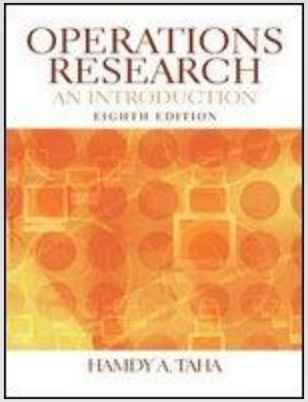
Chapter 3: Simplex methods [Big M method and special cases]



Hamdy A. Taha, Operations Research: An introduction, 8th Edition

Mjdah Al Shehri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Simplex method when some constraints are not "$\leq$" constraints

- We employ a mathematical " trick" to jumpstart the problem by adding artificial variables to the equations.

Simplex method when some constraints are not "≤" constraints (cont.)

<u>Example:</u>

Max $16x_1+15x_2+20x_3-18x_4$

ST

$2x_1 + x_2 + 3x_3 \leq 3000$ [1]

$3x_1 + 4x_2 + 5x_3 - 60x_4 \leq 2400$ [2]

$x_4 \leq 32$ [3]

$X_2 \geq 200$ [4]

$X_1 + x_2 + x_3 \geq 800$ [5]

$X_1 - x_2 - x_3 = 0$ [6]

$X_j \geq 0$ for all J

Simplex method when some constraints are not "≤" constraints (cont.)

Example:

Max 16x1+15x2+20x3-18x4

We assign a very large negative objective function coefficient , **-M** , (**+M** for minimization problem) to each artificial variable

ST

2x1 + x2 + 3x3 ≤ 3000 [1]

3x1 + 4x2 + 5x3 – 60x4 ≤ 2400 [2]

x4 ≤ 32 [3]

X2 **≥** 200 [4]

X1 + x2 + x3 **≥** 800 [5]

X1 – x2 –x3 **=**0 [6]

Xj ≥ 0 for all J

We add artificial : **R4, R5, R6**, respectively to the fourth, fifth, and sixth equations.

Simplex method when some constraints are not "$\leq$" constraints (cont.)

The solution

Max $16x_1+15x_2+20x_3-18x_4$ $-MR_4 - MR_5 - MR_6$

ST

$2x_1 + x_2 + 3x_3 + s_1 = 3000$ [1]

$3x_1 + 4x_2 + 5x_3 - 60x_4 + s_2 = 2400$ [2]

$x_4 + s_3 = 32$ [3]

$X_2 - s_4 + R_4 = 200$ [4]

$X_1 + x_2 + x_3 - s_5 + R_5 = 800$ [5]

$X_1 - x_2 - x_3 + R_6 = 0$ [6]

$X_j \geq 0$, $S_j \geq 0$, $R_j \geq 0$ for all J

The simplex algorithm can then be used to solve this problem

Solving For the optimal solution of **[Maximization]** when there are artificial variables

Example # 1:

MAX $2x1+ 5x2$

ST

$X1 \geq 4$

$x1 + 4x2 \leq 32$

$3x1+ 2x2 = 24$

Solving For the optimal solution of **[Maximization]** when there are artificial variables (cont.)

The Solution

- By adding the appropriate slack, surplus, and artificial variables, we obtain the following:

$$\text{MAX } 2x_1 + 5x_2 - MR_1 - MR_3$$

ST

$$X_1 - s_1 + R_1 = 4$$

$$X_1 + 4x_2 + s_2 = 32$$

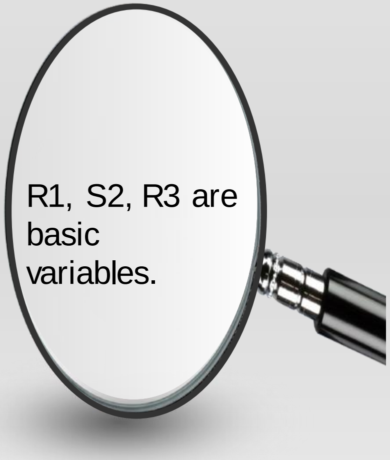
$$3x_1 + 2x_2 + R_3 = 24$$

$$X_1, x_2, s_1, s_2, R_1, R_3 \geq 0$$

Solving For the optimal solution of **[Maximization]** when there are artificial variables (cont.)

The initial table :

Basis	X1	X2	S1	S2	R1	R3	RHS
R1	1	0	-1	0	1	0	4
S2	1	4	0	1	0	0	32
R3	3	2	0	0	0	1	24
Z	-2	-5	0	0	+ M	+ M	0



- Make z consistent; (R1, R3) in z-row coefficient (+M,+M) it must be zero; By apply:

New z-row = old z-row + (-M * R1 row – M * R3 row) MAX objective function

New z-row = old z-row + (M * R1 row +M * R3 row) MIN objective function

Solving For the optimal solution of **[Maximization]** when there are artificial variables (cont.)

- <u>Starting table:</u>

Basis	X1	X2	S1	S2	R1	R3	RHS
R1	1	0	-1	0	1	0	4
S2	1	4	0	1	0	0	32
R3	3	2	0	0	0	1	24
Z	-2-4M	-5-2M	+M	0	- M	- M	-28M

Solving For the optimal solution of **[Maximization]** when there are artificial variables (cont.)

- To determine **Entering Variable**; We should look to the largest negative number in z-row.

Entering Variable

Basis	X1	X2	S1	S2	R1	R3	RHS
R1	1	0	-1	0	1	0	4
S2	1	4	0	1	0	0	32
R3	3	2	0	0	0	1	24
Z	-2-4M	-5-2M	+M	0	- M	- M	-28M

Largest negative number

Solving For the optimal solution of **[Maximization]** when there are artificial variables (cont.)

- Calculate the ratio; then, determine the smallest positive number as Leaving Variable

Leaving Variable

Basis	X1	X2	S1	S2	R1	R3	RHS	Ratio
R1	1	0	-1	0	1	0	4	4
S2	1	4	0	1	0	0	32	32
R3	3	2	0	0	0	1	24	8
Z	-2-4M	-5-2M	+M	0	- M	- M	-28M	

- Pivot element = (1, 0, -1, 0, 1, 0, 4)/ (1)

(1, 0, -1, 0, 1, 0, 4)

Solving For the optimal solution of **[Maximization]** when there are artificial variables (cont.)

- First iteration

Entering Variable

Leaving Variable

Basis	X1	X2	S1	S2	R1	R3	RHS	Ratio
X1	1	0	-1	0	1	0	4	....
S2	0	4	1	1	-1	0	28	28
R3	0	2	3	0	-3	1	12	4
Z	0	-5-2M	-2-3M	0	2+3M	-M	8-12M	

Solving For the optimal solution of **[Maximization]** when there are artificial variables (cont.)

- <u>Second iteration</u>

Entering Variable

Leaving Variable

Basis	X1	X2	S1	S2	R1	R3	RHS	Ratio
X1	1	2/3	0	0	0	1/3	8	12
S2	0	10/3	0	1	0	-1/3	24	7.2
S1	0	2/3	1	0	-1	1/3	4	6
Z	0	-11/3	0	0	0	2/3	+16	

Solving For the optimal solution of **[Maximization]** when there are artificial variables (cont.)

- <u>Third iteration</u>

Basis	X1	X2	S1	S2	R1	R3	RHS	Ratio
X1	1	0	-1	0	1	0	4	
S2	0	0	-5	1	5	-2	4	
X2	0	1	3/2	0	-3/2	1/2	6	
Z	0	0	11/3	0	-11/2	5/2	38	

Solving For the optimal solution of **[Maximization]** when there are artificial variables (cont.)

points	Classification	Reason
X1=0, X2=0	Not Feasible	R1, R3 both Positive (4, 24)
X1=4, X2=0	Not Feasible	R3 positive= 12
X1=8, X2=0	Feasible but not optimal	X2 is negative
X1=4, X2=6	Feasible and optimal	All x1,X2 ≥0

Solving For the optimal solution of **[Minimization]** when there are artificial variables

<u>Example # 2:</u>

Min $4x1 + x2$

ST

$3x1 + x2 = 3$

$4x1 + 3x2 \geq 6$

$X1 + 2x2 \leq 4$

$X1, x2 \geq 0$

Solving For the optimal solution of **[Minimization]** when there are artificial variables (cont.)

The Solution

- By adding the appropriate slack, surplus, and artificial variables, we obtain the following:

Min $4x1 + x2 + MR1 + MR2$

ST

$3x1 + x2 + R1 = 3$

$4x1 + 3x2 - s1 + R2 = 6$

$X1 + 2x2 + s2 = 4$

$X1, x2, s1, s2, R1, R2 \geq 0$

Solving For the optimal solution of **[Minimization]** when there are artificial variables (cont.)

- The initial table:

Basis	X1	X2	S1	R1	R2	S2	RHS
R1	3	1	0	1	0	0	3
R2	4	3	-1	0	1	0	6
S2	1	2	0	0	0	1	4
Z	-4	-1	0	-M	-M	0	0

- New z-row = old z-row +(M * R1 row +M * R3 row)

Solving For the optimal solution of **[Minimization]** when there are artificial variables (cont.)

Leaving Variable

- Starting table:

Entering Variable

Basis	X1	X2	S1	R1	R2	S2	RHS
R1	3	1	0	1	0	0	3
R2	4	3	-1	0	1	0	6
S2	1	2	0	0	0	1	4
Z	-4+7M	-1+4M	-M	0	0	0	9M

Solving For the optimal solution of **[Minimization]** when there are artificial variables (cont.)

- First iteration

Entering Variable

Leaving Variable

Basis	X1	X2	S1	R1	R2	S2	RHS
X1	1	1/3	0	1/3	0	0	1
R2	0	5/3	-1	-4/3	1	0	2
S2	0	5/3	0	-1/3	0	1	3
Z	0	(1+5M)/3	-M	(4-7M)/3	0	0	4+2M

Solving For the optimal solution of **[Minimization]** when there are artificial variables (cont.)

- Second iteration

Entering Variable

Leaving Variable

Basis	X1	X2	S1	R1	R2	S2	RHS
X1	1	0	1/5	3/5	-1/5	0	3/5
X2	0	1	-3/5	-4/5	3/5	0	6/5
S2	0	0	1	1	-1	1	1
Z	0	0	1/5	8/5 - M	-1/5 -M	0	18/5

Solving For the optimal solution of **[Minimization]** when there are artificial variables (cont.)

- Third iteration

Basis	X1	X2	S1	R1	R2	S2	RHS
X1	1	0	0	2/5	0	-1/5	2/5
X2	0	1	0	-1/5	0	3/5	9/5
s1	0	0	1	1	-1	1	1
Z	0	0	0	7/5 – M	-M	-1/5	17/5

- Optimal solution : x1= 2/5, x2= 9/5, z= 17/5

Simplex Algorithm – Special cases

- There are four special cases arise in the use of the simplex method.

1. Degeneracy
2. Alternative optima
3. Unbounded solution
4. Nonexisting (infeasible) solution

Simplex Algorithm – Special cases (cont.)

1. Degeneracy (no improve in objective)

- It typically occurs in a simplex iteration when in the minimum ratio test more than one basic variable determine *0, hence two or more variables go* to 0, whereas only one of them will be leaving the basis.

- This is in itself not a problem, but making simplex iterations from a degenerate solution may give rise to cycling, meaning that after a certain number of iterations without improvement in objective value the method may turn back to the point where it started.

Simplex Algorithm – Special cases (cont.)

Example:

Max $3x1 + 9x2$

ST

$X1 + 4x2 \leq 8$

$X1 + 2x2 \leq 4$

$X1, x2 \geq 0$

Simplex Algorithm – Special cases (cont.)

<u>The solution:</u>

- The constraints:

$$X1 + 4x2 + s1 = 8$$

$$X1 + 2x2 + s2 = 4$$

$$X1,\ x2,\ s1, s2 \geq 0$$

Simplex Algorithm – Special cases (cont.)

Entering Variable

Leaving Variable

Basis	X1	X2	S1	S2	RHS
s1	1	4	1	0	8
s2	1	2	0	1	4
Z	-3	-9	0	0	0

Simplex Algorithm – Special cases (cont.)

Entering Variable

Leaving Variable

Basis	X1	X2	S1	S2	RHS
X2	1/4	1	1/4	0	2
s2	½	0	-1/2	1	0
Z	-3/4	0	2/4	0	18

Simplex Algorithm – Special cases (cont.)

Basis	X1	X2	S1	S2	RHS
X2	0	1	½	-1/2	2
X1	1	0	-1	2	0
Z	0	0	3/2	3/2	18

Same objective

- Same objective no change and improve (cycle)
- It is possible to have no improve and no termination for computation.

Simplex Algorithm – Special cases (cont.)

2. Alternative optima

- If the z-row value for one or more nonbasic variables is 0 in the optimal tubule, alternate optimal solution is exist.

Simplex Algorithm – Special cases (cont.)

Example:

Max $2x1+ 4x2$

ST

$X1 + 2x2 \leq 5$

$X1 + x2 \leq 4$

$X1, x2 \geq 0$

Simplex Algorithm – Special cases (cont.)

<u>The solution</u>

Max $2x1+ 4x2$

ST

$X1 + 2x2 + s1= 5$

$X1 + x2 + s2 = 4$

$X1, x2, s1, s2 \geq 0$

Simplex Algorithm – Special cases (cont.)

Entering Variable

Leaving Variable

Basis	X1	X2	S1	S2	RHS
s1	1	2	1	0	4
s2	1	1	0	1	5
Z	-2	-4	0	0	0

Simplex Algorithm – Special cases (cont.)

- Optimal solution is 10 when x2=5/2, x1=0.

Basis	X1	X2	S1	S2	RHS
x2	1/2	1	1/2	0	5/2
s2	1/2	0	-1/2	1	3/2
Z	0	0	2	0	10

- How do we know from this tubule that alternative optima exist ?

Simplex Algorithm – Special cases (cont.)

- By looking at z-row coefficient of the nonbasic variable

Entering Variable

Leaving Variable

Basis	X1	X2	S1	S2	RHS
x2	1/2	1	1/2	0	5/2
s2	1/2	0	-1/2	1	3/2
Z	0	0	2	0	10

- The coefficient for x1 is 0, which indicates that x1 can enter the basic solution without changing the value of z.

Simplex Algorithm – Special cases (cont.)

- The second alternative optima is:

Basis	X1	X2	S1	S2	RHS
x2	0	1	1	-1	1
x1	1	0	-1	2	3
Z	0	0	2	0	10

- The new optimal solution is 10 when x1=3, x2=1

Simplex Algorithm – Special cases (cont.)

3. Unbounded solution

- It occurs when nonbasic variables are zero or negative in all constraints coefficient (max) and variable coefficient in objective is negative

Simplex Algorithm – Special cases (cont.)

Example

Max $2x_1 + x_2$

ST

$X_1 - x_2 \leq 10$

$2x_1 \leq 40$

$X_1, x_2 \geq 0$

Simplex Algorithm – Special cases (cont.)

The solution

Max 2x1+ x2

ST

X1 – x2 +s1= 10

2x1 +s2= 40

X1, x2,s1,s2≥0

Simplex Algorithm – Special cases (cont.)

Basis	X1	X2	S1	S2	RHS
x2	1	-1	1	0	10
x1	2	0	0	1	40
Z	-2	-1	0	0	0

- All value if x2(nonbasic variable) either zero or negative.
- So, solution space is unbounded

Simplex Algorithm – Special cases (cont.)

4. Infeasible solution

- R coefficient at end ≠ 0

- This situation can never occur if all the constraints are of the type "≤" with nonnegative RHS