MATH 142: Mathematical Modeling, Quiz 4

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Question 1

(i)

For A: At $x = 0$, the temperature is held constant. At $x = 5$, there is a heat source.

For B: At $x = 0$, the bar is insulated. At $x = 5$, the temperature is held constant.

(ii)

For A: $T(0, t) = 290$ and $\left.\frac{\partial T}{\partial x}\right|_{x=5} = \frac{320-290}{5-0} = 6$

For B: $\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0$ and $T(5, t) = 300$

(iii)

For A: $\left.\frac{\partial T}{\partial x}\right|_{x=5} = 6 = \frac{-q(5,t)}{k} \implies q(5, t) = -k * 6 = 59 * 6 = -354 \ \frac{\text{watt}}{\text{meter}^2}$

For B: not applicable

Question 2

Part A

Since morphogens are removed at $x = 0$, $c(0,t) = 0$. Combining the fact that morphogens flow into the embryo with a flux of magnitude β at $x = L$ and morphegens diffuse with diffusion coefficient D, $\frac{\partial c}{\partial x}\Big|_{x=L} = \frac{\beta}{D}$.

Part B

The equilibrium solution $c_\infty(x)$ must satisfy $D\frac{d^2c_\infty}{dx^2} - \lambda c_\infty = 0$.
The characteristic equation is:

$$Dr^2 - \lambda = 0$$

Solving for r:

$$r_1 = \sqrt{\frac{\lambda}{D}}, r_2 = -\sqrt{\frac{\lambda}{D}}$$

Thus, we have:

$$c_\infty(x) = Ae^{\gamma x} + Be^{-\gamma x}, \tag{1}$$

where $\gamma = \sqrt{\frac{\lambda}{D}}$.
Applying our first initial condition:

$$c(0,t) = 0 \implies Ae^0 + Be^0 = A + B = 0 \tag{2}$$

Applying our second initial condition:

$$\frac{\partial c}{\partial x}\Big|_{x=L} = \frac{\beta}{D} \implies A\gamma e^{\gamma L} - B\gamma e^{-\gamma L} = \frac{\beta}{D} \tag{3}$$

Using both (2) and (3):

$$A\gamma e^{\gamma L} + A\gamma e^{-\gamma L} = \frac{\beta}{D}$$

Factoring:

$$A\gamma(e^{\gamma L} + e^{-\gamma L}) = \frac{\beta}{D}$$

Solving for A:

$$A = \frac{\beta}{D\gamma(e^{\gamma L} + e^{-\gamma L})}$$

From (2):

$$B = \frac{-\beta}{D\gamma(e^{\gamma L} + e^{-\gamma L})}$$

Plugging A and B into (1):

$$c_\infty(x) = \frac{\beta}{D\gamma(e^{\gamma L} + e^{-\gamma L})}e^{\gamma x} - \frac{\beta}{D\gamma(e^{\gamma L} + e^{-\gamma L})}e^{-\gamma x}$$

We factor, and our equilibrium solution becomes:

$$c_\infty(x) = \frac{\beta}{D\gamma(e^{\gamma L} + e^{-\gamma L})}[e^{\gamma x} - e^{-\gamma x}], \tag{4}$$

where $\gamma = \sqrt{\frac{\lambda}{D}}$.

Part C

We plug $L = 1, D = .01, \lambda = .04, \beta = 1000, \gamma = \sqrt{\frac{.04}{.01}} = 2$ into (4):

$$c_\infty(x) = \frac{1000}{.01 * 2 * (e^{2*1} + e^{-2*1})}[e^{2x} - e^{-2x}]$$

Evaluating:

$$c_\infty(x) = 6645.056[e^{2x} - e^{-2x}] \tag{5}$$

We set up an integral of (5):

$$\int_0^1 6645.056[e^{2x} - e^{-2x}]dx$$

Taking out the constant:

$$6645.056 \int_0^1 [e^{2x} - e^{-2x}]dx$$

Integrating:

$$6645.056\Big[\frac{e^{2x}}{2} + \frac{e^{-2x}}{2}\Big]_{x=0}^{x=1}$$

Evaluating at the limits:

$$6645.056\Big[\frac{e^2}{2} + \frac{e^{-2}}{2} - (\frac{e^0}{2} + \frac{e^0}{2})\Big] = 18354.945$$

The total number of morphogen molecules in the embryo at equilibrium is 18354.945.