MATH 142: Mathematical Modeling, Quiz 2

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Produced on Tuesday, May. 05 2020 @ 10:18:31 AM

Question 1

Part A

$\frac{de}{dt} = -\alpha e\sigma_1 + \beta_1 p_1 - \gamma e\sigma_2 + \delta p_2$

$\frac{dp_1}{dt} = \alpha e\sigma_1 - \beta p_1$

$\frac{dp_2}{dt} = \gamma e\sigma_2 - \delta p_2$

Part B

We notice that $\frac{de}{dt} + \frac{dp_1}{dt} + \frac{dp_2}{dt} = 0$, which means $e + p_1 + p_2 = P_0$, where P_0 is a constant. We rearrange the equation to get $e = P_0 - p_1 - p_2$. With that, we can reduce the sysetm to two differential equations:

$\frac{dp_1}{dt} = \alpha\sigma_1(P_0 - p_1 - p_2) - \beta p_1$

$\frac{dp_1}{dt} = \gamma\sigma_2(P_0 - p_1 - p_2) - \delta p_2$

Part C

The dimensions of the variables and parameters are shown below.

variable	dimension	parameter	dimension
P_1, P_2	concentration	P_0, σ_1, σ_2	concentration
t	time	α, γ	$\frac{1}{\text{concentration} \cdot \text{time}}$
		β, δ	$\frac{1}{\text{time}}$

Let $\rho_1 = \frac{p_1}{P_0}$ and $\rho_2 = \frac{p_2}{P_0}$. Rearranging, we get $p_1 = P_0\rho_1$ and $p_2 = P_0\rho_2$.

Also, let $\tau = \alpha\sigma_1 t$. Differentiate to get $\frac{d\tau}{dt} = \alpha\sigma_1$

Plugging $p_1 = P_0\rho_1$ and $p_2 = P_0\rho_2$ in to $\frac{dp_1}{dt}$:

$$P_0\frac{d\rho_1}{dt} = \alpha\sigma_1(P_0 - P_0\rho_1 - P_0\rho_2) - \beta P_0\rho_1$$

Dividing both sides by P_0:

$$\frac{d\rho_1}{dt} = \alpha\sigma_1(1 - \rho_1 - \rho_2) - \beta\rho_1$$

Using chaing rule:

$$\frac{d\rho_1}{dt} = \frac{d\rho_1}{d\tau}\frac{d\tau}{dt} = \alpha\sigma_1\frac{d\rho_1}{d\tau} = \alpha\sigma_1(1 - \rho_1 - \rho_2) - \beta\rho_1$$

Dividing both sides by $\alpha\sigma_1$:

$$\frac{d\rho_1}{d\tau} = (1 - \rho_1 - \rho_2) - \frac{\beta}{\alpha\sigma_1}\rho_1 \tag{1}$$

Plugging $p_1 = P_0\rho_1$ and $p_2 = P_0\rho_2$ in to $\frac{dp_2}{dt}$:

$$P_0\frac{d\rho_2}{dt} = \gamma\sigma_2(P_0 - P_0\rho_1 - P_0\rho_2) - \delta P_0\rho_2$$

Dividing both sides by P_0:

$$\frac{d\rho_2}{dt} = \gamma\sigma_2(1 - \rho_1 - \rho_2) - \delta\rho_2$$

Using chain rule:

$$\frac{d\rho_2}{dt} = \frac{d\rho_2}{d\tau}\frac{d\tau}{dt} = \alpha\sigma_1\frac{d\rho_2}{d\tau} = \gamma\sigma_2(1 - \rho_1 - \rho_2) - \delta\rho_2$$

Dividing both sides by $\alpha\sigma_1$:

$$\frac{d\rho_2}{d\tau} = \frac{\gamma\sigma_2}{\alpha\sigma_1}(1 - \rho_1 - \rho_2) - \frac{\delta}{\alpha\sigma_1}\rho_2 \tag{2}$$

From (1) and (2), the nondimensionalized system is

$$\frac{d\rho_1}{d\tau} = (1 - \rho_1 - \rho_2) - \kappa\rho_1,$$

$$\frac{d\rho_2}{d\tau} = \lambda(1 - \rho_1 - \rho_2) - \mu\rho_2,$$

where $\rho_1 = \frac{p_1}{P_0}, \rho_2 = \frac{p_2}{P_0}, \tau = \alpha\sigma_1 t, \kappa = \frac{\beta}{\alpha\sigma_1}, \lambda = \frac{\gamma\sigma_2}{\alpha\sigma_1},$ and $\mu = \frac{\delta}{\alpha\sigma_1}$

Question 2

Part A

When developing the differential equations, we are interested in the different ways the population can change.

For rodents: They have a birth rate b_ρ, which means the population will increase by $b_\rho \rho$ rodents/year just from births. The rodents can die from natural causes at rate m_ρ, which means the population will decrease by $m_\rho \rho$ rodents/year from natural causes. Additionally, the rodents interact with cats at rate β, which means the population will decrease by $\beta \rho c$ rodents/year from their interactions with cats. Finally, α_ρ rodents enter per year, which means the population will increase by α_ρ rodents/year by them being added.

For cats: They have a birth rate of b_c, which means the population will increase by $b_c c$ cats/year just from births. Their death rate is m_c, which means the population will decrease by $m_c c$ cats/year from deaths. Lastly, α_c are added, which means the population will be increasing by α_c cats/year from being added.

From above, the system of differential equations are:

$$\frac{d\rho}{dt} = b_\rho \rho - m_\rho \rho - \beta \rho c + \alpha_\rho,$$

$$\frac{dc}{dt} = b_c c - m_c c + \alpha_c$$

Part B

(i)

Setting the differential equation to 0:

$$rn(1-n) - \frac{nh_{max}}{1+n} = 0$$

Multiplying both sides by $(1+n)$:

$$rn(1-n)(1+n) - nh_{max} = 0$$

Doing some factoring:

$$n(r(1-n^2) - h_{max}) = 0$$

Simplifying:

$$n(r - rn^2 - h_{max}) = 0$$

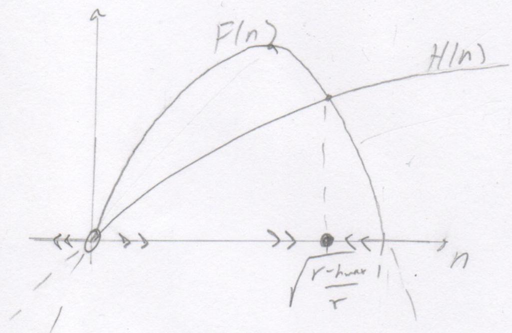
We can see that the fixed points are:

$$n^* = 0, \pm\sqrt{\frac{r - h_{max}}{r}}$$

(ii)

2bii. let $F(n) = rn(1-n)$ and $H(n) = \frac{h_{max} n}{1+n}$

when $h_{max} < r$

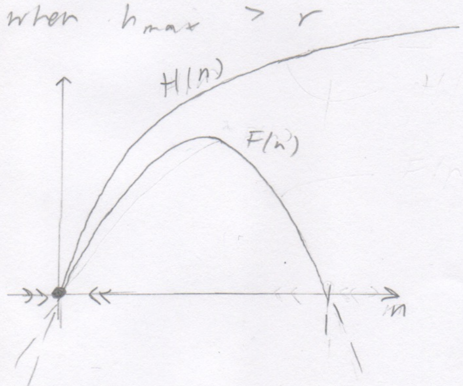


two fixed points $\in [0, 1]$

0 is unstable

$\sqrt{\frac{r-h_{max}}{r}}$ is stable

when $h_{max} > r$



only one fixed point $\in [0, 1]$

0 is stable

(iii)

When $h_{max} > r$, $\frac{dn}{dt} < 0$, which means the proportion of the carrying capacity of the hare population will be decreasing until the proportion reaches 0. Since $n = \frac{N}{K} = 0$, we can see that $N = 0$, meaning the hares have gone extinct.