MATH 164 - Optimization - Quiz 1

Darren Tsang, 405433124

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Note: I referenced the course textbook as I was completing this quiz. Furthermore, I collaborated with Tatiana Rosenberg for both of these problems.

Question 1

We are given the following function $f(x)$:

$$f(x) = f(x_1, x_2) = (x_1 + x_2^2)^2 = x_1^2 + 2x_1x_2^2 + x_2^4 \tag{1}$$

The gradient of $f(x)$ is:

$$\nabla f(x) = \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}\right]^T = [2x_1 + 2x_2^2, 4x_1x_2 + 4x_2^3]^T \tag{2}$$

The Hessian of $f(x)$ is:

$$Hess(f) = \boldsymbol{F}(x) = \begin{bmatrix} 2 & 4x_2 \\ 4x_2 & 12x_2^2 \end{bmatrix} \tag{3}$$

Part A

As we can see above, the gradient of $f(x)$ is:

$$\nabla f(x) = [2x_1 + 2x_2^2, 4x_1x_2 + 4x_2^3]^T$$

Part B

$$d^T \nabla f(x) = [1, -1] \begin{bmatrix} 2(0) + 2(1)^2 \\ 4(0)(1) + 4(1)^3 \end{bmatrix} = [1, -1] \begin{bmatrix} 2 \\ 4 \end{bmatrix} = -2 < 0$$

Since $-2 < 0$, $f(x^{(0)})$ decreases into the direction d, as desired.

Part C

We define w as the following:

$$w = x^{(0)} + \alpha d = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \alpha \\ 1 - \alpha \end{bmatrix}$$

Then:

$$f(w) = f([\alpha, 1 - \alpha]^T) = (\alpha + (1 - \alpha)^2)^2 = (\alpha^2 - \alpha + 1)^2 = \alpha^4 - 2\alpha^3 - 3\alpha^2 - 2\alpha + 1$$

We take the derivative of $f(w)$ respect to α:

$$\frac{df(w)}{d\alpha} = 4\alpha^3 - 6\alpha^2 - 6\alpha - 2$$

Setting the derivative equal to 0 and solving:

$$4\alpha^3 - 6\alpha^2 - 6\alpha - 2 = 0 \longrightarrow \alpha = \frac{1}{2}$$

Now that we know α, we can find $f(x^{(0)} + \alpha d) = f(w)$:

$$f(w) = \left(\frac{1}{2}\right)^4 - 2\left(\frac{1}{2}\right)^3 - 3\left(\frac{1}{2}\right)^2 - 2\left(\frac{1}{2}\right) + 1 = .5625$$

Part D

To find the minimum, we set the gradient to 0:

$$[2x_1 + 2x_2^2, 4x_1x_2 + 4x_2^3]^T = [0, 0]^T$$

Solving for x_1 and x_2, we see that x_2 is a free variable and $x_1 = -(x^2)$. Thus, the following points in the set S obtain the minimum:

$$S = \{[x_1, x_2]^T, \text{ where } x_2 \in \mathbb{R},\ x_1 = -(x_2^2)\} \subset \mathbb{R}^2$$

To find the minimum value, we find $f(d)$, for any $d \in S$. Let $d = [-1, 1]^T$:

$$f([-1, 1]^T) = (-1 + 1^2)^2 = 0$$

Thus, the minimum of $f(x)$ is 0, which is obtained when $x \in S$, as defined above.

Part E

We check the first condition for SONC:

$$\nabla f([0, 0]^T) = [2(0) + 2(0)^2, 4(0)(0) + 4(0)^3]^T = [0, 0]^T = 0$$

Let $d = [d_1, d_2]^T \in \mathbb{R}^2$. We check the second condition for SONC:

$$d^T Hess([0, 0]^T)d = [d_1, d_2] \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = [d_1, d_2] \begin{bmatrix} 2d_1 \\ 0 \end{bmatrix} = 2d_1^2 \geq 0$$

Since the two conditions are satisfied, $[0, 0]^T$ does satisfy the SONC.

Part F

We check the first condition for SOSC:

$$\nabla f([-1, 1]^T) = [2(-1) + 2(1)^2, 4(-1)(1) + 4(1)^3]^T = [0, 0]^T = 0$$

We check the second condition for SOSC:

$$\boldsymbol{F}([-1, 1]^T) = \begin{bmatrix} 2 & 4(1) \\ 4(1) & 12(1)^2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 4 & 12 \end{bmatrix} > 0$$

Since the two conditions are satisfied, $[-1, 1]^T$ does satisfy the SOSC.

Question 2

```r
newtonMethod <- function(f, fprime, initial, iter, tol){
  results <- c(initial)

  count <- 1
  while(count <= iter & abs(f(results[count])) > tol){
    results <- c(results, results[count] - f(results[count])/fprime(results[count]))
    count <- count + 1
  }
  cbind("n" = 0:(length(results) - 1), "p_n" = results, "|f(p_n)|" = abs(f(results)))
}
```

```r
f <- function(x){ x^2 }
fprime <- function(x) { 2*x }

newtonMethod(f = f, fprime = fprime, initial = 10^20, iter = Inf, tol = 10^-10)
```

```
##        n          p_n     |f(p_n)|
##  [1,]  0 1.000000e+20 1.000000e+40
##  [2,]  1 5.000000e+19 2.500000e+39
##  [3,]  2 2.500000e+19 6.250000e+38
##  [4,]  3 1.250000e+19 1.562500e+38
##  [5,]  4 6.250000e+18 3.906250e+37
##  [6,]  5 3.125000e+18 9.765625e+36
##  [7,]  6 1.562500e+18 2.441406e+36
##  [8,]  7 7.812500e+17 6.103516e+35
##  [9,]  8 3.906250e+17 1.525879e+35
## [10,]  9 1.953125e+17 3.814697e+34
## [11,] 10 9.765625e+16 9.536743e+33
## [12,] 11 4.882812e+16 2.384186e+33
## [13,] 12 2.441406e+16 5.960464e+32
## [14,] 13 1.220703e+16 1.490116e+32
## [15,] 14 6.103516e+15 3.725290e+31
## [16,] 15 3.051758e+15 9.313226e+30
## [17,] 16 1.525879e+15 2.328306e+30
## [18,] 17 7.629395e+14 5.820766e+29
## [19,] 18 3.814697e+14 1.455192e+29
## [20,] 19 1.907349e+14 3.637979e+28
## [21,] 20 9.536743e+13 9.094947e+27
## [22,] 21 4.768372e+13 2.273737e+27
## [23,] 22 2.384186e+13 5.684342e+26
## [24,] 23 1.192093e+13 1.421085e+26
## [25,] 24 5.960464e+12 3.552714e+25
## [26,] 25 2.980232e+12 8.881784e+24
## [27,] 26 1.490116e+12 2.220446e+24
## [28,] 27 7.450581e+11 5.551115e+23
## [29,] 28 3.725290e+11 1.387779e+23
## [30,] 29 1.862645e+11 3.469447e+22
## [31,] 30 9.313226e+10 8.673617e+21
## [32,] 31 4.656613e+10 2.168404e+21
## [33,] 32 2.328306e+10 5.421011e+20
```

```
## [34,] 33 1.164153e+10 1.355253e+20
## [35,] 34 5.820766e+09 3.388132e+19
## [36,] 35 2.910383e+09 8.470329e+18
## [37,] 36 1.455192e+09 2.117582e+18
## [38,] 37 7.275958e+08 5.293956e+17
## [39,] 38 3.637979e+08 1.323489e+17
## [40,] 39 1.818989e+08 3.308722e+16
## [41,] 40 9.094947e+07 8.271806e+15
## [42,] 41 4.547474e+07 2.067952e+15
## [43,] 42 2.273737e+07 5.169879e+14
## [44,] 43 1.136868e+07 1.292470e+14
## [45,] 44 5.684342e+06 3.231174e+13
## [46,] 45 2.842171e+06 8.077936e+12
## [47,] 46 1.421085e+06 2.019484e+12
## [48,] 47 7.105427e+05 5.048710e+11
## [49,] 48 3.552714e+05 1.262177e+11
## [50,] 49 1.776357e+05 3.155444e+10
## [51,] 50 8.881784e+04 7.888609e+09
## [52,] 51 4.440892e+04 1.972152e+09
## [53,] 52 2.220446e+04 4.930381e+08
## [54,] 53 1.110223e+04 1.232595e+08
## [55,] 54 5.551115e+03 3.081488e+07
## [56,] 55 2.775558e+03 7.703720e+06
## [57,] 56 1.387779e+03 1.925930e+06
## [58,] 57 6.938894e+02 4.814825e+05
## [59,] 58 3.469447e+02 1.203706e+05
## [60,] 59 1.734723e+02 3.009266e+04
## [61,] 60 8.673617e+01 7.523164e+03
## [62,] 61 4.336809e+01 1.880791e+03
## [63,] 62 2.168404e+01 4.701977e+02
## [64,] 63 1.084202e+01 1.175494e+02
## [65,] 64 5.421011e+00 2.938736e+01
## [66,] 65 2.710505e+00 7.346840e+00
## [67,] 66 1.355253e+00 1.836710e+00
## [68,] 67 6.776264e-01 4.591775e-01
## [69,] 68 3.388132e-01 1.147944e-01
## [70,] 69 1.694066e-01 2.869859e-02
## [71,] 70 8.470329e-02 7.174648e-03
## [72,] 71 4.235165e-02 1.793662e-03
## [73,] 72 2.117582e-02 4.484155e-04
## [74,] 73 1.058791e-02 1.121039e-04
## [75,] 74 5.293956e-03 2.802597e-05
## [76,] 75 2.646978e-03 7.006492e-06
## [77,] 76 1.323489e-03 1.751623e-06
## [78,] 77 6.617445e-04 4.379058e-07
## [79,] 78 3.308722e-04 1.094764e-07
## [80,] 79 1.654361e-04 2.736911e-08
## [81,] 80 8.271806e-05 6.842278e-09
## [82,] 81 4.135903e-05 1.710569e-09
## [83,] 82 2.067952e-05 4.276424e-10
## [84,] 83 1.033976e-05 1.069106e-10
## [85,] 84 5.169879e-06 2.672765e-11
```

From above, we can see it took 84 iterations to obtain the root (which is 0) with an accuracy of 10^{-10}.