	0
0	$x_6 = 2/3$	0	0	$-2/3$	$1/3$	0	1
2	$x_1 = 4/3$	1	0	$2/3$	$-1/3$	0	0
3	$x_2 = 10/3$	0	1	$-1/3$	$2/3$	0	0
	$Z_j - C_j$	0	0	$1/3$	$4/3$	0	0

Since all $Z_j - C_j \geq 0$, soln is optimal.

$x_1 = \frac{4}{3}$, $x_2 = \frac{10}{3}$, $x_3 = x_4 = 0$, $x_5 = 3$, $x_6 = \frac{2}{3}$

$\therefore Z = 2 \times \frac{4}{3} + 3 \times \frac{10}{3} + 0 + 0 + 0 + 0$

$= \frac{8+30}{3} = \frac{38}{3}$.

Max $Z = \frac{38}{3}$

Q. Obtain all the basic feasible solns of

(a) $x_1 + 2x_2 + x_3 = 4$

$2x_1 + x_2 + 5x_3 = 5$

→ No. of variables, $n = 3$

No. of constraints, $m = 2$

$n - m = 3 - 2 = 1$

Case 1: $x_1 = 0$

$2x_2 + x_3 = 4$

$x_2 + 5x_3 = 5 \quad \times 2$

$2x_2 + x_3 = 4 \quad -$

$2x_2 + 10x_3 = 10$

$-9x_3 = -6$

$9x_3 = 6 \Rightarrow x_3 = 2/3$

$x_2 + 5 \times 2/3 = 5 \Rightarrow x_2 + \frac{10}{3} = 5$

$\Rightarrow x_2 = 5 - \frac{10}{3}$

$= \frac{5}{3}$

$\therefore$ Sol^n, $X_1 = \begin{bmatrix} 0 \\ 5/3 \\ 2/3 \end{bmatrix}$ non-degenerate basic feasible sol^n.

Case 2: $x_2 = 0$.

$x_1 + x_3 = 4 \quad \times 2$

$2x_1 + 5x_3 = 5$

$2x_1 + 2x_3 = 8 \quad -$

$2x_1 + 5x_3 = 5$

$-3x_3 = 3 \Rightarrow x_3 = -1$

$x_1 - 1 = 4 \Rightarrow x_1 = 5$

Sol^n:- $X_2 = \begin{bmatrix} 5 \\ 0 \\ -1 \end{bmatrix}$ is a basic sol^n.

Case 3: $x_3 = 0$

$x_1 + 2x_2 = 4 \quad \times 2$

$2x_1 + x_2 = 5$

$2x_1 + 4x_2 = 8 \quad -$

$2x_1 + x_2 = 5$

$3x_2 = 3 \Rightarrow x_2 = 1$

$x_1 + 2 \times 1 = 4 \Rightarrow x_1 = 4 - 2 = 2$

$\therefore$ sol^n: $X_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ non-degenerate basic feasible sol^n.

$2 - \frac{5}{3} = \frac{6-5}{3}$

(b) $2x_1 + x_2 - x_3 = 2$

$3x_1 + 2x_2 + x_3 = 3$

$\rightarrow$ $n = 3$; $m = 2$; $n - m = 3 - 2 = 1$

Case 1: $x_1 = 0$.

$x_2 - x_3 = 2 \quad +$

$2x_2 + x_3 = 3$

$3x_2 = 5 \Rightarrow x_2 = 5/3$

$2x_2 + x_3 = 3 \Rightarrow 2 \times \frac{5}{3} + x_3 = 3$

$\Rightarrow \frac{10}{3} + x_3 = 3 \Rightarrow x_3 = 3 - \frac{10}{3} = \frac{-1}{3}$

Sol^n :- $X_1 = \begin{bmatrix} 0 \\ 5/3 \\ -1/3 \end{bmatrix}$ is a basic sol^n.

Case 2: $x_2 = 0$.

$2x_1 - x_3 = 2 \quad +$

$3x_1 + x_3 = 3$

$5x_1 = 5 \Rightarrow x_1 = 1$

$2x_1 - x_3 = 2 \Rightarrow 2\times1 - x_3 = 2$

$\Rightarrow 2-2 = x_3 \Rightarrow x_3 = 0$ //

Sol^n:- $X_2 = \begin{bmatrix}1\\0\\0\end{bmatrix}$ // is a degenerate basic feasible sol^n

Case 3:- $x_3 = 0$.

$2x_1 + x_2 = 2 \quad \times 2$

$3x_1 + 2x_2 = 3$

$4x_1 + 2x_2 = 4 \quad -$

$3x_1 + 2x_2 = 3$

$x_1 = 1$ //

$2x_1 + x_2 = 2 \Rightarrow 2 + x_2 = 2 \Rightarrow x_2 = 0$ //

$\therefore$ Sol^n:- $X_3 = \begin{bmatrix}1\\0\\0\end{bmatrix}$ // is a degenerate basic feasible sol^n.

(c) $2x_1 + 3x_2 + 4x_3 + x_4 = 2$

$x_1 + x_2 + 7x_3 + x_4 = 4$

$\rightarrow$ $n = 4$; $m = 2$; $4C_2 = 6$.

$n - m = 4 - 2 = 2$ //

Case 1:- $x_1 = x_2 = 0$.

$4x_3 + x_4 = 2$

$7x_3 + x_4 = 4 \quad -$

$-3x_3 = -2 \Rightarrow x_3 = 2/3$ //

$4 \times \frac{2}{3} + x_4 = 2 \Rightarrow \frac{8}{3} + x_4 = 2$

$\Rightarrow x_4 = 2 - \frac{8}{3} = -2/3$ //

Sol^n :- $X_1 = \begin{bmatrix} 0 \\ 0 \\ 2/3 \\ -2/3 \end{bmatrix}$ is a basic sol^n.

Case 2: $x_1 = x_3 = 0$

$$3x_2 + x_4 = 2$$
$$x_2 + x_4 = 4$$
$$2x_2 = -2 \Rightarrow x_2 = -1$$

$-1 + x_4 = 4 \Rightarrow x_4 = 4+1 = 5$

Sol^n :- $X_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 5 \end{bmatrix}$ is a basic sol^n

Case 3: $x_1 = x_4 = 0$

$$3x_2 + 4x_3 = 2$$
$$x_2 + 7x_3 = 4 \quad \times 3$$

$$3x_2 + 4x_3 = 2$$
$$3x_2 + 21x_3 = 12$$
$$-17x_3 = -10 \Rightarrow x_3 = 10/17$$

$x_2 + \frac{7 \times 10}{17} = 4 \Rightarrow x_2 = 4 - \frac{70}{17} = \frac{-2}{17}$

$\therefore \text{Sol}^n$:- $X_3 = \begin{bmatrix} 0 \\ 10/17 \\ -2/17 \\ 0 \end{bmatrix}$ is a basic sol^n.

Case 4: $x_2 = x_3 = 0$.

$$2x_1 + x_4 = 2$$
$$\underline{x_1 + x_4 = 4}$$
$$x_1 = -2$$

$-2 + x_4 = 4 \Rightarrow x_4 = 6$

$\therefore$ sol^n :- $X_4 = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 6 \end{bmatrix}$ is a basic sol^n.

Case 5:- $x_2 = x_4 = 0$.

$$2x_1 + 4x_3 = 2$$
$$\underline{x_1 + 7x_3 = 4} \quad \times 2$$
$$2x_1 + 4x_3 = 2$$
$$\underline{2x_1 + 14x_3 = 8}$$
$$-10x_3 = -6 \Rightarrow x_3 = \frac{6}{10} = \frac{3}{5}$$

$x_1 + 7 \times \frac{3}{5} = 4 \Rightarrow x_1 = 4 - \frac{21}{5} = -1/5$

$\therefore$ sol^n :- $X_5 = \begin{bmatrix} -1/5 \\ 0 \\ 3/5 \\ 0 \end{bmatrix}$ is a basic sol^n.

Case 6:- $x_3 = x_4 = 0$.

$$2x_1 + 3x_2 = 2$$
$$\underline{x_1 + x_2 = 4} \quad \times 2$$
$$2x_1 + 3x_2 = 2$$
$$\underline{2x_1 + 2x_2 = 8}$$
$$x_2 = -6$$

$x_1 - 6 = 4 \Rightarrow x_1 = 10$.

$\therefore$ sol^n, $X_B = \begin{bmatrix} 10 \\ -6 \\ 0 \\ 0 \end{bmatrix}$ is a basic sol^n.

24/11/2021.

Q. Solve Max $Z = 2x_1 + 2x_2 + 4x_3$

subjected to

$2x_1 + 3x_2 + x_3 \leq 300$.

$x_1 + x_2 + 3x_3 \leq 300$

$x_1 + 3x_2 + x_3 \leq 240$

where $x_1, x_2, x_3 \geq 0$. Use Simplex Method.

→ Step 1:-

Max $Z = 2x_1 + 2x_2 + 4x_3 + 0x_4 + 0x_5 + 0x_6$.

subjected to

$2x_1 + 3x_2 + x_3 + x_4 = 300$

$x_1 + x_2 + 3x_3 + x_5 = 300$

$x_1 + 3x_2 + x_3 + x_6 = 240$

$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$.

→ Step 2:- $n = 6$

$m = 3$

$n - m = 6 - 3 = 3$

Assume 3 variables to zero.

put $x_1 = x_2 = x_3 = 0$.

$\therefore x_4 = 300$; $x_5 = 300$; $x_6 = 240$.

→ Step 3:-

	C_j	2	2	4	0	0	0		
C_B	X_B	x_1	x_2	x_3	x_4	x_5	x_6	θ	
0	$x_4 = 300$	2	3	1	1	0	0	300	$R_1 \to R_1 - R_2$
0	$x_5 = 300$	1	1	3	0	1	0	100	$R_2 \to R_2/3$
0	$x_6 = 240$	1	3	1	0	0	1	240	$R_3 \to R_3 - R_2$
	$Z_j - C_j$	-2	-2	-4	0	0	0		
0	$x_4 = 200$	5/3	8/3	0	1	-1/3	0	120 ←	$R_1 \to R_1 \div \frac{5}{3}$
4	$x_3 = 100$	1/3	1/3	1	0	1/3	0	300	$R_2 \to R_2 - \frac{1}{3}R_1$
0	$x_6 = 140$	2/3	8/3	0	0	-1/3	1	210	$R_3 \to R_3 - \frac{2}{3}R_1$
	$Z_j - C_j$	-2/3	-2/3	0	0	4/3	0	1	
2	$x_1 = 120$	1	8/5	0	3/5	-1/5	0		
4	$x_3 = 60$	0	-1/5	1	-1/5	2/5	0		
0	$x_6 = 60$	0	8/5	0	-2/5	-1/5	1		
	$Z_j - C_j$	0	2/5	0	2/5	6/5	0		

All $Z_j - C_j \geq 0$.

$x_1 = 120$; $x_2 = 0$; $x_3 = 60$; $x_4 = x_5 = 0$; $x_6 = 60$

$\therefore$ Max $Z = 2x_1 + 2x_2 + 4x_3 + 0x_4 + 0x_5 + 0x_6$

$= 2 \times 120 + 2 \times 0 + 4 \times 60 + 0 + 0 + 0$

$= 240 + 240$

$= \underline{\underline{480}}$.

DATE: 25/11/2019

Q. Solve Max $z = 5x_1 + 7x_2$

subjected to

$2x_1 + x_2 \le 6$.

$3x_1 + 4x_2 \le 12$.

$x_1, x_2 \ge 0$.

→ Std form:-

Max $z = 5x_1 + 7x_2 + 0x_3 + 0x_4$,

subjected to

$2x_1 + x_2 + x_3 = 6$.

$3x_1 + 4x_2 + x_4 = 12$

$x_1, x_2, x_3, x_4 \ge 0$

$n = 4$

$m = 2$

$n - m = 4 - 2 = 2$

Assign any two variables to zero.

ie; Put $x_1 = x_2 = 0$.

$x_3 = 6$.

$x_4 = 12$

	C_j	5	7	0	0	
C_B	X_B	X_1	X_2	X_3	X_4	θ
0	$X_3 = 6$	2	1	1	0	$6/1 = 6$
0	$X_4 = 12$	3	4	0	1	$12/4 = 3$ ←
	$Z_j - C_j$	-5	-7 ↑	0	0	

$R_1 \to R_1 - R_2$

$R_2 \to \frac{R_2}{4}$

$2-\frac{3}{4}$

PAGE NO.
DATE $7 \cdot \frac{3}{4} = \frac{24}{4}$

	C_j	5	7	0	0
C_B	X_B	X_1	X_2	X_3	X_4
0	$X_3 = 3$	$5/4$	0	1	$-1/4$
7	$X_2 = 3$	$3/4$	1	0	$1/4$
	$Z_j - C_j$	$1/4$	0	0	$7/4$

$\therefore$ All $z_j - c_j \geq 0$.

$x_2 = 3$; $x_3 = 3$.

$$\text{Max } Z = 5x_1 + 7x_2 + 0x_3 + 0x_4$$
$$= 5 \times 0 + 7 \times 3 + 0 + 0$$
$$= \underline{\underline{21}}.$$

Q: Max $Z = 2x_1 + 4x_2$

subject to,

$2x_1 + x_2 \leq 10$.

$2x_1 + 2x_2 \leq 6$

$2x_1 + 2x_2 \geq -4$ $-2x_1 - 2x_2 \leq 4$

$-2x_1 + x_2 \leq 1$

$x_1, x_2 \geq 0$.

$\rightarrow$ Max $Z = 2x_1 + 4x_2 + 0x_3 + 0x_4 + 0x_5 + 0x_6$

$2x_1 + x_2 + x_3 = 10$

$2x_1 + 2x_2 + x_4 = 6$

$-2x_1 - 2x_2 + x_5 = 4$

$-2x_1 + x_2 + x_6 = 1$

$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$.

$n = 6$ $m = 4$

$n - m = 2$

$x_1 = x_2 = 0$;

$\Rightarrow x_3 = 10;\ x_4 = 6;\ x_5 = 4;\ x_6 = 1$

C_B	C_j → X_B	2 (x_1)	4 (x_2)	0 (x_3)	0 (x_4)	0 (x_5)	0 (x_6)	θ	
0	$x_3 = 10$	2	1	1	0	0	0	$\frac{10}{1} = 10$	$R_1 \to R_1 - R_4$
0	$x_4 = 6$	2	2	0	1	0	0	$\frac{6}{2} = 3$	$R_2 \to R_2 - 2R_4$
0	$x_5 = 4$	-2	-2	0	0	1	0	$\frac{4}{-2} = -2$	$R_3 \to R_3 + 2R_4$
0	$x_6 = 1$	-2	1	0	0	0	1	1 ←	
	$z_j - c_j$	-2	-4 ↑	0	0	0	0		
0	$x_3 = 6$	4	0	1	0	0	-1	$\frac{6}{4} = \frac{3}{2} = 1.5$	$R_1 \to R_1 - 4R_2$
0	$x_4 = 4$	6	0	0	1	0	-2	$\frac{4}{6} = \frac{2}{3} = 0.67$ ←	$R_2 \to \frac{R_2}{6}$
0	$x_5 = 6$	-6	0	0	0	1	2	$\frac{6}{-6} = -1$	$R_3 \to R_3 + 6R_2$
4	$x_2 = 1$	-2	1	0	0	0	1	$\frac{4}{-2} = -2$	$R_4 \to R_4 + 2R_2$
	$z_j - c_j$	-10 ↑	0	0	0	0	4		
0	$x_3 = 10/3$	0	0	1	$-2/3$	0	$1/3$		
2	$x_1 = 2/3$	1	0	0	$1/6$	0	$-1/3$		
0	$x_5 = 10$	0	0	0	1	1	0		
4	$x_2 = 7/3$	0	1	0	$1/3$	0	$1/3$		
	$z_j - c_j$	0	0	0	$4/3$	0	$4/3$		

$x_1 = 2/3 \quad x_2 = 7/3 \quad x_3 = 10/3 \quad x_5 = 10.$

$Z = 2 \times \frac{2}{3} + 4 \times \frac{7}{3} = \frac{4}{3} + \frac{28}{3} = \frac{32}{3}$

simplex:- IBFS in slack.
Big M :- IBFS in slack & artificial

30/11/2021

Big-M Method or Charnes Penalty Method:

9 Solve the following LPP

Max $z = 6x_1 - 3x_2 + 2x_3$

Subject to

$2x_1 + x_2 + x_3 \le 16$

$3x_1 + 2x_2 + x_3 \le 18$

$x_2 - 2x_3 \ge 8$

$x_1, x_2, x_3 \ge 0$.

→ Step 1: Convert given problem to std form.

Max $z = 6x_1 - 3x_2 + 2x_3 + 0x_4 + 0x_5 + 0x_6 - MA_1$

Subject to

$2x_1 + x_2 + x_3 + x_4 = 16$

$3x_1 + 2x_2 + x_3 + x_5 = 18$

$x_2 - 2x_3 - x_6 + A_1 = 8$

$x_1, x_2, x_3, x_4, x_5, x_6, A_1 \ge 0$

m - big M value for slack & artificial

$A_1 \to$ artificial variable

→ Step 2:- find initial basic feasible sol^n.

n = no. of variables = 7.

m = no. of eqns = 3

$\therefore n - m = 7 - 3 = 4$

Let $x_1 = x_2 = x_3 = x_6 = 0$.

Then $x_4 = 16$.

$x_5 = 18$

$A_1 = 8$

Assume big value, say 10,000 to m. And find which most -ve

C_B	X_B	x_1	x_2	x_3	x_4	x_5	x_6	A_1	θ	
	$\rightarrow C_j$	6	-3	2	0	0	0	$-m$		
0	$x_4=16$	2	1	1	1	0	0	0	16	$R_1 \to R_1 - R_3$
0	$x_5=18$	3	2	1	0	1	0	0	0	$R_2 \to R_2 - 2R_3$
$-m$	$A_1=8$	0	1	-2	0	0	-1	1	8 $\leftarrow$	
	$z_j - c_j$	-6	$3-m$ $\uparrow$	$2m-2$	0	0	m	0		
0	$x_4=8$	2	0	3	1	0	1	-1	4	$R_1 \to R_1 - 2R_2$
0	$x_5=2$	3	0	5	0	1	2	-2	$2/3$ $\leftarrow$	$R_2 \to \frac{R_3}{3}$
-3	$x_2=8$	0	1	-2	0	0	-1	1	∞	
	$z_j - c_j$	-6 $\uparrow$	0	4	0	0	3	$-3+m$		
0	$x_4=20/3$	0	0	$-1/3$	1	$-2/3$	$-1/3$	$1/3$		
6	$x_1=2/3$	1	0	$5/3$	0	$1/3$	$2/3$	$-2/3$		
-3	$x_2=8$	0	1	-2	0	0	-1	1		
	$z_j - c_j$	0	0	14	0	2	7	$m-7$		

$x_1 = \frac{2}{3}$; $x_2 = 8$; $x_4 = \frac{20}{3}$

$$\text{Max } Z = 6x_1 - 3x_2 + 2x_3 + 0x_4 + 0x_5 + 0x_6 - mA_1$$
$$= 6 \cdot \frac{2}{3} - 3 \cdot 8 + 0 + 0 + 0 + 0 - mA_1$$
$$= 4 - 24 - mA_1$$
$$= -20 - mA_1 \quad [\because A_1 = 0]$$
$$= -20$$

PAGE NO.
DATE:

Q. Solve ~~Max~~ Min $z = 2x_1 + 9x_2 + x_3$

subject to

$x_1 + 4x_2 + 2x_3 \geq 5$

$3x_1 + x_2 + 2x_3 \geq 4$

$x_1, x_2, x_3 \geq 0$.

→ Std form:-

Max $z = -(\text{Min } z)$

Max $z = -2x_1 - 9x_2 - x_3 + 0x_4 - mA_1 + 0x_5 - mA_2$

subject to constraints

$x_1 + 4x_2 + 2x_3 - x_4 + A_1 = 5$

$3x_1 + x_2 + 2x_3 - x_5 + A_2 = 4$

$x_1, x_2, x_3, x_4, x_5, A_1, A_2 \geq 0$

$n = 7$.

$m = 2$.

$n - m = 7 - 2$

$= 5$

Let $x_1 = x_2 = x_3 = x_4 = x_5 = 0$.

$A_1 = 5$

$A_2 = 4$.

$x_3 = 5/2 \quad x_5 = 1$

Max $z = -5/2$

min $z = 5/2$

C_B	X_B	X_1	X_2	X_3	X_4	X_5	A_1	A_2	θ	
	C_j	-2	-9	-1	0	0	$-m$	$-m$		
$-m$	$A_1=5$	1	4	2	-1	0	1	0	$5/4$	$R_1 \to \frac{R_1}{4}$
$-m$	$A_2=4$	3	1	2	0	-1	0	1	4	$R_2 \to R_2 - R_1$
	Z_j-C_j	$2-4m$	$9-5m$	$1-4m$	m	m	0	0		
-9	$X_2=5/4$	$1/4$	1	$1/2$	$-1/4$	0	$1/4$	0	5	
$-m$	$A_2=\frac{11}{4}$	$11/4$	0	$3/2$	$1/4$	-1	$-1/4$	1	1	
	Z_j-C_j	$\frac{-11m-1}{4}$	0	$\frac{-3m-1}{2}$	$\frac{9-m}{4}$	m	$-\frac{9+5m}{4}$	0		
-9	$X_2=1$	0	1	$4/11$	$-3/11$	$1/11$	$3/11$	$-1/11$	$11/4$	$R_1 \to R_1 - \frac{1}{4}R_2$
-2	$X_1=1$	1	0	$6/11$	$1/11$	$-4/11$	$-1/11$	$4/11$	$11/6$	$R_2 \to \frac{R_2}{11/4}$
	Z_j-C_j	0	0	$-37/11$	$25/11$	$-1/11$	$m-\frac{25}{11}$	$m+1/11$		
-9	$X_2=1/3$	$-2/3$	1	0	$-1/3$	$1/3$	$1/3$	$-1/3$	1	
-1	$X_3=11/6$	$11/6$	0	1	$1/6$	$-2/3$	$-1/6$	$2/3$	$-11/4$	
	Z_j-C_j	$37/6$	0	0	$17/6$	$-7/3$	$-17/6$	$2/3$		
0	$X_5=1$	-2	3	0	-1	1	1	-1		
-1	$X_3=5/2$	$1/2$	2	1	$-1/2$	0	$1/2$	0		
	Z_j-C_j	$3/2$	7	0	$1/2$	0	$m-1/2$	m		

$x_3 = 5/2$; $x_5 = 1$

Max $Z = -2x_1 - 9x_2 - x_3 + 0x_4 - mA_1 + 0x_5 - mA_2$

$= -5/2$

$\text{Min } z = 2x_1 + 9x_2 + x_3$

$= \underline{\underline{5/2}}$

Q. Solve Minimum $z = 4x_1 + x_2$

subject to the constraints

$3x_1 + x_2 = 3$

$4x_1 + 3x_2 \geq 6$

$x_1 + 2x_2 \leq 4$

$x_1, x_2 \geq 0.$

→ Standard form:-

$\text{Max } z = -4x_1 - x_2 + 0x_3 + 0x_4 - mA_1 - mA_2$

subject to

$3x_1 + x_2 + A_1 = 3$

$4x_1 + 3x_2 - x_3 + A_2 = 6$

$x_1 + 2x_2 + x_4 = 4$

$x_1, x_2, x_3, x_4, A_1, A_2 \geq 0.$

$n = 6$

$m = 3$

$n - m = 6 - 3 = 2.$

Let $x_1 = x_2 = x_3 = 0$

$A_1 = 3$

$A_2 = 6$

$x_4 = 4.$

C_B	X_B	x_1	x_2	x_3	x_4	A_1	A_2	θ
	C_j	-4	-1	0	0	$-M$	$-m$	
$-m$	$A_1=3$	$\boxed{3}$	1	0	0	1	0	1 ←
$-m$	$A_2=6$	4	3	-1	0	0	1	$3/2$
0	$x_4=4$	1	2	0	1	0	0	4
	Z_j-C_j	$-7m+4$ ↑	$-4m+1$	m	0	0	0	
-4	$x_1=1$	1	$1/3$	0	0	$1/3$	0	3
$-m$	$A_2=6$	0	$5/3$	-1	0	$-4/3$	1	$6/5$ ←
0	$x_4=4$	0	$5/3$	0	1	$-1/3$	0	$9/5$
	Z_j-C_j	0	$\frac{-5m-1}{3}$ ↑	m	0	$\frac{7m-4}{3}$	0	
-4	$x_1=3/5$	1	0	$1/5$	0	$3/5$	$-1/5$	3
-1	$x_2=6/5$	0	1	$-3/5$	0	$-4/5$	$3/5$	—
0	$x_4=1$	0	0	$\boxed{1}$	1	1	-1	1 ←
	Z_j-C_j	0	0	$-1/5$ ↑	0	$m-\frac{4}{5}$	$m+1/5$	
-4	$x_1=3/5$	1	0	0	$-2/5$	$2/5$	0	
-1	$x_2=9/5$	0	1	0	$3/5$	$-1/5$	0	
0	$x_3=1$	0	0	1	1	1	-1	
	Z_j-C_j	0	0	0	1	$m-7/5$	m	

$\frac{-8-9}{5}$

$= -17/5$

$\therefore x_1 = 3/5\ ;\ x_2 = 9/5\ ;\ x_3 = 1$

$\text{Max } z = -4 \cdot \frac{3}{5} - \frac{9}{5} = -17/5$

$\therefore \text{Min } z = \frac{17}{5}$ //

6/12/2021

Module - 2.

Duality in LPP:

Statement of Duality Theorem

Statement of complementary slackness theorem.

The primal-duality solns using simplex method.

Revised simplex method.

→ Duality in LPP :-

Associated with every LPP there is another problem called duality problem. The original LPP is called the primal problem. The optimal solns to the dual gives complete infmn about the optimum soln of the primal and vice versa.

Canonical Form :-

The canonical form

$$\text{Max } Z = CX$$

subject to,

$$AX \le B$$

$$X \ge 0.$$

Here B can be both +ve & -ve.

Q. Convert the given LPP into canonical form.

(a) $\text{Min } Z = 2x_1 + 3x_2$

subject to

$$x_1 + x_2 \le 4$$

$$2x_1 - x_2 \ge 3$$

$$3x_1 + 4x_2 = 10$$

$x_1 \ge 0$, x_2 unrestricted in sign.

[if $a = b$ then $a \le b$ & $a \ge b$.]

→ Since x_2 is unrestricted in sign.

Let $x_2 = x_2' = x_2''$, where $x_2', x_2'' \ge 0$.

Primal form:-

$\text{Min } z = 2x_1 + 3x_2' - 3x_2''$

subject to

$x_1 + x_2' - x_2'' \le 4$

$2x_1 - x_2' + x_2'' \ge 3$

$3x_1 + 4x_2' - 4x_2'' \le 10$

$3x_1 + 4x_2' - 4x_2'' \ge 10.$

$x_1, x_2', x_2'' \ge 0.$

Canonical form:-

$\text{Max } z = -(\text{Min } z)$

$\text{Max } z = -2x_1 - 3x_2' + 3x_2''$

Subject to

$x_1 + x_2' - x_2'' \le 4$

$-2x_1 + x_2' - x_2'' \le -3$

$3x_1 + 4x_2' - 4x_2'' \le 10$

$-3x_1 - 4x_2' + 4x_2'' \le -10.$

$x_1, x_2', x_2'' \ge 0.$

(b) $\text{Max } z = 3x_1 + 17x_2 + 9x_3$

Subject to

$x_1 - x_2 + x_3 \ge 3$

$-3x_1 + 2x_2 \le 1$

$x_1, x_2 \ge 0$; x_3 - unrestricted in sign.

$\longrightarrow x_3 = x_3' - x_3''$

Primal form:-

Max $Z = 3x_1 + 17x_2 + 9x_3' - 9x_3''$

subject to

$x_1 - x_2 + x_3' - x_3'' \geq 3$

$-3x_1 + 2x_2 \leq 1$

$x_1, x_2, x_3', x_3'' \geq 0$

Canonical form:-

Max $Z = 3x_1 + 17x_2 + 9x_3' - 9x_3''$

subject to,

$-x_1 + x_2 - x_3' + x_3'' \leq -3$

$-3x_1 + 2x_2 \leq 1$

$x_1, x_2, x_3', x_3'' \geq 0.$

(c) Min $Z = 3x_1 - 2x_2 + 4x_3$

subject to

$3x_1 + 5x_2 + 4x_3 \geq 7.$

$6x_1 + x_2 + 3x_3 \geq 4$

$7x_1 - 2x_2 - x_3 = 10$

$x_2 \leq 20$

$x_1 - 2x_2 + 5x_3 \geq 3$

$4x_1 + 7x_2 - 2x_3 \geq 2$

$x_1 \geq 0$, x_2 & x_3 unrestricted.

$\rightarrow$ $x_2 = x_2' - x_2''$

$x_3 = x_3' - x_3''$

Primal form:-

$$\text{Min } z = 3x_1 - 2x_2' + 2x_2'' + 4x_3' - 4x_3''$$

subject to

$$3x_1 + 5x_2' - 5x_2'' + 4x_3' - 4x_3'' \geq 7$$
$$6x_1 + x_2' - x_2'' + 3x_3' - 3x_3'' \geq 4$$
$$7x_1 - 2x_2' + 2x_2'' - x_3' + x_3'' = 10$$
$$x_2' - x_2'' \leq 20$$
$$x_1 - 2x_2' + 2x_2'' + 5x_3' - 5x_3'' \geq 3$$
$$4x_1 + 7x_2' - 7x_2'' - 2x_3' + 2x_3'' \geq 2$$
$$x_1, x_2', x_2'', x_3', x_3'' \geq 0.$$

Canonical form:-

$$\text{Max } z = -3x_1 + 2x_2' - 2x_2'' - 4x_3' + 4x_3''$$

subject to

$$-3x_1 - 5x_2' + 5x_2'' - 4x_3' + 4x_3'' \leq -7$$
$$-6x_1 - x_2' + x_2'' - 3x_3' + 3x_3'' \leq -4$$
$$7x_1 - 2x_2' + 2x_2'' - x_3' + x_3'' \leq 10.$$
$$7x_1 - 2x_2' + 2x_2'' - x_3' + x_3'' \geq 10.$$
$$-7x_1 + 2x_2' - 2x_2'' + x_3' - x_3'' \leq -10$$
$$x_2' - x_2'' \leq 20$$
$$-x_1 + 2x_2' - 2x_2'' - 5x_3' + 5x_3'' \leq -3$$
$$-4x_1 - 7x_2' + 7x_2'' + 2x_3' - 2x_3'' \leq -2$$
$$x_1, x_2', x_2'', x_3', x_3'' \geq 0.$$

Dual:-

Let the primal be in canonical form.

Max $Z = c_1x_1 + c_2x_2 + \dots + c_nx_n$

subject to

$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \le b_1$

$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \le b_2$

$\vdots$

$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \le b_m$

where

$x_1, x_2, x_3 \dots x_m \ge 0$

Then dual of the problem is

Min $W = b_1y_1 + b_2y_2 + \dots + b_my_m$

subject to

$a_{11}y_1 + a_{21}y_2 + a_{31}y_3 + \dots + a_{m1}y_m \ge c_1$

$a_{12}y_1 + a_{22}y_2 + a_{32}y_3 + \dots + a_{m2}y_m \ge c_2$

$\vdots$

$a_{1n}y_1 + a_{2n}y_2 + a_{3n}y_3 + \dots + a_{mn}y_m \ge c_n$

$y_1, y_2 \dots y_m \ge 0$.

* Canonical form $\Rightarrow$ Dual form

Max $Z = CX$	Min $W = B^TY$
$AX \le B$	$A^TY = C^T$
$X \ge 0$	$Y \ge 0$.

9/12/2021

? find the given dual of

$\text{Min } z = 12x_1 + 3x_2 + 7x_3$

Subjected to

$6x_1 - 2x_2 + 5x_3 \geq 3$

$-2x_1 - 3x_2 + 4x_3 \leq 2$

$3x_1 + 9x_2 + x_3 \geq 8$

$x_1, x_2, x_3 \geq 0.$

→ Primal problem can be converted to canonical form :-

$\text{Max } z = -(\text{Min } z)$

$\text{Max } z = -12x_1 - 3x_2 - 7x_3$

subject to

$-6x_1 + 2x_2 - 5x_3 \leq -3$

$-2x_1 - 3x_2 + 4x_3 \leq 2$

$-3x_1 - 9x_2 - x_3 \leq -8$

$x_1, x_2, x_3 \geq 0.$

The dual of the problem is

$\text{Min } w = -3y_1 + 2y_2 - 8y_3$

subject to

$-6y_1 - 2y_2 - 3y_3 \geq -12$

$2y_1 - 3y_2 - 9y_3 \geq -3$

$-5y_1 + 4y_2 - y_3 \geq -7$

$y_1, y_2, y_3 \geq 0.$

Q. find constructed dual of

$$\text{Max } z = 3x_1 + 5x_2$$

subject to

$$2x_1 + 6x_2 \le 50$$
$$3x_1 + 2x_2 \le 35$$
$$5x_1 + 3x_2 \le 10$$
$$x_2 \le 20$$
$$x_1, x_2 \ge 0.$$

→ The primal problem is in canonical form. so dual is

$$\text{Min } W = 50y_1 + 35y_2 + 10y_3 + 20y_4$$

subject to

$$2y_1 + 3y_2 + 5y_3 \ge 3$$
$$6y_1 + 2y_2 + 3y_3 + y_4 \ge 5$$
$$y_1, y_2, y_3, y_4 \ge 0.$$

Q. Construct dual of

$$\text{Min } z = 3x_1 - 2x_2 + 4x_3$$

subject to

$$3x_1 + 5x_2 + 4x_3 \ge 7$$
$$6x_1 + x_2 + 3x_3 \ge 4$$
$$7x_1 - 2x_2 - x_3 \le 10.$$
$$x_2 \le 20$$
$$x_1 - 2x_2 + 5x_3 \ge 3$$
$$4x_1 + 7x_2 - 2x_3 \ge 2$$
$$x_1, x_2, x_3 \ge 0.$$

→ Max $z = -3x_1 + 2x_2 - 4x_3$

subject to

$-3x_1 - 5x_2 - 4x_3 \le -7$

$-6x_1 - x_2 - 3x_3 \le -4$

$7x_1 - 2x_2 - x_3 \le 10$

$x_2 \le 20$

$-x_1 + 2x_2 - 5x_3 \le -3$

$-4x_1 - 7x_2 + 2x_3 \le -2$

$x_1, x_2, x_3 \ge 0$.

Dual is :-

Min $W = -7y_1 - 4y_2 + 10y_3 + 20y_4 - 3y_5 - 2y_6$.

subject to.

$-3y_1 - 6y_2 + 7y_3 + 0y_4 - y_5 - 4y_6 \ge -3$

$-5y_1 - y_2 - 2y_3 + y_4 + 2y_5 - 7y_6 \ge 2$

$-4y_1 - 3y_2 - y_3 + 0y_4 - 5y_5 + 2y_6 \ge -4$

$y_i \ge 0$; $i = 1, 2, 3, 4, 5, 6$.

⇒ Dual Problem when primal is in Standard form :-

Q. Write the dual of

Max $z = x_1 - 2x_2 + 3x_3$

subjected to

$-2x_1 + x_2 + 3x_3 = 2$

$2x_1 + 3x_2 + 4x_3 = 1$

$x_1, x_2, x_3 \ge 0$.

→ Primal is in canonical form:-

Max $z = x_1 - 2x_2 + 3x_3$

subjected to

$-2x_1 + x_2 + 3x_3 \le 2$

$2x_1 - x_2 - 3x_3 \le -2$

$2x_1 + 3x_2 + 4x_3 \le 1$

$2x_1 + 3x_2 + 4x_3 \ge 1$

$-2x_1 - 3x_2 - 4x_3 \le -1$

$x_1, x_2, x_3 \ge 0.$

Dual is

Min $z = 2y_1 - 2y_2 + y_3 - y_4$

Subjected to

$-2y_1 + 2y_2 - 2y_3 - 2y_4 \ge 1$

$y_1 - y_2 + 3y_3 - 3y_4 \ge -2$

$3y_1 - 3y_2 + 4y_3 - 4y_4 \ge 3$

$y_1, y_2, y_3, y_4 \ge 0.$

Let $y_1 - y_2 = k_1$

$y_3 - y_4 = k_2$

Then Min $z = 2k_1 + k_2$

Subject to

$-2k_1 + 2k_2 \ge 1$

$k_1 + 3k_2 \ge -2$

$3k_1 + 4k_2 \ge 3$

where k_1 & k_2 are unrestricted in sign.

Note:-

If the primal constraint is an eqn corresponding dual variable will be unrestricted in sign. Also if the primal variable is unrestricted in sign, the corresponding dual constraint will be an eqn.

Primal	Dual
• one unrestricted	→ one eqn
• one eqn	→ one unrestricted variable
• 2 unrestricted	→ 2 eqn (=)
• 2 eqns	→ 2 unrestricted
• one eqn, one unrestricted	→ one unrestricted, one eqn.

* ? find dual of

$\text{Min } z = 2x_1 + 3x_2 + 4x_3$

subjected to

$2x_1 + 3x_2 + 5x_3 \geq 2$

$3x_1 + x_2 + 7x_3 = 3$

$x_1 + 4x_2 + 6x_3 \leq 0$

$x_1, x_2 \geq 0$; x_3 unrestricted.

→ Canonical form of primal is

$\text{Max } z = -2x_1 - 3x_2 - 4x_3$

subjected to

$-5(x_3' - x_3'')$
$-5x_3' + 5x_3''$
$x_3 = x_3' - x_3''$

$-2x_1 - 3x_2 - 5x_3 \leq -2$
$3x_1 + x_2 + 7x_3 \leq 3$
$3x_1 + x_2 + 7x_3 \geq 3$
$\rightarrow -3x_1 - x_2 - 7x_3 \leq -3$
$x_1 + 4x_2 + 6x_3 \leq 0$
$x_1, x_2 \geq 0$; x_3 - unrestricted.

Max $z = -2x_1 - 3x_2 - 4x_3' + 4x_3''$
subjected to
$-2x_1 - 3x_2 - 5x_3' + 5x_3'' \leq -2$
$3x_1 + x_2 + 7x_3' - 7x_3'' \leq 3$
$-3x_1 - x_2 - 7x_3' + 7x_3'' \leq -3$
$x_1 + 4x_2 + 6x_3' - 6x_3'' \leq 0.$
$x_1, x_2, x_3', x_3'' \geq 0.$
Dual is :-
Min $W = -2y_1 + 3y_2 - 3y_3 + 0y_4$
subject to.
$-2y_1 + 3y_2 - 3y_3 + y_4 \geq -2$
$-3y_1 + y_2 - y_3 + 4y_4 \geq -3$
$-5y_1 + 7y_2 - 7y_3 + 6y_4 \geq -4$
$5y_1 - 7y_2 + 7y_3 - 6y_4 \geq 4$ $\quad y_2 - y_3 = y_5.$
$y_1, y_2, y_3, y_4 \geq 0.$
Let $y_2 - y_3 = y_5$
Then;

Min $w = -2y_1 + 3y_5 + 0y_4$

Subjected to

$-2y_1 + 3y_5 + y_4 \geq -2$

$-3y_1 + y_5 + 4y_4 \geq -3$

$5y_1 - 7y_5 - 6y_4 = 4$ $\begin{cases} -5y_1 + 7y_5 + 6y_4 \geq -4 \\ 5y_1 - 7y_5 - 6y_4 \geq 4 \end{cases}$

$y_1, y_4 \geq 0$; $y_5 \rightarrow$ unrestricted.

14/10/2021

⇒ Solution of primal & dual problem using Simplex Method:-

Acco. to fundamental theorem of duality if either the primal or dual problem has a optimum solⁿ then the other problem also have an optimum solⁿ. And optimum value of the objective funcⁿ in both problems are same.

If either problem has a unbounded solⁿ, then the other has no feasible or has an unbounded solⁿ.

The value of the primal (dual) variable can be directly read off from the dual (primal) simplex table as follows:

1. If the primal (dual) variable corresponds to a slack & or surplus variable in the

dual (primal), its value is read off as the net evaluation of the slack / surplus variable from optimal dual (primal) simplex table.

2. If the primal (dual) corresponds to a artificial variable in dual (primal) problems its value is read off as the net evaluation of the artificial variable after deleting the factor containing M from the original dual (primal) simple table.

Q. Solve using principle of duality

Min $z = 3x_1 + 5x_2$

subjected to

$2x_1 + 8x_2 \geq 40$

$3x_1 + 4x_2 \geq 50$

$x_1, x_2 \geq 0$.

→ Canonical form is

Max $z = -(\text{Min } z)$

Max $z = -3x_1 - 5x_2$

subjected to

$-2x_1 - 8x_2 \leq -40$

$-3x_1 - 4x_2 \leq -50$

$x_1, x_2 \geq 0$.

Dual is

Min $W = -40y_1 - 50y_2$

subjected to

$-2y_1 - 3y_2 \geq -3 \rightarrow 2y_1 + 3y_2 \leq 3$

$-8y_1 - 4y_2 \geq -5 \rightarrow 8y_1 + 4y_2 \leq 5$

$y_1, y_2 \geq 0.$

Standard form is

Max $W = -$Min W

Max $W = 40y_1 + 50y_2$

subjected to

$2y_1 + 3y_2 + y_3 = 3$

$8y_1 + 4y_2 + y_4 = 5$

$y_1, y_2, y_3, y_4 \geq 0.$

$n = 4 \; ; \; m = 2$

$n - m = 4 - 2 = 2$

$y_1 = y_2 = 0$

$\Rightarrow y_3 = 3$

$y_4 = 5$

Max $W = 40y_1 + 50y_2$ $\quad y_1 = 3/16$

$= 40 \times \frac{3}{16} + 50 \times \frac{7}{8}$ $\quad y_2 = 7/8$

$= \frac{120}{16} + \frac{350}{8} = \frac{205}{4}$

Min $Z = \frac{-205}{4}$

PAGE NO.
DATE

C_B	x_B	y_1	y_2	y_3	y_4	θ	
	C_j	40	50	0	0		
0	$y_3 = 3$	2	3	1	0	1	$\leftarrow$
0	$y_4 = 5$	8	4	0	1	$5/4$	
	$z_j - c_j$	-40	-50 $\uparrow$	0	0		
50	$y_2 = 1$	$2/3$	1	$1/3$	0	$3/2$	$R_1 \to R_1/3$
0	$y_4 = 1$	$16/3$	0	$-4/3$	1	$3/16$ $\leftarrow$	$R_2 \to R_2 - 4R_1$
	$z_j - c_j$	$-20/3$ $\uparrow$	0	$50/3$	0		
50	$y_2 = 7/8$	0	1	$1/2$	$-1/8$		$R_1 \to R_1 - \frac{2}{3} R_2$
40	$y_1 = 3/16$	1	0	$-1/4$	$3/16$		$R_2 \to R_2 \times \frac{3}{16}$
	$z_j - c_j$	0	0	15	$5/4$		

16/12/2021.

From the dual table, the primal sol^n is obtained from

$x_1 = 15$; $x_2 = 5/4$

$\therefore$ Min $z = 3x_1 + 5x_2$

$= 3 \times 15 + 5 \times 5/4$

$= \frac{205}{4}$.

? Solve the following using dual principle

Max $z = 40x_1 + 35x_2$

subject to

$$2x_1 + 3x_2 \leq 60$$
$$4x_1 + 3x_2 \leq 96$$
$$x_1, x_2 \geq 0.$$

→ Dual is

Min $W = 60y_1 + 96y_2$

Subjected to

$$2y_1 + 4y_2 \geq 40$$
$$3y_1 + 3y_2 \geq 35$$
$$y_1, y_2 \geq 0.$$

Std form of dual is

Max $W = -60y_1 - 96y_2 + 0y_3 + 0y_4 - mA_1 - mA_2$

subject to

$$2y_1 + 4y_2 - y_3 + A_1 = 40$$
$$3y_1 + 3y_2 - y_4 + A_2 = 35$$
$$y_1, y_2, y_3, y_4, A_1, A_2 \geq 0.$$

Initial basic feasible solⁿ is:

$n = 6$; $m = 2$; $n - m = 6 - 2 = 4$

put $y_1 = y_2 = y_3 = y_4 = 0$

$\therefore A_1 = 40$

$A_2 = 35$

C_B	X_B \ C_j	-60	-96	0	0	$-m$	$-m$	
	X_B	y_1	y_2	y_3	y_4	A_1	A_2	θ
$-m$	$A_1=40$	2	4	-1	0	1	0	$\frac{40}{4}=10 \leftarrow$ $R_1/4$
$-m$	$A_2=35$	3	3	0	-1	0	1	$\frac{35}{3}$ $R_2 \to R_2-3R_1$
	Z_j-C_j	$60-5m$	$96-7m$ $\uparrow$	m	m	0	0	
-96	$y_2=10$	$1/2$	1	$-1/4$	0	$1/4$	0	20
$-m$	$A_2=5$	$3/2$	0	$3/4$	-1	$-3/4$	1	$10/3 \leftarrow$ $R_2 \to R_2 \cdot \frac{2}{3}$
	Z_j-C_j	$\frac{12-3m}{2}$ $\uparrow$	0	$\frac{24-3m}{4}$	m	$\frac{-24+7m}{4}$	0	$R_1 \to R_1-\frac{1}{2}R_2$
-96	$y_2=25/3$	0	1	$-1/2$	$1/3$	$1/2$	$-1/3$	
-60	$y_1=10/3$	1	0	$1/2$	$-2/3$	$-1/2$	$2/3$	
	Z_j-C_j	0	0	18	8	$m-18$	$m-8$	

$$\begin{array}{r} 32\times \\ 25 \\ \hline 160 \\ 64 \\ \hline 800 \end{array}$$

$y_2 = 25/3 \quad ; \quad y_1 = 10/3$.

$$\text{Max } W = -60 \times \frac{10}{3} - 96 \times \frac{25}{3}$$

$$= -200 - 800 = \underline{-1000}$$

Also $A_1 = m-18 \Rightarrow x_1 = -18$

$A_2 = m-8 \Rightarrow x_2 = -8$.

$\therefore$ Max $z = 40x_1 + 35x_2$

$= 40 \times -18 + 35 \times -8$

$= \underline{-1000}$.

20/12/2021

⇒ Duality Theorems:-

Imp ① Theorem 1:-

The dual of the dual is primal.

Proof:-

Let the primal problem in canonical form is Max $z = CX$

Subject to $AX \leq B$

$X \geq 0$

Then the dual of the problem is

$$\text{Min } W = B^T Y$$

Subject to

$$A^T Y \geq C^T$$

$$Y \geq 0$$

The canonical form of above dual is

$$\text{Max } W' = -B^T Y$$

Subject to

$$-A^T Y \leq -C^T$$

$$Y \geq 0$$

The dual of dual is

$$\text{Min } W'' = (-C^T)^T P$$

Subject to

$$(-A^T)^T \geq (-B^T)^T$$

$$P \geq 0$$

ie; Min $W'' = -CP$ $\quad (A^T)^T = 0$

Subject to

$-AP \geq -B$

$P \geq 0$.

ie; Max $W'' = CP$

Subject to $AP \leq B$

$P \geq 0$.

ie; The canonical form of dual of dual is primal.

(2) Theorem 2 :-

If either the primal or the dual problem has an unbounded soln, then the solns to the other problem is infeasible.

(3) Theorem 3 :-

If the dual problem has a degenerate soln, the primal problems has alternate optimums.

(4) Theorem 4 :- Strong Duality Theorem/- Fundamental Theorem of Duality :-

If both the primal and dual problem has feasible soln, then both have ~~optimum~~ optimal solns

OR

If the primal problem have an optimal soln, then the dual also have an optimal soln and vice versa.

Q. Max $Z = 5x_1 + 8x_2$

Subject to

$x_1 - 2x_2 \le 2$

$-x_1 + 4x_2 \le 1$

$x_1 + x_2 \le 2$

$x_1, x_2 \ge 0$.

→ Dual is

Min $W = 2y_1 + y_2 + 2y_3$

Subject to

$y_1 - y_2 + y_3 \ge 5 \Rightarrow -y_1 + y_2 - y_3 \le -5$

$-2y_1 + 4y_2 + y_3 \ge 8 \Rightarrow 2y_1 - 4y_2 - y_3 \le -8$

$y_1, y_2, y_3 \ge 0$.

Standard form is

Max $W = -$ Min W

Max $W = -2y_1 - y_2 - 2y_3$

Subject to

$-y_1 + y_2 - y_3 = -5$

$2y_1 - 4y_2 - y_3 = -8$

21/12/2021

⑤ Weak Duality Theorem:-

For a maximisation problem, every feasible solⁿ to the dual has a objective funcⁿ value greater than or equal to every feasible solⁿˢ to the primal.

⑥ Optimality Criterion Theorem:-

If the primal and dual have a feasible solⁿ with same value of the objective funcⁿ, then both are optimal to the primal and dual respectively.

⑦ Main Duality Theorem:-

If the primal and dual have feasible solⁿˢ, then both have optimal solⁿˢ with same value of the objective funcⁿ

⑧ Fundamental Theorem of LPP:-

Every LPP is either feasible or unbounded or infeasible.

Imp ⇒ Complementary Slackness Theorem:-

If X^* and Y^* are the optimal solⁿˢ to the primal and dual respectively and U^* and V^* are the values of the primal and dual slack variables at the optimum, then $X^*V^* + Y^*U^* = 0$

Module - 3.

Transportation Problem - Soln of transportation problem - finding an IBFS - North West Corner method - Matrix Minima method - Vogel's approxn method - Test for Optimality - MODI method - Unbalanced transportation problem - Maximisation of TP -

Assignment Problem - Optimal Soln - Hungarian method of assignment - Maximisation in assignment problem.

Module-3.

Koenig's Theorem:-

It states that in a matrix consisting of elements of 2 types, the max no. of independent elements of any one type is equal to the min no. of lines drawn through rows & columns which cover all elements of that type.

OR.

Max no. of independent zeroes in any matrix is equal to min no. of lines through rows & columns which cover all zeroes in the matrix.

⇒ Assignment Problem:-

Let c_{ij} denote cost if i^{th} job is assigned to j^{th} machine. The $n \times n$ matrix $[c_{ij}]$ is called cost matrix.

Let $x_{ij} = \begin{cases} 1 ; \text{ if } i^{th} \text{ job assigned to } j^{th} \text{ machine} \\ 0 ; \text{ otherwise.} \end{cases}$

Mathematically,

$$\text{Min } Z = \sum_{j=1}^{n} \sum_{i=1}^{n} c_{ij} x_{ij}$$

subject to

- $\sum_{i=1}^{n} x_{ij} = 1, \quad j = 1, 2 \dots n$ (one machine is assigned only one job)
- $\sum_{j=1}^{n} x_{ij} = 1, \quad i = 1, 2 \dots n$ (one job is assigned to only one machine)

$; x_{ij} = 0 \text{ or } 1$

⇒ Assignment Problem
[Hungarian Technique - Minimization Case] :-

Steps:-

1. Check whether no. of rows & columns of the cost matrix are equal (ie; problem is balanced). If not balance it by adding suitable no. of dummy rows or columns with zero cost.
2. Subtract from every row its smallest element. Subtract from every column its smallest element.
3. Draw least no. of horizontal & vertical lines to cover all the zeroes. If no. of lines is equal to no. of rows or columns, then go to step 5.
4. Identify smallest uncovered element (not covered by the line). Subtract it from all the uncovered elements and add the same to the element at the intersection of lines. We get a reduced matrix. Go to step 3.
5. Examine rows successively to find one with exactly one unmarked zero. Make an assignment there by encircling the zero by O to show that they can't be considered for future assignment. Continue until

all rows have been examined. Repeat this also for columns.

6. Repeat step 5 successively until -
   (i) no unmarked zero is left or
   (ii) There lie more than one unmarked zero in one row or column.

In case (i) algorithm stops. In case (ii), encircle any one zero arbitrarily and cancel all zeroes in that row & column. Go to step 5.

7. We now have exactly one encircled zero in each row & column. The assignment schedule corr. to these zeroes is the optimal assignment.

Note → In case of maximisation problem, it can be converted to minimisation problem by any one of the following:-
① Multiply every element of matrix by -1
② Subtract every element of matrix from max element.

Q. Solve

	Machines				
	I	II	III	IV	V
Jobs 1	11	17	8	16	20
2	9	7	12	6	15
3	13	16	15	12	16
4	21	24	17	28	26
5	14	10	12	11	15

→ (1) no. of rows = no. of columns
∴ given problem is balanced.

(2) (i) Subtract smallest of ~~each~~ element of each row from all its elements.

$$\begin{array}{c|c} \begin{bmatrix} 3 & 9 & 0 & 8 & 12 \\ 3 & 1 & 6 & 0 & 9 \\ 1 & 4 & 3 & 0 & 4 \\ 4 & 7 & 0 & 11 & 9 \\ 4 & 0 & 2 & 1 & 5 \end{bmatrix} & \begin{matrix} -8 \\ -6 \\ -12 \\ -17 \\ -10 \end{matrix} \end{array}$$

(ii) Subtract smallest element of each COLUMN from all its elements.

$$\begin{array}{c} \begin{bmatrix} 2 & 9 & 0 & 8 & 8 \\ 2 & 1 & 6 & 0 & 5 \\ 0 & 4 & 3 & 0 & 0 \\ 3 & 7 & 0 & 11 & 5 \\ 3 & 0 & 2 & 1 & 1 \end{bmatrix} \\ \begin{matrix} -1 & -0 & -0 & -0 & -4 \end{matrix} \end{array}$$

(3) Draw min no. of lines to cover zeroes. If all rows & cols have more than one zero, draw line which covers most no. of zeroes.

$$\begin{array}{ccccc|c} 2 & 9 & 0 & 8 & 8 & \\ 2 & 1 & 6 & 0 & 5 & \\ 0 & 4 & 3 & 0 & 0 & L_4 \\ 3 & 7 & 0 & 11 & 5 & \\ 3 & 0 & 2 & 1 & 1 & \\ \hline & L_3 & L_1 & L_2 & & \end{array}$$

PAGE NO.
DATE:

(4) no. of lines = 4 ≠ no. of rows = 5.
Min among uncovered element = 1.
Subtract it from all uncovered elements and add it where lines cross.

step no: 4:

$$\begin{array}{ccccccl} 1 & 9 & 0 & 8 & 7 & \\ 1 & 1 & 6 & 0 & 4 & \\ \cancel{0} & 5 & 1 & 1 & 0 & L_3 \\ 2 & 7 & 0 & 11 & 4 & \\ \cancel{2} & 0 & 2 & 1 & 0 & L_4 \\ & & L_1 & L_2 & & \end{array}$$

add 1 to intersected elements

subtract 1 from uncovered elements.

no. of lines = 4 ≠ 5. Repeat step 4.
Min among uncovered = 4

$$\begin{array}{cccccl} 0 & 8 & 0 & 8 & 6 & \\ 0 & 0 & 6 & 0 & 3 & L_2 \\ 0 & 5 & 5 & 2 & 0 & L_5 \\ 1 & 6 & 0 & 11 & 3 & \\ 2 & 0 & 3 & 2 & 0 & L_4 \\ L_3 & & L_1 & & & \end{array}$$

(5) no. of lines = no. of rows = 5.
Now make assignments. Write matrix with only zeroes in their positions.

0	-	~~0~~	-	-
~~0~~	~~0~~	-	0	-
~~0~~	-	-	-	0
-	-	0	-	-
-	0	-	-	~~0~~

→ contains only one zero.

Row 4 - only one zero. Mark it & cancel all other zeroes in that column.
Now in Row 1 - only one unmarked zero. Mark it & cancel all other zeroes in that column.

Finally,

0		x		
x	x		0	
x				0
–		0		
–	0			x

(6) Assignment & corresponding costs are :-

Assignment	Cost.
(1, I)	11
(2, IV)	6
(3, V)	16
(4, III)	17
(5, II)	10
	60. ⟹ Total cost.

Qn → To find max
→ To convert max problem to min problem, subtract every element of given cost matrix from the max element.

* Unbalanced → add a dummy row or column (that job is declined since it is assigned to dummy machine)

→ Transportation Problem :-

Basic steps of soln procedure for a TP :-

i) find an initial basic feasible soln

ii) Test for optimality

iii) If optimal, stop. Otherwise

iv) find an improved basic feasible soln.

v) Go to step (ii).

* To find IBFS 2 methods :-

(NWCR) ① Northwest Corner Rule [For balanced TP] :-

north west corner of transportation table is the cell (1,1).

* Procedure :-

The north west corner of transportation table is cell (1,1). Allocate as many as possible to the cell (1,1). Max amt that can be allocated is the smallest of availability or requirement. The satisfied row/column is deleted and a new table is obtained. If a col & a row are satisfied simultaneously delete both. Adjust amts of availability & requirement. Consider north west corner of new table for allotment. Repeat the process till all the rim requirements are satisfied. This method is quick & easy way to find an IBFS.

Q. Obtain an IBFS to the following TP using NWCR.

	D_1	D_2	D_3	D_4	a_i
O_1	6	4	1	5	14
O_2	8	9	2	7	16
O_3	4	3	6	2	5
b_j	6	10	15	4	35

→ ∵ total supply = total demand = 35

∴ problem is balanced.

min

(6) 6	(8) 4	1	5	~~14~~ ~~8~~ 0 14 - 6 = 8
8	(2) 9	(14) 2	7	~~16~~ ~~14~~ 0
4	3	(1) 6	(4) 2	~~5~~ ~~4~~ 0
~~6~~ 0	~~10~~ ~~2~~ 0	~~15~~ ~~1~~ 0	~~4~~ 0	35

$$\text{Total cost} = (6\times6) + (8\times4) + (2\times9) + (14\times2) + (4\times2) + (1\times6)$$
$$= 36 + 32 + 18 + 28 + 8 + 6$$
$$= 128$$

m → rows
n → cols

no. of allocations = 6.

$m + n - 1 = 3 + 4 - 1 = 6$.

∴ above BFS is nondegenerate.

	D_1	D_2	D_3	D_4	
O_1	2	4	3	6	20
O_2	7	3	8	9	10
O_3	2	2	9	11	15
b_j	15	15	8	7	45

→ Total supply = total demand = 45

∴ balanced.

(15) 2	(5) 4	3	6	~~20~~ ~~5~~ 0
7	(10) 3	8	9	~~10~~ 0
2	2	(8) 9	(7) 11	~~15~~ ~~7~~ 0
~~15~~ 0	~~15~~ ~~10~~ 0	~~8~~ 0	~~7~~ 0	

Total cost = $(2\times15) + (5\times4) + (3\times10) + (8\times9) + (7\times11)$

$= 30+20+30+77+72$

$= 229$

no. of allocations = 5

$m+n-1 = 3+4-1 = 6$ } not equal

∴ soln is degenerate

(2) Vogel's Approximation Method (VAM):-

Transportation cost is also considered. Here we make an allotment to reduce the max penalty. Thus IBFS will be near optimum soln.

Steps:-

1. Calculate row & column penalty as

difference b/w smallest & next smallest costs for each row & column.

2. Select row or column with the largest penalty. Choose cell with least cost in selected row (or col). Make an allotment there. Max allotment = min of the row availability and column requirement. Delete row or column which is completely satisfied.

3. Repeat 1 & 2 till all rim requirements are satisfied.

Q. Find IBFS using VAM.

	D_1	D_2	D_3	D_4	Supply			
O_1	8 (6)	(6) 4	(17) 6	(5) 6	1 ~~34~~ ~~28~~ ~~11~~ 6	2	2	2
O_2	6 (15)	6	8	7	~~15~~ 0	1	1	1
O_3	9	7	7	(12) 6	~~12~~ 0	1	1	3
O_4	7	(19) 2	7	5	~~19~~ 0	3		
Demand	~~21~~ 6	~~25~~ ~~6~~ 0	~~17~~ 0	~~17~~ 5	80			
	1	2	1	1				
	2	2	1	1				
	2			1				

→ Total availability = total requirements = 80

∴ balanced.

no. of allocations = 7

$m+n-1 = 4+4-1 = 7$

∴ non degenerate

Total cost = $(8\times6) + (6\times4) + (17\times6) + (5\times6) + (6\times15) + (12\times6) + (19\times2) = \underline{\underline{404}}$

PAGE NO.
DATE:

⇒ Least Cost Method / Matrix Minima Method:-

This method takes into account the minimum unit cost. Choose the cell having lowest cost in the matrix. Allocate to that cell as much as possible. Thus either a row total or column total is exhausted. Cross-off the corresponding row or column. From reduced matrix locate the cell having lowest cost. Allocate to that cell max possible. Continue the process until all available quantities are exhausted.

Q. Find IBFS to the following TP by lowest cost Method.

(i)

F_1	2	7	4	5
F_2	3	3	1	8
F_3	5	4	7	7
F_4	1	6	2	14
	7	9	18	

→ The lowest cost is 1 in cells (2,3) and (4,1). Select any one say (2,3). Allocate min of 18 and 8, ie 8. Thus total row F_2 is exhausted. Cross off the row F_2.

F_1	2	7	4	5
~~F_2~~	3	3	8 1	$\cancel{8}$ 0
F_3	5	4	7	7
F_4	1	6	2	14
	7	9	$\cancel{18}$ 10	

Now, lowest cost = 1 at (4,1).

2	7	4	5
3	3	8 1	$\cancel{8}$ 0
5	4	7	7
7 1	6	2	$\cancel{14}$ 7
$\cancel{7}$ 0	9	$\cancel{18}$ 10	

Now, lowest cost = 2 at (4,3)

2	7	4	5
3	3	8 1	$\cancel{8}$ 0
5	4	7	7
7 1	6	7 2	$\cancel{14}$ $\cancel{7}$ 0
$\cancel{7}$ 0	9	$\cancel{18}$ $\cancel{10}$ 3	

lowest cost = 4 at (1,3). Then 4 at (4,2). Then 7 at (1,2)

2	2 7	3 4	$\cancel{5}$ $\cancel{2}$ 0
3	3	8 1	$\cancel{8}$ 0
5	7 4	7	$\cancel{7}$ 0
7 1	6	7 2	$\cancel{14}$ $\cancel{7}$ 0
$\cancel{7}$ 0	$\cancel{9}$ $\cancel{2}$ 0	$\cancel{18}$ $\cancel{10}$ $\cancel{3}$ 0	

PAGE NO.
DATE:

Sol^n $\Rightarrow$

f_1	2	(2) 7	(3) 4	5
f_2	3	3	(8) 1	8
f_3	5	(7) 4	7	7
f_4	(7) 1	6	(7) 2	14
	7	9	18	

Total cost $= (2\times 7) + (3\times 4) + (8\times 1) + (7\times 4) + (7\times 1)\ (7\times 2)$

$= 83$ //

Q iii)

Destinations	A	B	C	D	Supply
Origins 1	1	5	3	3	34
2	3	3	1	2	15
3	0	2	2	3	12
4	2	7	2	4	19
Demand	21	25	17	17	

	A	B	C	D	
$\rightarrow$ 1	(9) 1	(8) 5	3	(17) 3	~~34~~ ~~25~~ ~~8~~ 0
2	3	3	(15) 1	2	~~15~~ 0
3	(12) 0	2	2	3	~~12~~ 0
4	2	(17) 7	(2) 2	4	~~19~~ ~~17~~ 0
	~~21~~ ~~9~~ 0	~~25~~ 8	~~17~~ ~~2~~ 0	~~17~~ 0	

Total cost $= (9\times 1) + (8\times 5) + (17\times 3) + (15\times 1) + (12\times 0) + (17\times 7) + (2\times 2)$

$= 238$

Module - 3.
(- contd -).

Q Solve the transportation problem

	D_1	D_2	D_3	D_4	Supply
O_1	5	2	4	3	22
O_2	4	8	1	6	15
O_3	4	6	7	5	8
Demands	7	12	17	9	45

→ Total supply = total demand

∴ given transportation problem is balanced.

Step 1:- To find IBFS of given transportation problem by VAM.

5	12 2	2 4	8 3	~~22~~ ~~10~~ ~~8~~ 0	(1)	(1)	(1)	(2)	-
4	8	15 1	6	~~15~~ 0	(3)	(3)	-	-	-
7 4	6	7	1 5	~~1~~ ~~8~~	(2)	(1)	(1)	(1)	(1)
~~7~~ 0	~~12~~ 0	~~17~~ ~~2~~ 0	~~9~~ ~~1~~ 0	0					
(1)	(4)	(3)	(2)						
(1)	-	(3)	(2)						
(1)	-	(3)	(2)						
(1)	-	-	(2)						
	-	-							

By VAM, IBFS of the given TP is

$$\text{Min TP} = (12 \times 2) + (2 \times 4) + (8 \times 3) + (15 \times 1) + (7 \times 4) + (5 \times 1)$$

$$= 24 + 8 + 24 + 15 + 28 + 5$$

$$= 104$$

3

Here no. of allocations = 6

$n + m - 1 = 4 + 3 - 1$

$= 7 - 1 = 6$

Here no. of allocations = $n + m - 1$

∴ IBFS obtained from VAM is non-degenerate.

∴ There is a possibility of checking the soln is final or not by using MODI method (u-v method).

Step 2:- Checking the optimality by MODI method.

Step 1:- Occupied cells.

	D_1	D_2	D_3	D_4	
O_1	-	(12) 2	(2) 4	(8) 3	$u_1 = 0$
O_2	-	-	(15) 1	-	$u_2 = -3$
O_3	(7) 4	-	-	(1) 5	$u_3 = 2$
	$v_1 = 2$	$v_2 = 2$	$v_3 = 4$	$v_4 = 3$	

We find u_i and v_j values using the result $u_i + v_j = c_{ij}$

$\Rightarrow u_1 + v_2 = 2$

$u_1 + v_3 = 4$

$u_1 + v_4 = 3$

$u_2 + v_3 = 1$

$u_3 + v_1 = 4$

$u_3 + v_4 = 5$

Assume $u_1 = 0$.

$\Rightarrow v_2 = 2$; $v_3 = 4$; $v_4 = 3$

$u_2 + 4 = 1 \Rightarrow u_2 = -3$

$u_3 + 3 = 5 \Rightarrow u_3 = 5 - 3 = 2$

$u_3 = 2 \Rightarrow 2 + v_1 = 4 \Rightarrow v_1 = 4 - 2 = 2$

Step 2:- Find the net evaluations of the unocupied cells using the relation,

net evaluation $= u_i + v_j - c_{ij}$

Unocupied cells	$u_i + v_j - c_{ij}$
(1,1)	$0 + 2 - 5 = -3$
(2,1)	$-3 + 2 - 4 = -5$
(2,2)	$-3 + 2 - 8 = -9$
(2,4)	$-3 + 3 - 6 = -6$
(3,2)	$2 + 2 - 6 = -2$
(3,3)	$2 + 4 - 7 = -1$

All net evaluations of unocupied cells are -ve (non +ve)

$\therefore$ sol^n is optimum.

$\therefore$ final sol^n = 104. (VAM).

Q. Solve the transportation problem:-

	D_1	D_2	D_3	D_4	Supply
O_1	19	30	50	10	7
O_2	70	30	40	60	9
O_3	40	8	70	20	18
Demand	5	8	7	14	35

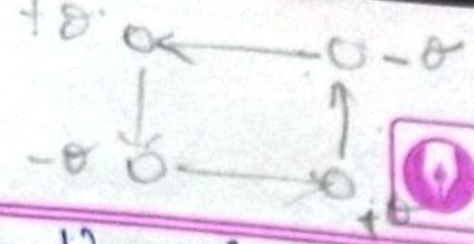
→ Total supply = total demand
∴ given TP is balanced.

? To find IBFS by VAM.

(5) 19	30	50	(2) 10	~~7~~ ~~2~~ 0	(9) (9) (40) (40) -
70	30	(7) 40	(2) 60	~~9~~ ~~2~~ 0	(10) (20) (20) (20) (20)
40	(8) 8	70	(10) 20	~~18~~ ~~10~~ 0	(12) (20) (50) - -
~~5~~ 0	~~8~~ 0	~~7~~ 0	~~14~~ ~~4~~ ~~2~~ 0		
(21)	(22)	(10)	(10)		
(21)	-	(10)	(10)		
-	-	(10)	(10)		
-	-	(10)	(50)		
-	-	-	-		

Min TP $= (5\times19) + (2\times10) + (7\times40) + (2\times60) + (8\times8) + (10\times20)$
$= 779$

no. of allocations $= 6$.
$n+m-1 = 4+3-1 = 6$
∴ non-degenerate



step 2:- Checking optimality by MODI method

① Occupied cells.

	D_1	D_2	D_3	D_4	
O_1	(5) 19	–	–	(2) 10	$u_1 = 0$
O_2	–	–	(7) 40	(2) 60	$u_2 = 50$
O_3	–	(8) 8	–	(10) 20	$u_3 = 10$
	$v_1 = 19$	$v_2 = -2$	$v_3 = -10$.	$v_4 = 10$	

$u_i + v_j = c_{ij}$.   Assume $u_1 = 0$.

$\Rightarrow u_1 + v_1 = 19$ $\Rightarrow v_1 = 19$.

$u_1 + v_4 = 10$ $\Rightarrow v_4 = 10$

$u_2 + v_3 = 40$ $\Rightarrow 50 + v_3 = 40 = v_3 = -10$ //

$u_2 + v_4 = 60$ $\Rightarrow u_2 = 60 - 10 = 50$

$u_3 + v_2 = 8$ $\Rightarrow 10 - 8 = -v_2 \Rightarrow v_2 = -2$ //

$u_3 + v_4 = 20$ $\Rightarrow u_3 = 20 - 10 = 10$.

② net evaluations of unoccupied cells:-

net evaluations $= u_i + v_j - c_{ij}$.

Unoccupied cells	$u_i + v_j - c_{ij}$.
(1, 2)	$0 - 2 - 30 = -32$
(1, 3)	$0 - 10 - 50 = -60$.
(2, 1)	$50 + 19 - 70 = -1$
(2, 2)	$50 - 2 - 30 = 18$
(3, 1)	$10 + 19 - 40 = -11$
(3, 3)	$10 - 10 - 70 = -70$.

Since all the net evaluations are non +ve, the IBFS is not optimum.

The largest possible net evaluation is +ve of the cell (2,2). Include that in the basis.

- Draw a loop starting from (2,2)
  Mark $+\theta$ and $-\theta$ alternatively starting with (2,2).

Note:- A loop is a sequence of cells with the property that

i) 2 adjacent cells are in a row or column.

ii) 3 or more adjacent cells are not in a row or column.

	D_1	D_2	D_3	D_4
O_1	(5) 19	–	–	(2) 10
O_2	–	$(+\theta)$ 30	(7) 40	(2) $-\theta$ 60
O_3	–	(8) $-\theta$ 8	–	(10) $+\theta$ 20

$\theta = \min\{8, 10, 2\}$
$= 2$

	D_1	D_2	D_3	D_4	
O_1	(5) 19	–	–	(2) 10	$u_1 = 0$
O_2	–	(2) 30	(7) 40	–	$u_2 = 32$
O_3	–	(6) 8	–	(12) 20	$u_3 = 10$
	v_1	v_2	v_3	v_4	
	19	−2	8	10	

$\theta^{-\theta}$ 60
θ $60 = -$

Assume $u_1 = 0$.

$u_1 + v_1 = 19 \Rightarrow v_1 = 19.$

$u_2 + v_2 = 30 \Rightarrow u_2 = 30 + 2 = 32$

$u_2 + v_3 = 40 \Rightarrow 32 + v_3 = 40 \Rightarrow 40 - 32 = 8 = v_3$

$u_3 + v_2 = 8 \Rightarrow 10 - 8 = -v_2 = v_2 = -2$

$u_3 + v_4 = 20 \Rightarrow u_3 + 10 = 20 \Rightarrow u_3 = 10.$

$u_1 + v_4 = 10. \Rightarrow v_4 = 10.$

Min cost $= (19 \times 5) + (10 \times 2) + (2 \times 30) + (7 \times 40) + (8 \times 6) + (12 \times 20)$

$= 743$

Unocupied cells	Net Evaluation $(u_i + v_j - c_{ij})$
(1,2)	-32
(1,3)	-42
(2,1)	-19
(2,4)	-18
(3,1)	-11
(3,3)	-52.

All the net evaluations are non +ve

$\therefore$ Min cost = 743 is optimum.

Module-4.

Network Analysis - Project scheduling
construction of project networks -
Critical path Method (CPM) -
Identification of Critical Path using
CPM
Estimation of floats
Total Float
Independent Float
Project Evaluation and Review Technique
(PERT)
Computation of expected completion times
by PERT.

12/09/2022

Module-4
Network Analysis

A project is composed of a no. of jobs, activities that are related one to another and all of these should be completed in order to complete the project. An activity or a project can be started only at the completion of many other activities. A network is a combⁿ of activities and events of a project.

⇒ Objectives of Network Analysis:-

1. Minimisation of total cost of a project
2. Minimisation of total time of a project
3. Minimisation of cost of a project for a given total time
4. Minimisation of time of a project for a given total cost
5. Minimisation of idle resources.
6. Planning, scheduling and controlling projects.

⇒ Network Diagram:-

Fulkersons Rule for Numbering of Events

The numbering of events is necessary

in a n/w. Event activity has 2 events known as tail and head events. These 2 events are identified by the numbers given to them. The following steps may be adopted for numbeeing of events.

1. The initial event of the n/w diagram is numbered 1.
2. The arrows emerging from the events 1 are then considered. Those arrows end in new events treat them as initial events and number them as 2, 3, 4 ...

   From these new initial arrow emerge which end in new events. They may be treated as new initial events and number them as 5, 6, 7 ... Follow step 2 until last events which has no emerging errors.

=) Rules for Constructing Network Diagram:-

1. Each activity is represented by one and only one arrow in the n/w.
2. No 2 activities can be identified by the same head and tail events.

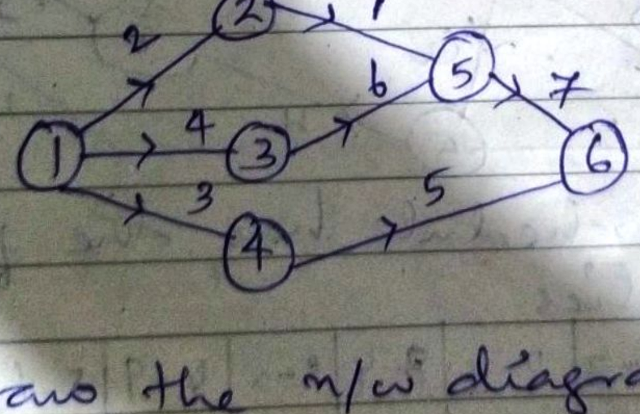
3. Except the beginning and ending nodes every node must have atleast one activity proceeding it and atleast one following it.

4. Only one activity may connect any 2 nodes.

Q. Draw the network diagrams for the following activities.

Activity	Duration
1-2	2
1-3	4
1-4	3
2-5	1
3-5	6
4-6	5
5-6	7

→



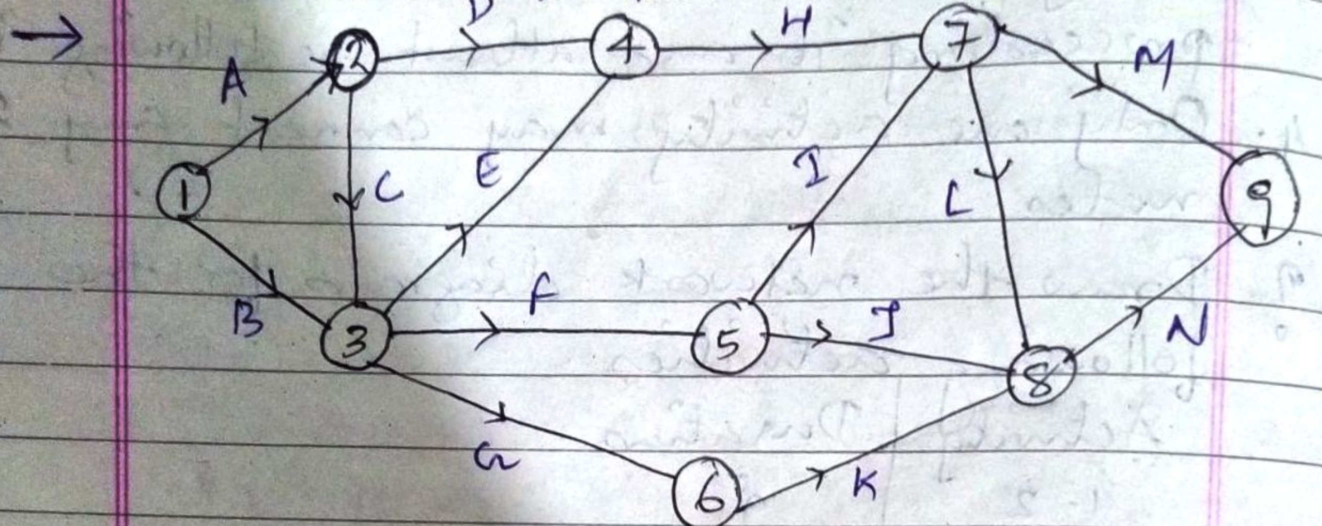
18/01/2022

Q. Draw the n/w diagrams for following activities.

Activity	A	B	C	D	E	F	G	H	I	J
Preceeding activities	-	-	A	A	B & C	B & C	B & C	D & E	F	F

K	L	M	N
G	H & I	H & I	J, K & L

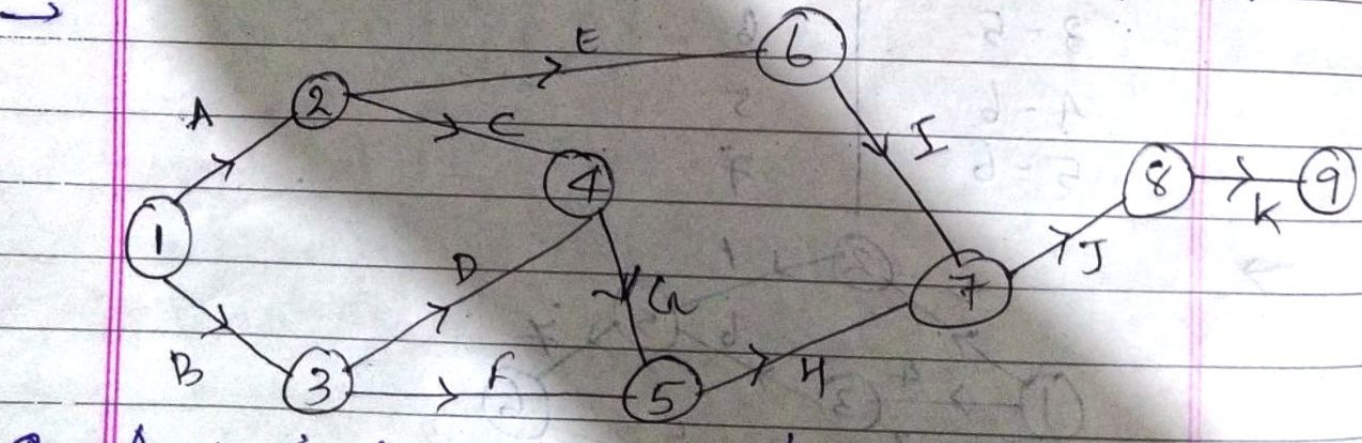
→



Q. Draw the network diagram

Activity	A	B	C	D	E	F	G	H	I	J	K
Prerequisites	–	–	A	B	A	B	C, D	G, F	E	H, I	J

→



Q. A project schedule has the following characteristics

Activity	1-2	1-3	2-4	3-4	3-5	4-9	5-6	5-7	6-8	7-8	8-10	9-10
Time	4	1	1	1	6	5	4	8	1	2	5	7

a) Construct a network diagram

b) Compute LST, EST, LFT, EFT of all activities

c) find critical path & project duration

d) find FF, TF, IF for each activity.

FF → free float

TF → Total float

IF → Independent float.

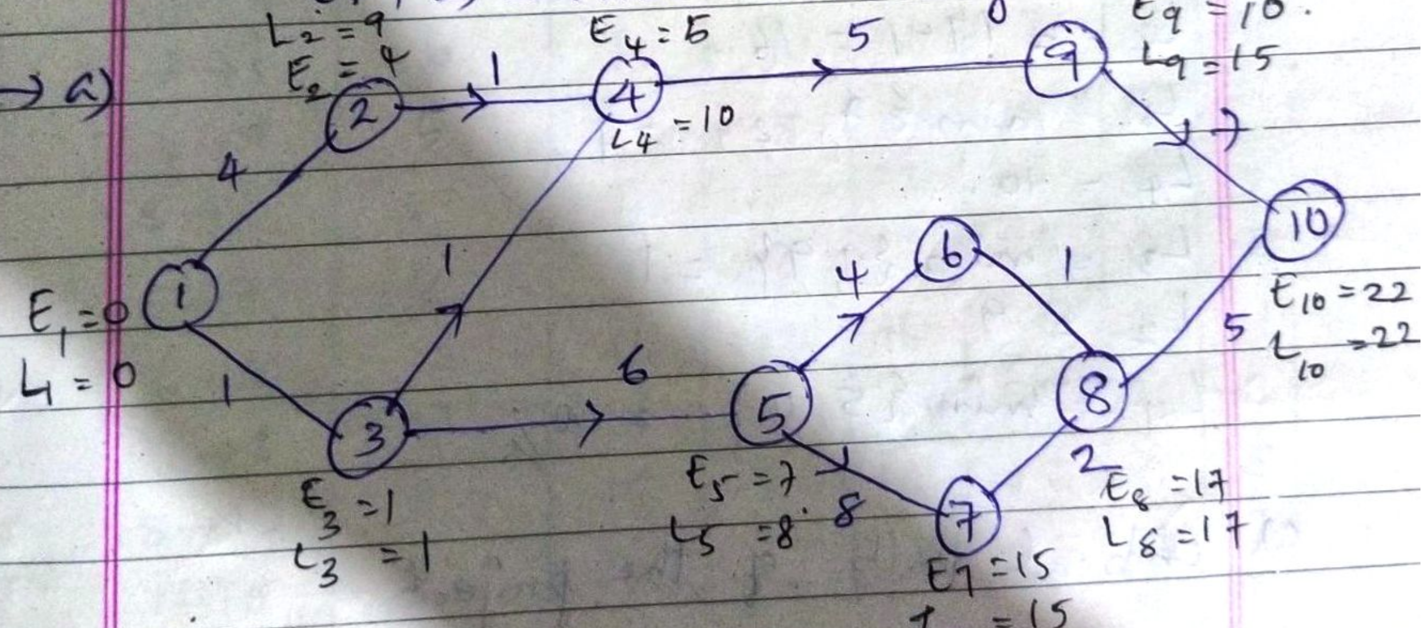
LST → Latest Starting Time

EST → Earliest Starting Time

LFT → Latest Finishing Time

EFT → Earliest Finishing Time

→ a)



b) <u>EST</u>:-

$ES1 = 0$

$ES2 = ES1 + t_{1-2} = 0 + 4 = 4$

$ES3 = ES_1 + t_{1-3} = 0 + 1 = 1$

20/01/2022

$ES_4 = \text{Max}\{E_2 + t_{2-4}, E_3 + t_{1-4}\}$
$= \text{Max}\{5, 2\} = 5$

ie; $E_1 = 0$ $\quad E_6 = 11$
$E_2 = 4$ $\quad E_7 = 15$
$E_3 = 1$ $\quad E_8 = 17$
$E_4 = 5$ $\quad E_9 = 10$
$E_5 = 7$ $\quad E_{10} = 22$

<u>LFT</u> :-

$L_{10} = 22$
$L_9 = 22 - 7 = 15$
$L_8 = 22 - 5 = 17$
$L_7 = 17 - 2 = 15$
$L_6 = 17 - 1 = 16.$
$L_5 = \min\{7, 12\} = 7$
$L_4 = 10.$
$L_3 = \min\{1, 9\} = 1$
$L_2 = 9$
$L_1 = \min\{5, 0\} = 0$

c) critical path of the project

$1 \to 3 \to 5 \to 7 \to 8 \to 10$

project duration $= 1 + 6 + 8 + 2 + 5$
$= \underline{22}.$

EST = E(i)   LFT = E(j)

EFT = E(i) + duration

LST = E(i) - duration

TF = LFT - EST or TF = LST - EST.

Table:-

Activity	Duration	Earliest time EST	Earliest time EFT	Latest time LST	Latest time LFT	Floats TF	Floats FF	Floats IF
1→2	4	0	4	5	9	5	-4	
1→3	1	0	1	0	1	0	-1	
2→4	1	4	5	9	10	5	-1	
3→4	1	1	2	9	10	8	-1	
3→5	6	1	7	1	7	0	-6	
4→9	5	5	10	10	15	5	-5	
5→6	4	7	11	12	16	5	-4	
5→7	8	7	15	7	15	0	-8	
6→8	1	11	12	16	17	5	-1	
7→8	2	15	17	15	17	0	-2	
8→10	5	17	22	17	22	0	-5	
9→10	7	10	17	15	22	5	-7	

FF = EST - EFT

31/01/2022

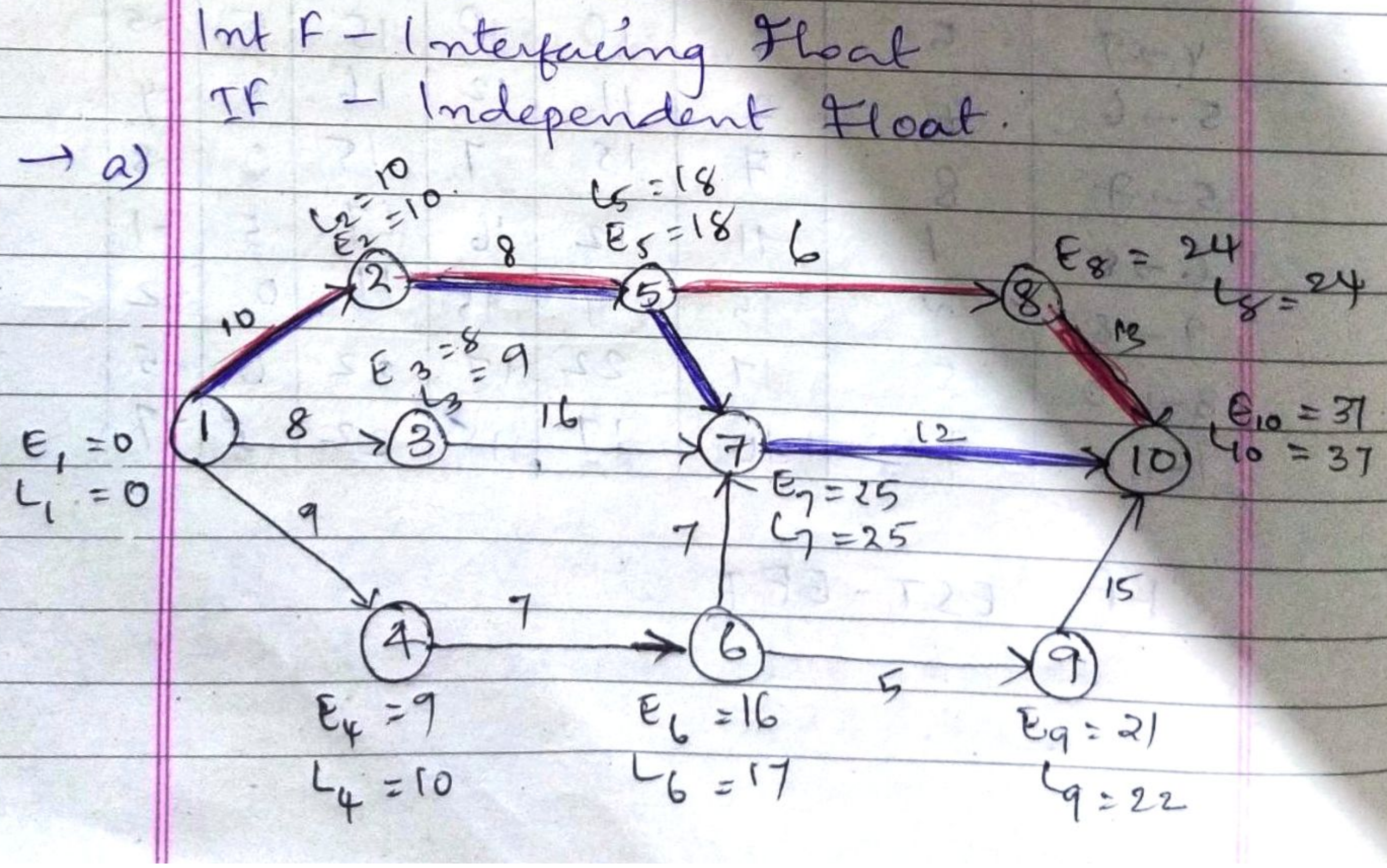
9.

Activity	1-2	1-3	1-4	2-5	4-6	3-7	5-7	6-7	5-8	6-9	7-10	8-10	9-10
Time	10	8	9	8	7	16	7	7	6	5	12	13	15

a) Draw the network diagram
b) Identify the critical path
c) Find the project duration
d) Compute EST, EFT, LST, LFT of all activations
e) Compute TF, FF, Int F, IF for all activations

TF – Total Float
FF – Free Float
Int F – Interfacing Float
IF – Independent Float

→ a)



b) Critical paths are:-

① → ② → ⑤ → ⑧ → ⑩

① → ② → ⑤ → ⑦ → ⑩

c) project duration

$= 10 + 8 + 6 + 13 = 37$

or

$= 10 + 8 + 7 + 12 = 37$ (e)

d)

Activity	Time	EST	EFT	LST	LFT	Remark	TF	FF	IntF	IF
1-2	10	0	10	0	10	CA	0	0	0	0
1-3	8	0	8	1	9	–	1	0		1
1-4	9	0	9	1	10	–	1	0		1
2-5	8	10	18	10	18	CA	0	0	0	0
4-6	7	9	16	10	17	–	1	0		1
3-7	16	8	24	9	25	–	1	1		0
5-7	7	18	25	18	25	CA	0	0	0	0
6-7	7	16	23	18	25	–	2	2		0
5-8	6	18	24	18	24	CA	0	0	0	0
6-9	5	16	21	17	22	–	1	0		1
7-10	12	25	37	25	37	CA	0	0	0	0
8-10	13	24	37	24	37	CA	0	0	0	0
9-10	15	21	36	22	37	–	1	1		0

$\text{Int F} = E_j - L_j - t_{ij}$

$\text{IF} = \text{TF} - \text{FF}$

1/02/2022

PAGE NO.
DATE:

⇒ PERT (Project Evaluation & Review Technique)

Q For the project given below; find
a) expected time for each activities
b) EST, EFT, LST, LFT
c) critical path and project duration
d) variance of the project and its SD.
e) find probability ~~that~~ of completing the project by 39 days?

Task	1-2	1-3	2-3	1-4	3-5	4-5	4-6	5-7	5-6	6-8	7-8
t_o	4	5	8	2	4	7	8	4	3	5	6
t_p	6	9	12	6	10	15	16	8	7	11	12
t_m	5	7	10	4	7	8	12	6	5	8	9

$t_o \rightarrow$ optimistic time
$t_p \rightarrow$ pessimistic time
$t_m \rightarrow$ most likely time

→ a) $t_e = \frac{t_o + t_p + 4t_m}{6}$

t_e	5	7	10	4	7	9	12	6	5	8	9

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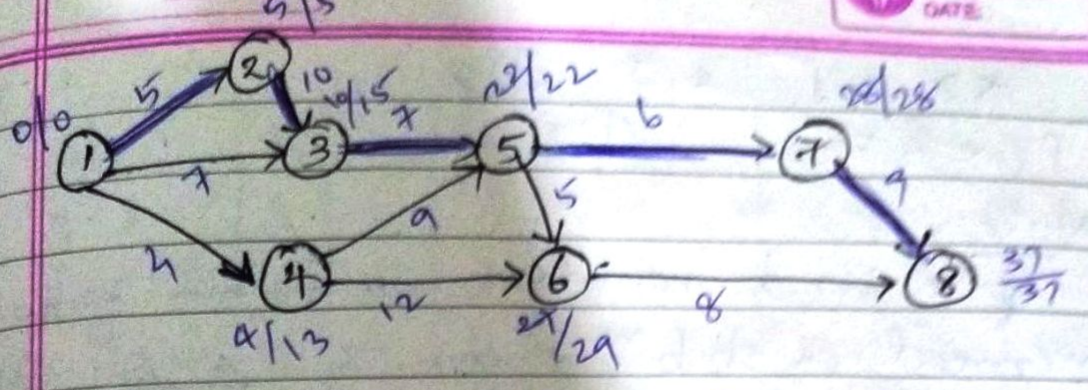


1 — 2 — 3 — 5 — 7 — 8

project duration = 37.

b)

Activity	t_e	EST (E_i)	EFT ($E_i + t_e$)	LST ($L_j - t_e$)	LFT (L_j)
1-2	5	0	5	0	5
1-3	7	0	7	8	15
2-3	10	5	15	5	15
1-4	4	0	4	9	13
3-5	7	15	22	15	22
4-5	9	4	13	13	22
4-6	12	4	16	17	29
5-7	6	22	28	22	28
5-6	5	22	27	24	29.
6-8	8	27	35	29	37
7-8	9	28	37	28	37

TF = LFT − EFT

or

= LST − EST

FF = $E_j - E_i - k_j$

Int F = $E_j - L_j - k_j$

IF = TF − FF

9/02/2022

d) Variance of project is

$$\sigma^2 = \left(\frac{t_p - t_o}{6}\right)^2$$

= sum of variance of all critical activity

$$= \frac{1}{9} + \frac{4}{9} + 1 + \frac{4}{9} + 1$$

$= 3$

SD $= \sqrt{3} = \pm 1.73$

e) $P(x \le 39)$ — ①

Note:-

If x is a random variable follows normal distrb^n then $z = \frac{x-\mu}{\sigma}$ is a new random variable follows std normal distrb^n.

$\mu \rightarrow$ project duration

$\sigma \rightarrow$ SD of project

$$P(x \le 39) = P\left(\frac{x-\mu}{\sigma} \le \frac{39-\mu}{\sigma}\right)$$

$$= P\left(z \le \frac{39-37}{1.73}\right)$$

$$= P(z \le 1.15) \quad \text{— ②}$$

Std normal table

1) $P(z \le a) = F(a)$

2) $P(a < z < b) = P(a \le z \le b) = P(a < z \le b)$
$= P(a \le z < b) = F(b) - F(a)$

3) $P(z > a) = P(z \ge a) = 1 - F(a)$

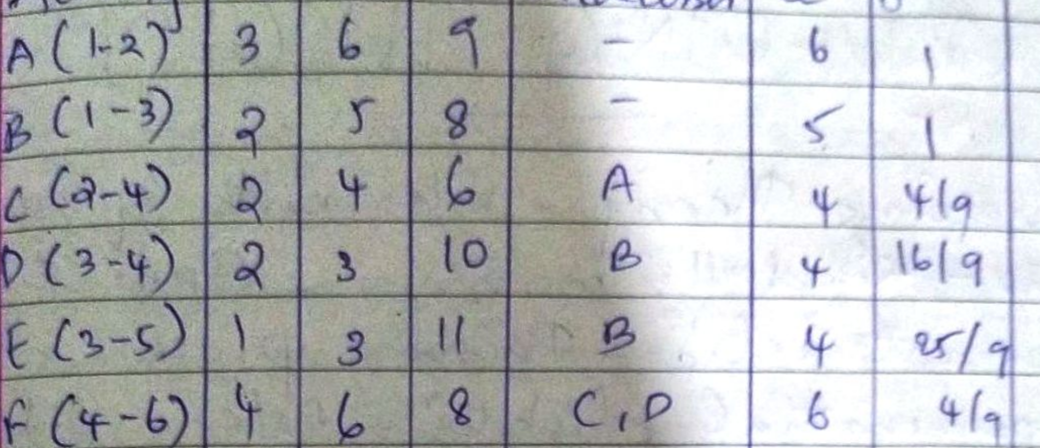
$\therefore P(z \le 1.5) = F(1.15)$ from std normal table
$= 0.87493$

Q. Consider following activities of the project

Activity	t_o	t_m	t_p	Predecessor	t_e	σ^2
A (1-2)	3	6	9	–	6	1
B (1-3)	2	5	8	–	5	1
C (2-4)	2	4	6	A	4	4/9
D (3-4)	2	3	10	B	4	16/9
E (3-5)	1	3	11	B	4	25/9
F (4-6)	4	6	8	C, D	6	4/9
G (5-6)	1	5	15	E	6	49/9

find the path and SD also find the pb of computing the project by 18 days.

→



critical path = ① — ② — ④ — ⑥

project duration, $\mu = 16$.

$$\sigma^2 = 1 + \frac{4}{9} + \frac{4}{9} = 1.888$$

$$SD = \sqrt{1.888} = 1.374$$

$$P(x < 18) = P\left(z \le \frac{18-\mu}{\sigma}\right) = P(z \le 1.45)$$

$$= F(1.45)$$

$$= 0.92647$$

Module - 4.

=> Network Analysis :-

A project is composed of no. of jobs, activities, that are related are to other and all of these should be completed in order to complete the project.

An activity of a project can be start only at the completion of many other activities.

A n/w is a comb^n of activities & events of a project.

=> Fulkersons rule for numbering of Events:-

The numbering of events is necessary in a n/w. Every activity has 2 events known as tail and head events. These 2 events are identified by the numbers given to them. The steps adopted for numbering of events are:-

① The initial event of n/w diagram is numbered 1.

② The arrows emerging from events 1 are then considered. Those arrows end in new events treat them as initial events and number them as 2, 3, 4 -- from these new initial arrow emerge which end in new events. They may be treated as new initial events and number them as 5, 6 --- Follow step 2 until last events which has no emerging arrows.

⇒ Rules for constructing n/w Diagram :-

1. Each activity is represented by one & only one arrow in the n/w
2. No 2 activities can be identified by same head and tail events
3. Except the beginning and ending nodes every node must have atleast one activity proceeding it and atleast one following it
4. Only one activity may connect any 2 nodes.

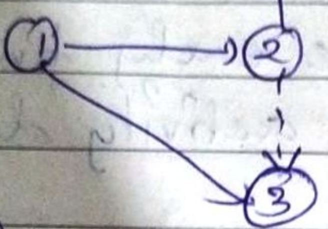
⇒ Activity

It is a task associated with a project. It is physically identifiable past of a project which consumes time & resources. An activity is represented by an arrow, the tail of which represents its start and the head, its finish.

eg :- (1) ——→ (2)

⇒ Dummy Activity

Certains activities which doesn't take time or resources are called Dummy Activity. These are used to represents a situation where one event cannot take place until a previous event has taken place. Denoted by Dotted arrows.

eg:- 

⇒ Event:-

Represents instants in time when certain activities have been started or completed ie, Event describes start / completion of a task.

2 types of event - tail event & head event.

eg:- activity 2-3 ⇒ 2 - tail $ 3 - head event

⇒ Earliest Event Time (T_E):-

The earliest occurence time / earliest event time is the earliest at which an event can occur. Denoted by $E_1, E_2 \ldots$

⇒ Latest Event Time (T_E):-

It is the latest time by which event must occur to keep project on schedule.

=) Earliest Starting Time (EST):-
It is the earliest time by it can comments.
eg:- ② ⟶ ③ EST is E_2
=) EFT :- Earliest finishing Time:-
If activity preceeds at its early time and takes estimated duration for completion, then it will have early finish.
EFT = EST + activity duration.

=) Slack :-
difference b/w Earliest event time and latest event time.
ie; Slack of event 2 = $L_2 - E_2$
=) Critical Activity:-
An activity is said to be critical if a delay in its start will cause a further delay in completion of entire project
=) Steps Involved in CPM:-
1. List all activities (tasks) & draw n/w diagram.
2. Find Earliest event time & latest event time of each event & show in diagram.
3. Calculate EST, EFT, LST, LFT for each activity

4. Determine float for each activity
5. Identify critical activities
6. Draw double lines in the n/w diagram passing through critical activities. Double lines show critical paths.
7. Calculate total project duration.

27/02/2022.

Module - 5

QUEUING THEORY

Queuing Theory :- Elements of queuing s/ms - Kendall's notation - Operating characteristics - Poisson process - Exponential distbn - Mean & Variance - Birth and Death process.

Queuing models based on Poisson process - Single server models with finite and infinite capacity - Multi server model with finite and infinite capacity.

A flow of customers from finite or infinite popl^{n} towards the service facility from a queue (waiting line) on account of lack of capacity to serve them all at a time.

The arriving unit that requires some service to be performed is called a customer. The customer may be persons, machines, vehicles etc.

Queue stands for the no. of customers waiting to be service. This doesn't include the customer being serviced.

The process or system that performs services to the customer is termed by service channel or service facility.

⇒ Transcient State & Steady State of the System:-

Queuing Theory involves the study of system's behaviour over time. If the behaviour of systems vary with time, it is said to be in transcient state. Usually a s/m is transcient during early stages of its op^{n} when its behaviour still depends upon the initial condition.

A system said to be a steady state condition its behaviour becomes independent of time.

⇒ Queue S/m or Characteristics of Queuing:-

The queue s/m can be completely described by:

1) The input [arrival pattern]
2) Service Mechanism [service pattern]
3) The queue discipline
4) Customer's behaviour.

Module - 5
(- contd -).

Customer generally behave in the following 4 ways:

1. Balking :-
A customer who leave the queue bcz the queue is too long and he has no time to wait or has no sacrifice space.

2. Reneging :-
This occurs when a waiting customer leaves the queue due to impatience

3. Priorities :-
In certain appl^n some customers are served before other regardless of their arrivals. These customers have priority over others.

4. ~~Joo~~ Jockeying :-
Customers may jockey from one waiting line to another. This is most common in a supermarket.

⇒ Queue_Length :-
It may be defined as a line length plus no. of customers being served.

Notations:-

notations used in the analysis of a queuing systems are:-

- × $n \rightarrow$ number of customers in a s/m (waiting and in service)
- × $P_n \rightarrow$ pb of 'n' customers in the s/m.
- × $\lambda \rightarrow$ mean customer arrival rate or avg no. of arrivals in queuing s/m per unit time
- × $\mu \rightarrow$ mean service rate or avg no. of customers completing service per unit time
- × $\frac{\lambda}{\mu}$ or $\rho = \frac{\text{Avg service completion time}}{\text{Avg inter-arrival time}} = \frac{1/\mu}{1/\lambda}$
  $\rightarrow$ traffic intensity or server utilization factor (expected fraction of time in which server is busy).
- × $S \rightarrow$ no. of service channels (service facilities)
- × $N \rightarrow$ max no. of customers allowed in s/m
- × $L_s \rightarrow$ mean no. of customers in s/m (waiting and in service)
- × $L_q \rightarrow$ mean no. of customers in the queue
- × $L_b \rightarrow$ mean length of non-empty queue
- × $W_s \rightarrow$ means waiting time in s/m (waiting and in service)

PAGE NO.
DATE:

× W_q → means waiting time in queue
× P_w → Pb that an arriving customer has to wait.

⇒ Kendall's Notation for Representing Queuing Models:-

Kendall's notation
$(a/b/c):(d/e)$

a = pb law of arrival line (inner arrival line)

b = pb law acco. to which customers are being served.

c = no. of channels or service ~~channels~~ stations

d = capacity of a s/m (in service and waiting) i.e, max no. allowed to the s/m.

e = queue discipline

⇒ Queuing Models:-

I) Model-1 $(M/M/1):(\infty/FCFS)$

λ = mean arrival rate

μ = mean service rate

The traffic intensity, $\rho = \frac{\lambda}{\mu}$

1) The idle time of server, $P_0 = 1-\rho$

2) $L_s = \frac{\rho}{1-\rho} = \frac{\lambda/\mu}{1-\lambda/\mu} = \frac{\lambda/\mu}{\frac{\mu-\lambda}{\mu}} = \frac{\lambda}{\mu-\lambda}$

3) $L_q = \frac{\rho^2}{1-\rho} = \frac{(\lambda/\mu)^2}{1-\lambda/\mu} = \frac{\lambda^2}{\mu(\mu-\lambda)}$

4) $W_s = \frac{1}{\mu-\lambda} = \frac{1}{\lambda} L_s$

5) $W_q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{1}{\lambda} L_q$

6) Pb that no. of customers the slm exceeds N,

$P(n > N) = \left(\frac{\lambda}{\mu}\right)^{N+1}$

Q. A barbarshop with one man takes exactly takes 25 mins to complete haircut. If customers arriving in Poisson fashion at an avg rate of every 40 mins. How long on the avg must a customer wait for service?

$\lambda = n/time$

$\mu = n/time$

→ This is a problem in $(M/M/1):(\infty/FCFS)$ model.

$\lambda = 1/40$; $\mu = 1/25$

$\therefore \rho = \frac{\lambda}{\mu} = \frac{1/40}{1/25} = \frac{25}{40} = \frac{5}{8}$

$$W_q = \frac{1}{\lambda} L_q$$

$$= \frac{1}{\lambda} \frac{\rho^2}{1-\rho}$$

$$= \frac{1}{1/40} \times \frac{(5/8)^2}{1-5/8} = 40 \times \frac{25/40}{\frac{8-5}{8}}$$

$$= \cancel{40} \times \frac{25}{\cancel{40}} \times \frac{8}{3} = \frac{200}{3} = 41.67 \text{ mins}$$

Q. Customers arrive at 1 person barber shop acc. to poisson distrib^n with mean inter arrival time 20 mins. Customers spend an avg 15 mins in the barber chair.

a) what is the pb that a customer will not have to wait for a haircut?

b) what is the no. of customers expected in the barbershop?

c) How much time can a customer expected to spent in barbershop?

→ $\lambda = 1/20$.

$\mu = 1/15$

$$\rho = \frac{\lambda}{\mu} = \frac{15}{20} = \frac{3}{4}$$

a) $P_0 = 1 - \rho = 1 - \frac{3}{4} = \frac{1}{4} = 0.25$

b) $L_s = \frac{\rho}{1-\rho} = \frac{0.25}{1-0.25} = 3$ $\left(\frac{3/4}{1-3/4}\right)$.

c) $W_s = \frac{1}{\mu - \lambda} = \frac{1}{\frac{1}{15} - \frac{1}{20}} = \frac{1}{\frac{20-15}{300}} = \frac{300}{5}$

$= 60$ mins

Q. A self service store employees one cashier at counter. 9 customers arrive on an avg at every 5 mins. While the cashier can serve 10 customers in 5 mins. Assuming Poisson distribⁿ for arrival rate and exponential distribⁿ for service rate, find:

a) avg no. of customers in the s/m.

b) avg no. of customers in queue

c) avg waiting time of customers in the s/m.

d) avg waiting time of customers in the queue

→ $\lambda = \frac{9}{5}$, $\mu = \frac{10}{5}$

$\rho = \frac{9}{10} = 0.9$

a) $L_s = \frac{\rho}{1-\rho} = \frac{9/10}{1-9/10} = \frac{9/10}{1/10} = 9$ //

b) $L_q = \frac{\rho^2}{1-\rho}$

$= \frac{9^2/10^2}{1/10} = \frac{9^2}{100} \times 10 = \frac{810}{100} = \underline{\underline{0.81}}$

c) $W_s = \frac{1}{\mu-\lambda} = \frac{1}{\frac{10}{5}-\frac{9}{5}} = \frac{1}{1/5} = 5$ //

d) $W_q = \frac{1}{\lambda} L_q = \frac{1}{9/5} \times 0.81$

$= \frac{5}{9} \times 0.81 = \underline{\underline{4.5 \text{ min}}}$

Q: In a supermarket the avg arrival rate of customers is 10 in every 30 mins, following poisson process. The avg time taken by a cashier to list is 2.5 min following exponential distribⁿ.

a) what is the pb that the queue length exceeds 6.?

b) what is the expected time spent by a customer in the s/m?

→ $\lambda = \frac{10}{30} = \frac{1}{3}$; $\mu = \frac{1}{2.5}$

$$\rho = \frac{\lambda}{\mu} = \frac{1/3}{1/2.5} = \frac{2.5}{3} = 0.83$$

a) $N = 6$.

$$P(N>6) = \left(\frac{\lambda}{\mu}\right)^N = \left(\frac{2.5}{3}\right)^6$$

$$= \cancel{0.279}\ 0.33$$

b) $$W_s = \frac{1}{\mu-\lambda} = \frac{1}{\frac{1}{2.5}-\frac{1}{3}} = \frac{1}{1/15} = 15 \text{ min.}$$

Q. A TV repairman finds that time spend on his jobs has an exponential distrib^n with mean 30 minutes. If he repaires sets in the order in which they came in. If the arrival of sets is approximately with an avg rate of 10 per 8 hour day,

(a) what is the repairmen's expected idle time each day?

(b) How many jobs are ahead of the avg set just brought in?

→ $\mu = \frac{1}{30}$; $\lambda = \frac{10}{8 \times 60} = \frac{1}{48}$

$$\rho = \frac{\lambda}{\mu} = \frac{1}{48} \times \frac{30}{1} = \frac{5}{8}$$

(a) Idle time, $P_0 = 1 - \rho$

$$= 1 - \frac{5}{8} = \frac{3}{8}$$

(b) $$L_s = \frac{\rho}{1-\rho} = \frac{5/8}{3/8} = \frac{5}{3}$$

Q. On an avg. 96 patients per 24 hour day require service of an emergency clinic. Also on an avg, a patient requires 10 min of active attention. Assume that the facility can handle only one emergency at a time. Suppose that it costs the clinic ₹100 per patient treated to obtain an avg servicing time of 10 mins, and that each min of decrease in this avg time would cost ₹10 per patient treated. How much would have to be budgeted by the clinic to dse the avg size of the queue from $1\frac{1}{3}$ patients to $\frac{1}{2}$ patient ?

→ $$\lambda = \frac{96}{24 \times 60} = \frac{1}{15}$$

$$\mu = \frac{1}{10}$$

Expected no. of patients in waiting time:

$$L_q = \cancel{\frac{\lambda^2}{\mu(\lambda-}} \frac{\lambda^2}{\mu(\mu-\lambda)}$$

$$= \frac{(1/15)^2}{\frac{1}{10}\left(\frac{1}{10}-\frac{1}{15}\right)} = 1\frac{1}{3}$$

But, $L_q = 1\frac{1}{3}$ is reduced to $L_q' = \frac{1}{2}$

$\therefore$ Substituting $L_q' = \frac{1}{2}$, $\lambda' = \lambda = \frac{1}{15}$

$$\Rightarrow L_q' = \frac{\lambda_1^2}{\mu'(\mu'-\lambda_1)}$$

$$\Rightarrow \frac{1}{2} = \frac{(1/15)^2}{\mu'(\mu'-\frac{1}{15})} \Rightarrow \mu' = \frac{2}{15}$$

$\therefore$ avg rate of treatment is $\frac{1}{\mu'} = 7.5$ min.

Consequently, the decrease in avg rate of treatment is $= 10 - \frac{15}{2} = \frac{5}{2}$ min.

budget per patient $= 100 + \frac{5}{2} \times 10 = ₹\,125$.

So in order to get required size of the queue, the budget should be increased from ₹100 to ₹125 per patient.

PAGE NO.
DATE:

Q. In a railway marshalling yard, goods trains arrive at a rate of 30 trains per day. Assuming that the inter-arrival time follows an exponential distrbn and service time (time taken to hump a train) distrbn is also exponential with an avg of 36 minutes. Calculate

(a) Expected queue size (line length)

(b) probability that the queue size exceeds 10.

If the input of trains increases to an avg of 33 per day what will be the change in (i) and (ii) ?

→ $\lambda = \frac{30}{60 \times 24} = \frac{1}{48}$

$\mu = \frac{1}{36}$

$\rho = \frac{\lambda}{\mu} = \frac{1}{48} \times \frac{36}{1} = \frac{3}{4}$

(a) Expected queue size, $L_s = \frac{\rho}{1-\rho}$

$= \frac{3/4}{1-3/4} = \frac{3/4}{1/4} = \underline{3}$

(b) $P(n \geq 10) = \rho^{10} = \left(\frac{3}{4}\right)^{10} = (0.75)^{10}$

$= \underline{0.06.}$

Now, if input increases to 33 trains per day, then we have

$$\lambda = \frac{33}{60 \times 24} = \frac{11}{480}$$

$$\mu = \frac{1}{36}$$

$$\rho = \frac{\mu}{\lambda} = \frac{11/480}{1/36} = 0.83$$

(i) $L_s = \frac{\rho}{1-\rho} = \frac{0.83}{1-0.83} = 4.88 \approx 5$

(ii) $P(n \geq 10) = \rho^{10} = (0.83)^{10} = 0.155$
≈ 0.2

Model II : (M/M/I) : (N/FCFS)

- limited or finite queue model.
- Here max no. of customers is limited to N.

* $P_0 = \frac{1-\rho}{1-\rho^{N+1}}$; $\rho \neq 1$; $\rho < 1$; $\rho = \frac{\lambda}{\mu}$

* $P_n = \begin{cases} \left(\frac{1-\rho}{1-\rho^{N+1}}\right)\rho^n, & 0 \leq n \leq N ; \rho = \frac{\lambda}{\mu} \neq 1 \\ \frac{1}{N+1} ; & \rho = 1 \ (\lambda = \mu) \end{cases}$

* L_s =

PAGE NO.
DATE:

$\times$ $$L_s = \sum_{n=1}^{N} n P_n = \sum_{n+1}^{N} n \left(\frac{1-\rho}{1-\rho^{N+1}} \right) \rho^n$$

$$= \begin{cases} \dfrac{\rho}{1-\rho} - \dfrac{(N+1)\rho^{N+1}}{1-\rho^{N+1}} ; & \rho \neq 1 (\lambda \neq \mu) \\ \dfrac{N}{2} ; & \rho = 1 (\lambda = \mu) \end{cases}$$

$\times$ $$L_q = L_s - \frac{\lambda}{\mu}$$

$\times$ $$W_s = \frac{L_s}{\lambda(1-P_N)} = \frac{1}{\lambda'} L_s \quad \left[\lambda' = \lambda(1-P_N) \right]$$

$\times$ $$W_q = W_s - \frac{1}{\mu}$$

Q. In a railway marshelling yard goods trains arrived at a rate of 30 trains per day. Assume that is the arrival time follows an exponential distrbn and the service time is also to be assumed as exponential with mean of 36 minute.

(a) Calculate pb that yard is empty.

(b) The avg queue length. Assume that the line capacity of yard is 9 trains.

$\rightarrow$ $$\lambda = \frac{30}{1 \times 24 \times 60} = \frac{1}{48}$$

$\mu = \frac{1}{36}$

$\rho = \frac{\lambda}{\mu} = \frac{1}{48} \times \frac{36}{1} = \frac{3}{4} = 0.75 \; (\rho < 1)$

$P_0 = \frac{1-\rho}{1-\rho^{N+1}} = \frac{1-0.75}{1-0.75^{9+1}} = \frac{0.25}{0.94} = \underline{\underline{0.26}}.$

$L_s = \frac{\rho}{1-\rho} - \frac{(N+1)\,\rho^{N+1}}{1-\rho^{N+1}}$

$= \frac{0.75}{0.25} - \frac{(9+1)\,0.75^{9+1}}{1-0.75^{9+1}}$

$= 3 - \frac{0.56}{0.94} = \underline{\underline{2.4}}.$

III Model III : $(M/M/s) : (\infty/FCFS)$

- Multiple servers, Unlimited queue model

* $\rho = \frac{\lambda}{\mu s}$

* $P_n = \begin{cases} \left(\frac{\lambda}{\mu}\right)^n P_0 & ; 1 \le n < s \\ \frac{\left(\frac{\lambda}{\mu}\right)^n P_0}{s!\, s^{n-s}} & ; n \ge s \end{cases}$

* $P_s = \frac{1}{s}\left(\frac{\lambda}{\mu}\right)^s P_0$.

PAGE NO.
DATE:

$\times \quad P_0 = \left[\sum_{n=0}^{s-1} \frac{\left(\frac{\lambda}{\mu}\right)^n}{n!} + \frac{\left(\frac{\lambda}{\mu}\right)^s}{s!\,(1-\rho)} \right]^{-1}$

$\times \quad L_q = P_s \left(\frac{\rho}{(1-\rho)^2} \right)$

$\times \quad L_s = 1 - \frac{\lambda}{\mu s}$

$\times \quad W_s = \frac{1}{\lambda} L_s$

$\times \quad W_q = \frac{1}{\lambda} L_q$