UCLA final, Math 164, Summer 2020

Student name and ID number: ______________________________

Instructions:

- This is a 24 hours open-notes exam.
- Clarity will also be considered in grading.
- Please upload directly your solutions to gradescope.

Question	Points	Score
1	10	
2	8	
3	10	
4	8	
5	10	
6	11	
Total:	57	

1. Consider the problem

$$\text{minimize} f(x_1, x_2) = x_1^3 + 2x_2^3 - x_1 - 4x_2 + 2.$$

   (a) (4 points) Find all of the points $(x_1, x_2)^T$ that satisfy the first-order necessary condition (FONC).

   (b) (4 points) For each of the points in the above question, identify whether it a local minimizer, local maximizer, or saddle point.

   (c) (2 points) Is there a global minimizer?

2. **Steepest descent for unconstrained quadratic function minimization**

The steepest descent method for

$$\text{minimize } f(\mathbf{x})$$

is the gradient descent method using exact line search, that is, the step size of the kth iteration is chosen as

$$\alpha_k = \operatorname*{argmin}_{\alpha \geq 0} f(\mathbf{x}^k - \alpha \nabla f(\mathbf{x}^k)).$$

(a) (3 points) Consider the objective function

$$f(\mathbf{x}) := \frac{1}{2}\mathbf{x}^T A\mathbf{x} - \mathbf{c}^T\mathbf{x} + d,$$

where $A \in \mathbb{R}^{n\times n}, \mathbf{c} \in \mathbb{R}^n, d \in \mathbb{R}$ are given. Assume that A is symmetric positive definite and, at $\mathbf{x}^k$, $\nabla f(\mathbf{x}^k) \neq 0$. Give a formula of α_k in terms $\mathbf{x}^k, A, \mathbf{c}, d$.

(b) (5 points) Consider $\mathbb{R}^2$. Starting from $\mathbf{x}^0 = (0, 0)$, perform *two iterations* of the steepest descent method for

$$\text{minimize} \quad f(x_1, x_2) := \frac{1}{2}(x_1^2 + x_2^2) + 2x_2 + 1.$$

3. **Newton's method**

Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) := \frac{1}{3}|x-a|^3$, where $a \in \mathbb{R}$ is a constant. The minimizer is obviously $x^* = a$.

Suppose that we apply Newton's method to the following problem:

$$\text{minimize} \quad f(x) := \frac{1}{3}|x-a|^3$$

from an initial point $x^0 \in \mathbb{R} \setminus \{a\}$.

(a) (3 points) Write down $f'(x)$ and $f''(x)$. You need to consider two cases: $x \geq a$ and $x < a$.

(b) (2 points) Write down the update equation for Newton's method applied to the problem.

(c) (2 points) Let x^k be the kth iterate in Newton's method. Provide a formula for $|x^k - a|$ in terms of x_0.

(d) (3 points) Does $x^k \to a$ for any initial point $x^0 \in \mathbb{R} \setminus \{a\}$? If so, what is the order of convergence?

4. (8 points) **Broyden's method**

Use Broyden's method with starting point $x^{(0)} = (1,1)^T$ to compute the second step $x^{(2)}$ of the approximation of the solution of the following nonlinear system:

$$
\begin{aligned}
3x_1^2 - x_2^2 &= 0 \\
3x_1 x_2^2 - x_1^3 - 1 &= 0.
\end{aligned}
$$

5. (10 points) **Optimization in neural network**

Consider a very simple neural network with two input values, one output value, and a single neuron with sigmoid activation. Each input to the neuron has an associated weight, and the neuron has a bias. So the network represents functions of the form $\sigma(w_1 x_1 + w_2 x_2 + b)$. We train the neural network using least squares loss on a single piece of training data $((1, -1), 0)$. Initially all weights and biases are set to 1. Carry out one iteration of gradient descent using a step size of 2.

6. **Problem on Linear programming and Simplex method**

The ℓ_1 norm of a vector $v \in \mathbb{R}$ is defined by

$$\|v\|_1 := \sum_{i=1}^{n} |v_i|$$

Problems of the form Minimize $\|v\|_1$ subject to $v \in \mathbb{R}^n$ and $Av = b$ arise very frequently in applied math, particularly in the field of compressed sensing.

Consider the special case of this problem whith $n = 3$,

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad b = \begin{pmatrix} 3 \\ 8 \end{pmatrix}.$$

(a) (3 points) Explain how to transform this into the following equivalent linear program in standard form (no need for a complete proof of equivalence)

$$\text{Minimize } x_1 + x_2 + x_3 + x_4 + x_5 + x_6$$

under the constraints

$$\begin{aligned} x_1 - x_2 + x_3 - x_4 &= 3 \\ 3x_1 - 3x_2 + x_5 - x_6 &= 8 \\ x_1, x_2, x_3, x_4, x_5, x_6 &\geq 0. \end{aligned}$$

(b) (6 points) Solve the linear program of part (a) using the simplex method.

(c) (2 points) What $v \in \mathbb{R}^3$ minimizes the original problem?