Math 142 - Spring 2020
Super Quiz 4

Please read the following instructions carefully before taking this quiz:

1. You have a 24 hour window to complete this quiz, from 1:00 PM Pacific time on Monday, June 1st to 1:00 PM Pacific time on Tuesday, June 2nd. You **must** upload the quiz to Gradescope before the deadline.
2. This quiz is open book, open note, open internet, but it **must** be completed on your own. You may not ask other students/tutors for help or use forums such as Stack Exchange to ask questions. You must show work on each problem to receive full credit.
3. If you have any clarification questions about the quiz, you may post your questions on Piazza, but you may not use Piazza to discuss how to do quiz problems.
4. You may scan or take pictures of the quiz in order to upload. Regardless of what option you are using, make sure that your work is legible and that pictures/scans are in focus. I would recommend that you use a free scanning app such as Scannable.
5. Please select the correct pages of your solution associated with each question when you upload to Gradescope.
6. If you wish, you may print out the quiz and write the answers on the test paper. You may also write answers on your own paper, on a tablet, or on a computer.

By signing below, you acknowledge that you have read the instructions on the previous page and that you agree to the following statement:

"I assert, on my honor, that I have not received assistance of any kind from any other person while working on this quiz and that I have not used any non-permitted resources during the period of this evaluation."

(If you are answering the questions on separate paper, please copy the statement above onto your paper, and write your name, student ID, and signature on the same page.)

NAME: ____

STUDENT ID:____

SIGNATURE:____

SHOW ALL WORK AND BOX YOUR ANSWERS

1. [**14 points**] *Heat equation on a bounded domain*

   Suppose you are measuring the temperature along a thin bar of length 5 meters (m). The bar is made of iron, which has thermal conductivity $k = 59$ W/(m · K) (watts per meter kelvin). Heat flow inside the bar is governed by the one-dimensional heat equation:

   $$\frac{\partial T}{\partial t} = D\frac{\partial^2 T}{\partial x^2}, \qquad 0 < x < 5,$$

   where $T(x,t)$ is temperature in kelvin (K) and D is the thermal diffusivity of iron.

   The graphs in parts (a) and (b) (on the following pages) show the evolution of temperature along the bar, at times $t = 0$ (the initial temperature profile), two intermediate times, and as $t \to \infty$ (the equilibrium temperature profile). On each graph, the ordered pairs for each endpoint are indicated.

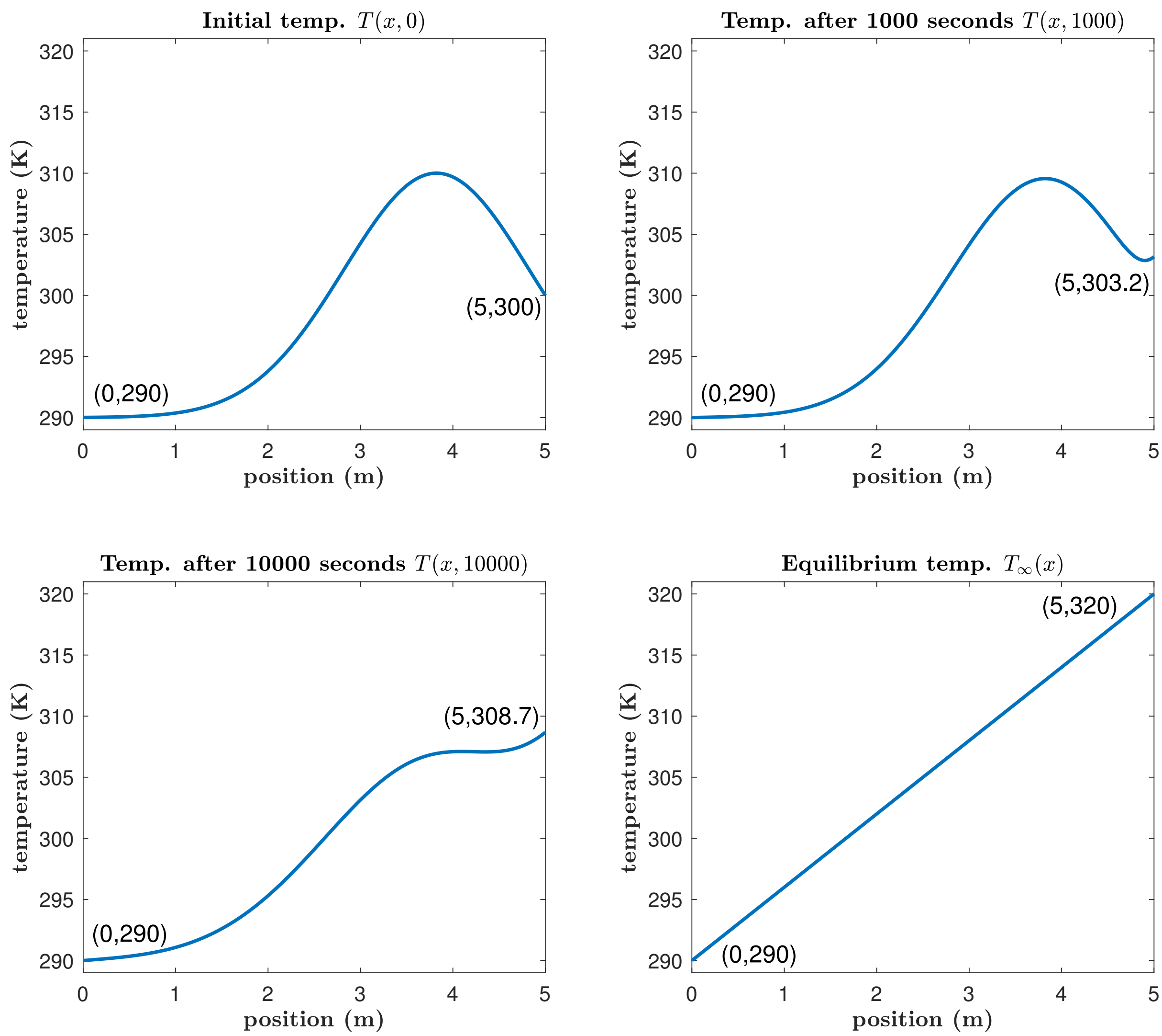
   By observing each set of graphs, determine the following:

   (i) The boundary conditions at $x = 0$ and $x = 5$, in words. Each boundary condition will be one of the following: temperature held constant, heat source, heat sink, or insulated. **Justify your answers.**

   (ii) The boundary conditions for $T(x,t)$, written mathematically. **Show work when the boundary condition is a heat source or sink.**

   (iii) For a heat source or sink, find the heat flux q. **Show work, and include units in your answer.** If there is no heat source or sink, skip this part.

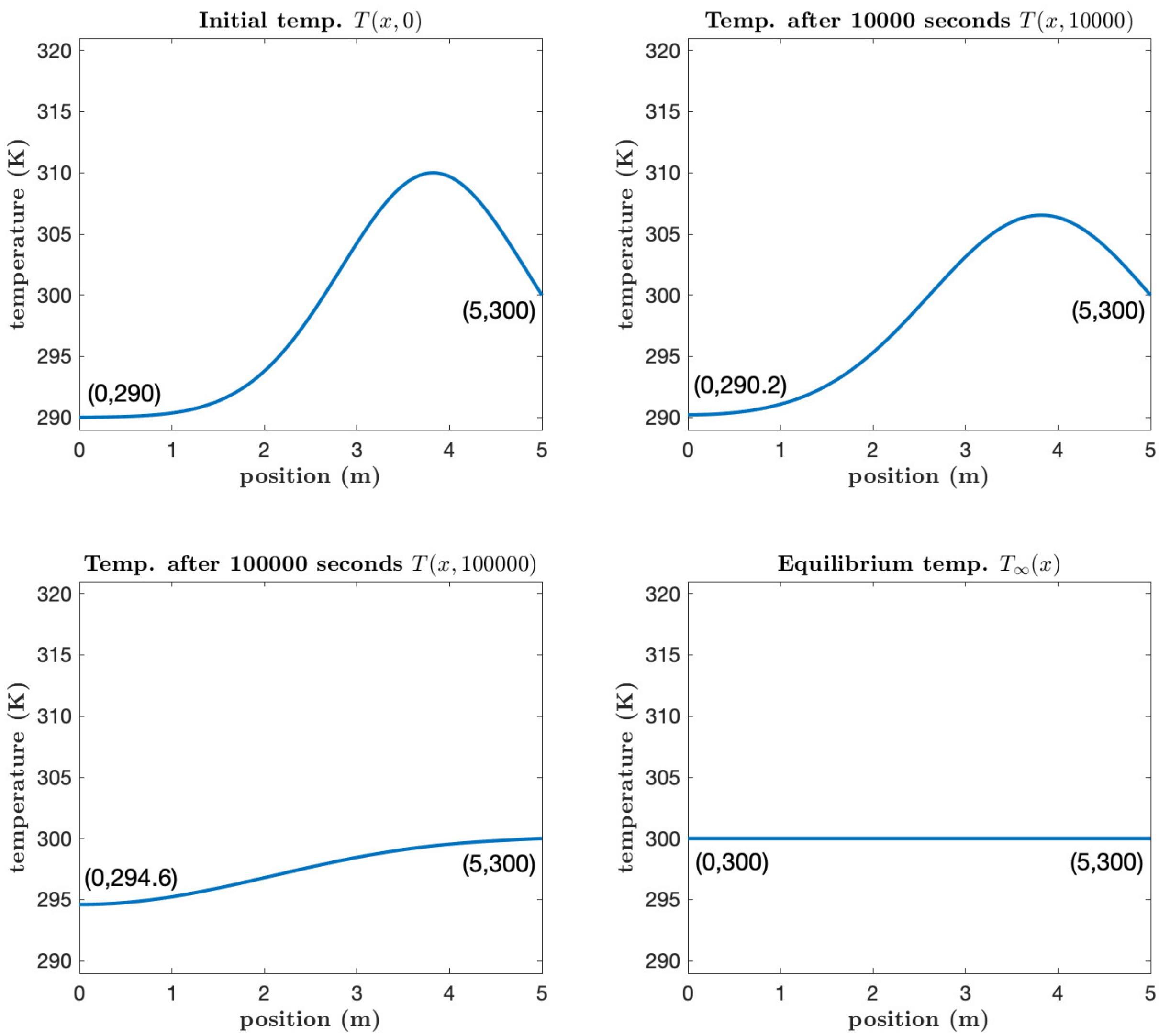
(a)



Initial temp. $T(x,0)$
temperature (K)
position (m)
(0,290)
(5,300)
Temp. after 1000 seconds $T(x,1000)$
temperature (K)
position (m)
(0,290)
(5,303.2)
Temp. after 10000 seconds $T(x,10000)$
temperature (K)
position (m)
(0,290)
(5,308.7)
Equilibrium temp. $T_\infty(x)$
temperature (K)
position (m)
(0,290)
(5,320)

(Space to continue your answer to Question 1(a))

(b)



Initial temp. $T(x,0)$
Temp. after 10000 seconds $T(x,10000)$
Temp. after 100000 seconds $T(x,100000)$
Equilibrium temp. $T_\infty(x)$
temperature (K)
position (m)
(0,290)
(5,300)
(0,290.2)
(0,294.6)
(0,300)

(Space to continue your answer to Question 1(b))

2. [**11 points**] *Reaction-diffusion equation on a bounded domain*

Morphogens are chemicals that govern the pattern of tissue development in embryos. Suppose that we are studying the distribution of morphogens in a fish embryo of length L. Let $c(x,t)$ be the concentration (density) of morphogen molecules in the embryo, where $0 \le x \le L$. Morphogens diffuse (with diffusion coefficient D) and are consumed by cells in the embryo at rate λ per morphogen molecule. Thus, $c(x,t)$ obeys the reaction-diffusion equation

$$\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2} - \lambda c, \qquad 0 < x < L.$$

Morphogens are produced at the tail of the embryo ($x = L$), flowing into the embryo with a flux of magnitude β. At the head of the embryo ($x = 0$), morphogens are absorbed (removed).

(a) (2 points) Explain why the boundary conditions are

$$c(0,t) = 0, \quad \left.\frac{\partial c}{\partial x}\right|_{x=L} = \frac{\beta}{D}.$$

(b) (6 points) Using the boundary conditions specified in part (a), find the equilibrium solution $c_\infty(x)$.

(c) (3 points) For $L = 1$, $D = 0.01$, $\lambda = 0.04$, and $\beta = 1000$, find the total number of morphogen molecules in the embryo at equilibrium.

(Space to continue your answer to Question 2)