MATH 151B - Applied Numerical Methods - Homework 4

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Question 1

We are given the following equation:

$$y(t_i) + h[af(t_i, y(t_i)) + bf(t_{i-1}, y(t_{i-1})) + cf(t_{i-2}, y(t_{i-2})) + df(t_{i-3}, y(t_{i-3}))]$$

Since $y'(t_i) = f(t_i, y(t_i))$:

$$y(t_i) + h[ay'(t_i) + by'(t_{i-1}) + cy'(t_{i-2}) + dy'(t_{i-3})] \tag{1}$$

We expand $y'(t_{i-1})$:

$$y'(t_{i-1}) = y'(t_i - h) \approx y'(t_i) - hy''(t_i) + \frac{1}{2}h^2 y^{(3)}(t_i) - \frac{1}{6}h^3 y^{(4)}(t_i) \tag{2}$$

We expand $y'(t_{i-2})$:

$$y'(t_{i-2}) = y'(t_i - 2h) \approx y'(t_i) - 2hy''(t_i) + 2h^2 y^{(3)}(t_i) - \frac{8}{6}h^3 y^{(4)}(t_i) \tag{3}$$

We expand $y'(t_{i-3})$:

$$y'(t_{i-3}) = y'(t_i - 3h) \approx y'(t_i) - 3hy''(t_i) + \frac{9}{2}h^2 y^{(3)}(t_i) - \frac{27}{6}h^3 y^{(4)}(t_i) \tag{4}$$

We plug (2), (3), and (4) into (1) and do some simplifications to get:

$$y(t_i)+h\left[(a+b+c+d)y'(t_i)+(-bh-2ch-3dh)y''(t_i)+\left(\frac{bh^2}{2}+2ch^2+\frac{9dh^2}{2}\right)y^{(3)}(t_i)+\left(-\frac{ah^3}{6}-\frac{8bh^3}{6}-\frac{27dh^3}{6}\right)y^{(4)}(t_i)\right]$$

Doing some further simplification:

$$y(t_i)+(a+b+c+d)hy'(t_i)+(-b-2c-3d)h^2y''(t_i)+\left(\frac{b}{2}+2c+\frac{9d}{2}\right)h^3y^{(3)}(t_i)+\left(-\frac{a}{6}-\frac{8b}{6}-\frac{27d}{6}\right)h^4y^{(4)}(t_i) \tag{5}$$

Since we are trying to approximate $y(t_{i+1})$:

$$y(t_{i+1}) = y(t_i + h) \approx y(t_i) + hy'(t_i) + \frac{1}{2}h^2 y''(t_i) + \frac{1}{6}h^3 y^{(3)}(t_i) + \frac{1}{24}h^4 y^{(4)}(t_i) \tag{6}$$

We set (5) and (6) to each other and get the following equations:

$$(a+b+c+d) = 1, \qquad (-b-2c-3d) = \frac{1}{2}, \qquad \left(\frac{b}{2} + 2c + \frac{9d}{2}\right) = \frac{1}{6}, \qquad \left(-\frac{a}{6} - \frac{8b}{6} - \frac{27d}{6}\right) = \frac{1}{24}$$

Solving the equations using a calculator, we get:

$$a = \frac{55}{24}, \qquad b = -\frac{59}{24}, \qquad c = \frac{37}{24}, \qquad d = -\frac{9}{24}$$

Question 2

Part A

We begin with the following equation:

$$w_{i+1} = w_i + af(t_{i+1}, w_{i+1}) + bf(t_i, w_i)$$

This can be rewritten as:

$$y(t_{i+1}) = y(t_i) + ay'(t_{i+1}) + by'(t_i) \tag{1}$$

We approximate $y(t_{i+1})$:

$$y(t_{i+1}) = y(t_i + h) \approx y(t_i) + hy'(t_i) + \frac{h^2}{2}y''(t_i) \tag{2}$$

We approximate $y'(t_{i+1})$:

$$y'(t_{i+1}) = y'(t_i + h) \approx y'(t_i) + hy''(t_i) \tag{3}$$

Plugging in (2) and (3) into (1):

$$y(t_i) + hy'(t_i) + \frac{h^2}{2}y''(t_i) = y(t_i) + a\Big[y'(t_i) + hy''(t_i)\Big] + by'(t_i)$$

Simplifying the right side:

$$y(t_i) + hy'(t_i) + \frac{h^2}{2}y''(t_i) = y(t_i) + (a+b)y'(t_i) + ahy''(t_i)$$

We obtain the following two equations:

$$(a+b) = h, \qquad ah = \frac{h^2}{2}$$

We solve the two equations to get:

$$a = b = \frac{h}{2}$$

Thus, the one-step implicit method is:

$$w_{i+1} = w_i + \frac{h}{2}f(t_{i+1}, w_{i+1}) + \frac{h}{2}f(t_i, w_i) = w_i + \frac{h}{2}\Big[f(t_{i+1}, w_{i+1}) + f(t_i, w_i)\Big]$$

Part B

```
def predictor_corrector(func, a, b, initial, h):
  N = round((b-a)/h)
  t = [a]
  w_tilde = [initial]
  w = [initial]

  for i in range(0, N):
    w_tilde.append(w[i] + h*func(t[i] + h/2, w[i] + h*func(t[i], w[i])/2))
    w.append(w[i] + h/2*(func(t[i] + h, w_tilde[i+1]) + func(t[i], w[i])))
    t.append(t[i] + h)
  return(t, w_tilde, w)
```

```
import math
import numpy as np
import pandas as pd

def func(t, u):
  A = [[0,1], [4,0]]
  b = [[0], [6*math.exp(-t)]]
  return(np.dot(A, u) + b)
```

Part C

```
t, w_tilde, w = predictor_corrector(func, 0, 1, np.array([[0], [0]]), .1)

r = pd.DataFrame({'i':range(0,len(t)), 't_i':t, '[[w(t_i)], [w\'(t_i)]]':w}).to_string(index=False)
print(r)
```

```
##   i  t_i                         [[w(t_i)], [w'(t_i)]]
##   0  0.0                                    [[0], [0]]
##   1  0.1  [[0.028536882735021418], [0.577451225410788]]
##   2  0.2  [[0.11325143344896317], [1.1230286263863505]]
##   3  0.3    [[0.252306376875106], [1.6640198546738365]]
##   4  0.4    [[0.446559152426227], [2.2270184173641536]]
##   5  0.5    [[0.6995480401771843], [2.839045815292328]]
##   6  0.6    [[1.0175911226366678], [3.528686012530031]]
##   7  0.7      [[1.410001605657765], [4.32727849810804]]
##   8  0.8    [[1.8894277626620628], [5.270218508951276]]
##   9  0.9    [[2.4723308352778446], [6.398417256187824]]
##  10  1.0     [[3.179619825562007], [7.759981406995886]]
```

As we can see above, we estimate $y(1)$ to be 3.179619825562007.

Question 3

We are given the following:

$$w_{i+1} = w_i + \frac{h}{24}\Big(9f(t_{i+1}, w_{i+1}) + 19f(t_i, w_i) - 5f(t_{i-1}, w_{i-1}) + f(t_{i-2}, w_{i-2})\Big)$$

Since $y'(t) = yg(t)$:

$$w_{i+1} = w_i + \frac{h}{24}\Big(9w_{i+1}(g(t_{i+1})) + 19w_i(g(t_i)) - 5w_{i-1}(g(t_{i-1})) + w_{i-2}(g(t_{i-2}))\Big)$$

Rearranging:

$$w_{i+1}\Big(1 - \frac{9}{24}hg(t_{i+1})\Big) = w_i + \frac{h}{24}\Big(19w_i(g(t_i)) - 5w_{i-1}(g(t_{i-1})) + w_{i-2}(g(t_{i-2}))\Big)$$

Getting a common denominator on the left side:

$$w_{i+1}\Big(\frac{24 - 9hg(t_{i+1})}{24}\Big) = w_i + \frac{h}{24}\Big(19w_i(g(t_i)) - 5w_{i-1}(g(t_{i-1})) + w_{i-2}(g(t_{i-2}))\Big)$$

Isolating w_{i+1}:

$$w_{i+1} = \Big(\frac{24}{24 - 9hg(t_{i+1})}\Big)\Big[w_i + \frac{h}{24}\Big(19w_i(g(t_i)) - 5w_{i-1}(g(t_{i-1})) + w_{i-2}(g(t_{i-2}))\Big)\Big]$$

Simplifying:

$$w_{i+1} = \frac{24w_i + h(19w_i(g(t_i)) - 5w_{i-1}(g(t_{i-1})) + w_{i-2}(g(t_{i-2})))}{24 - 9hg(t_{i+1})}, \text{ for } i = 2, ..., N-1,$$

which is an explicit version of the Adams-Moulton 3-step implicit method, as desired