Math 142 - Spring 2020
Super Quiz 3

Please read the following instructions carefully before taking this quiz:

1. You have a 24 hour window to complete this quiz, from 1:00 PM Pacific time on Monday, May 18th to 1:00 PM Pacific time on Tuesday, May 19th. You **must** upload the quiz to Gradescope before the deadline.

2. This quiz is open book, open note, open internet, but it **must** be completed on your own. You may not ask other students/tutors for help or use forums such as Stack Exchange to ask questions. You must show work on each problem to receive full credit.

3. If you have any clarification questions about the quiz, you may post your questions on Piazza, but you may not use Piazza to discuss how to do quiz problems.

4. You may scan or take pictures of the quiz in order to upload. Regardless of what option you are using, make sure that your work is legible and that pictures/scans are in focus. I would recommend that you use a free scanning app such as Scannable.

5. Please select the correct pages of your solution associated with each question when you upload to Gradescope.

6. If you wish, you may print out the quiz and write the answers on the test paper. You may also write answers on your own paper, on a tablet, or on a computer.

By signing below, you acknowledge that you have read the instructions on the previous page and that you agree to the following statement:

"*I assert, on my honor, that I have not received assistance of any kind from any other person while working on this quiz and that I have not used any non-permitted resources during the period of this evaluation.*"

(If you are answering the questions on separate paper, please copy the statement above onto your paper, and write your name, student ID, and signature on the same page.)

NAME: ______________________________

STUDENT ID: ______________________________

SIGNATURE: ______________________________

SHOW ALL WORK AND BOX YOUR ANSWERS

1. [**14 points**] *Stochastic birth-death process*

   (a) (10 points) In class, we derived the differential equation for a stochastic birth process. On homework 6, you derived the differential equations for a stochastic death process. In this problem, you will derive the differential equations for a stochastic birth-death process.

   Suppose that a population has an average birth rate of λ and an average death rate of μ. Let $P_n(t)$ be the probability that there are n individuals in the population at time t. Derive the following system of differential equations for $P_n(t)$:

$$\frac{dP_0}{dt} = \mu P_1,$$
$$\frac{dP_1}{dt} = 2\mu P_2 - (\lambda + \mu)P_1,$$
$$\frac{dP_n}{dt} = \lambda(n-1)P_{n-1} + \mu(n+1)P_{n+1} - (\lambda + \mu)nP_n, \quad n = 2, 3, 4, ...$$

   In your derivation, use the same procedure that we used in class to derive the differential equation for a birth process, starting from a recurrence equation. **You must show all your steps to receive full credit.** You need to also explain why the equations for P_0 and P_1 have fewer terms than the general equation for P_n ($n = 2, 3, 4, ...$).

   (b) (4 points) It can be shown that, for an initial population size of N_0,

$$P_0(t) = \left(\frac{\mu e^{(\mu-\lambda)t} - \mu}{\mu e^{(\mu-\lambda)t} - \lambda}\right)^{N_0}.$$

   (You do not need to show this.)

   $\lim_{t\to\infty} P_0(t)$ represents that probability that the population will eventually go extinct.

   i. Show that if $\lambda < \mu$, then the population is guaranteed to go extinct eventually.

   ii. Find the probability of extinction when $\lambda > \mu$.

(Space to continue your answer to Question 1)

(More space to continue your answer to Question 1)

2. [**11 points**] *Random walks*

(a) (7 points) Suppose that a population of bacteria are "run and tumble" swimming in a long, thin tube. At time $t = 0$, all bacteria are at the center of the time ($x = 0$). Every second, each bacterium can either:

- move 10 μm to the right, with probability 1/4,
- move 10 μm to the left, with probability 1/4, or
- perform a "tumble" (remain stationary), with probability 1/2.

i. The recurrence equation for the position of bacteria x (μm) after k seconds is

$$x_{k+1} = x_k + d_{k+1}.$$

Specify d_{k+1}.

ii. Suppose that at time $t = 3$ seconds, a food source appears at position $x = 10$ μm (10 micrometers to the right of $x = 0$). What is the probability that a bacterium is at the location of the food at this time? **Show all work.**

(b) (4 points) A population of *Dictyostelium* cells (slime molds) are searching for food as well (bacteria, incidentally). These cells are also all at the center of the tube ($x = 0$) at time $t = 0$. Their run length (distance covered at each time step) is l, and they are never stationary.

When a food source appears to their right at time $t = 0$, the cells perform a biased random walk in its direction, with probability 2/3 of moving right and probability 1/3 of moving left, for k_1 time steps. At this time, the food has been consumed, so, in order to search for more food, the cells perform an unbiased random walk (equal probability of moving left and right) for k_2 time steps. Finally, the cells discover another food source to their left, and perform a biased random walk in its direction, with probability 1/5 of moving right and probability 4/5 of moving left, for k_3 time steps.

At this point (that is, after $k_1 + k_2 + k_3$ time steps), what is the expected (average) position of the *Dictyostelium* population? **Explain your reasoning.**

(Space to continue your answer to Question 2)