MATH 142: Mathematical Modeling, Homework 6

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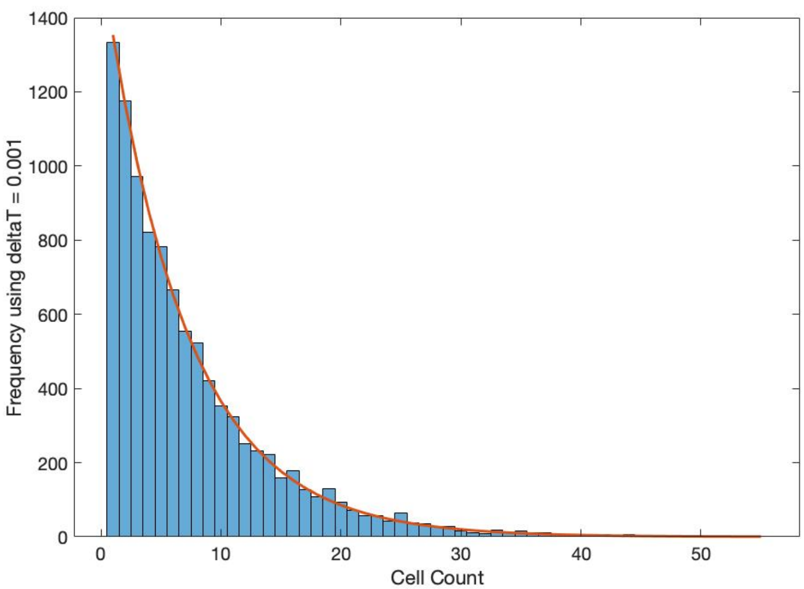
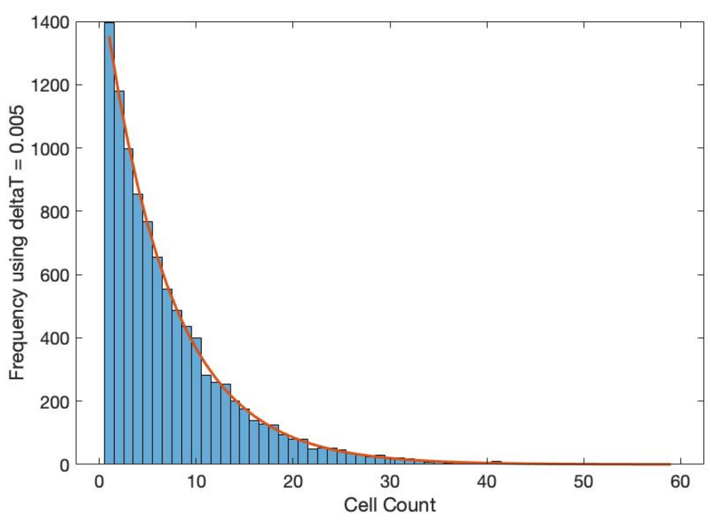
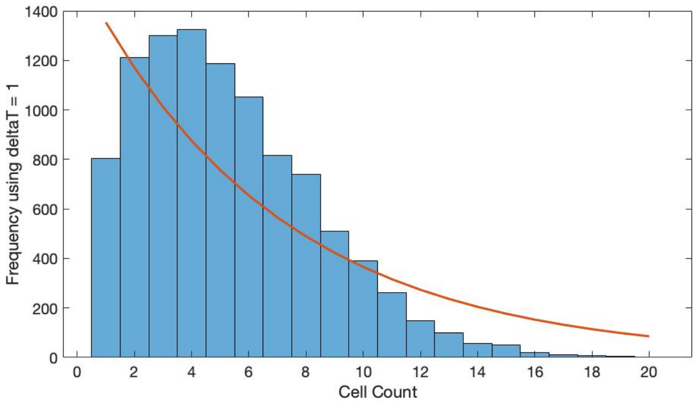
Question 1

Part A



Part B





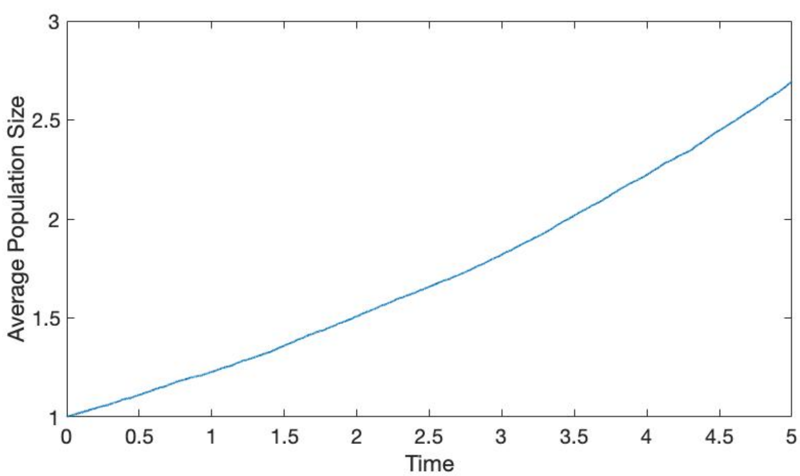
We can see that using $\Delta t = .005$ does not seem too problematic. On the other hand, using $\Delta t = 1$ is problematic. This is because using $\Delta t = 1$ violates our assumption that only one birth happens in each time step; $\Delta t = 1$ is big enough that multiple births can happen in one time step.

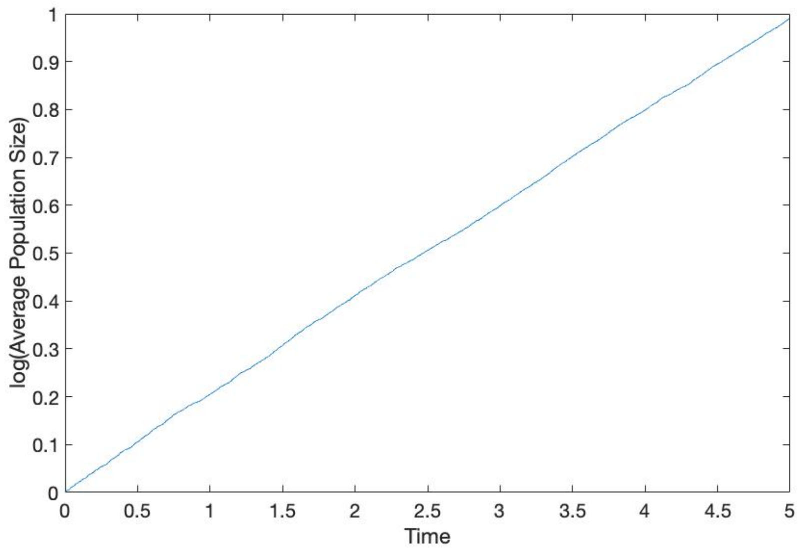
Part C

To account for cell deaths, we use a similar method for how we accounted for cell births. We create another random vector. Then, we find how many values in the newly created random vector less than $\mu\Delta t$. Finally, we subtract that amount found, so our recurrence equation will be $N(i+1,j) = N(i,j) + cellBorn - cellDeath$.

We do not worry about the probability of a cell both dividing and dying in the same time interval because of the same logic to assume that only one birth happens in each time step.

Part D





The average growth seems to be exponential. The rate of growth is the slope of the second graph: $\frac{1-0}{5-0} = .2$.

Question 2

Part A

$$P_n(0) = \begin{cases} 1 & , \quad n = N \\ 0 & , \quad n \neq N \end{cases}$$

Part B

Similar to what we did in class,

$$P_n(t + \Delta t) = \alpha_{n+1} P_{n+1}(t) + \beta_n P_n(t),$$

where α_{n+1} is the probability that exactly one death occurs among $n+1$ individuals, and β_n is the probability that no deaths occurs among n individuals.

Since Δt is small:

$$\beta_n = (1 - \mu\Delta t)^n \approx 1 - \mu n \Delta t$$

$$\alpha_{n+1} = 1 - (1 - \mu\Delta t)^{n+1} \approx 1 - (1 - \mu(n+1)\Delta t) = \mu(n+1)\Delta t$$

Plugging in β_n and α_{n+1}:

$$P_n(t + \Delta t) \approx (\mu(n+1)\Delta t)P_{n+1}(t) + (1 - \mu n \Delta t)P_n(t)$$

Dividing both sides by Δt:

$$\frac{P_n(t + \Delta t)}{\Delta t} \approx \mu(n+1)P_{n+1}(t) + \frac{P_n(t)}{\Delta t} - \mu n P_n(t)$$

Rearranging:

$$\frac{P_n(t + \Delta t)}{\Delta t} - \frac{P_n(t)}{\Delta t} \approx \mu(n+1)P_{n+1}(t) - \mu n P_n(t)$$

Taking the limit and dropping the $\approx$:

$$\frac{dP_n}{dt} = \mu(n+1)P_{n+1} - \mu n P_n \tag{1}$$

Intuitively:

$$\frac{dP_N}{dt} = -\mu N P_N \tag{2}$$

because $\frac{dP_N}{dt}$ can not increase, it can only decrease. Thus, it is similar to $\frac{dP_n}{dt}$, just without the positive part.

Part C

To find $P_N(t)$, we rearrange (2):

$$\frac{dP_N}{P_N} = -\mu N dt$$

Taking the integral:

$$ln(P_N) = -\mu N t + C$$

Rearranging:

$$P_N(t) = Ce^{-\mu N t}$$

Applying the initial condition $P_N(0) = 1$:

$$C = \frac{P_N(t)}{e^{-\mu N t}} = 1$$

Finally,

$$P_N(t) = e^{-\mu N t}$$

To find $P_{N-1}(t)$, we plug in $n = N - 1$ into (1):

$$\frac{dP_{N-1}}{dt} = \mu N P_N - \mu(N-1)P_{N-1}$$

Rearranging amd plugging in P_N:

$$\frac{dP_{N-1}}{dt} + \mu(N-1)P_{N-1} = \mu N e^{-\mu N t}$$

Let $\rho = e^{\int (N-1)\mu dt} = e^{(N-1)\mu t}$. Multiply the equation by ρ:

$$e^{(N-1)\mu t}\frac{dP_{N-1}}{dt} + e^{(N-1)\mu t}\mu(N-1)P_{N-1} = \mu N e^{-\mu t}$$

Using integration by parts and integrating:

$$\int \frac{d}{dt} e^{(N-1)\mu t} P_{N-1} = \int \mu N e^{-\mu t}$$

We get:

$$e^{(N-1)\mu t} P_{N-1} = -N e^{-\mu t} + C$$

Rearranging:

$$P_{N-1}(t) = -N e^{-N\mu t} + C e^{-(N-1)\mu t}$$

Plugging in the initial condition of $P_{N-1}(0) = 0$

$$C = \frac{P_{N-1} + N e^{-N\mu t}}{e^{-(N-1)\mu t}} = \frac{N}{1} = N$$

Thus,

$$P_{N-1}(t) = -N e^{-N\mu t} + N e^{-(N-1)\mu t} = N e^{-N\mu t}(e^{\mu t} - 1)$$

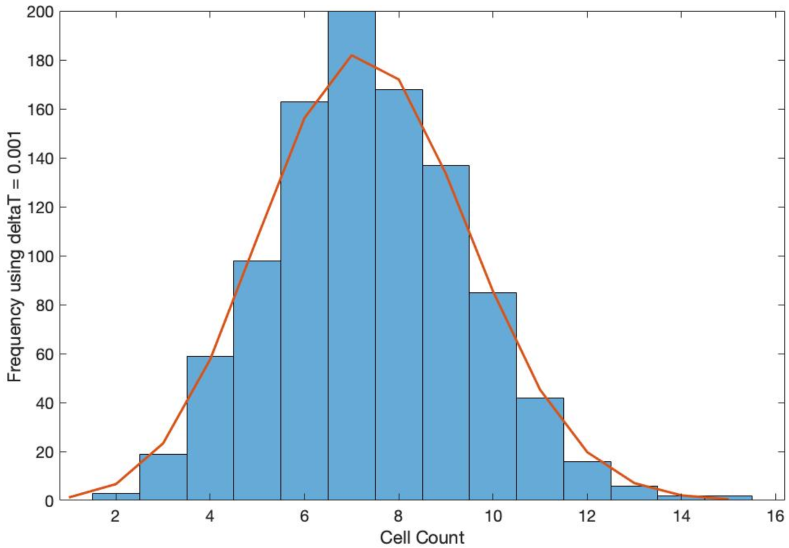
For P_{N-2}, we do the same steps as P_{N-1} (with a lot more messy algebra) and get:

$$P_{N-2}(t) = \frac{N(N-1)}{2} e^{-N\mu t}(e^{\mu t} - 1)^2$$

From P_N, P_{N-1}, and P_{N-2},

$$P_n(t) = \binom{N}{N-n} e^{-N\mu t}(e^{\mu t} - 1)^{N-n}$$

Part D



My solution for $P_n(t)$ seems to be correct.

Part E

We obtain:

$$\bar{n}(t + \Delta t) = (1 - \mu \Delta t)\bar{n}$$

Dividing by Δt:

$$\frac{\bar{n}(t + \Delta t)}{\Delta t} = \frac{\bar{n}}{\Delta t} - \mu \bar{n}$$

Rearranging:

$$\frac{\bar{n}(t + \Delta t)}{\Delta t} - \frac{\bar{n}}{\Delta t} = -\mu \bar{n}$$

Taking the limit:

$$\frac{d\bar{n}}{dt} = -\mu \bar{n}$$

Part F

Rearranging the terms:

$$\frac{d\bar{n}}{\bar{n}} = -\mu dt$$

Integrating both sides:

$$ln(\bar{n}) = -\mu t + C$$

Simplifying:

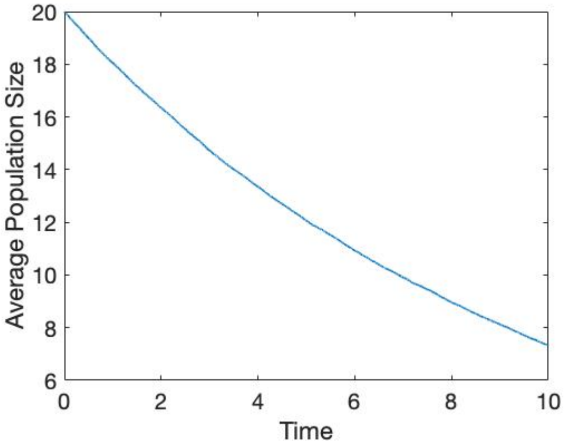
$$\bar{n}(t) = Ce^{-\mu t}$$

Plugging in our initial condition of $\bar{n}(0) = N$:

$$C = \frac{\bar{n}(t)}{e^{-\mu t}} = \frac{N}{1} = N$$

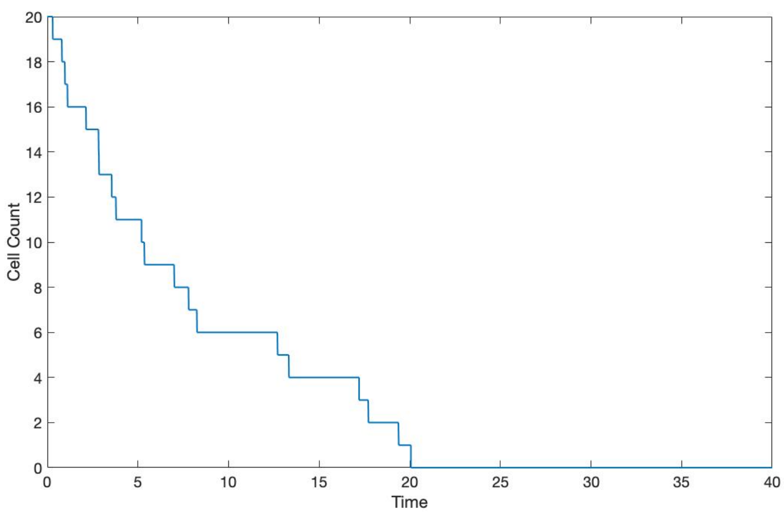
Finally,

$$\bar{n}(t) = Ne^{-\mu t}$$



The above plot is from running simulations, and we can see that it is the same as $\bar{n}(t) = 20e^{-.1t}$, as desired.

Part G



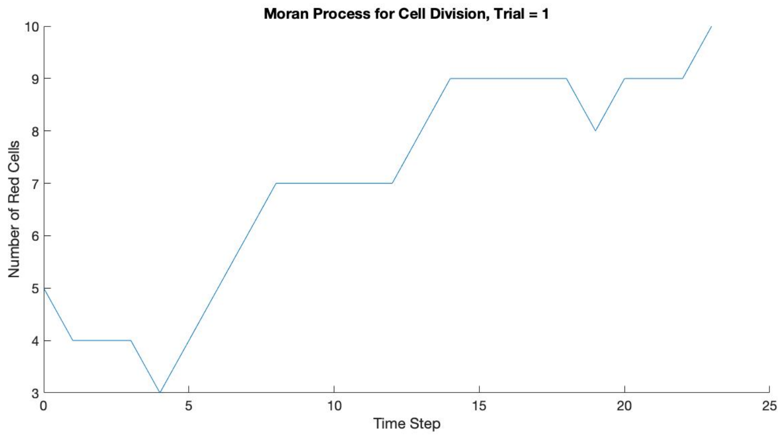
My solution from Part F never actually reaches 0; it just gets very, very close as t increases. The stochastic model is more realistic when the amount of bacteria is very small.

Question 3

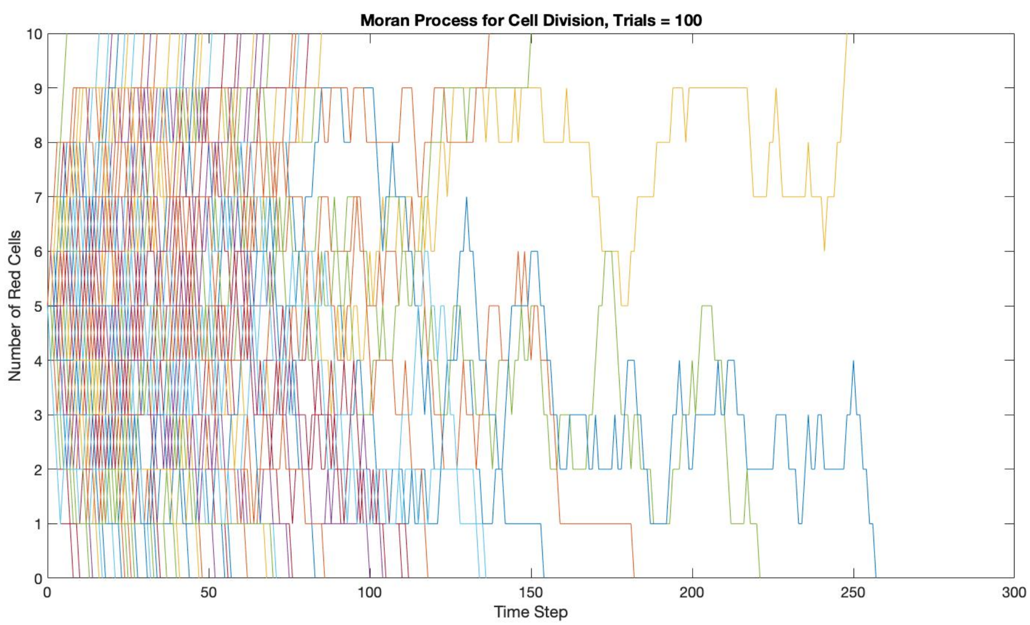
Part A

I would expect the average number of red cells to remain constant over time because the probabilty of increasing is the same as the probability of decreasing, which means the average number should remain constant.

Part B



Part C



Part D

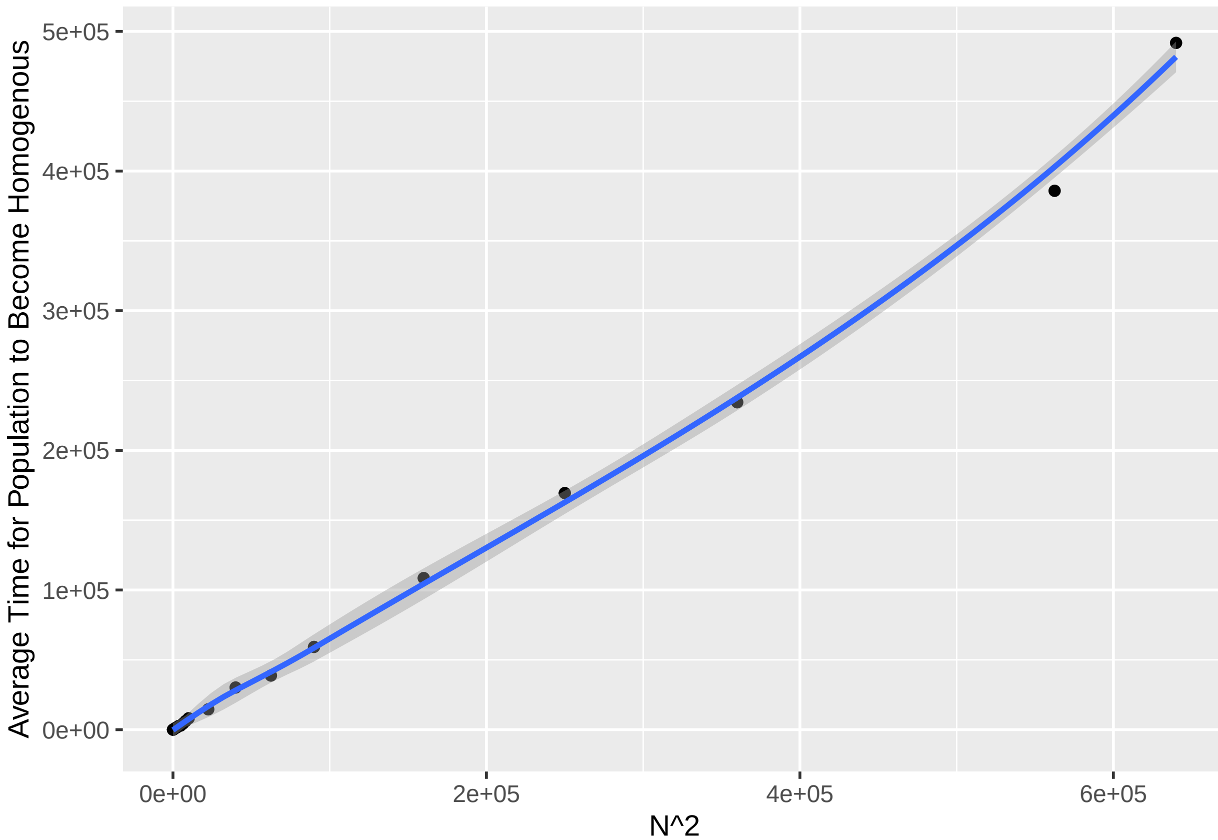
The fact that every simulation becomes homogeneous does not affect what I wrote for Part A; it is true that the probability of increasing or decreasing the number of red cells (as long as the amount of red cells is not N or 0) is the same, thus the average should not change.

The populations eventually homogenize because there is always a chance (albeit sometimes very small) that the number of red cells will either increase or decrease, as long as there are 0 or N red cells; thus, if there is enough time, the red cells will either equal 0 or N.

Part E

After running simulations with different N values, here are the results:

```
##          N    N^2 Average.Time
##   [1,]   0      0         0.00
##   [2,]  10    100        66.92
##   [3,]  20    400       265.79
##   [4,]  30    900       633.67
##   [5,]  40   1600      1189.50
##   [6,]  50   2500      1644.90
##   [7,]  60   3600      2591.40
##   [8,]  70   4900      2947.80
##   [9,]  80   6400      4365.10
##  [10,]  90   8100      6225.60
##  [11,] 100  10000      8144.40
##  [12,] 150  22500     14565.00
##  [13,] 200  40000     30197.00
##  [14,] 250  62500     38762.00
##  [15,] 300  90000     59257.00
##  [16,] 400 160000    108550.00
##  [17,] 500 250000    169420.00
##  [18,] 600 360000    234430.00
##  [19,] 750 562500    385860.00
##  [20,] 800 640000    491760.00
```


5e+05
4e+05
3e+05
2e+05
1e+05
0e+00
Average Time for Population to Become Homogenous
0e+00
2e+05
4e+05
6e+05
N^2