MATH 151A - Numerical Methods - Homework 3

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Please note that the code for the functions newtonMethod() and modifiedNewtonMethod() are shown at the end of this document.

Question 1

```r
f1 <- function(x){1 - 4*x*cos(x) + 2*x^2 + cos(2*x)}
f1prime <- function(x){-2*sin(2*x) + 4*x*sin(x) - 4*cos(x) + 4*x}
f1primeprime <- function(x){-4*cos(2*x) + 8*sin(x) + 4*x*cos(x) + 4}
```

```r
newtonMethod(f = f1, fprime = f1prime, initial = .5, iter = Inf, tol = 10^-5)
```

```
##       n       p_n     |f(p_n)|
##  [1,] 0 0.5000000 2.851372e-01
##  [2,] 1 0.6276112 6.611800e-02
##  [3,] 2 0.6848881 1.605692e-02
##  [4,] 3 0.7123298 3.962535e-03
##  [5,] 4 0.7257887 9.845643e-04
##  [6,] 5 0.7324567 2.454052e-04
##  [7,] 6 0.7357758 6.126069e-05
##  [8,] 7 0.7374317 1.530393e-05
##  [9,] 8 0.7382587 3.824583e-06
```

```r
modifiedNewtonMethod(f = f1, fprime = f1prime, fprimeprime = f1primeprime,
                     initial = .5, iter = Inf, tol = 10^-5)
```

```
##      n       p_n     |f(p_n)|
## [1,] 0 0.5000000 2.851372e-01
## [2,] 1 0.7216635 1.687142e-03
## [3,] 2 0.7390162 2.662984e-08
```

With the original Newton's Method, it took 8 iterations to obtain an estimate of 0.7382587. With the modified Newton's Method for multiple roots, it took 2 iterations to obtain an estimate of 0.7390162.

Question 2

Part A

Finding the interpolation polynomial of **degree one**:

$$P_1(x) = \sin(1.25\pi)\frac{x-1.6}{1.25-1.6} + \sin(1.6\pi)\frac{x-1.25}{1.6-1.25}$$

Simplifying $P_2(x)$:

$$P_1(x) = \frac{-20\sin(1.25\pi)}{7}(x-1.6) + \frac{20\sin(1.6\pi)}{7}(x-1.25)$$

Approximating $f(1.4)$ with $P_1(x)$:

$$P_1(1.4) = \frac{-20\sin(1.25\pi)}{7}(1.4-1.6) + \frac{20\sin(1.6\pi)}{7}(1.4-1.25) \approx -.81166$$

The absolute error is:

$$|f(1.4) - P_1(1.4)| \approx |-.95106 - (-.81166)| = .1394$$

Finding the interpolation polynomial of **degree two**:

$$P_2(x) = \sin(\pi)\frac{(x-1.25)(1-1.6)}{(1-1.25)(1-1.6)} + \sin(1.25\pi)\frac{(x-1)(x-1.6)}{(1.25-1)(1.25-1.6)} + \sin(1.6\pi)\frac{(x-1)(x-1.25)}{(1.6-1)(1.6-1.25)}$$

Simplifying $P_2(x)$:

$$P_2(x) = \frac{-80\sin(1.25\pi)}{7}(x-1)(x-1.6) + \frac{100\sin(1.6\pi)}{21}(x-1)(x-1.25)$$

Approximating $f(1.4)$ with $P_2(x)$:

$$P_2(1.4) = \frac{-80\sin(1.25\pi)}{7}(1.4-1)(1.4-1.6) + \frac{100\sin(1.6\pi)}{21}(1.4-1)(1.4-1.25) \approx -.91823$$

The absolute error is:

$$|f(1.4) - P_2(1.4)| \approx |-.95106 - (-.91823)| = .03283$$

Part B

We know that $f(x) = \sin(\pi x)$. Thus:

$$f'(x) = \pi\cos(\pi x), f''(x) = -\pi^2\sin(\pi x), f'''(x) = -\pi^3\cos(\pi x)$$

For the interpolation with **one degree** $P_1(x)$, the error is:

$$\text{Error} = \frac{f''(\xi(x))}{2!} \cdot |(x-1.25)(1-1.6)|, \text{ for } \xi(x) \in (1.25, 1.6)$$

The maximal value of $-\pi^3\cos(\pi\xi(x))$ over the interval $(1.25, 1.6)$ is $-\pi^3\cos(1.25\pi)$, and the maximal value of $|(x-1.25)(1-1.6)|$ over the interval $(1.25, 1.6)$ is $|(1.425-1.25)(1.425-1.6)|$. Thus, the error is:

$$\text{Error} \leq \frac{-\pi^3\cos(1.25\pi)}{2!} \cdot |(1.425-1.25)(1.425-1.6)| \approx .33572$$

For the interpolation with **two degrees** $P_2(x)$, the error is:

$$\text{Error} = \frac{f''(\xi(x))}{3!} \cdot |(x-1)(x-1.25)(1-1.6)|, \text{ for } \xi(x) \in (1.25, 1.6)$$

The maximal value of $-\pi^3 \cos(\pi\xi(x))$ over the interval $(1.25, 1.6)$ is $-\pi^3 \cos(1.25\pi)$, and the maximal value of $|(x-1)(x-1.25)(1-1.6)|$ over the interval $(1.25, 1.6)$ is $|(1.45725-1)(1.45725-1.25)(1.45725-1.6)|$. Thus, the error is:

$$\text{Error} \leq \frac{-\pi^3 \cos(1.25\pi)}{3!} \cdot |(1.45725-1)(1.45725-1.25)(1.45725-1.6)| = .04943$$

Question 3

Part A

$$f[x_0] = f(x_0), f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}, f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

Part B

From the equation we are given, we plug in x_1 to obtain:

$$P_n(x_1) = f[x_0] + a_1(x_1 - x_0) + 0 + ... + 0$$

Rearranging:

$$a_1 = \frac{P_n(x_1) - f[x_0]}{x_1 - x_0}$$

Using the definition of $P_n(x_1)$ and $f[x_0]$:

$$a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = f[x_0, x_1],$$

as desired.

Part C

From the equation we are given, we plug in x_2 to obtain:

$$P_n(x_2) = f[x_0] + f[x_0, x_1](x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1) + 0 + ... + 0$$

Rearranging:

$$a_2(x_2 - x_0)(x_2 - x_1) = P_n(x_2) - f[x_0] - f[x_0, x_1](x_2 - x_0)$$

Using the definition of $P_n(x_2)$ and $f[x_0]$:

$$a_2(x_2 - x_0)(x_2 - x_1) = f(x_2) - f(x_0) - f[x_0, x_1](x_2 - x_0)$$

Dividing both sides and using the definition of $f[x_0, x_1]$:

$$a_2(x_2 - x_1) = \frac{f(x_2) - f(x_0)}{x_2 - x_0} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

Doing some algebra on the right:

$$a_2(x_2 - x_1) = \frac{x_1 f(x_2) - x_1 f(x_0) - x_0 f(x_2) + x_0 f(x_0) + x_0 f(x_1) - x_0 f(x_0) - x_2 f(x_1) + x_2 f(x_0)}{(x_2 - x_0)(x_1 - x_0)}$$

Dividing both sides:

$$a_2 = \frac{x_1 f(x_2) - x_1 f(x_0) - x_0 f(x_2) + x_0 f(x_0) + x_0 f(x_1) - x_0 f(x_0) - x_2 f(x_1) + x_2 f(x_0)}{(x_2 - x_0)(x_1 - x_0)(x_2 - x_1)}$$

Using properties of fractions:

$$a_2 = \frac{\frac{x_1 f(x_2) - x_1 f(x_0) - x_0 f(x_2) + x_0 f(x_0) + x_0 f(x_1) - x_0 f(x_0) - x_2 f(x_1) + x_2 f(x_0)}{(x_1 - x_0)(x_2 - x_1)}}{(x_2 - x_0)} \tag{1}$$

For now, we can focus on the numerator n of (1):

$$n = \frac{x_1 f(x_2) - x_1 f(x_0) - x_0 f(x_2) + x_0 f(x_0) + x_0 f(x_1) - x_0 f(x_0) - x_2 f(x_1) + x_2 f(x_0)}{(x_1 - x_0)(x_2 - x_1)}$$

Canceling like terms:

$$n = \frac{x_1 f(x_2) - x_1 f(x_0) - x_0 f(x_2) + x_0 f(x_1) - x_2 f(x_1) + x_2 f(x_0)}{(x_1 - x_0)(x_2 - x_1)}$$

Adding $x_1 f(x_1) - x_1 f(x_1) = 0$ to the numerator of n:

$$n = \frac{x_1 f(x_2) - x_1 f(x_0) - x_0 f(x_2) + x_0 f(x_1) - x_2 f(x_1) + x_2 f(x_0) + x_1 f(x_1) - x_1 f(x_1)}{(x_1 - x_0)(x_2 - x_1)}$$

Simply rearranging the terms:

$$n = \frac{x_1 f(x_2) - x_1 f(x_1) - x_0 f(x_2) + x_0 f(x_1) - x_2 f(x_1) + x_2 f(x_0) + x_1 f(x_1) - x_1 f(x_0)}{(x_1 - x_0)(x_2 - x_1)}$$

Factoring the numerator of n:

$$n = \frac{[x_1(f(x_2) - f(x_1)) - x_0(f(x_2) - f(x_1))] + [x_2(f(x_0) - f(x_1)) - x_1(f(x_0) - f(x_1))]}{(x_1 - x_0)(x_2 - x_1)}$$

Factoring the numerator of n further:

$$n = \frac{(x_1 - x_0)(f(x_2) - f(x_1)) + (x_2 - x_1)(f(x_0) - f(x_1))}{(x_1 - x_0)(x_2 - x_1)}$$

Factoring out a negative:

$$n = \frac{(x_1 - x_0)(f(x_2) - f(x_1)) - (x_2 - x_1)(f(x_1) - f(x_0))}{(x_1 - x_0)(x_2 - x_1)}$$

Using properties of fractions:

$$n = \frac{(x_1 - x_0)(f(x_2) - f(x_1))}{(x_1 - x_0)(x_2 - x_1)} - \frac{(x_2 - x_1)(f(x_1) - f(x_0))}{(x_1 - x_0)(x_2 - x_1)}$$

Doing some canceling:

$$n = \frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0} = f[x_1, x_2] - f[x_0, x_1]$$

Plugging n back into (1):

$$a_2 = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = f[x_0, x_1, x_2],$$

as desired.

Question 4

From what we are given, we immediately know:

$$P_0(x) = 1, P_1(x) = 1.64872, P_2(x) = 2.71828, P_3(x) = 4.48169$$

We find $P_{0,1}$:

$$P_{0,1}(x) = \frac{(x - x_0)P_1(x) - (x - x_1)P_0(x)}{x_1 - x_0} = \frac{(x - 0)1.64872 - (x - .25)1}{.25 - 0} = \frac{.64872x + .25}{.25}$$

$$P_{0,1}(x) = 2.59488x + 1$$

We find $P_{1,2}$:

$$P_{1,2}(x) = \frac{(x - x_1)P_2(x) - (x - x_2)P_1(x)}{x_2 - x_1} = \frac{(x - .25)2.71828 - (x - .5)1.64872}{.5 - .25} = \frac{1.06956x + .14479}{.25}$$

$$P_{1,2}(x) = 4.27824x + .57916$$

We find $P_{2,3}$:

$$P_{2,3}(x) = \frac{(x - x_2)P_3(x) - (x - x_3)P_2(x)}{x_3 - x_2} = \frac{(x - .5)4.48169 - (x - .75)2.71828}{.75 - .5} = \frac{1.76341x - .20214}{.25}$$

$$P_{2,3}(x) = 7.05364x - .80854$$

We find $P_{0,1,2}$

$$P_{0,1,2}(x) = \frac{(x - x_0)P_{1,2}(x) - (x - x_2)P_{0,1}(x)}{x_2 - x_0} = \frac{(x - 0)(4.27824x + .57916) - (x - .5)(2.59488x + 1)}{.5 - 0}$$

$$P_{0,1,2}(x) = 3.36672x^2 + 1.7532x + 1$$

We find $P_{1,2,3}$:

$$P_{1,2,3}(x) = \frac{(x - x_1)P_{2,3}(x) - (x - x_3)P_{1,2}(x)}{x_3 - x_1} = \frac{(x - .25)(7.05364x - .80854) - (x - .75)(4.27824x + .57916)}{.75 - .25}$$

$$P_{1,2,3}(x) = 5.5508x^2 + .11514x + 1.27301$$

Finally, we can find $P_{0,1,2,3}$:

$$P_{0,1,2,3}(x) = \frac{(x - x_0)P_{1,2,3}(x) - (x - x_3)P_{0,1,2}(x)}{x_3 - x_0}$$

$$P_{0,1,2,3}(x) = \frac{(x - 0)(5.5508x^2 + .11514x + 1.27301) - (x - .75)(3.36672x^2 + 1.7532x + 1)}{.75 - 0}$$

$$P_{0,1,2,3}(x) = 2.91210x^3 + 1.18264x^2 + 2.11721x + 1$$

Finally, the approximations are:

$$P_{0,1}(.43) = 2.59488(.43) + 1 = 2.11580$$

$$P_{1,2}(.43) = 4.27824(.43) + .57916 = 2.41880$$

$$P_{2,3}(.43) = 7.05364(.43) - .80854 = 2.22441$$

$$P_{0,1,2}(.43) = 3.36672(.43)^2 + 1.7532(.43) + 1 = 2.37638$$

$$P_{1,2,3}(.43) = 5.5508(.43)^2 + .11514(.432.34886) + 1.27301 = 2.34886$$

$$P_{0,1,2,3}(.43) = 2.91210(.43)^3 + 1.18264(.43)^2 + 2.11721(.43) + 1 = 2.36060$$

Question 5

We begin by finding $P_{0,1,2}(x)$:

$$P_{0,1,2}(x) = \frac{(x - x_2)P_{0,1}(x) - (x - x_1)P_{0,2}(x)}{x_1 - x_2} = \frac{(x-2)(2x+1) - (x-1)(x+1)}{1-2} = -x^2 + 3x + 1$$

We calculate $P_{0,1,2}(2.5)$:

$$P_{0,1,2}(2.5) = -(2.5^2) + 3(2.5) + 1 = 2.25$$

Then, we can find $P_{0,1,2,3}(x)$:

$$P_{0,1,2,3}(x) = \frac{(x - x_3)P_{0,1,2}(x) - (x - x_0)P_{1,2,3}(x)}{x_0 - x_3} = \frac{(x-3)(-x^2 + 3x + 1) - (x-0)P_{1,2,3}(x)}{-3}$$

We calculate $P_{0,1,2,3}(2.5)$:

$$P_{0,1,2,3}(2.5) = \frac{(2.5-3)(2.25) - (2.5-0)(3)}{-3} = 2.875$$

Question 6

From what we are given, we immediately know:

$$f[x_0] = 16.94410, f[x_1] = 17.56492, f[x_2] = 18.50515, f[x_3] = 18.82091$$

We calculate $f[x_0, x_1]$:

$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0} = \frac{17.56492 - 16.94410}{8.3 - 8.1} = 3.10410$$

We calculate $f[x_1, x_2]$:

$$f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1} = \frac{18.50515 - 17.56492}{8.6 - 8.3} = 3.13410$$

We calculate $f[x_2, x_3]$:

$$f[x_2, x_3] = \frac{f[x_3] - f[x_2]}{x_3 - x_2} = \frac{18.82091 - 18.50515}{8.7 - 8.6} = 3.15760$$

We calculate $f[x_0, x_1, x_2]$:

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{3.13410 - 3.10410}{8.6 - 8.1} = .06000$$

We calculate $f[x_1, x_2, x_3]$:

$$f[x_1, x_2, x_3] = \frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1} = \frac{3.15760 - 3.13410}{8.7 - 8.3} = .05875$$

We calculate $f[x_0, x_1, x_2, x_3]$:

$$f[x_0, x_1, x_2, x_3] = \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0} = \frac{.05875 - .06000}{8.7 - 8.1} = -.00208$$

With degree one:

$$P_1(x) = 16.94410 + 3.10410(x - 8.1)$$
$$P_1(8.4) = 17.87533$$

With degree two:

$$P_2(x) = 16.94410 + 3.10410(x - 8.1) + .06000(x - 8.1)(x - 8.3)$$
$$P_2(8.4) = 17.87713$$

With degree three:

$$P_3(x) = 16.94410 + 3.10410(x - 8.1) + .06000(x - 8.1)(x - 8.3) - .00208(x - 8.1)(x - 8.3)(x - 8.6)$$

$$P_3(8.4) = 17.87714$$

```r
newtonMethod
```

```
## function(f, fprime, initial, iter, tol){
##   results <- c(initial)
##
##   count <- 1
##   while(count <= iter & abs(f(results[count])) > tol){
##     results <- c(results, results[count] - f(results[count])/fprime(results[count]))
##     count <- count + 1
##   }
##
##   cbind("n" = 0:(length(results) - 1),
##         "p_n" = results,
##         "|f(p_n)|" = abs(f(results)))
## }
## <bytecode: 0x7fb48b8f9d00>
```

```r
modifiedNewtonMethod
```

```
## function(f, fprime, fprimeprime, initial, iter, tol){
##   results <- c(initial)
##
##   count <- 1
##   while(count <= iter & abs(f(results[count])) > tol){
##     top <- f(results[count]) * fprime(results[count])
##     bottom <- (fprime(results[count]))^2 - f(results[count])*fprimeprime(results[count])
##     results <- c(results, results[count] - top/bottom)
##     count <- count + 1
##   }
##
##   cbind("n" = 0:(length(results) - 1),
##         "p_n" = results,
##         "|f(p_n)|" = abs(f(results)))
## }
## <bytecode: 0x7fb48375ef88>
```