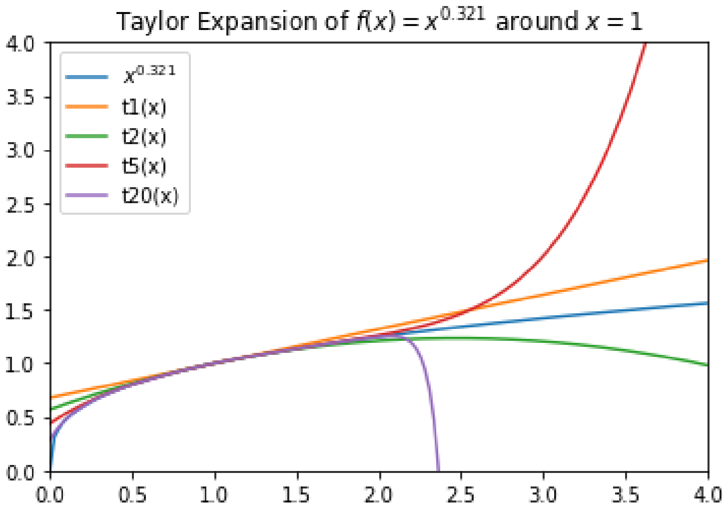
Quantitative Macroeconomics – Homework 2

Taylor Approximation of the power function

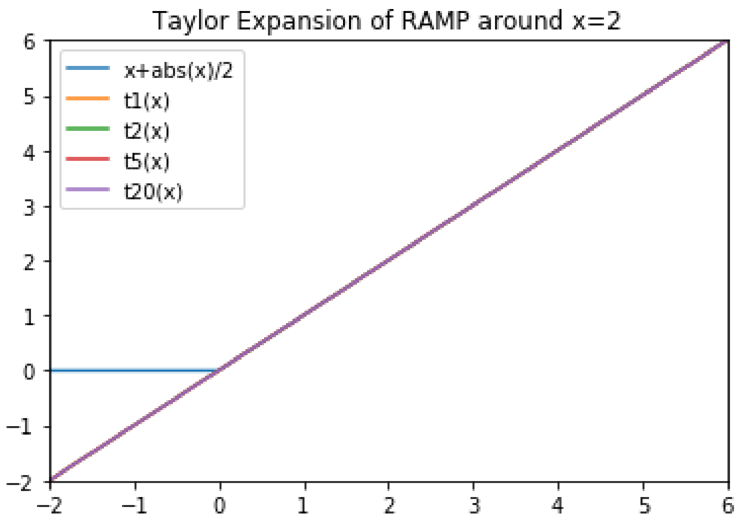
We use Taylor series approximations of orders 1,2,5 and 20 over the domain (0, 4) of the function f(x) = x $^{.321}$ around a x_0=1



For values near to one any order Taylor approximation approximates well the real function. As we move away from x=1 the approximations are less similar to the real function. The best approximation is the first order approximations, while the higher order approximations are the worst.

Taylor Approximation of the Ramp function

We use Taylor series approximations of orders 1,2,5 and 20 over the domain (-2, 6) of the Ramp function f(x) = (x+jxj)/2 around x_0=2.



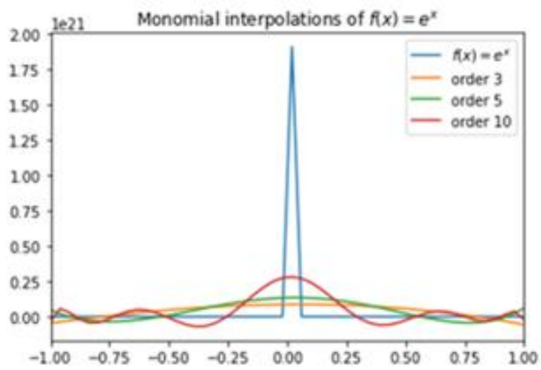
As we can see that any Taylor approximation fits exactly the real function for values higher than 0. Since we have a kink at x=0 all Taylor approximations will generate an increasing approximation error.

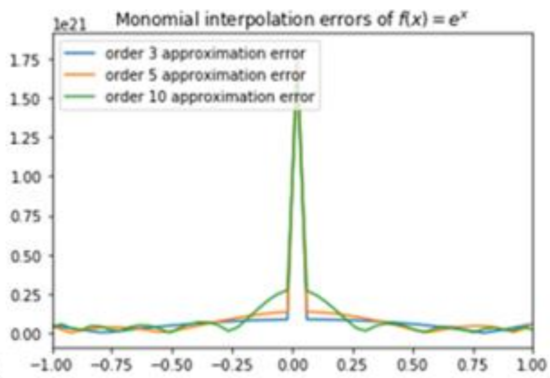
Approximation of the exponential, runge and ramp functions through interpolation methods

<u>Method 1: Evenly spaced interpolation nodes and monomials:</u>

Exponential function:

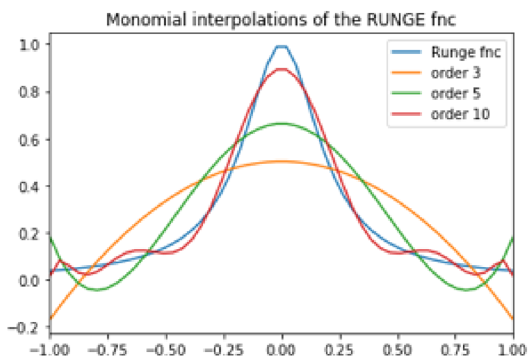
As we can see in the graph when we are not around zero all approximations are good, whereas when the function takes values around zero the approximation errors increase with the order of the approximation.

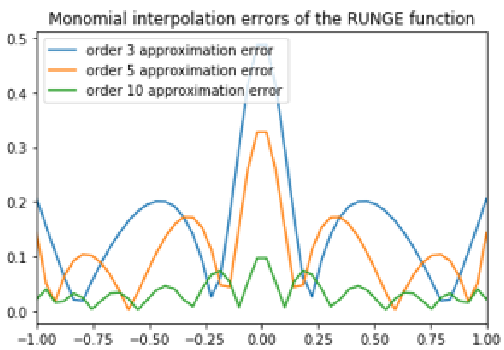




The runge function:

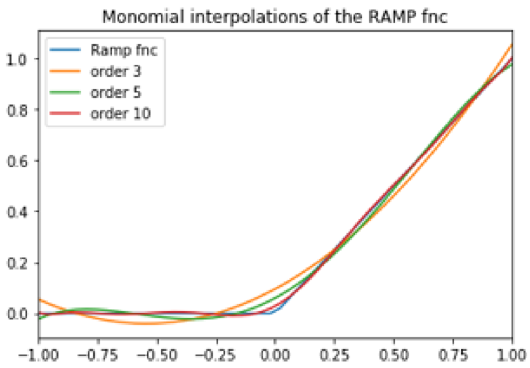
As we can see the best approximation for the runge function is given by the polynomial of order 10, and the higer error is given by the order 3 approximation.

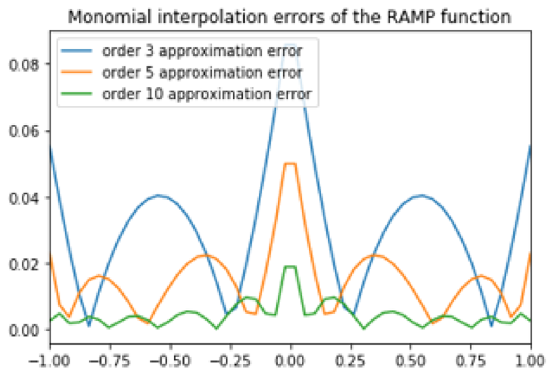




The ramp function:

In this case the polynomial of order 10 is the best approximation to the real function. The higher errors of each of the order happen for values around zero.

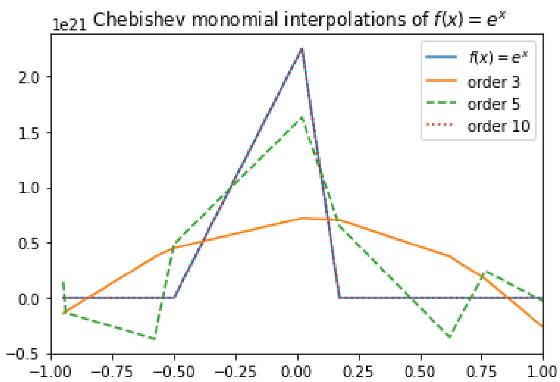


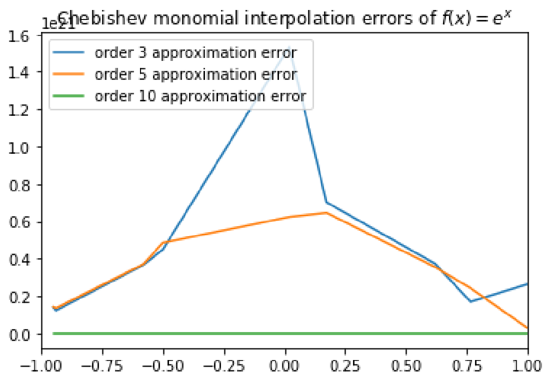


<u>Method 2:Chebyshev interpolation nodes and monomials:</u>

Exponential function

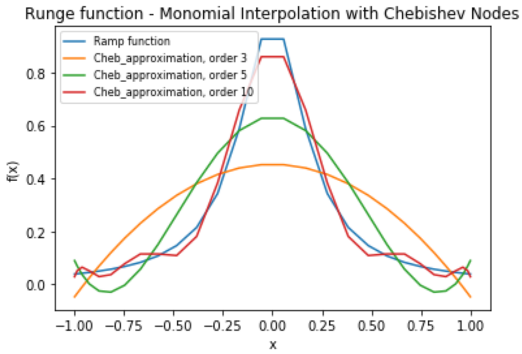
As we can see in the graphs below order 10 approximation is practically same as the original function, whereas order 3 approximation is far away for well approximating the real function. Looking at the graph of the errors we can see that the approximation error of order 10 is constant over the hall domain.

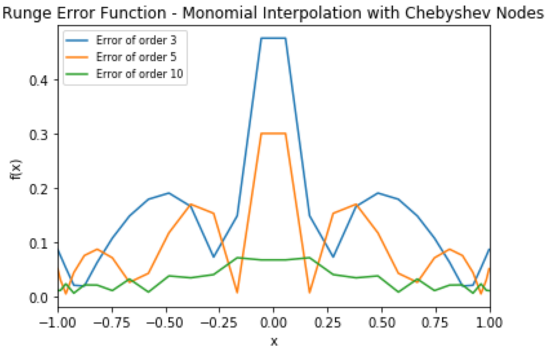




The runge function:

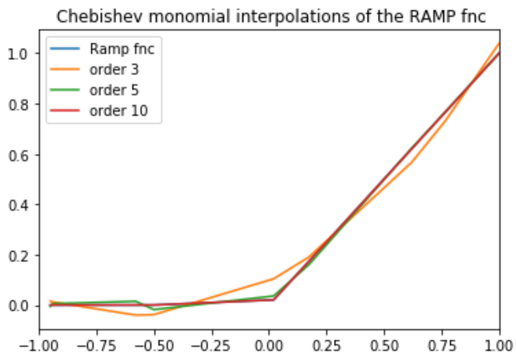
In this case order 10 approximation is the best one, followed(by far) by approximation of order 5. As we can see in the graph higher errors happen when we take values around zero.

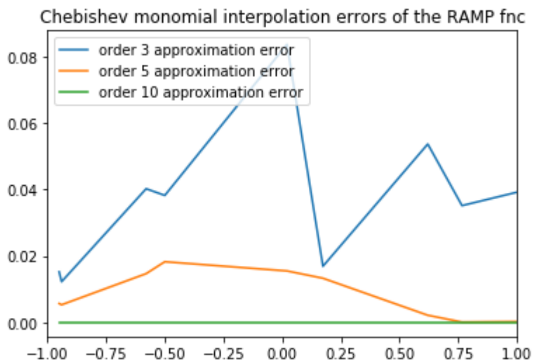




The ramp function:

In this case order 10 approximation fits very well the real function. Order 5 approximation fits better for higher x, at point 0.75 equals the error approximation of order 10. Order 3 approximation is the worst one, with the higher and less regular approximation error.

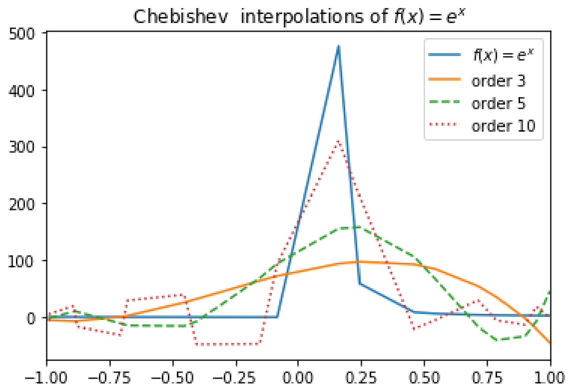


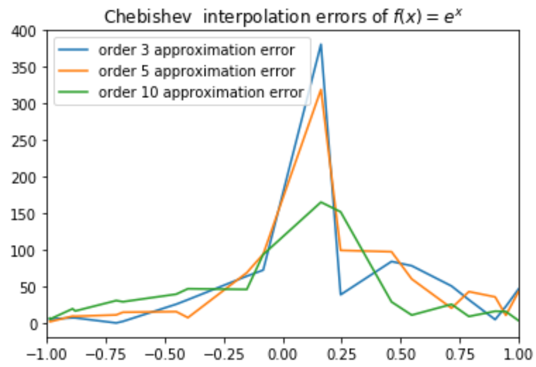


Method 3: Chebyshev interpolation nodes and Chebyshev polynomial:

Exponential function:

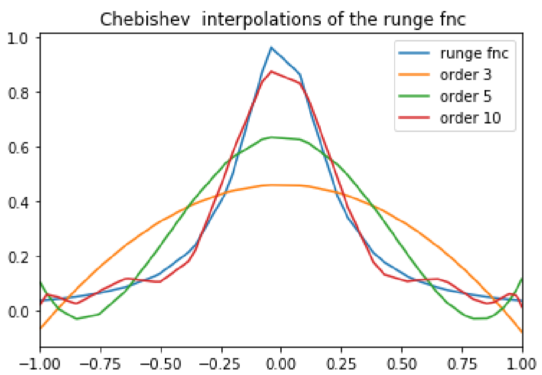
As we can see in the graphs higher order approximations led to better approximations of the real function. Here also higher errors are obtained for values around zero, whereas at extreme points approximation errors of the three orders are smaller and simillar between them.

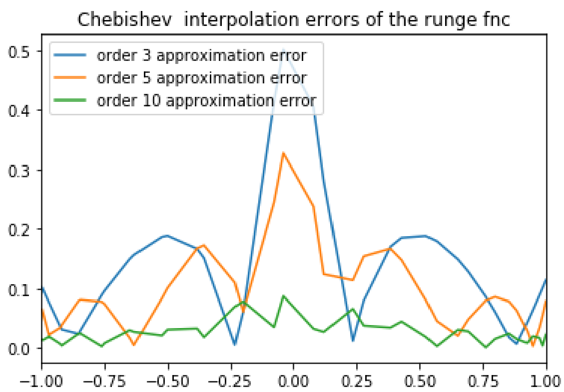




The runge function:

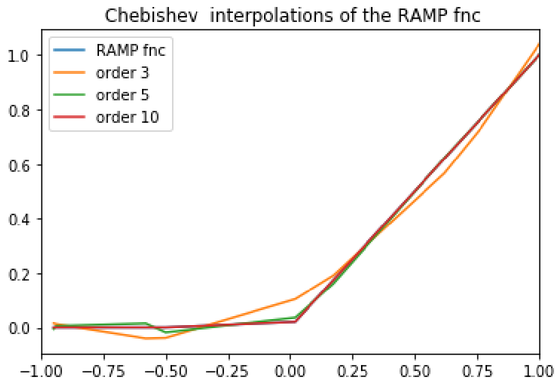
Order 10 approximation is the best one in this case. Errors for this order are more irregular than with the second method. Order 3 approximation is not a good choice for approximating the runge function by the chebyshev interpolation method.

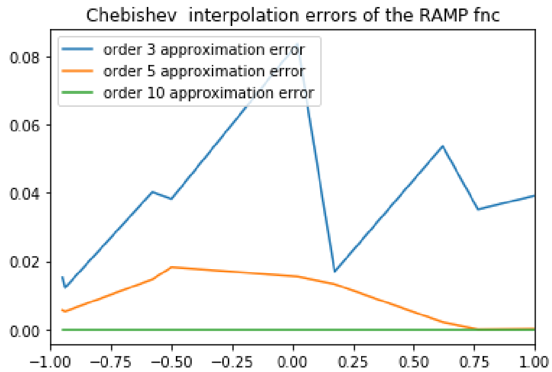




The ramp function:

In this case this graphics look very similar to the ones obtained when we approximate the function by the second method, so conclusions are same. Therefore, approximating the RAMP function with chebyshev nodes and monomials or with chebyshev nodes and the chebyshev polynomial led to the same result.





CES production function: 2d Chebyshev regression algorithm

CES production function:

$$f(k,h) = \left((1-\alpha)k^{\frac{\sigma-1}{\sigma}} + \alpha h^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}$$

Elasticity of substitution:

A we are going to show in the following lines, σ is the elasticity of substitution between capital and labour.

First we are going to compute the derivative of the production function with respect to capital and with respect to capital, and then dividen the first one by the second one in order to obtain the marginal rate of transformation.

$f'_h = ((1-\alpha)k^{\frac{\sigma-1}{\sigma}} + \alpha h^{\frac{\sigma-1}{\sigma}})^{\frac{1}{\sigma-1}} \alpha h^{\frac{-1}{\sigma}}$

$f'_k = ((1-\alpha)k^{\frac{\sigma-1}{\sigma}} + \alpha h^{\frac{\sigma-1}{\sigma}})^{\frac{1}{\sigma-1}} (1-\alpha)k^{\frac{-1}{\sigma}}$

MRT = $\frac{\alpha}{1-\alpha}\left(\frac{k}{h}\right)^{\frac{1}{\sigma}}$

From this expression we can get:

$$\left(\frac{k}{h}\right) = \left(\frac{1-\alpha}{\alpha} * MRT\right)^{\sigma}$$

Taking logs:

ln MRT = ln $\left(\frac{\alpha}{1-\alpha}\right) + \left(\frac{1}{\sigma}\right)ln\left(\frac{k}{h}\right)$

ln $\left(\frac{k}{h}\right)$ = σ $ln\left(\frac{1-\alpha}{\alpha}\ MRT\right)$

Taking the following derivative:

$$\frac{d\ln MRT}{d\ln(\frac{k}{h})} = \frac{d\ln\left(\frac{\alpha}{1-\alpha}\right) + \left(\frac{1}{\sigma}\right)\ln\left(\frac{k}{h}\right)}{d\ln(\frac{k}{h})} = 1/\sigma$$

And taking the inverse we obtain σ, so we have shown that sigma is the elasticity of substitution.

Labor share with competitive markets:

Labor share is given by:

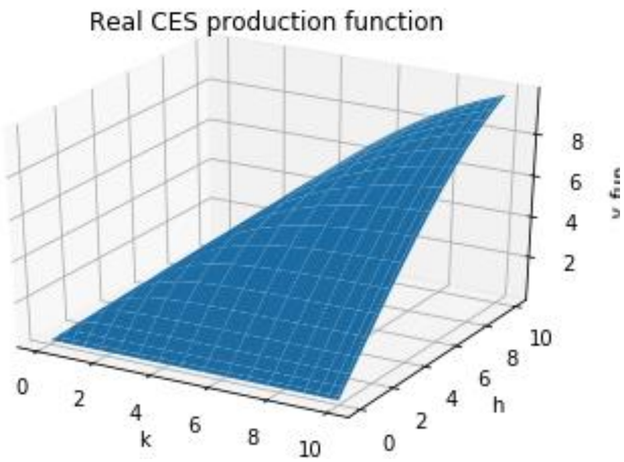
Labor_share = mgh / f (k,h)

Therefore we can compute it as follows:

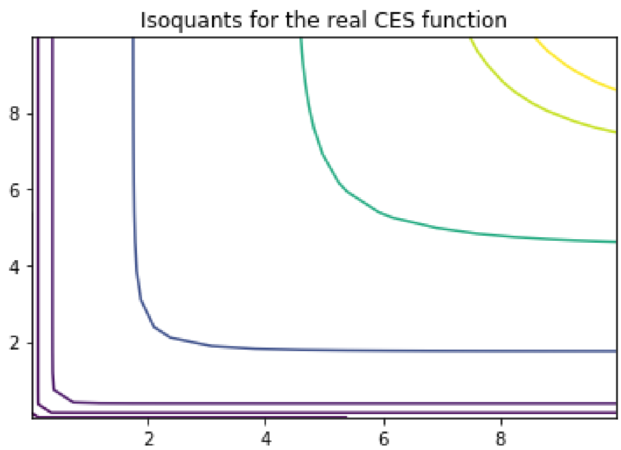
$$\text{Labor share} = \frac{((1-\alpha)k^{\frac{\sigma-1}{\sigma}} + \alpha h^{\frac{\sigma-1}{\sigma}})^{\frac{1}{\sigma-1}} \alpha h^{\frac{-1}{\sigma}}}{((1-\alpha)k^{\frac{\sigma-1}{\sigma}} + \alpha h^{\frac{\sigma-1}{\sigma}})^{\frac{\sigma}{\sigma-1}}} = \frac{\alpha}{((1-\alpha)k^{\frac{\sigma-1}{\sigma}} + \alpha h^{\frac{\sigma-1}{\sigma}}) h^{\frac{1}{\sigma}}}$$

Approximations:

In the figure below we can see the real CES production function, for the given parameters.

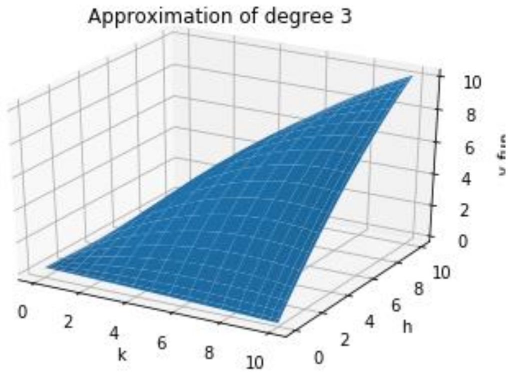


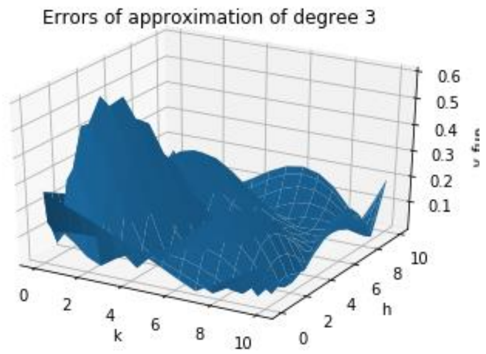
We also plot the exact isoquants associated with the percentiles 5, 10, 25, 50, 75, 90 and 95 of the output.

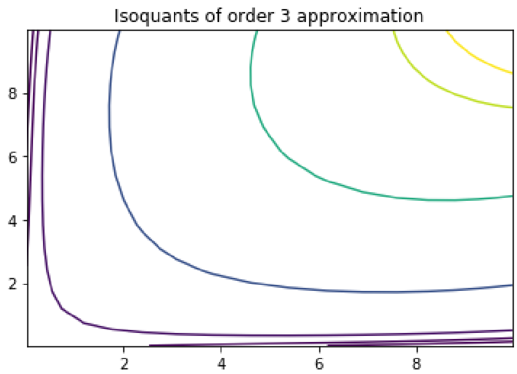


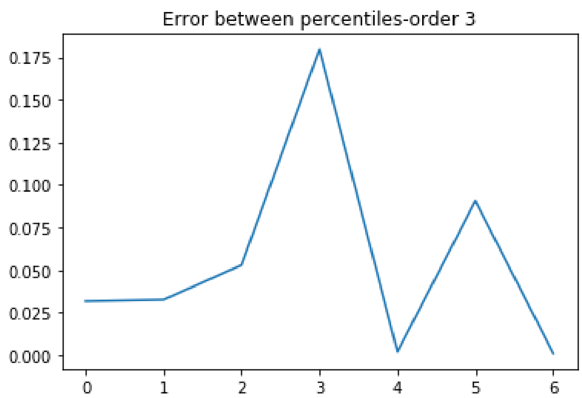
In order to approximate this function we use a 2-dimensional Chebyshev regression algorithm. For each approximation degree we obtain the approximated function and the errors. In addition we plot the isoquants of the approximation and the associated errors per each of the isoquants.

Approximation of degree 3:

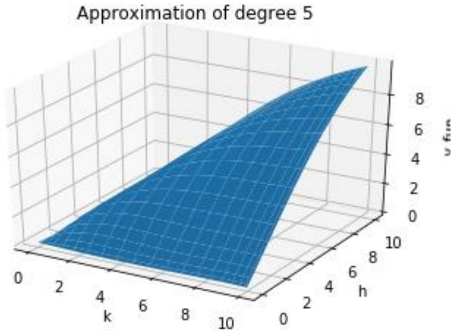


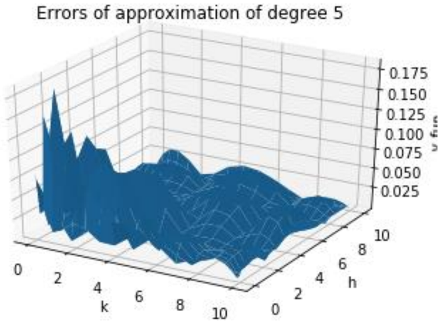


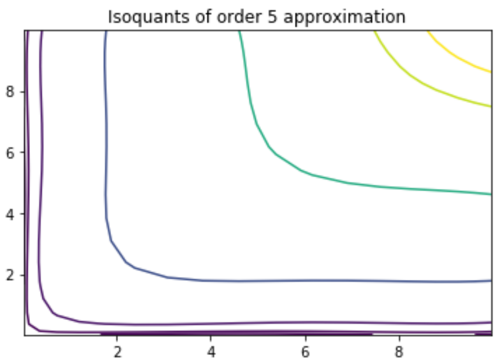


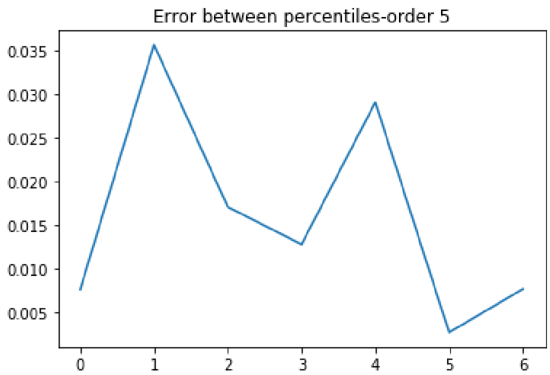


Approximation of degree 5:

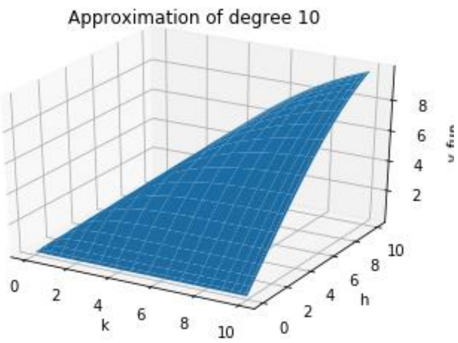


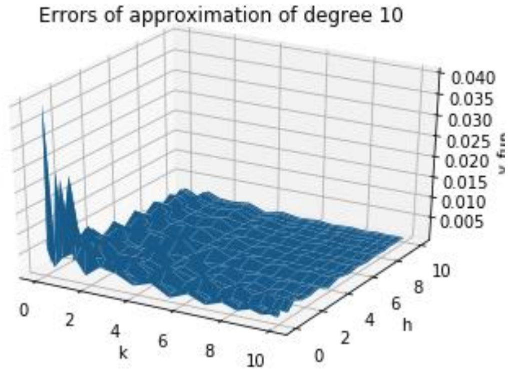


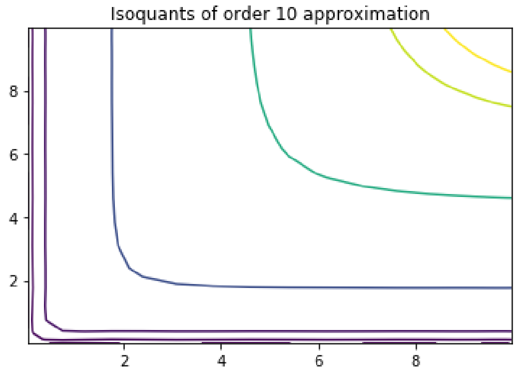


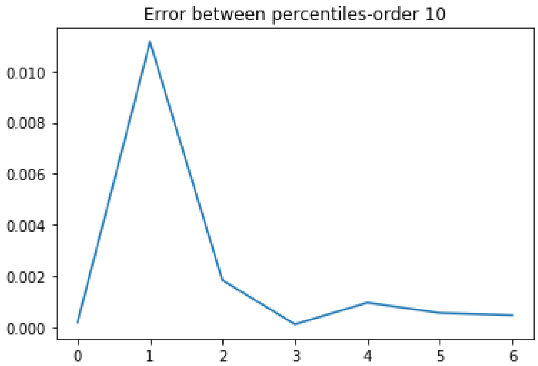


Approximation of degree 10:









Approximation of degree 15: