UCLA midterm, Math 164, Summer 2020

Student name and ID number: ____

Instructions:

- This is a 24 hours open-notes exam.
- Clarity will also be considered in grading.
- Please upload directly your solutions to gradescope.

Question	Points	Score
1	10	
2	16	
3	10	
4	8	
Total:	44	

1. Consider the function $f(x) = e^x + x^2$ on the interval $[-1, 1]$.
   (a) (6 points) Carry out the golden-ratio and the bisection method until the uncertainty interval is less than 0.2.

   (b) (4 points) What is the order of the convergence of the bisection method? Justify your answer.

2. Let A be an arbitrary $m \times n$ matrix, and let $b \in \mathbb{R}^n$ a fixed vector. Consider the function $f : \mathbb{R}^n \to \mathbb{R}$ given by

$$f(x) = \frac{1}{2}\|Ax\|^2 + \frac{1}{2}\|x - b\|^2.$$

(a) (5 points) Show that the function $f : \mathbb{R}^n \to \mathbb{R}^n$ has a global minimizer and find it in terms of A and b.

(b) (5 points) For the rest of the problem, consider the special case

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad b = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

For which step sizes does the gradient descent algorithm converge?

(c) (3 points) Write down the update for the steepest descent in this cases. Does it converge?

(d) (3 points) Does Newton's method globally converge for this function?

3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x) = \frac{1}{2}x_1^2 + \frac{1}{3}x_2^2 - \frac{2}{3}x_1 x_2 - 2x_1 - x_2.$$

(a) (3 points) Express $f(x)$ in the form of $f(x) = \frac{1}{2}x \cdot Qx + x \cdot b$.

(b) (4 points) Find the minimizer of f using the conjugate gradient algorithm. Use $x^{(0)} = (0, 0)^T$ as the starting point.

(c) (3 points) Calculate the minimizer of f analytically from Q and b.

4. (8 points) As you know, Newton's method can also be used to approximate solutions to systems of nonlinear equations. Use Newton's method with starting point $x^{(0)} = (0,0)^T$ to compute the second step $x^{(2)}$ of the approximation of the solution of the following nonlinear system:

$$\begin{aligned} 5x_1^2 - x_2^2 &= 0 \\ x_2 - 0.25(\sin x_1 + \cos x_2) &= 0. \end{aligned}$$