Math 142 - Homework 2
Due Wednesday, April 15th

(1) The continuous time population growth ODE that we derived in class (on Friday 4/10) is usually called Malthusian growth, after Thomas Robert Malthus, who first used it to predict the growth of human populations (Malthus was concerned about food production failing to keep up with population growth).

(a) Briefly describe how to derive the ODE $\left(\dfrac{dN}{dt} = R_0 N\right)$ from a recurrence equation.

(b) Using separation of variables, solve the ODE for the growth rate $R_0 = .03$ and initial population $N(0) = 1000$.

(c) Plot the solution you obtained in part (b) from t = 0 to 100. On the same graph, plot the solution to the recurrence equation with the same growth rate and initial population, with $\Delta t = 1$. What do you observe?

(2) Consider a discrete time model for the growth of two linked geographic subpopulations on two islands:

$$\mathbf{N}_{k+1} = \begin{pmatrix} 1 & 1 \\ \frac{1}{2} & \frac{3}{2} \end{pmatrix} \mathbf{N}_k,$$

with initial conditions $\mathbf{N}_0 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$.

(a) Without implementing the recurrence equation (i.e., just by studying the Leslie matrix), explain whether the population will go extinct or grow without bound.

(b) For the given initial condition, calculate an explicit expression for $\mathbf{N}_k$. (Hint: use the eigenvalues and eigenvectors of the Leslie matrix. See your class notes.) Please find your expression by hand and show work, then verify that your expression works by using a computer and your explicit formula to compute $\mathbf{N}_4$.

(c) Calculate the stable distribution of organisms between the two subpopulations as $k \to \infty$.

(3) You are monitoring the population of female wildebeest on an African grassland. The census period (length of time between population measurements) is two years, and the wildebeest are divided into two age classes: juveniles (under two years old) and adults (at least two years old). Between each census:

- A fraction m of adults die.
- A fraction s of juveniles survive to adulthood.
- An average of b juveniles are birthed from each adult. Assume that births occur before adults die. Juveniles do not reproduce.

Now, let $N_j^{(k)}$ be the number of juvenile wildebeest at the k^{th} census, and let $N_a^{(k)}$ be the number adults at the k^{th} census.

(a) You want to write down a Leslie matrix model of the form

$$\begin{pmatrix} N_j^{(k+1)} \\ N_a^{(k+1)} \end{pmatrix} = L \begin{pmatrix} N_j^{(k)} \\ N_a^{(k)} \end{pmatrix}$$

for the situation described above. Specify the matrix L in terms of m, s, and b.

(b) Suppose you are able to measure the following Leslie matrix:

$$L = \begin{pmatrix} 0 & 0.9 \\ 0.4 & 0.9 \end{pmatrix}.$$

Find (by hand) the long-term growth factor and stable age distribution of the wildebeest population.

(4) This question is about developing a model for the life-cycle of a type of nanoflagellates, a type of swimming cell that lives in water. The nanoflagellate cells go through three different life history stages. All cells start as swimming larvae (we call this the **juvenile** phase). They swim around looking for a surface (e.g., a plant) to settle on. When they settle they become **tethered**. Tethered cells feed for a while, and then when they have exhausted the food nearby they break their tethers, and start to swim around in search of another place to settle (we call these cells **freely swimming**). *Only tethered cells are capable of reproducing.* At all stages the organism can also die, in which case it won't transition to another stage. You are modeling how the three populations vary with time: at the kth census there are $N_k^{(F)}$ freely swimming cells, $N_k^{(J)}$ juvenile cells and $N_k^{(T)}$ tethered cells. Measurements are taken daily. Our model needs to incorporate the following information:

- In each day, a fraction m of cells in all of the stages die.
- Of the freely swimming cells that do not die, a fraction t will become tethered.
- Of the tethered cells that do not die, a fraction u will become untethered (i.e., enter the freely swimming phase).
- Of the tethered cells that do not die, a fraction b will divide in two, producing a larval cell as a daughter.
- Of the juvenile cells that do not die, a fraction c will become tethered.

(a) Show that the changes in this population from one census to another can be modeled by the following Leslie matrix equation:

$$\begin{pmatrix} N_{k+1}^{(F)} \\ N_{k+1}^{(J)} \\ N_{k+1}^{(T)} \end{pmatrix} = \begin{pmatrix} (1-m)(1-t) & 0 & (1-m)u \\ 0 & (1-m)(1-c) & (1-m)b \\ (1-m)t & (1-m)c & (1-m)(1-u) \end{pmatrix} \begin{pmatrix} N_k^{(F)} \\ N_k^{(J)} \\ N_k^{(T)} \end{pmatrix}$$

(b) Let's assume that you measure all of the parameters m, t, u, etc. to obtain the following matrix:

$$L \equiv \begin{pmatrix} 0.6 & 0 & 0.7 \\ 0 & 0.4 & 0.5 \\ 0.3 & 0.5 & 0.2 \end{pmatrix}$$

(i) Using Matlab, find the eigenvalues and eigenvectors of L. Does the cell population grow without bound, level out, or decay? Explain. What is the stable fraction of juvenile, tethered, and freely swimming cells?

(ii) Starting with 100 juvenile cells (and no cells in the other two classes), compute the vectors $\mathbf{N}_1$, $\mathbf{N}_2$, $\mathbf{N}_3$, ..., $\mathbf{N}_{10}$. Display your results in a table that tracks the number of cells in each class after each day. For the eighth, ninth, and tenth days, compute the growth factor (the factor by which cell population increased in each class from the previous day) and the fraction of cells in each class. Compare your results with what you obtained in part (i).