MATH 142: Mathematical Modeling, Quiz 3

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Question 1

Part A

Similar to what we did in class and on the homework:

$$P_n(t+\Delta t) = \sigma_{n+1}P_{n+1}(t) + \nu_n\alpha_n P_n(t) + \beta_{n-1}P_{n-1}(t),$$

where σ_{n+1} is the probability that a death occurred from a population of $n+1$ individuals, ν_n is the probability that no deaths occurred from a population of n individuals, α_n is the probability that no births occurred from a population of n individuals, and β_{n-1} is the probability that a birth occurred from a population of $n-1$ individuals

Since Δt is small:

$$\nu_n = (1-\mu\Delta t)^n \approx 1 - n\mu\Delta t$$
$$\sigma_{n+1} = 1-(1-\mu\Delta t)^{n+1} \approx 1-(1-(n+1)\mu\Delta t) = (n+1)\mu\Delta t$$
$$\alpha_n = (1-\lambda\Delta t)^n \approx 1 - n\lambda\Delta t$$
$$\beta_{n-1} = 1-(1-\lambda\Delta t)^{n-1} \approx 1-(1-(n-1)\lambda\Delta t) = (n-1)\lambda\Delta t$$

Plugging in:

$$P_n(t+\Delta t) \approx (n+1)\mu\Delta t P_{n+1}(t) + (1-n\mu\Delta t)(1-n\lambda\Delta t)P_n(t) + (n-1)\lambda\Delta t P_{n-1}(t)$$

Expanding:

$$P_n(t+\Delta t) \approx (n+1)\mu\Delta t P_{n+1}(t) + (1-n\lambda\Delta t - n\mu\Delta t + n^2\mu\lambda\Delta t^2)P_n(t) + (n-1)\lambda\Delta t P_{n-1}(t)$$

Dividing both sides by Δt:

$$\frac{P_n(t+\Delta t)}{\Delta t} \approx (n+1)\mu P_{n+1}(t) + (\frac{1}{\Delta t} - n\lambda - n\mu + n^2\mu\lambda\Delta t)P_n(t) + (n-1)\lambda P_{n-1}(t)$$

Rearranging:

$$\frac{P_n(t+\Delta t)}{\Delta t} - \frac{P_n(t)}{\Delta t} \approx (n+1)\mu P_{n+1}(t) + (-n\lambda - n\mu + n^2\mu\lambda\Delta t)P_n(t) + (n-1)\lambda P_{n-1}(t)$$

Taking the limit and dropping the $\approx$:

$$\frac{dP_n}{dt} = (n+1)\mu P_{n+1} + (-n\lambda - n\mu)P_n(t) + (n-1)\lambda P_{n-1}$$

Further simplifying:

$$\frac{dP_n}{dt} = (n-1)\lambda P_{n-1} + (n+1)\mu P_{n+1} - (\lambda+\mu)nP_n, \text{ for } n = 2, 3, 4, ... \tag{1}$$

From (1), we derive $\frac{dP_0}{dt}$:

$$\frac{dP_0}{dt} = \mu P_1, \tag{2}$$

noting that P_{-1} does not make sense in this context and $(\lambda + \mu) * 0 * P_0 = 0$.
From (1), we derive $\frac{dP_1}{dt}$:

$$\frac{dP_1}{dt} = 2\mu P_2 - (\lambda + \mu)P_1, \tag{3}$$

noting that $(1-1)\lambda P_{1-1} = 0$.

Part B

(i)

Let $\lambda < \mu$, and taking the limit:

$$\lim_{t\to\infty} P_0(t) = \lim_{t\to\infty}((\frac{\mu e^{(\mu-\lambda)t} - \mu}{\mu e^{(\mu-\lambda)t} - \lambda})^{N_0})$$

Factor out $e^{(\mu-\lambda)t}$:

$$\lim_{t\to\infty}((\frac{e^{(\mu-\lambda)t}(\mu - \frac{\mu}{e^{(\mu-\lambda)t}})}{e^{(\mu-\lambda)t}(\mu - \frac{\lambda}{e^{(\mu-\lambda)t}})})^{N_0})$$

Canceling out $e^{(\mu-\lambda)t}$:

$$\lim_{t\to\infty}((\frac{\mu - \frac{\mu}{e^{(\mu-\lambda)t}}}{\mu - \frac{\lambda}{e^{(\mu-\lambda)t}}})^{N_0})$$

Using property of limits:

$$\frac{\lim_{t\to\infty}((\mu - \frac{\mu}{e^{(\mu-\lambda)t}})^{N_0})}{\lim_{t\to\infty}((\mu - \frac{\lambda}{e^{(\mu-\lambda)t}})^{N_0})}$$

Taking the limit of the numerator and denominator:

$$1$$

Thus, when $\lambda < \mu$, the probability that the population will eventually go extinct is 1.

(ii)

Let $\lambda > \mu$, which means $\mu - \lambda < 0$. We take the limit of P_0

$$(\lim_{t\to\infty}(\frac{\mu e^{(\mu-\lambda)t} - \mu}{\mu e^{(\mu-\lambda)t} - \lambda}))^{N_0}$$

We factor out $e^{(\mu-\lambda)t}$ and it cancels out to get:

$$(\lim_{t\to\infty}(\frac{\mu - \frac{\mu}{e^{(\mu-\lambda)t}}}{\mu - \frac{\lambda}{e^{(\mu-\lambda)t}}}))^{N_0}$$

Using exponential property:

$$(\lim_{t\to\infty}(\frac{\mu - \mu e^{(\lambda-\mu)t}}{\mu - \lambda e^{(\lambda-\mu)t}}))^{N_0}$$

Using L'Hopital's Rule:

$$(\lim_{t\to\infty}(\frac{-\mu(\lambda-\mu)e^{(\lambda-\mu)t}}{-\lambda(\lambda-\mu)e^{(\lambda-\mu)t}}))^{N_0}$$

Taking the limit:

$$(\frac{\mu}{\lambda})^{N_0}$$

Thus, when $\lambda > \mu$, the probability that the population will eventually go extinct is $(\frac{\mu}{\lambda})^{N_0}$.

Question 2

Part A

(i)

d_{k+1} is the distance traveled by the bacterium during its $(k+1)^{st}$ run.

$$d_{k+1} = \begin{cases} 10\ \mu m & , \text{ with probability } \frac{1}{4} \\ -10\ \mu m & , \text{ with probability } \frac{1}{4} \\ 0\ \mu m & , \text{ with probability } \frac{1}{2} \end{cases}$$

(ii)

Let L = moving 10 μm to the left, R = moving 10 μm to the right, and T = tumbling. We can find the probability that a bacterium is at the location of the food ($x = 10\ \mu m$) at $t = 3$ seconds through brute force:

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position (μm)

time (seconds)	-30	-20	-10	0	10	20	30
1			L	T	R		
2		LL	LT TL	LR TT RL	RT TR	RR	
3	LLL	LLT LTL TLL	LLR LTT TLT LRL TTL RLL	LTR TLR LRT TTT RLT RTL TRL	LRR TTR RLR RTT TRT RRL	RTR TRR RRT	RRR

Prob of

LRR $= \frac{1}{4}\cdot\frac{1}{4}\cdot\frac{1}{4} = \frac{1}{64}$

TTR $= \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{4} = \frac{1}{16}$

RLR $= \frac{1}{4}\cdot\frac{1}{4}\cdot\frac{1}{4} = \frac{1}{64}$

RTT $= \frac{1}{4}\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{16}$

TRT $= \frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{2} = \frac{1}{16}$

RRL $= \frac{1}{4}\cdot\frac{1}{4}\cdot\frac{1}{4} = \frac{1}{64}$

$= \boxed{\frac{15}{64}}$

Part B

We are starting from $x = 0$. After k_1 time steps, the average position is $k_1(\frac{2}{3}l - \frac{1}{3}l) = \frac{k_1}{3}l$. After $k_1 + k_2$ time steps, the average position is $\frac{k_1}{3}l + k_2(\frac{1}{2}l - \frac{1}{2}l) = \frac{k_1}{3}l$. Finally, after $k_1 + k_2 + k_3$ time steps, the average position is $\frac{k_1}{3}l + k_3(\frac{1}{5}l - \frac{4}{5}l) = \frac{k_1}{3}l - \frac{3k_3}{5}l = (\frac{k_1}{3} - \frac{3k_3}{5})l$.