MATH 142: Mathematical Modeling, Homework 7

Darren Tsang 405433124

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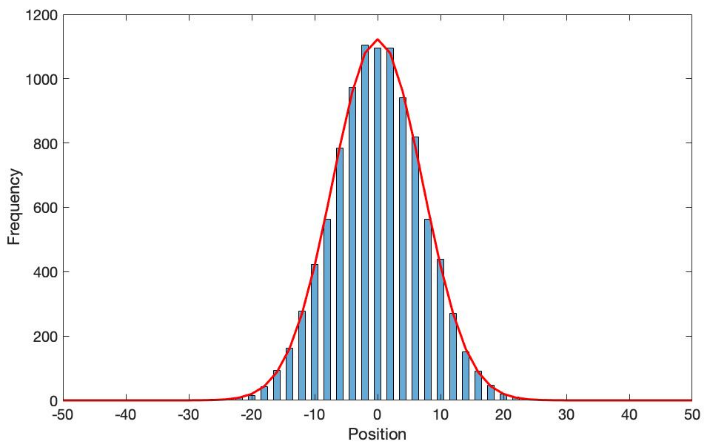
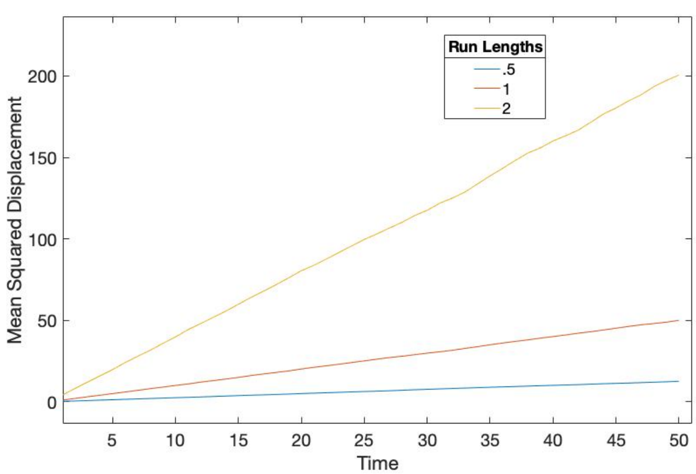
Question 1

Part A



There are gaps in the histogram because your position can only be a factor of l; for example, if $l = 2$ you can't be at position 1.5 or -2.5.

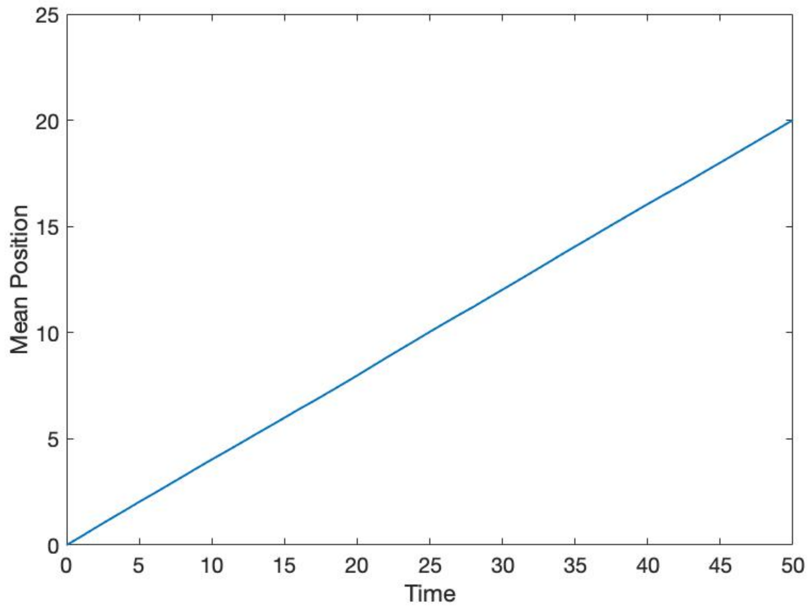
Part B



From the graph above, I can see that the mean squared displacement depends linearly on time because the 3 lines are linear. Futhermore, I can see that the mean squared displacement depends quadratically on run length because the rate of increase gets larger (quadratically) as the run length is getting larger (for example, the space between the three lines at t = 10 is much smaller than the space between the three lines at t = 50.

Question 2

Part A



Part B

The average distance traveled for one time step is $(p-(1-p)) = (2p-1)l$. Thus, after k time steps, $\overline{x_k} = k(2p-1)l$

When we plug in the given values, we get $\overline{x_{50}} = 50(\frac{2*7}{10} - 1)1 = 20$, which agrees with the simulation.

Question 3

Part A

We are given

$$\rho(x,t) = \frac{1}{\sqrt{4\pi Dt}} exp(\frac{-(x-x_0)^2}{4Dt})$$

and want to show

$$\frac{\partial \rho}{\partial t} = D\frac{\partial^2 \rho}{\partial x^2}, \ -\infty < x < \infty$$

To find $\frac{\partial \rho}{\partial t}$, we start with:

$$\frac{\partial \rho}{dt} = \frac{-4\pi D}{2}(4\pi Dt)^{-3/2} exp(\frac{-(x-x_0)^2}{4Dt}) + \frac{1}{(4\pi Dt)^{1/2}} exp(\frac{-(x-x_0)^2}{4Dt})\frac{(x-x_0)^2}{4Dt^2}$$

Factoring out $exp(\frac{-(x-x_0)^2}{4Dt})$:

$$\frac{\partial \rho}{dt} = exp(\frac{-(x-x_0)^2}{4Dt})(\frac{-4\pi D}{2}(4\pi Dt)^{-3/2} + \frac{1}{(4\pi Dt)^{1/2}}\frac{(x-x_0)^2}{4Dt^2})$$

Simplifying:

$$\frac{\partial \rho}{\partial t} = exp(\frac{-(x-x_0)^2}{4Dt})(\frac{-1}{4\pi^{1/2}D^{1/2}t^{3/2}} + \frac{(x-x_0)^2}{8D^{3/2}\pi^{1/2}t^{5/2}})$$

Simplifying futher:

$$\frac{\partial \rho}{\partial t} = exp(\frac{-(x-x_0)^2}{4Dt})(\frac{(x-x_0)^2 - 2Dt}{8D^{3/2}\pi^{1/2}t^{5/2}}) \tag{1}$$

To find $\frac{\partial \rho}{\partial x}$, we start with:

$$\frac{\partial \rho}{\partial x} = \frac{1}{(4\pi Dt)^{1/2}}(exp(\frac{-(x-x_0)^2}{4Dt})(\frac{-2(x-x_0)}{4Dt}))$$

Simplifying:

$$\frac{\partial \rho}{\partial x} = \frac{1}{(4\pi Dt)^{1/2}}(-exp(\frac{-(x-x_0)^2}{4Dt})(\frac{(x-x_0)}{2Dt}))$$

To find $\frac{\partial^2 \rho}{\partial x^2}$, take the derivative of $\frac{\partial \rho}{\partial t}$:

$$\frac{\partial^2 \rho}{\partial x^2} = \frac{-1}{(4\pi Dt)^{1/2}}(exp(\frac{-(x-x_0)^2}{4Dt})(\frac{-2(x-x_0)}{4Dt})(\frac{x-x_0}{2Dt}) + exp(\frac{-(x-x_0)^2}{4Dt})\frac{1}{2Dt})$$

Factoring:

$$\frac{\partial^2 \rho}{\partial x^2} = \frac{-1}{2Dt(4\pi Dt)^{1/2}} exp(\frac{-(x-x_0)^2}{4Dt})(\frac{-(x-x_0)^2}{2Dt} + 1)$$

Distributing and simplifying:

$$\frac{\partial^2 \rho}{\partial x^2} = exp(\frac{-(x-x_0)^2}{4Dt})(\frac{(x-x_0)^2 - 2Dt}{(2Dt)^2(4\pi Dt)^{1/2}})$$

Simplifying:

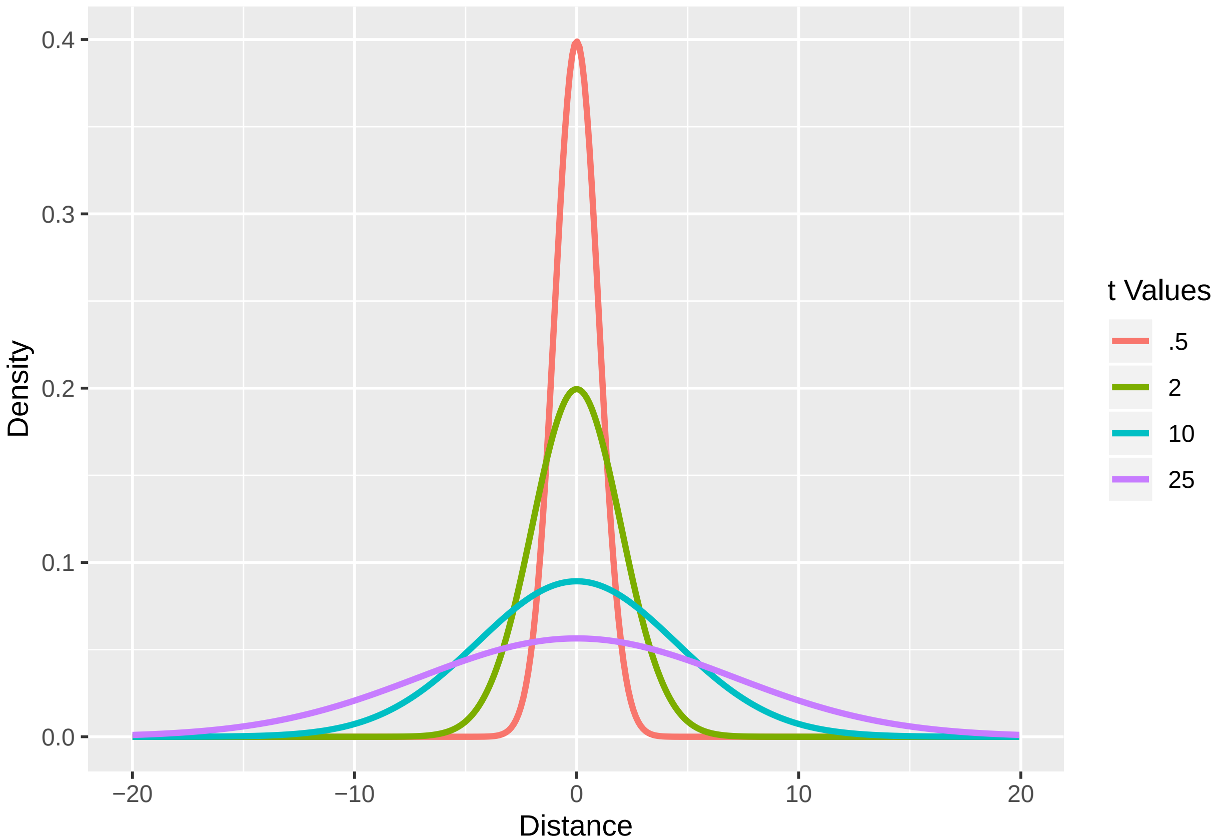
$$\frac{\partial^2 \rho}{\partial x^2} = exp(\frac{-(x-x_0)^2}{4Dt})(\frac{(x-x_0)^2 - 2Dt}{8D^{5/2}t^{5/2}\pi^{1/2}}) \tag{2}$$

Using (1) and (2):

$$D\frac{\partial^2 \rho}{\partial x^2} = exp(\frac{-(x-x_0)^2}{4Dt})(\frac{(x-x_0)^2 - 2Dt}{8D^{3/2}t^{5/2}\pi^{1/2}}) = \frac{\partial \rho}{\partial t},$$

as desired. Additionally, x_0 is the starting position of the bacteria.

Part B



As t increases, the curve become less pointy, which means the spread of the density increases as t increases.

Question 4

Part A

We start with:

$$P(x,t+\Delta t) = pP(x-\Delta x,t) + (1-p)P(x+\Delta x,t)$$

Subtracting $P(x,t)$ from both sides:

$$P(x,t+\Delta t) - P(x,t) = pP(x-\Delta x,t) + (1-p)P(x+\Delta x,t) - P(x,t)$$

Factoring out a Δt on the left and a $\frac{1}{2}$ on the right:

$$\Delta t(\frac{P(x,t+\Delta t) - P(x,t)}{\Delta t}) = \frac{1}{2}(2pP(x-\Delta x,t) + 2(1-p)P(x+\Delta x,t) - 2P(x,t))$$

Splitting up the right side:

$$\Delta t(\frac{P(x,t+\Delta t) - P(x,t)}{\Delta t}) = \frac{1}{2}(P(x-\Delta x,t) - 2P(x,t) + P(x+\Delta x,t)) + \frac{1}{2}(P(x-\Delta x,t)(2p-1) - P(x+\Delta x,t)(2p-1))$$

Factoring out $(2p-1)$ from the right side:

$$\Delta t(\frac{P(x,t+\Delta t) - P(x,t)}{\Delta t}) = \frac{1}{2}(P(x-\Delta x,t) - 2P(x,t) + P(x+\Delta x,t)) + \frac{2p-1}{2}(P(x-\Delta x,t) - P(x+\Delta x,t))$$

Multiplying the right side by $\frac{\Delta x^2}{\Delta x^2}$:

$$\Delta t(\frac{P(x,t+\Delta t) - P(x,t)}{\Delta t}) = \frac{\Delta x^2}{2}(\frac{P(x-\Delta x,t) - 2P(x,t) + P(x+\Delta x,t)}{\Delta x^2}) + \frac{(2p-1)\Delta x^2}{2}(\frac{P(x-\Delta x,t) - P(x+\Delta x,t)}{\Delta x^2})$$

Dividing both sides by Δt:

$$\frac{P(x,t+\Delta t) - P(x,t)}{\Delta t} = \frac{\Delta x^2}{2\Delta t}(\frac{P(x-\Delta x,t) - 2P(x,t) + P(x+\Delta x,t)}{\Delta x^2}) + \frac{(2p-1)\Delta x^2}{2\Delta t}(\frac{P(x-\Delta x,t) - P(x+\Delta x,t)}{\Delta x^2})$$

Rearranging the last term on the right:

$$\frac{P(x,t+\Delta t) - P(x,t)}{\Delta t} = \frac{\Delta x^2}{2\Delta t}(\frac{P(x-\Delta x,t) - 2P(x,t) + P(x+\Delta x,t)}{\Delta x^2}) - \frac{(2p-1)\Delta x^2}{\Delta t \Delta x}(\frac{P(x+\Delta x,t) - P(x-\Delta x,t)}{2\Delta x})$$

Taking $\lim_{\Delta t, \Delta x \to 0}$:

$$\frac{\partial P}{\partial t} = D\frac{\partial^2 P}{\partial x^2} - u\frac{\partial P}{\partial x},$$

where $D = \frac{\Delta x^2}{2\Delta t}$ and $u = \frac{(2p-1)\Delta x}{\Delta t}$.

Then, using the same reasoning from class, we can rewrite as:

$$\frac{\partial \rho}{\partial t} = D\frac{\partial^2 \rho}{\partial x^2} - u\frac{\partial \rho}{\partial x},$$

where $D = \frac{\Delta x^2}{2\Delta t}$ and $u = \frac{(2p-1)\Delta x}{\Delta t}$, as desired.

Part B

From above, we can see that $u = \frac{(2p-1)\Delta x}{\Delta t}$; the dimensions of u are $\frac{\text{distance}}{\text{time}}$.