MATH 151A - Numerical Methods - Homework 1

Darren Tsang, Discussion 1B

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Question 1

$$f(x) = \frac{1}{3}x^2 - \frac{123}{4}x + \frac{1}{6} \tag{1}$$

The roots of (1) are $x_1 = 92.24457962$ and $x_2 = .00542037$.

Method 1

To find an approximation x_1^* for the first root of (1):

$$x_1^* = \frac{-b + \sqrt{b^2 - 4ac}}{2a} = \frac{\frac{123}{4} + \sqrt{(\frac{-123}{4})^2 - 4(\frac{1}{3})(\frac{1}{6})}}{2(\frac{1}{3})} = \frac{30.75 + \sqrt{(-30.75)^2 - 4(.3333)(.1667)}}{.6666}$$

$$= \frac{30.75 + \sqrt{945.6 - 1.333(.1667)}}{.6666} = \frac{30.75 + \sqrt{945.6 - .2222}}{.6666} = \frac{30.75 + \sqrt{945.4}}{.6666}$$

$$= \frac{30.75 + 30.75}{.6666} = \frac{61.50}{.6666} = 92.26$$

To find an approximation x_2^* for the second root of (1) (we can reuse some of our work from above):

$$x_2^* = \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{30.75 - 30.75}{.6666} = 0$$

Method 2

To find an approximation x_1^{**} for the first root of (1):

$$x_1^{**} = \frac{-2c}{b + \sqrt{b^2 - 4ac}} = \frac{-2(\frac{1}{6})}{\frac{-123}{4} + \sqrt{(\frac{-123}{4})^2 - 4(\frac{1}{3})(\frac{1}{6})}} = \frac{-.3333}{-30.75 + \sqrt{(-30.75)^2 - 1.333(\frac{1}{6})}}$$

$$= \frac{-.3333}{-30.75 + \sqrt{945.6 - .2222}} = \frac{-.3333}{-30.75 + \sqrt{945.4}} = \frac{-.3333}{-30.75 + 30.75} = NaN$$

To find an approximation x_2^{**} for the second root of (1) (we can reuse some of our work from above):

$$x_2^{**} = \frac{-.3333}{-30.75 - 30.75} = .005420$$

Here are the absolute errors and the relative errors for Method 1 ("original" quadratic forumla) and Method 2 (modified quadratic forumla):

Method 1	Absolute Error	Relative Error
x_1^*	$\lvert 92.24457962 - 92.26\rvert = .0154$	$\frac{\lvert 92.24457962-92.26\rvert}{\lvert 92.24457962\rvert} = 1.6716 \cdot 10^{-4}$
x_2^*	$\lvert .00542037 - 0\rvert = .00542037$	$\frac{\lvert .00542037-0\rvert}{\lvert .00542037\rvert} = 1$

Method 2	Absolute Error	Relative Error
x_1^{**}	NaN	NaN
x_2^{**}	$\lvert .00542037 - .005420\rvert = 3.700 \cdot 10^{-7}$	$\frac{\lvert .00542037-.005420\rvert}{\lvert .00542037\rvert} = 6.826 \cdot 10^{-5}$

Using Method 2, we can not even find an estimate for x_1 because of rounding error. Furthermore, when finding an estimate for x_2, the errors we obtained from Method 1 are much larger than the errors we obtained from Method 2; this is because there there is subtraction of two (nearly) equal numbers. Thus, when estimating x_1, it is best to use Method 1, and when estimating x_2, it is best to use Method 2.

Question 2

$$f(x) = 1.01e^{4x} - 4.62e^{3x} - 3.11e^{2x} + 12.2e^{x} - 1.99 \tag{1}$$

Part A

Rewriting (1) using the polynomial nesting technique:

$$f(x) = (((1.01e^{x} - 4.62)e^{x} - 3.11)e^{x} + 12.2)e^{x} - 1.99 \tag{2}$$

Part B

$$\begin{gathered}
f(1.53) = 1.01(e^{1.53})^4 - 4.62(e^{1.53})^3 - 3.11(e^{1.53})^2 + 12.2e^{1.53} - 1.99 \\
= 1.01(456) - 4.62(98.6) - 3.11(21.3) + 12.2(4.62) - 1.99 \\
= 461 - 456 - 66.2 + 56.4 - 1.99 = -6.79
\end{gathered}$$

Part C

$$\begin{gathered}
f(1.53) = (((1.01e^{1.53} - 4.62)e^{1.53} - 3.11)e^{1.53} + 12.2)e^{1.53} - 1.99 \\
= (((1.01(4.62) - 4.62)(4.62) - 3.11)(4.62) + 12.2)(4.62) - 1.99 \\
= (((4.67 - 4.62)(4.62) - 3.11)(4.62) + 12.2)(4.62) - 1.99 \\
= (((.0500)(4.62) - 3.11)(4.62) + 12.2)(4.62) - 1.99 \\
= ((.231 - 3.11)(4.62) + 12.2)(4.62) - 1.99 \\
= ((-2.88)(4.62) + 12.2)(4.62) - 1.99 \\
= (-13.3 + 12.2)(4.62) - 1.99 \\
= (-1.10)(4.62) - 1.99 \\
= -5.08 - 1.99 \\
= -7.07
\end{gathered}$$

Part D

	Absolute Error	Relative Error
Part B	$\lvert 7.61 - 6.79 \rvert = .82$	$\frac{\lvert 7.61-6.79 \rvert}{\lvert -7.61 \rvert} = .108$
Part C	$\lvert 7.61 - 7.07 \rvert = .54$	$\frac{\lvert 7.61-7.07 \rvert}{\lvert 7.61 \rvert} = .0710$

We notice that the polynomial nesting technique is much better because it yields a smaller absolute and realtive error.

Question 3

Part A

$$\lim_{n\to\infty}\frac{|p_{n+1}-p|}{|p_n-p|^\alpha}=\lim_{n\to\infty}\frac{|(\frac{1}{10})^{n+1}-0|}{|(\frac{1}{10})^n-0|^1}=\lim_{n\to\infty}\frac{(\frac{1}{10})^{n+1}}{(\frac{1}{10})^n}=\lim_{n\to\infty}\frac{1}{10}=\frac{1}{10}=\lambda$$

We see that $\alpha=1$ and $0<\lambda=\frac{1}{10}<\infty$, which means the sequence $p_n=(\frac{1}{10})^n$ converges linearly to $p=0$.

Part B

$$\lim_{n\to\infty}\frac{|p_{n+1}-p|}{|p_n-p|^\alpha}=\lim_{n\to\infty}\frac{|10^{-2^{n+1}}-0|}{|10^{-2^n}-0|^2}=\lim_{n\to\infty}\frac{10^{-2^{n+1}}}{10^{-2^{n+1}}}=\lim_{n\to\infty}1=1=\lambda$$

We see that $\alpha=2$ and $0<\lambda=1<\infty$, which means the sequence $p_n=10^{-2^n}$ converges quadratically to $p=0$.

Question 4

Using the Bisection Method on $f(x) = \sqrt{x} - cos(x)$ over the interval $[0, 1]$:

n	a_n	b_n	p_n	$f(a_n)$	$f(b_n)$	$f(p_n)$
1	0	1	.5	$\sqrt{0} - cos(0) = -1$	$\sqrt{1} - cos(1) = .45970$	$\sqrt{.5} - cos(.5) = -.17048$
2	.5	1	.75	$-.17048$	.45970	$\sqrt{.75} - cos(.75) = .13434$
3	.5	.75	.625	$-.17048$	.13434	$\sqrt{.625} - cos(.625) = -.02039$

Question 5

Here is the output when I run the Bisection Method on $x^2 - 3$ over the interval $[1, 2]$:

```
Bisection Method
  I    P                  F(P)
  1    1.50000000e+00    -7.5000000e-01
  2    1.75000000e+00     6.2500000e-02
  3    1.62500000e+00    -3.5937500e-01
  4    1.68750000e+00    -1.5234375e-01
  5    1.71875000e+00    -4.5898438e-02
  6    1.73437500e+00     8.0566406e-03
  7    1.72656250e+00    -1.8981934e-02
  8    1.73046875e+00    -5.4779053e-03
  9    1.73242188e+00     1.2855530e-03
 10    1.73144531e+00    -2.0971298e-03
 11    1.73193359e+00    -4.0602684e-04
 12    1.73217773e+00     4.3970346e-04
 13    1.73205566e+00     1.6823411e-05
 14    1.73199463e+00    -1.9460544e-04

Approximate solution P =  1.73199463
with F(P) =  -0.00019461
Number of iterations =  14 Tolerance =  1.00000000e-04
```

As we can see above, after 14 iterations, we obtained an approximation of 1.73199463.

Question 6

We can use the following formula derived in class:

$$n \geq log_2(\frac{b-a}{\epsilon}) = log_2(\frac{2-1}{10^{-4}}) = 13.29$$

Thus, the bound for the number of iterations to achieve an approximation with accuracy 10^{-4} to the solution of $x^3 - x - 1 = 0$ is 14 iterations.

Here is the output when I run the Bisection Method on $x^3 - x - 1$ over the interval $[1, 2]$:

```
Bisection Method
  I    P                  F(P)
  1    1.50000000e+00      8.7500000e-01
  2    1.25000000e+00     -2.9687500e-01
  3    1.37500000e+00      2.2460938e-01
  4    1.31250000e+00     -5.1513672e-02
  5    1.34375000e+00      8.2611084e-02
  6    1.32812500e+00      1.4575958e-02
  7    1.32031250e+00     -1.8710613e-02
  8    1.32421875e+00     -2.1279454e-03
  9    1.32617188e+00      6.2088296e-03
 10    1.32519531e+00      2.0366507e-03
 11    1.32470703e+00     -4.6594883e-05
 12    1.32495117e+00      9.9479097e-04
 13    1.32482910e+00      4.7403882e-04
 14    1.32476807e+00      2.1370716e-04

Approximate solution P =  1.32476807
with F(P) =   0.00021371
Number of iterations =  14 Tolerance =  1.00000000e-04
```

As we can see above, after 14 iterations, we obtained an approximation of 1.32476807.