Estimation of Profit Functions When Profit Is Not Maximum

Subal C. Kumbhakar

This paper deals with derivation and implications of profit functions when profit is not maximum due to the presence of either technical inefficiency or allocative inefficiency, or both. We show that input demand and output supply, elasticities, and returns to scale are, in general, affected by these inefficiencies. We also show that the overall profit efficiency is not necessarily the product of technical and allocative efficiencies, meaning that technical and allocative inefficiencies are not necessarily independent. Estimation techniques are developed for both cross-sectional and panel data models. Working of the model is illustrated using a panel of 60 salmon farms.

Key words: allocative efficiency, frontier, inputs and output elasticities, non-maximum profit function, profit efficiency, returns to scale, technical efficiency.

The textbook analysis of producer behavior is almost entirely based on the assumption that firms make no mistakes either in allocating their inputs or producing outputs. That is, the standard microeconomic theory of producer behavior is based on the neoclassical approach which assumes firms to be efficient both technically and allocatively. There is, however, an extensive literature on the measurement of productive efficiency that goes back to Debreu and Farrell. These studies extended the basic neoclassical approach by allowing firms to make mistakes in the production of outputs and use of inputs. Since then, numerous papers have been published on this subject, some of which are listed in the surveys by Lovell, Greene, and in the book by Kumbhakar and Lovell.

At least three distinct methods are now used to measure efficiency. The data envelopment analysis developed by Charnes, Cooper, and Rhodes uses a linear programming technique to estimate efficiency for each production unit nonparametrically. The shadow price approach, first used by Hopper and then extended by Lau and Yotopoulos, Yotopoulos and Lau, and many others, focuses mainly on allocative inefficiency, which is estimated parametrically. The stochastic frontier approach (SFA), developed by Aigner, Lovell, and Schmidt, and Meeusen and van den Broeck, and many recent extensions of it use econometric techniques to estimate efficiency for each production unit based on some specific distributional assumptions on technical and allocative inefficiency (at least in cross-sectional models) and statistical noise components. Some of the distributional assumptions used in the SFA can be avoided if panel data are available (Schmidt and Sickles, Cornwell and Schmidt, Atkinson and Cornwell).

Although techniques are well developed in the SFA to estimate both technical and allocative efficiencies using cross-sectional and panel data in both primal and dual settings, attention in empirical studies is primarily focused on estimation of technical inefficiency only using parameter estimates from production functions. There are, however, some well-known limitations in the estimation of production functions using a single equation method (e.g., inputs are treated as exogenous and often assumed to be independent of technical inefficiency). These problems can be avoided in a profit maximizing approach where both output and inputs are choice (endogenous) variables (Kumbhakar 1987). This approach is more general than the minimum cost function approach which assumes output to be exogenous. It is also more suitable for many industries since both inputs and output are choice variables.

This paper focuses on estimation of the dual production technology when inputs and

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output are endogenous. We use a profit maximizing framework in which a producer is unable to attain the profit frontier, defined as the maximum profit function, due to the presence of either technical or allocative inefficiency or both.[1] Our main focus is on the economic implications of using profit functions when producers are inefficient. In this case actual profit is less than the maximum profit and hence we use the name non-maximum profit. We show that input demands and output supplies, their elasticities, and returns to scale derived from the non-maximum profit functions are not *always* the same as those derived from the neoclassical profit function (profit frontier). This result follows from the fact that the profit frontier[2] is not *always* a neutral transformation[3] of the non-maximum profit function (i.e., the profit function augmented to incorporate inefficiency). Consequently, the presence of inefficiency affects not only input demands and output supply functions, but also elasticities and returns to scale.[4] This finding gives a compelling reason for accommodating inefficiency in estimating the production technology, not for efficiency measurement per se.[5] We also show that the overall profit efficiency (the ratio of actual to maximum profit) cannot be expressed, in general, as the product of profit technical efficiency (the ratio of actual profit with *only* technical inefficiency to maximum profit) and profit allocative efficiency (the ratio of actual profit with *only* allocative inefficiency to maximum profit).[6] There are some special cases where the decomposition result holds.

We consider estimation of non-maximum profit functions when either cross-sectional or panel data are available. While Kumbhakar (1987) used the first-order conditions and the production function in estimation (which is possible for some simpler functional form of the production technology), here we use the dual profit system. This approach is more general and, in principle, can handle any functional form for the dual profit function. For cross-sectional models we derive maximum likelihood (ML) estimators based on the standard distributional assumptions on technical inefficiency and the noise components. An alternative to the ML method is suggested for the panel data models, viz., a non-linear iterative seemingly unrelated regression (NLITSUR) method. As an illustration of the methodology, we use the NLITSUR method to estimate some panel data models based on a sample of 60 Norwegian salmon farms observed during 1986–1992.

The basic idea of the paper is related to Kumbhakar (1987) in which technical and allocative inefficiencies were first introduced in a profit maximizing framework. However, the present paper is more general and provides new results for production functions that are not homogeneous. For example, if the underlying production technology is homogeneous, the technical inefficiency term appears additively in the log profit function. This result (worked out in Kumbhakar 1987 for the Cobb–Douglas case) perhaps led others to believe that the technical inefficiency term is additive for translog profit functions as well (e.g., Ali and Flinn). We show that this result is generally not true. Similarly, we find that some important features of the technology, such as measures of returns to scale, input and output elasticities, etc., can be affected by the presence of technical inefficiency.

In the next section we introduce the non-maximum profit functions and derive the implications, first, with only technical inefficiency, and then with both technical and allocative inefficiencies. We discuss issues related to both homogeneous and non-homogeneous production technologies. This is followed by a discussion of the estimation issues (estimation of both parameters and inefficiency), with cross-sectional and panel data models. The results from the Norwegian salmon farms follow and the main findings are summarized in the final section.

[1] We do not model scale inefficiency (e.g., Førsund, Lovell, and Schmidt, Kumbhakar, Biswas, and Bailey) separately here because it is subsumed in our definition of allocative inefficiency. For a discussion on relative allocative inefficiency and scale inefficiency the reader is referred to Kumbhakar, Biswas, and Bailey.

[2] Throughout this paper the word "frontier" is used to refer to a production or profit relationship without any inefficiency.

[3] By neutral transformation we mean a shift of the profit function independent of input and output prices.

[4] If the underlying production function is homogeneous and there is only technical inefficiency, then elasticities of input demands and output supplies, and returns to scale, are independent of technical inefficiency. This result is, however, not true for non-homogeneous production functions.

[5] Schmidt in the conclusion of his survey noted that the "... only compelling reason to estimate production frontier is to measure efficiency" (p. 320). Our specification shows that features of the technology vary with distance from the frontier.

[6] This decomposition result, however, always holds in a cost minimizing framework (i.e., overall cost efficiency is the product of cost technical efficiency and cost allocative inefficiency as shown in Farrell).

Profit Functions

In this section we first derive the non-maximum profit function with only technical inefficiency, and discuss some implications under homogeneous and non-homogeneous production functions. Then we derive the non-maximum profit function with both technical and allocative inefficiencies, and discuss some of the implications including decomposition of overall profit efficiency into profit technical and profit allocative efficiencies. Finally, we consider non-maximum profit functions with quasi-fixed inputs.

Only Technical Inefficiency

First we consider the case where a producer's profit is not maximum due to the presence of technical inefficiency. There are two ways to specify technical inefficiency in a production function, viz., input- and output-oriented approaches (Färe and Lovell, Atkinson and Cornwell). With output-oriented technical inefficiency, the production relationship is specified as

$$(1) \qquad y = f(x)e^{u} \qquad u \leq 0,$$

where a negative value on u shows that actual output (y) is less than the maximum possible output $f(x)$ (the production frontier), given a vector of inputs x. Alternatively, actual output of firms with $u \leq 0$ can be increased by $-100u\%(\geq 0)$, holding x fixed, when inefficiency is eliminated. Thus, u is called output-oriented technical inefficiency and $e^{u} = y/f(x) \leq 1$ is a measure of output technical efficiency.[7]

One can estimate technical efficiency from (1) by estimating a parametric form of $f(x)$. There are mainly two problems in estimating the production function directly. These problems are related to the assumptions that: (i) the inputs (x) are not decision variables (exogenously given) and (ii) inputs are independent of technical inefficiency. Neither assumption may be appropriate. First, the producers may choose input quantities to maximize profit. Second, input use may be affected by the presence of technical efficiency (Mundlak), thereby violating the independent assumption. Consequently, the estimators (parameters and inefficiency) based on a single equation production function will be inconsistent. We correct both these problems by using a profit maximization approach. In this vein, first we derive the non-maximum profit function in the presence of technical inefficiency.

Result 1. The profit function in the presence of technical inefficiency corresponding to the production function in (1) can be written as

$$(2a) \qquad \pi(w, p, u) = \pi(w, pe^{u}),$$

where the input price vector is $w = (w_1, \ldots, w_J)$ and p is the output price (Lau, Theorem II-3, p. 154). The rationale behind the above result is as follows. The first-order conditions of profit maximization are $f_j(x) = w_j/pe^{u}$, $j = 1, \ldots, J$ where $f_j(x) = \partial f(x)/\partial x_j$, and the production function in (1) can be rewritten as $ye^{-u} = f(x)$. Thus, if one substitutes p by pe^{u} and y by ye^{-u}, the above first-order conditions and the production function look as if we are back to the standard neo-classical world. Consequently, the solutions of input demand and output supply functions (adjusted for inefficiency) can be expressed as $x_j = \psi_j(w, pe^{u})$, $j = 1, \ldots, J$ and $ye^{-u} = \varphi(w, pe^{u})$. Therefore, the profit function, conditional on u, is defined as

$$\pi(w, pe^{u}) = \max_{ye^{-u},\, x} \{py - w'x \mid y = f(x)e^{u}\}.$$

Implications of Result 1.

1. *Relationship among various profit functions:* Define actual (observed) profit π^{a} as

$$(2b) \qquad \begin{aligned} \pi^{a} &= py(w, pe^{u}) - w \cdot x(w, pe^{u}) \\ &= pe^{u} f(x(.)) - w \cdot x(.) \\ &= \pi(w, pe^{u}). \end{aligned}$$

This means that profit when price is p and output equals $f(x)e^{u}$ is the same as profit when output equals $f(x)$ but price equals pe^{u}. That is, a 10% reduction in output given inputs has the same effect on profit

[7] With input-oriented technical inefficiency the production relationship is specified as $y = f(xe^{\eta})$, $\eta \leq 0$. Thus the actual input vector x is worth xe^{η}, x and therefore, every input is used by $-\eta 100\% \geq 0$ more than what is necessary to produce a given level of output (y) without being inefficient. Consequently, η is labeled as input-oriented (or input-saving) technical inefficiency. This measure of technical inefficiency is also labeled as radial measure of technical inefficiency (Farrell). To avoid repetition we only consider the output-oriented measure of technical inefficiency. Results from the input-oriented inefficiency formulation are very similar to those derived here. Details can be obtained from the author upon request.

as a 10% reduction in output price holding output constant. It is, however, to be noted that $\pi^a \le \pi(w, p)$ when $\pi(w, p)$ is maximum profit (the standard neoclassical profit function which is also labeled as the profit frontier) defined as

$$\text{(2c)} \quad \pi(w, p) = \max_{y,\, x}\{py - w'x \mid y = f(x)\}.$$

We label π^a as non-maximum profit since $pe^u \le p \Rightarrow \pi^a = \pi(w, pe^u) \le \pi(w, p)$.

To show the relationship among $\pi(w, pe^u)$, π^a, and the profit frontier $\pi(w, p)$, we rewrite (2b) as

$$\text{(2d)} \quad \pi^a = \pi(w, pe^u) = \pi(w, p) \cdot h(p, w, u),$$

when $h(w, p, u)$ is profit technical efficiency defined as $h(w, p, u) = \pi(w, pe^u)/\pi(w, p)$. Because $pe^u \le p \Rightarrow \pi(w, pe^u) \le \pi(w, p)$, which in turn implies that $h(p, w, u) \le 1$. Note that equation (2d) is a restatement of (2b). It shows the relationship between actual profit π^a and the profit frontier $\pi(w, p)$. This relationship is important because we focus more on the efficiency part, i.e., the $h(w, p, u)$ function, especially in deriving implications of $h(w, p, u)$ not being independent of w and p. Profit efficiency $h(w, p, u) = 1$ only if $u = 0$.

2. *Derivation of input demands and output supply functions, elasticities, and returns to scale*: Applying Hotelling's lemma to the non-maximum profit function π^a in (2b) gives the following input demands and output supply functions[8]:

$$\text{(3)} \quad \frac{\partial \pi^a}{\partial w_j} = -x_j(w, pe^u), \quad j = 1, \ldots, J, \quad \text{and} \quad \frac{\partial \pi^a}{\partial pe^u} = ye^{-u}(w, pe^u).$$

Thus input demands and output supply functions, in general, are affected by technical inefficiency. That is, in general, $\partial\pi^a/\partial w_j \neq \partial\pi(w, p)/\partial w_j \forall j$ and $\partial\pi^a/\partial p \neq \partial\pi(w, p)/\partial p$ unless $h(w, p, u)$ is independent of w and p.

If the input demands and output supply functions are affected by the presence of inefficiency, elasticities (i.e., $\partial \ln x_j(w, pe^u)/\partial \ln w_k$, $j, k = 1, \ldots, J$, $\partial \ln x_j(w, pe^u)/\partial \ln p \forall j$, $\partial \ln y(w, pe^u)/\partial \ln p$, and $\partial \ln y(w, pe^u)/\partial \ln w_j \forall j$) are also affected unless $x_j(w, pe^u)$ and $y(w, pe^u)$ differ from $x_j(w, p)$ and $y(w, p)$ by some multiplicative constants independent of w and p (i.e., $h(w, p, u)$ depends only on u). This point is also discussed later.

We now examine returns to scale (RTS) which can be measured from the profit function using the following formula:

$$\text{(4)} \quad \text{RTS} = \frac{\partial \ln f(\lambda x)}{\partial \ln \lambda}\Big|_{\lambda=1} = \sum_j \frac{\partial f(x)}{\partial x_j}\frac{x_j}{f(x)} = \sum_j \frac{w_j x_j}{pf(x)} = -\frac{\sum_j \partial \ln \pi(w, p)/\partial \ln w_j}{\partial \ln \pi(w, p)/\partial \ln p}.$$

In the presence of technical inefficiency we define RTS using the non-maximum profit function, which is not necessarily the profit frontier, i.e.,

$$\text{(5)} \quad \text{RTS} = -\frac{\sum_j \partial \ln \pi(w, pe^u)/\partial \ln w_j}{\partial \ln \pi(w, pe^u)/\partial \ln pe^u} = \frac{\sum_j x_j(w, pe^u) f_j(w, pe^u)}{f(x)},$$

since $\pi^a = \pi(w, pe^u)$. Thus RTS evaluated at a point (w^0, p^0), for example, using (4) will differ from (5).

3. *Model misspecification:* If there is only technical inefficiency, then the profit function written as $\ln \pi^a = \ln \pi(w, pe^u) = \ln \pi(w, p) + \ln h(w, p, u) \equiv \ln \pi(w, p) + u^*$, when u^* is a constant multiple of u, is misspecified unless the underlying production function is homogeneous (see equation (6) below). This misspecification arises due to the erroneous assumption that $\ln h(w, p, u)$ is independent of w and p.
4. *Relationship between production efficiency and profit efficiency*: There is an exact relationship between technical inefficiency (u) and profit technical efficiency ($h(w, p, u)$), the form of which depends on the parametric form of profit functions. Thus, one can go back and forth from output technical efficiency to profit

[8] Note that since $\pi(w, pe^u)$ equals π^a, one can substitute $\pi(w, pe^u)$ by π^a from (2a)–(5).

technical efficiency without assuming a self-dual production function.[9]

The case of a homogeneous production function.

1. If the production function is homogeneous of degree $r(r < 1)$, then the corresponding profit function can be written as[10]

$$(6)\quad \ln \pi^a = \ln \pi(w, pe^u) = \ln \pi(w, p) + \ln h(w, p, u) = \frac{1}{1-r} \ln p + \ln G(w) + \frac{1}{1-r} u,$$

where $G(w)$ is homogeneous of degree $-\frac{r}{1-r}$ in w. Thus, $h(w, p, u) = h(u) = \frac{1}{1-r} u$, and the profit function in (6) can be expressed as $\ln \pi(w, pe^u) = \ln \pi(w, p) + u^*$ where $u^* = \frac{1}{1-r} u$ is profit technical inefficiency (which can be interpreted as the percentage loss in profit due to output technical inefficiency). Consequently, profit technical inefficiency is a constant multiple of u; i.e., the profit frontier is a neutral transformation of the profit function. Thus, technical inefficiency in the profit function appears in a multiplicative form (independent of w and p) if the underlying production function is homogeneous. Homogeneity of the underlying production function is a sufficient condition for this result.

2. If the production function is homogeneous, then the input demands and output supply elasticities $(\partial \ln x_j(w, pe^u)/\partial \ln w_k,\ j, k = 1, \ldots, J,\ \partial \ln\ x_j(w, pe^u)/\partial \ln p \forall j,\ \partial \ln y(w, pe^u)/\partial \ln p$, and $\partial \ln y \times (w, pe^u)/\partial \ln w_j \forall j)$ are not affected by the presence of technical inefficiency. This is because the profit frontier is a neutral transformation of the profit function (as noted in comment 1 above). Similarly, returns to scale, calculated from (6) (using the formula in (4)), are independent of technical inefficiency. These findings corroborate Schmidt's observation that none of the important features of the technology are affected by the presence of technical inefficiency. This is, however, not the case if the underlying production technology is non-homogeneous.

The case of a non-homogeneous production function. To illustrate the non-homogeneous production function case, we consider a translog form on $\pi(w, pe^u)$, viz.,

$$(7)\quad \ln \pi^a = \ln \pi(w, pe^u) = \beta_0 + \sum \beta_j \ln w_j + \beta_p \ln(pe^u) + \frac{1}{2}\Big[\beta_{pp} \ln(pe^u) \ln(pe^u) + \sum_j \sum_k \beta_{jk} \ln w_j \ln w_k\Big] + \sum_j \beta_{jp} \ln w_j \ln(pe^u),$$

which can be rewritten as

$$(8)\quad \ln \pi^a = \ln \pi(w, pe^u) = \ln \pi(w, p) + \ln h(w, p, u),$$

where $\ln \pi(w, p)$ is the translog profit frontier; i.e., $\ln \pi(w, p) = \ln \pi(w, pe^u)|_{u=0}$ in (7). Finally, profit inefficiency $\ln h(w, p, u)$ becomes

$$(9)\quad \ln h(w, p, u) = \ln \pi(w, pe^u) - \ln \pi(w, pe^u)|_{u=0} = \Big[\beta_p + \beta_{pp} \ln p + \sum_j \beta_{jp} \ln w_j\Big] u + \frac{1}{2} \beta_{pp} u^2 \neq g(u).$$

The non-maximum profit function in (7) is assumed to be homogeneous in degree one in prices (i.e., in w and pe^u). This requires the following parametric restrictions, viz., $\sum_j \beta_j + \beta_p = 1$, $\sum_j \beta_{jp} + \beta_{pp} = 0$, $\sum_k \beta_{jk} + \beta_{jp} = 0 \,\forall j$. We also assume symmetry restrictions, viz., $\beta_{jk} = \beta_{kj}$. The homogeneity restrictions can

[9] For the Cobb–Douglas production function, $\ln f(x) = \ln \alpha_0 + \sum \alpha_j \ln x_j$, the profit technical inefficiency is $\ln h(w, p, u) = u/(1-r)$. This result can be intuitively explained as follows. Technical inefficiency reduces potential output by u%, which is equivalent to a price reduction of u% holding output unchanged. For a 1% reduction in price, profit (in the Cobb–Douglas case) is reduced by $\{1/(1-r)\}$% (i.e., $\partial \ln \pi(w, p) \partial \ln p = \{1/(1-r)\}$). Therefore, for a u% change in price, profit is reduced by $\{1/(1-r)\}u$ %. Dividing u by $(1-r)$ essentially captures the scale effects. The relationship for the translog case is derived next. See Kumbhakar (1996) for a similar result for the translog cost function.

[10] This result follows from Theorem II-1 in Lau (p. 151) when output price p in Lau is replaced by pe^u. More specifically, for the Cobb–Douglas production function $\ln f(x) = \ln \alpha_0 + \sum \alpha_j \ln x_j$, the $G(w)$ function is $\ln G(w) = \alpha_0^* - \sum\{\alpha_j/(1-r)\} \ln w_j$, where $\sum_j \alpha_j = r$ (returns to scale) and $\alpha_0^* = \{1/(1-r)\}\{\ln \alpha_0 + \sum \ln \alpha_j\}$ (Lau and Yotopoulos, Kumbhakar 1987). For other forms of homogeneous production functions, the link between $f(.)$ and $G(.)$ can similarly be established.

be built in the non-maximum profit function by expressing (7) in the normalized form

$$\ln(\pi^a/p) = \beta_0 + \sum_j \beta_j \ln(w_j/pe^u) + \frac{1}{2}\sum_j\sum_k \beta_{jk}\ln(w_j/pe^u) \times \ln(w_k/pe^u) + u.$$

One advantage of rewriting (7) as the sum of $\ln \pi(w, p)$ and $\ln h(w, p, u)$ is that from (9) one can see the exact link between u and $\ln h(w, p, u)$ in the case of a translog profit function. That is, equation (9) shows that one can go back and forth from production efficiency to profit efficiency without assuming a self-dual production function.[11]

Applying Hotelling's lemma to (7) and using the homogeneity condition $\beta_{jp} + \sum_k \beta_{jk} = 0$, we obtain the cost share equations

$$\text{(10a)}\quad \frac{\partial \ln \pi^a}{\partial \ln w_j} = -\frac{w_j x_j(w, pe^u)}{\pi(w, pe^u)} \equiv -S_j = \beta_j + \sum_k \beta_{jk}\ln(w_k/pe^u), \quad j = 1, \ldots, J,$$

in which the homogeneity restrictions (in prices) are already imposed. Similarly, the revenue share equation S_p is obtained from

$$\text{(10b)}\quad \frac{\partial \ln \pi^a}{\partial \ln(pe^u)} = \frac{pe^u.ye^{-u}(w, pe^u)}{\pi(w, pe^u)} = \frac{p.y(w, pe^u)}{\pi(w, pe^u)} = \beta_p + \beta_{pp}\ln(pe^u) + \sum_j \beta_{jp}\ln w_j.$$

The above result can also be obtained from the relationship $S_p - \sum_j S_j = 1$ which follows from the linear homogeneity property (in pe^u and w) of the profit function $\pi(w, pe^u)$, noted beneath equation (9).

The share equations in (10) are then used to derive the following input demands and output supply functions, viz.,

$$\text{(11)}\quad \ln x_j(w, pe^u) = \ln \pi(w, pe^u) + \ln S_j - \ln w_j \quad j = 1, \ldots, J,$$

$$\text{(12)}\quad \ln y(w, pe^u) = \ln \pi(w, pe^u) + \ln S_p - \ln p.$$

These input demands and output supply functions can be used to calculate the elasticities

$$\text{(13)}\quad \frac{\partial \ln x_j(w, pe^u)}{\partial \ln w_k} = -S_k - \frac{\beta_{jk}}{S_j} - \delta_{jk} \quad \forall j, k,$$

$$\frac{\partial \ln x_j(w, pe^u)}{\partial \ln p} = S_p - \frac{\beta_{jp}}{S_j} \quad \forall j,$$

$$\frac{\partial \ln y(w, pe^u)}{\partial \ln p} = S_p + \frac{\beta_{pp}}{S_p} - 1,$$

$$\frac{\partial \ln y(w, pe^u)}{\partial \ln w_j} = -S_j + \frac{\beta_{jp}}{S_p} \quad \forall j,$$

where $\delta_{jk} = 1$ for $j = k$, and $\delta_{jk} = 0$ when $j \neq k$. Finally, using (5), RTS can be expressed as

$$\text{(14)}\quad \text{RTS} = \sum_j S_j \Big/ \Big(1 + \sum_j S_j\Big) = 1 - 1/S_p.$$

It is clear from (13) and (14) that input demands and output supply, their elasticities, and RTS all depend on technical inefficiency via S_p and S_j ($j = 1, \ldots, J$). Alternatively, measures of input demands, output supply, elasticities, RTS, etc., based on actual profit, in general, will differ from those based on the profit frontier. These results show that the study of efficiency "… holds more promise than the Holy Grail of efficiency measurement" (Schmidt, p. 322).

Both Technical and Allocative Inefficiency

If producers are allocatively inefficient as well, then the first-order conditions of profit maximization can be written as (Kumbhakar 1987)

$$\text{(15)}\quad f_j(\cdot)pe^u = w_j^s, \qquad j = 1, \ldots, J,$$

where w_j^s, commonly labeled as shadow price of input j, is specified as $w_j^s = \theta_j w_j$ (Atkinson and Halvorsen, Toda, Yotopoulos

[11] Note that here technical inefficiency is output-oriented. A similar result can be derived for the input-oriented measure of technical inefficiency.

and Lau). The shadow prices can be viewed as a first-order Taylor series expansion of an arbitrary shadow price function $w_j^s(w_j)$ with the property that $w_j^s(0) = 0$ and $\partial w_j^s/\partial w_j > 0$. Thus, $\theta_j \neq 1$ implies the presence of allocative inefficiency. For example, if $\theta_j > 1$ then the value of the marginal product of input j $\{pf_j(.)e^u\}$ exceeds its actual price (w_j). Consequently, input j is under-used. Similarly, if $\theta_j < 1$ then input j is over-used. Alternatively, one can define relative allocative inefficiency μ_j as $f_j/f_1 = \{w_j\theta_j\}/\{w_1\theta_1\} = \{w_j\mu_j\}/w_1$, where $\mu_j = \theta_j/\theta_1$, $j = 2, \ldots, J$. If $\mu_j < 1\,(> 1)$ then input j is over- (under-) utilized relative to input 1.[12]

RESULT 2. The profit function in the presence of both technical and allocative inefficiencies can be expressed as

$$\pi(w, p, u, \theta) = \pi(w^s, pe^u), \tag{16}$$

where $w^s = (w_1^s, \ldots, w_J^s)$ is the shadow input price vector.

This result is similar to Result 1 except that w is replaced by w^s, the reason being that the first-order conditions of profit maximization in (15) show that the relevant input prices are given by the vector w^s, and the production function is $ye^{-u} = f(x)$. Thus, if we substitute p by pe^u, w by w^s, and y by ye^{-u} in the first-order conditions and the production function, we get back to the familiar neo-classical solution, and the profit function can be written as $\pi(w^s, pe^u)$. Hotelling's lemma can then be applied to $\pi(w^s, pe^u)$ in (16) to derive the input demands and output supply functions, viz.,

$$\begin{aligned} \frac{\partial \pi(w^s, pe^u)}{\partial pe^u} &= ye^{-u} \\ &\Rightarrow \frac{\partial \pi(w^s, pe^u)}{\partial p} \\ &= y(w^s, pe^u) \\ \frac{\partial \pi(w^s, pe^u)}{\partial w_j^s} &= -x_j(w^s, pe^u). \end{aligned} \tag{17}$$

Similarly, the input demand and output supply elasticities as well as returns to scale can be obtained following the same procedure as in (13) and (14). The S_j and S_p functions in (13) and (14) will now be replaced by

$$S_j^s = -\frac{\partial \ln \pi(w^s, pe^u)}{\partial \ln w_j^s} \quad \text{and}$$

$$S_p^s = \frac{\partial \ln \pi(w^s, pe^u)}{\partial \ln(pe^u)}.$$

Implications of Result 2.

1. *Relationships among various profit functions:* With both technical and allocative inefficiency, actual profit π^a can be expressed (using (17)) as

$$\begin{aligned} \pi^a &= p \cdot y - w \cdot x \\ &= pe^u \cdot \frac{\partial \ln \pi(w^s, pe^u)}{\partial \ln(pe^u)} \times \frac{\pi(w^s, pe^u)}{pe^u} \\ &\quad + \sum_j w_j \cdot \frac{\partial \ln \pi(w^s, pe^u)}{\partial \ln w_j^s} \times \frac{\pi(w^s, pe^u)}{w_j^s} \\ &= \pi(w^s, pe^u)\left\{S_p^s - \sum_j S_j^s \theta_j^{-1}\right\} \\ &= \pi(w^s, pe^u) \times \left\{1 + \sum_j (1 - \theta_j^{-1}) S_j^s\right\} \\ &\equiv \pi(w^s, pe^u) \cdot H(w, p, u, \theta), \end{aligned} \tag{18}$$

because $S_p^s - \sum_j S_j^s = 1$. Thus, $\pi^a \neq \pi(w^s, pe^u)$. It can be shown that π^a in the presence of both technical and allocative inefficiency is less than maximum profit $\pi(w, p)$ (see Appendix A). Thus, π^a in (18) is a non-maximum profit function in relation to $\pi(w, p)$.

[12] It is worth noting here that allocative inefficiency in the above formulation arises due to non-fulfillment of the first-order conditions of profit maximization. Such a case might arise due to failure on the part of producers to allocate inputs optimally, or due to some external constraints (such as regulations, noncompetitive markets, etc.) faced by producers. While the former is an error on the part of producers, the latter is not under the control of producers. Because of this, the θ_j are often labeled as distortion factors as opposed to allocative inefficiency, and w_j^s are labeled as shadow (virtual) prices. Since our focus here is on modeling, we do not make this distinction explicit. We also do not model scale inefficiency explicitly. It is subsumed in the allocative inefficiency parameters θ_j. One can, however, recover scale inefficiency parameter κ (defined as p = MC κ) from θ_j, $j = 1, \ldots, J$, where MC is the marginal cost defined either using the cost frontier (Førsund, Lovell, and Schmidt), or total cost including inefficiencies (Kumbhakar, Biswas, and Bailey).

2. *Decomposition of profit efficiency*: Define overall profit efficiency $A(w, p, u, \theta)$ as $A(w, p, u, \theta) = \pi^a/\pi(w, p)$ and define profit allocative efficiency $\delta(w, p, \theta)$ as $\delta(w, p, \theta) = \pi^a|_{u=0}/\pi(w, p)$. Rewrite $A(w, p, u, \theta)$, using (18), as

$$\text{(19)}\quad A(w, p, u, \theta) = \frac{\pi(w, pe^u)}{\pi(w, p)} \times \frac{\pi(w^s, pe^u)\cdot H(w, p, u, \theta)}{\pi(w, pe^u)} = h(w, p, u) \times h_1(w, p, \theta, u) \neq h(w, p, u) \times \delta(w, p, \theta).$$

The implication of the (19) is that one cannot *always* express overall profit efficiency as the product of profit technical efficiency and profit allocative efficiency. Alternatively, in a profit function formulation one cannot *always* separate technical inefficiency completely from allocative inefficiency. However, if the production function is homogeneous, then the overall profit efficiency is the product of profit technical and allocative efficiency [i.e., $A(w, p, u, \theta) = h(w, p, u) \cdot \delta(w, p, \theta)$] (see Appendix B).[13]

It is worth mentioning here that if the production technology is homogeneous then $H(w, p, u, \theta)$ is independent of u and $h(w, p, u)$ is independent of w and p. This means that the technical inefficiency term separates out from the rest of the log profit function in (18). However, the allocative inefficiency terms are not separated from everything else unless some special forms of $G(w^s)$ (such as the Cobb–Douglas) are chosen. In contrast, in the case of cost minimization technical inefficiency is always separated from everything else in the log cost function, and the overall cost efficiency is the product of technical and allocative efficiencies (Farrell, Kumbhakar 1991).

3. *Input demand, output supply, and elasticities:* Since π^a is not a constant multiple of $\pi(w^s, pe^u)$,

$$\frac{\partial \pi^a}{\partial(pe^u)} \neq ye^{-u}, \quad \text{and} \quad \frac{\partial \ln \pi^a}{\partial \ln(pe^u)} \neq \frac{\partial \ln \pi(w^s, pe^u)}{\partial \ln(pe^u)}$$

unless $H(w, p, u, \theta)$ is independent of u, which is the case if the production function is homogeneous. Similarly,

$$\frac{\partial \pi^a}{\partial w_j^s} \neq -x_j, \quad \text{and} \quad \frac{\partial \ln \pi^a}{\partial \ln w_j^s} \neq \frac{\partial \ln \pi(w^s, pe^u)}{\partial \ln w_j^s}$$

unless $H(w, p, u, \theta)$ is independent of u. Thus, one cannot apply Hotelling's lemma to π^a to derive the input demands and output supply functions. The same thing is true for the cross elasticities and returns to scale. Another important difference between the present case and the case with only technical inefficiency is that elasticities and returns to scale measures defined using π^a are affected by the presence of allocative inefficiencies even if the underlying production function is homogeneous. That is,

$$\frac{\partial \ln \pi^a}{\partial \ln w_j^s} = \frac{\partial \ln \pi(w^s, pe^u)}{\partial \ln w_j^s} + \frac{\partial \ln H(w, p, u, \theta)}{\partial \ln w_j^s} = \frac{\partial \ln G(w^s)}{\partial \ln w_j^s} + \frac{\partial \ln H_0(w, p, \theta)}{\partial \ln w_j^s},$$

which is not independent of θ. Note that the above expression is independent of u, and $H_0(.) = H(.)|_{u=0}$.

Non-Maximum Variable Profit Functions

So far our analysis is based on the assumption that all inputs used in the production process are variable. If there are some quasi-fixed inputs (z), the use of a variable profit function is recommended. With only technical inefficiency such a profit function can be written as $\pi(w, p, z, u) = \pi(w, pe^u, z)$. This follows from Result 1 in which the production function with quasi-fixed inputs becomes $y = f(x, z)e^u$. The actual (observed) variable profit is not maximum since $\pi^a = py - w\cdot x = \pi(w, pe^u, z) < \pi(w, p, z)$, when $\pi(w, p, z)$ is

[13] Banker and Maindiratta assumed that such a relationship always holds and defined allocative efficiency as the ratio of overall and technical efficiency measures. Note that here we defined allocative efficiency at the technically inefficient point.

the variable profit frontier. The last inequality follows from the result that $pe^u < p \Rightarrow \pi(w, p, z, u) < \pi(w, p, z)$.

With both technical and allocative inefficiencies such a profit function can be written as $\pi(w^s, p, z, u) = \pi(w^s, pe^u, z)$. Following the line of argument above without the z variables, we can write actual variable profit as $\pi^a = \pi(w^s, pe^u, z) \cdot H(w, p, z, \theta, u) < \pi(w, p, z)$. The last inequality can be proved by adding the z variables in the production function drawn in figure A.1 (Appendix A). Therefore, the observed variable profit (π^a) is non-maximum when compared with the variable profit frontier $\pi(w, p, z)$ (also known as the neoclassical variable profit function).

Parametric Specification and Estimation

In this section we deal initially with parametric specification of profit functions. Then we discuss estimation of parameters of the profit function and estimation of technical inefficiency under both cross-sectional and panel data settings. Finally, estimation of the parameters of profit function (including allocative inefficiency parameters) and technical inefficiency are discussed for both cross-sectional and panel data models.

Parametric Specification

Econometric estimation of the profit functions derived in the preceding section requires (i) parametric specification of functional forms on $\pi(w, pe^u)$ if one wants to model only technical inefficiency, and on $\pi(w^s, pe^u)$ when both technical and allocative inefficiencies are present, and (ii) linking actual profit with either $\pi(w, pe^u)$ or $\pi(w^s, pe^u)$. We use translog forms which impose minimal a priori restrictions on the underlying production technology. We first start with the case where firms are only technically inefficient to link actual profit with different profit functions. In this case, actual profit $\pi^a = \pi(w, pe^u)$. Thus, assuming a translog form on $\pi(w, pe^u)$, the estimable form of the profit function can be written as

$$(20)\quad \ln(\pi^a/p) = u + \beta_0 + \sum \beta_j \ln(w_j/pe^u) + \frac{1}{2}\sum_j \sum_k \beta_{jk} \times \ln(w_j/pe^u)\ln(w_k/pe^u),$$

where $\beta_{jk} = \beta_{kj}$. Applying Hotelling's lemma to (20) leads to the following observed cost share equations:

$$(21)\quad \frac{\partial \ln \pi(w, pe^u)}{\partial \ln w_j} = -\frac{w_j x_j(w, pe^u)}{\pi(w, pe^u)} \equiv -S_j^a = \beta_j + \sum_k \beta_{jk} \ln(w_k/pe^u), \quad j = 1, \ldots, J.$$

If (20) is rewritten as $\ln(\pi^a/p) = \ln \pi(w, p) + \ln h(w, p, u)$ where $\ln \pi(w, p)$ is the translog profit frontier, and

$$\ln h(w, p, u) = u\left\{1 - \sum_j \beta_j - \sum_j \sum_k \beta_{jk} \times \ln(w_j/p) + (u/2) \times \sum_j \sum_k \beta_{jk}\right\},$$

then it is obvious that profit technical inefficiency $\ln h(w, p, u)$ is not a constant multiple of u unless $\sum_j \beta_{jk} = 0 \,\forall\, k$ (which implies that the underlying production function is homogeneous). Thus assuming $\ln h(.)$ to be a constant multiple of u in a translog profit function is incorrect. The estimators of this misspecified model are likely to be inconsistent. Similarly, if technical inefficiency is completely neglected, the profit function suffers from omitted variables bias. Note that the share equations are also dependent on u. Thus use of a system approach omitting u from either the profit or share equations is likely to generate inconsistent parameter estimates.

We now consider the case where firms are both technically and allocatively inefficient. Since $\pi^a = py - w \cdot x$ and $\pi(w^s, pe^u) = pe^u ye^{-u} - w^s \cdot x$, combining them yields

$$(22)\quad \pi^a = \pi(w^s, pe^u) + \sum_j \left(w_j^s - w_j\right)x_j = \pi(w^s, pe^u)\left[1 - \sum_j \left(\theta_j^{-1} - 1\right)S_j^s\right],$$

where

$$S_j^s = -\frac{\partial \ln \pi(w^s, pe^u)}{\partial \ln w_j^s}.$$

Similar to (20) we assume a translog functional form on $\pi(w^s, pe^u)$. The observed cost shares are then

$$(23)\quad S_j^a = \frac{w_j x_j(w^s, pe^u)}{\pi^a} = \left\{\frac{w_j^s x_j(w^s, pe^u)}{\pi(w^s, pe^u)}\right\} \times \left\{\frac{\pi(w^s, pe^u)}{\pi^a}\right\}.\left\{\frac{w_j}{w_j^s}\right\} = \frac{S_j^s \theta_j^{-1}}{\left\{1+\sum_k (1-\theta_k^{-1}) S_k^s\right\}}.$$

Again u appears both additively and interactively in $\ln \pi^a$ and S_j^a. Thus neglecting u or θ from (22) or (23) is likely to produce inconsistent parameter estimates. It is also incorrect to assume that u enters additively in $\ln \pi^a$.

Estimation with Cross-Sectional Data

With only technical inefficiency, we use the cost share equations from (21) in estimation.[14] We add classical error term η to each of these share equations in (21) and write them as

$$(24)\quad S_j^a = -\left[\beta_j + \sum_k \beta_{jk} \ln(w_k/p)\right] + \beta_{j0}\, u + \eta_j \qquad j = 1, \dots, J,$$

where $\beta_{j0} = \sum_k \beta_{jk}$.

In a cross-sectional model, estimation of firm-specific technical inefficiency requires that u is random. Here we follow the standard practice in the stochastic frontier literature and assume u to be distributed as half-normal (i.e., $u \sim$ i.i.d. $N(0, \sigma_u^2)$ truncated at zero from above) and $\eta = (\eta_1, \dots, \eta_J)'$ to be distributed as $N(0, \Sigma)$. The elements of η are assumed to be independent of u. The justification for this assumption is that u is under the control of a firm whereas η is outside the control of any firm.

Based on the above distributional assumptions on u and η, the log-likelihood function for a sample of N firms (firm subscript i is added) can then be written as

$$(25)\quad \mathcal{L} = N \ln 2 - NJ/2 \ln(2\pi) - N/2 \ln|\Sigma| + N \ln \sigma + \sum_i \Phi(-Z_i' \Sigma^{-1} b\, \sigma) - N \ln \sigma_u - (1/2) \sum_i a_i,$$

where

$$(26)\quad \sigma^2 = \left(1/\sigma_u^2 + b'\Sigma^{-1} b\right)^{-1}$$
$$a = Z'\Sigma^{-1} Z - \sigma^2 (Z'\Sigma^{-1} b)^2,$$

$Z = (bu + \eta)'$, $b = (\beta_{10}, \dots, \beta_{J0})'$, and $\Phi(\cdot)$ is the cumulative distribution function of a standard normal variable (see Appendix C). Maximization of the above log-likelihood function gives consistent estimates of the parameters of the profit function, σ_u^2, and the parameters in Σ. Note that the profit function is not used in estimation directly, and therefore β_0 cannot be estimated.

After obtaining estimates of the parameters, technical inefficiency for each firm can be obtained using the decomposition formula in Kumbhakar (1987), which is a generalization of the Jondrow et al. result to a simultaneous equation system. This formula is based on the conditional mean or mode of u given $(b\,u + \eta)$. Since the conditional distribution of u given $(b\,u + \eta)$ is $N(Z'\Sigma^{-1} b \sigma^2, \sigma^2)$ truncated at zero from above, the following two estimators of u are suggested. These are

$$(27a)\quad \hat{u} = E(u \mid (bu + \eta)) = \mu - \sigma \frac{\phi(\mu/\sigma)}{\Phi(-\mu/\sigma)}$$

$$(27b)\quad \tilde{u} = \text{Mode } (u \mid (bu + \eta)) = \begin{cases} \mu & \text{if } \mu \le 0 \\ 0 & \text{otherwise,} \end{cases}$$

where $\phi(\cdot)$ is the pdf of a standard normal variable and $\mu = Z'\Sigma^{-1} b \sigma^2$.

Once u is estimated from either the mean or the mode, profit efficiency can be calculated from $h(w, p, u)$. For the translog profit function $h(w, p, u)$ can be obtained from

$$(28)\quad \ln h(w, p, u) = u\left[1 - \sum_j \beta_j - \sum_j \sum_k \beta_{jk} \ln(w_j/p) + (u/2) \sum_j \sum_k \beta_{jk}\right].$$

Thus, in the present formulation we can estimate both technical inefficiency u (or technical efficiency e^u) and profit technical

[14] One could, in principle, use the profit function as well in estimation. We are not including it here because the derivation of the likelihood function involves the square of u coming from (20). This makes derivation of the likelihood function very complicated, if not intractable.

inefficiency $\ln h(w, p, u)$ (or profit technical efficiency $h(w, p, u)$).

Now we consider the case where firms are both technically and allocatively inefficient. For this we rewrite the shadow share equations as

$$(29)\quad S_j^s = -\frac{\partial \ln \pi(w^s, pe^u)}{\partial \ln w_j^s} = -\Big[\beta_j + \sum_k \beta_{jk} \ln(w_k^s/p)\Big] + \sum_k \beta_{jk}\, u \equiv A_j(w^s, p) + \beta_{j0}\, u \qquad j = 1, \ldots, J.$$

The observed share equations defined in (23) can then, after some algebraic manipulations, be expressed as

$$(30)\quad \frac{S_j^a \theta_j \Big[1 - \sum_k (\theta_k^{-1} - 1) A_k\Big] - A_j}{\beta_{j0} + \theta_j\, m\, S_j^a} = u \qquad j = 1, \ldots, J,$$

where $m = \sum_k (\theta_k^{-1} - 1)\beta_{k0}$. These share equations, in implicit form, are used to derive the likelihood function reported in Appendix D.

Once the ML estimates of the parameters are obtained, technical inefficiency u can be obtained from either (27a) or (27b). The only difference is that μ will be replaced by $Z'\Sigma^{-1}\iota\,\sigma_0^2$ and σ will be replaced by σ_0 (defined in Appendix D) in (27a) and (27b). Using the estimated value of u along with the other parameters of the model, the overall profit efficiency can be calculated from $A(w, p, u, \theta)$ defined in (19). In the present case [i.e., with a translog from assumed on $\pi(w^s, pe^u)$] $A(w, p, u, \theta)$ can be obtained from

$$\begin{aligned}(31)\quad \ln A(w, p, u, \theta) &= u\Big[1 - \sum_j \beta_j - \sum_j \sum_k \beta_{jk} \ln(w_j/p) + u\,(1/2) \sum_j \sum_k \beta_{jk}\Big] \\ &\quad + \Big[\sum_j \beta_j \ln \theta_j + (1/2) \times \sum_j \sum_k \beta_{jk} \ln \theta_j \ln \theta_k \\ &\quad + \sum_j \sum_k \beta_{jk} \ln \theta_j \ln(w_k/p) \\ &\quad + \ln \Big\{1 - \sum_j (\theta_j^{-1} - 1) S_j^s\Big\}\Big].\end{aligned}$$

Estimation with Panel Data

If panel data are available and one is willing to make the assumption that technical inefficiency is time-invariant,[15] then (i) it is possible to estimate technical inefficiency without making any distributional assumptions, and (ii) the estimation procedure is quite simple and straightforward.

In the presence of technical inefficiency only, we use the system consisting of the profit function as well as the observed cost share equations, viz.,

$$\begin{aligned}\ln(\pi_{it}^a/p_{it}) &= \beta_0 + \sum_j \beta_j \ln(w_{jit}/p_{it} e^{u_i}) \\ &\quad + \frac{1}{2} \sum_j \sum_k \beta_{jk} \times \ln(w_{jit}/p_{it} e^{u_i}) \\ &\quad \times \ln(w_{kit}/p_{it} e^{u_i}) + u_i + v_{it}\end{aligned}$$

$$(32)\quad S_{jit}^a = -\Big[\beta_j + \sum_k \beta_{jk} \ln(w_{kit}/p_{it} e^{u_i})\Big] + \eta_{jit} \qquad j = 1, \ldots, J,$$

where v and η_j are stochastic noise components. Firm and time are indexed by the subscripts i and t, respectively, and u_i's are assumed to be firm-specific parameters (fixed-effects). Under the assumption that v and η_j have zero means and constant variance–covariance matrix, one can estimate the above system using the NLITSUR procedure. Technical inefficiency u can be estimated relative to the best firm in the sample as in Schmidt and Sickles. Once u is estimated, profit technical inefficiency can be calculated for each firm using (28).

It is to be noted that u_i appears both additively and interactively with the input prices. Furthermore, technical inefficiency appears in the share equations. Consequently, the models that fail to include the interaction terms, and use only firm-specific intercepts in the

[15] It is, however, possible to accommodate time-varying technical inefficiency along the line suggested by Cornwell, Schmidt, and Sickles without making any distributional assumption on u. This issue is not considered in the present study.

normalized profit function to capture technical inefficiency (e.g., Ali and Flinn), are misspecified and give inconsistent parameter estimates.

We now consider the case where firms can be inefficient both technically and allocatively. The profit function and observed cost share equations are

$$\begin{aligned}\ln\left(\pi^a_{it}/p_{it}\right) &= \beta_0 + \sum_j \beta_j \ln(w^s_{jit}/p_{it}e^{u_i}) \\ &\quad + \frac{1}{2}\sum_j\sum_k \beta_{jk} \ln(w^s_{jit}/p_{it}e^{u_i}) \\ &\quad \times \ln(w^s_{kit}/p_{it}e^{u_i}) \\ &\quad + \ln\left[1 - \sum_j (\theta_j^{-1} - 1)S^s_{jit}\right] \\ &\quad + u_i + v_{it},\end{aligned}$$

$$(33)\quad S^a_{jit} = \frac{S^s_j \theta_j^{-1}}{\{1 + \sum_k (1 - \theta_k^{-1})S^s_{kit}\}} + \eta_{jit} \qquad j = 1, \ldots, J,$$

where $w^s_j = \theta_j w_j$ and S^s_j is given in (29). As before, v and η_j are stochastic noise components associated with the profit and observed cost share equations, respectively.[16]

The model specified in (33) can be estimated using the NLITSUR procedure. Technical inefficiency, relative to the best firm in the sample, can be obtained from the firm-specific coefficients u_i. The overall profit efficiency can then be calculated using (31).

Empirical Results

Some of the models outlined in the preceding section are estimated using a panel data on 60 Norwegian salmon farms observed during 1988–1992. Norway has long been the world leader in farmed salmon (Bjørndal). Since 1982 the Norwegian Directorate of Fisheries has compiled salmon farm production data. In the present study we use a balanced panel of 60 such farms observed during 1988–1992.[17] The output variable (y) is sales (in thousand kilograms) of salmon. The input variables are: feed (F), fish (K), labor (L), and materials (M). Feed is a composite measure of salmon feed measured in thousand kilograms. Fish input is the stock of fish (measured in thousand kilograms) at the beginning of the year (January 1st). Labor is total hours of work (in thousand hours). Material is measured in thousand real Norwegian Kronor (NOK). It constitutes expenditure on repair and maintenance, electricity, office equipment, rents on equipment and building, etc. Price of salmon is the market price of salmon per kilogram in real NOK. The wage rate (in real NOK) is obtained by dividing labor cost by hours of hired labor. Summary statistics of these variables are given in table 1 by year. Although the production figures increased during this period, prices of salmon declined. Labor hours declined after attaining a peak in 1990. Real wages increased very little during this period. Material use increased substantially, except for the last year (1992). The fish input figures show a decline in the last year.

In the present study we treat labor as the only variable input. Reasons for not treating other inputs as variable are: (i) feed and material prices do not vary much across farms and (ii) no price data (or price index) are available for fingerling (fish input) used for next year's output. Because of these we use conditional factor specifications for fish, feed, and materials. Finally, actual variable profit π^a is calculated from $\pi^a = py - w_L L$.

Three different panel data models, which include inefficiency in one form or another, are estimated. The main advantage of the panel data models is that distributional assumption on technical inefficiency component is avoided. However, technical inefficiencies in all these models are assumed to be fixed parameters instead of random variables (as is the case with cross-sectional models). Model I considers only technical inefficiency while Model II accommodates both technical and allocative inefficiencies. In Model III we generalize Model II by allowing the allocative inefficiency parameter (θ_L) to vary over time. Finally, in Model IV we estimate the neoclassical profit function (i.e., farms are assumed to be both technically and allocatively efficient). In Models I–IV we used the profit function along with the labor share equation, and estimated the system using the NLITSUR technique. The NLITSUR estimators

[16] Although we assume θ_j parameters to be invariant across firms and over time, it is possible to estimate either firm- or time-specific allocative inefficiency parameters. Another approach to make allocative inefficiency input-, firm-, and time-specific, without estimating too many parameters, is to specify them as functions of some variables which are firm- and time-specific.

[17] The data used here are a subset of the data used by Tveterås in his PhD thesis. I am grateful to him for letting me use his data. Details on the sample and construction of the variables used here can be found in his thesis.

Table 1. Summary Statistics

Year	Output	Labor	Fish	Feed	Material	Price	Wage
1988	336.01	7.57	125.83	346.97	960.84	44.07	120.22
1989	449.97	8.09	181.44	435.84	1352.58	35.45	126.03
1990	478.17	8.14	232.88	462.19	1499.78	32.17	117.14
1991	477.42	7.66	227.89	409.52	1749.36	29.52	122.70
1992	481.80	7.80	208.95	466.30	1424.12	29.94	127.27

(when converged) are equivalent to the ML estimators.

Likelihood ratio (LR) tests are performed, first, to test the hypothesis that there is no technical inefficiency ($u = 0$). That is, Model I is tested against Model IV. The results are reported in table 2. The LR test rejects the null hypothesis of no technical inefficiency at the 1% level of significance. Thus, we find evidence in favor of technical inefficiency and conclude that Model IV is misspecified. We then test the hypothesis that there is technical inefficiency but no allocative inefficiency (Model I) against the alternative that allocative inefficiency is time-invariant (Model II). The LR test again rejects the null hypothesis ($\theta_L = 1$) at the 1% level of significance. Finally, Model III is tested against Model II to examine whether allocative inefficiency in labor use is time-invariant ($\theta_L = \theta_0$ against $\theta_L = \theta_0 + \theta_1 t$). The LR test rejects time-invariant allocative inefficiency ($\theta_1 = 0$) at the 1% level of significance. Thus from the above test results, we find evidence in support of (i) both technical and allocative inefficiencies and (ii) time-varying allocative inefficiency. Based on these results we argue that estimated elasticities from models which fail to include either technical and/or allocative inefficiency are inappropriate for two reasons. First, wrong formulas are used [the correct formulas are shown in (13) and (14) which depend on both technical and allocative inefficiency]. Second, the parameter estimates obtained from estimating the maximum profit function [equating π^a with $\pi(w, p)$] might be inconsistent due to omitting inefficiency terms in the profit function and share equations in (33).

In both Models II and III we find θ_L greater than unity, implying that the salmon farms under-used labor (marginal physical product of labor being greater than the real wage (w_L/p)). Model III, which allows θ_L to vary over time, shows an increasing trend (θ_1 is found to be positive and statistically significant) in labor under-utilization.

Table 2. Hypothesis Testing

Model/Feature	Log-Likelihood	Hypothesis and LR Statistics	Degrees of Freedom	Critical Value at the 1% Level	Reject
I: only TI	−1149.75	No TI: Models I vs. IV: LR = 104.98	59	87.16	yes
II: both TI and AI	−1141.99	No AI: Models I vs. II: LR = 15.52	1	6.63	yes
III: time-varying AI and TI	−1137.48	AI is time-invariant: Models III vs. II: LR = 9.02	1	6.63	yes
IV: no Inefficiency	−1202.24	–	–	–	–
I: homogeneous	−1171.29	Production technology homogeneous: LR = 43.08	4	13.28	yes
II: homogeneous	−1160.51	Production technology homogeneous: LR = 37.04	4	13.28	yes
III: homogeneous	−1146.82	Production technology homogeneous: LR = 18.68	4	13.28	yes
IV: homogeneous	−1213.07	Production technology homogeneous: LR = 21.66	4	13.28	yes

Notes: (i) AI = allocative ineffeciency, TI = technical inefficiency. Model IV is estimated without farm dummies while all other models include 59 farm dummies (one farm dummy is suppressed since an intercept is kept in each model). (ii) Homogeneous means that the models are estimated with the restrictions that the underlying production technology is homogeneous.

We now examine homogeneity of the underlying production technology because it plays an important role in the estimation of non-maximum profit functions. We therefore test this hypothesis in all four models. The test results are reported in the last four rows of table 2. Note that if the production function is homogeneous, technical inefficiency component $\ln h(w, p, z, u)$ in the log profit functions $\ln \pi^a$ becomes additive and independent of w, p, and z. With one variable input and three conditional inputs, the parametric restrictions necessary for the production to be homogeneous in Models I–IV are: $\beta_{LL} = 0$, $\gamma_{Lq} = 0$ for $q =$ feed, fish, and materials, where β_{LL} is the coefficient of $\ln(w_L/pe^u).\ln(w_L/pe^u)$ and γ_{Lq} are the coefficient of the cross-product terms $\ln(w_L/pe^u).\ln z_q$ in the normalized non-maximum variable profit function.[18] Thus in each of the Models I–IV we imposed four parametric restrictions to test homogeneity of the underlying production function. These restrictions are rejected in each model at the 1% level of significance.

In table 3 we report estimates of technical inefficiency (u) and profit inefficiency ($\ln A$ which is profit loss due to both technical and allocative inefficiencies) calculated using the formula given in (31) for Models I–III. The average yearly technical inefficiency in Model I is 17.38%, which can be interpreted as the average output loss by the salmon farms due to technical inefficiency. Alternatively, given the input quantities, average output level could be 17.38% more if all farms were technically efficient. We observe substantial variation in the efficiency level among farms. These variations can be observed from the frequency distribution of u which shows the number of farms in each inefficiency interval. The average yearly profit loss due to technical inefficiency is 20.17%. The frequency distribution of farms in terms of percentage profit losses (when multiplied by 100) due to inefficiency (either technical or allocative or both depending on the model) for each model can be seen from the columns under $\ln A$.

In table 4 we report the own- and cross-price elasticities (for labor demand and output supplies) as well as estimates of returns to scale derived from Models I–IV. There is substantial variation in these elasticities across alternative models as well as among farms. To save space we are reporting these elasticities for the first (1988) and last year (1992). All the elasticities in Models I–III are found to have the correct sign at every data point. These results indicate that profit functions in Models I–III are well behaved. Wrong signs on some of the elasticities are, however, found in Model IV. No such violations are, however, observed at the mean values of the data at every year. This finding may be viewed as another warning that Model IV is misspecified because it fails to accommodate technical and allocative inefficiencies.

Responsiveness of output supply with respect to its own price (E_{YP}) shows an increase over time in all three models. On the contrary, there is a decrease (in absolute value) in responsiveness of labor demand with respect to the wage rate (E_{LW}). This result is observed in Models I–III. Output supply responded inversely with a change in the wage rate (E_{YW}). The absolute value of E_{YW} has increased over time in all models. Finally, labor demand responded positively with changes in output price (E_{LP}), and such responsiveness has not changed much over time. This is evidenced by all three models. The returns to scale measure is not reported because we have only one variable input.

Summary and Conclusions

simp

This paper dealt with derivation of economic implications of using non-maximum profit functions. Specifically we considered a situation when firms fail to attain the profit frontier due to the presence of either technical or allocative inefficiency, or both, and showed that important features of the technology such as input demands and output supplies, their elasticities, and returns to scale derived from the non-maximum profit function are, in general, not the same as those from the profit frontier. Neglecting inefficiency, if any, can provide incorrect estimates on some important features of the technology such as returns to scale and input elasticities. Contrary to the widely held view that the only compelling reason for the study of production frontiers is efficiency measurement, our results show that the study of production frontiers "··· holds more promise than the Holy Grail of efficiency measurement" (Schmidt, p. 322).

[18] For the model with J variable inputs and Q quasi-fixed inputs, the homogeneity restrictions are $\sum_j \beta_{jk} = 0 \, \forall k$ and $\sum_j \gamma_{jq} = 0 \, \forall q$.

Table 3. Frequency Distribution of Inefficiency

	Technical Inefficiency (u)			Profit Inefficiency (ln A)		
Interval	Model I	Model II	Model III	Model I	Model II	Model III
0.00–0.05	6	8	10	4	5	7
0.05–0.10	10	6	8	8	7	9
0.10–0.15	8	9	7	10	6	5
0.15–0.20	10	12	9	8	10	6
0.20–0.25	9	7	10	7	8	8
0.25–0.30	8	5	6	12	7	11
0.30–0.35	5	6	5	6	9	6
>–0.035	4	7	5	5	8	8
Mean	**17.38**	**19.04**	**21.35**	**20.17**	**23.79**	**25.33**

Table 4. Output and Input Elasticities (Standard Errors in Parentheses)

	Year 1988			
Elasticity	Model I	Model II	Model III	Model IV
E_{YP}	0.101 (0.004)	0.097 (0.003)	0.068 (0.003)	0.0958 (0.005)
E_{LW}	−1.427 (0.632)	−0.508 (0.296)	−0.443 (0.241)	−1.504 (0.804)
E_{YW}	−0.101 (0.004)	−0.097 (0.003)	−0.068 (0.003)	−0.0958 (0.005)
E_{LP}	1.427 (0.632)	0.508 (0.296)	0.443 (0.241)	1.504 (0.804)
	Year 1992			
Elasticity	Model I	Model II	Model III	Model IV
E_{YP}	0.109 (0.005)	0.114 (0.006)	0.124 (0.006)	0.103 (0.045)
E_{LW}	−1.378 (0.719)	−0.549 (0.027)	−0.556 (0.238)	−1.434 (0.698)
E_{YW}	−0.109 (0.005)	−0.114 (0.006)	−0.124 (0.006)	−0.103 (0.045)
E_{LP}	1.378 (0.719)	0.549 (0.027)	0.556 (0.238)	1.434 (0.698)

We used a translog profit function augmented to incorporate both technical and allocative inefficiencies, and showed how to derive input demands and output supply functions, elasticities, and returns to scale. We also showed how to calculate profit losses due to technical inefficiencies and due to technical and allocative inefficiencies combined. Furthermore, it is shown that the overall (profit) efficiency *cannot always* be expressed as the product of technical and allocative efficiencies. The decomposition result holds if the underlying production function is homogeneous. Thus, technical and allocative inefficiencies in a profit function framework cannot *always* be independent—an assumption that is widely used in the literature.

Econometric estimation of non-maximum profit functions is discussed for both cross-sectional and panel data models. Four models are estimated using a panel of 60 salmon farms from Norway. Likelihood tests rejected the model without any inefficiency as well as models with either only technical or only allocative inefficiency. The homogeneity hypothesis of the underlying production function is also rejected in all four models. Estimates of elasticities from models which fail to include either technical inefficiency or allocative inefficiency or both are found to be quite different from the models which include both technical and allocative inefficiencies. These findings (including the test results) suggest that results based on models without inefficiency might be incorrect. Similarly results from models in which technical inefficiency is assumed to be independent of input and output prices might also be incorrect, unless the underlying production function is homogeneous.

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Appendix A

Profit Loss Due to Inefficiency

Here we show graphically that $\pi^a < \pi(w, p)$ when there is either only technical inefficiency, or only allocative inefficiency, or both technical and allocative inefficiency. From the definition of π^a we obtain

(A.1) $$y = \pi^a/p + (w/p)x.$$

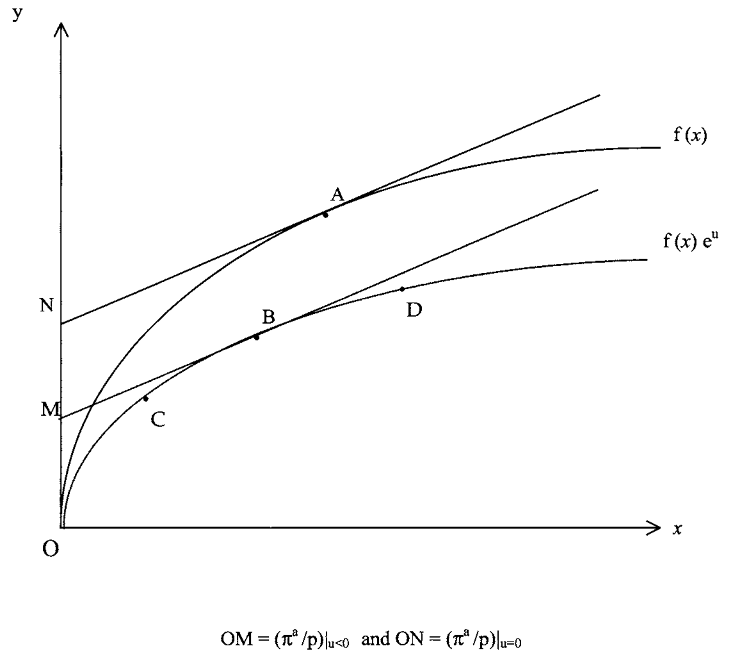

Figure A.1. Profit loss due to inefficiency

This equation, along with the production frontier $y = f(x)$ and the production function $y = f(x)e^u$, is drawn in figure A.1. The point of tangency of (A.1) with the production frontier $y = f(x)$ (point A) gives the optimum allocation of input x, and the vertical intercept gives π^a/p. Similarly, the point of tangency of (A.1) with the production function $y = f(x)e^u$ (point B) gives the optimum allocation of input x in the presence of only technical inefficiency, and the vertical intercept gives π^a/p.[19] Since $e^u < 1$ the optimum x with $u < 0$ is less when compared with $u = 0$, and the vertical intercept with $u < 0$ is less than its counterpart with $u = 0$. This shows that $\pi^a|_{u<0} < \pi^a|_{u=0} = \pi(w, p)$.

When there is only allocative inefficiency the first-order conditions are $f_j = w_j^s/p$, which can be greater (less) than w_j/p depending on the value of θ_j. Thus the optimal point can be to the left ($\theta > 1$) or to the right ($\theta < 1$) of point A on the production frontier. However, drawing lines parallel to the tangent at A through these points show that the vertical intercepts for these lines are smaller than the one corresponding to the tangent at point A. Thus, $\pi^a|_{\theta\neq 1,\, u=0} < \pi^a|_{\theta=1,\, u=0} = \pi(w, p)$.

Finally when a firm is both technically and allocatively inefficient the first-order conditions are $f_j e^u = w_j^s/p >$ or $< w_j/p$. Consequently, the optimum point can be either to the left or right of point B on the production function $y = f(x)e^u$. Lines drawn through these points (parallel to the tangent at B) show that vertical intercepts for these lines are smaller than the parallel line at A. Consequently, $\pi^a|_{\theta\neq 1,\, u<0} < \pi^a|_{\theta\neq 1,\, u=0} < \pi^a|_{\theta=1,\, u=0} = \pi(w, p)$.

[19] Note that optimality conditions with technical inefficiency are $f_j e^u = w_j/p$ which determines point B. To calculate profit corresponding to this point we draw a line parallel to the tangent at A passing through point B. The vertical intercept of this line gives π^a/p.

Appendix B

Decomposition of Profit Efficiency

Here we prove that the overall profit efficiency is the product of profit technical and profit allocative efficiencies. Since the underlying production function is homogeneous, $\pi(w, pe^u) = G(w) \cdot h(p) \cdot g(u)$ as shown in (6), and $\pi(w^s, pe^u) = G(w^s) \cdot h(p) \cdot g(u)$, which follows from (6) when w is replaced by w^s. Therefore, $\pi(w^s, pe^u)/\pi(w, pe^u) = G(w^s)/G(w)$. Similarly, using the homogeneity property in the definition of $H(w, p, u, \theta)$ in (18), we get

$$\text{(B.1)} \qquad H(w, p, u, \theta) = 1 + \sum_j \left(1 - \theta_j^{-1}\right) \times \left\{ -\frac{\partial \ln G(w^s)}{\partial \ln w_j^s} \right\},$$

since

$$S_j^s = -\frac{\partial \ln \pi(w^s, pe^u)}{\partial \ln w_j^s} = -\frac{\partial \ln G(w^s)}{\partial \ln w_j^s}.$$

Thus,

$$\text{(B.2)}\quad h_1(w, p, u, \theta) = \frac{G(w^s)}{G(w)} \cdot \Big[1 + \sum_j \Big(1 - \theta_j^{-1}\Big) \times \Big\{ - \frac{\partial \ln G(w^s)}{\partial \ln w_j^s} \Big\}\Big].$$

From (18), $\pi^a|_{u=0} = \pi(w^s, p) \cdot H_0(w, p, \theta)$, where $H_0(w, p, \theta)$ is obtained from $H(w, p, u, \theta)$ by setting $u = 0$. Using the homogeneity property $H_0(w, p, \theta)$ can be expressed as

$$\text{(B.3)}\quad H_0(w, p, \theta) = 1 + \sum_j \Big(1 - \theta_j^{-1}\Big) \times \Big\{ - \frac{\partial \ln G(w^s)}{\partial \ln w_j^s} \Big\},$$

which is nothing but $H(w, p, u, \theta)$ given in (B.1). That is, $H(w, p, u, \theta) = H_0(w, p, \theta)$.

Finally, using homogeneity property on the $\pi(w^s, p)$ and $\pi(w, p)$ functions, we get $\pi(w^s, p)/\pi(w, p) = G(w^s)/G(w)$. Therefore,

$$\text{(B.4)}\quad \delta(w, p, \theta) = \frac{G(w^s)}{G(w)} \cdot H_0(w, p, \theta) = \frac{G(w^s)}{G(w)} \cdot H(w, p, u, \theta),$$

which is nothing but $h_1(w, p, u, \theta)$ given in (B.2). Thus $h_1(w, p, \theta) = \delta(w, p, \theta)$ which proves the result.

Appendix C

Derivation of the Log-Likelihood Function with Technical Inefficiency

This appendix deals with derivation of the log-likelihood function reported in equation (25) in the text.

Assuming a sample of N firms, we wish to find the probability density function (pdf) of the error vector $(\beta_{10}u_i + \eta_{1i}, \ldots, \beta_{J0}u_i + \eta_{Ji})'$, where the subscript i denotes firm ($i = 1, \ldots, N$). Define $Z_i = (bu_i + \eta_i)'$ where $b = (\beta_{10}, \ldots, \beta_{J0})'$. Since both u and η are i.i.d. across firms, we drop the firm subscript in the following derivation. The pdf of Z, $f(Z)$, can be expressed as

$$f(Z) = \int_{-\infty}^{0} f(Z, u)du = \int_{-\infty}^{0} f(Z \mid u)h(u)du,$$

where $f(Z, u)$ is the joint pdf of Z and u, and $h(u)$ is the pdf of u. Using the distributional assumptions on u and η, the above integral can be expressed as

$$\text{(C.1)}\quad f(Z) = \frac{2}{(2\pi)^{(J+1)/2}|\Sigma|^{1/2}\sigma_u} \int_{-\infty}^{0} \exp \times \Big\{-\frac{1}{2}\Big[(Z - bu)'\Sigma^{-1} \times (Z - bu) + u^2/\sigma_u^2\Big]\Big\} du = \frac{2\sigma \exp(-a/2)}{(2\pi)^{(J/2)}|\Sigma|^{1/2}\sigma_u} \times \Phi(-Z'\Sigma^{-1}b\sigma),$$

where $\sigma^2 = (1/\sigma_u^2 + b'\Sigma^{-1}b)^{-1}$, $a = Z'\Sigma^{-1}Z - \sigma^2(Z'\Sigma^{-1}b)^2$, and $\Phi(\cdot)$ is the cumulative distribution function of a standard normal variable. The log-likelihood function reported in equation (25) in the text is the sum (over N firms) of $\ln f(Z)$.

Appendix D

Derivation of the Log-Likelihood Function with Technical and Allocative Inefficiencies

This appendix deals with derivation of the likelihood function corresponding to the system defined in (30). To do so we add classical error terms η_j to (30) for which the error vector is $(\iota u + \eta)$. This is a special case of $(b\,u + \eta)$ (the density function of which is derived in Appendix C), where ι is a column vector $(J \times 1)$ of ones.

Write (30) in implicit form as

$$q_j(S_j^a, w/p, \theta) = u + \eta_j, \qquad j = 1, \ldots, J,$$

or in matrix form as

$$\text{(D.1)}\quad q(\cdot) = \iota\, u + \eta \equiv \Xi,$$

where $q(\cdot) = (q_1(\cdot), \ldots, q_J(\cdot))'$, and $\Xi = \iota\, u + \eta$. Then the density function of Ξ, $f(\Xi)$, using the same assumptions on u and η as before (in Appendix C), becomes

$$\text{(D.2)}\quad f(\Xi) = \frac{2\sigma_0 \exp(-a_0/2)}{(2\pi)^{(J/2)}|\Sigma|^{1/2}\sigma_u} \times \Phi(-Z'\Sigma^{-1}\iota\,\sigma_0),$$

where $\sigma_0^2 = (1/\sigma_u^2 + \iota'\Sigma^{-1}\iota)^{-1}$ and $a_0 = Z'\Sigma^{-1}Z - \sigma_0^2(Z'\Sigma^{-1}\iota)^2$. Finally, the log-likelihood function is

$$\text{(D.3)}\quad \mathcal{L}_0 = N \ln 2 - NJ/2 \ln(2\pi) - N/2 \ln|\Sigma| + N \ln \sigma_0 + \sum_i \Phi(-Z_i'\Sigma^{-1}\iota\sigma_0) - N \ln \sigma_u - (1/2)\sum_i a_{0i} + \sum_i \ln|D_i|,$$

where D_i is the Jacobian of the transformation from $(\Xi_{1i}, \ldots, \Xi_{Ji})$ to $(S_{1i}^a, \ldots, S_{Ji}^a)$. The above log-likelihood function can be maximized to obtain the estimates of all parameters, including σ_u^2 and the elements of the Σ matrix.

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