DISCRETE MATHEMATICS AND GRAPH THEORY (21CIDS31)

MODULE – II: PRINCIPLES OF COUNTING AND INCLUSION-EXCLUSION

- ❖ Pigeonhole Principle
- ❖ Relations and Digraphs
- ❖ Equivalence Relations
- ❖ Partially Ordered Set
- ❖ Hasse Diagram
- ❖ Lattices
- ❖ Paths and Related Problems
- ❖ Transitive Closure and Wars hall's Algorithm
- ❖ Principle Of Inclusion-Exclusion
- ❖ Generalizations of the Principles
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- ❖ Rook Polynomials

PRINCIPLES OF COUNTING

The Rules of Sum and Product

In many situations of computational work, we employ two basic rules of counting, called the Sum Rule and the Product Rule. These rules are restated and illustrated in the following paragraphs.

The Sum Rule

Suppose two tasks T_1 and T_2 are to be performed. If the task T_1 can be performed in m different ways and the task T_2 can be performed in n different ways and if these two tasks cannot be performed simultaneously, then one of the two tasks (T_1 and T_2) can be performed in $m + n$ ways.

More generally, if T_1, T_2, T_3…….T_k are k tasks such that no two of these tasks can be performed at the same time and if the task T_i can be performed in n_i different ways, then one of the k tasks (namely T_1 or T_2 or T_3…….or T_k) can be performed in $n_1 + n_2 + n_3 +$…….$+n_k$ different ways.

PRINCIPLES OF COUNTING

Example 1: Suppose there are 16 boys and 18 girls in a class and we wish to select one of these students (either a boy or a girl) as the class representative.

The number of ways of selecting a boy is 16 and the number of ways of selecting a girl is 18. Therefore, the number of ways of selecting a student (boy or girl) is 16 + 18 = 34.

Example 2: Suppose a Hostel library has 12 books on Mathematics, 10 books on Physics, 16 books on Computer Science and 11 books on Electronics. Suppose a student wishes to choose one of these books for study.

The number of ways in which he can choose a book is 12 + 10 + 16 + 11 = 49.

PRINCIPLES OF COUNTING

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The Product Rule

Suppose that two tasks T_1 and T_2 are to be performed one after the other. If T_1 can be performed in n_1 different ways, and for each of these ways T_2 can be performed in n_2 different ways, then both of the tasks can be performed in $n_1\ n_2$ different ways.

More generally, suppose that k tasks T_1, T_2, T_3…….T_k are to be performed in a sequence. If T_1 can be performed in n_1 different ways and for each of these ways T_2can be performed in n_2 different ways, and for each of n_2 different ways of performing T_1 and T_2 in that order, T_3 can be performed in n_3 different ways, and so on, then the sequence of tasks T_1, T_2, T_3…….T_k can be performed in $n_1 n_2 n_3$…….n_k different ways.

PRINCIPLES OF COUNTING

Example 4. Suppose a person has 8 shirts and 5 ties. How many different ways of choosing a shirt and a tie?

Then he has $8 \times 5 = 40$ different ways of choosing a shirt and a tie.

Example 5. Suppose we wish to construct sequences of four symbols in which the first 2 are English letters and the next 2 are single digit numbers.

If no letter or digit can be repeated then the number of different sequences that we can construct is

$26 \times 25 \times 10 \times 9 = 58500$.

If repetition of letters and digits are allowed then the number of different sequences that we can construct is

$26 \times 26 \times 10 \times 10 = 67600$.

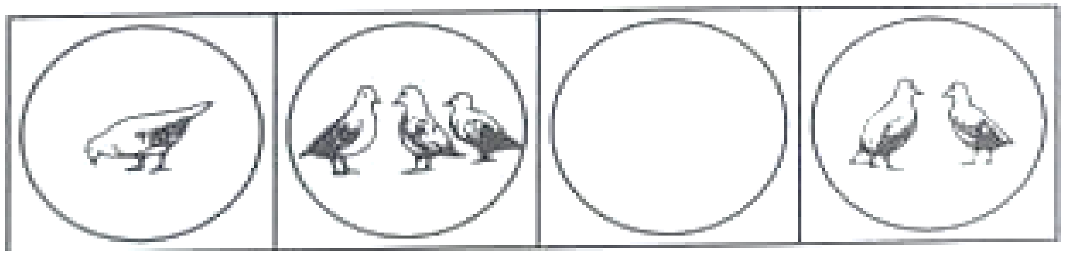
PIGEONHOLE PRINCIPLE

Statement: - If n pigeons occupy m pigeonholes and if $m < n$ then at least one pigeonhole contains two or more pigeons

The use of pigeonhole principle is to

- Identify the pigeons (objects)
- Identify the pigeonhole (categories the desired characteristics)

A simple Illustration of above principle is that, If 6 pigeons occupy 4 pigeon holes, then at least one pigeonhole must contain two or more pigeons in it.



EXTENDED PIGEONHOLE PRINCIPLE: -

Statement: - If n pigeon are assigned to m pigeonholes then one of the pigeonholes must contains at least $\left\lfloor \frac{n-1}{m} \right\rfloor + 1$ pigeons.

PROBLEMS

1. Show that if you pick any five numbers from the integers 1-8, then two of them must add up to 9.

Solution: - Let us first write all the numbers from 1 to 8 as (1,2,3,4,5,6,7,8)

Now let's take any 5 numbers from 1 to 8 such as (1,3,4,7,8)

As it given that any two of the numbers out of the 5 numbers we have chosen should be equal to sum 9.

Let's add every two numbers so that we can get one such pair of numbers whose sum would be 9.

Case 1>. 1 + 3 = 4

Case 2>. 3 + 4 = 7

Case 3>. 4 + 7 = 11

Case 4>. 7 + 8 = 15

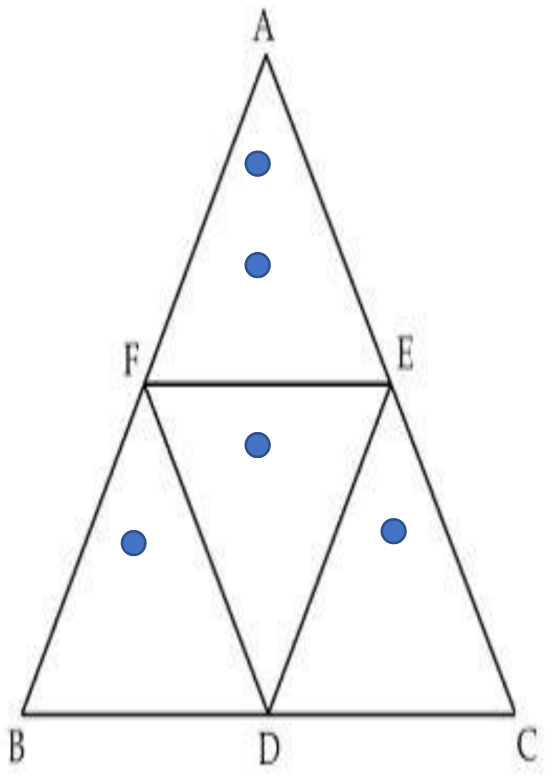
Case 5>. 8 + 1 = [9]

PROBLEMS

Hence in Case 5 we get a pair of numbers 8 and 1 whose sum is equal to 9 , so we present them together in a same set as {8,1}.So according to Pigeonhole Principle ,We can take any 5 numbers and there will always exist one pair whose sum is equal to 9.

PROBLEMS

2. ABC is an equilateral triangle whose sides are of the length 1cm each. If we select 5 points inside the triangles, Prove that at least two of these points are such that the distance is less than ½ cm.



Solution: -

Consider a Δ DEF formed by midpoint of the sides BC, CA, AB divide ABC into four small equilateral triangle(position) each of which has sides equal to ½ cm.

Treating each of the triangle as pigeonhole and 5 points chosen inside the triangle as a pigeon.

We find by using pigeonhole principle that at least one portion must contain two or more points.

Evidently the distance between such points is less than ½ cm

PROBLEMS

3. Show that if 30 dictionaries in a library contain a total of 61,327 pages, then one of the dictionaries must have at least 2045 pages.

Solution: - Given that pigeon(n) = 61327 and pigeonhole(m) = 30

We need to prove $p + 1 = 2045$

Therefore $$p + 1 = \left\lfloor \frac{n-1}{m} \right\rfloor + 1$$

$$= \left\lfloor \frac{61327 - 1}{30} \right\rfloor + 1$$

$$= \lfloor 2044.2 \rfloor + 1$$

$$= 2044 + 1$$

$$p + 1 = 2045$$

PROBLEMS

4. Show that if seven colors are used to paint 50 bicycles, at least eight bicycles will be the same color.

Solution: - Given that pigeon(n) = 50 and pigeonhole(m) = 7

We need to prove $p + 1 = 8$

Therefore
$$p + 1 = \left\lfloor \frac{n-1}{m} \right\rfloor + 1$$

$$= \left\lfloor \frac{50-1}{7} \right\rfloor + 1$$

$$= 7 + 1$$

$$p + 1 = 8$$

PROBLEMS

5. If 13 people are assembled in a room, show that at least two of them must have their birthdays in the same month.

Solution: - Given that pigeon(n) = 13 and pigeonhole(m) = 12

We need to prove $p + 1 = 2$

Therefore $$p + 1 = \left\lfloor \frac{n-1}{m} \right\rfloor + 1$$

$$= \left\lfloor \frac{13-1}{12} \right\rfloor + 1$$

$$= 1 + 1$$

$$p + 1 = 2$$

PROBLEMS

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6. How many friends must you have to guarantee that at least 5 of them will have birthdays in the same month?

Solution: - Given that pigeon(n) = ? and pigeonhole(m) = 12, p + 1 = 5

We have to find n = ?

Therefore $p + 1 = \left\lfloor \frac{n-1}{m} \right\rfloor + 1$

$$5 = \left\lfloor \frac{n-1}{12} \right\rfloor + 1$$

$$5 - 1 = \left\lfloor \frac{n-1}{12} \right\rfloor$$

$$4 = \left\lfloor \frac{n-1}{12} \right\rfloor$$

$$4 \times 12 = n - 1$$

48 = n – 1

n = 48 +1

n = 49

PROBLEMS

7. Six books each of physics, chemistry, mathematics and four books of biology totally contains 12225 pages. Find the number of pages contained in a book. [Homework].

8.Find how many of a sample size of 1000 people,

a. **Are born in the same month.**
b. **Born on a particular day are born in the same hour. [Homework].**