Math 142 - Homework 4
Due Wednesday, April 29th

(1) **Model for an autocatalytic reaction – from Strogatz**

A chemical species A combines with a species X to form two molecules of X. This means that the chemical X is involved in its own production, a process that chemists call *autocatalysis*. However, the reaction is reversible: two molecules of X can also combine and disassociate back into X and A. We represent this reaction by the chemical equation:

$$A+X \underset{k_b}{\overset{k_f}{\rightleftharpoons}} 2X$$

Here, k_f and k_b are the rate constants for the forward reaction and for the backward reaction, respectively. Assume that there is a very large surplus of the chemical A, so that we can treat the concentration of A as a constant: $[A] = a$.

(a) Use the mass-action laws to write down a differential equation for $x(t)$, the concentration of X in the reaction.

(b) Find the fixed points and evaluate their stability using a vector field plot. Explain what the stability of the fixed points is telling you about the behavior of the biological system.

(c) Sketch solutions $x(t)$ for various initial values x_0. Include inflections in your graphs of $x(t)$ where they occur.

(2) **A Hawk-Dove game**

In class, we discussed the "snowdrift" model as an example for interaction of organisms. In this problem, we will use a "Hawk-Dove" model instead. In the Hawk-Dove game, we assume that when two organisms interact, they compete for some resource (e.g., territory or food). Only one organism can win the competition, and the winner receives benefit b. However, if the organisms fight, one could be injured, incurring a cost.

Organisms can adopt two different strategies when they interact. *Hawks* always fight for the resource, while *Doves* always back down from a fight. When two doves meet, both back down, and they share the resource equally (i.e., each gets benefit $b/2$). When a hawk meets a dove, the dove surrenders the resource to the hawk. The hawk receives benefit b, while the dove gets nothing. When two hawks meet, they fight, and the victor receives the benefit b. The loser is injured in the fight and incurs a cost c.

(a) Construct the payoff matrix for the Hawk-Dove game. Your matrix should represent the average payoff obtained for the "player" in each of the interactions outlined above. It should have the following form:

		Opponent	
		Hawk	Dove
Player	Hawk		
	Dove		

(b) Assuming that the birth rates of hawks and doves are proportional to payoff and that the death rate is chosen so that the total population is constant (i.e., the assumptions we made in class), derive the following differential equation for the fraction of the population that are hawks (x):

$$\frac{dx}{dt} = knx(1-x)\left(\frac{b}{2} - \frac{cx}{2}\right), \tag{1}$$

with $y = 1 - x$ being the fraction that are doves.

(c) (i) Show that if $b > c$, then the only fixed points of (1) within $0 \leq x \leq 1$ are $x = 0$ and $x = 1$. Which fixed point is stable and which is unstable?

(ii) Show that if $b < c$, then there are three fixed points in this interval, and evaluate their stability.

(iii) Suppose that $b = c$; then

$$\frac{dx}{dt} = \frac{knb}{2}x(1-x)^2.$$

Again, we only have two fixed points. Evaluate their stability.

(iv) Give an interpretation of what happens to the population of hawks and doves long-term in each of the scenarios in (i)-(iii).

Nondimensionalization

(3) The following model describes the outbreak of the spruce budworm:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right) - \frac{BP^2}{A^2 + P^2}, \quad P(0) = P_0.$$

(a) Specify the dimensions of all variables and parameters in the model.

(b) Show that the model can be rewritten in the dimensionless form

$$\frac{du}{ds} = \alpha u\left(1 - \frac{u}{\beta}\right) - \frac{u^2}{1 + u^2}, \quad u(0) = \gamma,$$

and specify all dimensionless variables and parameters.

(4) The differential equation for the velocity v of a falling object with mass m is

$$\frac{dv}{dt} = g - \frac{k}{m}v^2. \tag{2}$$

(a) Specify the dimensions of all variables and parameters.

(b) Nondimensionalize the equation.

(c) Find the nondimensional terminal velocity of the object, and show that this agrees with the terminal velocity calculated from the original model (2).