Math 142 - Homework 7
Due Wednesday, May 20th

(1) **Discrete (unbiased) random walk**
The code provided with this assignment (randomWalk.m) simulates a discrete random walk. Use it to answer the following questions.

(a) Run the code for a population of 10,000 bacteria and total time = 50.

(i) Print out the histogram of positions of bacteria at the end of the simulation, compared with the predictions from the binomial distribution.

(ii) Explain why there are gaps in the histogram between the bars.

(b) Run the code for the run lengths $l = 0.5$, $l = 1$, and $l = 2$. Then plot the mean squared displacement as a function of time for each run length, all on the same plot (make sure to include a legend). By referring to your plot, verify that the mean squared displacement depends linearly on time and quadratically on run length: that is, $\overline{x_k^2} = kl^2$, where k is the number of time steps.

(2) **Discrete biased random walk**
Suppose that the bacteria are attracted to a food source to their right, so they are more likely to make a "run" to the right than to the left. We will assume that the probability of moving right remains constant in time, but is now greater than 1/2. This is a very simple model of chemotaxis, the movement of a cell or organism in the direction of increasing or decreasing concentrations of a chemical signal. (Note: this is not really how bacteria chemotax! Ask me if you are curious.)

(a) Run the provided code again, but this time with probability of moving right $p = 7/10$ (and hence, probability of moving left $= 1 - p = 3/10$). Use 10,000 bacteria, total time $= 50$, and $l = 1$. Plot the mean position of bacteria $\overline{x}$ (note: **not** the mean squared displacement) as a function of time.

(b) Recall that the mean position $\overline{x_k}$ of bacteria is always 0 for an unbiased random walk. What is the mean position $\overline{x_k}$ for a biased random walk, in terms of the time step k, the run length l, and the probability of moving right p? Explain your reasoning. Then, verify that your predicted form for $\overline{x_k}$ works by plugging in $p = 7/10$ and $l = 1$ and comparing it with the simulation.

(3) **Continuous (unbiased) random walk (the diffusion equation)**
In class (on Monday 5/18), we derived that the density of swimming bacteria, $\rho(x,t)$, satisfies the diffusion equation:

$$\frac{\partial \rho}{\partial t} = D\frac{\partial^2 \rho}{\partial x^2}, \quad -\infty < x < \infty. \tag{1}$$

(a) Verify that

$$\rho(x,t) = \frac{1}{\sqrt{4\pi Dt}} \exp\left(-\frac{(x-x_0)^2)}{4Dt}\right)$$

is a solution of the diffusion equation (1) for any constant x_0. What does x_0 represent?

(b) Using Matlab or other graphing software, plot $\rho(x,t)$ for $D = 1$ and $x_0 = 0$, for $t = 0.5,\ 2,\ 10,\ 25$. Use $-20 \leq x \leq 20$ for the bounds on x, and plot all four curves on the same graph (don't forget to include a legend and axis labels). What happens as t increases?

(4) **Continuous biased random walk (the advection-diffusion equation)**
Let's derive the partial differential equation for a biased random walk. Begin, as we did in class, with the following recursive equation for the probability $P(x,t)$ of being at position x at time t:

$$P(x, t + \Delta t) = pP(x - \Delta x, t) + (1 - p)P(x + \Delta x, t). \tag{2}$$

For an unbiased random walk, $p = 1 - p = 1/2$, while for a biased random walk, $p \neq 1/2$.

(a) Show that for a biased random walk, we obtain the **advection-diffusion equation**

$$\frac{\partial \rho}{\partial t} = \underbrace{D\frac{\partial^2 \rho}{\partial x^2}}_{\text{diffusion}} - \underbrace{u\frac{\partial \rho}{\partial x}}_{\text{advection}}.$$

Hint for the derivation: Manipulate equation (2) to look like the definitions of various partial derivatives (see below), and then take $\lim_{\Delta t, \Delta x \to 0}$.

$$\frac{\partial P}{\partial t} = \lim_{\Delta t \to 0}\left(\frac{P(x, t + \Delta t) - P(x,t)}{\Delta t}\right),$$

$$\frac{\partial P}{\partial x} = \lim_{\Delta x \to 0}\left(\frac{P(x + \Delta x, t) - P(x - \Delta x, t)}{2\Delta x}\right),$$

$$\frac{\partial^2 P}{\partial x^2} = \lim_{\Delta x \to 0}\left(\frac{P(x + \Delta x, t) - 2P(x,t) + P(x - \Delta x, t)}{\Delta x^2}\right).$$

(b) This equation contains a diffusive term, representing the spreading out of bacteria, and an advective term, representing the flow (transport) of bacteria. What is u in terms of Δx, Δt, and p, and what is the dimension of u?