Chapter 4

The Building Blocks: Binary Numbers, Boolean Logic, and Gates

INVITATION TO

Computer Science

6TH EDITION

Objectives

After studying this chapter, students will be able to:

- Translate between base-ten and base-two numbers, and represent negative numbers using both sign-magnitude and two's complement representations
- Explain how floating-point numbers, character, sounds, and images are represented inside the computer
- Build truth tables for Boolean expressions and determine when they are true or false
- Describe the relationship between Boolean logic and computer hardware/circuits

Objectives (continued)

After studying this chapter, students will be able to:

- Construct circuits using the sum-of-products circuit design algorithm, and analyze simple circuits to determine their truth tables
- Explain how the compare-for-equality (CE) circuit works and its construction from one-bit CE circuits, and do the same for the adder circuit and its one-bit adder parts
- Describe the purpose and workings of multiplexor and decoder control circuits

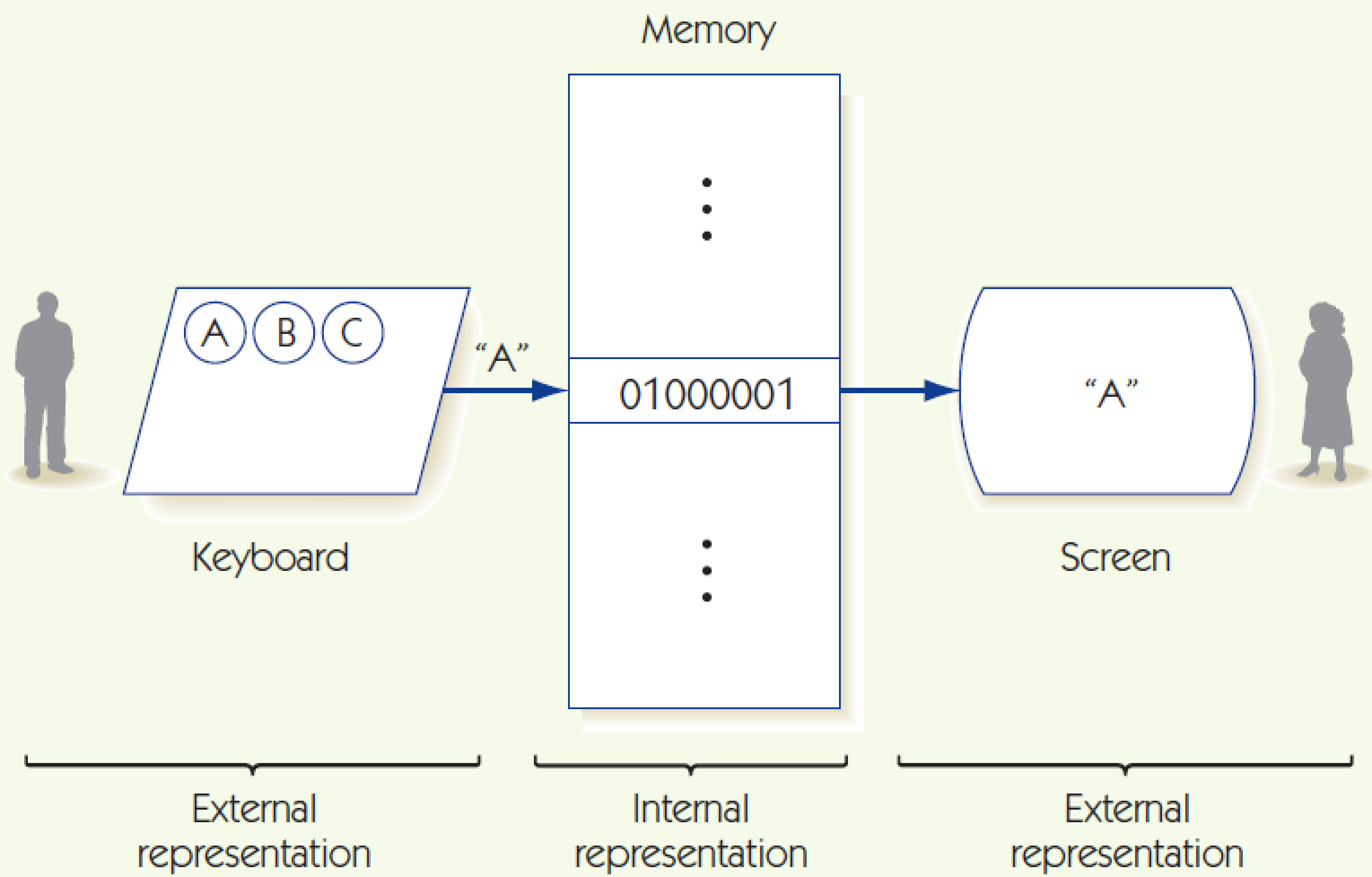
Introduction

- This chapter is about how computers work
- All computing devices are built on the ideas in this chapter
  - Laptops, desktops
  - Servers, supercomputers
  - Game systems, cell phones, MP3 players
  - Calculators, singing get-well cards
  - Embedded systems, in toys, cars, microwaves, etc.

The Binary Numbering System

- How can an electronic (or magnetic) machine represent information?
- Key requirements: clear, unambiguous, reliable
- External representation is human-oriented
  - base-10 numbers
  - keyboard characters
- Internal representation is computer-oriented
  - base-2 numbers
  - base-2 codes for characters

FIGURE 4.1



Distinction between external and internal representation of information

The Binary Numbering System (continued)

- The **binary numbering system** is a base-2 **positional numbering system**
- Base ten:
  - Uses 10 digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9
  - Each place corresponds to a power of 10
  - $1,943 = 1 * 10^3 + 9 * 10^2 + 4 * 10^1 + 3 * 10^0$
- Base two:
  - Uses 2 digits: 0, 1
  - Each place corresponds to a power of 2
  - $1101 = 1 * 2^3 + 1 * 2^2 + 0 * 2^1 + 1 * 2^0 = 13$

FIGURE 4.2

Binary	Decimal	Binary	Decimal
0	0	10000	16
1	1	10001	17
10	2	10010	18
11	3	10011	19
100	4	10100	20
101	5	10101	21
110	6	10110	22
111	7	10111	23
1000	8	11000	24
1001	9	11001	25
1010	10	11010	26
1011	11	11011	27
1100	12	11100	28
1101	13	11101	29
1110	14	11110	30
1111	15	11111	31

Binary-to-decimal conversion table

The Binary Numbering System (continued)

- Converting from binary to decimal
  - Add up powers of two where a 1 appears in the binary number
- Converting from decimal to binary
  - Repeatedly divide by two and record the remainder
  - Example, convert 9:
    - 9/2 = 4 remainder 1, binary number = 1
    - 4/2 = 2 remainder 0, binary number = 01
    - 2/2 = 1 remainder 0, binary number = 001
    - 1/2 = 0 remainder 1, binary number = 1001

The Binary Numbering System (continued)

- Computers use fixed-length binary numbers for integers, e.g., with 4 bits could represent 0 to 15
- **Arithmetic overflow**: when computer tries to make a number that is too large, e.g. 14 + 2 with 4 bits
- Binary addition: 0+0=0, 0+1=1, 1+0=1, 1+1=0 with carry of 1
- Example: 0101 + 0011 = 1000

The Binary Numbering System (continued)

- Signed integers include negative numbers
- **Sign/magnitude notation** uses 1 bit for sign, the rest for value
  - +5 = 0101, -5 = 1101
  - 0 = 0000 and 1000!
- **Two's complement representation**: to make the negative of a number, flip every bit and add one
  - +5 = 0101, -5 = 1010 + 1 = 1011
  - 0 = 0000, -0 = 1111 + 1 = 0000

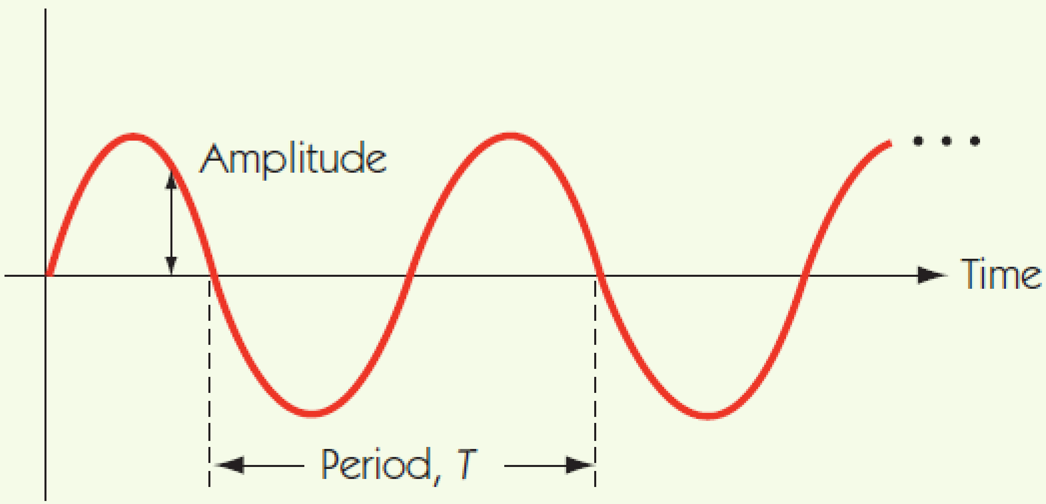
The Binary Numbering System (continued)

- Floating point numbers use binary scientific notation
  - **Scientific notation**, base 10: 1.35×10^{-5}
  - Base 2: $3.25_{10} = 11.01_2 = 1.101 \times 2^1$
- Characters and text: map characters onto binary numbers in a standard way
  - ASCII (8-bit numbers for each character)
  - Unicode (16-bit numbers for each character)

The Binary Numbering System (continued)

- Sounds and images require converting naturally **analog representations** to **digital representations**
- Sound waves characterized by:
  - **amplitude:** height of the wave at a moment in time
  - **period:** length of time until wave pattern repeats
  - **frequency:** number of periods per time unit

FIGURE 4.4

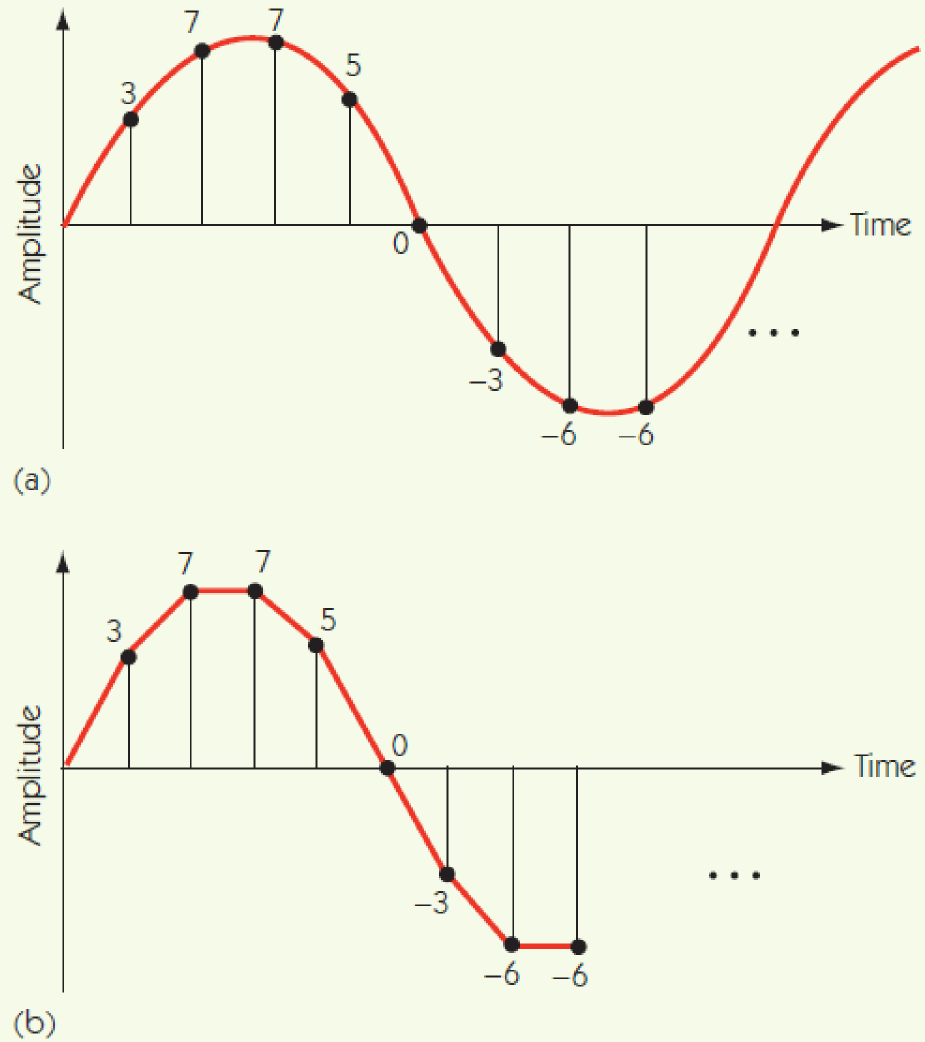


Example of sound represented as a waveform

The Binary Numbering System (continued)

- **Digitize**: to convert to a digital form
- **Sampling**: record sound wave values at fixed, discrete intervals
- To reproduce sound, approximate using samples
- Quality determine by:
  - **Sampling rate:** number of samples per second
    - More samples = more accurate wave form
  - **Bit depth:** number of bits per sample
    - More bits = more accurate amplitude

FIGURE 4.5

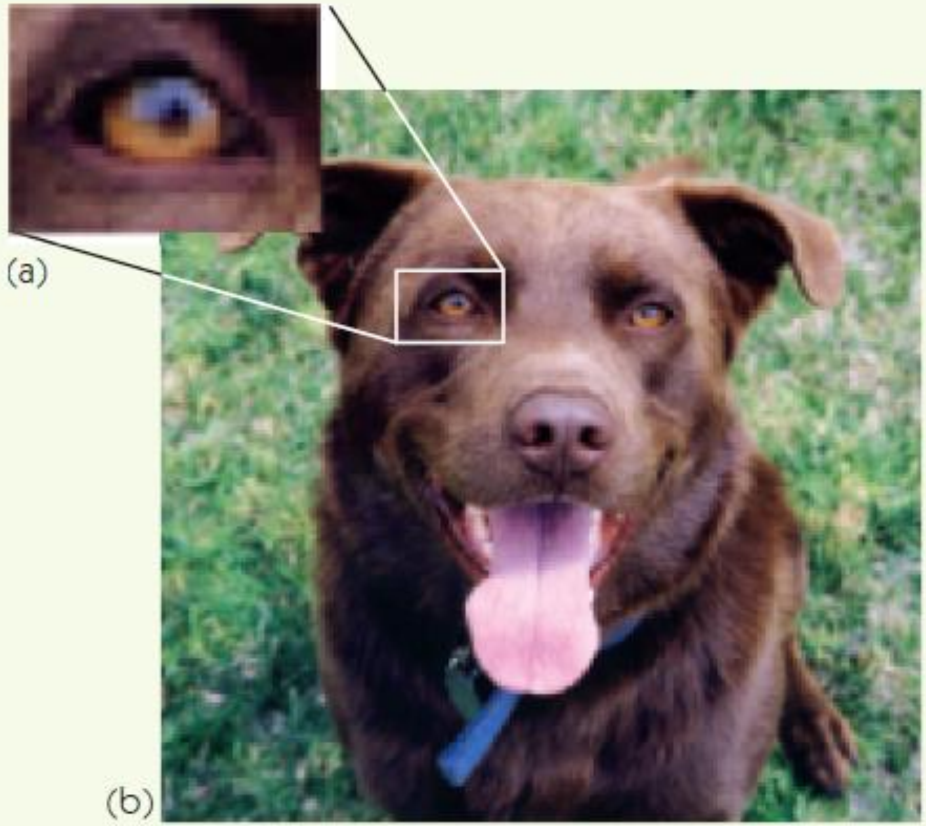


Digitization of an analog signal
(a) Sampling the original signal
(b) Recreating the signal from the sampled values

The Binary Numbering System (continued)

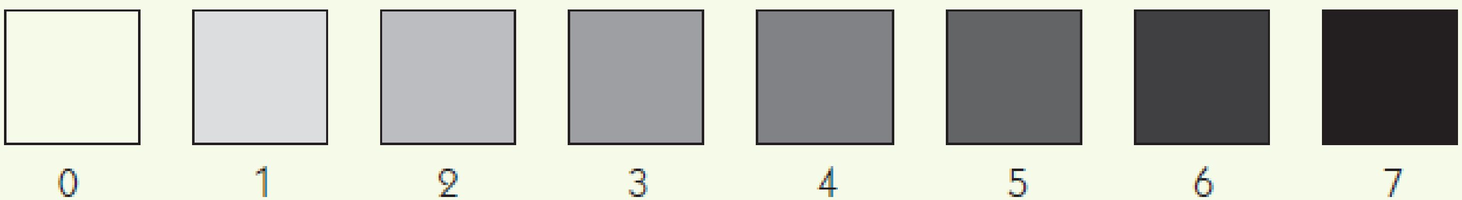
- Image sampling: record color or intensity at fixed, discrete intervals in two dimensions
- Pixels: individual recorded samples
- **RGB encoding scheme**:
  - Colors are combinations of red, green, and blue
  - One byte each for red, green, and blue
- **Raster graphics** store picture as two-d grid of pixel values

FIGURE 4.6



Example of a digitized photograph
(a) Individual pixels in the photograph
(b) Photograph

FIGURE 4.7



An eight-level gray scale

The Binary Numbering System (continued)

- Data size: how much to store:
  - 1000 integer values
  - 10-page text paper
  - 60-second sound file
  - 480 by 640 image
- **Data compression**: storing data in a reduced-size form to save space/time
  - Lossless: data can be perfectly restored
  - Lossy: data cannot be perfectly restored

FIGURE 4.8

Letter	4-bit Encoding	Variable Length Encoding
A	0000	00
I	0001	10
H	0010	010
W	0011	110
E	0100	0110
O	0101	0111
M	0110	11100
K	0111	11101
U	1000	11110
N	1001	111110
P	1010	1111110
L	1011	1111111
	(a)	(b)

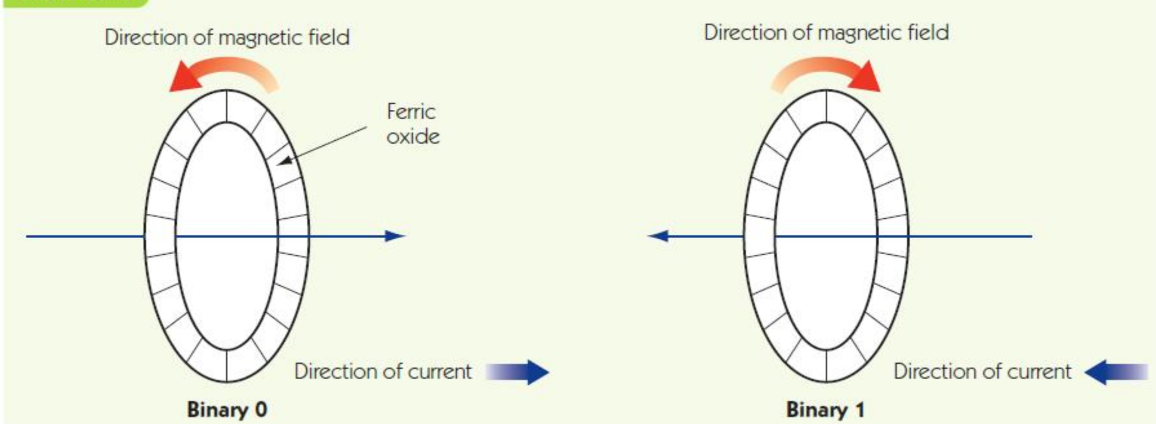
Using variable-length code sets

(a) Fixed length
(b) Variable length

The Binary Numbering System (continued)

- Computers use binary because "bistable" systems are reliable
  - current on/off
  - magnetic field left/right

FIGURE 4.9

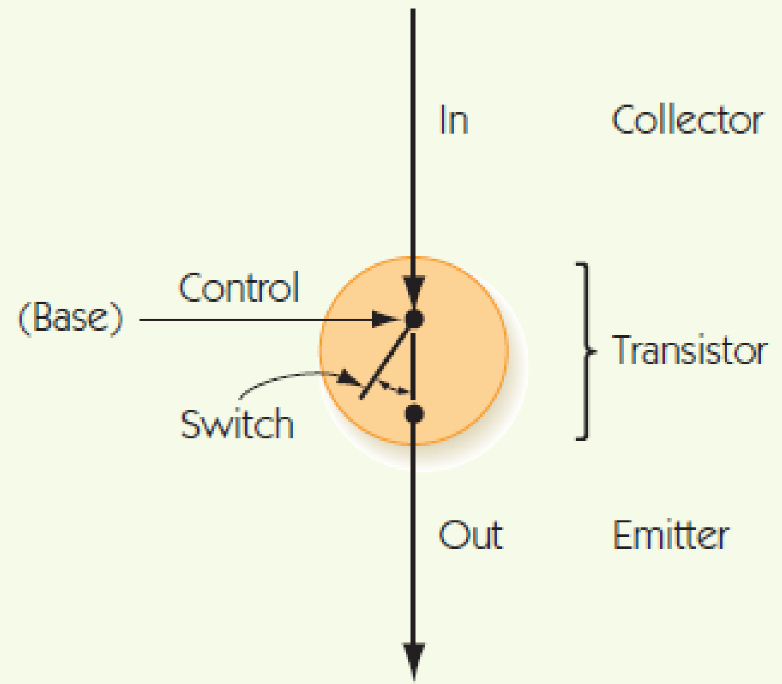


Using magnetic cores to represent binary values

The Binary Numbering System (continued)

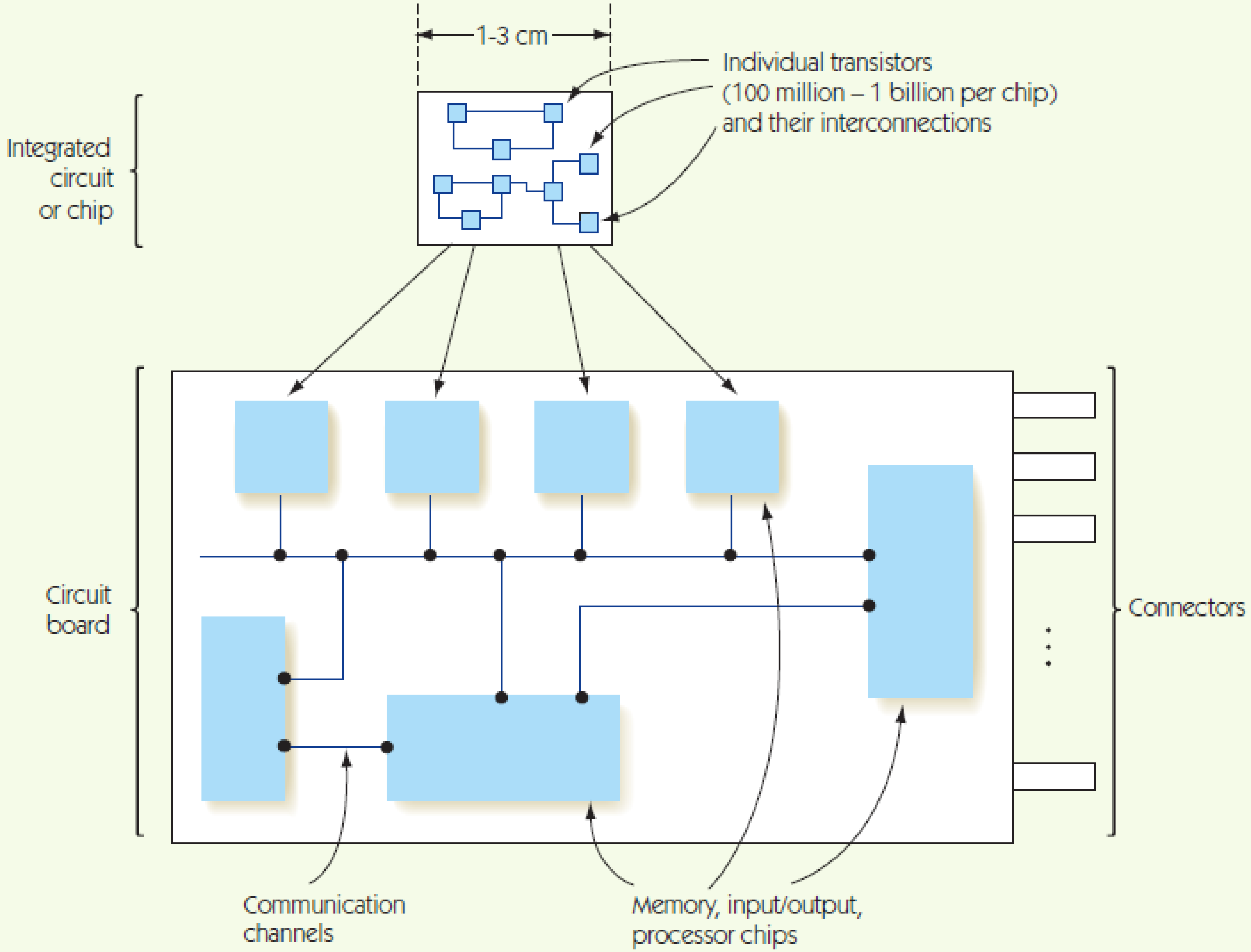
- **Transistors**
  - Solid-state switches
  - Change on/off when given power on control line
  - Extremely small (billions per chip)
  - Enable computers that work with **gigabytes** of data

FIGURE 4.11



Simplified model of a transistor

FIGURE 4.10



Relationships among transistors, chips, and circuit boards

Boolean Logic and Gates

- **Boolean logic:** rules for manipulating true/false
- Boolean expressions can be converted to circuits
- **Hardware design/logic design** pertains to the design and construction of new circuits
- Note that 1/0 of binary representations maps to true/false of Boolean logic
- Boolean expressions: $x \le 35$, $a = 12$
- Boolean operators: $(0 \le x)$ AND $(x \le 35)$, $(a = 12)$ OR $(a = 13)$, NOT $(a = 12)$

  $(0 \le x) \bullet (x \le 35)$, $(a = 12) + (a = 13)$, $\sim(a = 12)$

Boolean Logic and Gates (continued)

- **Truth tables** lay out true/false values for Boolean expressions, for each possible true/false input
- Example: (a • b) + (a • ~b)

a	b	~b	(a • b)	(a • ~b)	(a • b) + (a • ~b)
true	true	false	true	false	true
true	false	true	false	true	true
false	true	false	false	false	false
false	false	true	false	false	false

FIGURE 4.12

Inputs		Output a AND b
a	b	(also written a · b)
False	False	False
False	True	False
True	False	False
True	True	True

Truth table for the AND operation

FIGURE 4.13

Inputs		Output a OR b
a	b	(also written a + b)
False	False	False
False	True	True
True	False	True
True	True	True

Truth table for the OR operation

FIGURE 4.14

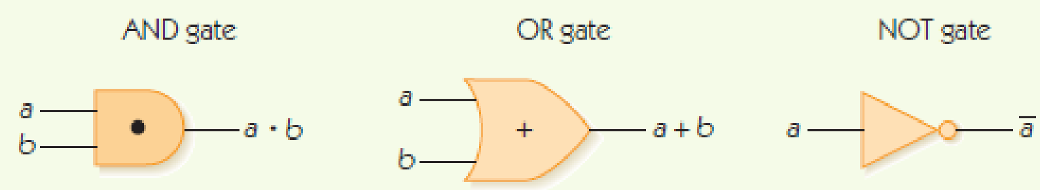
Input	Output NOT a
a	(also written $\bar{a}$)
False	True
True	False

Truth table for the NOT operation

Boolean Logic and Gates (continued)

- **Gate:** an electronic device that operates on inputs to produce outputs
- Each gate corresponds to a Boolean operator

FIGURE 4.15



a	b	a · b
0	0	0
0	1	0
1	0	0
1	1	1

a	b	a + b
0	0	0
0	1	1
1	0	1
1	1	1

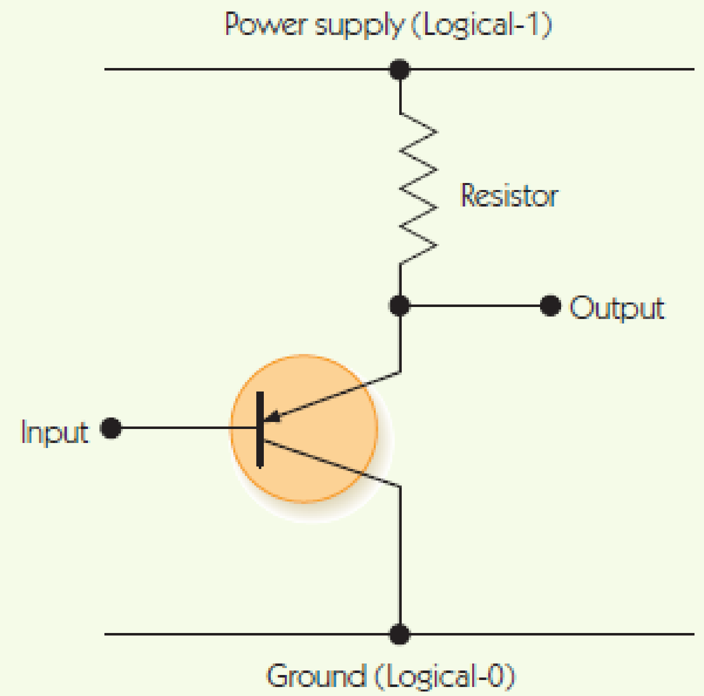
a	$\bar{a}$
0	1
1	0

The three basic gates and their symbols

Boolean Logic and Gates (continued)

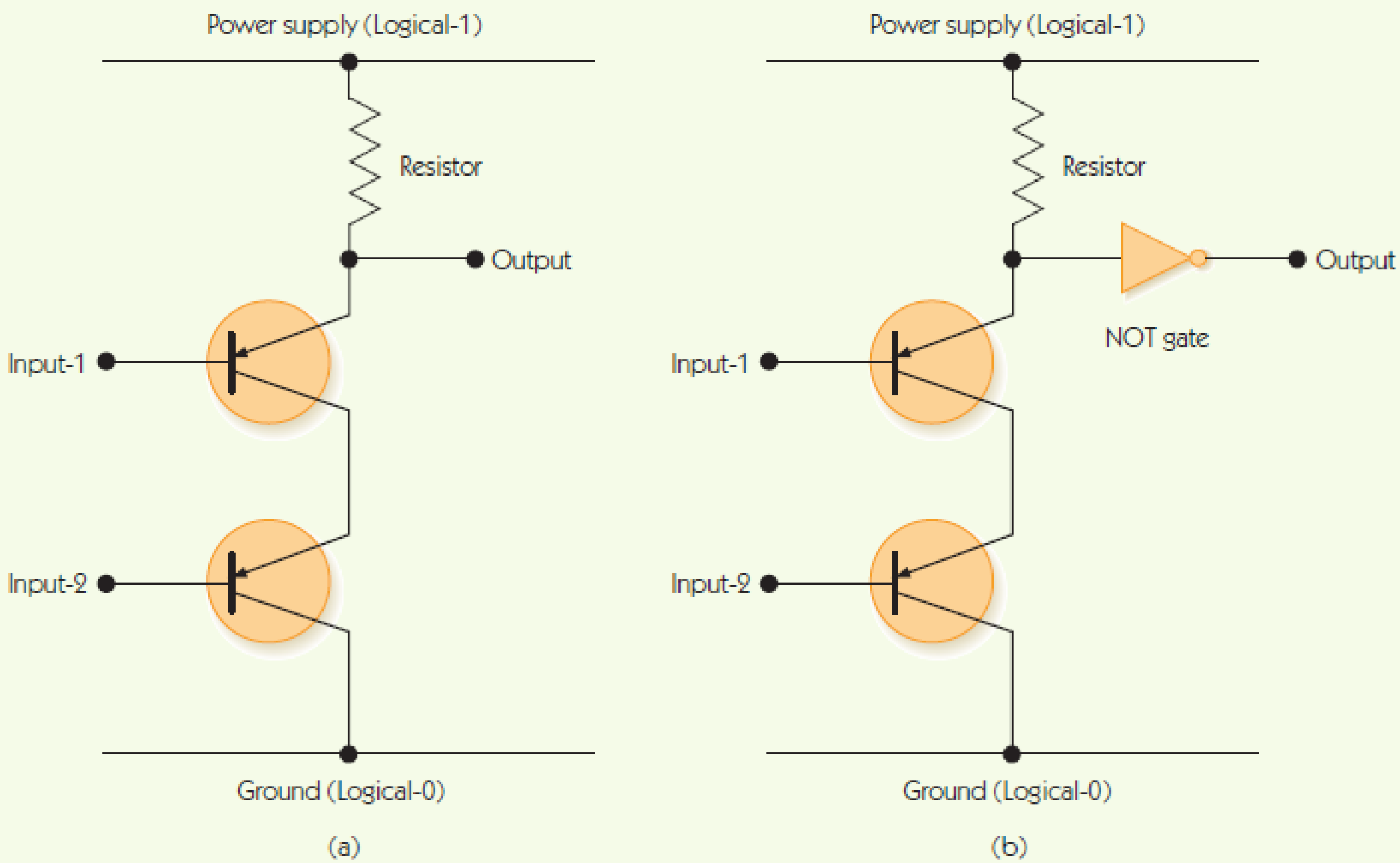
- Gates are built from transistors
- NOT gate: 1 transistor
- AND gate: 3 transistors
- OR gate: 3 transistors
- NAND and NOR: 2 transistors

FIGURE 4.16



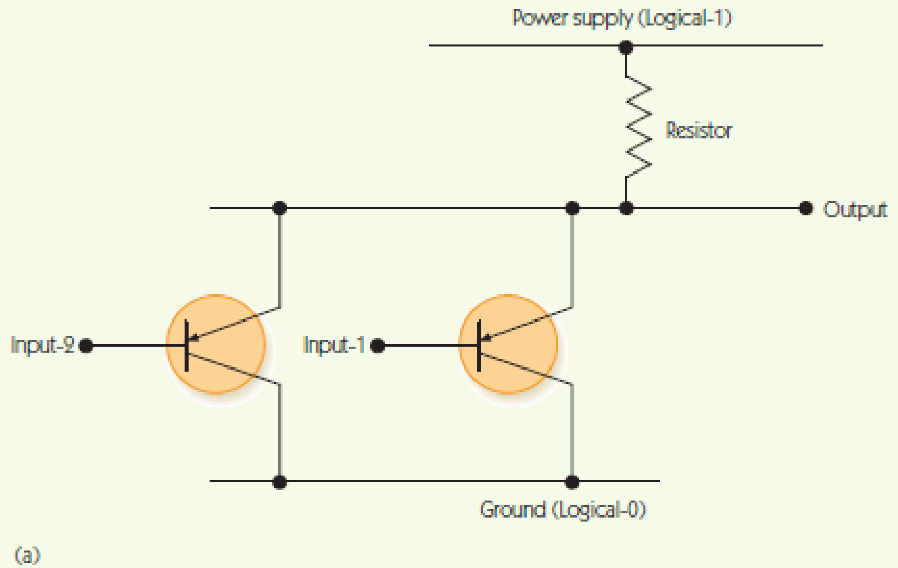
Construction of a NOT gate

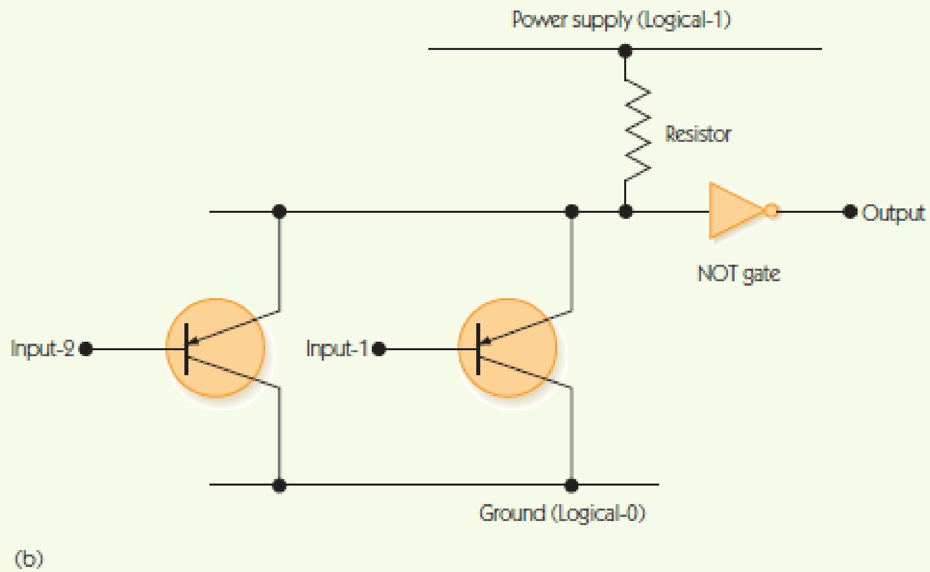
FIGURE 4.17



Construction of NAND and AND gates
(a) A two-transistor NAND gate
(b) A three-transistor AND gate

FIGURE 4.18





Construction of NOR and OR gates
(a) A two-transistor NOR gate
(b) A three-transistor OR gate

Building Computer Circuits

- **Circuit:** has input wires, contains gates connected by wires, and has output wires
- Outputs depend only on current inputs: no state

FIGURE 4.19

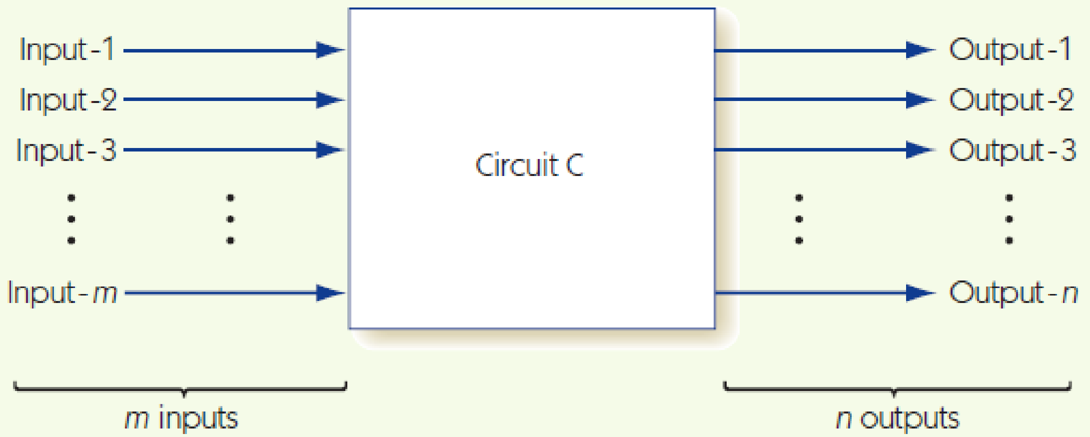


Diagram of a typical computer circuit

Building Computer Circuits (continued)

- To convert a circuit to a Boolean expression:
  - Start with output and work backwards
    - Find next gate back, convert to Boolean operator
    - Repeat for each input, filling in left and/or right side
- To convert a Boolean expression to a circuit:
  - Similar approach
- To build a circuit from desired outcomes:
  - Use standard **circuit construction algorithm**:
    - e.g., sum-of-products algorithm

Building Computer Circuits (continued)

Example from text

- Build truth table:

a	b	c	Output1	Output2
0	0	0	0	1
0	0	1	0	0
0	1	0	1	1
0	1	1	0	1
1	0	0	0	0
1	0	1	0	0
1	1	0	1	1
1	1	1	0	0

Building Computer Circuits (continued)

Example from text

- Find true rows for Output1

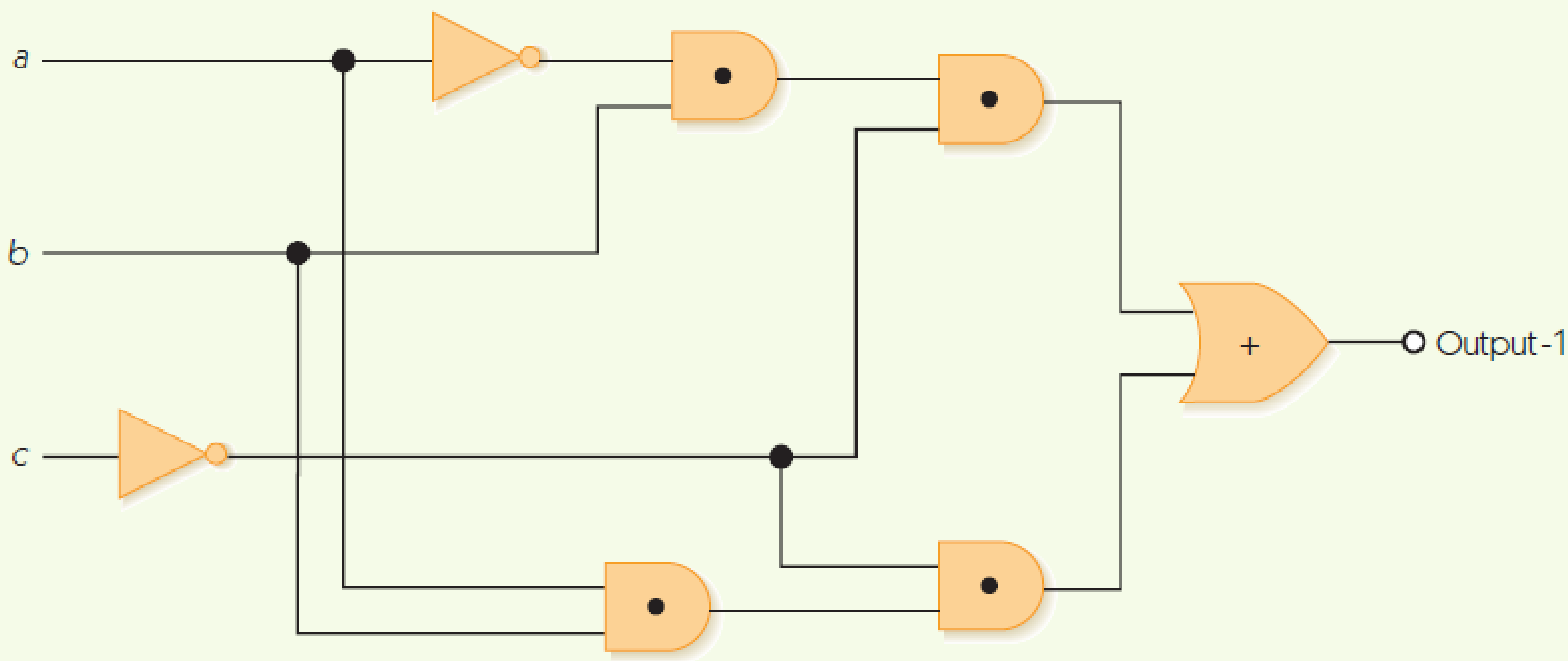
a	b	c	Output1	Output2
0	0	0	0	1
0	0	1	0	0
0	**1**	**0**	**1**	1
0	1	1	0	1
1	0	0	0	0
1	0	1	0	0
1	**1**	**0**	**1**	1
1	1	1	0	0

Building Computer Circuits (continued)

Example from text

- For each true row, AND input sto make 1
  - a = 0, b = 1, c = 0: (~a • b • ~c)
  - a = 1, b = 1, c = 0: (a • b • ~c)
- Combine row subexpressions with OR
  - (~a • b • ~c) + (a • b • ~c)
- Build circuit from expression
- (and repeat for other output)

FIGURE 4.20



Circuit diagram for the output labeled Output-1

FIGURE 4.21

1. Construct the truth table describing the behavior of the desired circuit
2. While there is still an output column in the truth table, do Steps 3 through 6
3. Select an output column
4. Subexpression construction using AND and NOT gates
5. Subexpression combination using OR gates
6. Circuit diagram production
7. Done

The sum-of-products circuit construction algorithm

Building Computer Circuits (continued)

Compare-for-equality (CE) circuit

- Input is two unsigned binary numbers
- Output is 1 if inputs are identical, and 0 otherwise
- Start with one-bit version (1-CE) and build general version from that

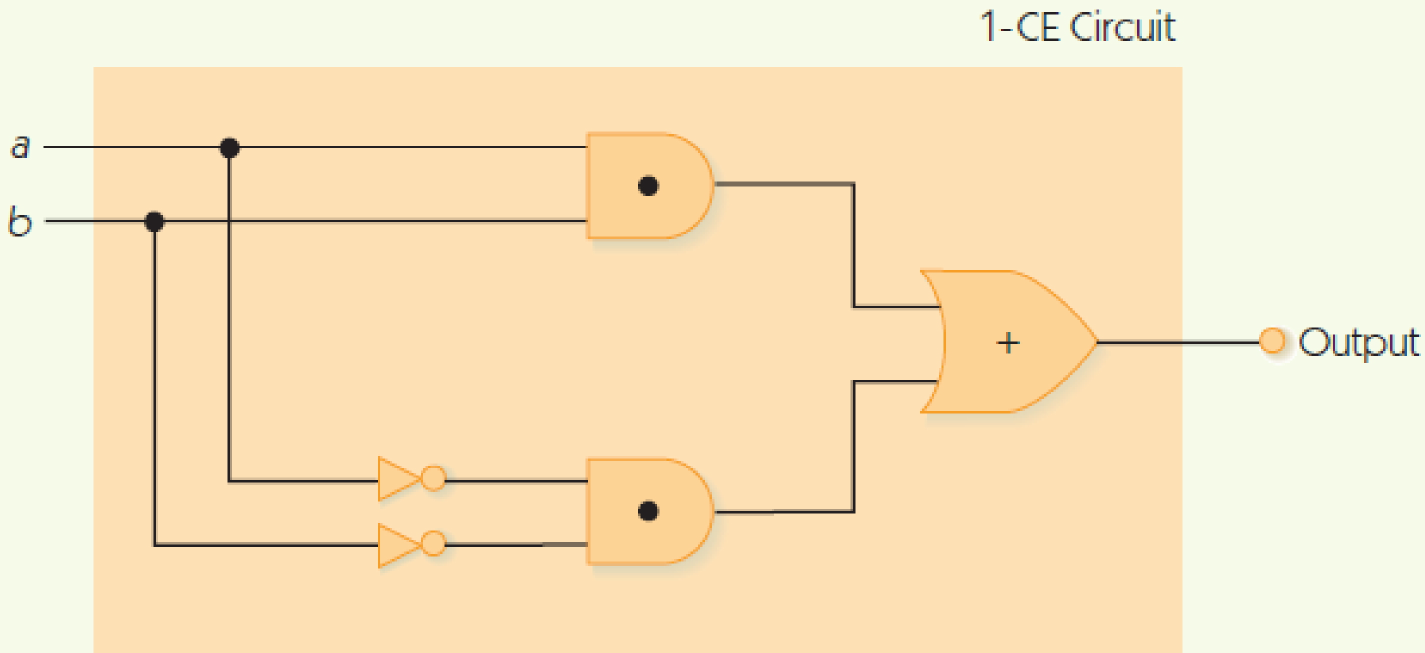
Building Computer Circuits (continued)

- 1-CE circuit: compare two input bits for equality
- Truth table:

a	b	Output
0	0	1
0	1	0
1	0	0
1	1	1

- Boolean expression: (a • b) + (~a • ~b)

FIGURE 4.22

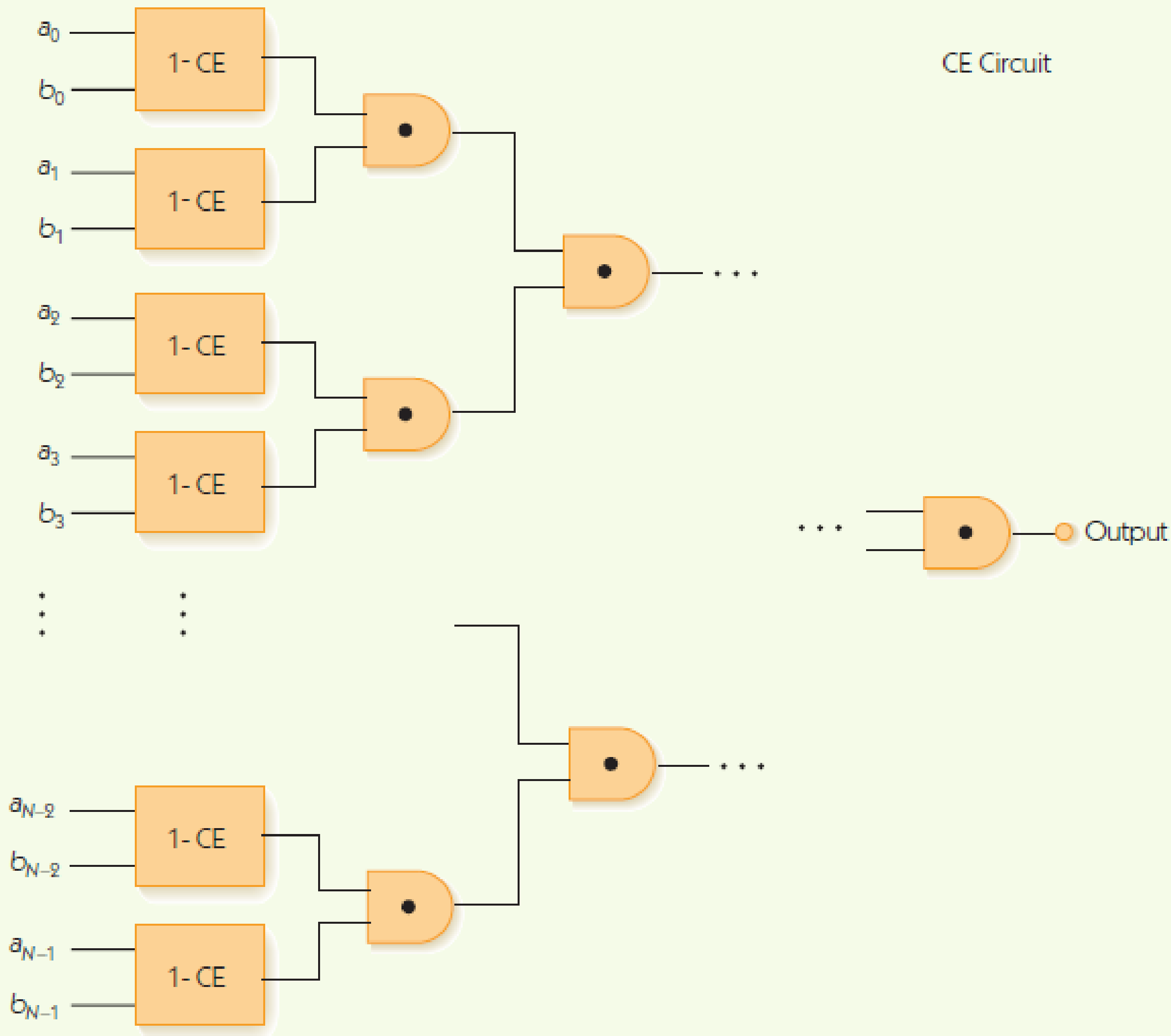


One-bit compare-for-equality circuit

Building Computer Circuits (continued)

- N-bit CE circuit
- Input: $a_0a_2\ldots a_{n-1}$ and $b_0b_2\ldots b_{n-1}$, where a_i and b_i are individual bits
- Pair up corresponding bits: a_0 with b_0, a_1 with b_1, etc.
- Run a 1-CE circuit on each pair
- AND the results

FIGURE 4.23



N-bit compare-for-equality circuit

Building Computer Circuits (continued)

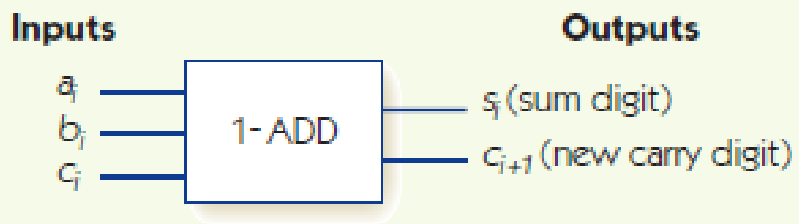
Full adder circuit

- Input is two unsigned N-bit numbers
- Output is one unsigned N-bit number, the result of adding inputs together
- Example:

				1	
	0	0	1	0	1
+	0	1	0	0	1
	0	1	1	1	0

- Start with one-bit adder (1-ADD)

FIGURE 4.24



Inputs			Outputs	
a_i	b_i	c_i	s_i	c_{i+1}
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1

The 1-ADD circuit and truth table

Building Computer Circuits (continued)

- Sum digit, s_i, has Boolean expression:

  $(\sim a_i \bullet \sim b_i \bullet c_i) + (\sim a_i \bullet b_i \bullet \sim c_i) +$

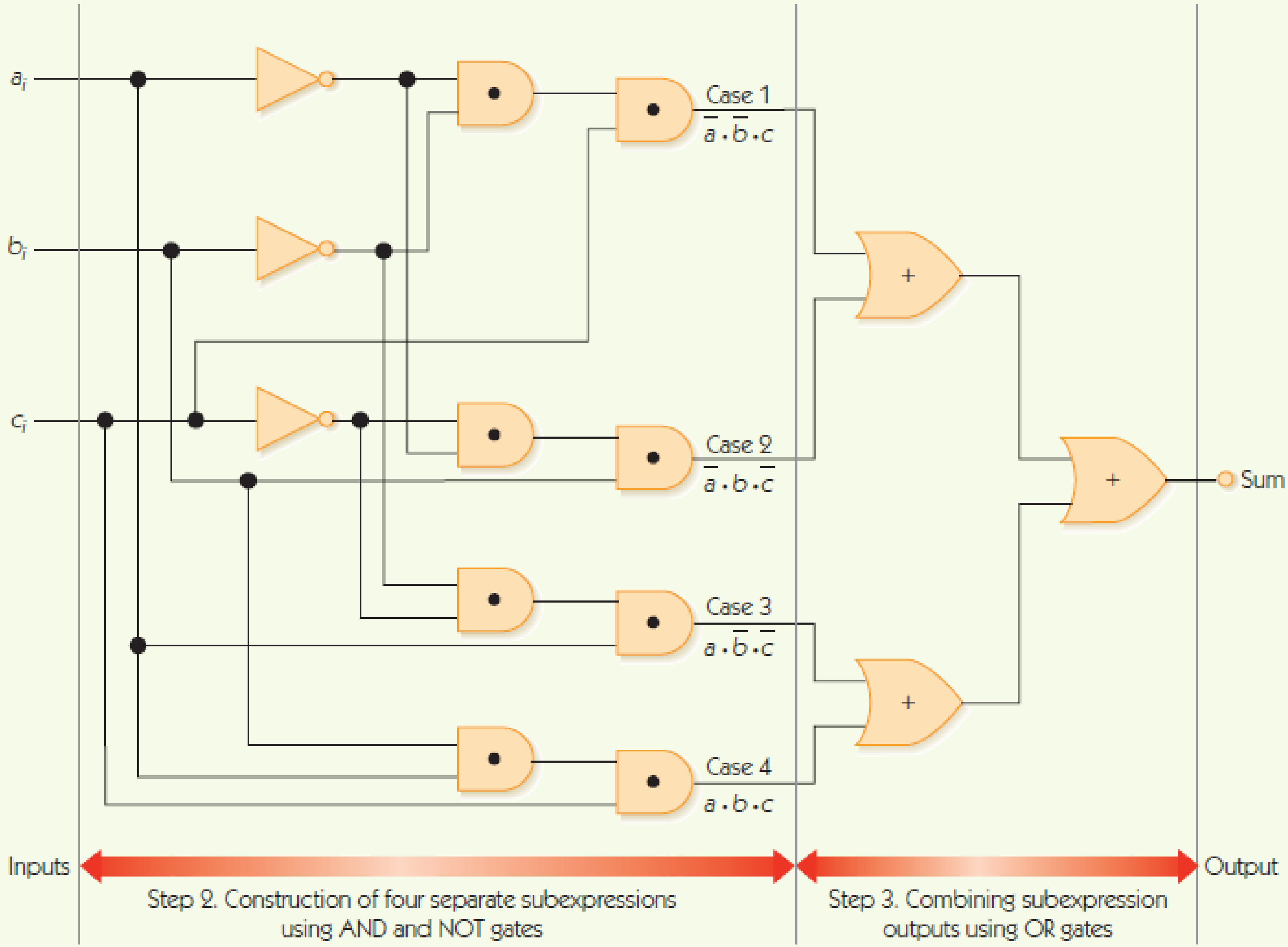
  $(a_i \bullet \sim b_i \bullet \sim c_i) + (a_i \bullet b_i \bullet c_i)$

- Carry digit, c_{i+1}, has Boolean expression:

  $(\sim a_i \bullet b_i \bullet c_i) + (a_i \bullet \sim b_i \bullet c_i) +$

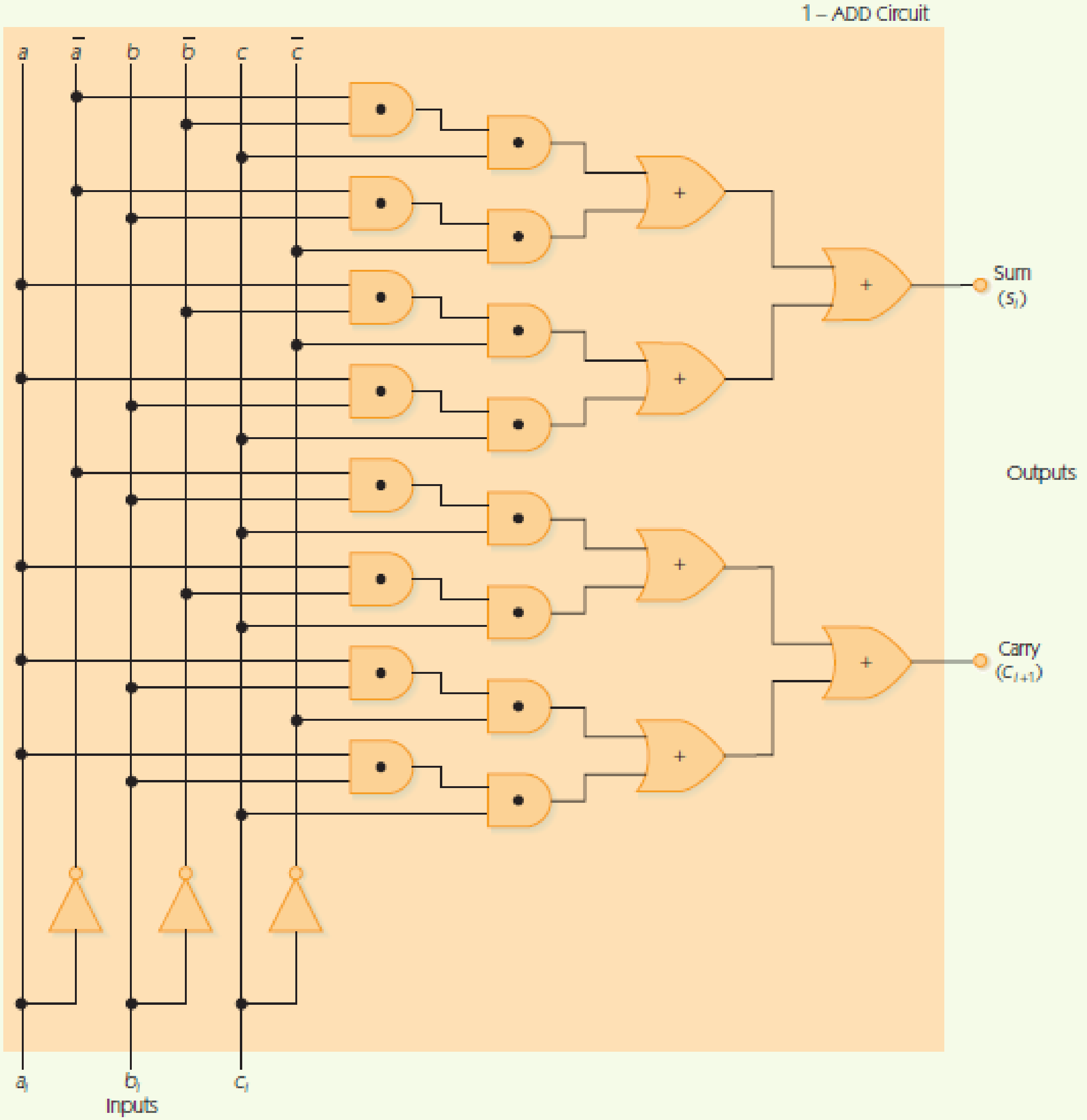
  $(a_i \bullet b_i \bullet \sim c_i) + (a_i \bullet b_i \bullet c_i)$

FIGURE 4.25



Sum output for the 1-ADD circuit

FIGURE 4.26

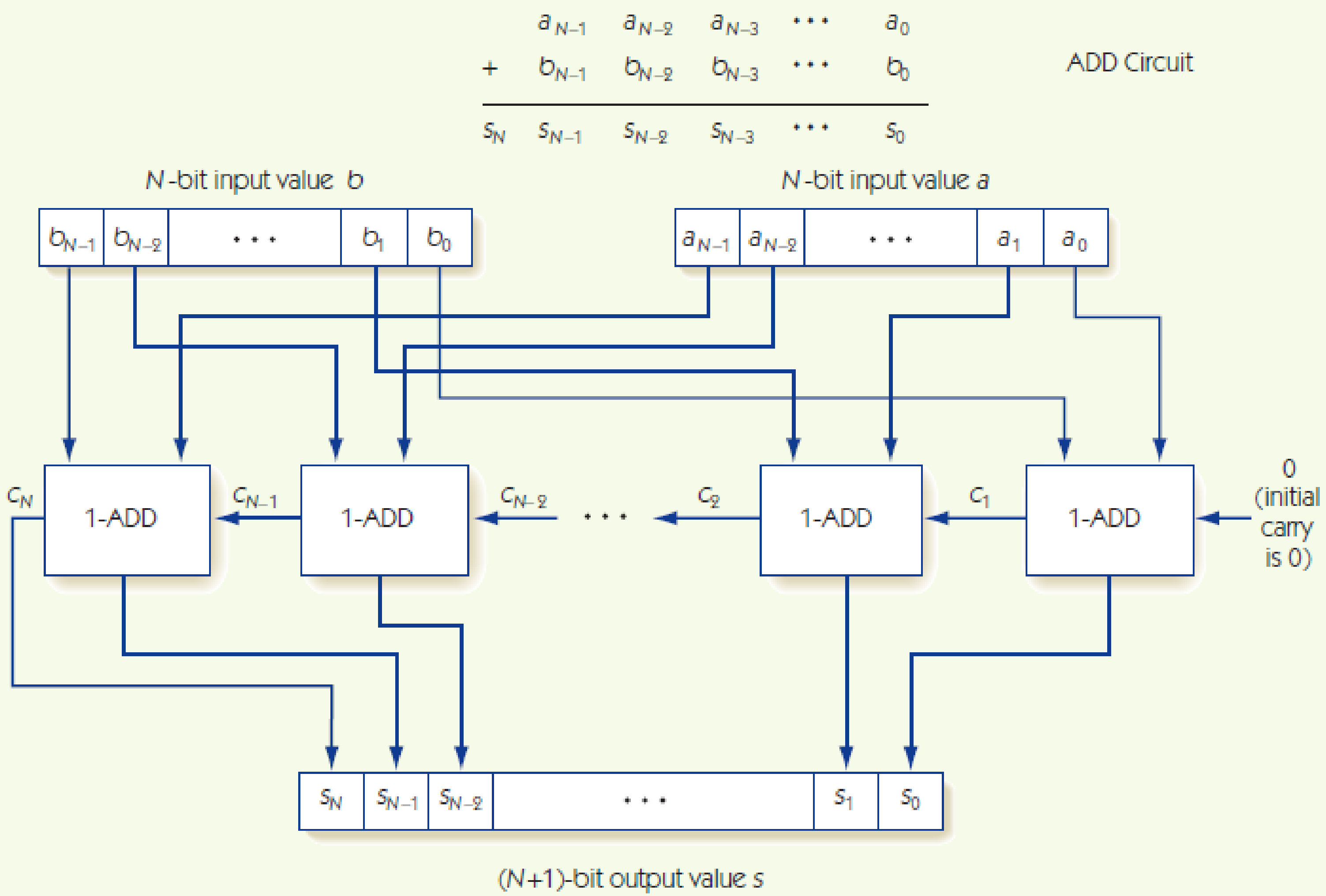


Complete 1-ADD circuit for 1-bit binary addition

Building Computer Circuits (continued)

- N-bit adder circuit
- Input: $a_0 a_2 \ldots a_{n-1}$ and $b_0 b_2 \ldots b_{n-1}$, where a_i and b_i are individual bits
- a_0 and b_0 are least significant digits: ones place
- Pair up corresponding bits: a_0 with b_0, a_1 with b_1, etc.
- Run 1-ADD on a_0 and b_0, with fixed carry in $c_0 = 0$
- Feed carry out c_1 to next 1-ADD and repeat

FIGURE 4.27

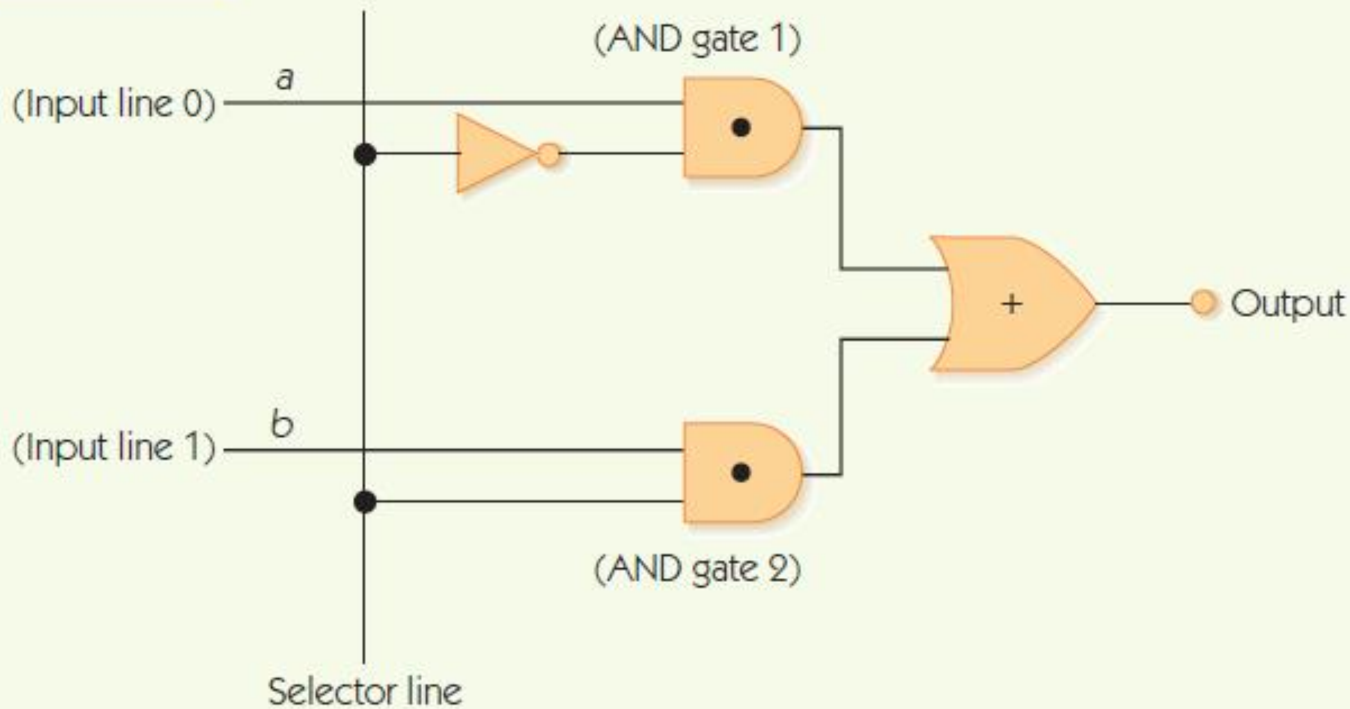


The complete full adder ADD circuit

Control Circuits

- **Control circuits** make decisions, determine order of operations, select data values
- **Multiplexor** selects one from among many inputs
  - 2^N input lines
  - N selector lines
  - 1 output line
- Each input line corresponds to a unique pattern on selector lines
- That input value is passed to output

FIGURE 4.28

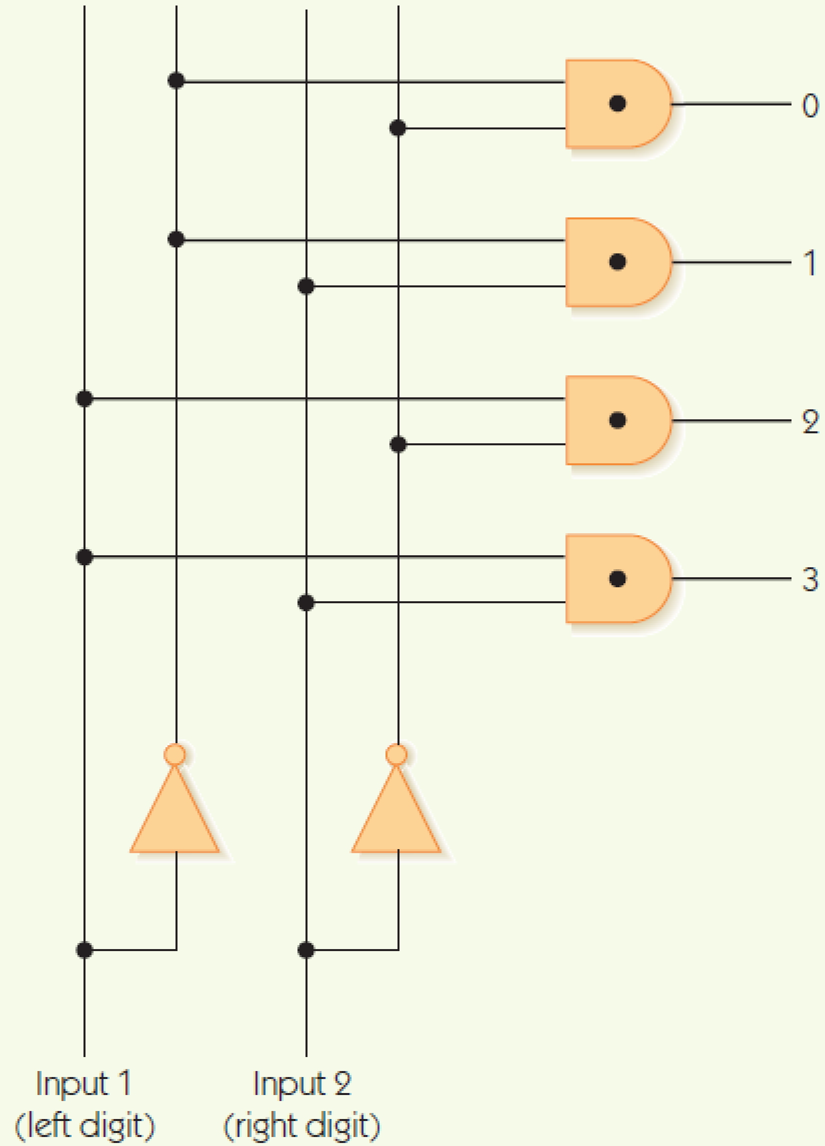


A two-input multiplexor circuit

Control Circuits (continued)

- **Decoder** sends a signal out only one output, chosen by its input
  - N input lines
  - 2^N output lines
- Each output line corresponds to a unique pattern on input lines
- Only the chosen output line produces 1, all others output 0

FIGURE 4.29

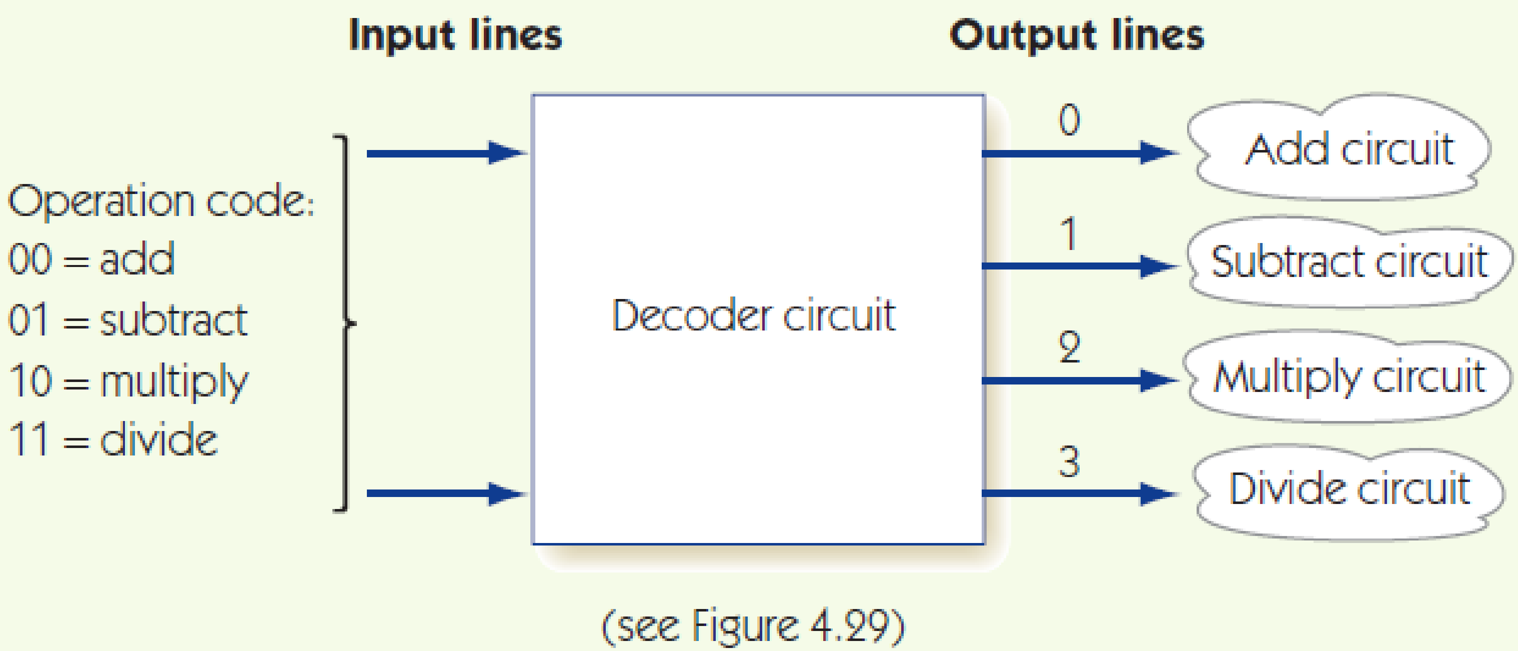


A 2-to-4 decoder circuit

Control Circuits (continued)

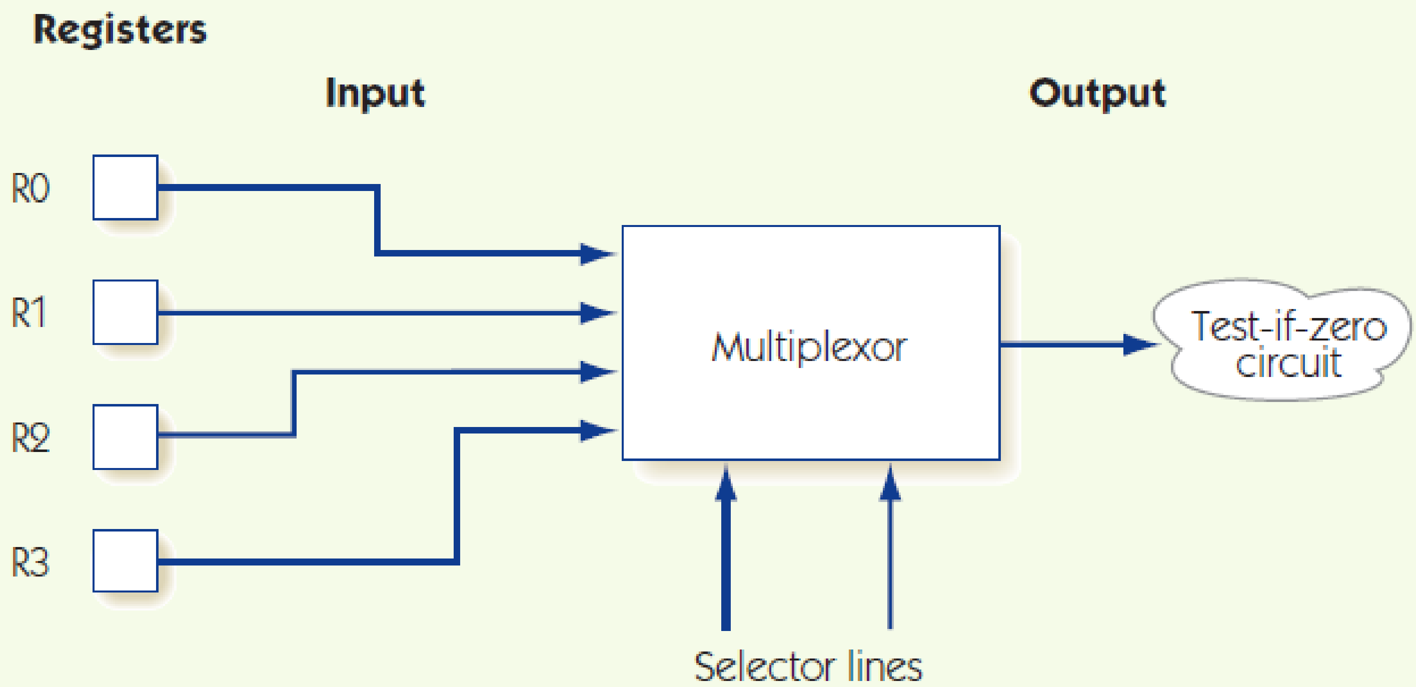
- Decoder circuit uses
  - To select a single arithmetic instruction, given a code for that instruction
  - Code activates one output line, that line activates corresponding arithmetic circuit
- Multiplexor circuit uses
  - To choose one data value from among a set, based on selector pattern
  - Many data values flow into the multiplexor, only the selected one comes out

FIGURE 4.30



Example of the use of a decoder circuit

FIGURE 4.31



Example of the use of a multiplexor circuit

Summary

- Computers use binary representations because they maximize reliability for electronic systems
- Many kinds of data may be represented at least in an approximate digital form using binary values
- Boolean logic describes how to build and manipulate expressions that are true/false
- We can build logic gates that act like Boolean operators using transistors
- Circuits may be built from logic gates: circuits correspond to Boolean expressions

Summary

- Sum-of-products is a circuit design algorithm: takes a specification and ends with a circuit
- We can build circuits for basic algorithmic tasks:
  - Comparisons (compare-for-equality circuit)
  - Arithmetic (adder circuit)
  - Control (multiplexor and decoder circuits)