Module: 3

Generating Functions and Recurrence Relations

1. Sequence generated by the Following functions:

(i) $(3+x)^3$

$$\boxed{\therefore (1+x)^n = 1+nx+\frac{n(n-1)}{1\times 2}x^2+\frac{n(n-1)(n-2)}{1\times 2\times 3}x^3+\cdots = \sum_{r=0}^{\infty}\binom{n}{r}x^r}$$

when n is a positive integer

Sol:- $(3+x)^3 = \left(3\left(1+\frac{x}{3}\right)\right)^3$

$$= (3)^3 \cdot \left(1+\frac{x}{3}\right)^3$$

$$= 27 \cdot \left\{ {}^3C_0\left(\frac{x}{3}\right)^0 + {}^3C_1\left(\frac{x}{3}\right)^1 + {}^3C_2\left(\frac{x}{3}\right)^2 + {}^3C_3\left(\frac{x}{3}\right)^3 \right\}$$

$$= 27 \cdot \left\{ 1+3\left(\frac{x}{3}\right)^1 + 3\left(\frac{x}{3}\right)^2 + 1\left(\frac{x}{3}\right)^3 \right\}$$

$$= 27+\cancel{3}\cdot 27\left(\frac{x}{\cancel{3}}\right)+27\cdot 3\left(\frac{x}{3}\right)^2+27\left(\frac{x}{3}\right)^3$$

$$= 27+27x+\overset{3}{\cancel{27}}\cdot 3\left(\frac{x^2}{\cancel{9}}\right)+\cancel{27}\left(\frac{x^3}{\cancel{27}}\right)$$

$$= 27+27x+9x^2+x^3$$

$\therefore$ The sequence generated by $(3+x)^3$ is constants coefficients

27, 27, 9, 1, 0, 0 ......

(ii) $2x^2(1-x)^{-1}$

$$\boxed{\therefore (1-x)^{-1} = 1+x+x^2+x^3+\cdots\cdots = \sum_{r=0}^{\infty} x^r}$$

Sol :- $2x^2(1+x+x^2+x^3+\cdots\cdots)$

$= 2x^2+2x^3+2x^4+2x^5+\cdots\cdots$

$= 0+0x+2x^2+2x^3+2x^4+\cdots\cdots$

$\therefore$ The sequence generated by $2x^2(1-x)^{-1}$ is $0, 0, 2, 2, 2\cdots\cdots$

(iii) $e^{2x} = 1+\frac{2x}{1!}+\frac{(2x)^2}{2!}+\frac{(2x)^3}{3!}+\cdots\cdots\cdots$

$$\boxed{\therefore e^x = \sum_{r=0}^{\infty} a_r \frac{(x)^r}{r!}}$$

$= 1+\frac{2}{1!}x+\frac{2^2}{2!}x^2+\frac{2^3}{3!}x^3+\cdots\cdots$

Given :-

$3x^3+e^{2x} = 1+\frac{2}{1!}x+\frac{2^2}{2!}x^2+\left(3+\frac{2^3}{3!}\right)x^3+\cdots\cdots$

$\therefore$ The sequence generated by $3x^3+e^{2x}$ is

$1, \frac{2}{1!}, \frac{2^2}{2!}, 3+\frac{2^3}{3!}, \frac{2^4}{4!}, \cdots\cdots$

(2) $f(x)$ is generating function for the sequences $\langle a_r \rangle$

$g(x)$ is generating function for the sequence $\langle b_r \rangle$

- Express $g(x)$ in terms of $f(x)$.

i) $b_3 = 3$, $b_7 = 7$, $b_n = a_n$ for $n \neq 3, 7$

Sol :- $g(x) = \sum_{r=0}^{\infty} b_r x^r$

$= b_0 + b_1 x^1 + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 + \cdots$

Scanned with OKEN Scanner

interms of $\langle a_r \rangle$

$$= a_0 + a_1 x + a_2 x^2 + 3x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + 7x^7 + a_8 x^8 + \dots$$

$$= \left\{ \sum_{r=0}^{\infty} a_r x^r - a_3 x^3 - a_7 x^7 \right\} + 3x^3 + 7x^7$$

$$= \sum_{r=0}^{\infty} a_r x^r + (3 - a_3) x^3 + (7 - a_7) x^7$$

in terms of $f(x)$

$$\boxed{f(x) = f(x) + (3 - a_3) x^3 + (7 - a_7) x^7}$$

3. Find the generating function for the following sequences

i) $0, 1, 2, 3, 4, \dots\dots$

By Binomial expansion

$$\boxed{\therefore (1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots}$$

$$0 + 1x + 2x^2 + 3x^3 + \dots = x(1 + 2x + 3x^2 + \dots)$$

$$= x(1-x)^{-2}$$

$\therefore$ The sequence generated by $f(x) = x(1-x)^{-2}$ is

$0, 1, 2, 3, 4 \dots\dots$

$\therefore$ $f(x) = x(1-x)^{-2}$ is generating function for the sequence

$0, 1, 2, 3, 4 \dots\dots$

Scanned with OKEN Scanner

3) ii) $0^2, 1^2, 2^2, 3^2, 4^2, \ldots\ldots$

Sol :- we have

$$0^2 + 1^2 x + 2^2 x^2 + 3^2 x^3 + \cdots = x\left(1^2 + 2^2 x + 3^2 x^2 + \cdots\right) \Rightarrow x\left(x(1-x)^{-2}\right)$$

Differentiating on both sides we get

$$\frac{d}{dx}\left\{x\left(x(1-x)^{-2}\right)\right\}$$

$$= \frac{d}{dx}\left\{x \cdot \frac{x}{(1-x)^2}\right\}$$

$$f(x) = \frac{x(1+x)}{(1-x)^3}$$

These is the generating sequence $0^2, 1^2, 2^2, 3^2, \ldots$ for generating function.

4. Determine the co-efficient of

i) $$\boxed{\sum_{r=0}^{\infty} a_r x^r = 1+x+x^2+\dots+x^n = \frac{1-x^{n+1}}{1-x}}$$

$$\boxed{f(x) = \frac{(1-x^{n+1})}{(1-x)}}$$

$$\left(x^2+x^3+x^4+\dots\right)^4$$

Sol :- we have

$$\left(x^2+x^3+x^4+\dots\right)^4 = \left(x^2\left(1+x+x^2+\dots\right)\right)^4$$

$$= x^8\left[1+x+x^2+\dots\right]^4$$

$$= x^8\left\{(1-x)^{-1}\right\}^4$$

$$= x^8 \times \sum_{r=0}^{\infty} \binom{n+r-1}{r} x^r$$

$$= x^8 \times \sum_{r=0}^{\infty} \binom{4+r-1}{r} x^r$$

$$= x^8 \times \sum_{r=0}^{\infty} \binom{3+r}{r} x^r$$

∴ The co-efficient of x^n in this is

$$\binom{3+n-8}{n-8} = \binom{n-5}{(n-5)-(n-8)} = \binom{n-5}{3}$$

$$\boxed{\therefore \text{ Formula :- } \binom{n+r-1}{r} = \binom{n+r-1}{(n+r-1)-r}}$$

4. ii) $\left(1+x^2+x^4+\dots\right)^7 = \left\{\left(1-x^2\right)^{-1}\right\}^7$

$$= \left(1-x^2\right)^{-7} \Rightarrow \sum_{r=0}^{\infty} \binom{n+r-1}{r} x^r$$

$$= \sum_{r=0}^{\infty} \binom{7+r-1}{r} \left(x^2\right)^r$$

$$= \sum_{r=0}^{\infty} \binom{6+r}{r} \left(x^2\right)^r$$

$\therefore$ From this we note that when n is odd the co-efficient of x^n is zero, and when n is even say 2m the coefficient is ~~2m~~.

$$\binom{6+m}{m} \Rightarrow \binom{6+m}{(6+m)-m} = \binom{6+\frac{n}{2}}{6}$$

$$\Rightarrow \binom{6+m}{6} = \binom{6+\frac{n}{2}}{6}$$

(5) Find the number integer solution of the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 30$ under the constraints $x_i \geq 0$ for $i = 1,2,3,4,5$ and further x_2 is even and x_3 is odd.

Sol :- The constraints for x_1, x_2, x_3, x_4, x_5 is ≥ 0. where x_i is non-negative integer. Take values $0, 1, 2, 3, \ldots\ldots$

$$F_1(x) = x^0 + x^1 + x^2 + \ldots\ldots$$

$$F_2(x) = x^0 + x^2 + x^4 + x^6 + \ldots\ldots$$

$$F_3(x) = x^1 + x^3 + x^5 + \ldots\ldots$$

Scanned with OKEN Scanner

$$F_4(x) = x^0 + x^1 + x^2 + x^3 + \cdots\cdots$$

$$F_5(x) = x^0 + x^1 + x^2 + x^3 + \cdots\cdots$$

$\therefore$ The generating function associated with the problem is

$$f(x) = F_1(x) \cdot F_2(x) \cdot f_3(x) \cdot F_4(x) \cdot F_5(x)$$

$$= \left(x^0 + x^1 + x^2 + x^3 + \cdots\right)^3 \cdot F_2(x) \cdot f_3(x)$$

$$= \left(1 + x + x^2 + x^3 + \cdots\right)^3 \cdot f_2(x) \cdot f_3(x)$$

$$= \left[(1-x)^{-1}\right]^3 \cdot f_2(x) \cdot f_3(x)$$

$$f_1(x) \cdot f_4(x) \cdot f_5(x) = (1-x)^{-3}$$

$$F_2(x) = x^0 + x^2 + x^4 + x^6 + \cdots\cdots = (1-x^2)^{-1}$$

$$F_3(x) = x^1 + x^3 + x^5 + x^7 + \cdots\cdots = x(1-x^2)^{-1}$$

$$f(x) = x(1-x^2)^{-2}(1-x)^{-3}$$

$$= x \times \sum_{r=0}^{\infty} \binom{2+r-1}{r} (x^2)^r \times \sum_{r=0}^{\infty} \binom{3+s-1}{s} x^s$$

$$= \sum_{r=0}^{\infty} \binom{1+r}{r} x^{2r+1} \times \sum_{r=0}^{\infty} \binom{2+s}{s} x^s$$

The co-efficient of x^{30} is

$$C_{30} = \binom{1}{0}\binom{31}{29} + \binom{2}{1}\binom{29}{27} + \cdots\cdots + \binom{14}{13}\binom{5}{3} + \binom{15}{14}\binom{3}{1}$$

$\therefore$ This is the required number.

6. Find the generating function for the number of integer solutions to the equation $c_1+c_2+c_3+c_4=20$, where $-3 \le c_1, -3 \le c_2$, $-5 \le c_3 \le 5$ and $0 \le c_4$. Also find the number of such solutions.

Sol :- Let, $x_1 = c_1+3$, $x_2 = c_2+3$, $x_3 = c_3+5$, $x_4 = c_4$

$x_1 \ge 0$, $x_2 \ge 0$, $0 \le x_3 \le 10$, $x_4 \ge 0$ and the given equation

$$x_1+x_2+x_3+x_4 = 31$$

Thus, the number of integer solution of the given equation under the given constraints is,

$f_1(x) = x^0+x^1+x^2+x^3+\dots\dots = (1-x)^{-1}$

$f_2(x) = x^0+x^1+x^2+x^3+\dots\dots = (1-x)^{-1}$

$f_3(x) = x^0+x^1+x^2+x^3+\dots\dots+x^{10}$

$f_4(x) = x^0+x^1+x^2+\dots\dots = (1-x)^{-1}$

$\therefore$ The generating function is

$F(x) = f_1(x) \cdot f_2(x) \cdot f_3(x) \cdot f_4(x)$

$$= \left(1+x+x^2+x^3+\dots+x^{10}\right) \times (1-x)^{-3}$$

$$= \left(1+x+x^2+x^3+\dots+x^{10}\right) \times \sum_{r=0}^{\infty} \binom{n+r-1}{r} x^r$$

$\therefore$ consider -3 as $+3$

$$= \left(1+x+x^2+x^3+\dots+x^{10}\right) \times \sum_{r=0}^{\infty} \binom{3+r-1}{r} x^r$$

$$= \left(1+x+x^2+x^3+\dots+x^{10}\right) \times \sum_{r=0}^{\infty} \binom{2+r}{r} x^r$$

The co-efficient of x^{31} is :-

$$\boxed{C_{31} = \binom{33}{31} + \binom{32}{30} + \dots\dots + \binom{23}{21}}$$

Scanned with OKEN Scanner

7. In how many ways can 12 oranges be distributed among three children A, B and C, so that A gets at least four, B and C get at least two, but C gets no more than five?

Sol:- Let x_1 be the number of oranges which A can get,

x_2 be the number of oranges which B can get,

x_3 be the number of oranges which C can get.

Then

$$x_1 + x_2 + x_3 = 12$$

Given constraints are

$$x_1 \geq 4, \quad x_2 \geq 2, \quad 2 \leq x_3 \leq 5$$

The required number is equal to the number of integer solutions of the equation (i) under the constraints

$$f_1(x) = x^4 + x^5 + x^6 + \cdots \Rightarrow x^4(1 + x + x^2 + \cdots) \Rightarrow x^4(1-$$

$$f_2(x) = x^2 + x^3 + x^4 + x^5 + \cdots \Rightarrow x^2(1 + x + x^2 + x^3 + \cdots) \Rightarrow x^2(1-x$$

$$f_3(x) = x^2 + x^3 + x^4 + x^5 \Rightarrow x^2(1 + x + x^2 + x^3)$$

∴ The generating function associated with the problem is

$$f(x) = f_1(x) \cdot f_2(x) \cdot f_3(x)$$

$$= x^8(1 + x + x^2 + x^3)(1-x)^{-2}$$

$$= x^8(1 + x + x^2 + x^3) \sum_{r=0}^{\infty} \binom{2+r-1}{r} x^r$$

$$= x^8(1 + x + x^2 + x^3) \sum_{r=0}^{\infty} \binom{1+r}{r} x^r$$

The co-efficient of x^{12} in this

$$C_{12} = \binom{1+4}{4} + \binom{1+3}{3} + \binom{1+2}{2} + \binom{1+1}{1}$$

$$= ({}^5C_4) + ({}^4C_3) + ({}^3C_2) + ({}^2C_1) = 5 + 4 + 3 + 2 \Rightarrow 14$$

Scanned with OKEN Scanner

8. using generating function, Find the number of partitions of n=6 into distinct summands.

Sol :- Formula : Rule: 1 The co-efficient of x^n.

$$\prod_{r=1}^{\infty} \frac{1}{(1-x^r)} = \frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdot \frac{1}{1-x^3} \cdot \frac{1}{1-x^4} \cdot \frac{1}{1-x^5} \cdot \frac{1}{1-x^6} \cdots$$

$$= (1-x)^{-1} \cdot (1-x^2)^{-1} \cdot (1-x^3)^{-1} \cdot (1-x^4)^{-1} \cdot (1-x^5)^{-1} \cdot (1-x^6)^{-1} \cdots$$

$$= (1+x+x^2+x^3+\cdots) \times (1+x^2+x^4\cdots) \times (1+x^3+x^6+x^9+\cdots) \times$$
$$(1+x^4+x^8+\cdots) \times (1+x^5+x^{10}+\cdots) \times (1+x^6+x^{12}+\cdots) \times \cdots$$

$$= (1+x+x^2+x^3+x^4+x^5+x^6+\cdots+x^2+x^3+x^4+x^5+x^6+\cdots+x^4+x^5+x^6+\cdots) \times$$
$$(1+x^3+x^6+x^9+\cdots+x^4+x^7+x^{10}+\cdots+x^8+x^{11}+x^{14}+\cdots) \times$$
$$(1+x^5+x^{10}+\cdots+x^6+x^{11}+x^{16}+\cdots) \times \cdots$$

$$= (1+x+2x^2+2x^3+3x^4+3x^5+3x^6+\cdots) \times (1+x^3+x^4+x^5+2x^6+\cdots) \times \cdots$$

$$= 1+x+2x^2+3x^3+6x^4+7x^5+11x^6+\cdots$$

The co-efficient of x in this is 11. Thus, there are $P(n) =$ 11 partitions for $n=6$.

we note that these partitions are as given below:

$6 = 1+1+1+1+1+1$

$6 = 1+1+1+1+2$

$6 = 1+1+1+3$

$6 = 1+1+4$

$6 = 1+5$

$6 = 1+1+2+2$

$6 = 1+2+3$

$6 = 2+4$

$6 = 2+2+2$

$6 = 3+3$

$6 = 6$

Scanned with OKEN Scanner

(9) using exponential generating function, Find the number of ways in which 4 of the letters in ENGINE be arranged.

Sol:-

$$\boxed{E(x) = 1 + \frac{n!\,x}{1!} + 2!\frac{x^2}{2!} + 3!\frac{x^3}{3!} + \cdots}$$

In word Engine there are 2 E's, 2 N's and 1 each of G and I.

we consider the function $E_1(x)$ corresponding to (the Letter) E as

$$E_1(x) = 1 + x + \frac{x^2}{2!}$$

The function $E_2(x)$ corresponding to N as

$$E_2(x) = 1 + x + \frac{x^2}{2!}$$

The function $E_3(x)$ corresponding to G as

$$E_3(x) = 1 + x$$

The function $E_4(x)$ corresponding to I as

$$E_4(x) = 1 + x$$

∴ The generating function

$$E(x) = E_1(x)\,E_2(x)\,E_3(x)\,E_4(x)$$

The co-efficient of $\left(\frac{x^4}{4!}\right)$ in this function gives the number of arrangements of four letters of the given word. we find that

$$E(x) = \left(1 + x + \frac{x^2}{2!}\right)^2 (1+x)^2$$

$$\boxed{\therefore (a+b+c)^2 = a^2 + b^2 + c^2 + 2(ab+bc+ca)}$$

$$= \left(1 + 4x + 3x^2 + 2x^3 + \frac{x^4}{4}\right)\left(1 + 2x + x^2\right)$$

the co-efficient of $\left(\frac{x^4}{4!}\right)$ is :

$$= (4!) \times \left(\frac{1}{4} + 4 + 3\right)$$

$$= 4! \times \frac{29}{4}$$

$$= 4!\ {}^{29}/_{4} \Rightarrow (3!) \times 29 = 165$$

Scanned with OKEN Scanner

(11). Solve the recurrence relation $a_n = na_{n-1}$ for $n \geq 1$ given that $a_0 = 1$.

Sol: $F(n) = 0$

so, it is Homogeneous

$a_n = na_{n-1}$

$n \Rightarrow n+1$

$a_{n+1} = ca_n$

solution

$\boxed{a_n = c^n a_0}$ $\therefore a_0 = 1$

$\boxed{a_n = c^n}$

$\boxed{a_n = n^n}$

(12) [Question read from Question Bank]

Sol Let a_n be the number of moves required to transfer n disks. Evidently $a_0 = 0$. Let us denote the peg on which the disks are originally located as P_1. To effect the transfer, for $n \geq 1$, we first transfer the top $n-1$ disks to a vaccant peg, say $P2$, in the prescribed manner. These involves a_{n-1} moves. Then we transfer the nth disk to the other vaccant peg, say $P3$. This involves 1 move. Lastly, we transfer the $n-1$ disks from peg $P2$ to the peg $P3$, in the prescribed manner. This involves a_{n-1} moves. Thus, the total

$$a_n = a_{n-1} + 1 + a_{n-1}$$

$$\boxed{a_n = 2a_{n-1} + 1} \quad \text{for } n \geq 1$$

or equivalently

$\boxed{n \Rightarrow n+1}$

$$a_{n+1} = 2a_n + 1, \quad \text{for } n \geq 0$$

Scanned with OKEN Scanner

The general solution for this non-homogeneous recurrence relatio

$$a_n = 2^n a_0 + \sum_{K=1}^{n} 2^{n-K} \cdot f(K)$$

$$= \sum_{K=1}^{n} 2^{n-K} \cdot 1 \quad \text{because } a_0 = 0$$

$$= 2^{n-1} + 2^{n-2} + \cdots + 2^2 + 2 + 1$$

$$= \frac{2^n - 1}{2-1} \Rightarrow 2^n - 1$$

$\therefore$ $2^n - 1$ is the required number of moves.

(13) Solve the recurrence relation $a_n + a_{n-1} - 6a_{n-2} = 0$ for $n \geq 2$, given that $a_0 = -1$ and $a_1 = 8$.

Sol: Given recurrence relation is second order Linear Homogeneous

Characteristic Roots Technique

$\alpha = 1 \qquad \beta = -6$

$$\boxed{x^2 + \alpha x + \beta = 0}$$

$$x^2 + x - 6 = 0$$

Characteristic roots are $x = 2, -3$. [Two distinct roots $a_n = ax_1^n + bx_2^n$]

Thus, The solution to reccurrence relation will have the form

$$a_n = ax_1^n + bx_2^n$$

$$a_n = a2^n - b3^n$$

To find a & b plug in n=0 & n=1 to get a system of two equation with two unknowns

$$a_0 = a(2)^0 + b(3)^0$$

$$-1 = a + b \longrightarrow (1)$$

$$a_1 = a(2)^1 - b(3)^1$$

$$8 = 2a - 3b \longrightarrow (2)$$

Scanned with OKEN Scanner

solving this system gives $a = (+1)$ and
$b = -(2)$

∴ solution to reccurrence relation is

$$a_n = +1(2)^n + (-(2)(3)^n)$$

$$\boxed{a_n = +(2)^n + -(2)(3)^n}$$

Scanned with OKEN Scanner

(2) $a_n = 2(a_{n-1} - a_{n-2})$

with $a_0 = 1$, $a_1 = 2$

Sol:- $a_n - 2a_{n-1} + 2a_{n-2} = 0$

$C_n K^2 + C_{n-1}K + C_{n-2} = 0$

$1 \cdot K^2 - 2K + 2 = 0$

use formula :

$$= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \times 1 \times 2}}{2 \times 1}$$

$$= \frac{2 \pm \sqrt{4 - 4 \times 2}}{2}$$

$$K = \frac{2 \pm \sqrt{4-8}}{2} = K = \frac{2 \pm 2i}{2}$$

$$\boxed{K = 1 \pm i}$$

$K = p \pm iq$

$r = \sqrt{p^2 + q^2}$

$r = \sqrt{(1)^2 + (1)^2}$

$\boxed{r = \sqrt{2}}$ $\boxed{\therefore \theta = \tan^{-1}\left(\frac{p}{q}\right)}$

$\theta = \tan^{-1}\left(\frac{1}{1}\right) = \pi/4$

$$a_n = (\sqrt{2})^n \left[A\cos n\pi/4 + B\sin n\pi/4 \right]$$

$n = 0$

$a_0 = (\sqrt{2})^0 \left[A \cdot 1 + B \cdot 0 \right]$

$\boxed{1 = A}$ $\rightarrow (1)$

$a_1 =$

$a_1 =$

$2 =$

$2 =$

$(A+$

$1+$

$F_{n+2} =$

Sol:- n

f_n

K^2

$K =$

$K_1 =$

$f_n =$

$n=0$

$f_0 =$

$f_0 =$

f_0

A

$n =$

$n=1$

$a_1 = (\sqrt{2})^1 \left[A \cos \pi/4 + B \sin \pi/4 \right]$

$a_1 = (\sqrt{2}) \left[A \left(1/\sqrt{2} \right) + B \left(1/\sqrt{2} \right) \right]$

$2 = (\sqrt{2}) \left[A \left(1/\sqrt{2} \right) + B \left(1/\sqrt{2} \right) \right]$

$2 = \frac{\cancel{\sqrt{2}}}{\cancel{\sqrt{2}}} \left[A + B \right]$

$A + B = 2 \longrightarrow (2)$

sub eq (1) in (2)

$1 + B = 2$

$\boxed{B = +1}$

G.s is $\boxed{a_n = (\sqrt{2})^n \left[1 \cos n\pi/4 + 1 \sin n\pi/4 \right]}$

Scanned with OKEN Scanner

(15) Solve the recurrence relation $F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$, given that $F_0 = 0$ and $F_1 = 1$.

Sol: Given:

rewrite the given reccurrence relation as

$$F_{n+2} - F_{n+1} - F_n = 0$$

$n \Rightarrow n-2$

$$F_n - F_{n-1} - F_{n-2} = 0$$

Now we form the characteristic equation

$$x^2 - x - 1 = 0$$

And we solve for x

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2 \times 1}$$

$$x = \frac{1 \pm \sqrt{1+4}}{2} \Rightarrow x = \frac{1 \pm \sqrt{5}}{2}$$

Scanned with OKEN Scanner

So $x = \frac{1+\sqrt{5}}{2}$ & $x = \frac{1-\sqrt{5}}{2}$ are characteristic roots.

Thus, the solution to reccurrence relation will have the form

$$F_n = a x_1^n + b x_2^n$$

$$F_n = a\left[\frac{1+\sqrt{5}}{2}\right]^n + b\left[\frac{1-\sqrt{5}}{2}\right]^n$$

To find a & b, put in n=0 & n=1 to get a system of two equation with two unknowns

$$F_0 = a\left[\frac{1+\sqrt{5}}{2}\right]^0 + b\left[\frac{1-\sqrt{5}}{2}\right]^0$$

$$0 = a+b \quad \text{— (1)}$$

$$F_1 = a\left[\frac{1+\sqrt{5}}{2}\right]^1 + b\left[\frac{1-\sqrt{5}}{2}\right]^1$$

$$1 = a\left[\frac{1+\sqrt{5}}{2}\right] + b\left[\frac{1-\sqrt{5}}{2}\right] \quad \text{— (2)}$$

Solving this system gives $a = 1/\sqrt{5}$ and $b = -1/\sqrt{5}$ so the solution to the reccurrence relation is

$$F_n = 1/\sqrt{5}\left[\frac{1+\sqrt{5}}{2}\right]^n + -1/\sqrt{5}\left[\frac{1-\sqrt{5}}{2}\right]^n$$

$$\boxed{F_n = 1/\sqrt{5}\left\{\left[\frac{1+\sqrt{5}}{2}\right]^n - \left[\frac{1-\sqrt{5}}{2}\right]^n\right\}}$$

16) Solve the reccurrence relation $a_n + 4a_{n-1} + 4a_{n-2} = 8$ for $n \geq 2$, given that $a_0 = 1$ and $a_1 = 2$.

Sol :- Given:

relation is Non-homogeneous recurrence relation

General solution is = Homogeneous + PI (particular integral)

$$= a_n^{(H)} + a_n^{(P)}$$

Scanned with OKEN Scanner

(16) $a_n + 4a_{n-1} + 4a_{n-2} = 8$ for $n \geq 2$, given that $a_0 = 1$ and $a_1 = 2$.

Sol:- Given:
relation is Non-homogeneous reccurrence relation

General solution is = Homogeneous + PI (particular Integral)

$= a_n^{(H)} + a_n^{(P)}$

Homogeneous characteristic equation is

$$C_n K^2 + C_{n-1} K + C_{n-2} = 0$$

$$K^2 + 4K + 4 = 0$$

The roots are $\boxed{K = -2, -2}$ repeated & Real $\boxed{K_1 = K_2}$

General solution $\Rightarrow \boxed{a_n = (A + Bn) K^n}$

$a_n = (A + Bn) K^n$

$a_0 = (A + B(0))(-2)^0$

$a_0 = A$

From Question $\boxed{a_0 = 1}$

$\boxed{1 = A} \rightarrow (1)$

$a_1 = (A + B(1))(-2)^1$

$2 = (A + B)(-2)$ $\boxed{\therefore a_1 = 2}$

substitute from eq (1) A

$2 = (1 + B)(-2)$

$1 + B = 2/-2$

$1 + B = -1$

$B = -2$

General solution for Homogeneous

$$\boxed{a_n = (1 + (-2)n)(-2)^n}$$

$\Rightarrow \boxed{a_n^{(P)} = A_0}$ $\therefore$ Because the degree is 1

substitute in Equation given

$A_0 + 4A_0 + 4A_0 = 8$

$9A_0 = 8$

$A_0 = 8/9$ $\therefore \boxed{a_n^{(P)} = 8/9}$

Non-homogeneous General solution is $a_n^{(H)} + a_n^{(P)}$

$$\boxed{G.s = a_n = (1 + (-2)n)(-2)^n + \frac{8}{9}}$$

Scanned with OKEN Scanner

(17) reccurrence relation $a_{n+2} + 3a_{n+1} + 2a_n = 3^n$ for $n \geq 0$

given that $a_0 = 0$ and $a_1 = 1$.

Sol

$\Rightarrow$ rewrite given equation into homogeneous equation

$\Rightarrow$ replace $\boxed{n \longrightarrow n-2}$

$$a_n + 3a_{n-1} + 2a_{n-2} = 0$$

Characteristic equation is

$$K^2 + 3K + 2 = 0$$

$$\boxed{K = -1, -2}$$

$\therefore$ roots are real & distinct i.e, $K_1 \neq K_2$

General solution $= a_n = AK_1^n + BK_2^n$

$$\boxed{G.S = a_n = A(-1)^n + B(-2)^n}$$

Scanned with OKEN Scanner

(particular Integral)

$$\boxed{a_n^{(P)} = A_0 b^n} \quad \because \boxed{K \neq b}$$

$$\boxed{a_n^{(P)} = A_0 \times 3^n} \rightarrow (1)$$

put $a_n^{(P)}$ in given relation, we get

$$A_0 \times 3^{n+2} + 3A_0 \times 3^{n+1} + 2A_0 \times 3^n = 3^n$$

$\underline{n=0}$

$$(A_0 \times 3^2) + (3A_0 \times 3) + 2A_0 = 1$$

then we get $9A_0 + 9A_0 + 2A_0 = 1$

$$20A_0 = 1$$

$$A_0 = 1/20 \rightarrow (2)$$

substitute eq (2) in (1) we get

$$\boxed{a_n^{(P)} = \frac{1}{20} \times 3^n}$$

General solution for given relation is

$$a_n = a_n^{(H)} + a_n^{(P)}$$

$$a_n = A \times (-1)^n + B(-1)^n + \frac{1}{20} \times 3^n$$

$a_0 = 0$ and $a_1 = 1$

$$a_0 = A \times (-1)^0 + B(-2)^0 + \frac{1}{20} \times 3^0$$

$$\boxed{a_0 = A + B + \frac{1}{20}} \Rightarrow \boxed{0 = A + B + 1/20}$$

$$a_1 = A \times (-1)^1 + B(-2)^1 + \frac{1}{20} \times 3^1$$

$$\boxed{1 = -A - 2B + \frac{3}{20}}$$

solving these, we get $\boxed{a_n = -4/5 \times (-2)^n + \frac{3}{4} \times (-1)^n + \frac{1}{20} \times 3^n}$

$A = -1/5$

$B = 1/4$

Scanned with OKEN Scanner

function.

(10) A company appoints 11 software engineers, each of whom is to be assigned to one of Four offices of the company. Each office should get at least one of these engineers. In how many ways can these assignments be made?

Sol :- Let A, B, C, D denote the four offices.

Then the problem is equivalent to counting the number of sequences of length 11 in which atleast one of A, B, C, D occurs.

Since in every sequence of the desired type each of A, B, C, D occurs atleast once, we take the exponential generating function for the problem as:

$$E(x) = \left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)^4 = (e^x - 1)^4$$

$$= e^{x} - 4e^{3x} + 6e^{2x} - 4e^x + 1$$

$$= \sum_{r=0}^{\infty} \frac{(4x)^r}{r!} - 4 \times \sum_{r=0}^{\infty} \frac{(3x)^r}{r!} + 6 \times \sum_{r=0}^{\infty} \frac{(2x)^r}{r!} - 4 \times \sum_{r=0}^{\infty} \frac{x^r}{r!} + 1$$

Scanned with OKEN Scanner

$\therefore$ The co-efficient of $\left(\dfrac{x^{11}}{11!}\right)$ in this gives the required number. The number is

$$= 4^{11} - (4\times 3^{11}) + (6\times 2^{11}) - (4\times 1)$$

$$= 34,98,000.$$

Scanned with OKEN Scanner