MATH 142: Mathematical Modeling, Homework 4

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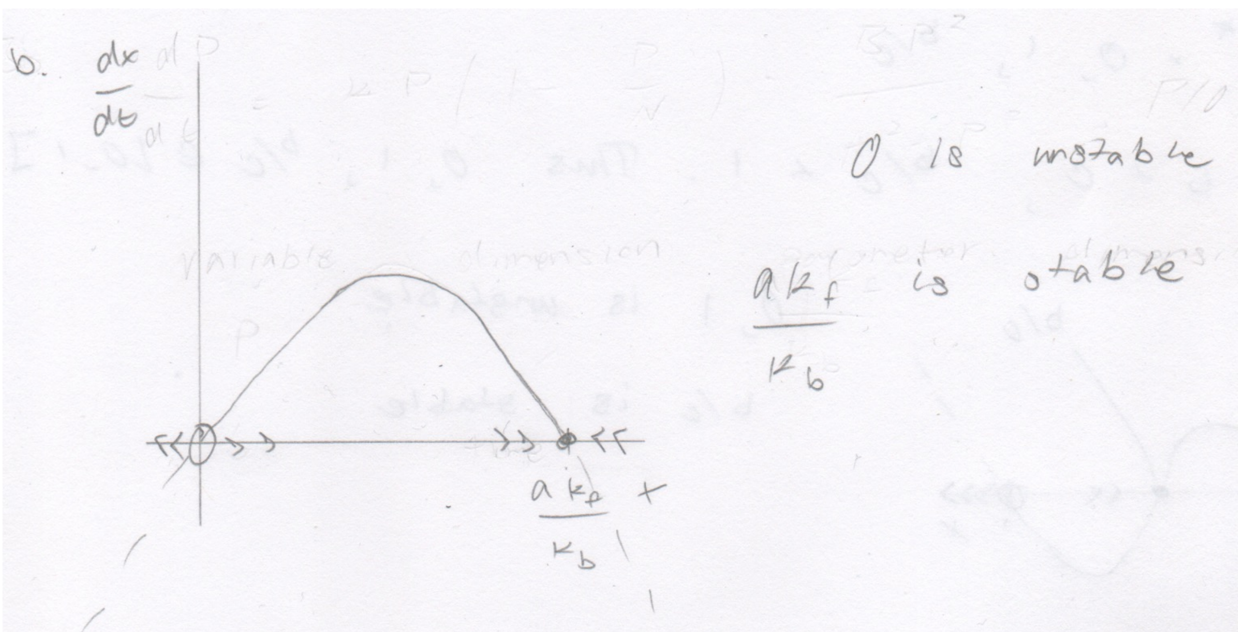
Question 1

Part A

$\frac{dx}{dt} = axk_f - x^2k_b$

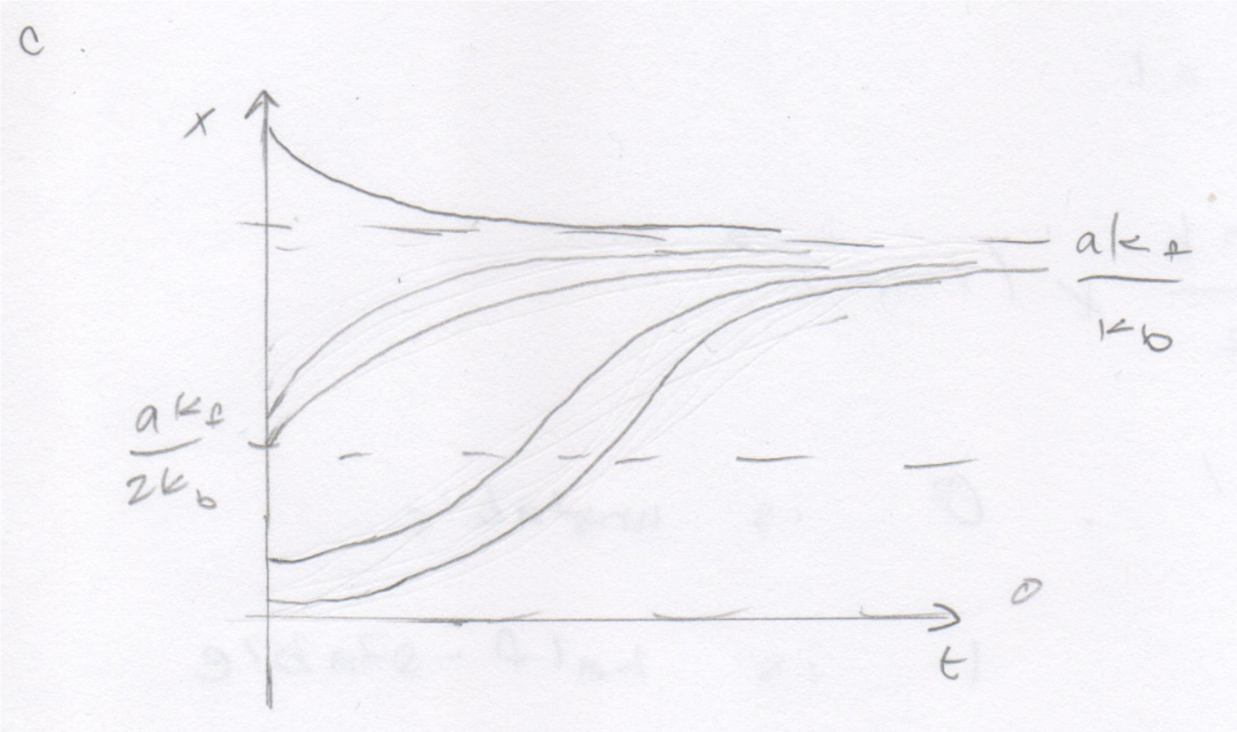
Part B

We set the differential equation equal to 0: $axk_f - x^2k_b = x(ak_f - xk_b) = 0$. Thus, the fixed points are $x^* = 0, \frac{ak_f}{k_b}$.



The stability of the fixed points tells use that the concentration of X will always stable out to $\frac{ak_f}{k_b}$ as long as there is some concentration of X to begin with.

Part C



Question 2

Part A

a.

		Opponent	
Player		Hawk	Dove
	Hawk	$\frac{b-c}{2}$	b
	Dove	0	$b/2$

Part B

b. Let N = total population of organisms

x = fraction that are hawks

y = " " " doves

$$N\frac{dx}{dt} = \cancel{N}x\,kn\left(x\left(\frac{b-c}{2}\right) + yb\right) - m\cancel{N}x$$

$$\cancel{N}\frac{dy}{dt} = \cancel{N}y\,kn\left(y\left(\frac{b}{2}\right)\right) - m\cancel{N}y$$

$$\hookrightarrow \frac{dx}{dt} = x\,kn\left(x\left(\frac{b-c}{2}\right) + yb\right) - mx$$

$$\frac{dy}{dt} = \frac{y^2knb}{2} - my$$

Population is constant, $Nx + Ny = N \rightarrow \frac{dx}{dt} + \frac{dy}{dt} = 0$

$$x\,kn\left(x\left(\frac{b-c}{2}\right) + yb\right) + \frac{y^2knb}{2} - m\left(x+y\right) = 0$$

$$m = x\,kn\left(x\left(\frac{b-c}{2}\right) + yb\right) + \frac{y^2knb}{2}$$

$$= kn\left(x^2\left(\frac{b-c}{2}\right) + xyb + \frac{y^2b}{2}\right)$$

$$\frac{dx}{dt} = x\,kn\left(x\left(\frac{b-c}{2}\right) + yb - x^2\left(\frac{b-c}{2}\right) - xyb - \frac{y^2b}{2}\right)$$

Plugging in $y = 1-x$:

$$\frac{dx}{dt} = knx\left(\frac{x(b-c)}{2} + (1-x)b - \frac{x^2(b-c)}{2} - x(1-x)b - \frac{(1-x)^2 b}{2} \right)$$

$$= knx\left(\frac{xb}{2} - \frac{cx}{2} + b - xb - \frac{x^2 b}{2} + \frac{x^2 c}{2} - xb + x^2 b - \frac{(1-x)^2 b}{2} \right)$$

$$= knx\left(-\frac{3xb}{2} + \cancel{\frac{x^2 b}{2}} + b - \frac{cx}{2} + \frac{cx^2}{2} - \frac{b}{2} + \frac{2xb}{2} - \cancel{\frac{bx^2}{2}} \right)$$

$$= knx\left(-\frac{xb}{2} + \frac{b}{2} - \frac{cx}{2} + \frac{cx^2}{2} \right)$$

$$= knx\left(\frac{b}{2}(1-x) - \frac{cx}{2}(1-x) \right)$$

$$= knx(1-x)\left(\frac{b}{2} - \frac{cx}{2} \right) \checkmark$$

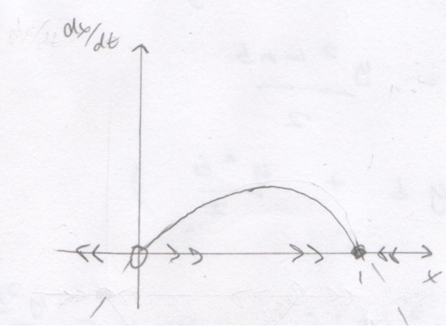
Part C

(i)

C. i. $b > c$

$x^* = 0,\ 1,\ \frac{b}{c}$

Since $b > c$, $b/c > 1$, thus only two fixed points $\in [0, 1]$.



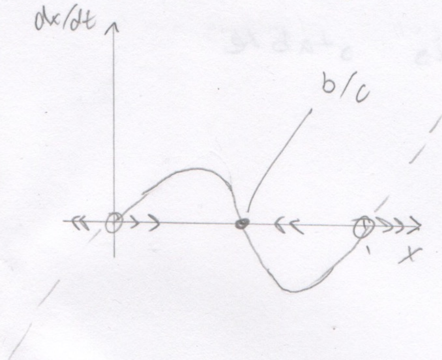
0 is unstable

1 is stable

(ii)

C. ii. $x^* = 0, 1, b/c$

Since $b < c$, $b/c < 1$. Thus $0, 1, b/c \in [0, 1]$



0, 1 is unstable

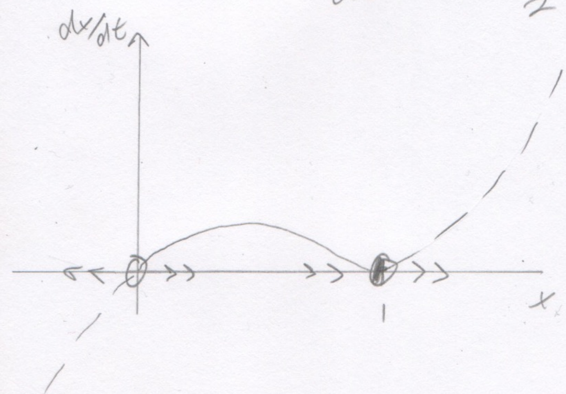
b/c is stable

(iii)

C. iii.

Since $b = c$, $b/c = 1$.

Then, $\frac{dx}{dt} = \frac{knb}{2} x(1-x)^2$



0 is unstable

1 is half-stable

(iv)

In the case where $b < c$, the hawks and doves will reach an eqilibrium. When $b \geq c$, the doves will eventually die off.

Question 3

3. $$\frac{dP}{dt} = kP\left(1-\frac{P}{N}\right) - \frac{BP^2}{A^2+P^2}, \quad P(0) = P_0$$

a.

variable	dimension	parameter	dimension
P	population	N, A, P_0	population
t	time	k	$\frac{1}{\text{time}}$
		B	$\frac{\text{population}}{\text{time}}$

b. $$\frac{dP}{dt} = kP\left(1-\frac{P}{N}\right) - \frac{B\left(\frac{P}{A}\right)^2}{1+\left(\frac{P}{A}\right)^2}$$

$u = P/A \rightarrow P = Au$

$$A\frac{du}{dt} = kAu\left(1-\frac{Au}{N}\right) - \frac{Bu^2}{1+u^2}$$

$$\tau = \frac{B}{A}t \rightarrow \frac{du}{dt} = \frac{du}{d\tau}\frac{d\tau}{dt} = \frac{B}{A}\frac{du}{d\tau}$$

$$\cancel{A}\left(\frac{B}{\cancel{A}}\right)\frac{du}{d\tau} = kAu\left(1-\frac{Au}{N}\right) - \frac{Bu^2}{1+u^2}$$

$$\frac{du}{d\tau} = k\frac{A}{B}u\left(1-\frac{A}{N}u\right) - \frac{u^2}{1+u^2}$$

$$\frac{du}{d\tau} = \alpha u\left(1-\frac{u}{\beta}\right) - \frac{u^2}{1+u^2}, \quad u(0) = \gamma,$$

where $\alpha = \frac{kA}{B}$, $\beta = \frac{N}{A}$, $u = \frac{P}{A}$, $\tau = \frac{B}{A}t$, and

$$\gamma = \frac{P_0}{A}$$

Question 4

4. $\frac{dv}{dt} = g - \frac{k}{m} v^2$ $\qquad \left(\frac{\frac{m}{d}}{m} \frac{d}{t^2} \right)^2 = \frac{d}{t^2}$

a.

variable	dimension	parameter	dimension
v	$\frac{\text{distance}}{\text{time}}$	g	$\frac{\text{distance}}{\text{time}^2}$
t	time	k	$\frac{\text{mass}}{\text{distance}}$
		m	mass

b. $\frac{1}{g} \frac{dv}{dt} = 1 - \frac{k}{mg} v^2$

only chain rule for independent variable

$u = \sqrt{\frac{k}{mg}}\, v \rightarrow v = \sqrt{\frac{mg}{k}}\, u$

$\frac{1}{g} \left(\sqrt{\frac{mg}{k}} \right) \frac{du}{dt} = 1 - \frac{k}{mg} \left(\frac{mg}{k} \right) u^2 = 1 - u^2$

$\tau = \frac{t}{\sqrt{\frac{m}{gk}}} \rightarrow \frac{du}{dt} = \frac{du}{d\tau} \frac{d\tau}{dt} = \frac{1}{\sqrt{\frac{m}{gk}}} \cdot \left(\frac{du}{d\tau} \right)$

$\rightarrow \frac{du}{d\tau} = 1 - u^2, \quad u = \sqrt{\frac{k}{mg}}\, v, \quad \tau = \frac{t}{\sqrt{\frac{m}{gk}}}$

c. $u^2 = 1 \rightarrow u = 1, -1 \xrightarrow{\text{stable}} 1 = \sqrt{\frac{k}{mg}}\, v \rightarrow v = \frac{1}{\sqrt{\frac{k}{mg}}}$ ✓

$g - \frac{k}{m} v^2 = 0 \rightarrow v = \sqrt{\frac{mg}{k}}$ ✓