Math 142 - Homework 3
Due Wednesday, April 22nd

(1) Make a vector field plot for each differential equation. Find the fixed points (equilibria) and use your vector field plot to classify whether each fixed point is stable or unstable. Then sketch the solution $y(t)$ for $y(0) = -1$, $y(0) = 0.5$, and $y(0) = 2$.

(a) $\frac{dy}{dt} = y - y^2$

(b) $\frac{dy}{dt} = y^3 - 2y$

(2) Break each equation into two parts that you can sketch. By sketching the two parts, determine how many fixed points the equation has, and classify them as stable or unstable.

(a) $\frac{dx}{dt} = e^{-x} + x - 2$

(b) $\frac{dx}{dt} = \frac{1}{2} - \frac{x^2}{x^2+1}$

(3) The **Beverton-Holt model** is a discrete model for density-dependent population growth. For time steps of length Δt, the recurrence equation for the Beverton-Holt model is specified as follows:

$$N(t + \Delta t) = \frac{1 + r_0 \Delta t}{1 + \alpha \Delta t N(t)} N(t). \tag{1}$$

(a) Find the fixed points N^* of the Beverton-Holt model (1).

(b) Show that if we take $\Delta t \to 0$ in equation (1), we obtain the **logistic differential equation**

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right), \tag{2}$$

and define r and K in terms of the original parameters in model (1). This shows that the logistic equation is the continuous-time analog of the Beverton-Holt model.

(c) Find the fixed points N^* of the logistic model (2), and verify that they are the same as the fixed points of model (1).

(4) Let's return to the yeast cell data that we studied in homework 1. Remember, the data can be downloaded from the BioQuest website:

`https://bioquest.org/numberscount/data-details/?product_id=31395`

The data consist of a count of yeast cells (measured in millions of cells per mL of the growth medium) as a function of time. Recall that an exponential growth model did not fit the data well. In this problem, we'll be constructing a logistic model to describe the growth of the yeast cell population.

(a) Recall that the logistic growth model is

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right),$$

where $N(t)$ is the population at time t, r is (approximately) the exponential growth rate when N is small, and K is the carrying capacity (the maximum sustainable population size).

(i) Find the growth rate from $t = 0$ to $t = 3$ hours (*Hint*: Solve

$$N(3) = e^{3r}N(0)$$

for r). Use this as an estimate for r for the yeast cell population.

(ii) By examining the yeast cell counts at the end of the experiment, estimate the carrying capacity K for the yeast cell population.

(b) Write down the logistic differential equation describing the yeast cell population growth, including your estimates for r and K. Include the initial condition with your differential equation.

(c) Solve the differential equation from part (b).

(d) Plot the solution on the same axes as the data. Use (unconnected) points for the data and a curve for the solution. Then explain why a logistic growth model works better to describe yeast cell population dynamics than an exponential growth model.

(5) An **enzyme** is a catalyst for a chemical reaction. The molecules upon which enzymes act are called **substrates**. When an enzyme binds to a substrate, it forms a complex, and converts the substrate into a different molecule called a **product**. When a product is formed, the enzyme is not consumed; it can catalyze further reactions.

Suppose that the enzyme (E) binds to the substrate (S) at rate k_f, forming a complex C. The complex dissociates (without forming a product) at rate k_b. Finally, the substrate is converted to a product at rate k_p, releasing the enzyme. The following diagram describes these reactions:

$$E+S \underset{\mathrm{k}_b}{\overset{\mathrm{k}_f}{\rightleftharpoons}} C \xrightarrow{\mathrm{k}_p} P+E$$

(a) Using the lass of mass action, write down a system of four differential equations describing how the concentrations of enzyme (e), substrate (s), complex (c), and product (p) change in time.

(b) Find two different conserved quantities for the system, and make use of them to reduce the system to two differential equations.