MATH 142: Mathematical Modeling, Homework 3

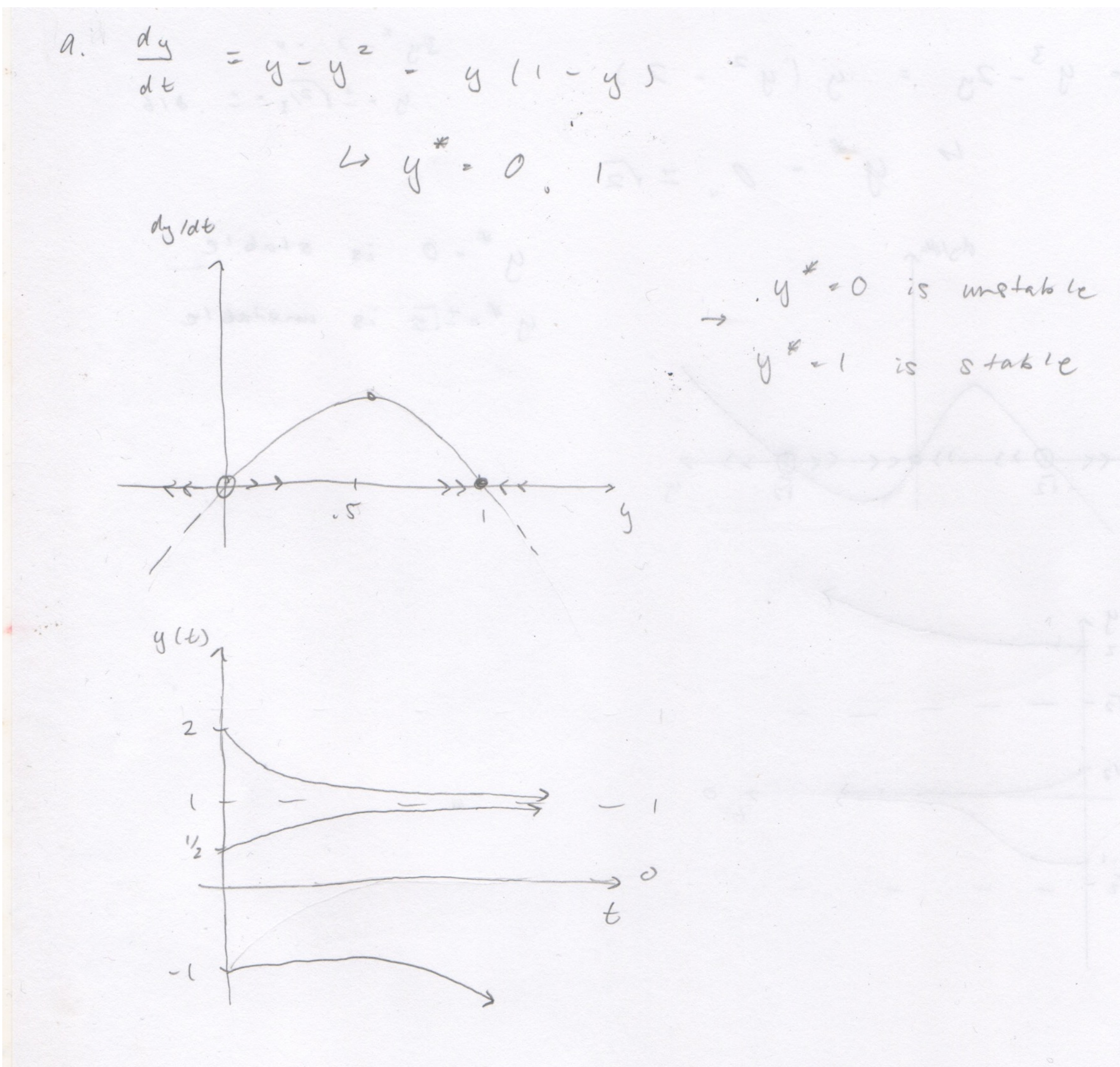
Darren Tsang 405433124

Produced on Wednesday, Apr. 22 2020 @ 07:42:32 PM

```r
library(readxl) #load this library to read in xls files
library(ggplot2) #load this library for plotting
```

Question 1

Part A



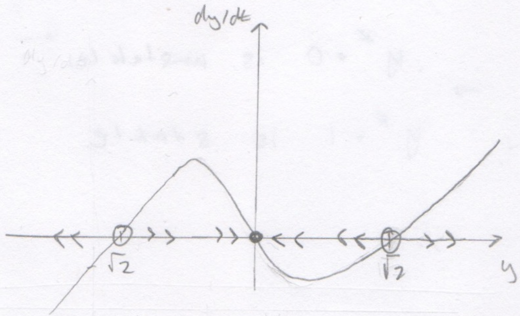
Part B

b. $\frac{dy}{dt} = y^3 - 2y = y(y^2 - 2)$

$\hookrightarrow y^* = 0, \pm\sqrt{2}$

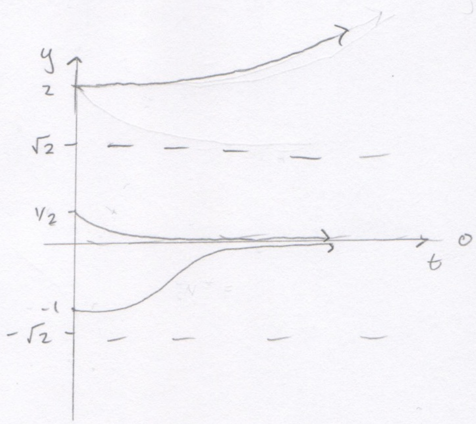
$3y^2 - 2 = 0$

$y = \pm\sqrt{2/3} = \pm .816$



$y^* = 0$ is stable

$y^* = \pm\sqrt{2}$ is unstable

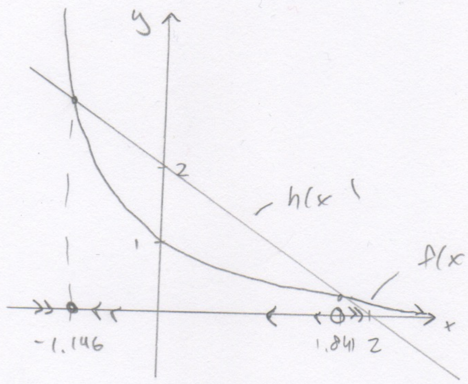


Question 2

Part A

2a. $\frac{dx}{dt} = e^{-x} + x - 2 = e^{-x} - (2 - x)$

$f(x) = e^{-x}$, $h(x) = 2 - x$



There are two fixed points:

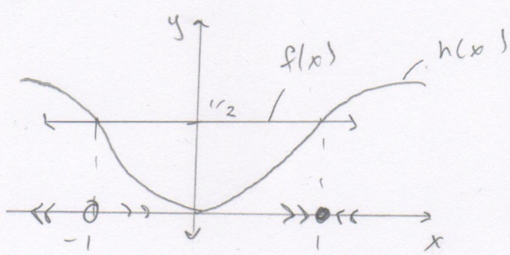
1.841 is unstable

-1.146 is stable

Part B

2b. $\frac{dx}{dt} = \frac{1}{2} - \frac{x^2}{x^2+1}$

$f(x) = \frac{1}{2}$, $h(x) = \frac{x^2}{x^2+1}$



There are two fixed points:

1 is stable

-1 is unstable

Question 3

$$N(t+\Delta t) = \frac{1+r_0\Delta t}{1+\alpha\Delta t N(t)}N(t)$$

Part A

We can find the fixed points by solving the following equation: $N(t) = \frac{1+r_0}{1+\alpha N(t)}N(t)$

$$N(t)(1+\alpha N(t)) = N(t)(1+r_0)$$
$$N(t)(1-1-r_0+\alpha N(t)) = N(t)(\alpha N(t) - r_0) = 0$$
$$N^* = 0, \frac{r_0}{\alpha}$$

Part B

3b. $N(t+\Delta t)(1+\alpha\Delta t N(t)) = N(t) + r_0 \Delta t N(t)$

$N(t+\Delta t) + \alpha \Delta t N(t) N(t+\Delta t) = N(t) + r_0 \Delta t N(t)$

$\frac{N(t+\Delta t)}{\Delta t} + \alpha N(t) N(t+\Delta t) = \frac{N(t)}{\Delta t} + r_0 N(t)$

$\frac{N(t+\Delta t)}{\Delta t} - \frac{N(t)}{\Delta t} = r_0 N(t) - \alpha N(t) N(t+\Delta t)$

limit of both sides:

$\frac{dN}{dt} = r N(t) - \alpha N(t) \cdot N(t)$

$= N(t)(r - \alpha N(t))$

$= r N(t)\left(1 - \frac{\alpha}{r_0} N(t)\right)$

$\frac{dN}{dt} = r N(t)\left(1 - \frac{N(t)}{K}\right)$

Part C

We can find the fixed points of the logistic model by setting the model equal to 0.

$$\frac{dN}{dt} = rN(1 - \frac{N}{K}) = 0$$
$$N^* = 0, K$$

where K is defined as $\frac{r_0}{\alpha}$ in the equation $r_0 - \alpha N(t)$. Thus, the two sets of fixed points are indeed the same.

Question 4

Part A

(i) When we solve $N(3) = e^{3r}N(0)$, we get $r = \frac{ln\frac{N(3)}{N(0)}}{3} = \frac{ln\frac{47.2}{9.6}}{3} = .5309$

(ii) I notice that the population seems to get really close to 662. Thus, I will estiamte the carrying capacity K to be 662.

Part B

The logistic differential equation is $\frac{dN}{dt} = .5309N(1 - \frac{N}{662})$, $N(0) = 9.6$.

Part C

Isolating the variables:

$$\frac{dN}{N(1-\frac{N}{K})} = rdt$$

Further simplifying:

$$\frac{dN}{N(K-N)} = \frac{r}{K}dt$$

We use partial fraction decomposition before integrating:

$$\frac{1}{K}\int(\frac{1}{N} + \frac{1}{K-N})dN = \int\frac{r}{K}dt$$

Taking the integral (no absolute value needed becuase the population will never be greater than the carrying capcacity):

$$ln(N) - ln(K-N) = rt + c$$

Using the initial conidition of $N(0) = 9.6$:

$$c = ln(9.6) - ln(662 - 9.6) = -4.1289$$

The equation is now:

$$ln(N) - ln(K-N) = rt - 4.1289$$

Solving for N:

$$ln(\frac{N}{K-N}) = rt - 4.1289$$

$$\frac{N}{K-N} = e^{rt-4.1289}$$

$$N = e^{rt-4.1289}(K-N)$$

$$N(1 + e^{rt-4.1289}) = Ke^{rt-4.1289}$$

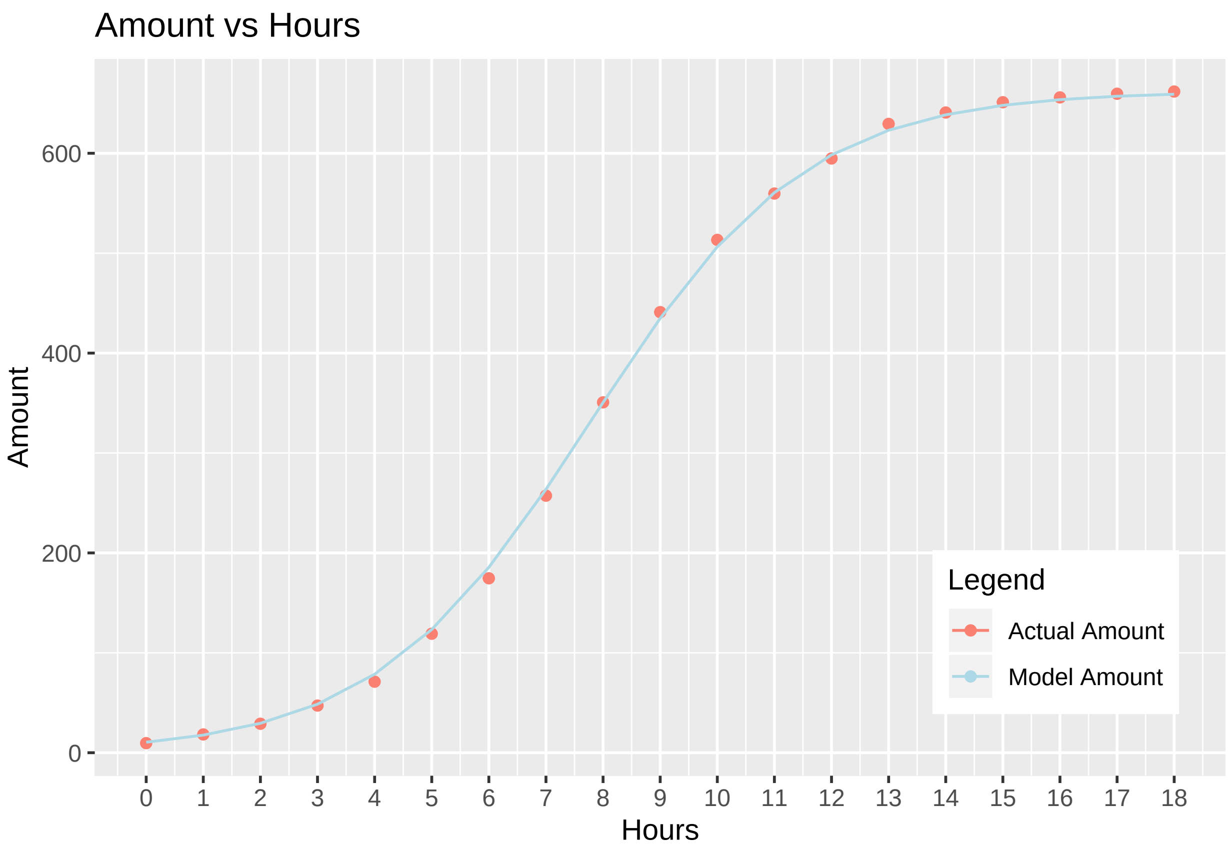
$$N(t) = \frac{Ke^{rt-4.1289}}{1+e^{rt-4.1289}} = \frac{662e^{.5309t-4.1289}}{1+e^{.5309t-4.1289}}$$

Part D

```r
cells <- read_excel("31395_Carlson_Yeast_Data.xls")

K <- 662
r <- .5309
c <- 4.1289
model_amount <- (K * exp(r*cells$Hours - c)) / (1 + exp(r*cells$Hours - c))

ggplot() +
  geom_point(aes(x = cells$Hours, y = cells$Amount, color = "Actual Amount")) +
  geom_line(aes(x = cells$Hours, y = model_amount, color = "Model Amount")) +
  scale_x_continuous(breaks = seq(0, 18, 1)) +
  labs(x = "Hours", y = "Amount", title = "Amount vs Hours") +
  scale_colour_manual("Legend",
                      values = c("Actual Amount" = "salmon",
                                 "Model Amount" = "lightblue")) +
  theme(legend.position = c(.85, .2))
```



An exponential growth model means that the population will grow without bounds, which is not realistic because there are not infinite resources. On the other hand, a logistic growth model can account for the carrying capacity. Thus, the logistic growth model follows the actual data points very closely.

Question 5

Part A

$\frac{de}{dt} = -k_f \cdot e \cdot s + k_b \cdot c + k_p \cdot c$

$\frac{ds}{dt} = -k_f \cdot e \cdot s + k_b \cdot c$

$\frac{dc}{dt} = k_f \cdot e \cdot s - k_b \cdot c - k_p \cdot c$

$\frac{dp}{dt} = k_p \cdot c$

Part B

I notice that $\frac{de}{dt} + \frac{dc}{dt} = 0$ and $\frac{ds}{dt} + \frac{dp}{dt} + \frac{dc}{dt} = 0$. We can say $e + c = X$ and $s + p + c = Y$, where X, Y are constants. Then, $c = X - e \rightarrow s + p + X - e = Y \rightarrow e = s + p + X - Y$. Plugging in the above equations, we can simplify the system to two differential equations:

$\frac{ds}{dt} = -k_f(s + p + X - Y)s + k_b(Y - s - p)$

$\frac{dp}{dt} = k_p(Y - s - p)$