MODULE- II

Duality in LPP

Statement of Duality Theorem

- The primal - Dual Solutions using simplex method.
- Revised Simplex Method.

Duality in LPP:-

Associated with every LPP there is another problem called duality problem. The original LPP is called the primal problem. The optimal solution to the dual gives complete information about the optimum soln of the ~~pri~~ ~~dual~~ primal & vice versa.

Canonical Form :-

The canonical form

$$\text{Max } Z = CX$$

Subject to,

$$AX \leq B$$

$$X \geq 0$$

Here B can be both +ve or -ve.

1. Convert the given LPP into canonical form -

$$\text{Min } z = 2x_1 + 3x_2$$

Subject to,

$$x_1 + x_2 \le 4$$

$$2x_1 - x_2 \ge 3$$

$$3x_1 + 4x_2 = 10$$

$x_1 \ge 0$, x_2 unrestricted in sign

$\therefore$ If $a = b$ then $a \le b$; $a \ge b$

Soln:-

Since x_2 is unrestricted in sign.

Let $x_2 = x_2' - x_2''$, where x_2' & $x_2'' \ge 0$.

Primal Form:-

$\therefore$ Min $z = 2x_1 + 3x_2' - 3x_2''$

Subject to

$$x_1 + x_2' - x_2'' \le 4$$

$$2x_1 - x_2' + x_2'' \ge 3$$

$$3x_1 + 4x_2' - 4x_2'' \le 10$$

$$3x_1 + 4x_2' - 4x_2'' \ge 10$$

$$x_1, x_2', x_2'' \ge 0$$

Canonical Form:-

$$\text{Max, } z' = -(\text{Min } z)$$

$$\text{Max, } z' = -2x_1 - 3x_2' + 3x_2''$$

Subject to,

$$x_1 + x_2' - x_2'' \le 4$$

$1\times \Rightarrow$ $-2x_1 + x_2' - x_2'' \le -3$

$3x_1 + 4x_2' - 4x_2'' \le 10$

$-1\times \Rightarrow$ $-3x_1 - 4x_2' + 4x_2'' \le -10$

$x_1, x_2', x_2'' \ge 0.$

2.) convert the given problems to canonical form:-

Max, $Z = 3x_1 + 17x_2 + 9x_3$

Subject to,

$x_1 - x_2 + x_3 \ge 3$

$-3x_1 + 2x_2 \le 1$

$x_1, x_2 \ge 0$ & x_3 unrestricted in sign.

Soln:- Since x_3 is unrestricted, $\boxed{x_3 = x_3' - x_3''}$

Primal:- Max, $Z = 3x_1 + 17x_2 + \cancel{9x_3}\ 9x_3' - 9x_3''$

Subject to,

~~$x_1 + x_2$~~

$x_1 - x_2 + x_3' - x_3'' \ge 3$

$-3x_1 + 2x_2 \le 1$

$x_1, x_2, x_3', x_3'' \ge 0.$

Canonical:- Max, $Z = 3x_1 + 17x_2 + 9x_3' - 9x_3''$

Subject to,

$-1\times \Rightarrow$ $-x_1 + x_2 - x_3' + x_3'' \le -3$

$-3x_1 + 2x_2 \le 1$

$x_1, x_2, x_3', x_3'' \ge 0.$

3) Min, $Z = 3x_1 - 2x_2 + 4x_3$

Subject to,

$$3x_1 + 5x_2 + 4x_3 \geq 7$$
$$6x_1 + x_2 + 3x_3 \geq 4$$
$$7x_1 - 2x_2 - x_3 = 10$$
$$x_2 \leq 20$$
$$x_1 - 2x_2 + 5x_3 \geq 3$$
$$4x_1 + 7x_2 - 2x_3 \geq 2$$
$$x_1 \geq 0, \; x_2 \,\&\, x_3 \text{ unrestricted}$$

Soln:-

Since x_2 & x_3 are unrestricted

$\therefore$ $x_2 = x_2' - x_2''$

$x_3 = x_3' - x_3''$

Primal:-

Min, $Z = 3x_1 - 2x_2' + 2x_2'' + 4x_3' - 4x_3''$

Subject to,

$$3x_1 + 5x_2' - 5x_2'' + 4x_3' - 4x_3'' \geq 7$$
$$6x_1 + x_2' - x_2'' + 3x_3' - 3x_3'' \geq 4$$
$$7x_1 - 2x_2' + 2x_2'' - x_3' + x_3'' = 10$$
$$x_2' - x_2'' \leq 20$$
$$x_1 - 2x_2' + 2x_2'' + 5x_3' - 5x_3'' \geq 3$$
$$4x_1 + 7x_2' - 7x_2'' - 2x_3' + 2x_3'' \geq 2$$
$$x_1, x_2', x_2'', x_3', x_3'' \geq 0.$$

Canonical form:-

$$\text{Max}, Z' = -3x_1 + 2x_2' - 2x_2'' - 4x_3' + 4x_3''$$

Subject to;

$$-3x_1 - 5x_2' + 5x_2'' - 4x_3' + 4x_3'' \leq -7$$

$$-6x_1 - x_2' + x_2'' - 3x_3' + 3x_3'' \leq -4$$

$$7x_1 - 2x_2' + 2x_2'' - x_3' + x_3'' \leq 10$$

$$7x_1 - 2x_2' + 2x_2'' - x_3' + x_3'' \geq 10$$

$$\cancel{x_2' - x_2''} \leq 20$$

$$\Rightarrow -7x_1 + 2x_2' - 2x_2'' + x_3' - x_3'' \leq -10.$$

$$x_2' - x_2'' \leq 20$$

$$-x_1 + 2x_2' - 2x_2'' - 5x_3' + 5x_3'' \leq -3$$

$$-4x_1 - 7x_2' + 7x_2'' + 2x_3' - 2x_3'' \leq -2$$

$$x_1, x_2', x_2'', x_3', x_3'' \geq 0.$$

Dual:-

Let the primal be in the canonical form

$$\text{Max}, Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

Subject to

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq b_m$$

where

$$x_1, x_2, x_3, \cdots, x_m \geq 0.$$

then the dual of the problem is,

$$\text{Min}, D = b_1 y_1 + b_2 y_2 + \cdots + b_m y_m$$

Subject to;

$$a_{11} y_1 + a_{21} y_2 + a_{31} y_3 + \cdots + a_{m1} y_m \geq C_1$$

$$a_{12} y_1 + a_{22} y_2 + a_{32} y_3 + \cdots + a_{m2} y_m \geq C_2$$

— — — — — —

— — — — — —

$$a_{1n} y_1 + a_{2n} y_2 + a_{3n} y_3 + \cdots + a_{mn} y_m \geq C_n$$

$$y_1, y_2, \cdots, y_m \geq 0$$

i.e, Canonical Form $\Longrightarrow$ Dual Form

Canonical Form	Dual Form
$\text{Max}, Z = CX$	$\text{Min}, D = B^T Y$
$AX \leq B$	$A^T Y \geq C^T$
$X \geq 0$	$Y \geq 0$

Only A's matrices transponse is taken not x_1, x_2 X's matrix

1) Find the dual of

Min, $Z = 12x_1 + 3x_2 + 7x_3$

Subject to;

$6x_1 - 2x_2 + 5x_3 \geq 3$

$-2x_1 - 3x_2 + 4x_3 \leq 2$

$3x_1 + 9x_2 + x_3 \geq 8$

$x_1, x_2, x_3 \geq 0.$

Soln:-

S1: The primal problem can be converted to canonical form.

Max, $Z' = -(\text{Min}, Z)$

Max, $Z' = -12x_1 - 3x_2 - 7x_3$

Subjected to;

$-6x_1 + 2x_2 - 5x_3 \leq -3$

$-2x_1 - 3x_2 + 4x_3 \leq 2$

$-3x_1 - 9x_2 - x_3 \leq -8$

$x_1, x_2, x_3 \geq 0.$

S2. The dual of the problem is:-

Min $W = -3y_1 + 2y_1 - 8y_3$

Subject to;

$-6y_1 - 2y_2 - 3y_3 \geq -12$

$2y_1 - 3y_2 - 9y_3 \geq -3$

$-5y_1 + 4y_2 - y_3 \geq -7$

$y_1, y_2, y_3 \geq 0.$

2.) Construct the dual of

$$\text{Max } z = 3x_1 + 5x_2$$

Subject to;

$$2x_1 + 6x_2 \leq 50$$

$$3x_1 + 2x_2 \leq 35$$

$$5x_1 + 3x_2 \leq 10$$

$$x_2 \leq 20$$

$$x_1, x_2 \geq 0.$$

Soln:-

The primal problem in canonical form.

∴ Dual is;

$$\text{Min } w = 50y_1 + 35y_2 + 10y_3 + 20y_4$$

Subject to;

$$2y_1 + 3y_2 + 5y_3 + 0y_4 \geq 3$$

$$6y_1 + 2y_2 + 3y_3 + y_4 \geq 5$$

$$y_i \geq 0 \ \forall i = 1, 2, 3, 4$$

8) Construct a dual of

$$\text{Min } Z = 3x_1 - 2x_2 + 4x_3$$

Subject to,

$$3x_1 + 5x_2 + 4x_3 \geq 7$$
$$6x_1 + x_2 + 3x_3 \geq 4$$
$$7x_1 - 2x_2 - x_3 \leq 10$$
$$x_2 \leq 20$$
$$x_1 - 2x_2 + 5x_3 \geq 3$$
$$4x_1 + 7x_2 - 2x_3 \geq 2$$
$$x_1, x_2, x_3 \geq 0.$$

Soln:-

1) Canonical Form:-

$$\text{Max, } Z' = -3x_1 + 2x_2 - 4x_3$$

Subject to,

$$-3x_1 - 5x_2 - 4x_3 \leq -7$$
$$-6x_1 - x_2 - 3x_3 \leq -4$$
$$7x_1 - 2x_2 - x_3 \leq 10$$
$$x_2 \leq 20$$
$$-x_1 + 2x_2 - 5x_3 \leq -3$$
$$-4x_1 - 7x_2 + 2x_3 \leq -2$$
$$x_1, x_2, x_3 \geq 0.$$

2) Dual form:-

$$Min, w = -7y_1 - 4y_2 + 10y_3 + 20y_4 - 3y_5 - 2y_6$$

Subject to;

$$-3y_1 - 6y_2 + 7y_3 + 0y_4 - y_5 - 4y_6 \geq -3$$

$$-5y_1 - y_2 - 2y_3 + y_4 + 2y_5 - 7y_6 \geq 2$$

$$-4y_1 - 3y_2 - y_3 + 0y_4 - 5y_5 + 2y_6 \geq -4$$

$$y_i \geq 0 \quad \forall i = 1,2,\dots,6$$

Dual problem when primal is in the standard form:-

1) Write the dual of:

$$Max, Z = x_1 - 2x_2 + 3x_3$$

Subject to;

$$-2x_1 + x_2 + 3x_3 = 2 \quad \begin{cases} -2x_1 + x_2 + 3x_3 \leq 2 \\ -2x_1 + x_2 + 3x_3 \geq 2 \end{cases}$$

$$2x_1 + 3x_2 + 4x_3 = 1 \quad \begin{cases} 2x_1 + 3x_2 + 4x_3 \leq 1 \\ 2x_1 + 3x_2 + 4x_3 \geq 1 \end{cases}$$

$$x_1, x_2, x_3 \geq 0$$

Soln:-

1) Canonical Form:-

$$Max, Z = x_1 - 2x_2 + 3x_3$$

Subject to;

$$-2x_1 + x_2 + 3x_3 \leq 2$$

$$2x_1 - x_2 - 3x_3 \leq -2$$

$$2x_1 + 3x_2 + 4x_3 \leq 1$$

$-2x_1 - 3x_2 - 4x_3 \leq -1$

$x_1, x_2, x_3 \geq 0.$

Dual form:—

$\text{Min}, W = 2y_1 - 2y_2 + y_3 - y_4$

Subject to;

$-2y_1 + 2y_2 + 2y_3 - 2y_4 \geq 1$

$y_1 - y_2 + 3y_3 - 3y_4 \geq -2$

$3y_1 - 3y_2 + 4y_3 - 4y_4 \geq 3$

$y_1, y_2, y_3, y_4 \geq 0.$

NOTE:

Let $y_1 - y_2 = k_1$ &

$y_3 - y_4 = k_2$, Then

$\text{Min}, W = 2k_1 + k_2$

Subject to;

$-2k_1 + 2k_2 \geq 1$

$k_1 + 3k_2 \geq -2$

$3k_1 + 4k_2 \geq 3$ $\quad k_1, k_2 \geq 0.$

NOTE:– If the primal constraint is an equatn, corresponding dual variable will be "un-restricted" in sign.

Also, if the primal variable is unrestricted in sign, the corresponding dual constraints will be an equatn. "{equality}"

Primal	Dual
If 1 eqn is present	1 unrestricted var
1 unrestricted var	1 eqn
2 eqn	2 unrestricted var
2 unrestricted var	2 eqn

Q.) Find the dual of

$$\text{Min } z = 2x_1 + 3x_2 + 4x_3$$

Subject to;

$$2x_1 + 3x_2 + 5x_3 \geq 2$$
$$3x_1 + x_2 + 7x_3 = 3$$
$$x_1 + 4x_2 + 6x_3 \leq 0$$
$$x_1, x_2 \geq 0$$

$x_3 \rightarrow$ unrestricted in sign.

Soln:-

i) Canonical Form:-

$$\text{Max } Z' = -2x_1 - 3x_2 - 4x_3$$

Subject to;

$$-2x_1 - 3x_2 - 5x_3 \leq -2$$
$$3x_1 + x_2 + 7x_3 \leq 3$$
$$-3x_1 - x_2 - 7x_3 \leq -3$$
$$x_1 + 4x_2 + 6x_3 \leq 0.$$

$x_1, x_2 \geq 0$ $\quad x_3 \rightarrow$ unrestricted.

$$\text{Max } Z' = -2x_1 - \cancel{3x_2'} \overset{3x_2}{} + \cancel{3x_2''} - 4x_3' + 4x_3''$$

Subject to;

$$-2x_1 - 3x_2 - 5x_3' + 5x_3'' \leq -2$$
$$3x_1 + x_2 + 7x_3' - 7x_3'' \leq 3$$
$$-3x_1 - x_2 - 7x_3' + 7x_3'' \leq -3$$
$$x_1 + 4x_2 + 6x_3' - 6x_3'' \leq 0$$
$$x_1, x_2, x_3', x_3'' \leq 0.$$

Dual Form:-

$$\text{Min } w = -2y_1 + 3y_2 - 3y_3 + 0y_4$$

Subject to;

$$-2y_1 + 3y_2 - 3y_3 + y_4 \geq -2$$

$$-3y_1 + y_2 - y_3 + 4y_4 \geq -3$$

$$-5y_1 + 7y_2 - 7y_3 + 6y_4 \geq -4$$

$$5y_1 - 7y_2 + 7y_3 - 6y_4 \geq 4$$

Let $y_2 - y_3 = y_5$, Then

$$\text{Min}, w = -2y_1 + 3y_5 + 0y_4$$

Subject to;

$$-2y_1 + 3y_5 + y_4 \geq -2$$

$$-3y_1 + y_5 + 4y_4 \geq -3$$

$$\left.\begin{array}{l} -5y_1 + 7y_5 + 6y_4 \geq -4 \\ 5y_1 - 7y_5 - 6y_4 \geq 4. \end{array}\right\} 5y_1 - 7y_5 - 6y_4 = 4$$

Solution of primal & dual problem using simplex method:-

⇒ The value of the primal (dual) variable can be directly read off from the dual (primal) simplex table as follows:-

- The primal (dual) variable corresponds to a slack & or ~~all~~ surplus variable in dual (primal) problem, its value is read off as ~~on~~ the net evaluation of the slack and or ~~or~~ surplus variable from the optimal dual (primal) simplex table.

- If the primal (dual) variable corresponds to an artificial variable in the dual (primal) problem. Its value is read off ~~the~~ as the net evaluation of artificial var. after deleting the factor containing 'M' from the original dual (primal) simplex table.

1) Solve using the principle of duality:-

Min, $z = 3x_1 + 5x_2$

Subject to,

$2x_1 + 8x_2 \geq 40$ $\quad x_1, x_2 \geq 0.$

$3x_1 + 4x_2 \geq 50$

Soln:- The ~~dual~~ primal is:-

$$\text{Min}, Z = 3x_1 + 5x_2$$

$$2x_1 + 8x_2 \geq 40$$

$$3x_1 + 4x_2 \geq 50$$

$$x_1, x_2 \geq 0.$$

$\boxed{\text{Primal soln:- } x_1 = 15,\ x_2 = 5/4}$

The canonical form is:-

$$\text{Max } Z' = -3x_1 - 5x_2$$

$$-2x_1 - 8x_2 \leq -40$$

$$-3x_1 - 4x_2 \leq -50$$

$$x_1, x_2 \geq 0.$$

The dual is:-

$$\text{Min}, D = -40y_1 - 50y_2$$

Subject to;

$$-2y_1 - 3y_2 \geq -3$$

$$-8y_1 - 4y_2 \geq -5$$

$$y_1, y_2 \geq 0.$$

In std form
B should be +ve.
So x by -1 ⇒
& then add slack for

The standard form of dual is:-

$$\text{Max } D' = 40y_1 + 50y_2 + 0y_3 + 0y_4$$

Subject to;

$$2y_1 + 3y_2 \leq 3 \longrightarrow 2y_1 + 3y_2 + y_3 = 3$$

$$8y_1 + 4y_2 \leq 5 \Longrightarrow 8y_1 + 4y_2 + y_4 = 5$$

$$y_1, y_2, y_3, y_4 \geq 0.$$

(CF of dual of dual is primal)

Duality Theorem:-

1. Theorem - I :-

Dual of the dual is primal.

Proof:-

Let the primal problem in canonical form is

Max $Z = CX$

Subject to;

$AX \leq B$

$X \geq 0$

Then the dual of the problem is

Min $W = B^T Y$

Subject to;

$A^T Y \geq C^T$

$Y \geq 0.$

Canonical form of the above dual is:-

Max, $W' = -(B^T Y)$

Subject to;

$-(A^T Y) \leq -C^T$

$Y \geq 0.$

Dual of dual is:-

$$\text{Min } w'' = (-C^T)^T P.$$

Subject to;

$$(-A^T)^T P \geq (-B^T)^T$$

$$P \geq 0.$$

ie, $\text{Min } w'' = -CP$

Subject to;

$$-AP \geq -B$$

$$P \geq 0.$$

i.e, the canonical form of the dual of dual is primal.

Theorem - II

If either the primal/dual problem has an unbounded solution, then the solutions to the other problem is infeasible.

Theorem - III

If the dual problem has a de-generate soln; the primal problem has alternate optimum

Theorem - IV {Fundamental theorem of Duality}

If both the primal & dual problem have feasible solutions, then both have optimal solutions.

⇒ In other words, if the primal problem have an optimal soln, then the dual also have optimum soln & vice versa. Also known as strong duality theorem.

*. Theorem - V {Weak Duality Theorem}

For a maximization problem, every feasible soln to the dual has a objective fn: value greater than or equal to every feasible soln in the primal.

Theorem - VI {Optimality criterion theorem}

If the primal & dual have a feasible soln with same value of the objective fn:, then both are optimal to the primal & dual respectively.

Theorem - VII {Main Duality Theorem}

If the primal & dual have feasible soln: then both have optimal solns with the same value of the objective fn.

Theorem – VIII {Fundamental theorem of LPP}

(no max. +ve or most −ve)

Every LPP is either feasible or unbounded or infeasible.

(pos. variable present)

Theorem – IX {Complementary slackness theorem}

If X^* (x_1, s_1) & Y^* (x_1, s_2) are the optimal solns to the primal & dual respectively & U^* & V^* $(-s_3, s_6)$ are the values of the primal & dual slack variables at the optimum, then

$$X^* V^* + Y^* U^* = 0$$

Revised Simplex Method:-

Q) Min, $Z = -3x_1 + x_2 + x_3$

Subject to;

$$x_1 - 2x_2 + x_3 \le 11$$

$$-4x_1 + x_2 + 2x_3 \ge 3$$

$$2x_1 - x_3 = -1$$

$$x_i \ge 0,\ i = 1, 2, 3$$

Soln: Max, $Z = 3x_1 - x_2 - x_3$

Subject to,

$$x_1 - 2x_2 + x_3 + x_4 = 11$$

$$-4x_1 + x_2 + 2x_3 - x_5 = 3$$

$$-2x_1 + x_3 = 1$$

$$x_i \geq 0, \quad i = 1, 2, 3, 4, 5$$

Max, $z = 3x_1 - x_2 - x_3 - Mx_6 - Mx_7$ where $M \to \infty$

Subject to,

$$x_1 - 2x_2 + x_3 + x_4 = 11$$

$$-4x_1 + x_2 + 2x_3 - x_5 + x_6 = 3$$

$$-2x_1 + x_3 + x_7 = 1$$

$x_i \geq 0$, $i = 1, 2, 3, 4, 5$ &

$x_i \geq 0$ for $i = 6, 7$.