Math 142 - Spring 2020
Super Quiz 2

Please read the following instructions carefully before taking this quiz:

1. You have a 24 hour window to complete this quiz, from 1:00 PM Pacific time on Monday, May 4th to 1:00 PM Pacific time on Tuesday, May 5th. You **must** upload the quiz to Gradescope before the deadline.

2. This quiz is open book, open note, open internet, but it **must** be completed on your own. You may not ask other students/tutors for help or use forums such as Stack Exchange to ask questions. You must show work on each problem to receive full credit.

3. If you have any clarification questions about the quiz, you may post your questions on Piazza, but you may not use Piazza to discuss how to do quiz problems.

4. You may scan or take pictures of the quiz in order to upload. Regardless of what option you are using, make sure that your work is legible and that pictures/scans are in focus. I would recommend that you use a free scanning app such as Scannable.

5. Please select the correct pages of your solution associated with each question when you upload to Gradescope.

6. If you wish, you may print out the quiz and write the answers on the test paper. You may also write answers on your own paper, on a tablet, or on a computer.

By signing below, you acknowledge that you have read the instructions on the previous page and that you agree to the following statement:

"*I assert, on my honor, that I have not received assistance of any kind from any other person while working on this quiz and that I have not used any non-permitted resources during the period of this evaluation.*"

(If you are answering the questions on separate paper, please copy the statement above onto your paper, and write your name, student ID, and signature on the same page.)

NAME: ______________________________

STUDENT ID:______________________________

SIGNATURE:______________________________

SHOW ALL WORK AND BOX YOUR ANSWERS

1. [**13 points**] An *enzyme* is a catalyst for a chemical reaction. The molecules upon which enzymes act are called *substrates*. When an enzyme binds to a substrate, it converts the substrate into a different molecule called a *product*.

   Suppose that the enzyme (E) can bind to two different substrates S_1 and S_2 to form products P_1 and P_2, respectively. The following diagrams describe these reactions:

$$E + S_1 \underset{\beta}{\overset{\alpha}{\rightleftharpoons}} P_1$$

$$E + S_2 \underset{\delta}{\overset{\gamma}{\rightleftharpoons}} P_2$$

   (a) (3 points) Assume that the substrates S_1 and S_2 are in such excess that we can treat their concentrations as constants σ_1 and σ_2, respectively. Using the law of mass action, write down a system of three differential equations describing how the concentrations of enzyme (e) and products (p_1 and p_2) change in time.

   (b) (2 points) Find a conserved quantity for your system in part (a), and use it to reduce the system to the following differential equations for p_1 and p_2:

$$\begin{aligned} \frac{dp_1}{dt} &= \alpha\sigma_1(P_0 - p_1 - p_2) - \beta p_1, \\ \frac{dp_2}{dt} &= \gamma\sigma_2(P_0 - p_1 - p_2) - \delta p_2, \end{aligned}$$

   where P_0 is a constant.

   (c) (8 points) Find the dimension (units) of the parameters α, β, γ, and δ. Then derive the following nondimensionalized system from part (b):

$$\begin{aligned} \frac{d\rho_1}{d\tau} &= (1 - \rho_1 - \rho_2) - \kappa\rho_1, \\ \frac{d\rho_2}{d\tau} &= \lambda(1 - \rho_1 - \rho_2) - \mu\rho_2. \end{aligned}$$

   Make sure to specify the dimensionless variables ρ_1, ρ_2, and τ and the dimensionless parameters κ, λ, and μ in terms of the original variables/parameters.

(Space to continue your answer to Question 1)

2. [**12 points**] *Modeling invasive species populations on islands*

(a) (5 points) Macquarie Island is a remote island in the southwestern Pacific Ocean. In the early 1800s, humans brought mice and rats to Macquarie Island on their ships. The population of rodents on the island quickly exploded. In 1818, in an effort to curb the rodent population, cats were introduced onto the island to eat the mice and rats, with disastrous results (feral cat population then exploded as a result).

Let $\rho(t)$ = population of rodents and $c(t)$ = population of cats, where t is time in years. Assume the following:

- Cats have a birth rate b_c/year and a death rate m_c/year.
- α_c cats per year are introduced (added) to the island to try to control the rodent population.
- Rodents have a birth rate b_ρ/year.
- Rodents can die in two different ways: by being eaten by cats or by natural causes. Suppose that rodents interact with cats (and are consumed) at rate β/year per cat and die by natural causes at rate m_ρ/year.
- α_ρ rodents per year arrive on the island from ships.

Using the information above, write down the system of differential equations describing how the population of rodents and cats changes in time. **Explain your process** in arriving at your equations (e.g., by starting with word equations).

(b) (7 points) On the islands in Maine's Bay of Fundy, the snowshoe hare was an invasive species for nearly 50 years, until hunters and trappers finally drove the hares to extinction in 2007. Suppose that the following differential equation describes the population of hares on the island:

$$\frac{dn}{dt} = \underbrace{rn(1-n)}_{\text{logistic growth}} - \underbrace{\frac{h_{\max} n}{1+n}}_{\text{hunting/trapping}}, \tag{1}$$

where $n = \dfrac{N}{K}$, the proportion of the carrying capacity of the hare population.

i. Find the fixed point(s) of equation (1).

ii. Find the stability of the (biologically relevant) fixed point(s) when $h_{\max} < r$ and when $h_{\max} > r$. Use a vector field plot in which you split up the right hand side of (1) into two functions you can sketch.

iii. In which scenario from part ii are hares driven to extinction? **Explain your reasoning.**

(Space to continue your answer to Question 2)