MATH 142: Mathematical Modeling, Homework 8

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Question 1

Part A

It is important to assume that the pool of cold water is very large because the temperature of the pool should not change; if it did, the temperature of the bar at $x = 0$ would not be a constant $T = \alpha_0$. The temperature of the water would change if there was only a small quantity.

Part B

The boundary conditions are:

$$T(x = 0, t) = \alpha_0,$$

$$q(x = L, t) = \beta_L \to \frac{\partial T}{\partial x}\Big|_{x=L} = \frac{-\beta_L}{k}$$

Part C

We begin with

$$T_\infty(x) = Ax + B$$

Applying boundary conditions:

$$T(0, t) = \alpha_0 \to T_\infty(0) = \alpha_0 = B$$

$$\frac{\partial T}{\partial x}\Big|_{x=L} = \frac{-\beta_L}{k} \to T'_\infty(L) = A = \frac{-\beta_L}{k}$$

Thus,

$$T_\infty(x) = \frac{-\beta_L}{k}x + \alpha_0$$

Part D

For balsa wood:

$$T_\infty(3) = \frac{-(-4)}{.048}(3) + 277 = 527$$

For gold:

$$T_\infty(3) = \frac{-(-4)}{327}(3) + 277 = 277.037$$

The wooden bar is heated to a much higher temperature than the gold bar because the gold bar is much better at allowing the heat to "move" from one end to the other. Thus, the temperature is lower for the gold bar when compared to the wooden bar.

Question 2

Part A

(i)

$$T(0,t) = 200, \ \frac{\partial T}{\partial x}\Big|_{x=10} = 0$$

(ii)

$$T(0,t) = 200, \ \frac{\partial T}{\partial x}\Big|_{x=10} = 10$$

(iii)

$$T(0,t) = 200, \ T(10,t) = 250$$

Part B

For all protoytpes, we will start with $T_\infty(x) = Ax + B$

(i)

Applying the first boundary condition:

$$T(0,t) = 200 \rightarrow T_\infty(0) = 200 = B$$

Applying the second boundary condition:

$$\frac{\partial T}{\partial x}\Big|_{x=10} = 0 \rightarrow T'_\infty(10) = 0 = A$$

Thus,

$$T_\infty(x) = 200$$

(ii)

Applying the first boundary condition:

$$T(0,t) = 200 \rightarrow T_\infty(0) = 200 = B$$

Applying the second boundary condition:

$$\frac{\partial T}{\partial x}\Big|_{x=10} = 10 \rightarrow T'_\infty(10) = 10 = A$$

Thus,

$$T_\infty(x) = 10x + 200$$

(iii)

Applying the first boundary condition:

$$T(0,t) = 200 \rightarrow T_\infty(0) = 200 = B$$

Applying the second boundary condition:

$$T(10,t) = 250 \rightarrow T_\infty(10) = 250 \rightarrow A = \frac{250-200}{10} = 5$$

Thus,

$$T_\infty(x) = 5x + 200$$

We can see from above that (ii) generates the most amount of heat because the temperature increases the most as you move along the curling iron (as x increases).

Part C

Based on the feedback, I did not choose the best prototype. I should have chosen (i) instead because the temperature will be 200° F no matter where on the curling iron you are at (no matter the value of x).

Question 3

The boundary condtions are:

$$T(0,t) = T_0 \text{ and } T(L,t) = T_L,$$

where $T_L > T_0$.

We begin by setting $\frac{\partial T}{\partial t}$ to 0 and replacing the partial derivative with ordinary derivative:

$$\frac{dT}{dt} = D\frac{d^2T_\infty}{dx^2} - c(T_\infty - T_0) = 0$$

Distributing and rearranging:

$$D\frac{d^2T_\infty}{dx^2} - cT_\infty = -cT_0$$

Writing the characteristic equation:

$$Dr^2 - c = 0$$

Solving the characteristic equation, we get:

$$r_1 = \sqrt{\frac{c}{D}}, r_2 = -\sqrt{\frac{c}{D}}$$

Then, the complementary function is (and letting $\lambda = \sqrt{\frac{c}{D}}$):

$$y_{CF} = c_1e^{\lambda x} + c_2e^{-\lambda x} \tag{1}$$

To find the particular solution:

$$y_{PS} = P \to y''_{PS} = y'_{PS} = 0 \to 0 - cP = -cT_0 \to P = T_0 \tag{2}$$

The general solution is (1) + (2):

$$T_\infty(x) = c_1e^{\lambda x} + c_2e^{-\lambda x} + T_0,$$

where $\lambda = \sqrt{\frac{c}{D}}$.

Applying the first boundary condition:

$$T_\infty(0) = c_1e^{\lambda * 0} + c_2e^{-\lambda * 0} + T_0 = T_0 \to c_1 + c_2 = 0 \tag{3}$$

Applying the second boundary condition:

$$T_\infty(L) = c_1e^{\lambda L} + c_2e^{-\lambda L} + T_0 = T_L \tag{4}$$

Using (3) and (4), we solve for c_1 and c_2:

$$c_1e^{\lambda L} - c_1e^{-\lambda L} = T_L - T_0$$

Factoring out c_1 and rearranging:

$$c_1 = \frac{T_L - T_0}{e^{\lambda L} - e^{-\lambda L}} \to c_2 = -c_1 = -\frac{T_L - T_0}{e^{\lambda L} - e^{-\lambda L}}$$

Thus,

$$T_\infty(x) = \frac{T_L - T_0}{e^{\lambda L} - e^{-\lambda L}}e^{\lambda x} - \frac{T_L - T_0}{e^{\lambda L} - e^{-\lambda L}}e^{-\lambda x} + T_0$$

Factoring:

$$T_\infty(x) = \frac{T_L - T_0}{e^{\lambda L} - e^{-\lambda L}}[e^{\lambda x} - e^{-\lambda x}] + T_0,$$

where $\lambda = \sqrt{\frac{c}{D}}$.

Question 4

Part A

We start by plugging in the advection-diffusion equation into the continuity equation:

$$D\frac{\partial^2 \rho}{\partial x^2} - u\frac{\partial \rho}{\partial x} + \frac{\partial q}{\partial x} = 0$$

Rearranging:

$$\frac{\partial q}{\partial x} = u\frac{\partial \rho}{\partial x} - D\frac{\partial^2 \rho}{\partial x^2}$$

Integrating both sides:

$$q(x,t) = u\rho - D\frac{\partial \rho}{\partial x} + f(t)$$

At $x = 0$:

$$q(0,t) = u\rho\Big|_{x=0} - D\frac{\partial \rho}{\partial x}\Big|_{x=0} = \beta_0,$$

as desired.

Futhermore, since ants are removed from the domain at $x = L$, $\rho(L,t) = 0$.

Part B

We begin by setting $\frac{\partial \rho}{\partial t}$ to 0 and replacing the partial derivative with ordinary derivative:

$$\frac{d\rho}{dt} = D\frac{d^2 \rho_\infty}{dx^2} - u\frac{d\rho_\infty}{dx} = 0$$

Writing the characteristic equation:

$$Dr^2 - ur = 0$$

Solving for r:

$$r_1 = 0, r_2 = \frac{u}{D}$$

Then,

$$\rho_\infty(x) = c_1 + c_2 e^{\frac{u}{D}x}, \tag{5}$$

Applying the first boundary condition:

$$-D\frac{\partial \rho}{\partial x}\Big|_{x=0} + u\rho\Big|_{x=0} = \beta_0 \rightarrow -Dc_2\frac{u}{D} + u(c_1 + c_2) = \beta_0$$

Simplifying:

$$-c_2 u + u(c_1 + c_2) = \beta_0$$

Distributing:

$$-c_2 u + c_1 u + c_2 u = \beta_0$$

Solving for c_1:

$$c_1 = \frac{\beta_0}{u}$$

Applying the second boundary condition:

$$\rho(L,t) = 0 \rightarrow c_1 + c_2 e^{\frac{u}{D}L} = 0$$

Plugging in c_1:

$$\frac{\beta_0}{u} + c_2 e^{\frac{u}{D}L} = 0$$

Solving for c_2:

$$c_2 = \frac{-\beta_0}{ue^{\frac{u}{D}L}}$$

Plugging in c_1 and c_2 into (5):

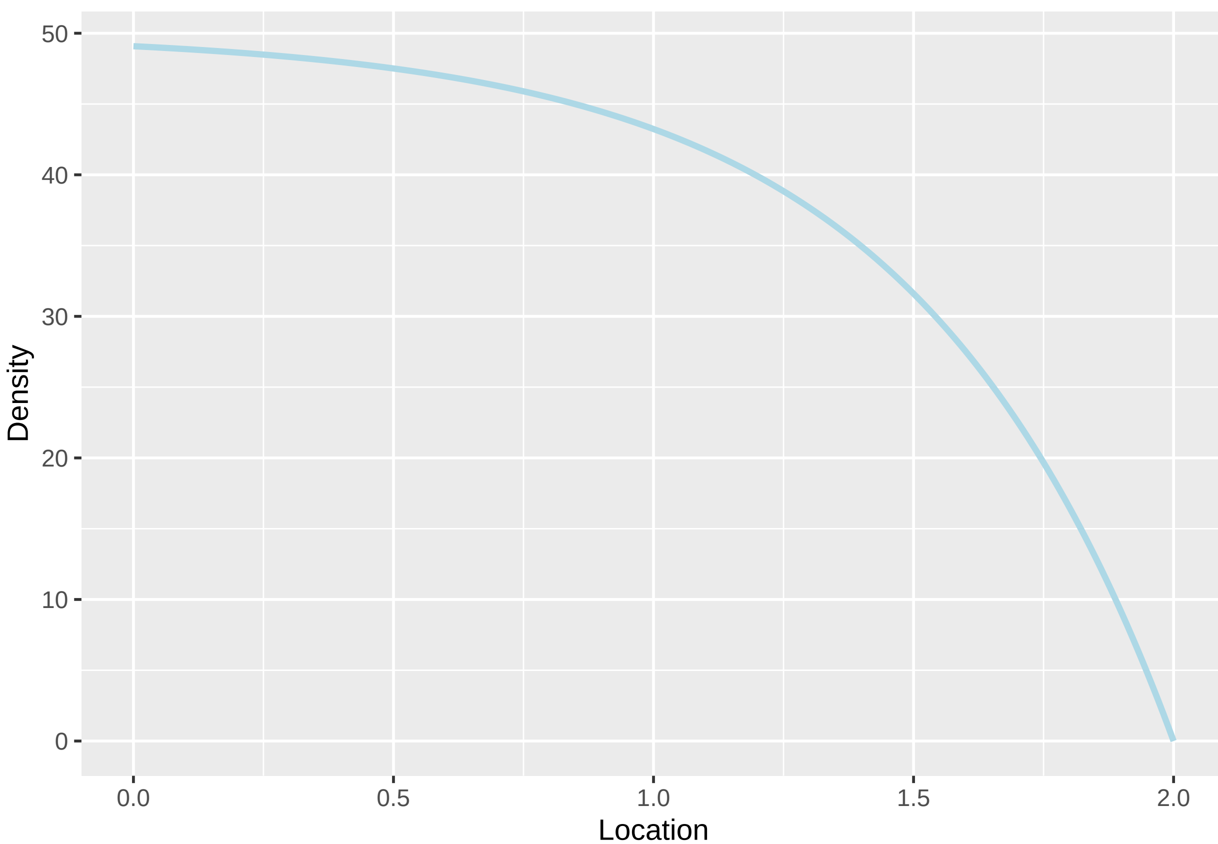
$$\rho_\infty(x) = \frac{\beta_0}{u} + \frac{-\beta_0}{ue^{\frac{u}{D}L}} e^{\frac{u}{D}x},$$

Simplifying, we finally get:

$$\rho_\infty(x) = \frac{\beta_0}{u}(1 - \frac{e^{\gamma x}}{e^{\gamma L}}),$$

where $\gamma = \frac{u}{D}$.

Part C



Part D

We integrate our equilibrium solution from Part B:

$$\int_0^2 \frac{\beta_0}{u}(1 - \frac{e^{\gamma x}}{e^{\gamma L}})dx$$

Taking the constant out of the integral:

$$\frac{\beta_0}{u}\int_0^2 (1 - \frac{e^{\gamma x}}{e^{\gamma L}})dx$$

Integrating:

$$\frac{\beta_0}{u}[x - \frac{e^{\gamma x}}{\gamma e^{\gamma L}}]_{x=0}^{x=2}$$

Evaluating:

$$\frac{\beta_0}{u}[(2 - \frac{e^{2\gamma}}{\gamma e^{\gamma L}}) - (0 - \frac{1}{\gamma e^{\gamma L}})]$$

Plugging in values from Part C (and noting that $\gamma = \frac{.1}{.05} = 2$):

$$\frac{5}{.1}[(2 - \frac{e^{2*2}}{2e^{2*2}}) - (0 - \frac{1}{2e^{2*2}})] = 75.457$$

The total number of ants in the domain at equilibrium is approximately 75.