Math 142 - Homework 8
Due Wednesday, May 27th

Heat flow on a bounded domain

(1) We discussed how temperature $T(x,t)$ in a one-dimensional bar of length L obeys the diffusion equation

$$\frac{\partial T}{\partial t} = D\frac{\partial^2 T}{\partial x^2}, \quad 0 < x < L,$$

where D is the thermal diffusivity of the bar. We typically refer to the diffusion equation as the **heat equation** in this context. Heat flux $q(x,t)$ is described by **Fourier's law**:

$$q = -k\frac{\partial T}{\partial x},$$

where k is the thermal conductivity of the bar (its ability to conduct heat). Suppose that heat flows into the bar such that the magnitude of the flux β_L is constant at $x = L$. The other end of the bar ($x = 0$) is immersed in a very large pool of cold water, which maintains the temperature of the bar at $T = \alpha_0$.

(a) Why is it important for the assumptions in this problem that the pool of cold water is very large? What would happen to the temperature of the water if there were a small quantity of it instead?

(b) Write down the boundary conditions at $x = 0$ and $x = L$.

(c) Find the equilibrium temperature profile $T_\infty(x)$.

(d) Suppose that $\alpha_0 = 277$ K (Kelvin), $L = 3$ m, and $\beta_L = 4$ W/m^2. Find the equilibrium temperature at $x = L$ when the bar is made out of balsa wood ($k = 0.048$ W/(m $\cdot$ K)) and gold ($k = 327$ W/(m $\cdot$ K)). Then explain (physically, not mathematically) why the wooden bar is heated to a much higher temperature than the gold bar.

(2) You are leading a team to develop Conair's new curling iron. You know that you can model the curling iron as a one-dimensional rod, where the temperature T of the curling iron (in degrees Fahrenheit) satisfies the heat equation

$$\frac{\partial T}{\partial t} = D\frac{\partial^2 T}{\partial x^2}.$$

Here, x is the position along the curling iron (in inches) and t is time (in seconds). Suppose the curling iron will be 10 inches long. When the curling iron is turned on, one end ($x = 0$) will be set at a temperature of 200° Fahrenheit.

Your team is considering three prototypes:

(i). The other end of the curling iron is perfectly insulated.

(ii). The other end of the curling iron maintains a positive temperature gradient of 10 degrees Fahrenheit per inch.

(iii). The other end of the curling iron is fixed at a temperature of 250° Fahrenheit.

(a) Write down the boundary conditions for each prototype.

(b) Which prototype should you choose to ensure that your curling iron generates the highest amount of total heat? Justify your response by finding the equilibrium solution for each prototype.

(c) After market testing your original design, the consumers tell you that the most important feature of the curling iron is actually that there is a uniform temperature along the length of the iron. Based on this feedback, did you choose the best prototype? If not, which should you choose? Justify your answer.

(3) **Reaction-diffusion on a bounded domain**
In the derivation of the one-dimensional heat equation (not shown in this class), there is an assumption made that the lateral surfaces of our thin rod (of length L) are insulated, so that heat cannot enter or leave through the sides of the rod (only through the ends). If this assumption is removed, then Newton's law of cooling applies, and the heat equation becomes the reaction-diffusion equation

$$\frac{\partial T}{\partial t} = D\frac{\partial^2 T}{\partial x^2} - c(T - T_0), \quad 0 < x < L,$$

where T_0 and c are constants (T_0 is the temperature of the surrounding environment and c is the rate of cooling).

Suppose that the temperature of the rod is held constant at both ends. One end of the rod ($x = 0$) has the same temperature as the surroundings, while the other end ($x = L$) has temperature $T_L > T_0$. Find the equilibrium solution $T_\infty(x)$ for these boundary conditions.

(4) **Advection-diffusion on a bounded domain**
Suppose that a large population of ants is following a pheromone trail to a new nesting site, located at $x = L$. Ants originate from their nest at $x = 0$, with flux $q(0, t) = \beta_0 =$ constant. Once an ant reaches the new nesting site, it stays there, and is removed from the domain. In the domain $x \in (0, L)$, the density of ants $\rho(x, t)$ obeys the advection-diffusion equation

$$\frac{\partial \rho}{\partial t} = D\frac{\partial^2 \rho}{\partial x^2} - u\frac{\partial \rho}{\partial x}.$$

(a) Explain why the boundary conditions are $-D\frac{\partial \rho}{\partial x}\Big|_{x=0} + u\rho\Big|_{x=0} = \beta_0$ and $\rho(L, t) = 0$.

(*Hint for x=0 boundary condition:* Rewrite the differential equation in the form of the continuity equation $\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0$ and find the flux $q(x, t)$).

(b) Find the equilibrium solution $\rho_\infty(x)$ for the boundary conditions in part (a).

(c) For $\beta_0 = 5$ ants/s, $u = 0.1$ m/s, $D = 0.05$ m^2/s, and $L = 2$ m, plot the equilibrium solution on a computer.

(d) For the same set of parameter values, find the total number of ants in the domain at equilibrium.