MATH 151A - Numerical Methods - Homework 4

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Produced on Friday, Jul. 24 2020 @ 09:30:09 PM

Question 1

There are two different 3-point formulas:

$$f'(x_0) = \frac{1}{2h}[-3f(x_0) + 4f(x_0+h) - f(x_0+2h)] + \frac{h^2}{3}f^{(3)}(\xi_0),\ \text{where } \xi_0 \in [x_0, x_0+2h] \qquad \text{(Endpoint)}$$

$$f'(x_0) = \frac{1}{2h}[f(x_0+h) - f(x_0-h)] - \frac{h^2}{6}f^{(3)}(\xi_1),\ \text{where } \xi_1 \in [x_0-h, x_0+h] \qquad \text{(Midpoint)}$$

Part A

For $x = -.3$, we will use the endpoint formula:

$$f'(-.3) \approx \frac{1}{2(.1)}[-3f(-.3) + 4f(-.2) - f(-.1)] = \frac{1}{.2}[-3(-.27652) + 4(-.25074) - (-.16134)] = -.0603$$

For $x = -.2$, we will use the midpoint formula:

$$f'(-.2) \approx \frac{1}{2(.1)}[f(-.1) - f(-.3)] = \frac{1}{.2}[-.16134 - (-.27652)] = .5759$$

For $x = -.1$, we will use the midpoint formula:

$$f'(-.1) \approx \frac{1}{2(.1)}[f(0) - f(-.2)] = \frac{1}{.2}[0 - (-.25074)] = 1.2537$$

For $x = 0$, we will use the endpoint formula:

$$f'(0) \approx \frac{1}{2(-.1)}[-3f(0) + 4f(-.1) - f(-.2)] = \frac{1}{-.2}[-3(0) + 4(-.16134) - (-.25074)] = 1.9731$$

Part B

We note that:

$$f'(x) = 2(\sin(2x) + e^{2x})$$

We also note that:

$$f^{(3)} = -8(\sin(2x) - e^{2x})$$

is decreasing over the interval $[-.3, 0]$.

At **x = -.3**, the actual value is:

$$f'(-.3) = 2(\sin(2(-.3)) + e^{2(-.3)}) = -.031662$$

This means that the actual error is:

$$\text{Error} = |-.0603 - (-.031662)| = .028638$$

The bound for the error is:

$$\text{Error} \leq \frac{.1^2}{3} f^{(3)}(-.3) = .029692$$

At **x = -.2**, the actual value is:

$$f'(-.2) = 2(\sin(2(-.2)) + e^{2(-.2)}) = .561803$$

This means that the actual error is:

$$\text{Error} = |.5759 - .561803| = .014097$$

The bound for the error is:

$$\text{Error} \leq \frac{.1^2}{6} f^{(3)}(-.3) = .014846$$

At **x = -.1**, the actual value is:

$$f'(-.1) = 2(\sin(2(-.1)) + e^{2(-.1)}) = 1.240123$$

This means that the actual error is:

$$\text{Error} = |1.2537 - 1.240123| = .013577$$

The bound for the error is:

$$\text{Error} \leq \frac{.1^2}{6} f^{(3)}(-.2) = .014130$$

At **x = 0**, the actual value is:

$$f'(0) = 2(\sin(2(0)) + e^{2(0)}) = 2$$

This means that the actual error is:

$$\text{Error} = |1.9731 - 2| = .026900$$

The bound for the error is:

$$\text{Error} \leq \frac{.1^2}{3} f^{(3)}(-.2) = .028260$$

Organized in a table:

x	Approximation of $f'(x)$	Actual Value of $f'(x)$	Actual Error	Error Bound
$-.3$	$-.0603$	$-.031662$	.028638	.029692
$-.2$	.5759	.561803	.014097	.014846
$-.1$	1.2537	1.240123	.013577	.014130
0	1.9731	2	.026900	.028260

As we can see from the table, the actual error is less than the error bound for all points, which is to be expected.

Question 2

The second derivative midpoint formula is:

$$f''(x_0) = \frac{1}{h^2}[f(x_0 - h) - 2f(x_0) + f(x_0 + h)] - \frac{h^2}{12}f^{(4)}(\xi), \text{ where } x_0 - h < \xi < x_0 + h$$

Using $h = .1$:

$$f''(1.3) \approx \frac{1}{.1^2}[f(1.2) - 2f(1.3) + f(1.4)] = \frac{1}{.1^2}[11.59006 - 2(14.04276) + 16.86187] = 36.6410$$

Using $h = .01$:

$$f''(1.3) \approx \frac{1}{.01^2}[f(1.29) - 2f(1.3) + f(1.31)] = \frac{1}{.01^2}[13.78176 - 2(14.04276) + 14.30741] = 36.5000$$

The second derivative of $f(x) = 3xe^x - \cos(x)$ is:

$$f''(x) = \cos(x) + (3x + 6)e^x$$

At $x = 1.3$:

$$f''(1.3) = \cos(1.3) + (3(1.3) + 6)e^{1.3} = 36.5935$$

Using $h = .1$, the absolute error is:

$$\text{Error} = |36.5935 - 36.6410| = .0475$$

Using $h = .01$, the absolute error is:

$$\text{Error} = |36.5935 - 36.5000| = .0935$$

We can see that when using the larger h, the approximation is closer to the actual value of $f''(1.3)$, which is surprising.

Question 3

Part A

We are given the following:

$$f'(x_0) = \frac{1}{h}[f(x_0 + h) - f(x_0)] - \frac{h}{2}f''(x_0) - \frac{h^2}{6}f'''(x_0) + O(h^3) \tag{1}$$

We set $h = 2h$ in Equation (1):

$$f'(x_0) = \frac{1}{2h}[f(x_0 + 2h) - f(x_0)] - hf''(x_0) - \frac{2h^2}{3}f'''(x_0) + O(h^3) \tag{2}$$

Then, we do $2(1) - (2)$:

$$f'(x_0) = \frac{2}{h}[f(x_0 + h) - f(x_0)] - \frac{h^2}{3}f'''(x_0) - \frac{1}{2h}[f(x_0 + 2h) - f(x_0)] + \frac{2h^2}{3}f'''(x_0) + O(h^3)$$

We can combine like terms:

$$f'(x_0) = \frac{2}{h}[f(x_0 + h) - f(x_0)] + \frac{h^2}{3}f'''(x_0) - \frac{1}{2h}[f(x_0 + 2h) - f(x_0)] + O(h^3) \tag{3}$$

We set $h = 2h$ in Equation (3):

$$f'(x_0) = \frac{1}{h}[f(x_0 + 2h) - f(x_0)] + \frac{4h^2}{3}f'''(x_0) - \frac{1}{4h}[f(x_0 + 4h) - f(x_0)] + O(h^3) \tag{4}$$

Then, we do $4(3) - (4)$:

$$3f'(x_0) = \frac{8}{h}[f(x_0+h)-f(x_0)]-\frac{2}{h}[f(x_0+2h)-f(x_0)]-\frac{1}{h}[f(x_0+2h)-f(x_0)]+\frac{1}{4h}[f(x_0+4h)-f(x_0)]+O(h^3)$$

Solving for $f'(x_0)$:

$$f'(x_0) = \frac{8}{3h}[f(x_0+h)-f(x_0)]-\frac{2}{3h}[f(x_0+2h)-f(x_0)]-\frac{1}{3h}[f(x_0+2h)-f(x_0)]+\frac{1}{12h}[f(x_0+4h)-f(x_0)]+O(h^3)$$

Part B

```r
derivative_approximation <- function(func, x, h){
  first <- 8/(3*h) * (f(x + h) - f(x))
  second <- 2/(3*h) * (f(x + 2*h) - f(x))
  third <- (f(x + 2*h) - f(x))/(3*h)
  fourth <- (f(x + 4*h) - f(x))/(12*h)

  first - second - third + fourth
}

f <- function(x){exp(2*x) - cos(2*x)}
derivative_approximation(f, -.2, .1)
```

```
## [1] 0.5613143
```

The error is:

$$\text{Error} = |f'(-.2) - .5613143| = |2(\sin(2 * -.2) + e^{2*-.2}) - .5613143| = |.561803 - .561314| = .000489$$

Question 4

We are given the following function:

$$f(x) = x^2 e^{-x}$$

We calculate the derivatives:

$$f'(x) = -(x-2)xe^{-x}, f''(x) = (x^2 - 4x + 2)e^{-x}, f^{(3)} = -(x^2 - 6x + 6)e^{-x}, f^{(4)} = (x^2 - 8x + 12)e^{-x}$$

Note that the second and fourth order derivative is decreasing over the interval [0,1].

First, we find:

$$\int_0^1 x^2 e^{-x} dx = .160603$$

Using the **Trapezoidal Rule**:

$$\int_0^1 x^2 e^{-x} dx \approx \frac{h}{2}[f(x_0) + f(x_1)] = \frac{1}{2}[(0)^2 e^{-(0)} + (1)^2 e^{-1}] = .183940$$

The bound for the error:

$$\text{Error} = \frac{h^3}{12} f''(\xi) \leq \frac{1^3}{12}(0^2 - 4(0) + 2)e^{-0} = .166667$$

The actual error is:

$$|.160603 - .183940| = .023337$$

Using **Simpson's Rule**:

$$\int_0^1 x^2 e^{-x} dx \approx \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)] = \frac{.5}{3}[0^2 e^{-0} + 4(.5^2)e^{-.5} + 1^2 e^{-1}] = .162402$$

The bound for the error:

$$\text{Error} = \frac{h^5}{90} f^{(4)}(\xi) \leq \frac{.5^5}{90}(0^2 - 8(0) + 12)e^{-0} = .004167$$

The actual error is:

$$|.160603 - .162402| = .001799$$

Using the **Midpoint Rule**, where $n = 0$:

$$\int_0^1 x^2 e^{-x} dx \approx 2hf(x_0) = 2(.5).5^2 e^{-.5} = .151633$$

The bound for the error:

$$\text{Error} = \frac{h^3}{3} f''(\xi) \leq \frac{.5^3}{3}(0^2 - 4(0) + 2)e^{-0} = .083333$$

The actual error is:

$$|.160603 - .151633| = .008997$$

Question 5

From the Trapezoidal Rule:

$$\int_0^2 f(x)dx \approx \frac{2}{2}[f(0) + f(2)] = 4 \longrightarrow f(0) + f(2) = 4$$

From Simpson's Rule:

$$\int_0^2 f(x)dx \approx \frac{1}{3}[f(x_0) + 4f(x_1) + f(x_2)] = 2 \longrightarrow f(0) + 4f(1) + f(2) = 6$$

We plug in our equation from the Trapezoidal Rule into our equation from Simpson's Rule:

$$f(0) + 4f(1) + f(2) = f(0) + f(2) + 4f(1) = 4 + 4f(1) = 6 \longrightarrow f(1) = \frac{1}{2}$$