GENERATING FUNCIONS AND RECURRENCE RELATIONS

MODULE-3

DISCRETE MATHEMATICS & GRAPH THEORY

21CIDS31

CONTENT

- **Generating Functions:**
  - Generating Functions: Definition & Examples
  - Calculational (Counting) Techniques
  - Partitions of Integers
  - Exponential Generating Functions, and Summation Operator. Method of Generating Functions.
- **Recurrence Relations:**
  - First - Order Linear Recurrence Relation.
  - Second - Order Linear Homogeneous Recurrence Relations with Constant Coefficients.
  - Non-Homogeneous Recurrence Relations.

Generating Function

A tool used for handling special constraints in selection and arrangement problems with repetition.

Suppose

- a_r is the number of ways to select 'r' objects from 'n' objects.
- $f(x)$ is a generating function for a_r.
- The Polynomial expansion of $f(x)$.

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_r x^r + \ldots + a_n x^n$$

Definition: Consider a sequence of real numbers a_0, a_1, a_2,…… Let us denote this sequence by $\langle a_r \rangle$, $r = 0, 1, 2, \ldots \ldots$ or just $\langle a_r \rangle$. Given this sequence, suppose there exists a function $f(x)$ whose expansion in a series of powers of x is as given below:

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_r x^r + \ldots + a_n x^n = \sum_{r=0}^{\infty} a_r x^r$$

$$f(x) = a_0 + a_1x + a_2x^2 + \dots + a_rx^r + \dots + a_nx^n = \sum_{r=0}^{\infty} a_r\, x^r \quad \text{--(1)}$$

If $f(x)$ is a generating function of $\langle a_r \rangle$,
That means,
$f(x)$ generates the sequence $\langle a_r \rangle$.

This is called power series expansion of $f(x)$.

Example:

- $(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots = \sum_{r=0}^{\infty} x^r$ ---------(2)

$f(x) = (1-x)^{-1}$ is a generating function for the sequences 1, 1, 1,1….

- $(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots = \sum_{r=0}^{\infty} (-1)^r\, x^r$ -------(3)

$f(x) = (1+x)^{-1}$ is a generating function for the sequences 1,-1, 1,-1….

Binomial Expansion: $\boxed{(1+x)^n = \sum_{r=0}^{\infty} \binom{n}{r} x^r}$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1\times 2}x^2 + \frac{n(n-1)(n-2)}{1\times 2\times 3}x^3 + \cdots$$

$$= \sum_{r=0}^{\infty} \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!} x^r \text{------(4)}$$

$$= \sum_{r=0}^{\infty} \binom{n}{r} x^r \text{---------(5)}$$

From this we note that for any real number n the function $f(x) = (1+x)^n$ is a generating function for the sequence

$$1, \frac{n}{1!}, \frac{n(n-1)}{2!}, \frac{n(n-1)(n-2)}{3!}, \ldots$$

$$\equiv \binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \binom{n}{4} \ldots\ldots \text{-------(6)}$$

When ***n is a positive integer***, all the terms in the RHS of the expression (6) beyond the term containing x^n are identically zero.

$$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \binom{n}{4} \ldots\ldots \binom{n}{n}, 0, 0, 0 \ldots.$$

- **Properties**

❑If $f(x)$ is a generating function for a sequence $\langle \boldsymbol{a_r} \rangle$ and $g(x)$ is a generating function for a sequence $\langle \boldsymbol{b_r} \rangle$,then $pf(x) + \ qg(x)$ is a generating function for the sequence $(p\boldsymbol{a_r} + \ q\boldsymbol{b_r})$. where p and q are any two real numbers.

❑If $f(x)$ is a generating function for a sequence $\langle \boldsymbol{a_r} \rangle$ then $xf'(x)$ (- where $f'(x)$ is a derivative At B of $f(x)$) is a generating function for the sequence $\langle \boldsymbol{ra_r} \rangle$.

PROBLEMS

1. Find the sequence generated by the following functions:

I. $(3+x)^3$

II. $2x^2(1-x)^{-1}$

III. $\frac{1}{1-x}+2x^3$

IV. $(1+3x)^{-1/3}$

V. $(1-4x)^{-1/2}$

VI. $3x^3+e^{2x}$

Solution:

$I.\ (3+x)^3 = 3^3\cdot\left(1+\frac{x}{3}\right)^3$

$= 27\cdot\left\{1+C(3,1)\left(\frac{x}{3}\right)+C(3,2)\left(\frac{x}{3}\right)^2+C(3,3)\left(\frac{x}{3}\right)^3\right\}$

$= 27\cdot\left\{1+x+\frac{x^2}{2}+\frac{x^3}{27}\right\}$

$= 27+27x+9x^2+x^3$

This shows that the sequence generated by $(3+x)^3$ is

$27,\ 27,\ 9,\ 1,\ 0,\ 0,\ 0\ldots$

$II.\ 2x^2(1-x)^{-1} = 2x^2\left(1+x+x^2+x^3+\cdots\right)$

$= 0+0x+2x^2+2x^3+2x^4+\cdots$

This shows that the sequence generated by $2x^2(1-x)^{-1}$ is

$0,\ 0,\ 2,\ 2,\ 2,\ \ldots$

$III.\ \frac{1}{1-x}+2x^3 = (1-x)^{-1}+2x^3$

$= \left(1+x+x^2+x^3+x^4+\cdots\right)+2x^3$

$= 1+x+x^2+3x^3+x^4+x^5+\cdots$

The sequence generated by $\frac{1}{1-x}+2x^3$ is

$1,\ 1,\ 1,\ 3,\ 1,\ 1,\ 1\ldots$

IV.

$$= 1 + \sum_{r=1}^{\infty} \frac{(-1/3)(-1/3-1)(-1/3-2)\cdots(-1/3-r+1)}{r!}(3x)^r$$

$$= 1 + \sum_{r=1}^{\infty} \frac{(-1/3)(-4/3)(-7/3)\ldots\{(-3r+2)/3\}}{r!}(3^r x^r)$$

$$= 1 + \sum_{r=1}^{\infty} \frac{(-1)(-4)(-7)\ldots(-3r+2)}{r!}x^r$$

$$= 1 - x + \frac{(-1)(-4)}{2!}x^2 + \frac{(-1)(-4)(-7)}{3!}x^3 + \cdots$$

The sequence generated is

$$1, -1, \frac{(-1)(-4)}{2!}, \frac{(-1)(-4)(-7)}{3!}, \ldots$$

$$VI.\ e^{2x} = 1 + \frac{(2x)}{1!} + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} + \cdots$$

$$= 1 + \frac{2}{1!}x + \frac{2^2}{2!}x^2 + \frac{2^3}{3!}x^3 + \frac{2^4}{4!}x^4 + \cdots$$

$$3x^3 + e^{2x} = 1 + \frac{2}{1!}x + \frac{2^2}{2!}x^2 + \left(3 + \frac{2^3}{3!}\right)x^3 + \frac{2^4}{4!}x^4 + \cdots$$

The generated sequence is

$$1, \frac{2}{1!}, \frac{2^2}{2!}, 3 + \frac{2^3}{3!}, \frac{2^4}{4!}, \ldots$$

$$V.\ (1-4x)^{-1/2} = 1 + \sum_{r=1}^{\infty} \frac{(-1/2)(-1/2-1)(-1/2-2)\ldots(-1/2-(r-1))}{r!}(-4x)^r$$

$$1 + \sum^{\infty} \frac{(-1)(-3)(-5)\ldots(-2r+1)}{r!}\frac{(-4)^r}{2^r}x^r$$

$$= 1 + \sum_{r=1}^{\infty} \frac{1\cdot 3\cdot 5\cdot\ldots(2r-1)}{r!}(2^r x^r)$$

$$= 1 + \sum_{r=1}^{\infty} \frac{1\cdot 3\cdot 5\ldots(2r-1)\cdot 2^r(r!)}{r\,r!}x^r$$

$$= 1 + \sum_{r=1}^{\infty} \frac{1\cdot3\cdot5\ldots(2r-1)(2\cdot4\cdot6\cdot.2r)}{r!r!}x^r$$

$$= 1 + \sum_{r=1}^{\infty} \frac{(2r)!}{r!r!}x^n = 1 + \sum_{r=1}^{\infty} \binom{2r}{r}x^r$$

The generated sequence is

$$1, \binom{2}{1}, \binom{4}{2}, \binom{6}{3} \ldots$$

PROBLEMS

2. In each of the following, $f(x)$ is a generating function for the sequence sequence $\langle \boldsymbol{a_r} \rangle$ and $g(x)$ is a generating function for a sequence $\langle \boldsymbol{b_r} \rangle$. Express $g(x)$ in terms of $f(x)$.

$(i)\ b_3 = 3, b_7 = 7, b_n = a_n \quad \text{for} \quad n \neq 3, 7$

$(ii)\ b_1 = 1, b_3 = 3, b_7 = 7, b_n = 2a_n + 5 \quad \text{for} \quad n \neq 1, 3, 7$

Solution:

$$
\begin{aligned}
(i)\ g(x) &= \sum_{r=0}^{\infty} b_r x^r \\
&= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + b_5 x^5 + b_6 x^6 + b_7 x^7 + b_8 x^8 + b_9 x^9 + \cdots \\
&= a_0 + a_1 x + a_2 x^2 + 3x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + 7x^7 + a_8 x^8 + a_9 x^8 + \cdots \\
&= \left\{ \sum_{r=0}^{\infty} a_r x^r - a_3 x^3 - a_7 x^7 \right\} + 3x^3 + 7x^7 = \sum_{r=0}^{\infty} a_r x^r + (3 - a_3) x^3 + (7 - a_7) x^7 \\
&= f(x) + (3 - a_3) x^3 + (7 - a_7) x^7.
\end{aligned}
$$

Solution:

$$(ii)\ g(x) = \sum_{r=0}^{\infty} b_r x^r$$

$$= \left\{ \sum_{r=0}^{\infty} (2a_r + 5)x^r - (2a_1 + 5)x - (2a_3 + 5)x^3 - (2a_7 + 5)x^7 \right\} + b_1 x + b_3 x^3 + b_r x_1$$

$$= 2\sum_{r=0}^{\infty} a_r x^r + 5\sum_{r=0}^{\infty} x^r + (b_1 - 2a_1 - 5)x + (b_3 - 2a_3 - 5)x^3 + (b_r - 2a_1 - 5)x^2$$

$$= 2f(x) + 5(1 - x)^{-1} - 2\,(a_1 + 2)x - 2\,(a_3 + 1)x^3 - 2\,(a_1 - 1)x^7.$$

PROBLEMS

3. Find the generating functions for the following sequences

(i) 1, 2, 3, 4,… (ii) 1, -2, 3, -4,… (iii) 0, 1, 2, 3,…

(iv) 0, 1, -2, 3, -4,…

Solution:

(i) By Binomial expansion

$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$

From this, it follows that $f(x) = (1-x)^{-2}$ is a generating function for the sequence $1, 2, 3, 4\ldots\ldots$

(ii) *We have*

$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$

As such that, $f(x) = (1+x)^{-2}$ is generarting function for the sequence $1, -2, 3, -4\ldots.$

(iii) *We note that*

$0 + 1x + 2x^2 + 3x^3 + \cdots = x\left(1 + 2x + 3x^2 + \cdots\right) = x(1-x)^{-2}$

This shows that $f(x) = x(1-x)^{-2}$ is generating function for the sequence $0, 1, 2, 3, \ldots.$

(iv). *We note that*

$0 + 1x - 2x^2 + 3x^3 - 4x^4 + \cdots = x\left(1 - 2x + 3x^2 - 4x^3 + \cdots\right) = x(1+x)^{-2}$

Accordingly, $f(x) = x(1+x)^{-2}$ is generating function for the sequence $0, 1, -2, 3, -4$

PROBLEMS

4. Find the generating function for the following sequences

$(i)\ 1^2, 2^2, 3^2, 4^2, \ldots \quad (ii)\ 0^2, 1^2, 2^2, 3^2 \ldots \quad (iii)\ 1^3, 2^3, 3^3, 4^3, \ldots (iv) 0^3, 1^3, 2^3, 3^3 \ldots$

Solution:

(i) We have

$0 + 1^2x + 2x^2 + 3x^3 + 4x^4 + \cdots = x\left(1 + 2x + 3x^2 + 4x^3 + \cdots\right) = x(1-x)^{-2}$

Differentiating both sides of this, we get

$1^2 + 2^2x + 3^2x^2 + 4^2x^3 + \cdots = \frac{d}{dx}\left\{x(1-x)^{-2}\right| = \frac{d}{dx}\left\{\frac{x}{(1-x)^2}\right\} = \frac{1+x}{(1-x)^3}$

shows that

$$f(x) = \frac{(1+x)}{(1-x)^3}$$

is a generating function for the sequence $1^2, 2^2, 3^2, 4^2,$

$(i)\ 1^2, 2^2, 3^2, 4^2, \ldots\quad (ii)\ 0^2, 1^2, 2^2, 3^2 \ldots\quad (iii)\ 1^3, 2^3, 3^3, 4^3, \ldots (iv) 0^3, 1^3, 2^3, 3^3 \ldots$

Solution:

(ii) We have

$$0^2 + 1^2x + 2^2x^2 + 3^2x^3 + \cdots = x\left(1^2 + 2^2x + 3^2x^2 + \cdots\right)$$

$$= x \cdot \frac{1+x}{(1-x)^3} \text{ by using the results got above}$$

$$Thus \quad f(x) = \frac{x(1+x)}{(1-x)^3}$$

is the generating function for the sequence $0^2, 1^2, 2^2, 3^2, \ldots$.

(iii) From what has been proved above (ii), we have

$$0^2 + 1^2x + 2^2x^2 + 3^2x^3 + \cdots = \frac{x(1+x)}{(1+x)^3}$$

Differentiating both th side we get

$$1 + 2^3x + 3^3x^2 + \cdots = \frac{d}{dx}\left\{\frac{x(1+x)}{(1-x)^3}\right\} = \frac{x^2+4x+1}{(1-x)^4}$$

$$\text{Thus } f(x) = \frac{(x^2+4x+1)}{(1-x)^4}$$

is a generating functiuon for the sequence $1^3, 2^3, 3^3, \ldots$

$(i)\ 1^2, 2^2, 3^2, 4^2, \ldots\ (ii)\ 0^2, 1^2, 2^2, 3^2 \ldots\ (iii)\ 1^3, 2^3, 3^3, 4^3, \ldots (iv) 0^3, 1^3, 2^3, 3^3 \ldots$

Solution:

(iv) We have

$$0^3 + 1^3x + 2^3x^2 + 3^3x^3 + \cdots = x\left(1^3 + 2^3x + 3^3x^2 + \cdots\right)$$

$$= x \cdot \frac{x^2 + 4x + 1}{(1-x)^4}, \text{ by (iii).}$$

$$\text{Thus } f(x) = \frac{x\left(x^2 + 4x + 1\right)}{(1-x)^4}$$

is a generating function for the sequence $0^3, 1^3, 2^3, 3^3, \ldots$

PROBLEMS

5. Find the generating function for the following sequences

(i) 1, 1, 0, 1, 1, 1... (ii) 0, 2, 6, 12, 20, 30, 42..... (iii) 8, 26, 54, 92,....

Solution:

(i) *We have*

$$1 + 1x + 0 \cdot x^2 + x^3 + x^4 + x^5 + \cdots$$

$$= \{1 + x + x^2 + x^3 + x^4 + x^5 + \cdots\} - x^2$$

$$= (1 - x)^{-1} - x^2$$

This shows that $f(x) = (1 - x)^{-1} - x^2$ is a generating function for the sequence 1, 1, 0, 1, 1, 1 (i)1, 1, 0, 1, 1, 1… (ii) 0, 2, 6, 12, 20, 30, 42….. (iii) 8, 26, 54, 92,….

Solution:

(ii) Denoting the given sequence by (a_r), we find that

$a_0 = 0 = 0 + 0$

$a_1 = 2 = 1 + 1$

$a_2 = 6 = 2 + 2^2$

$a_3 = 12 = 3 + 3^2$

$a_4 = 20 = 4 + 4^2 \ldots.$

So that

$a_r = r + r^2 \quad \text{for } r = 0, 1, 2$

We recall that a generating function for the sequence $(r) = 0, 1, 2, 3, \ldots.\ is$

$f(x) = \frac{x}{(1-x)^2}$ (see Example 3)

and a generating function for the sequence $(r^2) = 0^2, 1^2, 2^2, 3^2, \ldots.\ is$

$g(x) = \frac{x(1+x)}{(1-x)^3}$

Therefore, a generating function for the given sequence is

$$f(x) + g(x) = \frac{x}{(1-x)^2} + \frac{x(1+x)}{(1-x)^3} = \frac{2x}{(1-x)^3}$$

(i)1, 1, 0, 1, 1, 1… (ii) 0, 2, 6, 12, 20, 30, 42….. (iii) 8, 26, 54, 92,….

Solution:

(*iii*) Denoting the given sequence by (a_r), *we check that*

$a_r = 3(r+1) + 5(r+1)^2$ for $r = 0, 1, 2$

We recall that a generating function for the sequence $(r+1) =$ 1, 2, 3.….

$f(x) = \frac{1}{(1-x)^2}$

and a generating function for the sequence $(r+1)^2 = 1^2, 2^2, 3^2, \ldots$ *is*

$g(x) = \frac{(1+x)}{(1-x)^3}$

Therefore, a generating function for the given sequence is

$$3f(x) + 5g(x) = \frac{3}{(1-x)^2} + \frac{5(1+x)}{(1-x)^3} = \frac{8+2x}{(1-x)^3}$$

NOTE: Generating function for the sequence (a_r), $a_r = 1$ for $0 \le r \le n$ and $a_r = 0$ for $r \ge n+1$.

☆

$$\sum_{r=0}^{\infty} a_r x^r = 1 + x + x^2 + \cdots + x^n = \frac{1-x^{n+1}}{1-x}$$

$$f(x) = \frac{(1-x^{n+1})}{(1-x)}$$

PROBLEMS

6. If n is a positive integer, prove the following:

(i) $\binom{-n}{r} = (-1)^r \binom{n+r-1}{r} = (-1)^r \binom{n+r-1}{n-1}$

(ii) $\binom{n+r-1}{r} = \binom{n+r-1}{n-1} = (-1)^r \binom{-n}{r}$

Solution:

(i) Using definition, we find that

$$\binom{-n}{r} = \frac{(-n)(-n-1)(-n-2)\ldots(-n-r+1)}{r!}$$

$$= (-1)^r \frac{n(n+1)(n+2)\ldots(n+r-1)}{r!}$$

$$= (-1)^r \frac{(n+r-1)!}{(n-1)!r!} \quad = \binom{n+r-1}{n-1}$$

$$= (-1)^r \binom{n+r-1}{n-1} \quad = (-1)^r \binom{n+r-1}{r}$$

(ii) From the result proved above, we find that

$$(-1)^r \binom{-n}{r} = (-1)^{2r} \binom{n+r-1}{r} \quad = (-1)^{2r} \binom{n+r-1}{n-1}$$

$$= \binom{n+r-1}{r}$$

$$= \binom{n+r-1}{n-1}$$

PROBLEMS

7. If n is a positive integer ,PT

$$(1-x)^{-n} = \sum_{r=0}^{\infty} \binom{n+r-1}{r} x^r$$

Solution:

$$We\ have\ \binom{n+r-1}{r} = (-1)^r \binom{-n}{r},$$

Problem 6, (ii)

Therefore

$$\sum_{r=0}^{\infty} \binom{n+r-1}{r} x^r = \sum_{=0}^{\infty} (-1)^r \binom{-n}{r} x^r$$

$$= \sum_{r=0}^{\infty} \binom{-n}{r} (-x)^r = (1-x)^{-n}$$

Hence proved

PROBLEMS

8. Determine the coefficient of

$(i)\ x^{12}$ in $x^3(1-2x)^{10}$ $\quad (ii)\ x^0$ in $\left[3x^2-(2/x)\right]^{15}$ $\quad (iii)\ x^5$ in $(1-2x)^{-7}$

$(iv)\ x^{20}$ in $(x^2+x^3+x^4+x^5+x^6)^5$

Solution:

(i) *We have*

$$x^3(1-2x)^{10} = x^3 \times \sum_{r=0}^{10}\binom{10}{r}(-2x)$$

$$= \sum_{r=0}^{10}(-2)^r\binom{10}{r}x^{r+3}$$

Therefore the coefficient of x^{12} in R.H.S is

$$c_{12} = (-2)^9\binom{10}{9} = -\left(2^9\times 10\right) = -5120$$

(ii) We have

$$\left(3x^2-\frac{2}{x}\right)^{15} = (3x^2)^{15}\times\left(1-\frac{2}{3x^3}\right)^{15}$$

$$= (3^{15}x^{30})\times\sum_{r=0}^{15}\binom{15}{r}\left(-\frac{2}{3x^3}\right)^r$$

$$= 3^{15}\times\sum_{r=0}^{15}\binom{15}{r}\left(-\frac{2}{3}\right)^r x^{30-3r}$$

Therefore the coefficient of x^0 in R.H.S is

$$c_0 = 3^{15}\times\binom{15}{10}\times\left(-\frac{2}{3}\right)^{10} = 3^5\times 2^{10}\times\binom{15}{10}$$

$(iii)\ x^5$ in $(1-2x)^{-7}$

$(iv)\ x^{20}$ in $\left(x^2+x^3+x^4+x^5+x^6\right)^5$

Solution:

(iii) We have $(1-2x)^{-7} = \sum_{r=0}^{\infty}\binom{7+r-1}{r}(2x)$

$$= \sum_{r=0}^{\infty}\binom{6+r}{r}(2x)^r$$

Therefore the coefficient of x^5 in R.H.S is

$$c_5 = 2^5 \times \binom{11}{5} = 2^5 \times \frac{11!}{5!6!}$$

(iv) We have $\left(x^2+x^3+x^4+x^5+x^6\right)^5 = x^{10}\left(1+x+x^2+x^3+x^4\right)^5$

$$= x^{10}\left(\frac{1-x^5}{1-x}\right)^5 = x^{10}\cdot\left(1-x^5\right)^5\cdot(1-x)^{-5}$$

$$= x^{10}\cdot\sum_{r=0}^{5}\binom{5}{r}\left(-x^5\right)^r \times \sum_{s=0}^{\infty}\binom{5+s-1}{s}x^s$$

$$= \sum_{r=0}^{5}\sum_{s=0}^{\infty}(-1)^r\binom{5}{r}\binom{4+s}{s}x^{10+5r+s}$$

From this we find that the coefficient of x^{20} in R.H.S is

$$c_{20} = (-1)^0\binom{5}{0}\binom{4+10}{10} + (-1)^1\binom{5}{1}\binom{4+5}{5}(-1)^2\binom{5}{2}\binom{4+0}{0} \quad = \binom{14}{10} - 5\binom{9}{5} + \binom{5}{2}$$

PROBLEMS 9. Determine the coefficient of

$(i)\ x^{15}$ in $\dfrac{(1+x)^4}{(1-x)^4}$ $\quad (ii)\ x^{10} in \dfrac{(x^3-5x)}{(1-x)^3}$ $\quad (iii)\ x^8$ in $\dfrac{1}{(x-3)(x-2)^2}$

Solution:

$$(i)\quad \frac{(1+x)^4}{(1-x)^4} = (1+x)^4(1-x)^{-4}$$

$$= \sum_{r=0}^{4}\binom{4}{r}x^r\cdot(1-x)^{-4}$$

$$= \{1+4x+6x^2+4x^3+x^4\}\times(1-x)^{-4}$$

$$= (1+4x+6x^2+4x^3+x^4)\times\sum_{r=0}^{\infty}\binom{3+r}{r}x^r$$

Therefore the coefficient of x^{15} in R.H.S is

$$c_{15} = \binom{18}{15} + 4\times\binom{17}{14} + 6\times\binom{16}{13} + 4\times\binom{15}{12} + \binom{14}{11}$$

$$(ii)\quad \frac{(x^3-5x)}{(1-x)^3} = (x^3-5x)(1-x)^{-3}$$

$$= (x^3-5x)\times\sum_{r=0}^{\infty}\binom{3+r-1}{r}x^r$$

$$= (x^3-5x)\times\sum_{r=0}^{\infty}\binom{2+r}{r}x^r$$

$$= \sum_{r=0}^{\infty}\binom{2+r}{r}x^{r+3} - 5\times\sum_{r=0}^{\infty}\binom{2+r}{r}x^{r+1}$$

Therefore the coefficient of x^{10} in R.H.S is

$$c_{10} = \binom{9}{7} - 5\times\binom{11}{9} = \frac{9!}{7!2!} - \left(5\times\frac{11!}{9!2!}\right) = -239$$

$(iii)\ x^8$ in $\dfrac{1}{(x-3)(x-2)^2}$

Solution:

$$(iii)\ \frac{1}{(x-3)(x-2)^2} = \frac{1}{(-3)(1-x/3)(-2)^2(1-x/2)^2}$$

$$= -\frac{1}{12} \times (1-x/3)^{-1} \times (1-x/2)^{-2}$$

$$= -\frac{1}{12} \times \sum_{r=0}^{\infty} \left(\frac{x}{3}\right)^r \times \sum_{s=0}^{\infty} \binom{1+s}{-s} \left(\frac{x}{2}\right)^s$$

$$= -\frac{1}{12} \sum_{r=0}^{\infty} \left(\frac{1}{3}\right)^r x^r \times \sum_{s=0}^{\infty} \left(\frac{1}{2}\right)^x \binom{1+s}{s} x^r$$

Therefore the coefficient of x^8 in R.H.S is

$$c_8 \approx -\frac{1}{12} \times \left\{ \left(\frac{1}{3}\right)^0 \left(\frac{1}{2}\right)^8 \binom{9}{8} + \left(\frac{1}{3}\right)^1 \left(\frac{1}{2}\right)^7 \binom{8}{7} + \left(\frac{1}{3}\right)^2 \left(\frac{1}{2}\right)^6 \binom{7}{6} + \cdots\cdots + \left(\frac{1}{3}\right)^7 \left(\frac{1}{2}\right) \binom{2}{1} + \left(\frac{1}{3}\right)^8 \left(\frac{1}{2}\right)^0 \binom{1}{0} \right\}$$

$$= -\frac{1}{12} \times \sum_{k=0}^{8} \left(\frac{1}{3}\right)^k \left(\frac{1}{2}\right)^{8-k} \binom{9-k}{8-k}$$

PROBLEMS

10. Determine the coefficient of x^{27}

$(i)\ (x^4 + x^3 + x^5 + \dots)^5$

$(ii)\ (x^4 + 2x^5 + 3x^6 + \cdots)^5$

Solution:

$$(i)\ (x^4 + x^5 + x^6 + \cdots)^5 = x^{20}(1 + x + x^2 + \cdots)^5$$

$$= x^{20}\{(1-x)^{-1}\}^5$$

$$= x^{20} \times (1-x)^{-5}$$

$$= x^{20} \times \sum_{r=0}^{\infty} \binom{4+r}{r} x^r$$

The coefficient of $x^{27}$is

$$\binom{11}{7} = \frac{11!}{7!4!} = 330$$

$$(ii)\ (x^4 + 2x^5 + 3x^6 + \cdots)^5 = x^{20}(1 + 2x + 3x^2 + \cdots)^5$$

$$= x^{20} \times \{(1-x)^{-2}\}^5$$

$$= x^{20} \times (1-x)^{-10}$$

$$= x^{20} \times \sum_{r=0}^{\infty} \binom{9+r}{r} x^r$$

The coefficient of $x^{27}$is

$$\binom{16}{7} = \frac{16!}{9!7!} = 11,440$$

JAIN DEEMED-TO-BE UNIVERSITY FACULTY OF ENGINEERING AND TECHNOLOGY

PROBLEMS

11. Determine the coefficient of x^{18}

$(i)\ \left(x+x^2+x^3+x^4+x^5\right)\left(x^2+x^3+x^4+\cdots\right)^5$

$(ii)\ \left(x+x^3+x^5+x^7+x^9\right)\left(x^3+2x^4+3x^5+\cdots\right)^3$

Solution:

$$(i)\ \left(x+x^2+x^3+x^4+x^5\right)\left(x^2+x^3+x^4+\cdots\right)^5$$

$$=x^{11}\left(1+x+x^2+x^3+x^4\right)\left(1+x+x^2+\cdots\right)^5$$

$$=x^{11}\left(1+x+x^2+x^3+x^4\right)\left\{(1-x)^{-1}\right\}^5$$

$$=x^{11}\left(1+x+x^2+x^3+x^4\right)(1-x)^{-5}$$

$$=x^{11}\left(1+x+x^2+x^3+x^4\right)\times\sum_{r=0}^{\infty}\binom{4+r}{r}x^r$$

The coefficient of x^{18} is

$$\binom{11}{7}+\binom{10}{6}+\binom{9}{5}+\binom{8}{4}+\binom{7}{3}$$

$$(ii)\ \left(x+x^3+x^5+x^7+x^9\right)\left(x^3+2x^4+3x^5+\cdots\right)^3$$

$$=x^{10}\left(1+x^2+x^4+x^6+x^8\right)\left(1+2x+3x^2+\cdots\right)^3$$

$$=x^{10}\left(1+x^2+x^4+x^6+x^8\right)\left\{(1-x)^{-2}\right\}^3$$

$$=x^{10}\left(1+x^2+x^4+x^6+x^8\right)\times\sum_{r=0}^{\infty}\binom{5+r}{r}$$

The coefficient of x^{18} is

$$\binom{13}{8}+\binom{11}{6}+\binom{9}{4}+\binom{7}{2}+\binom{5}{0}$$

Counting Technique

Suppose we wish to determine the number of integer solutions of the equation

$$\boldsymbol{x_1 + x_2 + x_3 + \cdots + xn} \textbf{ =r where } \boldsymbol{n \geq r \geq 0}$$

under the constraints that

x_1 can take the integer values P11, P12, P13....

x_2 can take the integer values p21, P22, P23....

x_n can take the integer values Pn1, Pn2, Pn3,...

To solve this problem we first define the functions $f_1(x), f_2(x), \ldots + f_n(x)$ as follows

$$f_1(x) = x^{p11} + x^{p12} + x^{p13} + \cdots$$
$$f_2(x) = x^{p21} + x^{p22} + x^{p23} + \cdots$$
$$\cdots\cdots$$
$$f_n(x) = x^{pn1} + x^{pn2} + x^{pn3} + \cdots$$

We then consider the function $f(x)$

$$f(x) = f_1(x) \cdot f_2(x) \cdot f_3(x) \ldots f_n(x)$$

PROBLEMS 1. Find the generating function that determines the number of non-negative integer solution of the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 20$

Under the constraints $x_1 \leq 3,\ x_2 \leq 4,\ 2 \leq x_3 \leq 6,\ 2 \leq x_4 \leq 5,\ x_5$ is odd with $x_5 \leq 9$

Solution:

The constraints for x_1 is that x_1 is a nonnegative integer, and $x_1 \leq 3$. Hence x_1 can take the values 0, 1, 2, 3. Keeping this in mind, let us take

$$f_1(x) = x^0 + x^1 + x^2 + x^3$$

$$f_2(x) = x^0 + x^1 + x^2 + x^3 + x^4$$

$$f3(x) = x^2 + x^3 + x^4 + x^5 + x^6$$

$$f_4(x) = x^2 + x^3 + x^4 + x^5$$

$$f_5(x) = x^1 + x^3 + x^5 + x^7 + x^9$$

Hence, the generating function for the number of nonnegative integer solution of the given equation under the given constraints is

$$f(x) = f_1(x) \cdot f_2(x) \cdot f_3(x) \cdot f_4(x) \cdot f_5(x)$$

where $f_1(x), f_2(x), \ldots, f_5(x)$ are given above

PROBLEMS 2. Using generating function, find the number of (i) nonnegative, and (ii) Positive integer solutions of the equation $x_1 + x_2 + x_3 + x_4 = 25$

Solution: (i) In case of nonnegative integer solutions, x_i can take the values 0, 1, 2, 3…. Accordingly, we choose

$$f_i(x) = x^0 + x^1 + x^2 + x^3 + \cdots, \text{ for } i = 1, 2, 3, 4.$$

Therefore, the generating function associated with the problem is

$$\begin{aligned} f(x) = f_1(x) \cdot f_2(x) \cdot f_3(x) \cdot f_4(x) &= \left(x^0 + x^1 + x^2 + x^3 + \cdots\right)^4 \\ &= \left(1 + x + x^2 + x^3 + \cdots\right)^4 = \left((1-x)^{-1}\right|^4 \\ &= (1-x)^{-4} \\ &= \sum_{r=0}^{\infty} \binom{3+r}{r} x^r \end{aligned}$$

The coefficient of x^{25} in this is

$$\binom{3+25}{25} = \frac{28!}{25!3!} = 3276$$

(ii) In case of positive integer solutions

Solution:

(ii) In case of positive integer solutions, x_i can take the values 1, 2, 3…. Accordingly, we choose

$$f_i(x) = x^1 + x^2 + x^3 + \cdots, \text{ for } i = 1,2,3,4$$

Therefore, the generating function associated with the problem is

$$f(x) = x^1 + x^2 + x^3 + \cdots, \text{ for } i = 1,2,3,4$$

$$f(x) = f_1(x) \cdot f_2(x) \cdot f_3(x) \cdot f_4(x) = \left(x + x^2 + x^3 + \cdots\right)^4$$

$$= x^4\left(1 + x + x^2 + x^3 + \cdots\right)^4$$

$$= x^4\left[(1-x)^{-1}\right]^4$$

$$= x^4(1-x)^{-4}$$

$$= x^4 \sum_{r=0}^{\infty} \binom{3+r}{r} x^r$$

The coefficient of x^{25} in this is

$$\binom{3+21}{21} = \frac{24!}{21!3!} = 2024$$

PROBLEMS 3 Find the generating function for the number of integer solution to the equation $c_1 + c_2 + c_1 + c_4 = 20$ where $-3 \le c_1,\ -3 \le c_2,\ -5 \le c_3 \le 5$ and $0 \le c_4$ also find the number of such solution.

Solution: Let us set $x_1 = c_1 + 3,\ x_2 = c_2 + 3,\ x_3 = c_3 + 5,\ x_4 = c_4$ where

$x_1 \ge 0,\ x_2 \ge 0,\ 0 \le x_3 \le 10,\ x_4 \ge 0$ and the given equation reads

$$\boxed{x_1 + x_2 + x_3 + x_4 = 31}$$

Thus, the number of integer solution of the given equation under the given constraints is,

$$f_1(x) = x^0 + x' + x^2 + \cdots = (1-x)^{-1}$$

$$f_2(x) = x^0 + x^1 + x^2 + \cdots (1-x)^{-1}$$

$$f_3(x) = x^0 + x^1 + x^2 + \cdots + x^{10}$$

$$f_4(x) = x^0 + x^1 + x^2 + \cdots = (1-x)^{-1}$$

Therefore the generating function is

$$f(x) = f_1(x) \cdot f_2(x) \cdot f_3(x) \cdot f_4(x)$$

$$= (1 + x + x^2 + \cdots + x^{10})(1-x)^{-3}$$

$$= (1 + x + x^2 + \cdots + x^{10}) \times \sum_{i=10}^{\infty} \binom{2+r}{r} x^r$$

The coefficient of x^{31} in this is

$$c_{31} = \binom{33}{31} + \binom{32}{30} + \binom{31}{29} + \cdots + \binom{23}{21}$$

PROBLEMS 4. In how many ways can 12 oranges be distributed among three children A, B, C, so that A gets at least four, B and C get at least two, but C gets no more than five?

Solution:

- Let x_1 be the number of oranges which A can get, x2 be the number of oranges which B can get, and x3 be the number of oranges which C can get.

Then

x1 + x2 + x3 = 12

From the given constraints, we note that

X1 ≥ 4, x2 ≥ 2, 2≤ x3 ≤5,

The required number is equal to the number of integer solutions of the equation (i) under the constraints (ii).

Keeping the constraints (ii) in mind, let us take

$$f_1(x) = x^4 + x^5 + x^6 + \cdots = x^4\left(1 + x + x^2 + \cdots\right) = x^4(1-x)^{-1}$$

$$f_2(x) = x^2 + x^3 + x^4 + \cdots = x^2\left(1 + x + x^2 + \cdots\right) = x^2(1-x)^{-1}$$

$$f_3(x) = x^2 + x^3 + x^4 + x^2 = x^2\left(1 + x + x^2 + x^3\right)$$

Then the generating function associated with the problem is

$$f(x) = f_1(x) f_2(x) f_1(x)$$

$$= x^8\left(1 + x + x^2 + x^3\right)(1-x)^{-2}$$

$$= x^8\left(1 + x + x^2 + x^3\right) \times \sum_{r=0}^{\infty} \binom{1+r}{r} x^r$$

The coefficient of x^{12} in this is

$$c_{12} = \binom{4+1}{4} + \binom{3+1}{3} + \binom{2+1}{2} + \binom{1+1}{1}$$

$$= 5 + 4 + 3 + 2 = 14$$

PROBLEMS Find the number of ways of forming a committee of 9 students drawn from 3 different classes so that students from the same class do not have an absolute majority in the committee.

Solution: If any class is excluded, students from one of the other two classes will have an absolute majority. So, there must be at least 1 student from each class. Also, no class can have more than 4 representatives in the committee. Therefore, if x is the number of students drawn from class i, we have x1 + x2 + x3 = 9 and $1 \leq x \leq 4$ for i = 1, 2, 3. Therefore, we take

$$f_i(x) = x^2 + x^2 + x^3 + x^4 \quad \text{for } i = 1, 2, 3$$

So that the generating function associated with the problem is

$$f(x) = f_1(x) f_2(x) f_3(x)$$

$$= x^8 \left(1 + x + x^2 + x^3\right)(1 - x)^{-2} = x^8 \left(1 + x + x^2 + x^3\right) \times \sum_{r=0}^{\infty} \binom{1+r}{r} x^2$$

The coefficient of x^9 in this is

$$c_{12} = \binom{4+1}{4} + \binom{3+1}{3} + \binom{2+1}{2} + \binom{1+1}{1} = 5 + 4 + 3 + 2 = 14.$$

Partitions of Integers

Consider a positive integer n. Then an expression that gives as sum of $1's. 2's, 3's \ldots. n's$ without regard to order, is called a partition of n. The number of partitions of n is denoted by $\boldsymbol{p(n)}$.

For example, the following expressions give 4 as sum of 1's, 2's, 3's and 4's (without regard to order):

$\mathbf{4 = 1 + 1 + 1 + 1}$, $\mathbf{4 = 1 + 1 + 2}$, $\mathbf{4 = 2 + 2}$. $\mathbf{4 = 3 + 1}$. $\mathbf{4 = 4}$.

Each of the above is a partition of 4, and the number of these partitions is $\boldsymbol{p(4) = 5}$,

Similarly, the partitions of 5 are

$\mathbf{5 = 1 + 1 + 1 + 1 + 1}$, $\mathbf{5 = 1 + 1 + 1 + 2}$, $\mathbf{5 = 1 + 1 + 3}$, $\mathbf{5 = 2 + 2 + 1}$, $5 = 1 + 4$, $5 = 2 + 3$, $5 = 5$ and $\boldsymbol{p(5) = 7}$.

The following Rules give the number of partitions of positive integer n, under various restriction

Rule 1: For any integer n, the number of partition of n, namely $p(n)$, is the coefficient of x^n in the product

$$\frac{1}{1-x}\cdot\frac{1}{1-x^2}\cdot\frac{1}{1-x^3}\cdots\cdot\frac{1}{1-x^n}\cdots$$

This infinite product is denoted by $\prod_{r=1}^{\infty}\left(\frac{1}{1-x^r}\right)^*$

And it is called generating function for $P(n)$.

For example $p(4)$ is the coefficient of x^4 in the product

$$\prod_{r=1}^{\infty}\left(\frac{1}{(1-x^r)}\right)=\frac{1}{(1-x)}\cdot\frac{1}{(1-x^2)}\cdot\frac{1}{(1-x^3)}\cdot\frac{1}{(1-x^4)}\cdots$$

$$=(1-x)^{-1}(1-x^2)^{-1}(1-x^3)^{-1}(1-x^4)^{-1}\times\cdots$$

$$=(1+x+x^2+x^3+\cdots)(1+x^2+x^4+\cdots)\times\cdots$$
$$\times(1+x^3+x^6+\cdots)\times(1+x^4+x^8+\cdots)\times\cdots$$

$$=(1+x+2x^2+2x^3+3x^4+\cdots)\times(1+x^3+x^4+\cdots)\times\cdots$$

$$=(1+x+2x^2+2x^3+3x^4+\cdots)\times(1+x^3+x^4+\cdots)\times\cdots$$

$$=1+x+2x^2+3x^3+5x^4+\cdots$$

The coefficient of x^4 in this is 5. Thus $p(4)=5$.

Rule 2: For any positive integer n, the number of partitions of n into summands whose number does not exceed a fixed number k, is the coefficient of x" in the generating function

$$\prod_{r=1}^{k}\left(\frac{1}{1-x^r}\right) \equiv \frac{1}{1-x}\cdot\frac{1}{1-x^2}\cdots\frac{1}{1-x^k}$$

For example. Let $n = 4$ and $k = 3$, then

$$\prod_{r=1}^{3}\frac{1}{(1-x^r)} \equiv \frac{1}{1-x}\cdot\frac{1}{1-x^2}\cdot\frac{1}{1-x^3}$$

$$= (1-x)^{-1}(1-x^2)^{-1}(1-x^3)^{-1}$$

$$= (1+x+x^2+x^3+x^4+\cdots)\times(1+x^2+x^4+\cdots)$$
$$\times(1+x^3+x^6+\cdots)$$

$$= 1+x+2x^2+3x^3+4x^4+\cdots$$

The coefficient of $x^n = x^4$ in this is 4. This is the number of partitions of n = 4 in which the number of summands does not exceed k = 3. (These four partitions are 4=1+1+2, 4=2+2,4=3+1, 4 = 4).

Rule 3. For any positive integer n, the number of partitions of n into distinct summands, de noted by $p_d(n)$, is the coefficient of x^n in the generating function

$$P_d(x) = \prod_{r=1}^{\infty} (1 + x^n) = (1 + x)\,(1 + x^2)\,(1 + x^3) \ldots (1 + x^n) \ldots.$$

For example $p_d(4)$ is the coefficient of x^4 in the product

$$\prod_{r=1}^{\infty} (1 + x^n) \equiv (1 + x) \times (1 + x^2) \times (1 + x^3) \times (1 + x^4) \times \cdots$$

$$= (1 + x + x^2 + x^3) \times (1 + x^3 + x^4 + x^7) \times \cdots$$

$$= (1 + x + x^2 + 2x^3 + 2x^4 + \cdots)$$

Note: The generating function for the number of partitions of n with

(i) distinct even summands is $\prod_{r=1}^{\infty} (1 + x^{2r})$

(ii) distinct odd summands is $\prod_{r=1}^{\infty} (1 + x^{2r-1})$

The coefficient of x^4 in this is 2. Thus $p_d(4)$ = 2. This means that 4 has 2 partitions with distinct summands. (4=3+1, 4=4)

Rule 4. For any positive integer n, the number of partitions of n into odd summands, denoted by Po(n), is the coefficient of x^n in the generating function

$$P_o(x) = \prod_{r=0}^{\infty}\left(\frac{1}{1-x^{2r+1}}\right) = \frac{1}{1-x}\cdot\frac{1}{1-x^3}\cdot\frac{1}{1-x^5}\cdots$$

For example $p_o(5)$ is the coefficient of x^5 in the product

$$P_a(x) = \prod^{m}\left(\frac{1}{1-x^{2x+1}}\right) = \frac{1}{1-x}\cdot\frac{1}{1-x^3}\cdot\frac{1}{1-x^5}\cdots.$$

$$= (1+x+x^2+x^3+x^4+\cdots)\times(1+x^3+x^6+\cdots)$$
$$\times(1+x^5+x^{10}+\cdots)\times\cdots$$

$$= 1+x+x^2+2x^3+2x^4+3x^5+\cdots$$

Note: The generating function for the number of partitions of n into even summands is

$$\prod_{r=1}^{\infty}\left(\frac{1}{1-x^2}\right)$$

The coefficient of x in this is 3. Thus, Po(5) = 3. This means that there are 3 partitions of 5 into odd summands. (These are 5 = 1+1+1+1+1, 5=1+1+3, 5 = 5).

Rule 5. For any positive integer n, the number of partitions of n into odd summands wherein each such (odd) summand occurs an odd number of times - or not at all, is the coefficient of x^n in the generating function

$$f(x) = \prod_{k=0}^{\infty} \left\{ 1 + \sum_{r=0}^{\infty} x^{(2k+1)(2r+1)} \right\}$$

Expanding the RHS we find that

$$f(x) = \prod_{k=0}^{\infty} \left\{ 1 + x^{(2k+1)(0+1)} + x^{(2k+1)(2+1)} + x^{(2k+1)(4+1)} + \cdots \right\}$$

$$= \left(1 + x + x^3 + x^5 + \cdots\right) \times \left(1 + x^3 + x^9 + x^{15} + \cdots\right)$$
$$\times \left(1 + x^5 + x^{15} + x^{25} + \cdots\right) \times \cdots$$

The coefficient of x^3 in this infinite product is 2. This means that the number of partitions of n = 3 into odd summands wherein each such (odd) summand occurs an odd number of times (or not all) is 2. (These partitions are 3 = 1 + 1 + 1,3 = 3). For n = 4, we find that this number is 1. (Observe that the coefficient of x^4 in f(x) is 1. and the concerned partition is 4 = 1+3).

PROBLEMS 1. Using generating function, find the number of partitions of n = 6.

Solution: ► The generating function for the partition p(n) of n = 6 is (see Rule 1)

$$\prod_{r=1}^{\infty}\frac{1}{(1-x^r)} \equiv \frac{1}{1-x}\cdot\frac{1}{1-x^2}\cdot\frac{1}{1-x^3}\cdot\frac{1}{1-x^4}\cdot\frac{1}{1-x^5}\cdot\frac{1}{1-x^6}\cdots.$$

$$= (1-x)^{-1}(1-x^2)^{-1}(1-x^3)^{-1}(1-x^4)^{-1}(1-x^5)^{-1}(1-x^6)^{-1}\cdots$$

$$=\left(1+x+x^2+x^3+x^4+x^5+x^6+\cdots\right)\times\left(1+x^2+x^4+x^6+\cdots\right)$$
$$\times\left(1+x^3+x^6+x^9+\cdots\right)\times\left(1+x^4+x^8+\cdots\right)$$
$$\times\left(1+x^5+x^{10}+\cdots\right)\times\left(1+x^6+x^{12}+\cdots\right)\times\cdots$$

$$=\left(1+x+x^2+x^3+x^4+x^5+x^6+\cdots\right.$$
$$\left.+x^2+x^3+x^4+x^5+x^6+\cdots+x^4+x^5+x^6+\cdots\right)$$
$$\times\left(1+x^3+x^6+x^9+\cdots+x^4+x^7+x^{10}+\cdots+x^8+x^{11}+x^{14}+\cdots\right)$$
$$\times\left(1+x^5+x^{10}+\cdots+x^6+x^{11}+x^{16}+\cdots\right)\times\cdots$$

$$=\left(1+x+2x^2+2x^3+3x^4+3x^5+3x^6+\cdots\right)$$
$$\times\left(1+x^3+x^4+x^5+2x^6+\cdots\right)\times\cdots$$

$$=1+x+2x^2+3x^3+6x^4+7x^5+11x^6+\cdots$$

The coefficient of x in this is 11. Thus, there are p(n) = 11 partitions for n = 6.

We note that these partitions are as given below:

6=1+1+1+1 +1+1	6=1+1+1+3	
6=1+1+1+1+2	6=1+1+46=1+5	
6=2+2+2	6=3+3	
6=1+1+4,	6=1+1+2+2	
6=1+2+3	6=2+4	6 = 6.

PROBLEMS 2. Find all partitions of n = 7.

Solution: The generating function for the partition p(n) of n = 7 is

$$\prod_{r=}^{\infty} \frac{1}{(1-x^2)} = \frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdot \frac{1}{1-x^3} \cdot \frac{1}{1-x^4} \cdot \frac{1}{1-x^3} \cdot \frac{1}{1-x^6} \cdot \frac{1}{1-x^7}$$

$$= (1-x)^{-1}(1-x^2)^{-1}(1-x^3)^{-1}(1-x^4)^{-1}(1-x^5)^{-1}(1-x^6)^{-1}(1-x^7)^{-1}\ldots$$

Expanding each factor in the RHS of the above expression by binomial theorem and simplifying the resulting expression" (like in Example 1), we find that the coefficient of x^7 in this expression is 15. Thus, there are $p(n) = 15$ partitions of $n = 7$. These partitions are as given below.

7=1+1+1+1+1+1+1, 7=1+1+1+1+1+2, 7=1+1+1+2+2, 7=1+1+2+3,

7=1+3+3, 7=3+4, 7=1+6, 7=1+1+5,

7=1+1+1+1+3 7=1+1+1+4 7=1+2+2+2 7=1+2+4

7=2+2+3 7=2+5, 7=7

PROBLEMS 3. Prove that the number of partitions of a positive integer n into distinct summands is equal to the number of partitions of n into odd summands.

Solution: We have to prove that $p_d(n) = p_o(n)$ for every positive integer n.

First, we note that

$$P_d(x) \equiv \prod_{r=1}^{\infty} \left(1 + x^5\right) = (1 + x)\left(1 + x^2\right)\left(1 + x^3\right)\left(1 + x^4\right)\dots$$

$$= \frac{1 - x^2}{1 - x} \cdot \frac{1 - x^4}{1 - x^2} \cdot \frac{1 - x^6}{1 - x^3} \cdot \frac{1 - x^8}{1 - x^4} \dots$$

$$= \frac{1}{1 - x} \cdot \frac{1}{1 - x^3} \cdot \frac{1}{1 - x^5} \dots$$

$$= \prod_{r=0}^{\infty} \frac{1}{1 - x^{2r+1}} \equiv P_d(x)$$

Therefore, for any positive integer n, the coefficients ofx^n in $p_d(x) and\ p_o(x)$ must be equal; that is,

$p_d(n) = p_o(n)$(see Rules 3 and 4).

PROBLEMS 4. Using generating function, find the number of partitions of n = 6 into distinct summands.

Solution: The generating function for the required number, $p_d(n)$ for n=6, is (see Rule 3)

$$P_d(x) = \prod_{r=1}^{\infty}(1+x^r) = (1+x)(1+x^2)(1+x^3)(1+x^4)\times(1+x^5)(1+x^6)\times\cdots$$

$$= (1+x+x^2+x^3)(1+x^3+x^4+x^7)(1+x^5+x^6+x^{11})\times\cdots$$

$$= (1+x+x^2+2x^3+2x^4+2x^5+2x^6+2x^7+x^8+x^9+x^{10})$$
$$\times(1+x^5+x^6+x^{11})\times\cdots$$

The coefficient of x^6 in this infinite product is 4. Thus, $p_d(n) = 4$ for n = 6.

(The 4 partitions of n = 6 into distinct summands are

6=1+5,6=1+2+3,6=2+4, 6 = 6).

Exponential generating function

Given a sequence $\langle a_r \rangle$ suppose there exists a function $E(x)$ such that the expansion of $E(x)$ in a series of powers of x is given by

$$E(x) = a_0 + a_1 x + a_2 \frac{x^2}{2!} + a_3 \frac{x^3}{3!} + \cdots + a_n \frac{x^n}{n!} + \cdots$$
$$= \sum_{r=n}^{\infty} a_r \frac{x^r}{r!}$$

Then $E(x)$ is called the exponential generating function for the sequence $\langle a_r \rangle$.

In other words, given a sequence $\langle a_r \rangle$, if there exists a function $E(x)$ such that a_r, is the coefficient of $(x^r/r!)$ in the expansion of $E(x)$ in a series of powers of x, then *$E(x)$ is called the exponential generating function* of the sequence $\langle a_r \rangle$.

As an example, we note that, since

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{r=0}^{\infty} \frac{x^r}{r!}$$

the function e^x is the exponential generating function for the sequence 1, 1, 1, 1.....Similarly, since

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots = \sum_{r=0}^{\infty} (-1)^r \frac{x^r}{r!}$$

e^{-x} is the exponential generating function for the sequence 1,-1,1,-1,1 We note that

$$\frac{1}{2}(e^x + e^{-x}) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

$$\frac{1}{2}(e^x - e^{-x}) = x + \frac{x^3}{3!} + \frac{x^5}{5!}$$

Accordingly, $\frac{1}{2}(e^x + e^{-x})$ is the exponential generating function for the sequence 1,0,1,0,1,0,..., and

$\frac{1}{2}(e^x - e^{-x})$ is the exponential generating function for the sequence 0, 1, 0, 1, 0,1 ...

Note: From the definitions of ordinary generating function (considered in slide no. 3) and exponential generating function, it follows that:

(1) If E(x) is the exponential generating function for a sequence $\langle a_r \rangle$,then $f(x)$ is an ordinary generating function for the sequence **($\langle \boldsymbol{a_r} \rangle / \boldsymbol{r}!$).**

(2) If f(x) is an ordinary generating function for a sequence $\langle a_r \rangle$,then $f(x)$ is the exponential generating function for the sequence **($\langle \boldsymbol{a_r} \rangle\ (\boldsymbol{r}!)$).**

Properties: The following are some properties of exponential generating functions.

(1) If $E(x)$ is the exponential generating function for a sequence $\langle a_r \rangle$ then $xE'(x)$ is the exponential generating function for the sequence $\langle r a_r \rangle$

(2) (2) If $E1(x)$ is the exponential generating function for a sequence $\langle a_r \rangle$ and $E2(x)$ is the exponential generating function for a sequence $\langle b_r \rangle$ then $pE1(x) + qE2(x)$ is the exponential generating function for the sequence $(p\ a_r + q\ b_r)$ for all real numbers $p\ and\ q$.

PROBLEMS 1. Determine the sequences for which the following are the exponential generating functions:

$(i)\ 6e^{5x} - 3e^{2x}$ $(ii)\ \dfrac{1}{1-x}$ $(iii)\ \dfrac{3}{1-2x} + e^{x}$ $(iv)\ e^{2x} - 3x^3 + 5x^2 + 7x$

Solution: (i) we have

$$6e^{5x} - 3e^{2x} = 6\sum_{r=0}^{\infty}\frac{(5x)^r}{r!} - 3\sum_{r=0}^{\infty}\frac{(2x)^r}{r!}$$

$$= \sum_{r=0}^{\infty}(6\times 5^r - 3\times 2^r)\frac{x^r}{r!}$$

Therefore the required sequence is

$$\langle\langle a_r\rangle = \langle(6\times 5^r - 3\times 2^r)\rangle$$

(ii) We have

$$\frac{1}{1-x} = (1-x)^{-1}$$

$$= 1 + x + x^2 + x^3 + \cdots$$

$$= \sum_{r=0}^{\infty} x^r$$

$$= \sum_{r=0}^{\infty}(r!)\frac{x^r}{r!}$$

Therefore the required sequence is $\langle a_r\rangle = \langle r!\rangle$

$$(iii)\ \frac{3}{1-2x}+e^{x}$$

$$(iv)\ e^{2x}-3x^{3}+5x^{2}+7x$$

Solution:

(iii) We have

$$\frac{3}{1-2x}+e^{x}=3(1-2x)^{-1}+e^{x}$$

$$=3\times\sum_{r=0}^{\infty}(2x)^{r}+\sum_{r=0}^{\infty}\frac{x^{r}}{r!}$$

$$=\sum_{r=0}^{\infty}\{(3\times 2^{r}\times r!)+1\}\frac{x^{r}}{r!}$$

Therefore the required sequence is

$$\langle a_r\rangle=(1+(3\times 2^{r}\times r!))$$

(iv) We have

$$e^{2x}-3x^{3}+5x^{2}+7x=\sum_{r=0}^{\infty}\frac{(2x)^{r}}{r!}-3x^{3}+5x^{2}+7x$$

$$=1+(2+7)x+\left(\frac{2^{2}}{2!}+5\right)x^{2}+\left(\frac{2^{3}}{3!}-3\right)x^{3}+\sum_{n=1}^{\infty}\frac{2^{r}x^{r}}{r!}$$

$$=1\cdot\frac{x^{0}}{0!}+9\cdot\frac{x}{1!}+\{2^{2}+5\times(2!)\}\frac{x^{2}}{2!}+(2^{3}-3\times(3!))\frac{x^{3}}{3!}+\sum_{r=4}^{\infty}2^{n}\left(\frac{x^{r}}{r^{3}}\right)$$

Therefore the required sequence is (a_r), *where* $a_0=1, a_1=9, a_2=2^{2}+5\times(2!)=14,$

$a_3=2^{3}-(3\times 3!)=-10,$ and $a_r=2^{r}$ for $r\geq 4.$

JAIN DEEMED-TO-BE UNIVERSITY FACULTY OF ENGINEERING AND TECHNOLOGY

PROBLEMS 2. Find the exponential generating function for each of the following sequences:

(i) $1, 2, 3, 0, 0, 0, 0$ (ii) $0, 0, 1, 1, 1, 1.$ (iii) $0!, 1!, 2!!, 3!$

(iv) $1, a, a^2, a^3, a^4, \ldots$ (v) $1, -a, a^2, -a^3, a^4$ (vi) $0, 1, 2a, 3a^2, 4a^3$

Solution: The required function is

(i) $E(x) = 1 + 2x + 3 \cdot \frac{x^2}{2!} + 0 + 0 + 0 + \quad = \left(1 + 2x + \frac{3}{2}x^2\right)$

(ii) $E(x) = 0 + 0 + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) - (1 + x) \; = e^x - (1 + x)$

(iii) $E(x) = 0! + 1!x + 2!\frac{x^2}{2!} + 3!\frac{x^3}{3!} + \cdots \; = 1 + x + x^2 + x^3 + \cdots \; = (1 - x)^{-1}$

(iv) $E(x) = 1 + ax + a^2\frac{x^2}{2!} + a^3\frac{x^3}{3!} + a^4\frac{x^4}{4!} + \cdots = e^{ax}$

(v) $E(x) = 1 - ax + a^2\frac{x^2}{2!} - a^3\frac{x^3}{3!} + a^4\frac{x^4}{4!} + \quad = e^{-ax}$

(vi) $E(x) = 0 + x + 2a\frac{x^2}{2!} + 3a^2\frac{x^3}{3!} + 4a^3\frac{x^4}{4!} + \cdots = x\left(1 + ax + a^2\frac{x^2}{2!} + a^3\frac{x^3}{3!} + \cdots\right) \; = xe^{ax}$

PROBLEMS 2. Using exponential generating function, find the number of ways in which 4 of the letters in the words given below be arranged.(i) ENGINE(ii) HAWAII

Solution: (i) In the word ENGINE there are 2 E's, 2 N's and 1 each of G and I. In an arrangement of these letters E can appear 0, 1 or 2 times. Keeping this in mind, we consider the function $E_1(x)$ corresponding to (the letter) E as

$$E_1(x) = 1 + x + \frac{x^2}{2!}$$

Similarly, the function corresponding to N is taken as

$$E_2(x) = 1 + x + \frac{x^2}{2!}$$

Since G can appear 0 or 1 time in an arrangement, the function corresponding to G is taken as

$$E_3(x) = 1 + x$$

Similarly, the function corresponding to I is taken as

$$E_4(x) = 1 + x$$

Solution:

$$\text{Thus } E(x) = E_1(x)E_2(x)E_3(x)E_4(x)$$

serves as the exponential generating function for the problem in the sense that the coefficient of $(x^4/4!)$ in this function gives the number of arrangements of four letters of the given word. We find that

$$E(x) = \left(1 + x + \frac{x^2}{2!}\right)^2 (1+x)^2$$

$$= \left(1 + 2x + 2x^2 + x^3 + \frac{x^4}{4}\right)(1 + 2x + x^2)$$

the coefficient of $(x^4/4!)$ in this is

$$(4!) \times \left(\frac{1}{4} + 2 + 2\right) = 4! \times \frac{17}{4} = 17 \times (3!) = 102$$

This is the required number

Solution:

(ii) In the word HAWAII, there are two A's, two I's and one each of H and W. Keeping this in mind, we find (as in case (1)) that the exponential generating function for the problem is

$$E(x) = \left(1 + x + \frac{x^2}{2!}\right)^2 (1+x)^2$$

$$= \left(1 + 2x + 2x^2 + x^3 + \frac{x^4}{4}\right)\left(1 + 2x + x^2\right)$$

the coefficient of $(x^4/4!)$ in this is

$$(4!) \times \left(\frac{1}{4} + 2 + 2\right) = 4! \times \frac{17}{4} = 17 \times 3! = 102$$

This is the required number.

PROBLEMS 3. Find how many distinct 4-digit and S-digit integers one can make from the digits 1, 3, 3, 7, 7, 8.

Solution: The given digits consist of two each of 3 and 7 and one each of 1 and 8. This fact leads us to consider the following exponential generating function:

$$E(x) = \left(1 + x + \frac{x^2}{2!}\right)^2 (1+x)^2$$
$$= \left(1 + 2x + 2x^2 + x^3 + \frac{x^4}{4}\right)(1 + 2x + x^2)$$

the coefficient of $(x^4/4!)$ in this generating function is $(4!) \times \left(\frac{1}{4} + 2 + 2\right) = (4!) \times \frac{17}{4} = (3!) \times 17 = 102$

Thus, four of the given integers can be arranged in 102 distinct ways. This means that the number of distinct four-digit integers that can be formed from the given digits is 102.

Next, we find that the coefficient of $(x^5/5!)$ in the generating function considered above is

$$(5!) \times \left(\frac{2}{4} + 1\right) = (5!) \times \frac{3}{2} = 180$$

This shows that the number of five-digit integers that can be formed from the given digits is180.

PROBLEMS 4. There are 10 marbles of the same size but of different colours (1 red, 3 blue, 2 green, 2 orange, 1 white, and 1 black) in a bag. Find in how may ways six of these marbles can be arranged in a row.

Solution: In the given set of marbles, there are 3 marbles of one colour, 2 each of two colours and 1 each of 3 colours. This fact leads us to consider the exponential generating function

$$E(x) = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}\right)^1 \times \left(1 + x + \frac{x^2}{2!}\right)^2 \times (1+x)^3$$

$$= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6}\right)\left(1 + 2x + 2x^2 + x^3 + \frac{1}{4}x^4\right) \times \left(1 + 3x + 3x^2 + x^3\right)$$

$$= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6}\right)\left(1 + 5x + 11x^2 + 14x^3 + \frac{45}{4}x^4 + \frac{23}{4}x^5 + \frac{7}{4}x^6 + \frac{1}{4}x^7\right)$$

the coefficient of $(x^6/6!)$ in this is

$$(6!) \times \left(\frac{7}{4} + \frac{23}{4} + \frac{45}{8} + \frac{7}{3}\right) = \frac{(6!) \times 371}{24} = 11,130.$$

Thus, six of the given marbles can be arranged in a row in 11,130 ways

JGi JAIN DEEMED-TO-BE UNIVERSITY | FACULTY OF ENGINEERING AND TECHNOLOGY

PROBLEMS 5. For each of the words given below, find the exponential generating function for the number of ways to arrange ns 11 letters selected from the word. ***(i) MISSISSIPPI (ii) ISOMORPHISM(ii) MATHEMATICS (v) ENGINEERING***

Solution:

(i)Here, there are 4 each of I and S, 2 P's and 1 M. Therefore, the required exponential generating function is

$$E(x) = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}\right)^2 \left(1 + x + \frac{x^2}{2!}\right)(1 + x)$$

(ii) Here, there are 2 each of I, S, O and M, and 1 each of R, P and exponential generating function is

$$E(x) = \left(1 + x + \frac{x^2}{2!}\right)^4 (1 + x)^3$$

(iii) Here, there are two each of M, A and T, and one each of H, E, I, C, S. Therefore, the required exponential generating function is

$$E(x) = \left(1 + x + \frac{x^2}{2!}\right)^3 (1 + x)^5$$

Solution:

(iv) Here, there are three each of E and N, two each of G and I and one R. Therefore, the required exponential generating function is

$$E(x) = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}\right)^2 \left(1 + x + \frac{x^2}{2!}\right)^2 (1 + x)$$

6.If a leading digit 0 is permitted, find the number of r-digit binary sequences that can be formed using an even number of 0's and an odd number of 1's.

7. A company appoints 11 software engineers, each of whom is to be assigned to one of four offices of the company. Each office should get at least one of these engineers. In how many ways can these assignments be made?

THANK YOU

Recurrence Relation

First, we consider for solution recurrence relations of the form

$$a_n = ca_{n-1} + f(n), \quad \text{for } n \geq 1 \qquad \text{---------(1)}$$

where c is a known constant and $f(n)$ is a known function. Such a relation is called a linear recurrence relation of first-order with constant coefficient.

If $f(n) = 0$, the relation is called homogeneous; otherwise, it is called non-homogeneous (or inhomogeneous).

The relation (1) can be solved in a trivial way. First, we note that this relation may be rewritten as (by changing n to n + 1)

$$a_{n+1} = ca_n + f(n+1), \quad \text{for } n \geq 0 \qquad \text{--------(2)}$$

For n=0,1,2,3,..., this relation yields, respectively.

$$a_1 = ca_0 + f(1),$$

$$a_2 = ca_1 + f(2) = c\{ca_0 + f(1)\} + f(2)$$

$$= c^2a_0 + cf(1) + f(2),$$

$$a_3 = ca_2 + f(3) = c\{c^2a_0 + cf(1) + f(2)\} + f(3)$$
$$= c^3a_0 + c^2f(1) + cf(2) + f(3)$$

and so on. Examining these, we obtain, by induction,

$$a_n = c^n a_0 + c^{n-1}f(1) + c^{n-2}f(2) + \cdots + cf(n-1) + f(n)$$

$$= c^n a_0 + \sum_{k=1}^{n} c^{n-k} f(k), \quad \text{for } n \geq 1 \qquad \text{-----(3)}$$

This is the general solution of the recurrence relation (2) which is equivalent to the relation (If f(n) = 0, that is if the recurrence relation is homogeneous, the solution (3) becomes

$$a_n = c^n a_0 \quad \text{for } n \geq 1 \qquad \text{-----(4)}$$

The solutions (3) and (4) yield particular solutions if an is specified. The specified a_o is called the initial condition.

PROBLEMS 1. Solve the recurrence relation $a_{n+1} = 4a_n$, for given that $a_0 = 3$.

Solution: The given relation is homogeneous. Its general solution is

$$a_n = 4^n a_0 \quad \text{for } n \geq 1. \qquad \text{---------(i)}$$

It is given that $a_0 = 3$. Putting this into (i), we get

$$a_n = 3 \times 4^n \text{ for } n \geq 1. \qquad \text{---------(ii)}$$

This is the particular solution of the given relation, satisfying the initial condition $a_0 = 3$

PROBLEMS 2. Solve the recurrence relation $a_n = 7a_{n-1}$, where $n \geq 1$, given that $a_2 = 98$.

Solution:

The given relation may be rewritten $a_{n+1} = 7a_n$, för $n \geq 0$. The general solution of homogeneous relation is

$$a_n = 7^n a_0 \quad \text{for } n \geq 1 \qquad \text{---------(i)}$$

It is given that $a_2 = 98$. Using this in (i) we get $98 = a_2 = 7^n a_0$so that a = 2. Puttinginto the general solution (i) we get the particular solution

$$a_n = 2 \times 7^n \qquad \text{---------(ii)}$$

This is the solution of the given relation under the condition $a_2 = 98$..

PROBLEMS 3. Solve the recurrence relation $a_n - 3a_{n-1} = 5 \times 3^n$ for $n \geq 1$ given that $a_0 = 2$

Solution: The given relation may be rewritten as

$$a_{n+1} = 3a_n + 5 \times 3^{n+1} \text{ for } n \geq 0$$

$$= 3a_n + f(n+1), \quad \text{where } f(n) = 5 \times 3^n$$

The general solution for this relation

$$a_n = 3^n a_0 + \sum_{1=1}^{n} 3^{n-k} f(k)$$

$$= 3^n a_0 + 3^{n-1} f(1) + 3^{n-2} f(2) + 3^{n-3} f(3) + \cdots + 3^0 f(n)$$

Substituting for a_0 and $f(n), n = 1,2, \ldots n$ in this, we get

$$a_n = 2 \times 3^n + 3^{n-1} \times (5 \times 3^l) + 3^{n-2} \times (5 \times 3^2) + 3^{n-3} \times (5 \times 3^3) + \cdots + 3^0 \times (5 \times 3^n)$$

$$= 2 \times 3^n + 5 \times (3^n + 3^n + 3^n + \cdots + 3^n)$$

$$= 2 \times 3^n + 5 \times (n3^n)$$

$$= (2 + 5n)3^n$$

This is the required solution.

4. Solve the recurrence relation $a_n - 3a_{n-1} = 5 \times 7^n$ for $n \geq 1$, given that $a_0 = 2$. (Homework)

PROBLEMS 5. The number of virus affected files in a system is 1000 (to start with) and this increases 250% every two hours. Use a recurrence relation to determine the number of virus affected files in the system after one day.

Solution: In the beginning, the number of virus affected files is 1000. Let us denote this by a_0 Let a_n denote the number of virus affected files after 24 hours. Then the number increases by a_n x 250/100, in the next two hours.

Thus, after 2n + 2 hours, the number is

$$a_{n+1} = a_n + a_n \times \frac{250}{100}$$
$$= a_n(1 + 2.5) = a_n(3.5)$$

This is the recurrence relation for the number of virus affected files. Solving this relation, we get

$$a_n = (3.5)^n a_0 = 1000 \times (3.5)^n$$

This gives the number of virus affected files after 24 hours. From this, we get (for n =12)

$$a_{12} = 1000 \times (3.5)^{12} = 3379220508$$

This is the number of virus affected files after one day (24 hours).

PROBLEMS 6. Suppose that there are $n \geq 2$ person sat a party and that each of these persons shakes hands (exactly once) with all of the other persons present. Using a recurrence relation find the number of hand shakes.

Solution: Let $a_n - 2$ denote the number of hand shakes among the $n \geq 2$ persons present. (If n=2, the number of handshakes is 1: that is a_o = 1). If a new person joins the party, he will shake hands with each of the n persons already present. Thus, the number of hand shakes increases by n when the number of persons changes to $n + 1$ from n. Thus,

$$a_{(n+1)-2} = a_{n-2} + n \quad \text{for } n \geq 2,$$

$$\text{or} \qquad a_{m+1} = a_m + (m+2) \text{ for } \quad m \geq 0, \quad \text{where } m = n-2$$

Setting f(m) = m + 1, this reads

$$a_{m+1} = a_m + f(m+1) \qquad \text{for } m \geq 0$$

The general solution of this nonhomogeneous recurrence relation is

$$a_m = (1^m \times a_0) + \sum_{k=1}^{m} 1^{m-k} f(k) = a_0 + \sum_{k=1}^{m} (k+1)$$

Solution:

Since a_o = 1, this becomes

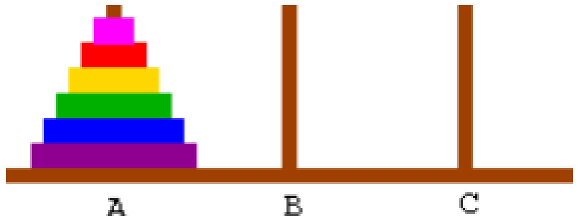
$$a_m = 1 + \{2 + 3 + 4 + \cdots + m + (m+1)\}$$

$$= \frac{1}{2}(m+1)(m+2) \quad \text{for } m \geq 0$$

$$\text{OR } a_{n-2} = \frac{1}{2}(n-1)n \quad \text{for } n \geq 2$$

This is the number of handshakes in the party when $n \geq 2$ persons are present."

PROBLEMS 7. There are 3 pegs fixed vertically on a table, and n circular disks having holes at their centers and having increasing diameters are slipped onto one of these pegs, with the largest disk at the bottom. The disks are to be transferred, one at a time, onto another peg with the condition that at no time a larger disk is put on a smaller disk. Determine the number of moves for the transfer of all the n disks, so that at the end the disks are in their original order.



Solution: Let a_n be the number of moves required to transfer n disks. Evidently, $a_0 = 0$. Let us denote the peg on which the disks are originally located as P_1. To effect the transfer, for $n \geq 1$, we fin transfer the top $n-1$ disks to a vacant peg, say P_2, in the prescribed manner. This involves a_{n-1} moves. Then we transfer the nth disk to the other vacant peg. say P_3. This involves 1 move. Lastly, we transfer the $n-$ 1 disks from peg P_2 to the peg P_3, in the prescribed manner. This involves a_{n-1} moves. Thus, the total number of moves involved in the transfer of n disks is

Solution:

$$a_n = a_{n-1} + 1 + a_{n-1} = 2a_{n-1} + 1 \qquad \text{for } n \geq 1.$$

or, equivalently

$$a_{n+1} = 2a_n + 1, \quad \text{for } n \geq 0$$

The general solution for this nonhomogeneous recurrence relation is

$$a_n = 2^n a_0 + \sum_{k=1}^{n} 2^{n-k} \cdot 1 = \sum_{k=1}^{n} 2^{n-k} \qquad \text{because } a_0 = 0.$$

$$= 2^{n-1} + 2^{n-2} + \cdots + 2^2 + 2 + 1$$

$$= \frac{2^n - 1}{2 - 1} = 2^n - 1$$

This is the required number of moves.

The Second-Order Linear Homogeneous Recurrence Relation with constant Coefficients

We now consider a method of solving recurrence relations of the form

$$C_n a_n + C_{n-1} a_{n-1} + C_{n-2} a_{n-2} = 0, \quad n \geq 2.\text{----------}(1)$$

where $Cn, Cn-1$ *and* $Cn-2$ are real constants with $Cn \neq 0$. A relation of this types is called ***second-order linear homogeneous recurrence relation with constant coefficients.***

We seek a solution of relation (1) in the form $a_n = ck^n$ where $c \neq 0$ and $k \neq 0$. Putting $a_n = ck^n$ in (1) we get ,

$$C_n ck^n + C_{n-1} ck^{n-1} + C_{n-2} ck^{n-2} = 0, \qquad n \geq 2.$$

$$C_n ck^2 + C_{n-1} k^1 + C_{n-2} = 0, \quad n \geq 2.\text{----------}(2)$$

Thus, $a_n = ck^n$ is a solution of (1) if k satisfies the quadratic equation (2). This quadratic equation is called the auxiliary equation or the characteristic equation for the relation (1).Now, the following three cases arise

Solution of The Second Order Linear Recurrence Relation

To find the solution, we follow the **Characteristic Roots Technique.**

Let given recurrence relation $a_n + \alpha a_{n-1} + \beta a_{n-2} = 0$, then the characteristic polynomial is $x^2 + \alpha x + \beta$, giving the characteristic equation $\boldsymbol{x^2 + \alpha x + \beta = 0}$

1. If x_1 & x_2 are two **distinct roots** of the characteristic polynomial (i.e., solution to the characteristic equation), then the solution to the recurrence relation is

$\boldsymbol{a_n = a x_1^n + b x_2^n}$ Where a & b are constant determined by the initial conditions

2. If x_1 & x_2 are **real and equal roots** of the characteristic polynomial (i.e., solution to the characteristic equation), then the solution to the recurrence relation is, $\boldsymbol{a_n = a x^n + n b x^n}$

3. If x_1 & x_2 are **Complex roots** [$x = p \pm iq$ of the characteristic polynomial (i.e., solution to the characteristic equation), then the solution to the recurrence relation is

$$\boldsymbol{a_n = r^n[a \cos n\theta + b \sin n\theta]}$$

Where $r = \sqrt{p^2 + q^2}$ & $\theta = \tan^{-1}\frac{q}{p}$

Where a & b are constant determined by the initial conditions

PROBLEMS 1. ***Solve the recurrence relation $a_n = 7a_{n-1} - 10a_{n-2}$ with $a_0 = 2$ and $a_1 = 3$.***

Solution: Rewrite the recurrence relation

$$a_n - 7a_{n-1} + 10a_n - 2 = 0$$

Now form the characteristic equation

$$x^2 - 7x + 10 = 0$$

And we solve for x i.e.,

$$(x-2)(x-5) = 0$$

So $x = 2$ & $x = 5$ are characteristic roots. Thus, the solution to recurrence relation will have the form

$$a_n = ax_1^n + bx_2^n$$

$$a_n = a2^n + b5^n$$

To find a & b, plug in $n = 0$ & $n = 1$ to get a system of two equation with two unknowns

$$a_0 = a2^0 + b5^0$$

$$2 = a + b ------(1)$$

$$a_1 = a2^1 + b5^1$$

$$3 = 2a + 5b -----(2)$$

Solving this system gives $a = \frac{7}{3}$ and $b = -\frac{1}{3}$

so the solution to the recurrence relation is

$$a_n = \frac{7}{3}2^n - \frac{1}{3}5^n$$

PROBLEMS ***2. Solve the recurrence relation $\boldsymbol{b_n = 2b_{n-1} - b_{n-2}}$ given $\boldsymbol{b_1 = 1.5}$ & $\boldsymbol{b_2 = 3}$***

Solution: Let us rewrite the given recurrence relation as

$$b_n - 2b_{n-1} + b_{n-2} = 0$$

Now we form the characteristic equation

$$x^2 - 2x + 1 = 0$$

And we solve for x i.e., $(x-1)^2 = 0$

So $x = 1,$ & $x = 1$ are characteristic roots. Thus, the solution to recurrence relation will have the form

$$a_n = ax^n + nbx^n$$
$$b_n = a1^n + nb1^n$$

To find a & b, plug in $n = 1$ & $n = 2$ to get a system of two equation with two unknowns

$$b_1 = a1^1 + 1b1^1$$
$$1.5 = a + b ----- -(1)$$
$$b_2 = a1^2 + 2b1^2$$
$$3 = a + 2b ----- -(2)$$

Solving this system gives $a = 0$ and $b = 1.5$ so the solution to the recurrence relation is

$$b_n = (0)1^n + n(1.5)1^n$$
$$b_n = 1.5n$$

3. Solve the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$ with the initial conditions $a_0 = 1$ and $a_1 = 4$

Solution

Let us rewrite the given recurrence relation as

$$a_n - 6a_{n-1} + 9a_{n-2} = 0$$

Now we form the characteristic equation

$$x^2 - 6x + 9 = 0$$

And we solve for x i.e.,

$$(x - 3)^2 = 0$$

So $x = 3,\ \&\ x = 3$ are characteristic roots. Thus, the solution to recurrence relation will have the form

$$a_n = ax^n + nbx^n$$
$$a_n = a3^n + nb3^n$$

To find a & b, plug in $n = 0$ & $n = 1$ to get a system of two equation with two unknowns

$$a_0 = a3^0 + (0)b3^0$$
$$1 = a$$
$$a_1 = a3^1 + 1b3^1$$
$$4 = 1(3) + 3b$$
$$b = \frac{1}{3}$$

solving this system gives $a = 1$ and $b = \frac{1}{3}$ so the solution to the recurrence relation is

$$a_n = 3^n + \frac{1}{3}n3^n$$

4. Solve the recurrence relation $\boldsymbol{a_n = 3a_{n-1} - 2a_n}$ $\forall \boldsymbol{n \geq 2}$ ***given*** $\boldsymbol{a_1 = 5}$ & $\boldsymbol{a_2 = 3}$ ***[homework]***

5. Solve the recurrence relation $\boldsymbol{D_n = bD_{n-1} - b^2 D_{n-2}}$ $\forall \boldsymbol{n \geq 3}$ ***given***

$\boldsymbol{D_1 = b > 0}$ & $\boldsymbol{D_2 = 0}$

Solution

Let us rewrite the given recurrence relation as

$$D_n - bD_{n-1} + b^2 D_{n-2} = 0$$

Now we form the characteristic equation

$$x^2 - bx + b^2 = 0$$

And we solve for x i.e.,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-b) \pm \sqrt{(-b)^2 - 4(1)(b^2)}}{2(1)}$$

$$x = \frac{b \pm \sqrt{b^2 - 4(b^2)}}{2(1)}$$

$$x = \frac{b \pm b\sqrt{1-4}}{2}$$

$$x = \frac{b \pm bi\sqrt{3}}{2}$$

$x = \frac{1}{2}b \pm i\frac{\sqrt{3}}{2}b$ Thus, the solution to recurrence relation will have the form

$$D_n = r^n[a\cos n\theta + b\sin n\theta]$$

To find a & b, put in $n = 1$ & $n = 2$ to get a system of two equation with two unknowns

$$r = \sqrt{p^2 + q^2} = \sqrt{\left(\frac{1}{2}b\right)^2 + \left(\frac{\sqrt{3}}{2}b\right)^2} = \sqrt{\frac{b^2}{4} + \frac{3b^2}{4}} = \sqrt{\frac{4}{4}b^2} = b$$

$$\theta = \tan^{-1}\frac{q}{p} = \tan^{-1}\frac{\frac{\sqrt{3}}{2}b}{\frac{1}{2}b} = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$$

$$D_n = r^n[A\cos n\theta + B\sin n\theta]$$

$$D_1 = r[A\cos\theta + B\sin\theta]$$

$$b = b\left[A\cos\frac{\pi}{3} + B\sin\frac{\pi}{3}\right]$$

$$1 = A\frac{1}{2} + B\frac{\sqrt{3}}{2}$$

$$2 = A + \sqrt{3}B -----(1)$$

$$D_n = r^n[A\cos n\theta + B\sin n\theta]$$

$$D_2 = r^2[A\cos 2\theta + B\sin 2\theta]$$

$$0 = b^2\left[A\cos 2\frac{\pi}{3} + B\sin 2\frac{\pi}{3}\right]$$

$$0 = A\left(-\frac{1}{2}\right) + B\left(\frac{\sqrt{3}}{2}\right)$$

$$0 = -A + B\sqrt{3} -----(2)$$

solving this system gives $A = 1$ and $B = \frac{1}{\sqrt{3}}$ so the solution to the recurrence relation is

$$D_n = b^n[\cos n\frac{\pi}{3} + \frac{1}{\sqrt{3}}\sin n\frac{\pi}{3}]$$

6. Solve the recurrence relation $F_{n+2} = F_{n+1} + F_n \; \forall n \geq 0$ given $F_0 = 0, F_1 = 1$

Solution

Let us rewrite the given recurrence relation as

$$F_{n+2} - F_{n+1} - F_n = 0$$
$$F_n - F_{n-1} - F_{n-2} = 0$$

Now we form the characteristic equation

$$x^2 - x - 1 = 0$$

And we solve for x i.e.,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2}$$
$$x = \frac{1 \pm \sqrt{5}}{2}$$

So $x = \frac{1+\sqrt{5}}{2}$, & $x = \frac{1-\sqrt{5}}{2}$ are characteristic roots. Thus, the solution to recurrence relation will have the form

$$F_n = a x_1^n + b x_2^n$$
$$F_n = a \left[\frac{1+\sqrt{5}}{2}\right]^n + b \left[\frac{1-\sqrt{5}}{2}\right]^n$$

To find a & b, put in $n = 0$ & $n = 1$ to get a system of two equation with two unknowns

$$F_0 = a\left[\frac{1+\sqrt{5}}{2}\right]^0 + b\left[\frac{1-\sqrt{5}}{2}\right]^0$$

$$0 = a + b ----- -(1)$$

$$F_1 = a\left[\frac{1+\sqrt{5}}{2}\right]^1 + b\left[\frac{1-\sqrt{5}}{2}\right]^1$$

$$1 = a\left[\frac{1+\sqrt{5}}{2}\right] + b\left[\frac{1-\sqrt{5}}{2}\right] ----- -(2)$$

solving this system gives $a = \frac{1}{\sqrt{5}}$ and $b = \frac{-1}{\sqrt{5}}$ so the solution to the recurrence relation is

$$F_n = \frac{1}{\sqrt{5}}\left[\frac{1+\sqrt{5}}{2}\right]^n + \frac{-1}{\sqrt{5}}\left[\frac{1-\sqrt{5}}{2}\right]^n$$

$$F_n = \frac{1}{\sqrt{5}}\left\{\left[\frac{1+\sqrt{5}}{2}\right]^n - \left[\frac{1-\sqrt{5}}{2}\right]^n\right\}$$

THANK YOU