Quantitative Macroeconomics – Homework 5

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Question 1: Factor Input Missallocation ($\gamma = 0.6$)

Simulating the data:

Firm specific output, capital and productivity are, respectively, y_i, k_i and z_i.

We know that ln z_i and ln k_i follow a joint normal distribution, amd the following momento conditions:

- Corr(lnz_i, lnk_i) = 0
- Var (lnz_i) = 1
- Var (lnk_i) = 1
- E(s) = 1
- E(k) = 1

Since the log variables follow a joint normal distribution we need the variance-covariance matrix and the average of each of the variables in order to describe the entire distribution.

Given that the log variables are uncorrelated and that the variance of both is equal to one, we obtain the following variance-covariance matrix:

$$\text{Cov(lnzi, lnki)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

When correlation betwen variables changes to 0.5 and to -0.5, since both have unit variance the covariance will be equal to the correlation coefficient in each case. In that cases the covariance matrix will change by adjusting out of diagonal elements to the covariance value.

Since lnz and lnk follow a normal distribution, the level variables will follow a log normal distribution.

The variable k comes from a log normal distribution, therefore we can write its expectation as:

$$E(k) = e^{\mu_k + 0.5\sigma^2} = 1$$

From this expression we obtain $\mu_k = -0.5$

The density function for z is the following:

$$f(z) = \frac{1}{z}\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{lnz-\mu_z}{\sigma}\right)^2}$$

Knowing that $s = z^{\frac{1}{1-\gamma}}$, we construct the density of s as:

$$f(s) = f_z(s^{1-\gamma})ds = f_z(s^{1-\gamma})|(1-\gamma)s^{-\gamma}|$$

Therefore, the expectation of s can be writen as:

$$E(s) = \int_0^{\infty} sf(s)ds = s\frac{1}{s^{1-\gamma}}\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{\ln(s^{1-\gamma})-\mu_z}{\sigma}\right)^2}(1-\gamma)s^{-\gamma}$$
$$= \frac{1-\gamma}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{\ln(s^{1-\gamma})-\mu_z}{\sigma}\right)^2} = 1$$

Solving for this we obtain $\mu_z = -1.25$.

With this information we can characterize the joint normal distribution for lnk and lnz as (in the case of no correlation):

$$\begin{pmatrix}\ln(k)\\ \ln(z)\end{pmatrix} \sim N\left[\begin{pmatrix}-0.5\\ -1.25\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right]$$

For correlation = 0.5:

$$\begin{pmatrix}\ln(k)\\ \ln(z)\end{pmatrix} \sim N\left[\begin{pmatrix}-0.5\\ -1.25\end{pmatrix}, \begin{pmatrix}1 & 0.5\\ 0.5 & 1\end{pmatrix}\right]$$

For correlation = - 0.5:

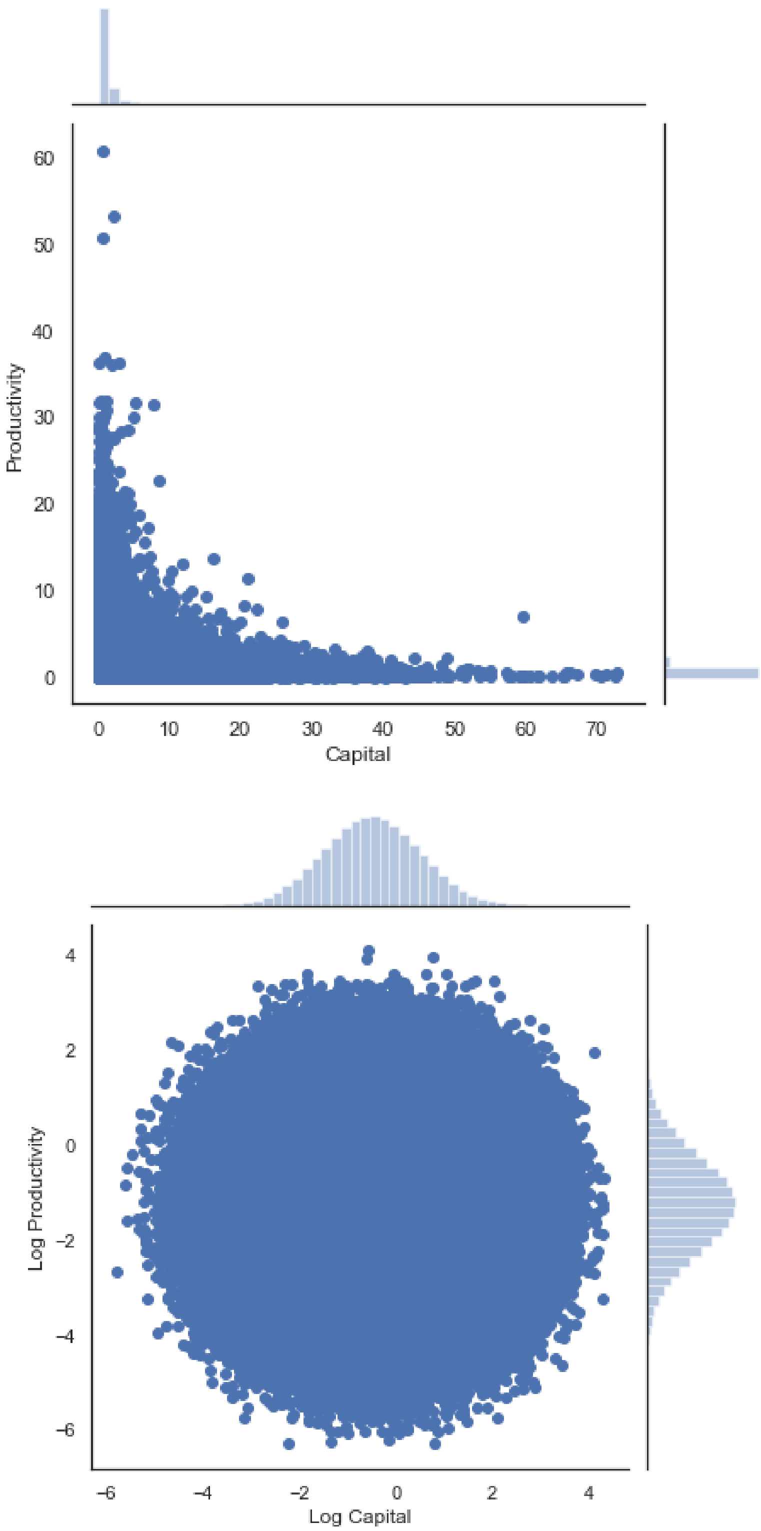
$$\begin{pmatrix}\ln(k)\\ \ln(z)\end{pmatrix} \sim N\left[\begin{pmatrix}-0.5\\ -1.25\end{pmatrix}, \begin{pmatrix}1 & -0.5\\ -0.5 & 1\end{pmatrix}\right]$$

Therefore the level variables follow a joint log normal distribution.

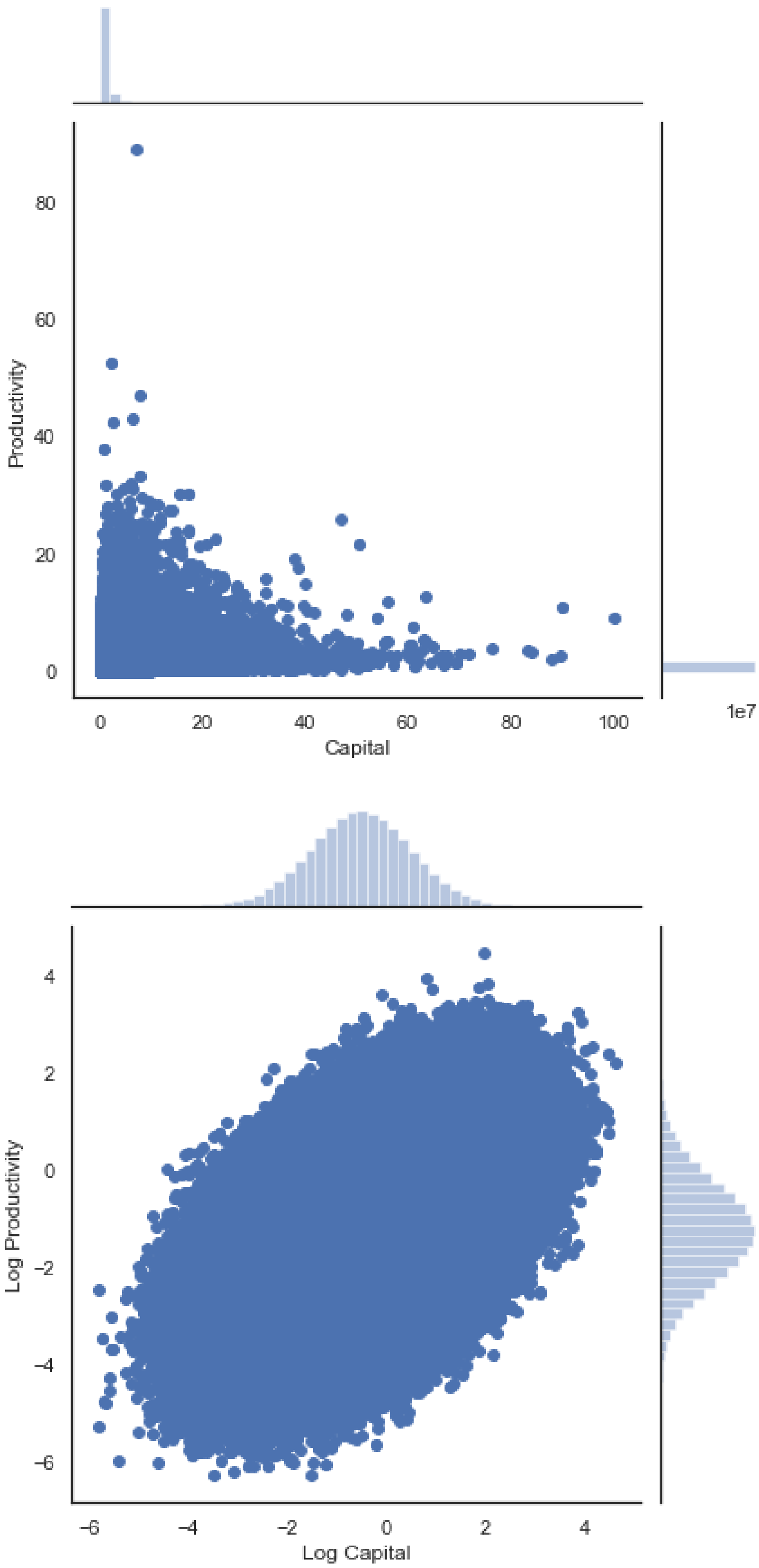
Knowing the entire distribution of both variables we simulate 10000000 observations, which will be our complete data that captures the entire population of firms in a given country.

In the graphs below we can see the joint density of capital and productivity in levels and in logs, for each value of correlation

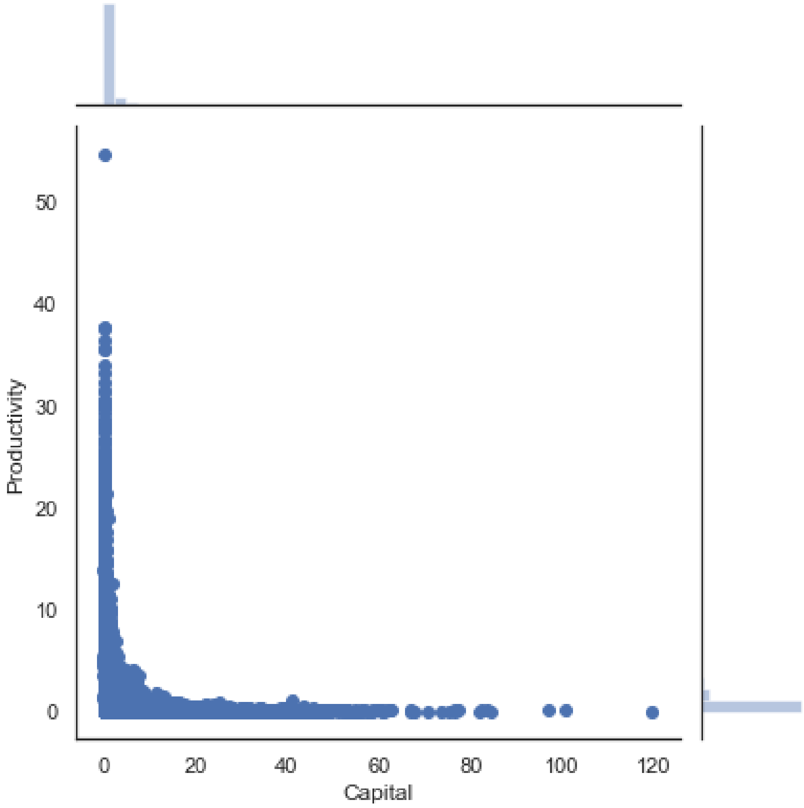
Correlation = 0

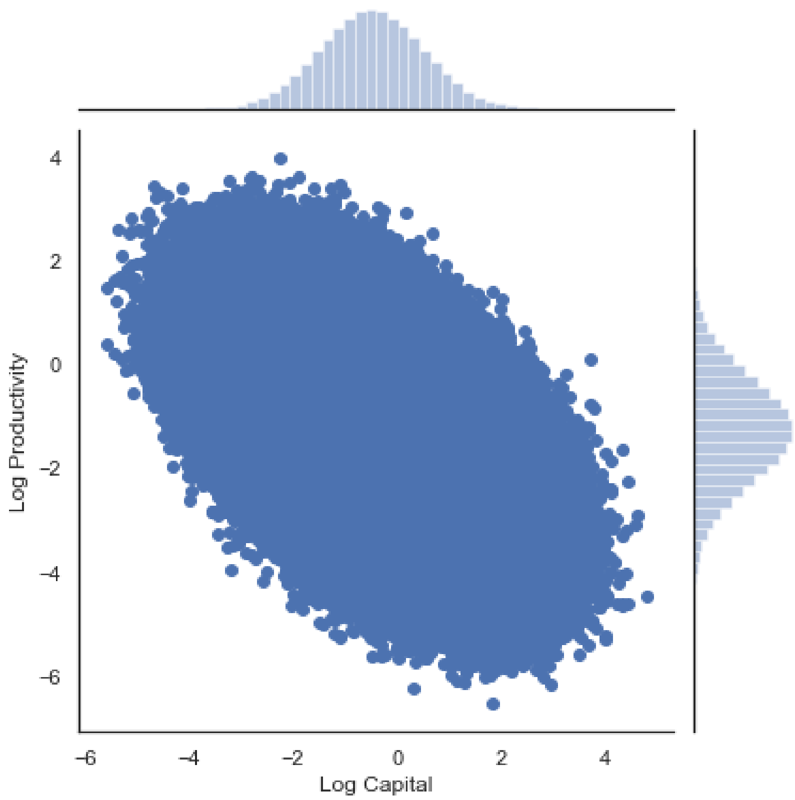


Correlation = 0.5



Correlation = - 0.5





Computing firm output:

The firm output is given by:

$$y_i = {s_i}^{1-\gamma} k_i^{\gamma} = z_i\, k_i^{\gamma}$$

Computing this we obtain the output of a particular firm i, for each of our observations.

Solving the maximization problem:

The maximization problem is the following:

$$Y^e = \max_{k_i} \sum_i s_i^{1-\gamma} k_i^{\gamma}$$

s. t

$$K = \sum_i k_i,$$

Where K is the aggregate capital.

Solving this problem we obtain the optimal capital for each firm, which gives the efficient aggregate output.

In order to find optimal capital we use the FOC together with the aggregate resource constraint, and we obtain that the effiecient level of capital is given by:

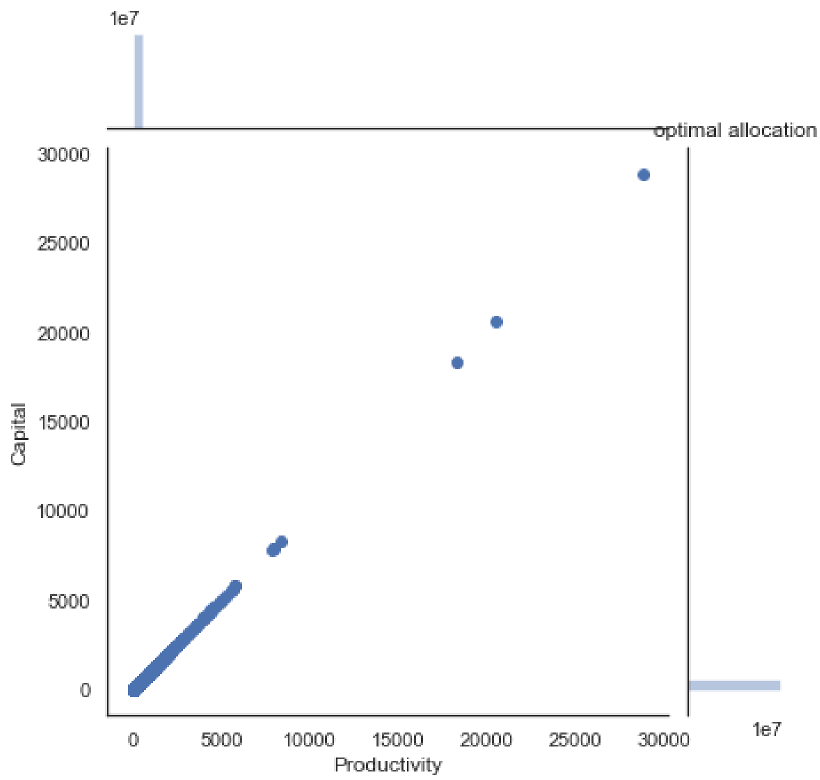
$$k_i^e = \frac{s_i}{S} K \text{ , where } S = \sum_i s_i$$

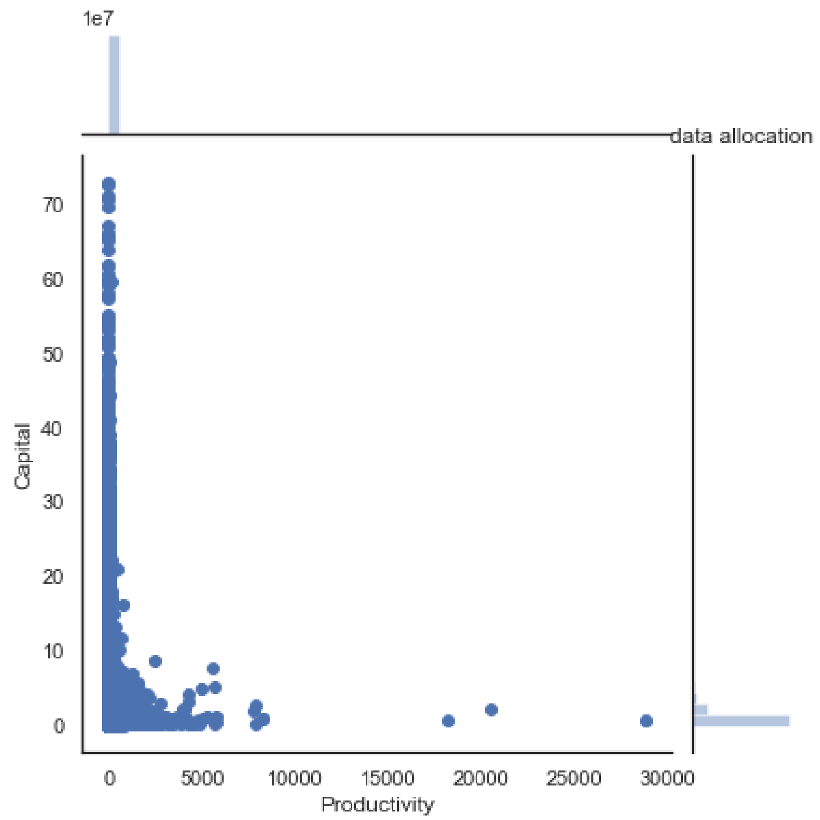
Comparing optimal allocations k^e against the data:

In the following graphs we plot productivity levels against optimal allocations and against data allocations.

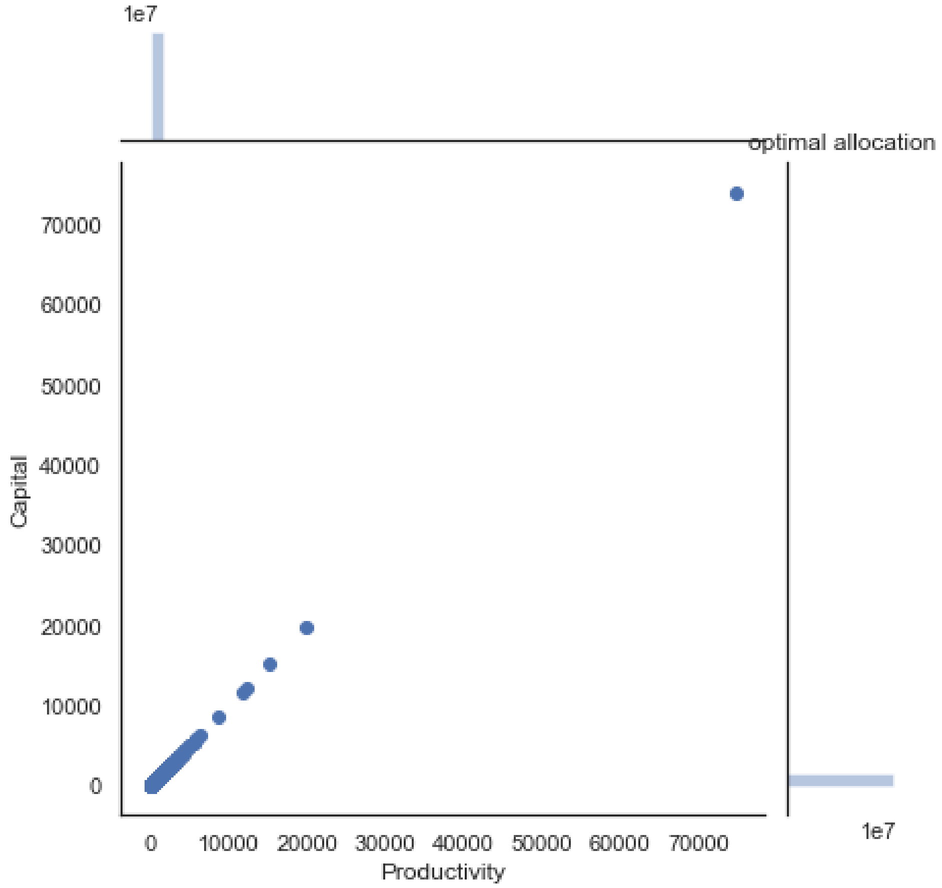
As we can see in the graphs, the optimal allocation is along a straight line of more or less 45 degrees, whereas the inefficient allocation accumulates at the lowest values and shows greater variance than the optimal one.

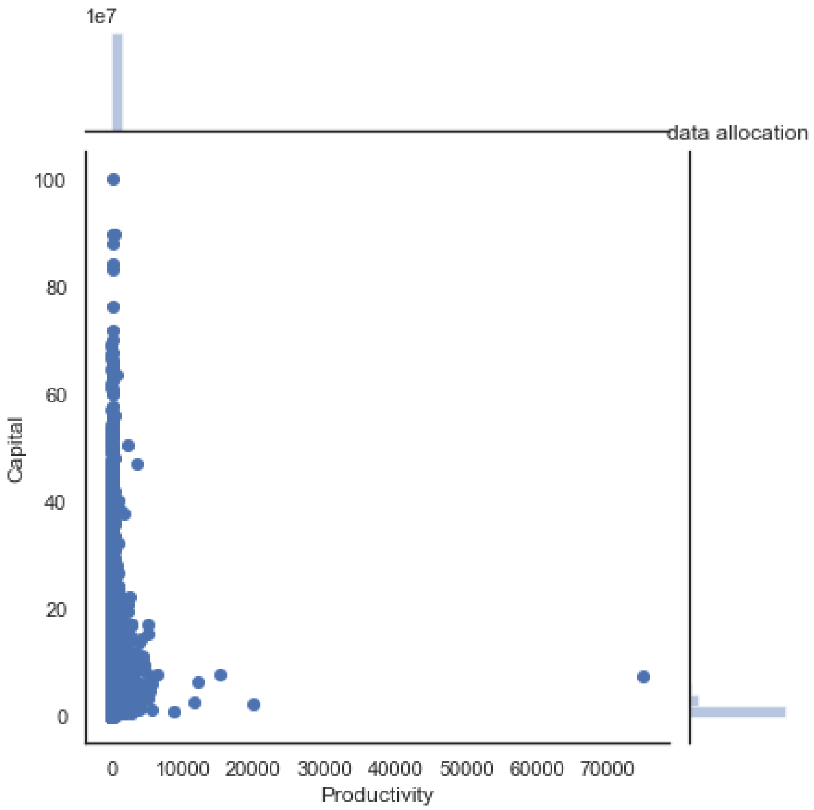
Correlation = 0



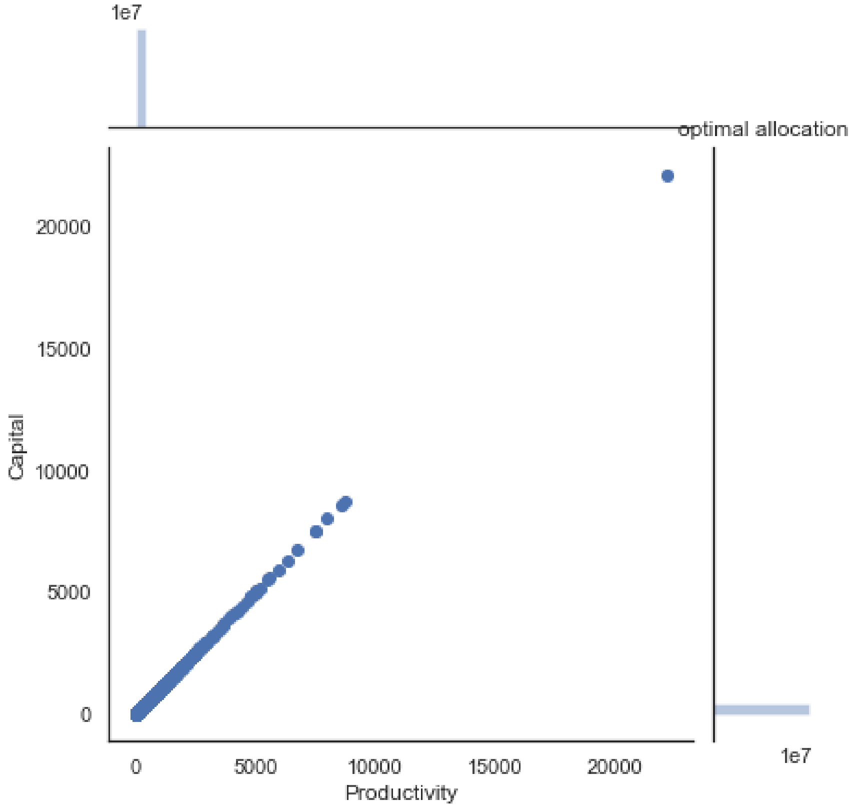


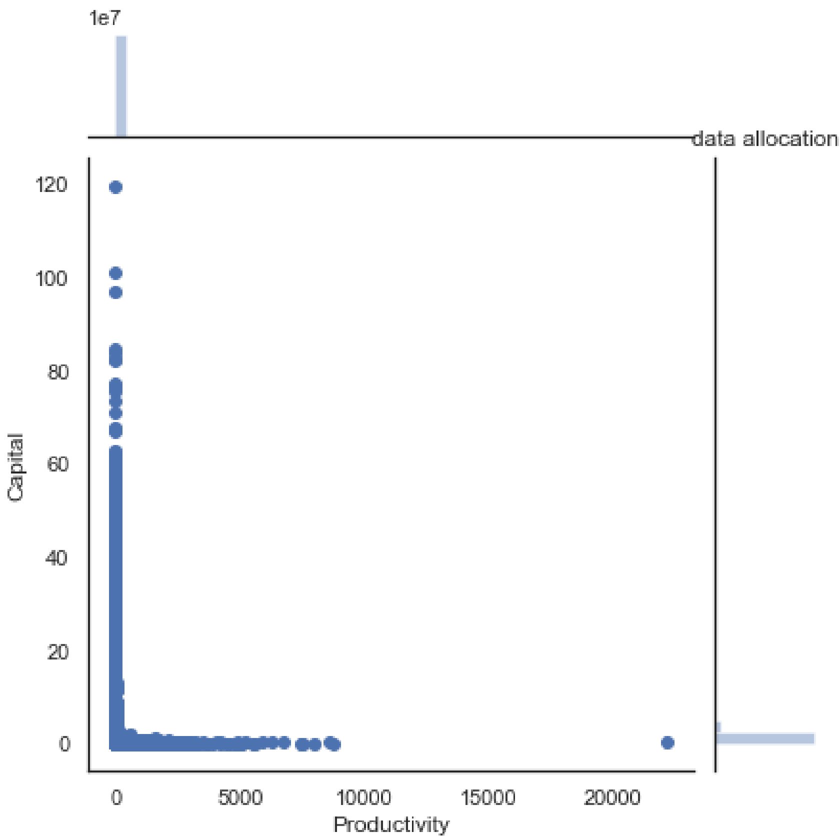
Correlation = 0.5





Correlation = - 0.5





Output gains from reallocation:

We compute the output gains from reallocation as follows:

$$\left(\frac{Y^e}{Y^a} - 1\right) * 100$$

where Y^a is the aggregate output in data.

Correlation Coefficient	0	0.5	-0.5
Output gains	138,629%	76,71%	223,079%

When capital and productivity are uncorrelated, if we move the economy to the efficient allocation, the aggregate output will increase by 138.384 %. If capital and productivity have a correlation coefficient of 0.5, the output gain will be the lowest one, whereas when they are negatively correlated the output gain is the highest one.

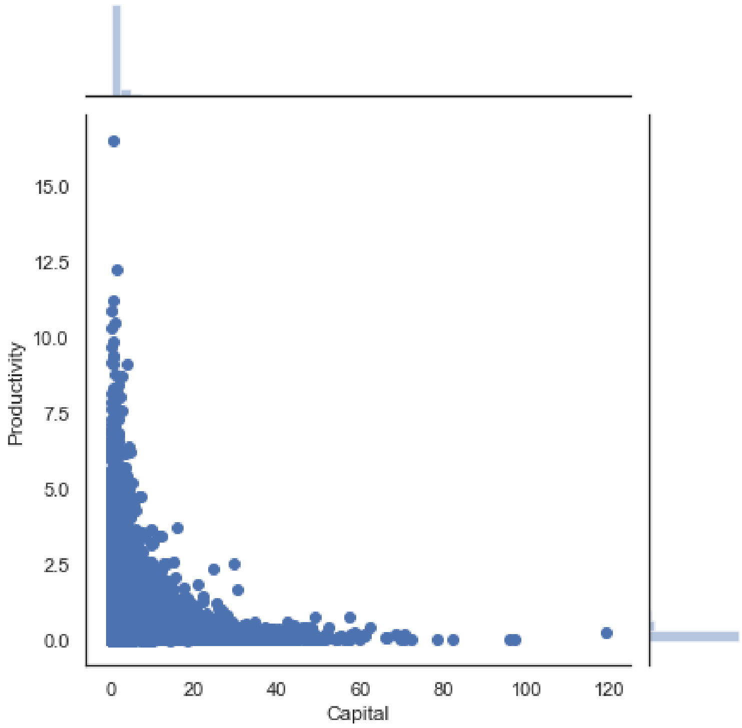
Question 2: Higher Span of Control ($\gamma = 0.8$)

Simulating the data:

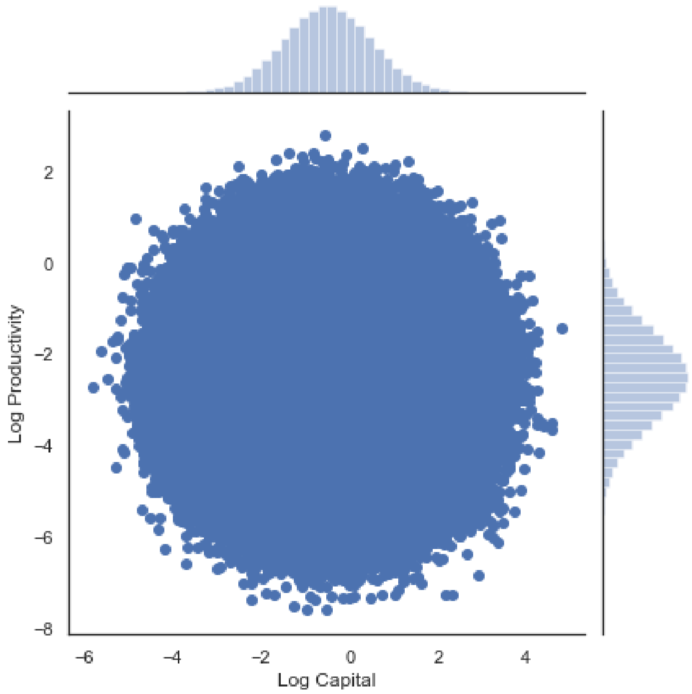
Now we are going to do the same as in exercise 1, but changing the value of gamma to 0.8. Therefore now μ_z will decrease to -2.5. Therefore our simulated data will change.

The new distribution will be now the following (in the case of no correlation):

$$\begin{pmatrix} \ln(k) \\ \ln(z) \end{pmatrix} \sim N\left[\begin{pmatrix} -0.5 \\ -2.5 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]$$



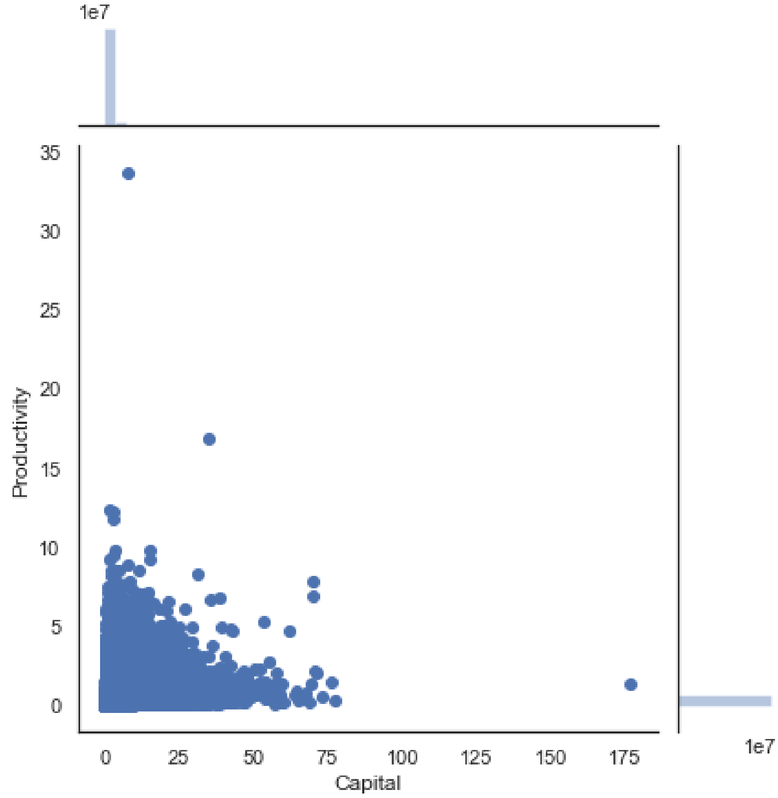
15.0
12.5
10.0
7.5
5.0
2.5
0.0
Productivity
0
20
40
60
80
100
120
Capital

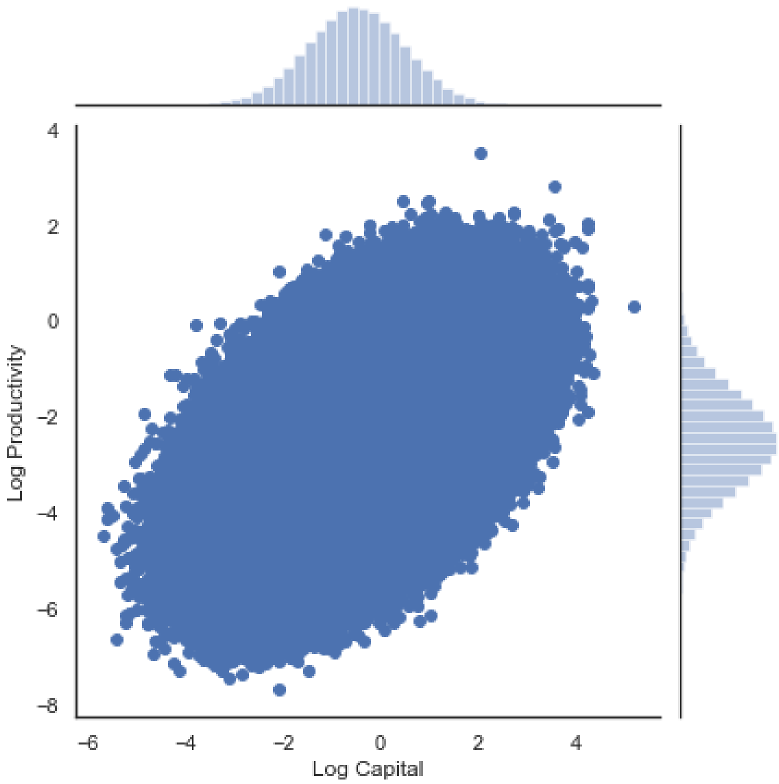


2
0
−2
−4
−6
−8
Log Productivity
−6
−4
−2
0
2
4
Log Capital

For correlation = 0.5:

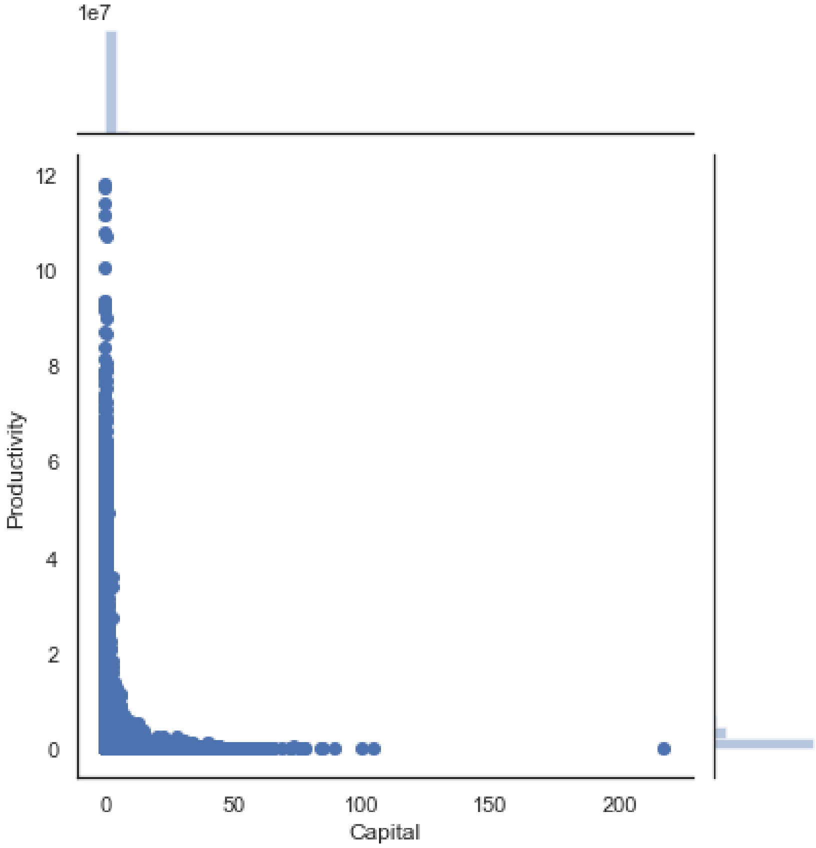
$$\begin{pmatrix} \ln(k) \\ \ln(z) \end{pmatrix} \sim N\left[\begin{pmatrix} -0.5 \\ -2.5 \end{pmatrix}, \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}\right]$$

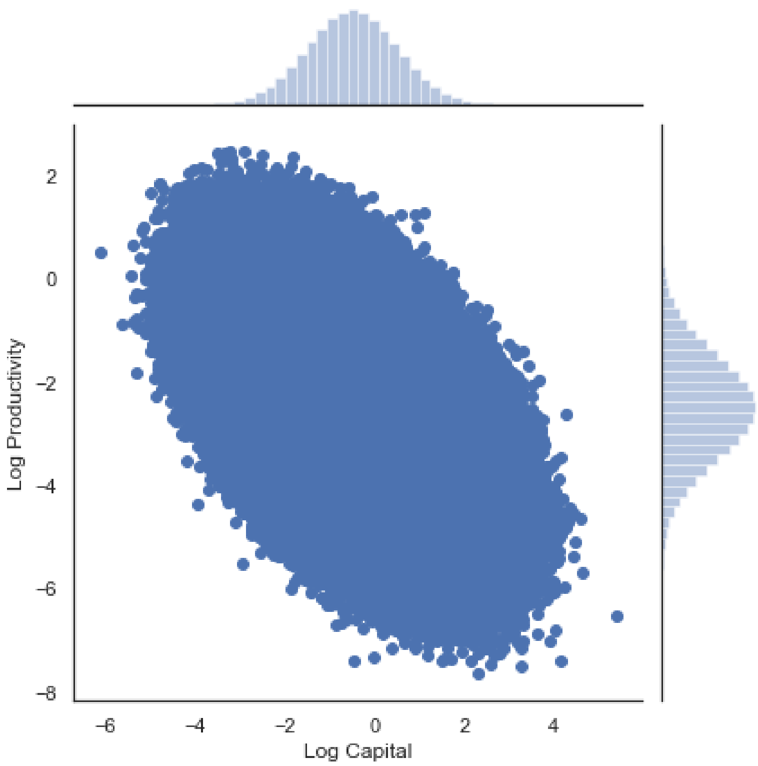




For correlation = - 0.5:

$$\begin{pmatrix} \ln(k) \\ \ln(z) \end{pmatrix} \sim N\left[\begin{pmatrix} -0.5 \\ -2.5 \end{pmatrix}, \begin{pmatrix} 1 & -0.5 \\ -0.5 & 1 \end{pmatrix}\right]$$

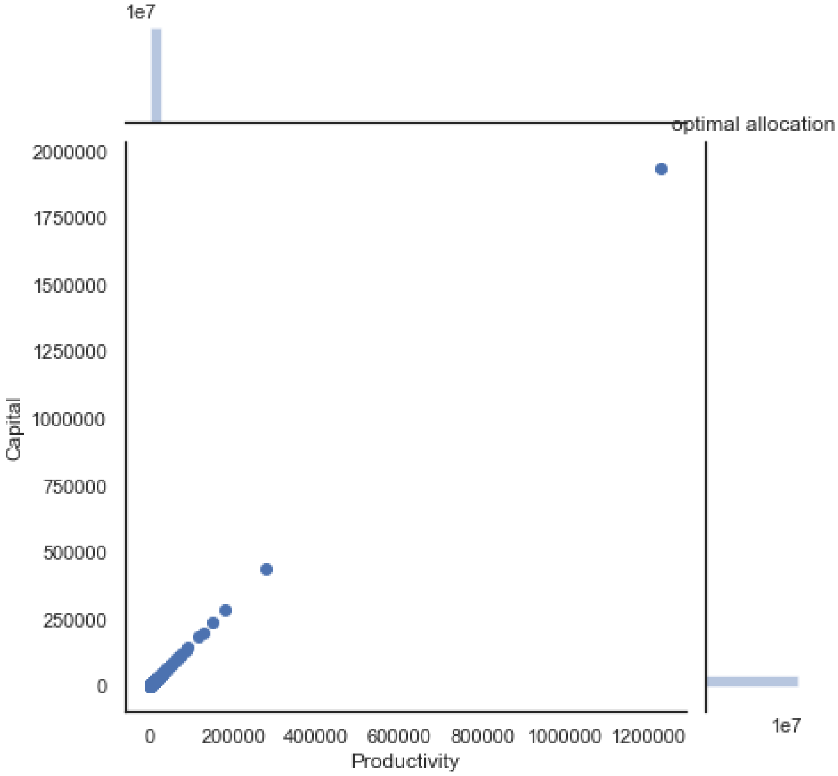


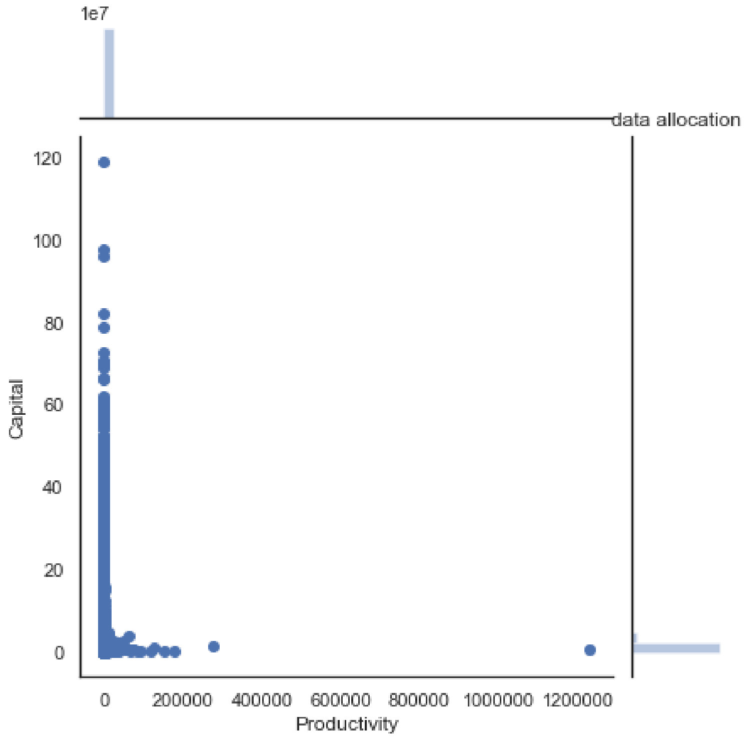


Comparing optimal allocations k^e against the data:

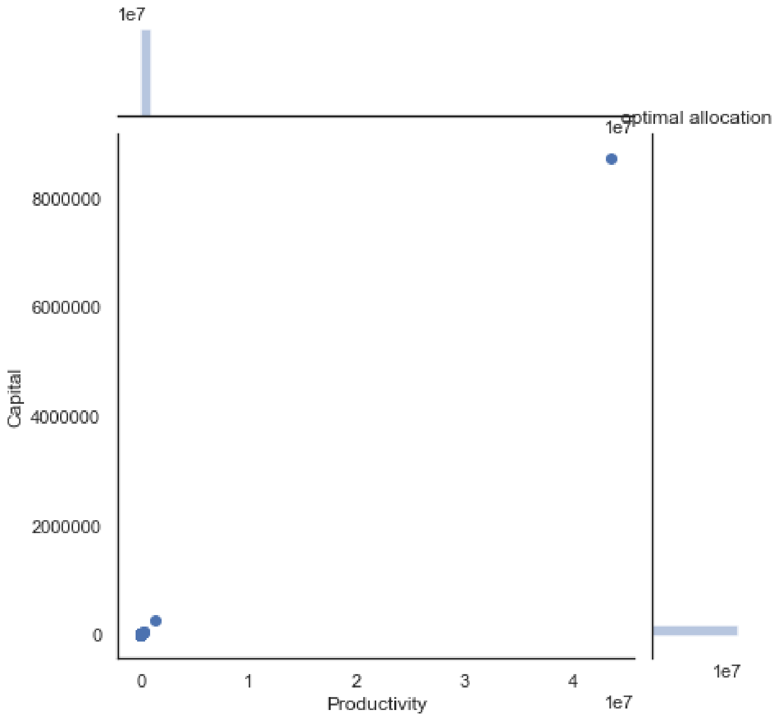
In the following graphs we plot productivity levels against optimal allocations and against data allocations.

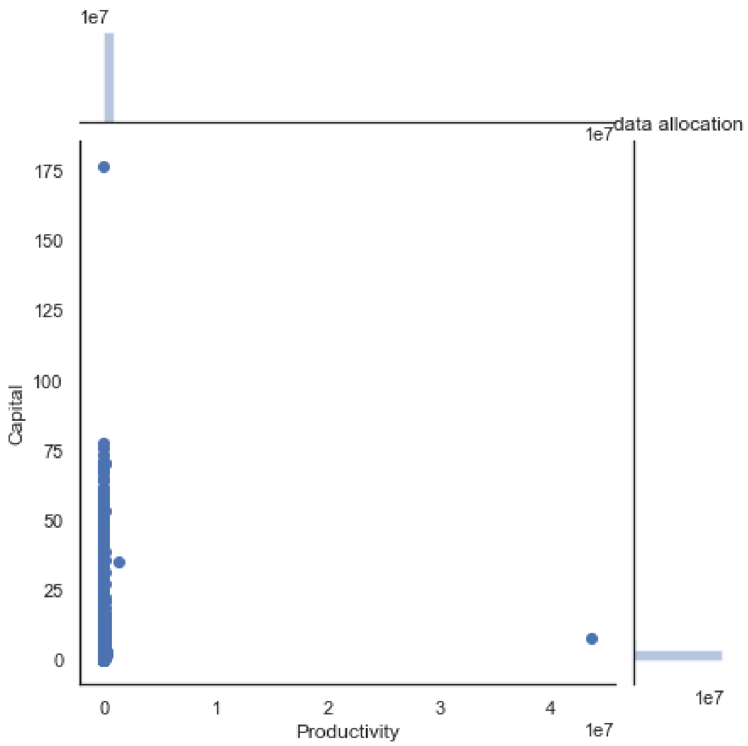
Correlation = 0



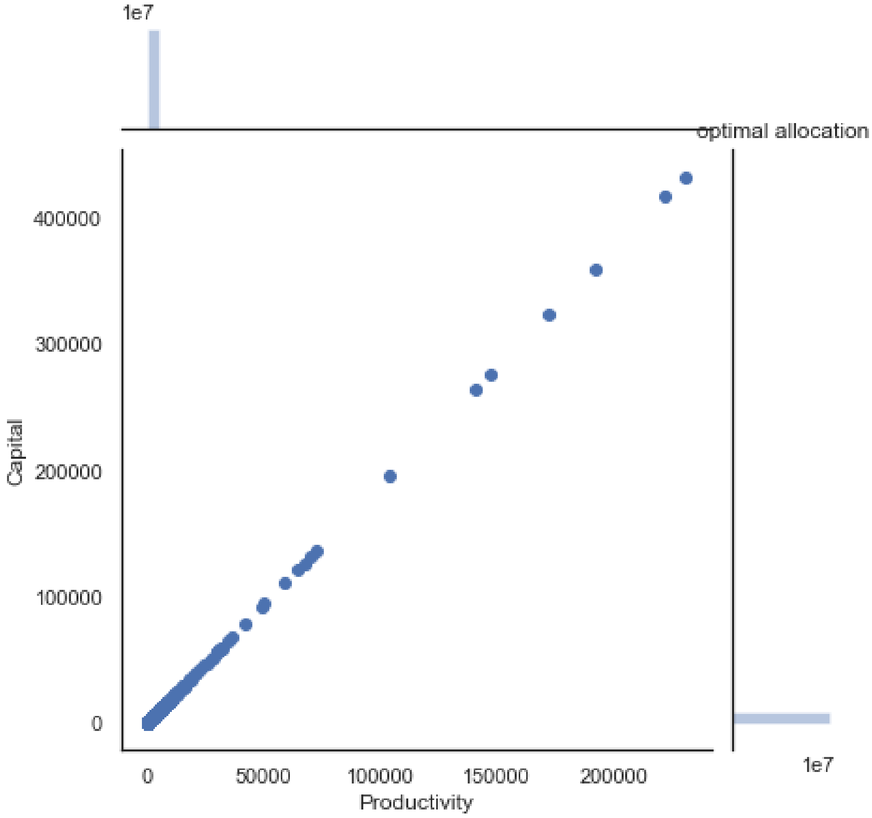


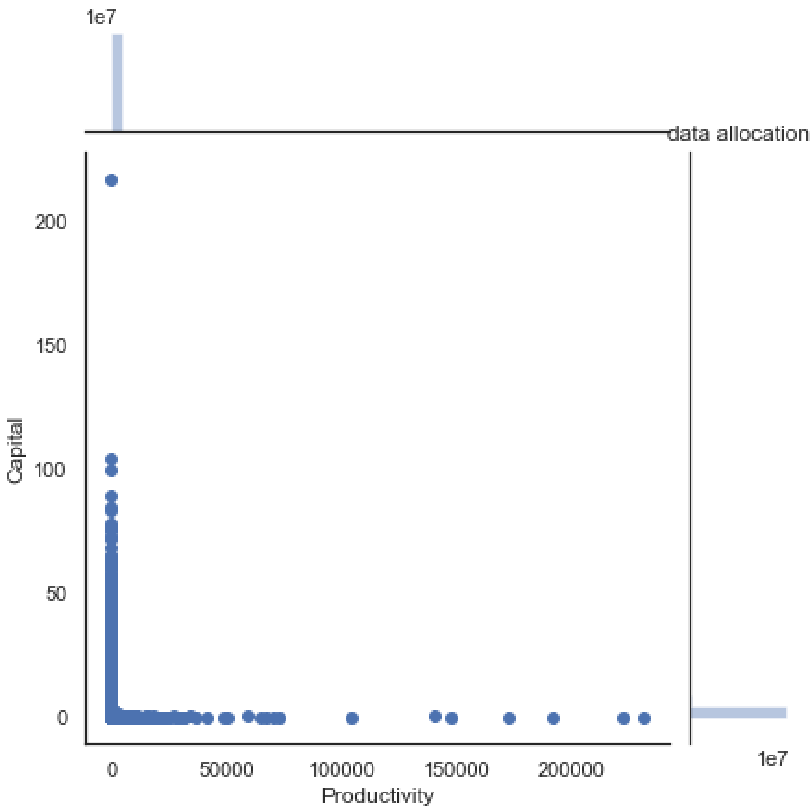
Correlation = 0.5





Correlation = - 0.5





Output gains from reallocation:

Correlation Coefficient	0	0.5	-0.5
Output gains	630,743%	640,167%	953,944%

When we increase the span of control output gains from reallocation increase significatively.

Question 3: From Complete Distributions to Random Samples

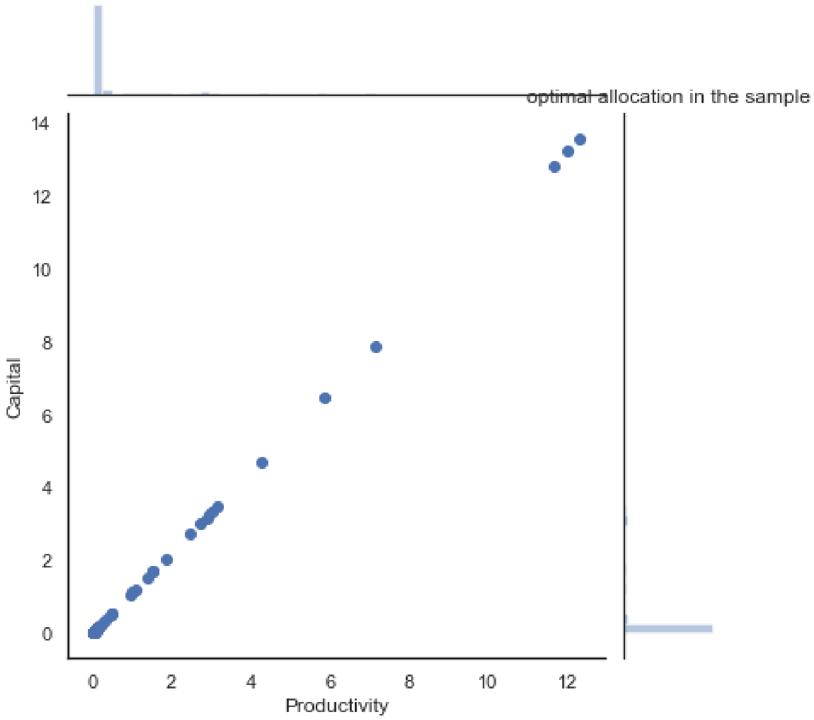
Now instead of usin the complete data we are going to use random samples (without replacement) 100000, 10000, 1000 and 100 observations. For this we are going to take $\gamma = 0.6$ and lnz and lnk are uncorrelated.

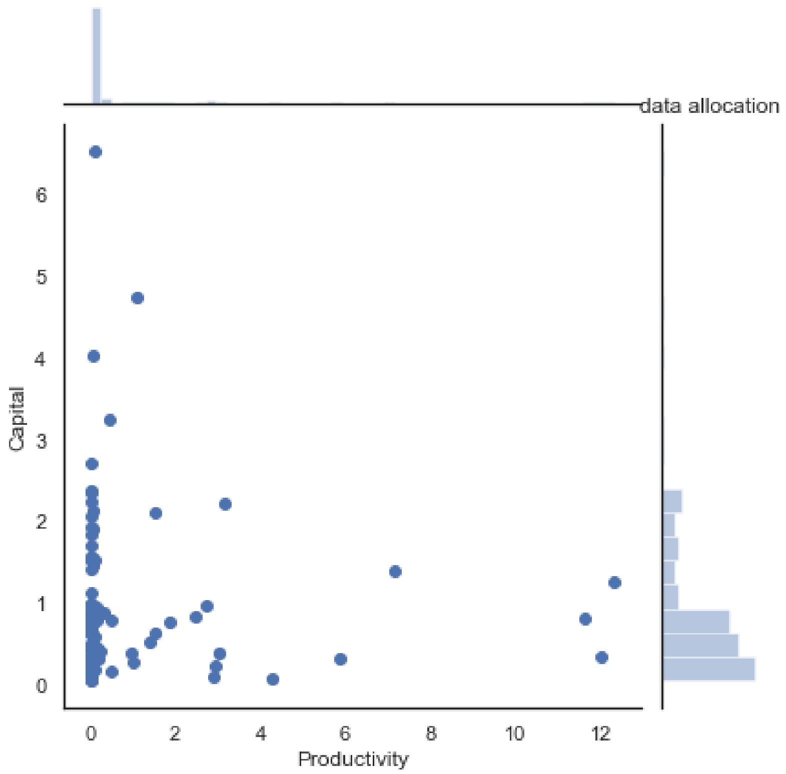
Descriptive statistics for different simple sizes:

Sample size	Var(lnz)	Var(lnk)	Correlation coef.
100	1.17	0.942	0.04
1000	0.932	1.021	-0.04
10000	1.009	0.971	-0.01
100000	1.003	1.001	0

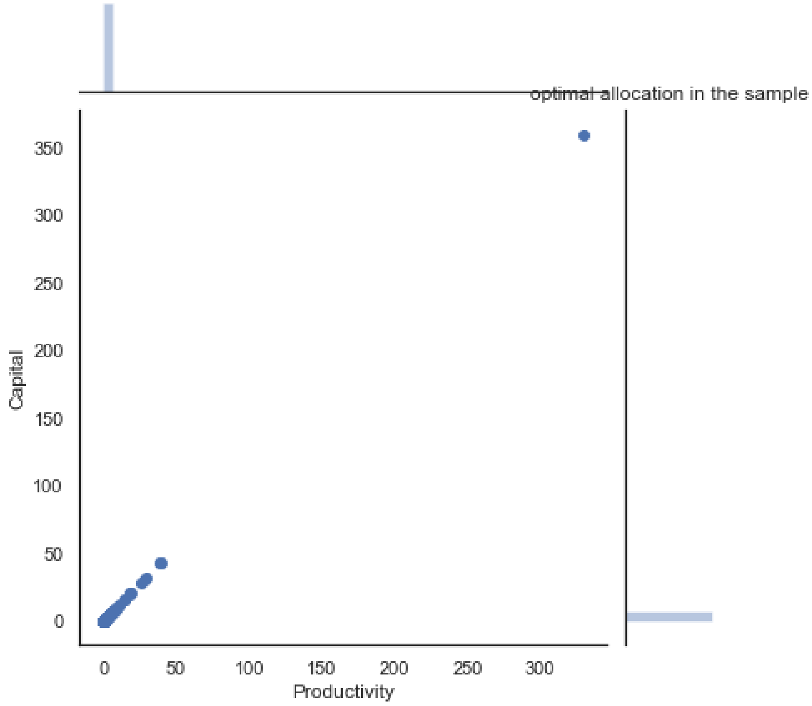
Comparing optimal allocations k^e against the data:

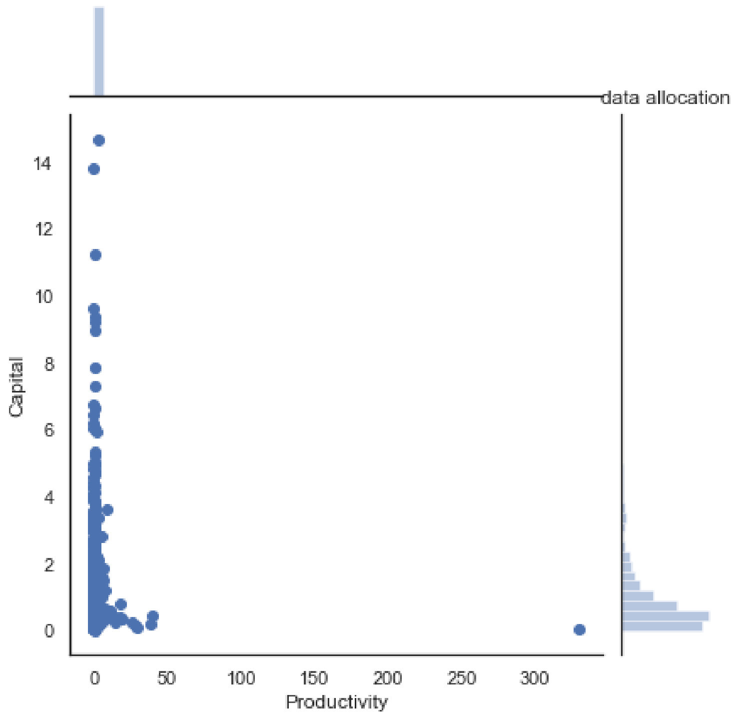
Sample size = 100



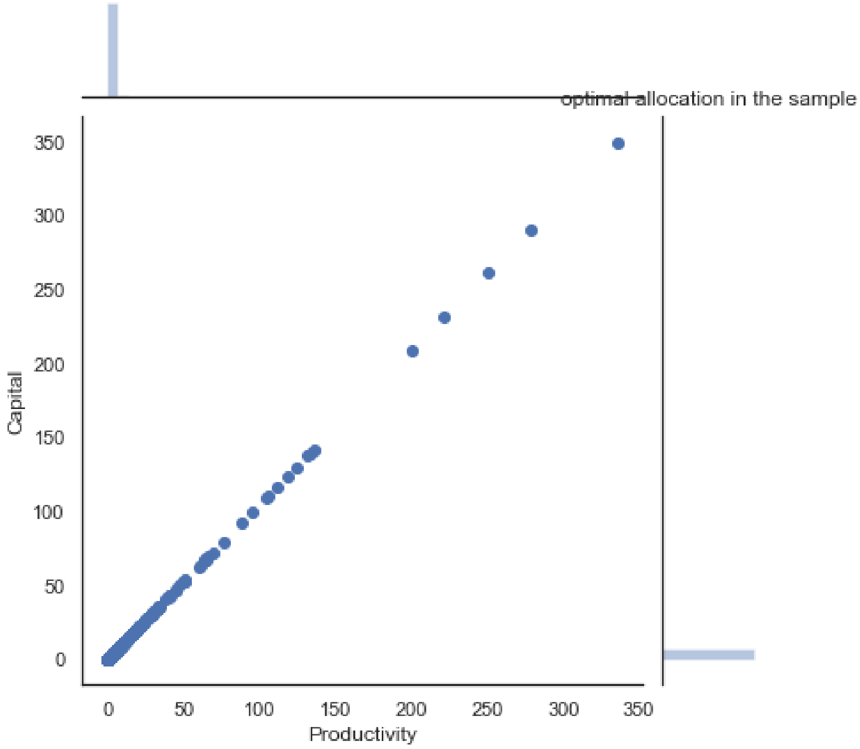


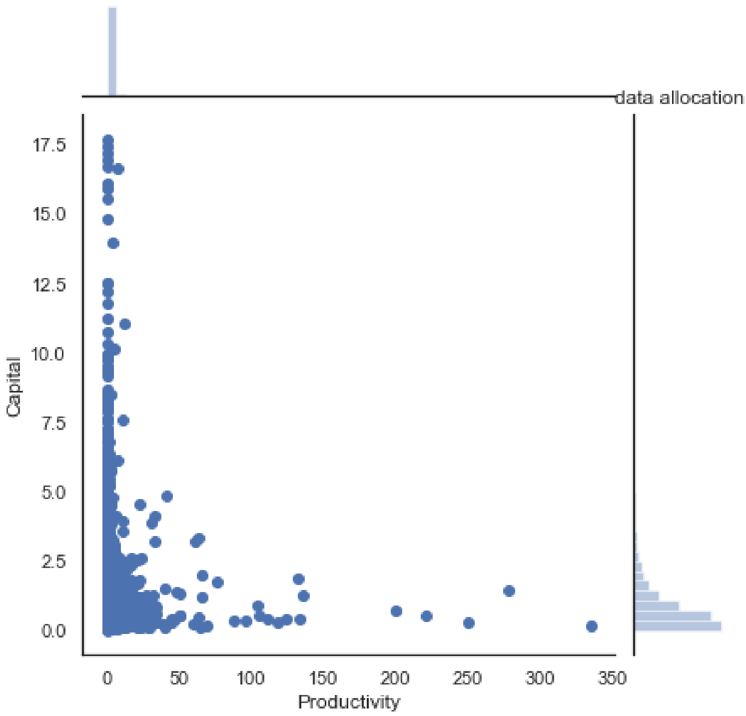
Sample size = 1000



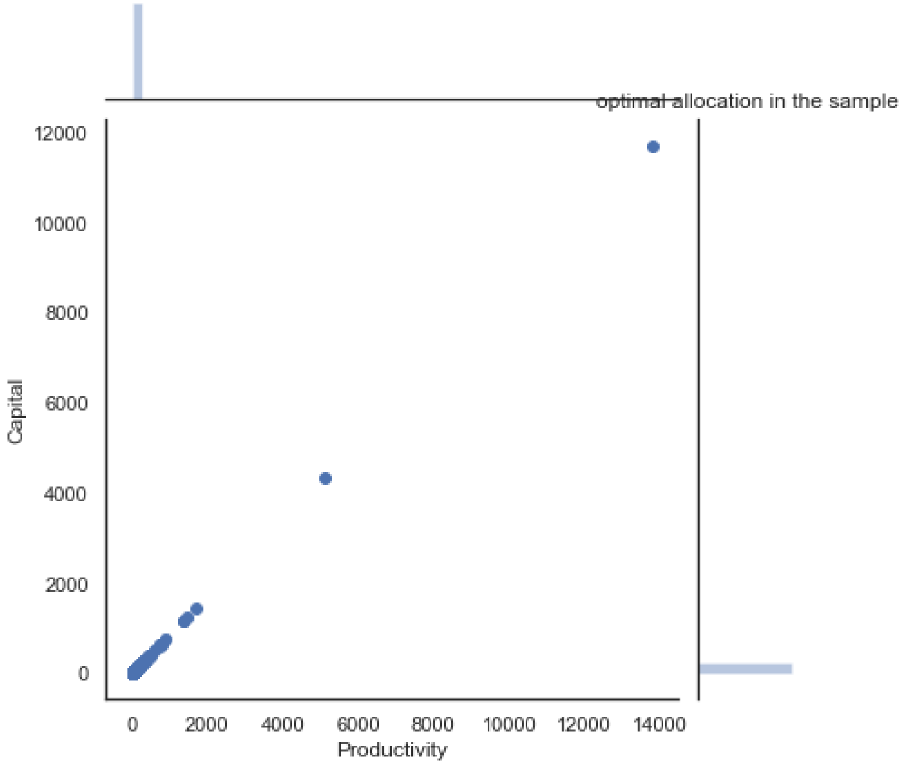


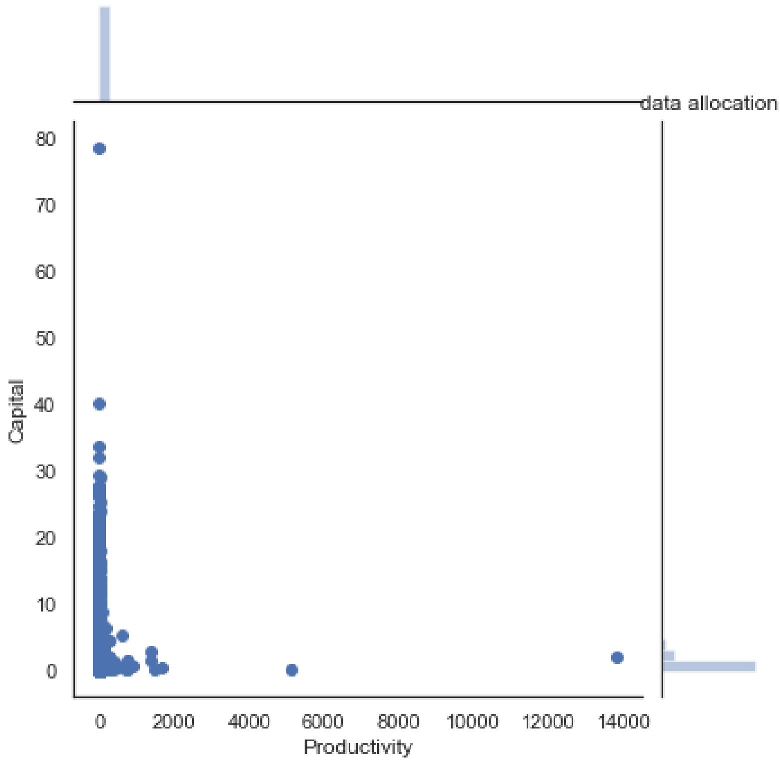
Sample size = 10000





Sample size = 100000



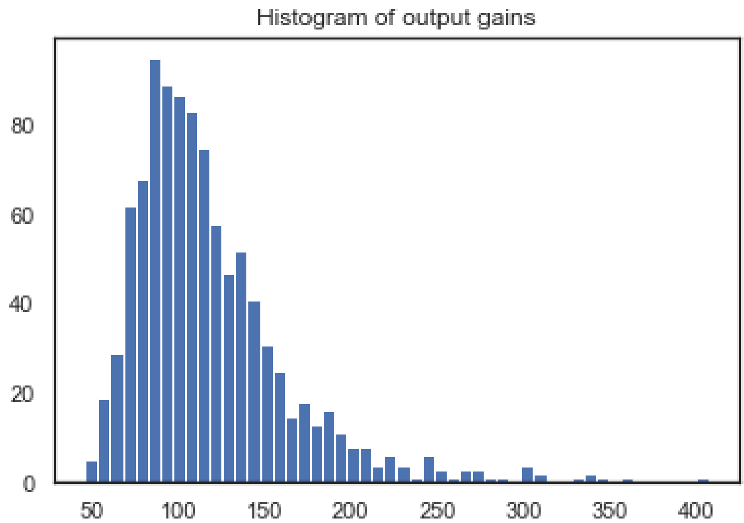


Output gains from reallocation:

Sample size	Output gains
100	101,526%
1000	153,74%
10000	133,384%
100000	155,299%

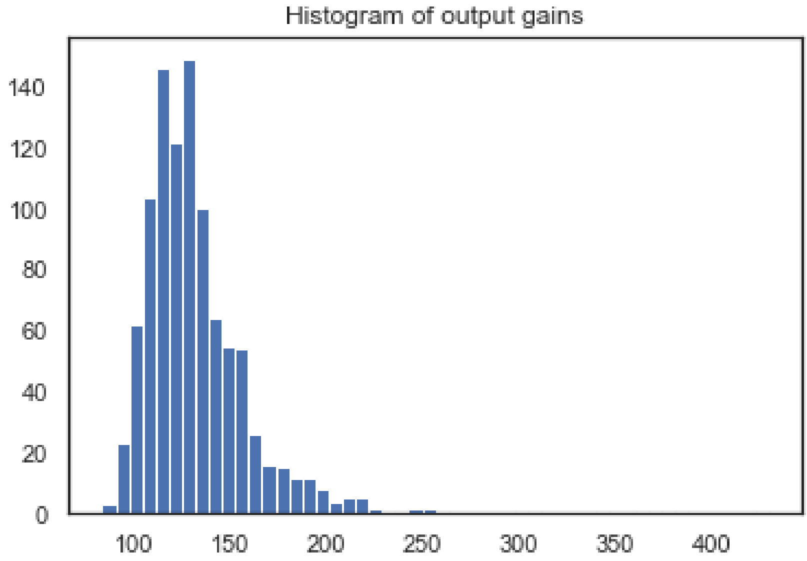
Taking samples 1000 times:

Sample size = 100



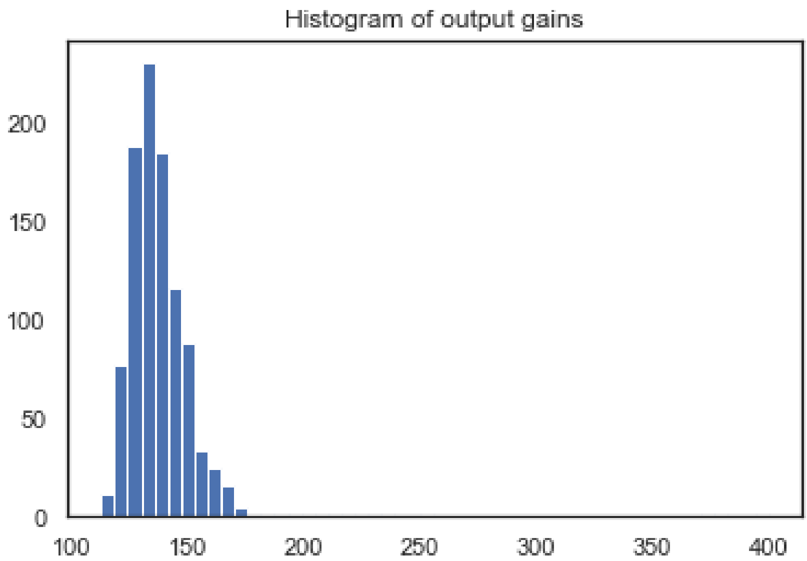
The median value of the distribution of output gains for a sample of size 100 is 108,049.

Sample size = 1000



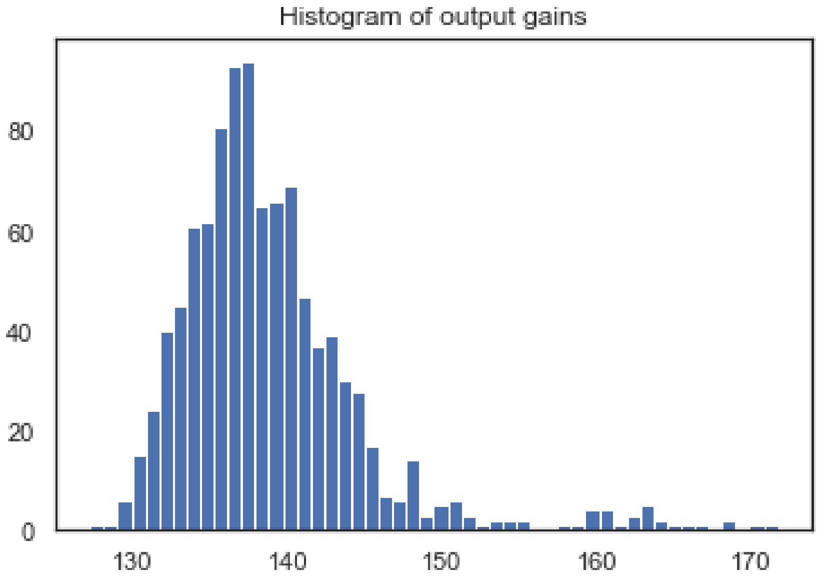
The median value of the distribution of output gains for a sample of size 1000 is 126,756.

Sample size = 10000



The median value of the distribution of output gains for a sample of size 10000 is 135,014.

Sample size = 100000



The median value of the distribution of output gains for a sample of size 100000 is 137,711.

Probability that a simple gain is is within 10% of actual misallocation:

Sample size	Probability
100	17,3%
1000	37,5%
10000	77,5%
100000	96,6%

As we can see, as we increase the size of the random sample, the probability that a sample gain is within an interval of 10% with respext to the gains obtained from complete data increases significatively.