MATH 151A - Numerical Methods - Homework 5

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Question 1

When $f(x) = 1$:

$$\int_0^1 1dx = 1 = c_0 + c_1 \tag{1}$$

When $f(x) = x$:

$$\int_0^1 xdx = \frac{1}{2} = c_1x_1 \tag{2}$$

When $f(x) = x^2$:

$$\int_0^1 x^2dx = \frac{1}{3} = c_1x_1^2 \tag{3}$$

From $(1), (2), (3)$:

$$\frac{1}{3} = c_1x_1x_1 = \frac{1}{2}x_1 \longrightarrow x_1 = \frac{2}{3} \longrightarrow c_1 = \frac{1}{2x_1} = \frac{3}{4} \longrightarrow c_0 = 1 - c_1 = \frac{1}{4}$$

Thus, the constants should be $c_0 = \frac{1}{4}, c_1 = \frac{3}{4},$ and $x_1 = \frac{2}{3}$.

Question 2

Using Composite Trapezoidal Rule:

$$\int_1^2 x\ln(x)dx \approx \frac{1/4}{2}[1\ln(1) + 2(1.25\ln(1.25) + 1.5\ln(1.5) + 1.75\ln(1.75)) + 2\ln(2)] = .6399005$$

```r
x <- log(1) + 2*(1.25*log(1.25) + 1.5*log(1.5) + 1.75*log(1.75)) + 2*log(2)
x / 8
```

```
## [1] 0.6399005
```

Using Composite Simpson's Rule:

$$\int_1^2 x\ln(x)dx \approx \frac{1/4}{3}[\ln(1) + 2[1.5\ln(1.5)] + 4[1.25\ln(1.25) + 1.75\ln(1.75)] + 2\ln(2)] = .6363098$$

```r
x <- log(1) + 2*(1.5*log(1.5)) + 4*(1.25*log(1.25) + 1.75*log(1.75)) + 2*log(2)
x / 12
```

```
## [1] 0.6363098
```

Using Composite Midpoint Rule:

$$\int_1^2 x\ln(x)dx \approx 2\left(\frac{1}{6}\right)[\frac{7}{6}\ln(7/6) + \frac{9}{6}\ln(9/6) + \frac{11}{6}\ln(11/6)] = .6330964$$

```r
x <- 7/6 * log(7/6) + 9/6 * log(9/6) + 11/6 * log(11/6)
x / 3
```

```
## [1] 0.6330964
```

Question 3

We are given:

$$f(x) = x\ln(x)$$

Taking the derivative:

$$f'(x) = \ln(x) + 1$$

Taking the derivative again:

$$f''(x) = \frac{1}{x}$$

We note that $f''(x)$ is decreasing over the interval $[1, 2]$.

Taking the derivative again:

$$f^{(3)}(x) = \frac{-1}{x^2}$$

Taking the derivative again:

$$f^{(4)}(x) = \frac{2}{x^3}$$

We note that $f^{(4)}(x)$ is decreasing over the interval $[1, 2]$.

Part A

We want the following to be true:

$$\left|\frac{h^2}{12}(b-a)f''(\xi)\right| < 10^{-5} \longrightarrow h < \sqrt{\frac{12 \cdot 10^{-5}}{(2-1)f''(1)}} = .010955 \longrightarrow n \geq \frac{b-a}{h} = \frac{2-1}{.010955} \approx 92$$

Part B

We want the following to be true:

$$\left|\frac{h^4}{180}(b-a)f^{(4)}(\xi)\right| < 10^{-5} \longrightarrow h < \left(\frac{180 \cdot 10^{-5}}{(2-1)f^{(4)}(1)}\right)^{1/4} = .173205 \longrightarrow n \geq \frac{b-a}{h} = \frac{2-1}{.173205} \approx 6$$

Part C

We want the following to be true:

$$\left|\frac{h^2}{6}(b-a)f''(\xi)\right| < 10^{-5} \longrightarrow h < \sqrt{\frac{6 \cdot 10^{-5}}{(2-1)f''(1)}} = .007746 \longrightarrow n \geq \frac{b-a}{h} - 2 = \frac{2-1}{.007746} - 2 \approx 128$$

Question 4

From class, we derived the following equations:

$$c_1 + c_2 = 2 \tag{1}$$

$$c_1 x_1 + c_2 x_2 = 0 \tag{2}$$

$$c_1 x_1^2 + c_2 x_2^2 = \frac{1}{3} \tag{3}$$

$$c_1 x_1^3 + c_2 x_2^3 = 0 \tag{4}$$

We arrange the equations to get:

$$c_1 = 2 - c_2$$

$$(2 - c_2)x_1 + c_2 x_2 = 0 \longrightarrow x_1 = \frac{c_2 x_2}{c_2 - 2}$$

$$(2 - c_2)\left(\frac{c_2 x_2}{c_2 - 2}\right)^2 + c_2 x_2^2 = \frac{2}{3} \longrightarrow c_2 = \frac{2}{3x_2^2 + 1}$$

$$\left(2 - \frac{2}{3x_2^2 + 1}\right)\left(\frac{\frac{2x_2}{3x_2^2+1}}{\frac{2}{3x_2^2+1} - 2}\right)^3 + \left(\frac{2}{3x_2^2 + 1}\right)x_2^3 = 0 \longrightarrow x_2 = \frac{\sqrt{3}}{3}$$

Now that we have $x_2 = \frac{\sqrt{3}}{3}$, we can solve for the rest:

$$c_2 = \frac{2}{3(\sqrt{3}/3)^2 + 1} = 1$$

$$x_1 = \frac{1(\sqrt{3}/3)}{1 - 2} = \frac{-\sqrt{3}}{3}$$

$$c_1 = 2 - 1 = 1$$

Thus,

$$\int_{-1}^{1} f(x)dx = f\left(\frac{\sqrt{3}}{3}\right) + f\left(\frac{-\sqrt{3}}{3}\right)$$

is exact when $f(x)$ is an polynomial of degree 3 or less.

To check, let $f(x) = x^4$:

$$\int_{-1}^{1} x^4 dx = \frac{x^5}{5}\bigg|_{-1}^{1} = \frac{2}{5} \neq \left(\frac{\sqrt{3}}{3}\right)^4 + \left(\frac{-\sqrt{3}}{3}\right)^4 = \frac{2}{9}$$

Question 5

We want change of variable $[a, b] \to [-1, 1]$. We let $t = Mx + N$.

When $x = a$ and $t = -1$:

$$-1 = Ma + N \tag{1}$$

When $x = b$ and $t = 1$:

$$1 = Mb + N \tag{2}$$

We do $(2) - (1)$:

$$Mb - Ma = 2 \longrightarrow M = \frac{2}{b-a}$$

We can find N:

$$N = 1 - Mb = 1 - \frac{2b}{b-a} = \frac{-(a+b)}{b-a}$$

We plug N and M into our equation for t:

$$t = \frac{2}{b-a}x + \frac{-(a+b)}{b-a} \longrightarrow x = \frac{b-a}{2}t + \frac{a+b}{2}$$

We take the derivative of x:

$$\frac{dx}{dt} = \frac{b-a}{2} \longrightarrow dx = \frac{b-a}{2}dt$$

Now,

$$\int_{-1}^{1} f(x)dx = \int_{-1}^{1} f\left(\frac{b-a}{2}t + \frac{a+b}{2}\right)\frac{b-a}{2}dt = \frac{b-a}{2}\left[f\left(\frac{b-a}{2}\left(\frac{\sqrt{3}}{3}\right) + \frac{a+b}{2}\right) + f\left(\frac{b-a}{2}\left(\frac{-\sqrt{3}}{3}\right) + \frac{a+b}{2}\right)\right]$$