Queueing Theory

A flow of customers from finite or infinite population towards the service facility forms a queue (making line). On account of lack of capability to serve them all at a time.

The arriving unit that requires some service to be performed is called customer. The customer may be persons, machines, vehicles etc. queue stands for the no of customers waiting to be service. This does not include the customer being serviced. The process or system that performs services to the customer is termed by service channel or service facility.

Transient state and steady state of the system

Queueing theory involves the study of systems behavior over time. If the behavior of the system vary with time, its is said to be in ***transient state***. Usually a system is transient during early stages of its operation, when its behavior still depends upon the initial condition. A system is said to be in ***steady state*** condition if its behavior becomes independent of time. An essential condition for reaching a steady state is that the total elapsed time since the start of the operation must be sufficiently large.

Queue system or characteristics of queueing

The queueing system can be completely described by

1) The input (arrival pattern)

2) Service mechanism (service pattern)

3) The queue discipline

4) Customers behaviour.

The customer generally behave in the following four ways

1) Balking: A customer who leave the queue because the queue is too long and he has no time to wait or has no sufficient space.

2) Reneging: This occurs when a waiting customer leaves the queue due to impatience.

3)Priorities: In certain application some customers are served before others regardless of their arrivals. These customers have priority over others.

4)Jockeying: Customers may jockey from one waiting line to another. This is most common in a supermarket.

Queue Length

Queue length may be defined as the line length plus number of customers being served.

Notations

The notations used in the analysis of a queuing system are as follows:

n= number f customers in the system (waiting and in service)

P_n = Probability of n customers in the system

ë = mean customer arrival rate or average number of arrivals in the queuing system per unit time

ì = mean service rate or average number of customers completing service per unit time

$\frac{\lambda}{\mu}$ or $\rho = \frac{\textit{Average service completion time}(1/\mu)}{\textit{Average inter - arrival time}(1/\lambda)}$

= traffic intensity or server utilization factor (the expected fraction of time in which server is busy)

S= number of service channels (service facilities)

N = Maximum number of customers allowed in the system

L_s = mean number of customers in the system (waiting and in service)

L_q = mean number of customers in the queue (queue length)

L_b = mean length of non-empty queue

W_s = means waiting time in the system (waiting and in service)

W_q = means waiting time in the queue

P_w = Probability that an arriving customer has to wait.

Kendall's Notation for representing queuing models.

Kendell's Notation

(a/b/c): (d/e)

a = probability law of the arrival line (inner arrival line)

b = probability law according to which the customers are being served.

c = Number of channels or service stations.

d = Capacity of the system (in service and waiting) ie, the maximum number allowed to the system.

e = queue discipline

Difference Equation

If the behavior of the system is independent of time, the system is said to be in steady state, otherwise it is said to be in transient state. If P(n) be the probability of 'n' customers in the system at time t, we derive an equation for P_n, the probability of 'n' customers at any time in the steady state. For that let λ_n be the average arrival rate and 'μ_n' be the average service rate when there are 'n' customers in the system.

Let $P_n(t+\Delta t)$ is the probability of 'n' customers at time $t+\Delta t$. There are four different ways of selecting 'n' they are

1. There is no arrival or departure in $(t, t+\Delta t)$
2. There are n-1 customers at t and one arrival, no departure during Δt time.
3. There are (n+1) customers at t and no arrival, one departure during Δt time.
4. There are n customers at t and one arrival one departure during Δt time.

More than one arrival or departure during Δt is ruled out. Since probability of arrival occurring 't' time is λt.

i.e.,

$$P_n(t+\Delta t) = P_n(t)[1-\lambda_n\Delta t](1-\mu_n\Delta t) + P_{n-1}(t)$$
$$\lambda_{n-1}\Delta t(1-\mu_{n-1}+\Delta t) + P_{n+1}(t)[1-\lambda_{n+1}\Delta t]$$
$$\mu_{n+1}\Delta t + P_n(t)\lambda_n\Delta t\,\mu_n\Delta t$$

i.e.,

$$P_n(t+\Delta t) = P_n(t) - (\lambda_n+\mu_n)P_n(t)\Delta t +$$
$$\lambda_{n-1}P_{n-1}(t)\Delta t + \mu_{n+1}P_{n+1}(t)\Delta t$$

Since Δt is very small $(\Delta t)^2$ can be neglected.

$$P_n(t+\Delta t) - P_n(t) = -(\lambda_n+\mu_n)P_n(t)\Delta t + \lambda_{n-1}P_{n-1}(t)\Delta t +$$
$$\mu_{n+1}P_{n+1}(t)\Delta t$$

$$\frac{P_n(t+\Delta t)-P_n(t)}{\Delta t} = \lambda_{n-1}P_{n-1}(t) - (\lambda_n+\mu_n)P_n(t)$$
$$\mu_{n+1}P_{n+1}(t)$$

When $\Delta t \to 0$

$$P_n'(t) = \lambda_{n-1}P_{n-1}(t) - (\lambda_n+\mu_n)P_n(t) + \mu_{n+1}P_{n+1}(t) \text{ ----------- } (1)$$

Since the above equation does not hold for n=0, we derive a differential equation satisfied by Po (t) independently proceeding as before,

$$P_o(t+\Delta t) = P_o(t)[1-\lambda_o\Delta t] + P_1(t)(1-\lambda_1+\Delta t)\mu_1\Delta t$$

ie.
$$\frac{P_o(t+\Delta t)-P_o(t)}{\Delta t} = -\lambda_o P_o(t)\mu_1 P_1(t)$$

as
$$\Delta t \to o,\ P_o'(t) = -\lambda_o P_o + \mu_1 P_1(t) \text{ -------- } (2)$$

For the steady state, $P_n(t)$ and $P_o(t)$ are independent of time,

$$\therefore P_n'(t) = o$$
$$P_n'(t) = o$$

Hence (1) and (2) becomes

$$\lambda_{n-1}P_{n-1} - (\lambda_n+\mu_n)P_n + \mu_{n+1}P_{n+1} = 0 \text{ ------------------- } (3)$$

and
$$-\lambda_o P_o + \mu_1 P_1 = o$$

$$\therefore P_1 = \frac{\lambda_o p_o}{\mu_1}$$

Put n=1 in 3 we get

$$\lambda_o P_o + \mu_2 P_2 = (\lambda_1 + \mu_1) P_1$$

$$\mu_2 P_2 = (\lambda_1 + \mu_1) P_1 - \lambda_o P_o$$

$$= (\lambda_1 + \mu_1) \frac{\lambda_o P_o}{\mu_1} - \lambda_o P_o$$

$$= \frac{\lambda_o \lambda_1}{\mu_1} P_o$$

$$P_2 = \frac{\lambda_o \lambda_1}{\mu_1 \mu_2} P_o$$

Putting n=2, 3,4, ... in 3 and proceeding like above we get

$$P_3 = \frac{\lambda_0 \lambda_1 \lambda_2}{\mu_1 \mu_2 \mu_3} P_0$$

$$P_n = \frac{\lambda_0 \lambda_1 \ldots \lambda_{n-1}}{\mu_1 \mu_2 \ldots \mu_n} P_0, n = 1, 2, \ldots$$

Since number of customers in the system can be o or 1 or 2 or 3 etc, we get $\sum_{n=0}^{\infty} P_n = 1$

ie $P_o + \sum_{n=1}^{\infty} \left(\frac{\lambda_0 \lambda_1 \lambda_2 \ldots\ldots \lambda_{n-1}}{\mu_1 \mu_2 \ldots\ldots \mu_n} \right) P_o = 1$

$$\therefore P_o = \frac{1}{1 + \sum_{n=1}^{\infty} \left(\frac{\lambda_o \lambda_1 \ldots\ldots \lambda_{n-1}}{\mu_1 \mu_2 \ldots \mu_n} \right)}$$

Classification of Queuing Models

Model -1 $(M / M / 1):(\infty / FCFS)$

This model denotes poisson's arrival, poisson's departure, single service, infinite capacity. M is used to denote the Markovian property of exponential process. Here $\lambda_n = \lambda$ and $\mu_n = \mu (\lambda < \mu)$

Measures of Model-1

Let N denote the total number of customers in the system [ie., in queue + service]

Since N can take discrete values 0,1,2,.........

such that $P(N = n) = P_n = \left(\frac{\lambda}{\mu} \right)^n P_o$

Also

$$P_0 = \frac{1}{1 + \sum_{n=1}^{\infty} \left(\frac{\lambda_0 \lambda_1 \ldots \lambda_{n-1}}{\mu_1 \mu_2 \ldots \mu_n} \right)} = \frac{1}{\sum_{n=1}^{\infty} \left(\frac{\lambda}{\mu} \right)^n}$$

$$= \frac{1}{\sum_{n=1}^{\infty} \left(\frac{\lambda}{\mu} \right)^n}$$

$$= \frac{1}{\frac{1}{1 - \frac{\lambda}{\mu}}} = 1 - \frac{\lambda}{\mu}$$

$$\therefore P_n = \left(\frac{\lambda}{\mu} \right)^n \left(1 - \frac{\lambda}{\mu} \right)$$

1. Average number of Customers in the system

$$Ls = E(N) = \sum_{n=0}^{\infty} n P_n$$

$$= \sum_{n=0}^{\infty} n\left(\frac{\lambda}{\mu}\right)^n\left(1-\frac{\lambda}{\mu}\right)$$

$$= \left(\frac{\lambda}{\mu}\right)\left(1-\frac{\lambda}{\mu}\right)\sum_{n=1}^{\infty} n\left(\frac{\lambda}{\mu}\right)^{n-1}$$

$$= \left(\frac{\lambda}{\mu}\right)\left(1-\frac{\lambda}{\mu}\right)\left[1+2\left(\frac{\lambda}{\mu}\right)+3\left(\frac{\lambda}{\mu}\right)^2+\ldots\ldots\right]$$

$$= \left(\frac{\lambda}{\mu}\right)\left(1-\frac{\lambda}{\mu}\right)\left[1-\frac{\lambda}{\mu}\right]^{-2}$$

$$= \left(\frac{\lambda}{\mu}\right)\left(1-\frac{\lambda}{\mu}\right)^{-1}$$

$$= \left(\frac{\lambda}{\mu}\right)\times\frac{1}{\left(1-\frac{\lambda}{\mu}\right)} = \frac{\lambda}{\mu-\lambda}$$

$\frac{\rho}{1-\rho}$ $\frac{\lambda}{\lambda\left(1-\frac{\lambda}{\mu}\right)}$

2. Average number of queue in the system

Since $(M/M/1):(\infty/FCFS)$ is a single server system, if N is the number of customers in the system then N-1 is the number of Customers in the queue.

$$Lq = E(n-1) = \sum_{n=1}^{\infty}(N-1)Pn$$

$$= \sum_{n=1}^{\infty}(n-1)\left(\frac{\lambda}{\mu}\right)^n\left(1-\frac{\lambda}{\mu}\right)$$

$$= \left(\frac{\lambda}{\mu}\right)^2\left(1-\lambda/\mu\right)\sum_{n=2}^{\infty}(n-1)\left(\frac{\lambda}{\mu}\right)^{n-2}$$

$$= \left(\frac{\lambda}{\mu}\right)^2\left(1-\frac{\lambda}{\mu}\right)\left[1+2\left(\frac{\lambda}{\mu}\right)+3\left(\frac{\lambda}{\mu}\right)^2+\ldots\ldots\right]$$

$$= \left(\frac{\lambda}{\mu}\right)^2\left(1-\frac{\lambda}{\mu}\right)\left(1-\frac{\lambda}{\mu}\right)^{-2}$$

$$= \frac{\left(\frac{\lambda}{\mu}\right)^2}{1-\frac{\lambda}{\mu}} = \frac{\lambda^2}{\mu(\mu-\lambda)}.$$

3. Average number of Customers in a nonempty queue (L_b)

Average number of Customers in a

Non empty queue $= E\{N-1/N-1=0\}$

$$\frac{E(N-1)}{P(N-1>0)} = \frac{\lambda^2}{\mu(\mu-\lambda)}\frac{1}{\sum_{n=2}^{\infty} Pn}$$

$$\frac{E(N-1)}{P(N-1>0)}=\frac{\lambda^2}{\mu(\mu-\lambda)}\frac{1}{\sum_{n=2}^{\infty}Pn}$$

$$=\frac{\lambda^2}{\mu(\mu-\lambda)}\times\frac{1}{\sum_{n=2}^{\infty}\left(\frac{\lambda}{\mu}\right)^n\left(1-\lambda/\mu\right)}$$

$$=\frac{\lambda^2}{\mu(\mu-\lambda)}\times\frac{1}{\left(1-\lambda/\mu\right)\left(\frac{\lambda}{\mu}\right)^2\sum_{n=2}^{\infty}\left(\frac{\lambda}{\mu}\right)^{n-2}}$$

$$=\frac{\lambda^2}{\mu(\mu-\lambda)}\times\frac{1}{\left(1-\lambda/\mu\right)\left(\frac{\lambda}{\mu}\right)^2\left(1-\frac{\lambda}{\mu}\right)^{-1}}\left[\sin ce\sum_{n=0}^{\infty}x^n=\frac{1}{1-x}\right]$$

$$=\frac{\mu}{\mu-\lambda}$$

4. Probability that the number of customers in the system exceeds N

$$P(n>N)=\sum_{n=N+1}^{\infty}P_n$$

$$=\sum_{n=N+1}^{\infty}\left(\frac{\lambda}{\mu}\right)^n\left(1-\frac{\lambda}{\mu}\right)$$

$P(n>10)=\sum_{n=11}^{\infty}P_n$

$=P_0\sum_{n=11}^{\infty}\rho^n$

$=P_0\,\rho^{11}\left[\rho^{12}+\rho^{13}+\dots\right]\ \sum_{n=11}^{\infty}\rho^{n-11}$

$=P_0\rho^{11}\left[\rho^0+\rho^1+\dots\right]$

$=P_0\rho^{11}(1-\rho)$

$$\left(\frac{\lambda}{\mu}\right)^{N+1}\left(1-\frac{\lambda}{\mu}\right)\sum_{n=N+1}^{\infty}\left(\frac{\lambda}{\mu}\right)^{n-(N+1)}$$

$$=\left(\frac{\lambda}{\mu}\right)^{N+1}$$

5. Probability that the waiting time of a customer in the system exceeds t

$$P(w_s>t)=\int_t^{\infty}f(w_s)dw$$

Where $f(w_s)$ is the p.d.f of w_s and is given by

$$f(w_s)=(\mu-\lambda)e^{-(\mu-\lambda)w}$$

$$\text{in } P(w_s>t)=\int_t^{\infty}(\mu-\lambda)e^{-(\mu-\lambda)w}dw$$

$$=(\mu-\lambda)\left[\frac{e^{-(\mu-\lambda)w}}{-(\mu-\lambda)}\right]_t^{\infty}$$

$$=e^{-(\mu-\lambda)t}$$

6. Average waiting time of a customer in the system

$$E(w_s)=\int_0^{\infty}w\mu\left(1-\frac{\lambda}{\mu}\right)e^{-(\mu-\lambda)w}\,dw$$

$$=\int_0^{\infty}w(\mu-\lambda)e^{-(\mu-\lambda)w}dw$$

$$=\frac{1}{\mu-\lambda}$$

7. Average waiting time of a customer in the queue

$$E(w_q) = \int_0^\infty w\, g(w_q)\, dw$$

$$= \frac{\lambda}{\mu}(\mu-\lambda)\int_0^\infty w\, e^{-(\mu-\lambda)w} dw$$

[where $g(w_q)$ is the p.d.f of w_q and is given by

$$g(w_q) = \frac{\lambda}{\mu}(\mu-\lambda)e^{-(\mu-\lambda)w}$$

$L_s = \frac{\rho}{1-\rho}$

$L_q = \frac{\rho^2}{1-\rho}$

$$= \frac{\lambda}{\mu}(\mu-\lambda)\left[\frac{w e^{-(\mu-\lambda)w}}{-(\mu-\lambda)} - \frac{e^{-(\mu-\lambda)w}}{(\mu-\lambda)^2}\right]_0^\infty$$

$$= \frac{\lambda}{\mu}(\mu-\lambda)\times\frac{1}{(\mu-\lambda)^2} = \frac{\lambda}{\mu(\mu-\lambda)}$$

$W_s = \frac{1}{\mu(1-\rho)}$

8. Average waiting time of a customer in the queue, if he has to wait or expected waiting time in the queue for busy system.

$$E(w_q / w_q > 0) = \frac{E(w_q)}{P(w_{q>0})}$$

$W_q = \frac{\rho}{\mu(1-\rho)}$

$W_s = \frac{1}{\mu(1-\rho)}$

$$= \frac{E(w_q)}{1-P(w_q=0)}$$

$$= \frac{\lambda}{\mu(\mu-\lambda)}\times\frac{1}{1-P_0}$$

$$= \frac{\lambda}{\mu(\mu-\lambda)}\times\frac{1}{\left[1-\left(1-\frac{\lambda}{\mu}\right)\right]}$$

$$= \frac{1}{\mu-\lambda}$$

$L_s = \frac{\rho}{1-\rho}$

$L_q = \frac{\rho^2}{1-\rho}$

$W_s = \frac{1}{\lambda} L_s = \frac{1}{\lambda}\left[\frac{\rho}{1-\rho}\right] = \frac{1}{\lambda}\frac{\lambda}{\mu(1-\frac{\lambda}{\mu})} = \frac{1}{\mu(\mu-\lambda)}$

10. Probability of Queue length $\geq n = \rho^n$

Little'e Formula & Inter relationship between Ls, Lq, Ws, Wq

a. $L_s = \lambda (W_s)$ b. $L_q = \lambda W_q$

$W_s = \frac{1}{\lambda} L_s$

c. $W_q = W_s - \frac{1}{\mu}$

$\frac{5}{9} \times 8 = \frac{40}{9} = 4.4$

Examples:

1. A barber shop with one man takes exactly 25 minutes to complete hair cut. If customers arrive in poisson's fashion at an average rate of every 40 minutes, how long on the average must a customer wait for service.

Mean arrival rate, λ = customer/minutes

= 1/40 customer/minutes

Mean service rate, μ = 1/25 customer/minutes

Waiting time in the queue,

$$W_q = \frac{\lambda}{\mu(\mu-\lambda)}$$

$$= \frac{\frac{1}{40}}{\frac{1}{25}(\frac{1}{25}-\frac{1}{40})}$$

$$= \textbf{41.67 minutes}$$

2. Customers arrive at one persons barber shop according to poisson's process with mean inter arrival line 20 minute. Customers spend an av 15 minute in the barber chair.

   a) What is the probability that a customer will not have to wait for a hair cut.

   b) What is the expected number of customers in the barber shop?

   c) How much line can a customer expected to spend in the barber shop?

$\lambda = \frac{1}{20}; \quad \mu = \frac{1}{15}$

a) The probability that the customer will not have to wait

$$= 1 - \frac{\lambda}{\mu} = \frac{1}{4} = 0.25$$

b) $$Ls = \frac{\rho}{1-\rho} = \frac{\frac{3}{4}}{1-\frac{3}{4}} = \frac{\frac{3}{4}}{\frac{1}{4}} = 3$$

c) $$Ws = Wq + \frac{1}{\mu}$$

$$= \frac{1}{\mu - \lambda}$$

$$= \frac{1}{\frac{1}{15}-\frac{1}{20}}$$

$$= \textbf{60 minutes}$$

3. Arrival at a telephone booth are considered to be a poisson with an average time of 10 minute into one arrival and next. The length of a phone call is assumed to be distributed exponentially with 3 minute.

   a) Find the average number of persons waiting in the system

   b) What is the time that a person arriving in the booth will have to wait in the queue.

   c) What is the probability that it will take more than 10 minutes all together to wait for phone and complete his call.

$P(W_s > 10) = e^{-P_0 \mu t}$

$= e^{-(1-0.3)\frac{1}{3}10}$

$= e^{-\frac{10}{3}(1-0.3)}$

$= e^{-10/3 \times 0.7}$

$= e^{-7/3}$

$= e^{-2.33}$

$\rho = \lambda/\mu = \frac{1/10}{1/3} = \frac{3}{10} = 0.3$

$3\overline{)7}^{2.3}$, $\frac{6}{10}$, $\frac{9}{10}$

$\lambda = \frac{1}{10}; \quad \mu = \frac{1}{3}$

$$Ls = \frac{\rho}{1-\rho}$$

$$= \frac{\frac{1}{10}}{\frac{1}{3}} = \frac{\frac{3}{10}}{\frac{7}{10}} = 0.428$$

b. $Wq = \frac{\lambda}{\mu(\mu-\lambda)}$

$= \frac{3}{10} = \frac{70}{7} = 1.28 \text{ minutes}$

c. Probability $= \int_{10}^{\infty} \rho(\mu-\lambda)e^{-(\mu-\lambda)w}\,dw$

Verify

$= \int_{10}^{\infty} 0.3x0.23e^{-0.23\omega}\,d\omega$

$= 0.3\ x0.23 \left[\frac{e^{-0.23\omega}}{-0.23}\right]_{10}^{\infty}$

$= -0.3\,[0 - e^{-2.3}] = 0.3\ e^{-2.3}$

$= 0.03$

4. A self service store employs one cashier at counter. 9 customers arrive on an average every 5 minute. while the cashier can serve 10 customers in 5 minute. Assuming poission distribution for arrival rate and exponential distribution for service rate. Find
   a) Average. No: of customers in the system
   b) Average. No. of customers in the queue or average queue length
   c) Average time a customer spend in the system
   d) Average time a customer waits for being served.

$\lambda = \frac{9}{5};\ \mu = \frac{10}{5} = 2$

a. $Ls = \frac{\rho}{1-\rho}$

$\rho = \frac{\frac{9}{5}}{\frac{10}{5}} = \frac{9}{10} = 0.9$

$Ls = \frac{0.9}{0.1} = 9$

b. $Lq = Ls - \frac{\lambda}{\mu} = 9 - 0.9 = 8.1$

c. $Ws = Wq + \frac{1}{\mu}$

$Wq = \frac{\lambda}{\mu(\mu-\lambda)}$

$= \frac{\frac{9}{5}}{\frac{10}{5}\left(\frac{10}{5}-\frac{9}{5}\right)}$

$= \frac{9}{10} \times \frac{5}{5} = 4.5$

$Ws = 4.5 + \frac{1}{2} = 5$

d. $Wq = 4.5$

5. A TV repair man finds that the time spend on his jobs has an exponential distribution with mean 30 minutes. If he

repairs sets in the order in which they come in. If the arrival of sets is approximately with an average rate of 10 per 8 hr day,

(a) What is the repairmen's expected idle time each day.

b) How many jobs are ahead of the average set jut brought in.

$$\mu = \frac{1}{30} \text{ sets/minutes}$$

$$\lambda = \frac{10}{8 \times 60} = \quad = \frac{1}{48} \text{ sets/minutes}$$

$$\text{Traffic intensity, } \rho = \frac{\lambda}{\mu} \times 8 \times 60 = \frac{\frac{1}{48} \times 8}{\frac{1}{30}} = 5$$

$P_0 = 1 - \rho$ 1. The idle time $= 8 - 5 = 3$ hrs.

b. $$L_s = \frac{\lambda}{\mu - \lambda}$$

$$= \frac{\frac{1}{48}}{\frac{1}{30} - \frac{1}{48}} = \frac{5}{3}$$

$$Ls \approx 2 sets$$

6. In a supermarket, the average arrival rate of customer is 10 every 30 minute following poisson process. The average time taken by a cashier to list and calculate the customers purchase is 2.5 minute following exponential distribution.

(a) What is the probability that the queue length exceed 6

(b) What is the expected time spend by a customer in the sytem

$$\lambda = \frac{10}{30} = \frac{1}{3} \text{ customer/minute}$$

$$\mu = \frac{1}{2.5} \text{ customer/minute}$$

$$\rho = \frac{\lambda}{\mu} = \frac{\frac{1}{3}}{\frac{1}{2.5}} = 0.83$$

Probability of queue size >6 P(L

1. $\rho^n = \rho^6 = 083^6 = 0.326$

2. $$Ws = \frac{1}{\mu - \lambda} = \frac{1}{0.067}$$

$$= 14.92 \text{ minutes}$$

7. On an average 96 patients per 24 hour day require the service of an emergency clinic. Also on average, a patient requires 10minutes of active attention. Assume that the facility can handle only one emergency at a time .Suppose that it costs the clinic Rs. 100 per patient treated to obtain an average servicing time of 10minutes, and that each minute of decrease in this average time would cost Rs.10 per patient treated. How much would have to be budgeted by the clinic to decrease the average size of the queue from $1\frac{1}{3}$ patients to ½ patient.

Here $\lambda = \frac{96}{24 \times 60} = \frac{1}{15}$ *patient / minute*,

$$\mu = \frac{1}{10} \text{ patient / minute}$$

Expected number of patients in the waiting time

$$Lq = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{\left(\frac{1}{15}\right)^2}{\frac{1}{10}\left(\frac{1}{10}-\frac{1}{15}\right)} = 1\frac{1}{3} \text{ patients}$$

But, $Lq = 1\frac{1}{3}$ is reduced to $Lq^1 = \frac{1}{2}$

Therefore, substituting $Lq^1 = \frac{1}{2}, \lambda^1 = \lambda = \frac{1}{15}$ in the formula

$$Lq^1 = \frac{\lambda^{1^2}}{\mu^1\left(\mu^1-\lambda^1\right)}, \text{ we get}$$

$$\frac{1}{2} = \frac{\left(\frac{1}{15}\right)^2}{\mu^1\left(\mu^1-\frac{1}{15}\right)}$$

Which gives $\mu^1 = \frac{2}{15}$ patient /minute.

Hence the average rate of treatment is $\frac{1}{\mu^1} = 7.5$ minutes.

Consequently, the decrease in the average rate of treatment

$= 10 - \frac{15}{2} = \frac{5}{2}$ minutes. and the budget per patient

$$100 + \frac{5}{2} \times 100 = Rs.125$$

So in order to get the required size of the queue, the budget should be increased from Rs. 100 to Rs. 125 per patient.

8. The mean rate of arrival of planes at an airport during the peak period is 20/hour but the actual number of arrivals in any hour follows a Poisson distribution. The airport can land 60 planes per hour on an average in good weather and 30 planes per hour inbad weather, but the actual number of landing in hour with respective averages. When there is congestion, the planes are forced to fly over the field in the stack awaiting the landing of other planes that arrived earlier.

(i) How many planes would be flying over the field in the stack on an average in good weather and in bad weather?

(ii) How long a plane would be in the stack and in the process of landing in good and in bad weather?

Here $\lambda = 20$ planes/ hour

$$\text{and } \mu = \begin{cases} 60 \text{ planes per hour in good weather} \\ 30 \text{ planes per hour in bad weather} \end{cases}$$

Answer:

(i) $$Lq = \frac{\lambda^2}{\mu(\mu-\lambda)} = \begin{cases} \frac{20^2}{60(60-20)} \\ \frac{20^2}{30(30-20)} \end{cases} = \begin{cases} \frac{1}{6} \\ \frac{4}{3} \end{cases}$$

(ii) $W_s = \frac{1}{\mu-\lambda} = \left\{ \begin{array}{l} \frac{1}{60-20} \\ \frac{1}{30-20} \end{array} \right. = \left\{ \begin{array}{l} \frac{1}{40} \\ \frac{1}{10} \end{array} \right.$

9. A refinery distributes its products by trucks, loaded at the loading dock. Both company trucks and independent distributors trucks are loaded. The independent firms complained that some times they must wait in time and thus lose money paying for a truck and driver, that is only waiting. They have asked the refinery either to put in a second loading dock or to discount prices equivalent to the waiting time. Extra loading dock cost Rs. 100/- per day whereas the waiting time for the independent firms cost Rs25/- per hour. The following data have been accumulated. Average arrival rate of all trucks is 2 per hour and average service rate is 3 per hour. Thirty per unit of all trucks are independent. Assuming that there rates are random according to the Poisson distributions, determine.:

(a) The probability that a truck has to wait

(b) the waiting time of a truck that wait and

(c) the expected cost of waiting time of independent trucks per day.

Is it advantageous to decide in favour of a second loading dock toward off the Complaints?

We are given that $\lambda = 2$ per hour and $\mu = 3$ per hour

(a) The probability that a truck has to wait for service is the utilization factor,

$$\rho = \frac{\lambda}{\mu} = \frac{2}{3} = 0.66$$

(b) The waiting time of a truck that waits is

$$\frac{1}{\mu-\lambda} = \frac{1}{3-2} = 1 \ hour$$

(c) The total expected waiting time of independent trucks per day is given by Expected waiting time

= Trucks per day × % independent truck × Expected Waiting time per truck

$$= (2\times 8)(0.3W_q) = 16\times 0.3\times \frac{\lambda}{\mu(\mu-\lambda)}$$

$$= 4.8\times \frac{2}{3(3-2)}$$

$$= 3.2 \ hour \ per \ day$$

Expected Cost = Rs (3.2 x 25) = Rs. 80

10. An airlines organization has one reservation clerk on duty in its local branch at any given time. The clerk handles information regarding passenger reservation and flight timings. Assume that the number of customers arriving during any given period is Poisson distributed with an arrival rate of 8 per hour and that the reservation clerk can serve a customer in 6 minutes on an average with an exponentially distributed service time.

(i) What is the probability that the system is busy?

(ii) What is the average time a customer spends in the system ?

(iii) What is the average length of the queue and what is the number of customers in the system?

According to the given information mean arrival rate, $\lambda = 8$ customers per hour Mean service rate, $\mu = \frac{60}{6} = 10$ customers per hour $\therefore p = \frac{\lambda}{\mu} = \frac{8}{10}$ or $\frac{4}{5}$

(i) The probability that the system is busy is given by

$$1-P_0=1-\left[1-\frac{\lambda}{\mu}\right]$$

$$=\frac{\lambda}{\mu}=0.8$$

ie, 80% of the time system is busy.

(ii) The average time a customer spends in the system is given by

$$W_s=\frac{1}{\mu-\lambda}=\frac{1}{10-8}$$

$$=\frac{1}{2}\text{ hours or 30 minutes}$$

(iii) The average length of the queue is given by :

$$Lq=\frac{\lambda}{\mu}\times\frac{\lambda}{\mu-\lambda}=\frac{8}{10}\times\frac{8}{10-8}$$

$$=3.2\text{ Customers.}$$

The average number of customers in the system is given by :

$$Ls=\frac{\lambda}{\mu-\lambda}=\frac{8}{10-8}\text{ or 4 customers}$$

11: Customers arrive at a one-window drive according to a Poisson distribution with mean of 10 minutes and service time per customer is exponential with mean of 6 minutes. The space in front of the window can accommodate only three vehicles including the serviced one. Other vehicles have to wait outside this space. Calculate:

(a) Probability that an arriving customer can drive directly to the space in front of the window.

(b) Probability that an arriving customer will have to wait outside the directed space.

(c) How long an arriving customer is expected to wait before getting the service?

Solution: we have $\lambda=6$ customers per hour; $\mu=10$ customers per hour (a) probability that arriving customer can drive directly to the space in front of the window $=P_0+P_1+P_2$

$$=\left(1-\frac{\lambda}{\mu}\right)+\frac{\lambda}{\mu}\left(1-\frac{\lambda}{\mu}\right)+\left(\frac{\lambda}{\mu}\right)^2\left(1-\frac{\lambda}{\mu}\right)$$

$$=\left(1-\frac{\lambda}{\mu}\right)\left[1+\frac{\lambda}{\mu}+\left(\frac{\lambda}{\mu}\right)^2\right]$$

$$=\left(1-\frac{6}{10}\right)\left[1+\frac{6}{10}+\left(\frac{6}{10}\right)^2\right]$$

$$=\frac{98}{125}=0.784$$

(b) Probability that an arriving customer will have to wait outside the directed space $=1-(P_0+P_1+P_2)=1-0.784=0.216$

(c) Expected waiting time of a customer before getting the service is :

$$W_q=\frac{\lambda}{\mu(\mu-\lambda)}=\frac{6}{10(10-6)}=\frac{3}{20}\text{ hr. or 9 minutes.}$$

12. In a railway marshaling yard, goods trains arrive at a rate of 30 trains per day. Assuming that the inter-arrival time follows an exponential distribution and the service time (time taken to hump a train) distribution is also exponential with an average of 36 minutes. Calculates

(a) Expected queue size (line length)

(b) Probability that the queue size exceeds 10.

If the input of trains increases to an average of 33 per day, what will be the change in (i) and (ii)?

Solution :

We have $\lambda = 30/60 \times 24 = 1/48$ trains per minute and $\mu = 1/36$ trains per minute traffic intensity and $\rho = \lambda/\mu = 36/48 = 0.75$.

(a) Expected queue size (line length) $L_s = \dfrac{\rho}{1-\rho}$

$$= \frac{0.75}{1-0.75} = 3 \text{ trains.}$$

(b) Probability that the queue size exceeds 10 is:

$$P(n \geq 10) = \rho^{10} = (0.75)^{10}$$

$$= 0.06$$

Now, if the input increases to 33 trains per day, then we have $\lambda = 33/60 \times 24 = 11/480$ trains per minute and $\mu = 1/36$ trains per minute; $\rho = \lambda/\mu = (11/480)(36) = 0.93$.

(i) $$L_s = \frac{\rho}{1-\rho} = \frac{0.83}{1-0.83} = 5 \text{ trains (approx.)}$$

(ii) $$P(n \geq 10) = \rho^{10} = (0.83)^{10} = 0.2 \text{ (approx.)}$$

13: Arrivals at telephone booth are considered to be Poisson with an average time of 10 minutes between one arrival and the next. The length of phone call is assumed to be distributed exponentially, with mean 3 minutes.

(a) What is the probability that a person arriving at the booth will have to wait?

(b) The telephone department will install a second booth when convinced that an arrival would expect waiting for at least 3 minutes for a phone call. By how much should the flow of arrivals increase in order to justify a second booth?

(c) What is the average length of the queue that forms from time to time?

(d) Find the average number of customers in the systems.

(e) What is the probability that it will take him more than 10 minutes altogether to wait for the phone and complete his call?

Solution: We have $\lambda = 1/10 = 0.10$ person per minute and $\mu = 1/3 = 0.33$ person per minute

(a) Probability that a person has to wait at the booth.

$$P(n > 0) = 1 - P_0 = \lambda/\mu = 0.10/0.33 = 0.3$$

(b) The installation of second booth will be justified only if the arrival rate is more than the waiting time. Let λ' be the increased arrival rate. Then expected waiting time in the queue will be $W_q = \dfrac{\lambda'}{\mu(\mu-\lambda')}$ or $3 = \dfrac{\lambda'}{0.33(0.33-\lambda')}$ or

$\lambda' = 0.16$ where $W_q = 3$ (given and $\lambda = \lambda'$ (say) for second booth. Hence, the increase in the arrival rate is $0.16 - 0.10 = 0.06$ arrivals per minute.

(c) Average length of non-empty queue

$$L_b = \frac{\mu}{\mu-\lambda} = \frac{0.33}{0.23} = 2 \text{ customers (approx.)}$$

(d) Average number of customers in the system

$$L_s = \frac{\lambda}{\mu-\lambda} = \frac{0.10}{0.33-0.10}$$

$$= \frac{0.10}{0.23} = 1 \text{ customer (approx.)}$$

(e) Probability of waiting for 10 minutes or more is given by

$$P(t \geq 20) \int_{10}^{\infty} \frac{\lambda}{\mu}(\mu-\lambda)e^{-(\mu-\lambda)t}\,dt$$

$$= \int_{10}^{\infty} (0.3)(0.23)e^{-0.23t}\,dt$$

$$= 0.069\left[\frac{e^{-0.23t}}{-0.23}\right]_{10}^{\infty} = 0.03$$

This show that 3 per cent of the arrivals on an average will have to wait for 10 minutes or more before they can use the phone.

14: At a public telephone booth in a Post Office arrivals are considered to be Poisson with an average inter-arrival time of 12 minutes. The length of a phone call may be assumed to be distributed exponentially with an average of 4 minutes. Calculate the following:

(a) What is the probability that a fresh arrival will not have to wait for the phone?

(b) What is the probability that an arrival will have to wait more than 10 minutes before the phone is free?

(c) What is the average length of queue that form from time to time?

Solution : We have $\lambda = 1/12$ per minute and $\mu = 1/4$ per minute; $\lambda/\mu = 4/12 = 0.33$.

(a) Probability that an arrival will not have to wait is given by $1 - P(w > 0) = 1 - \rho = 1 - 0.33 = 0.67$

(b) Probability that an arrival have to wait for at least 10 minutes is given by

$$P(w \geq 10) = \int_{10}^{\infty} \frac{\lambda}{\mu}(\mu - \lambda) e^{-(\mu-\lambda)t} dt$$

$$= \int_{10}^{\infty} (0.33)(0.165) e^{-0.165t} dt$$

$$= 0.054 \left[\frac{e^{-0.165t}}{-0.165} \right]_{10}^{\infty} = 0.063$$

This shows that 6.3 per cent of the arrivals on an average will have to wait for 10 minutes or more before they can use the phone.

(c) The average length of queue that form from time to time is

$$L_b = \frac{\mu}{\mu - \lambda} = \frac{0.25}{0.25 - 0.085}$$

$$= \frac{0.25}{0.165} = 1.5 = 2 \text{ customers.}$$

15: Consider a self-service store with one cashier. Assume Poisson arrivals and exponential service times. Suppose that 9 customers arrive on the average every 5 minute and the cashier can serve 10 in 5 minutes. Calculate the following:

(a) Average number of customers queuing for service.

(b) Probability of having more than 10 customers in the system.

(c) Probability that a customer has to queue for more 2 minutes.

If the service can be speeded upto 12 in 5 minutes by suing a different cash register, what will be the effect on the quantities (1), (b) and (c).

Solution: Case I : Give that λ=9/5 per minute and μ= 10/5 per minute

(a) Average number of customers queing for service:

$$L_s = \frac{\lambda}{\mu - \lambda} = \frac{1.8}{2 - 1.8} = 9 \text{ customers.}$$

(b) Probability of having more than 10 customers in the system is:

$$P(\geq 10) = \rho^{10} = \left(\frac{1.8}{2.0}\right)^{10} = (0.9)^{10}$$

(c) $$P(w \geq 2) = \int_{2}^{\infty} \frac{\lambda}{\mu}(\mu - \lambda) e^{-(\mu-\lambda)t} dt$$

$$= 0.9e^{-2(0.2)} = 0.9e^{-0.4} = 0.603$$

Case 2: When μ = 12/5 instead of μ=10/5,

$$L_s = \frac{\lambda}{\mu - \lambda} = 3 \text{ customers}$$

and $$P(w \geq 2) = \int_{2}^{\infty} \frac{\lambda}{\mu}(\mu - \lambda) e^{-2(\mu-\lambda)}$$

$$= 0.75\, e^{-1.2} = 0.225$$

Clearly, Case 2 is preferred because the average number of customers will be reduced to 3 and the probability that a customer has to wait for more than 2 minutes also reduces to 0.225 from 0.603.

Model II: {(M/M/I) : (N/FCFS)}

This is a Single Server, Finite (or limited) Queue Model. The difference between Model I and Model II is, in Model II the maximum number of customers is limited to N.

Measures of Model II.

$$P_0 = \frac{1-\rho}{1-\rho^{N+1}}; \rho \neq 1 \, and \, \rho < 1; \rho = \frac{\lambda}{\mu}$$

$$P_n = \begin{cases} \left(\frac{1-\rho}{1-\rho^{N+1}}\right)\rho^n; 0 \leq n \leq N; \rho = \frac{\lambda}{\mu} \neq 1 \\ \frac{1}{N+1} \quad ; \rho = 1 (\lambda = \mu) \end{cases}$$

$\frac{\rho - 1}{\rho^{N+1} - 1} \quad \rho > 1$

1. Expected number of customers in the system

$$L_s = \sum_{n=1}^{N} nP_n = \sum_{n+1}^{N} n\left(\frac{1-\rho}{1-\rho^{N+1}}\right)\rho^n$$

$$= \begin{cases} \frac{\rho}{1-\rho} - \frac{(N+1)\rho^{N+1}}{1-\rho^{N+1}}; \rho \neq 1 (\lambda \neq \mu) \\ \frac{N}{2} \quad ; \quad \rho = 1 (\lambda = \mu) \end{cases}$$

Expected queue length $L_q = L_s - \frac{\lambda}{\mu}$.

Expected waiting time of a customer in the system

$W_s = \frac{L_s}{\lambda(1-P_N)}$ $\qquad W_s = \frac{1}{\lambda'} L_s \qquad \lambda' = \lambda(1-P_N)$

Expected waiting time of a customer in the queue

$W_q = W_s - \frac{1}{\mu}$ $\qquad W_q = \frac{1}{\lambda'} L_q \qquad \lambda' = \lambda(1-P_N)$

Examples :

1. In a railway marshelling yard goods train arrived at the rate of 30 trains per day. Assume that in the arrival time follows an exponential distribution and the service time is also to be assumed as exponential with mean of 36 minute.

(a) Calculate the probability that the yard is empty.

(b) The average queue length. Assuming that the line capacity of yard is 9 trains.

λ $= \frac{30}{1 \times 24 \times 60} = \frac{1}{48}$ trains/minutes

μ $= \frac{1}{36}$ trains/minutes

ρ $= \frac{\lambda}{\mu} = \frac{1}{48} x36 = \frac{3}{4} = 0.75$ $\qquad \rho < 1$

P_0 $= \frac{\cancel{1.75}\ 0.25}{1-75^{10}} = 0.205 = 0.28 \qquad = \frac{1-\rho}{1-\rho^{N+1}} = \frac{1-0.75}{1-0.75^{10}}$

$= 0.25$

$Ls = \sum_{n=0} n P_n \quad \sum_{n=0} n\rho P_0 \quad P_0 \sum_{n=0}^{N} n\rho^n$

$= P_0 + P_1 + 2P_2 + 3P_3 + \cdots + NP_N$

$$Ls = P_0 \sum_{n=0}^{N} n\rho^n \quad =$$

$$Ls = \frac{0.75}{0.25} - \frac{10 \times (0.75)^{10}}{1 - 0.75^{10}} = 0.205 \sum_{n=0}^{9} n(0.75)^n$$

0.56368

3 -

$$= .205\left[.75 + 2(.75)^2 + 3(.75)^3 + \ldots + 9(.75)^9\right]$$

$\approx 3\,trains$

2. A barber shop has space to accommodate only 10 customers. He can serve only one person at a time. If a customer comes to his shop and finds its full. He goes to the next shop. Customers randomly arrive at an average rate 10/hr and the barber service time is –ve exponential with an average of 5minuts/customer

Find P_0 & P_n

$N = 10$

$\lambda = \frac{10}{60} = \frac{1}{6}$ customer/minutes

$\mu = \frac{1}{5}$ customer/minutes

$\rho = \frac{\lambda}{\mu} = \frac{1}{6} \times 5 = \frac{5}{6} = 0.83$ 4.1

$P_0 = \frac{1-\rho}{1-\rho^{N+1}} = \frac{1-0.83}{1-0.83^{11}} = 0.195$

$P_n = \rho^n P_0$

$= 0.195\,(0.83)^n, \quad n = 1, 2, \ldots 10$

3. A car park contains 5 car. The arrival of cars is poisson at a mean rate of 10/hr. The length of time each car spends in the car park is exponential distribution with mean of 5 hrs. How many cars are in the car park an average?

$N = 5$

$\lambda = \frac{10}{1} = 10$ cars/hrs

$\mu = \frac{1}{5}$ cars/hrs

$\rho = 10 \times 5 = 50$

$P_0 = 3.13 \times 10^{-9}$ $\quad \frac{\rho - 1}{\rho^{N+1} - 1} = \frac{49}{50^6 - 1}$

$Ls = P_0 \sum_{n=0}^{N} n\rho^n$

$= 3.13 \times 10^{-9}\,[50 + 2 \times 50^2 + 3 \times 50^3 + 4 \times 50^4 + 5 \times 50^5]$

$= 4.97 \approx 5\,cars$

4: Assume that the goods trains are coming in a yard at the rate of 30 trains per day and suppose that the inter-arrival times follow an exponential distribution. The service time for each train is assumed to be exponential with an average of 36 minutes. If the yard can admit 9 trains at a time (there being 10 lines, one of which is reserved for shunting purposes), calculate the probability that the yard is empty and find the average queue length.

Solution:

Given that $\lambda = \frac{30}{60 \times 24} = \frac{1}{48}$ and

$$\mu = \frac{1}{16} \text{ trains per minute ;}$$

$$\rho = \frac{\lambda}{\mu} = \frac{36}{48} = 0.75 \text{ and } N = 9$$

P_0 = The probability that the yard

$$\text{is empty } = \frac{1-\rho}{1-\rho^{N+1}} = = \frac{1-0.75}{1-(0.75)^{10}} = 0.2514$$

$$L_s = \text{Average queue length } = \frac{\rho}{1-\rho} - \frac{(N+1)\rho^{N+1}}{1-\rho^{N+1}}$$

$$= \frac{\rho^2\left[1 - N\rho^{N-1} + (N-1)\rho^N\right]}{(1-\rho)\left(1-\rho^{N+1}\right)}$$

$$= \frac{(0.75)^2\left[1-9(0.75)^8 + 8(0.75)^9\right]}{0.25\left[1-(0.75)^{10}\right]}$$

$$= \frac{0.5625[1-0.8982+0.5984]}{0.25(1-0.0056)}$$

$$= \frac{0.3938}{0.2486} = 1.584 \cong 2 \text{ trains}$$

5: If for a period of 2 hours in the day (8 to 10 a.m) trains arrive at the yard every 20 minutes but the service time continues to remain 36 minutes. Then calculate for this period.

(a) The probability that the yard is empty,

(b) Average number of trains in the system; on the assumption that the line capacity of the yard is limited to 4 trains only.

Solution

Given that $\lambda = 1/20$ per minute and $\mu = 1/36$ trains per minute ;

$$\rho = \frac{\lambda}{\mu} = \frac{36}{20} = 1.8(>1) ; \ N=4$$

(a) P_0 = Probability that the yard is empty $= \frac{1-\rho}{1-\rho^{N+1}}$

$$= \frac{1-1.8}{1-(1.8)^5} = 0.04$$

(b) L_S = Average number of trains in the system

$$= P_0 \sum_{n=0}^{4} n\rho^n = P_0\left(\rho + 2\rho^2 + 3\rho^3 + 4\rho^4\right)$$

$$= 0.04 \times 67.77 = 2.71 \text{ or } 3 \text{ trains}$$

6: At a railway station, only one train is handled at a time. The railway yard is sufficient only for two trains to wait while other is given signal to leave the station. Trains arrive at the station at an average rate of 6 per hour and railway station can handle them on an average of 12 per hour. Assuming Poisson arrivals and exponential service distribution, find the steady-state probabilities for the various number of trains in the system. Also find the average waiting time of a new trains coming into the yard.

Solution :

Given that λ = 6 and μ = 12 ; λ/μ = 6/12 = 0.5

Maximum queue length = 2 ; Maximum number of trains in the system, N = 3.

(a) P_0 = Probability that there is no train in the system (both waiting and in service)

$$= \frac{1-\rho}{1-\rho^{N+1}} = \frac{1-0.5}{1-(0.5)^{3+1}} = 0.53$$

$P_n = P_0 \rho^N$

(b) L_s = Average queue length

$$= P_0\sum_{n=0}^{3} n\rho^n = P_0\left(\rho+2\rho^2+3\rho^3\right)$$

$= 0.53 \times 1.375 = 0.728$

Thus the average number of trains in the systems is $0.728 \cong 1$ train. As the arrival of new train expects to find an average of 0.728 trains in the system before it, average waiting time of a new train is :

$$W_s = 0.728 \times \frac{1}{\mu} = 0.728 \times \left(\frac{1}{12}\right)$$

$W_s = \frac{1}{\lambda'} L_s$

$= 0.06$ hrs or 3.6 minutes

$= \frac{1}{\lambda(1-P_N)}$

$= \frac{1}{6(1-0.53\times(0.5)^3)}$

7: A petrol station has a single pump and space for not more than 3 cars (2 waiting, 1 being served). A car arriving when the space is filled to capacity goes elsewhere for petrol. Cars arrive according to a Poisson distribution at a mean rate of one every 8 minutes. Their service time has an exponential distribution with a mean of 4 minutes.

The owner has the opportunity of renting an adjacent piece of land, which would provide space for an additional car to wait. (he cannot build another pump). The rent would be Rs.100 per week. The expected net profit from each customer is Rs.5 and the station is open 10 hours every day. Would it be profitable to rent the additional space.

Solution

Given that λ = 1/8 per minute, μ = ¼ per minute and ρ = λ/μ = 4/8 = 0.5.

Maximum number of customers (cars) in the system is N = 3.

P_0 = Probability that there is no car in the system

$$= \frac{1-\rho}{1-\rho^{N+1}} = \frac{1-0.5}{1-(0.5)^{3+1}} = 0.533$$

P_3 = Probability that the system is full

$$= P_0\left(\frac{\lambda}{\mu}\right)^3 = 0.533\times(0.5)^3 = 0.067$$

Thus the proportion of lost customer is 0.067. Under the proposed system, if maximum number of cars in the system is increased to N=4, then the proportion of lost customers is :

$$P_0 = \frac{1-0.5}{1-(0.5)^{4+1}} = 0.516, \text{ and}$$

$$P_4 = 0.516\times(0.5)^4 = 0.032.$$

Hence, the increase in cars served per hour is : λ (0.067 – 0.032) = (1/8) × 0.035 = 0.0043 cars per minute or 0.0043 × 60 = 0.262 per hour.

Increase in cars served per week $= 0.262 \times 10 \times 7 = 18.34 \cong 19$ cars. Thus increase in profit per week will be $0.50 \times 18.34 =$ Rs.9.17.

Since rent for additional space would be Rs.10 per week, it is not economical to increase the existing space.

Excersise

1. In a railway mastering yard trains arrive at the rate of 30 per day and service time distribution is exponential with an average of 36 minutes. If the capacity of yard is 5. Calculate

   (a) Probability that the yard is empty

   (b) Average queue length

2. Arrivals at a telephone booth are considered to be poisson with an average time of 10 minutes between one arrival and the next. The length of a phone call is assumed to be distributed exponentially with mean 3 minutes. What is the probability that a person arriving at the booth will have to wait. The telephone department will install a second booth when convinced that an arrival would expect to have to wait at least 3 minutes for the phone. By how much must the flow of arrival be increased inorder to justify a second booth.

3. Assuming for a period of 2 hrs in a day trains arrive at the yard every 20 minutes. Calculate for this period (a) probability that yard is empty (b) average queue length assumed that capacity of yard is 4 trains only.

$$\frac{\frac{\lambda}{\mu s}}{\left(1-\frac{\lambda}{\mu s}\right)} = \frac{\lambda}{\mu s}\,\frac{\mu^2 s^2}{(\mu s-\lambda)^2} = \lambda \mu s \qquad \frac{\lambda s \mu}{(s\mu-\lambda)^2}.$$

Model III : {(M/M/S): (∞/FCFS)}

This is a Multiple Servers, Unlimited Queue Model. This model consisting 's' number of servers and capacity of system is infinitive.

Here $\rho = \dfrac{\lambda}{\mu s}$

i) $\rho = \dfrac{\lambda}{\mu s}$

ii) $P_n = \begin{cases} \left(\frac{\lambda}{\mu}\right)^n P_0 & 1 \le n < s \\ \dfrac{\left(\frac{\lambda}{\mu}\right)^n P_0}{s!\, s^{n-s}}, & n \ge s \end{cases}$

Measures of Model III

$$P_n = \begin{cases} \left(\dfrac{\lambda}{\mu}\right)^n P_0; 1 \le n < s \\ \dfrac{\left(\dfrac{\lambda}{\mu}\right)^n P_0}{s!\, s^{n-s}}; n \ge s \end{cases}$$

iii) $P_s = \frac{1}{s}\left(\frac{\lambda}{\mu}\right)^s P_0$

iv) $P_0 = \left[\sum_{n=0}^{s-1} \frac{\left(\frac{\lambda}{\mu}\right)^n}{n!} + \frac{\left(\frac{\lambda}{\mu}\right)^s}{s!(1-\rho)}\right]^{-1}$

v) $L_q = P_s \frac{\rho}{(1-\rho)^2}$

$$P_0 = \left[\sum_{n=0}^{s-1} \frac{(s\rho)^n}{n!} + \frac{1}{s!}\frac{(s\rho)^s}{(1-\rho)}\right]^{-1}; \rho = \frac{\lambda}{s\mu}$$

vi) $L_s = L_q + \frac{\lambda}{\mu}$

vi) $W_s = \frac{1}{\lambda} L_s$

vi) $W_q = \frac{1}{\lambda} L_q$

$$= \left[\sum_{n=0}^{s-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s \frac{s\mu}{s\mu-\lambda}\right]^{-1}$$

1. Expected (or average) waiting time of a customer in the queue

$$L_q = \left[\frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s \frac{\lambda . s\mu}{(s\mu-\lambda)^2}\right] P_0 = \left[\frac{1}{(s-1)!}\left(\frac{\lambda}{\mu}\right)^s \frac{\lambda\mu}{(s\mu-\lambda)^2}\right] P_0$$

$$\left(1-\frac{\lambda}{\mu s}\right)^2 \qquad \frac{\rho}{\left(1-\frac{\lambda}{\mu s}\right)^2} \qquad \frac{\frac{\lambda}{\mu s}}{(\mu s-\lambda)^2} = \frac{\lambda}{\mu s} \times \frac{\mu^2 s^2}{(\mu s-\lambda)^2}$$