Math 142 - Homework 1
Due Wednesday, April 8th

(1) The following questions will give you some practice on how to write word equations.

(a) You are building a mathematical model for the population of kakapo (an endangered species of flightless parrots that live in New Zealand). These slow-moving birds are endangered because they are eaten by predators such as rats. Write down a word equation relating the population N_k, in a specific wildlife reserve, in one year to the population N_{k+1} in the next year. Your equation will include the following terms:

- Number of kakapo born in the wild
- Number of kakapo removed for captive breeding
- Number of kakapo reintroduced into the wild from captive breeding
- Number of kakapo killed by rats

(b) Recurrence equation models can be used to model a wider range of phenomena than just population growth. One application area is pharmacokinetics: the study of how medications move through the human body. Suppose we are building a model for how a medication like ibuprofen ("Advil") passes from a patient's gut into their blood, and from there enters their cells or is eliminated from their body by their kidneys. Write down a word equation for the amount of ibuprofen in the patient's blood t hours after they start a course of treatment with ibuprofen. The word equation should include the following terms:

- Amount of ibuprofen in the patients blood after $t-1$ hours
- Amount of ibuprofen absorbed from the gut into the blood
- Amount of ibuprofen absorbed into tissues
- Amount of ibuprofen filtered from blood by kidneys
- Amount of ibuprofen reabsorbed into blood from tissues

(2) **Saving the Kakapo** You are modeling the size of the population of the kakapo in an island reserve in New Zealand. You want to use the mathematical model to predict the size of the population. The data in this question are taken from *Elliot et al. (2001)*.

(a) You start by writing a word equation relating the population size N_k in year k, that is, k years after the study began, to the population size N_{k+1} in the next year:

$$N_{k+1} = N_k + \text{\# of birds born in one year} - \text{\# of birds that die in one year}$$

We will now derive terms for the birth and death rates.

(i) To estimate the number of birds born, assume that half of the birds are female. A female bird lays one egg every four years. However, because of the large numbers of predators (mostly rats), only 29% of hatchlings survive the first year. Use this data to show that the number of births is $0.03625 \cdot N_k$.

(ii) Kakapo life expectancy is not well understood, but we will assume that they live around 50 years. That is, in a given year, one in 50 kakapo will die. What is the corresponding number of deaths?

(iii) Assume that the starting population size on this island is 50 birds (i.e., $N_0 = 50$). Calculate the predicted population size over the next five years (i.e., calculate N_1, N_2, ..., N_5).

(iv) When (if ever) will the population size reach 100 birds? What about 200 birds? (You will find it helpful to derive an explicit formula for the size of the population N_k).

(b) Using your model from part (a), you want to evaluate the effectiveness of two different conservation strategies:
(Strategy 1) If the kakapo are given supplementary food, then they will breed more frequently. If given supplementary food, rather than laying an egg every four years, a female will lay an egg every three years.
(Strategy 2) By hand-rearing kakapo chicks, it is possible to increase their one-year survival rate from 29% to 50%.

(i) Write down a recurrence equation for the population size N_k if strategy 1 is implemented. Assuming $N_0 = 50$, calculate N_1, N_2, ..., N_5.

(ii) Write down a recurrence equation for the population size N_k if strategy 2 is implemented. Assuming $N_0 = 50$, calculate N_1, N_2, ..., N_5.

(iii) Which conservation strategy gives the biggest increase in population size?

(3) You are trying to model the growth of a population of yeast cells, and to compare to real data. The real data we will use is from a paper by Carlson. You should start by downloading the data from the BioQuest website:

`https://bioquest.org/numberscount/data-details/?product_id=31395`

The data consist of a count of yeast cells (the count is measured in terms of millions of cells per mL of the growth medium) as a function of time.

(a) Start by discussing how, if you watched a group of cells for some interval of time, you could estimate their mortality (death) rate m.

(b) Show by plotting the data that the population size in Carlson's experiment does not grow exponentially. Why doesn't it?

(c) In fact, in the first few hours of the experiment, the population growth is approximately exponential. Explain how to show this from the data.

(d) Use the experimental data to estimate the growth rate R_0 of the yeast cells when the population is growing exponentially.

(4) **Growth of the European settler population in the United States.** In this exercise, we will develop a mathematical model for the population of European settlers in the United States. We will build a mathematical model to predict the year-to-year changes in the European settler population, starting in 1630, where the population size was 4646 individuals.

Here is some data that will be helpful to you: at the start of our study, life expectancy in this population was 35. On average, each woman had 6 children. We will start by determining how to use these data to fit the parameters b (birth rate) and m (mortality rate) in the following equation:

$$N_{k+1} = (1 + b - m)N_k,$$

where k is the number of years since 1630.

(a) Birth rate: assume for simplicity's sake that half of the population is female and that women in the population have 6 children distributed randomly over 35 years (their life expectancy). Explain how this translates into a per-person yearly birthrate of $b = \frac{3}{35}$.

(b) Let's interpret the life expectancy data as saying that a randomly chosen individual has a likelihood of one-half of reaching 35 years old. Equivalently, if we start with a large cohort of individuals (and ignore births, to only follow the individuals initially present), then if the initial population size is N_0, we expect that after 35 years, the population size will have dropped to $\frac{N_0}{2}$. Find the corresponding mortality rate m.

(c) A lot of assumptions and simplifications go into our model. Describe two main issues with the simplifications in our model (multiple answers are possible here). Discuss some ideas about how you might change the model to deal with these simplifications.

(d) Compare your model (the original model from parts a and b) with the real US settler population size data on the Wikipedia page:

`https://en.wikipedia.org/wiki/Demographic_history_of_the_United_States`

The data is in the table marked "Historical Population." Make a plot of your data with $\Delta t = 1$, and compare it with the real data for the years 1630, 1640, 1650, 1660, 1670, 1680 (i.e., plot your predicted population size as a function of time with the real data on the same axes).

(e) Our model does not include the contribution of immigration and emigration: i.e., of new settlers arriving in the US or of settlers leaving the US. Suppose that settlers arrive at a constant rate I_0 per year. The recurrence equation governing population growth now becomes:

$$N_{k+1} = (1 + b - m)N_k + I_0.$$

Try different values for the rate of immigration: $I_0 = 200, 500$, and 1000. For each value of I_0, plot the predicted population growth curve and the real data on the same graph. Use the same range of years as in part (d). For which value of I_0 do you see the best agreement with the real data?

(5) Suppose that we are measuring the concentration (in mg/liter) of ibuprofen in a patient's blood every hour. Consider:

- When the patient takes a pill, it takes an hour be absorbed in the blood.
- Each pill increases concentration of ibuprofen by 40 mg/liter.
- Ibuprofen has first-order elimination kinetics - 24.25% of ibuprofen in the blood is eliminated each hour.
- The patient takes a pill every 6 hours. They take the first pill at $t = 0$.

(a) The word equation for the concentration of ibuprofen c_t after t hours is

$$c_{t+1} = c_t + \text{amt. absorbed in blood} - \text{amt. eliminated in one hour}$$

Starting with the word equation, find the concentration of ibuprofen in the blood for the first seven hours (i.e., find $c_0, c_1, c_2, ..., c_7$).

(b) Write down a recurrence equation for C_n, the concentration of ibuprofen one hour after **pill** n is taken (note: not the same equation as in part a; this is a different quantity).

(c) Using your equation from part (b), compute C_1, C_2, ..., C_5.

(d) You should see that C_n appears to be reaching a limiting concentration as n increases. Find this limiting concentration by finding the fixed point of your equation in part (b).

(e) We can find the limiting concentration using an alternate method! Derive an explicit formula for C_n (*hint:* write down C_2, C_3, and C_4 in terms of C_1, and look for a pattern). Then take $\lim_{n\to\infty} C_n$.