Lecture 4

Operations Research

Simplex Method

Prepared by I Made Murwantara, Universitas Pelita Harapan, 2018

Simplex Method

- Graphical model is convenient for Linear Programming model that involved two variables.
- For two or more variables, we need to use method that adaptable to computers.
- Simplex Method developed by George Dantzig in 1946.
- It provides us with a systematic way of examining the vertices of the feasible region to determine the optimal value of the objective function.
- Maximization and Minimization
- Prerequisite: Gauss-Jordan Elimination

Simplex Method: Maximization

Suppose we want to find the maximum value of $z = 4x_1 + 6x_2$, where $x_1 \geq 0$ and $x_2 \geq 0$, subject to the following constraints.

$$\begin{aligned} -x_1 + x_2 &\leq 11 \\ x_1 + x_2 &\leq 27 \\ 2x_1 + 5x_2 &\leq 90 \end{aligned}$$

Simplex Method: Maximization

Since the left-hand side of each *inequality* is less than or equal to the right-hand side, there must exist nonnegative numbers s_1, s_2 and s_3 that can be added to the left side of each equation to produce the following system of linear *equations*.

$$\begin{aligned}
-x_1 + x_2 + s_1 \qquad\qquad &= 11 \\
x_1 + x_2 \qquad + s_2 \qquad &= 27 \\
2x_1 + 5x_2 \qquad\qquad + s_3 &= 90
\end{aligned}$$

The numbers s_1, s_2 and s_3 are called **slack variables** because they take up the "slack" in each inequality.

Simplex Method: Maximization

A linear programming problem is in **standard form** if it seeks to *maximize* the objective function $z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$ subject to the constraints

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\leq b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\leq b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\leq b_m
\end{aligned}
$$

where $x_i \geq 0$ and $b_i \geq 0$.

Simplex Method: Maximization

After adding slack variables, the corresponding system of **constraint equations** is

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + s_1 \qquad\qquad &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \qquad + s_2 \qquad &= b_2 \\
&\ \ \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \qquad\qquad + s_m &= b_m
\end{aligned}$$

where $s_i \geq 0$

Simplex Method: Maximization

A **basic solution** of a linear programming problem in standard form is a solution $(x_1, x_2, \ldots, x_n, s_1, s_2, \ldots, s_m)$ of the constraint equations in which *at most m* variables are nonzero—the variables that are nonzero are called **basic variables.** A basic solution for which all variables are nonnegative is called a **basic feasible solution.**

Simplex Tableau

- The simplex method is carried out by performing elementary row operations on a matrix **-- simplex tableau.**
- This tableau consists of the augmented matrix corresponding to the constraint equations together with the coefficients of the objective function written in the form shown below

$$-c_1x_1 - c_2x_2 - \cdots - c_nx_n + (0)s_1 + (0)s_2 + \cdots + (0)\,s_m + z = 0.$$

Simplex Tableau: Example

$$z = 4x_1 + 6x_2$$

Objective function

$$\left.\begin{aligned}
-x_1 + x_2 + s_1 \qquad\qquad &= 11 \\
x_1 + x_2 \qquad + s_2 \qquad &= 27 \\
2x_1 + 5x_2 \qquad\qquad + s_3 &= 90
\end{aligned}\right\}$$

Constraints

Simplex Tableau: Example

x_1	x_2	s_1	s_2	s_3	b	*Basic Variables*
-1	1	1	0	0	11	s_1
1	1	0	1	0	27	s_2
2	5	0	0	1	90	s_3
-4	-6	0	0	0	0	

$\uparrow$

Current z–value

Simplex Tableau: Example

For this **initial simplex tableau,** the **basic variables** are s_1, s_2, and s_3, and the **nonbasic variables** (which have a value of zero) are x_1 and x_2. Hence, from the two columns that are farthest to the right, we see that the current solution is

$$x_1 = 0, \quad x_2 = 0, \quad s_1 = 11, \quad s_2 = 27, \quad \text{and} \quad s_3 = 90.$$

This solution is a basic feasible solution and is often written as

$$(x_1, x_2, s_1, s_2, s_3) = (0, 0, 11, 27, 90)$$

Pivoting

- To improve the current solution, we bring a new basic variable into the solution— **entering variable.**
- This implies that one of the current basic variables must leave, otherwise we would have too many variables for a basic solution– **departing variable.**
- We choose the entering and departing variables as follows.
  - The **entering variable** corresponds to the smallest (the most negative) entry in the bottom row of the tableau.
  - The **departing variable** corresponds to the smallest nonnegative ratio of , in the column determined by the entering variable.
  - The entry in the simplex tableau in the entering variable’s column and the departing variable’s row is called the **pivot.**
- We apply Gauss-Jordan elimination to the column that contains the pivot

Pivoting

Note that the current solution ($x_1 = 0, x_2 = 0, s_1 = 11, s_2 = 27, s_3 = 90$) corresponds to a z–value of 0. To improve this solution, we determine that x_2 is the entering variable, because -6 is the smallest entry in the bottom row.

$$
\begin{array}{cccccc|c}
x_1 & x_2 & s_1 & s_2 & s_3 & b & \text{Basic Variables} \\
\hline
-1 & 1 & 1 & 0 & 0 & 11 & s_1 \\
1 & 1 & 0 & 1 & 0 & 27 & s_2 \\
2 & 5 & 0 & 0 & 1 & 90 & s_3 \\
\hline
-4 & -6 & 0 & 0 & 0 & 0 & \\
 & \uparrow & & & & & \\
 & \text{Entering} & & & & &
\end{array}
$$

Pivoting

To see *why* we choose x_2 as the entering variable, remember that $z = 4x_1 + 6x_2$. Hence, it appears that a unit change in x_2 produces a change of 6 in z, whereas a unit change in x_1 produces a change of only 4 in z.

To find the departing variable, we locate the b_i's that have corresponding positive elements in the entering variables column and form the following ratios.

$$\frac{11}{1} = 11, \quad \frac{27}{1} = 27, \quad \frac{90}{5} = 18$$

Here the smallest positive ratio is 11, so we choose s_1 as the departing variable.

Pivoting

x_1	x_2	s_1	s_2	s_3	b	*Basic Variables*	
-1	1	1	0	0	11	s_1	$\leftarrow$ *Departing*
1	1	0	1	0	27	s_2	
2	5	0	0	1	90	s_3	
-4	-6	0	0	0	0		
	$\uparrow$ *Entering*						

Pivoting

We use Gauss-Jordan elimination to obtain the following improved solution

$1-(S_1;X_1)$
$1-(-1)$

$1-(S_1;X_2)$
$1-(1)$

$27-(S_1;b)$
$27-(11)$

Before Pivoting *After Pivoting*

$$\begin{bmatrix} -1 & \boxed{1} & 1 & 0 & 0 & 11 \\ 1 & \boxed{1} & 0 & 1 & 0 & 27 \\ 2 & \boxed{5} & 0 & 0 & 1 & 90 \\ -4 & \boxed{-6} & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & 1 & 1 & 0 & 0 & 11 \\ 2 & 0 & -1 & 1 & 0 & 16 \\ 7 & 0 & -5 & 0 & 1 & 35 \\ -10 & 0 & 6 & 0 & 0 & 66 \end{bmatrix}$$

$0-(S_1;S_1)$
$0-(1)$

$1-(S_1;S_2)$
$1-(0)$

$0-(S_1;S_3)$
$0-(0)$

Pivoting

We use Gauss-Jordan elimination to obtain the following improved solution

2-(5*(-1))

5-(5*1)

90-(5*11)

Before Pivoting

$$\begin{bmatrix} -1 & 1 & 1 & 0 & 0 & 11 \\ 1 & 1 & 0 & 1 & 0 & 27 \\ 2 & 5 & 0 & 0 & 1 & 90 \\ -4 & -6 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow$$

After Pivoting

$$\begin{bmatrix} -1 & 1 & 1 & 0 & 0 & 11 \\ 2 & 0 & -1 & 1 & 0 & 16 \\ 7 & 0 & -5 & 0 & 1 & 35 \\ -10 & 0 & 6 & 0 & 0 & 66 \end{bmatrix}$$

0-(5*1)

0-(5*0)

1-(5*0)

Pivoting

x_1	x_2	s_1	s_2	s_3	b	Basic Variables
-1	1	1	0	0	11	x_2
2	0	-1	1	0	16	s_2
7	0	-5	0	1	35	s_3
-10	0	6	0	0	66	

Note that x_2 has replaced s_1 in the basis column and the improved solution

$$(x_1, x_2, s_1, s_2, s_3) = (0, 11, 0, 16, 35)$$

has a z-value of

$$z = 4x_1 + 6x_2 = 4(0) + 6(11) = 66.$$

Pivoting

We choose X_1 as entering variable.
The smallest nonnegative ratio of 11/(-1), 16/2=8 and 35/7 is 5, so S_3 is the departing variable.

$$\begin{bmatrix} -1 & 1 & 1 & 0 & 0 & 11 \\ 2 & 0 & -1 & 1 & 0 & 16 \\ 7 & 0 & -5 & 0 & 1 & 35 \\ -10 & 0 & 6 & 0 & 0 & 66 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & 1 & 1 & 0 & 0 & 11 \\ 2 & 0 & -1 & 1 & 0 & 16 \\ 1 & 0 & -\frac{5}{7} & 0 & \frac{1}{7} & 5 \\ -10 & 0 & 6 & 0 & 0 & 66 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 0 & 1 & \frac{2}{7} & 0 & \frac{1}{7} & 16 \\ 0 & 0 & \frac{3}{7} & 1 & -\frac{2}{7} & 6 \\ 1 & 0 & -\frac{5}{7} & 0 & \frac{1}{7} & 5 \\ 0 & 0 & -\frac{8}{7} & 0 & \frac{10}{7} & 116 \end{bmatrix}$$

Pivoting

$$\begin{bmatrix} -1 & 1 & 1 & 0 & 0 & 11 \\ 2 & 0 & -1 & 1 & 0 & 16 \\ 7 & 0 & -5 & 0 & 1 & 35 \\ -10 & 0 & 6 & 0 & 0 & 66 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & 1 & 1 & 0 & 0 & 11 \\ 2 & 0 & -1 & 1 & 0 & 16 \\ 1 & 0 & -\frac{5}{7} & 0 & \frac{1}{7} & 5 \\ -10 & 0 & 6 & 0 & 0 & 66 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 0 & 1 & \frac{2}{7} & 0 & \frac{1}{7} & 16 \\ 0 & 0 & \frac{3}{7} & 1 & -\frac{2}{7} & 6 \\ 1 & 0 & -\frac{5}{7} & 0 & \frac{1}{7} & 5 \\ 0 & 0 & -\frac{8}{7} & 0 & \frac{10}{7} & 116 \end{bmatrix}$$

7*(1/7)-1

0*(1/7)+1

-5*(1/7)+1

0*(1/7)+0

1*(1/7)+0

35*(1/7)+11

Pivoting

$$\begin{bmatrix} -1 & 1 & 1 & 0 & 0 & 11 \\ 2 & 0 & -1 & 1 & 0 & 16 \\ 7 & 0 & -5 & 0 & 1 & 35 \\ -10 & 0 & 6 & 0 & 0 & 66 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & 1 & 1 & 0 & 0 & 11 \\ 2 & 0 & -1 & 1 & 0 & 16 \\ 1 & 0 & -\frac{5}{7} & 0 & \frac{1}{7} & 5 \\ -10 & 0 & 6 & 0 & 0 & 66 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & \frac{2}{7} & 0 & \frac{1}{7} & 16 \\ 0 & 0 & \frac{3}{7} & 1 & -\frac{2}{7} & 6 \\ 1 & 0 & -\frac{5}{7} & 0 & \frac{1}{7} & 5 \\ 0 & 0 & -\frac{8}{7} & 0 & \frac{10}{7} & 116 \end{bmatrix}$$

7*-(2/7)+2

0*-(2/7)+0

-5*-(2/7)-1

0*-(2/7)+0

35*-(2/7)+16

1*-(2/7)+0

Pivoting

$$
\begin{bmatrix} -1 & 1 & 1 & 0 & 0 & 11 \\ 2 & 0 & -1 & 1 & 0 & 16 \\ 7 & 0 & -5 & 0 & 1 & 35 \\ -10 & 0 & 6 & 0 & 0 & 66 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & 1 & 1 & 0 & 0 & 11 \\ 2 & 0 & -1 & 1 & 0 & 16 \\ 1 & 0 & -\frac{5}{7} & 0 & \frac{1}{7} & 5 \\ -10 & 0 & 6 & 0 & 0 & 66 \end{bmatrix}
$$

$$
\rightarrow \begin{bmatrix} 0 & 1 & \frac{2}{7} & 0 & \frac{1}{7} & 16 \\ 0 & 0 & \frac{3}{7} & 1 & -\frac{2}{7} & 6 \\ 1 & 0 & -\frac{5}{7} & 0 & \frac{1}{7} & 5 \\ 0 & 0 & -\frac{8}{7} & 0 & \frac{10}{7} & 116 \end{bmatrix}
$$

7*(10/7)-10

35*(10/7)+66

0*(10/7)+0

-5*(10/7)+6

0*(10/7)+0

1*(10/7)+0

Pivoting

the new simplex tableau is

$$
\begin{array}{cccccc|c}
x_1 & x_2 & s_1 & s_2 & s_3 & b & \textit{Basic Variables} \\
\hline
0 & 1 & \frac{2}{7} & 0 & \frac{1}{7} & 16 & \boldsymbol{x_2} \\
0 & 0 & \frac{3}{7} & 1 & -\frac{2}{7} & 6 & \boldsymbol{s_2} \quad \leftarrow \textit{Departing} \\
1 & 0 & -\frac{5}{7} & 0 & \frac{1}{7} & 5 & \boldsymbol{x_1} \\
\hline
0 & 0 & -\frac{8}{7} & 0 & \frac{10}{7} & 116 & \\
 & & \uparrow & & & & \\
 & & \textit{Entering} & & & &
\end{array}
$$

In this tableau, there is still a negative entry in the bottom row. Thus, we choose s_1 as the entering variable and s_2 as the departing variable, as shown in the following tableau.

Pivoting

(3/7)*(7/3)

1*(7/3)

6*(7/3)

-(2/7)*(7/3)

x_1	x_2	s_1	s_2	s_3	b	*Basic Variables*
0	1	0	$-\frac{2}{3}$	$\frac{1}{3}$	12	x_2
0	0	1	$\frac{7}{3}$	$-\frac{2}{3}$	14	s_1
1	0	0	$\frac{5}{3}$	$-\frac{1}{3}$	15	x_1
0	0	0	$\frac{8}{3}$	$\frac{2}{3}$	132	← *Maximum z-value*

Pivoting

(3/7)*-(2/3)+(2/7)

1*-(2/3)+0

-(2/3)*-(2/7)+(1/7)

6*-(2/3)+16

x_1	x_2	s_1	s_2	s_3	b	Basic Variables
0	1	0	$-\frac{2}{3}$	$\frac{1}{3}$	12	x_2
0	0	1	$\frac{7}{3}$	$-\frac{2}{3}$	14	s_1
1	0	0	$\frac{5}{3}$	$-\frac{1}{3}$	15	x_1
0	0	0	$\frac{8}{3}$	$\frac{2}{3}$	132	← Maximum z-value

Pivoting

$$
\begin{array}{cccccc|l}
x_1 & x_2 & s_1 & s_2 & s_3 & b & \begin{array}{c}\textit{Basic}\\ \textit{Variables}\end{array} \\
\hline
0 & 1 & 0 & -\frac{2}{3} & \frac{1}{3} & 12 & x_2 \\
0 & 0 & 1 & \frac{7}{3} & -\frac{2}{3} & 14 & s_1 \\
1 & 0 & 0 & \frac{5}{3} & -\frac{1}{3} & 15 & x_1 \\
\hline
0 & 0 & 0 & \frac{8}{3} & \frac{2}{3} & 132 & \leftarrow \textit{Maximum z-value}
\end{array}
$$

(3/7)*(8/3)-(8/7)

1*(8/3)+0

-(2/7)*(8/3)+(10/7)

6*(8/3)+116

Pivoting

$$\begin{array}{cccccc|c}
x_1 & x_2 & s_1 & s_2 & s_3 & b & \text{Basic Variables} \\
\hline
0 & 1 & 0 & -\frac{2}{3} & \frac{1}{3} & 12 & x_2 \\
0 & 0 & 1 & \frac{7}{3} & -\frac{2}{3} & 14 & s_1 \\
1 & 0 & 0 & \frac{5}{3} & -\frac{1}{3} & 15 & x_1 \\
\hline
0 & 0 & 0 & \frac{8}{3} & \frac{2}{3} & 132 & \leftarrow \text{Maximum } z\text{-value}
\end{array}$$

(3/7)*(5/3)-(5/7)

1*(5/3)+0

-(2/7)*(5/3)+(1/7)

6*(5/3)+5

Pivoting

$$
\begin{array}{cccccc|c}
x_1 & x_2 & s_1 & s_2 & s_3 & b & \text{Basic Variables} \\
\hline
0 & 1 & 0 & -\frac{2}{3} & \frac{1}{3} & 12 & x_2 \\
0 & 0 & 1 & \frac{7}{3} & -\frac{2}{3} & 14 & s_1 \\
1 & 0 & 0 & \frac{5}{3} & -\frac{1}{3} & 15 & x_1 \\
\hline
0 & 0 & 0 & \frac{8}{3} & \frac{2}{3} & 132 & \leftarrow \textit{Maximum z-value}
\end{array}
$$

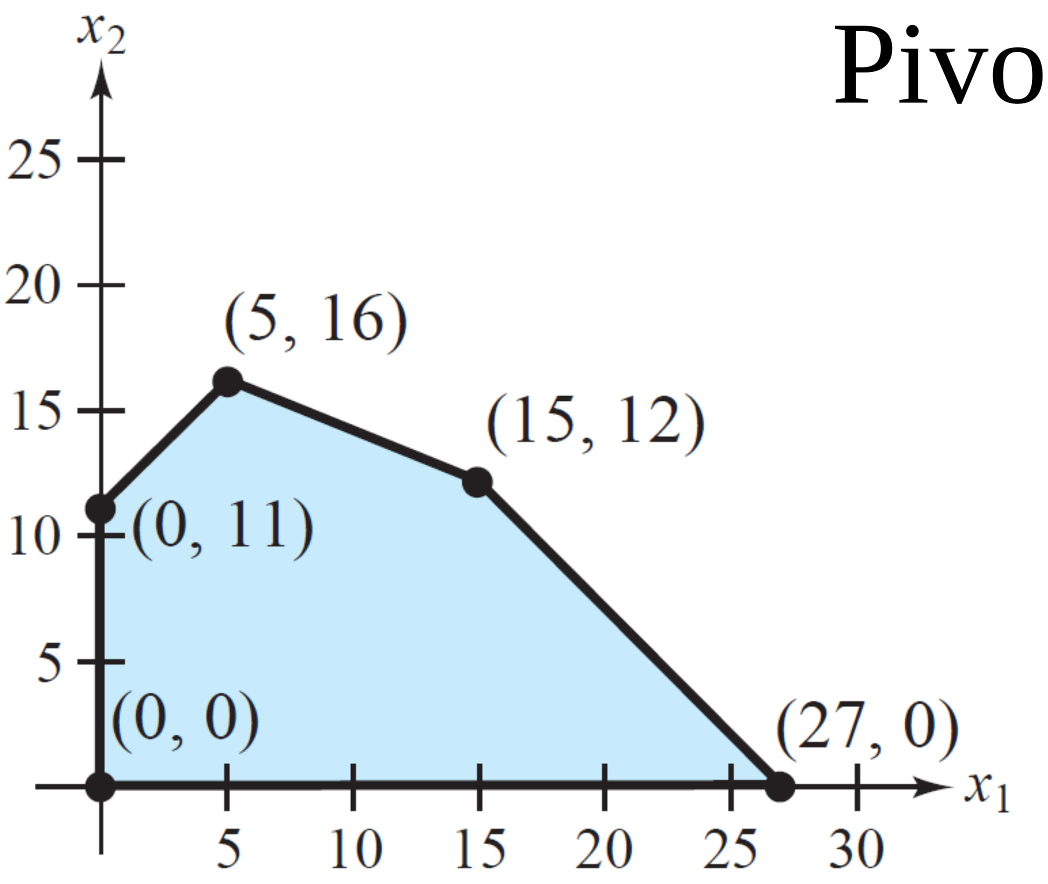
In this tableau, there are no negative elements in the bottom row. We have therefore determined the optimal solution to be

$$(x_1, x_2, s_1, s_2, s_3) = (15, 12, 14, 0, 0)$$

with

$$z = 4x_1 + 6x_2 = 4(15) + 6(12) = 132.$$

Pivoting



$(0, 0)$ → $(0, 11)$ → $(5, 16)$ → $(15, 12)$

$z = 0$ → $z = 66$ → $z = 116$ → $z = 132$

Simplex: Three Decision Variables

Use the simplex method to find the maximum value of

$$z = 2x_1 - x_2 + 2x_3 \qquad \text{Objective function}$$

subject to the constraints

$$\begin{aligned} 2x_1 + x_2 \phantom{{}- 2x_3} &\le 10 \\ x_1 + 2x_2 - 2x_3 &\le 20 \\ x_2 + 2x_3 &\le 5 \end{aligned}$$

where $x_1 \ge 0$, $x_2 \ge 0$, and $x_3 \ge 0$.

Simplex: Three Decision Variables

Using the basic feasible solution

$$(x_1, x_2, x_3, s_1, s_2, s_3) = (0, 0, 0, 10, 20, 5)$$

x_1	x_2	x_3	s_1	s_2	s_3	b	*Basic Variables*	
2	1	0	1	0	0	10	s_1	
1	2	−2	0	1	0	20	s_2	
0	1	2	0	0	1	5	s_3	← *Departing*
−2	1	−2	0	0	0	0		
		↑ *Entering*						

Simplex: Three Decision Variables

x_1	x_2	x_3	s_1	s_2	s_3	b	*Basic Variables*	
2	1	0	1	0	0	10	s_1	← *Departing*
1	3	0	0	1	1	25	s_2	
0	$\frac{1}{2}$	1	0	0	$\frac{1}{2}$	$\frac{5}{2}$	x_3	
-2	2	0	0	0	1	5		
↑ *Entering*								

Simplex: Three Decision Variables

$$
\begin{array}{ccccccc|c}
x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & b & \text{Basic Variables} \\
\hline
1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 5 & x_1 \\
0 & \frac{5}{2} & 0 & -\frac{1}{2} & 1 & 1 & 20 & s_2 \\
0 & \frac{1}{2} & 1 & 0 & 0 & \frac{1}{2} & \frac{5}{2} & x_3 \\
\hline
0 & 3 & 0 & 1 & 0 & 1 & 15 &
\end{array}
$$

This implies that the optimal solution is

$$(x_1, x_2, x_3, s_1, s_2, s_3) = (5, 0, \tfrac{5}{2}, 0, 20, 0)$$

and the maximum value of z is 15.

Business Application

A manufacturer produces three types of plastic fixtures. The time required for molding, trimming, and packaging is given in Table 9.1. (Times are given in hours per dozen fixtures.)

TABLE 9.1

Process	*Type A*	*Type B*	*Type C*	*Total time available*
Molding	1	2	$\frac{3}{2}$	12,000
Trimming	$\frac{2}{3}$	$\frac{2}{3}$	1	4,600
Packaging	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$	2,400
Profit	\$11	\$16	\$15	—

How many dozen of each type of fixture should be produced to obtain a maximum profit?

Business Application

Letting x_1, x_2, and x_3 represent the number of dozen units of Types A, B, and C, respectively, the objective function is given by

$$\text{Profit} = P = 11x_1 + 16x_2 + 15x_3.$$

constraints

$$\begin{aligned} x_1 + 2x_2 + \tfrac{3}{2}x_3 &\le 12{,}000 \\ \tfrac{2}{3}x_1 + \tfrac{2}{3}x_2 + x_3 &\le 4{,}600 \\ \tfrac{1}{2}x_1 + \tfrac{1}{3}x_2 + \tfrac{1}{2}x_3 &\le 2{,}400 \end{aligned}$$

$x_1 \ge 0$, $x_2 \ge 0$, and $x_3 \ge 0$

the basic feasible solution

$$(x_1, x_2, x_3, s_1, s_2, s_3) = (0, 0, 0, 12{,}000, 4{,}600, 2{,}400)$$

Business Application:Tableau

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables	
1	2	$\frac{3}{2}$	1	0	0	$12{,}000$	s_1	$\leftarrow$ Departing
$\frac{2}{3}$	$\frac{2}{3}$	1	0	1	0	$4{,}600$	s_2	
$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$	0	0	1	$2{,}400$	s_3	
-11	-16	-15	0	0	0	0		
	$\uparrow$ Entering							

Business Application: Tableau

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables	
$\frac{1}{2}$	1	$\frac{3}{4}$	$\frac{1}{2}$	0	0	6,000	x_2	
$\frac{1}{3}$	0	$\frac{1}{2}$	$-\frac{1}{3}$	1	0	600	s_2	
$\frac{1}{3}$	0	$\frac{1}{4}$	$-\frac{1}{6}$	0	1	400	s_3	← *Departing*
-3	0	-3	8	0	0	96,000		
↑ *Entering*								

Business Application: Tableau

x_1	x_2	x_3	s_1	s_2	s_3	b	*Basic Variables*	
0	1	$\frac{3}{8}$	$\frac{3}{4}$	0	$-\frac{3}{2}$	5,400	x_2	
0	0	$\frac{1}{4}$	$-\frac{1}{6}$	1	-1	200	s_2	$\leftarrow$ *Departing*
1	0	$\frac{3}{4}$	$-\frac{1}{2}$	0	3	1,200	x_1	
0	0	$-\frac{3}{4}$	$\frac{13}{2}$	0	9	99,600		
		$\uparrow$ *Entering*						

Business Application: Tableau

x_1	x_2	x_3	s_1	s_2	s_3	b	Basic Variables
0	1	0	1	$-\frac{3}{2}$	0	5,100	x_2
0	0	1	$-\frac{2}{3}$	4	-4	800	x_3
1	0	0	0	-3	6	600	x_1
0	0	0	6	3	6	100,200	

Business Application: Tableau

From this final simplex tableau, we see that the maximum profit is $100,200, and this is obtained by the following production levels.

Type A:	600 dozen units
Type B:	5,100 dozen units
Type C:	800 dozen units

Business Application: Media Selection

The advertising alternatives for a company include television, radio, and newspaper advertisements. The costs and estimates for audience coverage are given in Table 9.2

TABLE 9.2

	Television	*Newspaper*	*Radio*
Cost per advertisement	\$ 2,000	\$ 600	\$ 300
Audience per advertisement	100,000	40,000	18,000

The local newspaper limits the number of weekly advertisements from a single company to ten. Moreover, in order to balance the advertising among the three types of media, no more than half of the total number of advertisements should occur on the radio, and at least 10% should occur on television. The weekly advertising budget is \$18,200. How many advertisements should be run in each of the three types of media to maximize the total audience?

Business Application: Media Selection

To begin, we let x_1, x_2, and x_3 represent the number of advertisements in television, newspaper, and radio, respectively. The objective function (to be maximized) is therefore

$$z = 100{,}000x_1 + 40{,}000x_2 + 18{,}000x_3 \qquad \text{Objective function}$$

where $x_1 \geq 0$, $x_2 \geq 0$, and $x_3 \geq 0$. The constraints for this problem are as follows.

$$\begin{aligned}
2000x_1 + 600x_2 + 300x_3 &\leq 18{,}200 \\
x_2 &\leq 10 \\
x_3 &\leq 0.5(x_1 + x_2 + x_3) \\
x_1 &\geq 0.1(x_1 + x_2 + x_3)
\end{aligned}$$

Business Application: Media Selection

$$\left.\begin{aligned}
20x_1 + 6x_2 + 3x_3 &\le 182 \\
x_2 &\le 10 \\
-x_1 - x_2 + x_3 &\le 0 \\
-9x_1 + x_2 + x_3 &\le 0
\end{aligned}\right\} \quad \textbf{Constraints}$$

Business Application: Media Selection

$$
\begin{array}{cccccccc|c}
x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & s_4 & b & \text{Basic Variables} \\
\hline
20 & 6 & 3 & 1 & 0 & 0 & 0 & 182 & s_1 \leftarrow \text{Departing} \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 10 & s_2 \\
-1 & -1 & 1 & 0 & 0 & 1 & 0 & 0 & s_3 \\
-9 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & s_4 \\
\hline
-100{,}000 & -40{,}000 & -18{,}000 & 0 & 0 & 0 & 0 & 0 & \\
\uparrow & & & & & & & & \\
\text{Entering} & & & & & & & &
\end{array}
$$

Business Application: Media Selection

x_1	x_2	x_3	s_1	s_2	s_3	s_4	b	Basic Variables
1	$\frac{3}{10}$	$\frac{3}{20}$	$\frac{1}{20}$	0	0	0	$\frac{91}{10}$	x_1
0	1	0	0	1	0	0	10	s_2 ← *Departing*
0	$-\frac{7}{10}$	$\frac{23}{20}$	$\frac{1}{20}$	0	1	0	$\frac{91}{10}$	s_3
0	$\frac{37}{10}$	$\frac{47}{20}$	$\frac{9}{20}$	0	0	1	$\frac{819}{10}$	s_4
0	−10,000	−3,000	5,000	0	0	0	910,000	
	↑ *Entering*							

Business Application: Media Selection

x_1	x_2	x_3	s_1	s_2	s_3	s_4	b	Basic Variables
1	0	$\frac{3}{20}$	$\frac{1}{20}$	$-\frac{3}{10}$	0	0	$\frac{61}{10}$	x_1
0	1	0	0	1	0	0	10	x_2
0	0	$\frac{23}{20}$	$\frac{1}{20}$	$\frac{7}{10}$	1	0	$\frac{161}{10}$	s_3 ← *Departing*
0	0	$\frac{47}{20}$	$\frac{9}{20}$	$-\frac{37}{10}$	0	1	$\frac{449}{10}$	s_4
0	0	$-3{,}000$	5,000	10,000	0	0	1,010,000	
		↑ *Entering*						

Business Application: Media Selection

x_1	x_2	x_3	s_1	s_2	s_3	s_4	b	Basic Variables
1	0	0	$\frac{1}{23}$	$-\frac{9}{23}$	$-\frac{3}{23}$	0	4	x_1
0	1	0	0	1	0	0	10	x_2
0	0	1	$\frac{1}{23}$	$\frac{14}{23}$	$\frac{20}{23}$	0	14	x_3
0	0	0	$\frac{8}{23}$	$-\frac{118}{23}$	$-\frac{47}{23}$	1	12	s_4
0	0	0	$\frac{118{,}000}{23}$	$\frac{272{,}000}{23}$	$\frac{60{,}000}{23}$	0	1,052,000	

Business Application: Media Selection

From this tableau, we see that the maximum weekly audience for an advertising budget of \$18,200 is

$$z = 1{,}052{,}000$$ **Maximum weekly audience**

and this occurs when $x_1 = 4$, $x_2 = 10$, and $x_3 = 14$. We sum up the results here.

Media	*Number of Advertisements*	*Cost*	*Audience*
Television	4	\$ 8,000	400,000
Newspaper	10	\$ 6,000	400,000
Radio	14	\$ 4,200	252,000
Total	28	\$18,200	1,052,000

Exercise

1. Objective function:

$$z = x_1 + 2x_2$$

Constraints:

$$2x_1 + x_2 \leq 8$$

$$x_1 + x_2 \leq 5$$

$$x_1, x_2 \geq 0$$

Exercise

2. Objective function:

$$z = x_1 + 3x_2$$

Constraints:

$$x_1 + x_2 \leq 4$$

$$x_1 - x_2 \leq 1$$

$$x_1, x_2 \geq 0$$

Exercise

3. Objective function:

$$z = 2x_1 + 3x_2 + 4x_3$$

Constraints:

$$\begin{aligned} x_1 + 2x_2 \qquad\quad &\le 12 \\ x_1 \qquad\quad + x_3 &\le 8 \\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$

Exercise

4. Objective function:

$$z = 6x_1 - 9x_2$$

Constraints:

$$\begin{aligned} 2x_1 - 3x_2 &\leq 6 \\ x_1 + x_2 &\leq 20 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Exercise

A company has budgeted a maximum of $600,000 for advertising a certain product nationally. Each minute of television time costs $60,000 and each one-page newspaper ad costs $15,000. Each television ad is expected to be viewed by 15 million viewers, and each newspaper ad is expected to be seen by 3 million readers. The company's market research department advises the company to use at most 90% of the advertising budget on television ads. How should the advertising budget be allocated to maximize the total audience?

R: Simplex Method