# MODULE-4
# INTRODUCTION TO GRAPH THEORY-I

# Directed Graphs

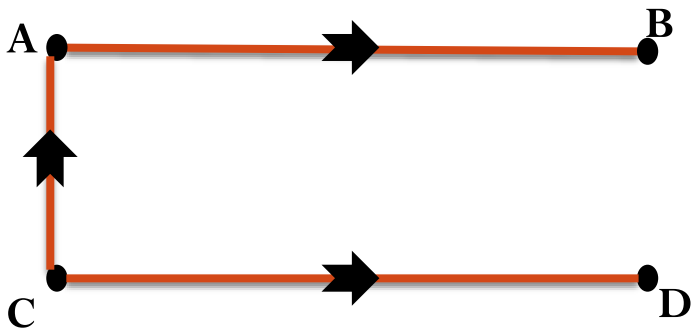


Fig 1.1

The diagram (Fig 1.1) consists of four vertices A, B, C, D and three edges AB, CD, CA *with directions attached to them*, the directions being indicated by arrows.

Because of attaching directions to the edges, the edge AB has to be interpreted as *an edge from the vertex A to B* and it cannot be written as BA.

Similarly, the edge CD is from C to D cannot be written as DC, and the edge CA from C to A cannot be written as AC. Thus, here, the edges ***AB, CD, CA*** are *directed edges*

# CONTD...

The directed edges AB , CD and CA may be represented by the *ordered pair (A, B), (C, D) and (C, A) respectively*.

The diagram in Fig 1.1 consists of a nonempty set of vertices, namely $\{A, B, C, D\}$, and set of directed pairs of vertices taken from this set, namely $\{(A, B), (C, D), (C, A)\}$. Such a diagram is called *diagram of a directed graph (or a diagraph).*

***Definition: A directed graph (or diagraph) is a pair (V, E), where V is a nonempty set and E is a set of ordered pairs of elements taken from the set V.***

# CONTD...

For directed graph (V, E), the ***elements of V are called vertices (points or nodes)*** and the ***elements od E are called directed edges***.

The set V is called the vertex set and the set E is called the directed edge set.

The directed graph (V, E) is also denoted as $D = (V, E)\ orD = D(V, E)$ or just D.

For diagram of Fig 1.1

*Vertex set* $V = \{A, B, C, D\}$ *Edge set* $E = \{AB, CD, CA\}$

# CONTD...

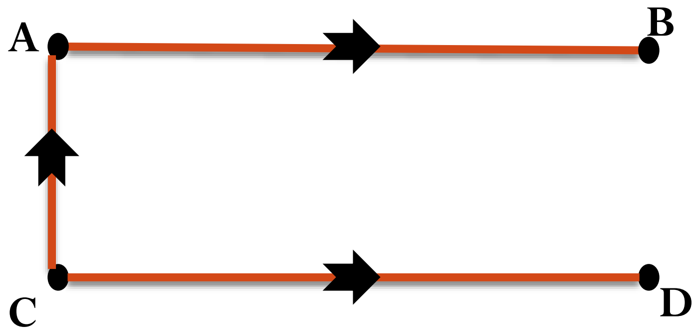


Fig 1.1

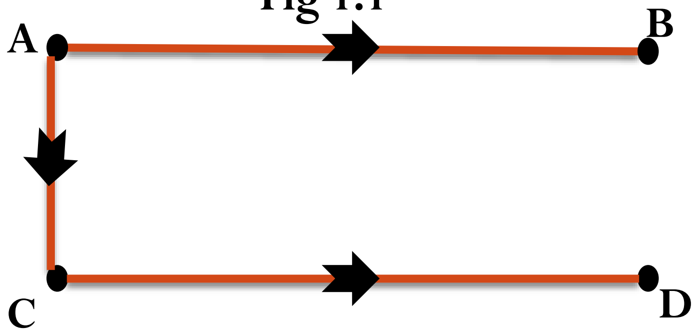


Fig 1.2


Fig 1.3

**Whether Fig 1.1 and Fig 1.2 are same?**

Fig 1.1 and Fig 1.2 are not same, although both of these two directed graph have the same vertex set, their directed edges sets are different.

**Whether Fig 1.3 is a directed graph?**

In the directed graph the directed edges need not to be straight line segments; they can be curved lines (arc) also.

# CONTD...

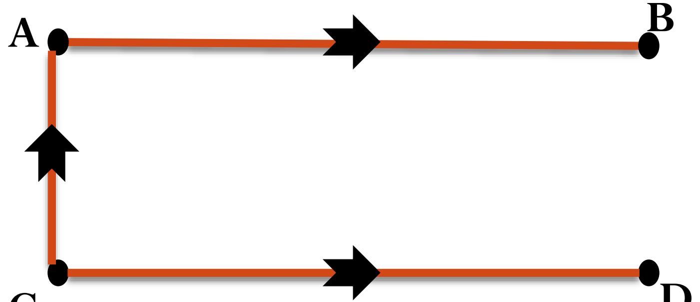


Fig 1.1

Every directed edge of a diagraph is determined by two vertices --- *a vertex from which it begins* and a *vertex at which it ends.*

*If AB is a directed edge of a diagraph D. A- Initial vertex and B- terminal vertex of AB. AB is incident out of A and incident into B.*

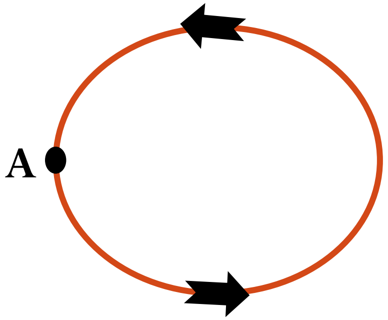


Fig 1.4

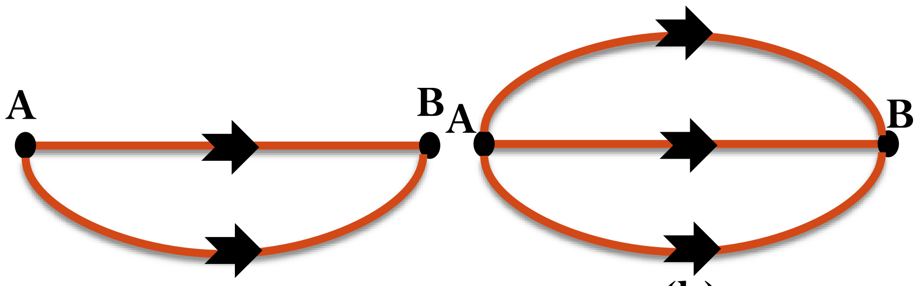


Fig 1.5

The directed edge shown in Fig 1.4 is ***directed loop*** which begins and ends at the vertex A.

Two directed edges having the same initial vertex and the same terminal vertex are called ***parallel directed edges fig 1.5 (a).***

Two or more directed edges having the same initial vertex and the same terminal vertex are called ***multiple directed edges fig 1.5 (a).***

# CONTD...

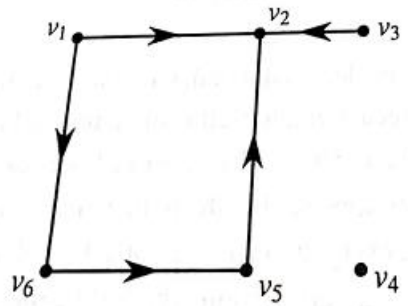


Figure 1.6

**Isolated Vertex: $(\boldsymbol{v_4})$:** A vertex of diagraph which is neither an initial vertex nor a terminal vertex of any directed edge.

**Non-isolated vertex** happens to be initial vertex or a terminal vertex for some directed edges.

A non-isolated vertex which is not a terminal vertex for any directed edge is called **source**.

A non-isolated vertex which is not a initial vertex for any directed edge is called **sink**.

$\boldsymbol{v_1}$ $\boldsymbol{and v_3}$ are sources. $\boldsymbol{v_2}$ $is\ a\ sink$.

# In degree and Out-degree

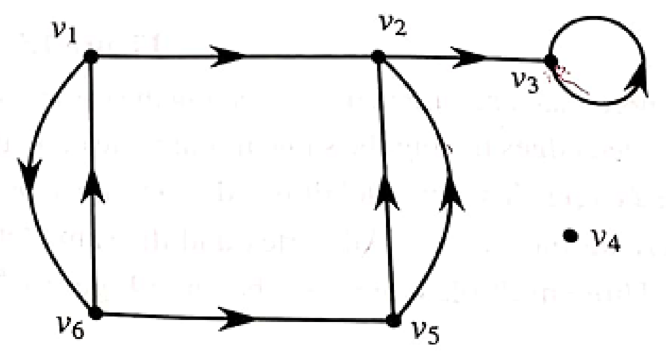



Figure 1.7

If $v$ is a vertex of a diagraph D, the *number of edges for which $v$ is the initial vertex* is called the **out-degree of $v$. $(d^+(v))$**

*The number of edges for which $v$ is the terminal vertex* is called the **in-degree of $v$. $(d^-(v))$**

$d^+(v_1) = 2$
$d^+(v_2) = 1$
$d^+(v_3) = 1$
$d^+(v_4) = 0$
$d^+(v_5) = 2$
$d^+(v_6) = 2$

$d^-(v_1) = 1$
$d^-(v_2) = 3$
$d^-(v_3) = 2$
$d^-(v_4) = 0$
$d^-(v_5) = 1$
$d^-(v_6) = 1$

The loop contributes a count 1 to each in-degree and out-degree.

***Property 1****: In every diagraph D, the sum of the out-degrees of all vertices is equal to the sum of the in-degrees of all vertices, each sum being equal to the number of edges in D.*

## Examples

1. **Find out the in degrees and out degrees of the vertices of the diagraph shown in fig 1.8.**

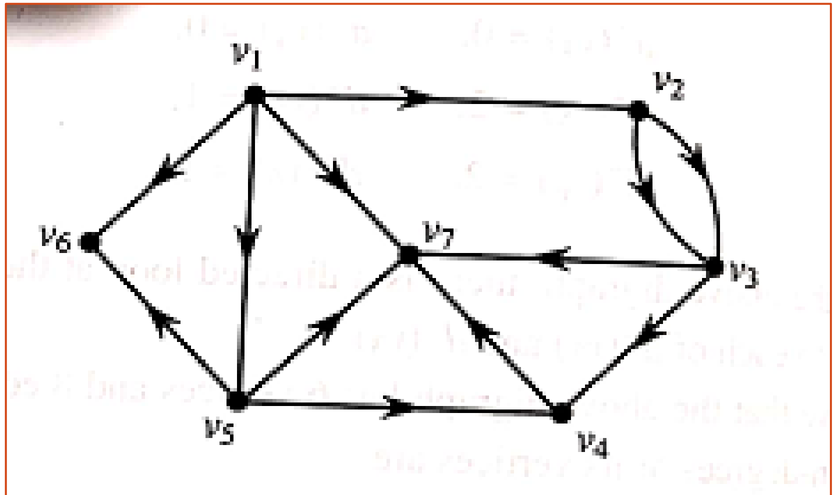



**Figure 1.8**

1. Write down the vertex set and the directed edge set of each of the following digraphs.

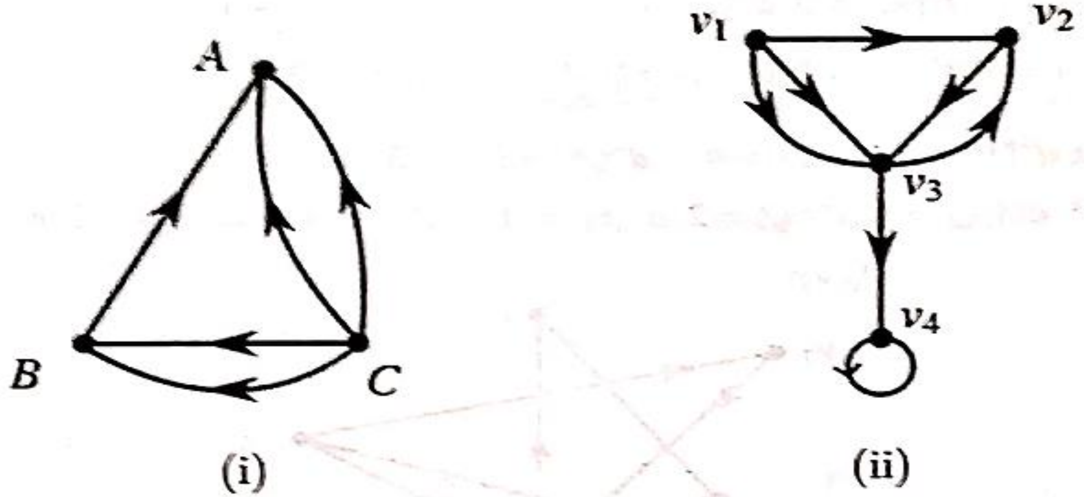



**Figure 1.9**

2. For the digraph shown in Figure 1.6, determine the out-degrees and in-degrees of all the vertices.

3. For the digraphs of Exercise 1 above, determine the out-degrees and in-degrees of all the vertices.

4. Let $D$ be the digraph whose vertex set is $V = \{v_1, v_2, v_3, v_4, v_5\}$ and the directed edge set is

$$E = \{(v_1, v_4), (v_2, v_3), (v_3, v_5), (v_4, v_2), (v_4, v_4), (v_4, v_5), (v_5, v_1)\}.$$

Write down a diagram of $D$ and indicate the out-degrees and in-degrees of all the vertices.

5. Verify the First theorem of Digraph theory for (i) the digraphs shown in Figures 1.6 and 1.9, and (ii) the digraphs shown below:

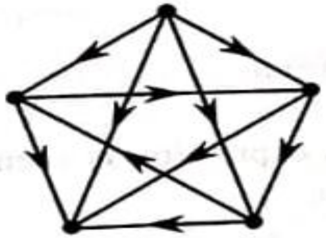


Figure 1.10

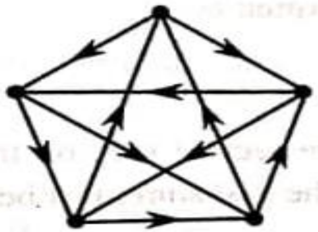


Figure 1.11

# Solutions of previous problems

**1.** (i) Vertex set is $V = \{A, B, C\}$ and the directed edge set is $E = \{(B, A), (C, A), (C, A), (C, B), (C, B)\}$.

(ii) Vertex set is $V = \{v_1, v_2, v_3, v_4\}$ and the directed edge set is $E = \{(v_1, v_2), (v_1, v_3), (v_1, v_3), (v_2, v_3), (v_3, v_2), (v_3, v_4), (v_4, v_4)\}$.

**2.** $d^-(v_1) = 0, d^-(v_2) = 3, d^-(v_3) = 0, d^-(v_4) = 0, d^-(v_5) = 1, d^-(v_6) = 1.$

$d^+(v_1) = 2, d^+(v_2) = 0, d^+(v_3) = 1, d^+(v_4) = 0, d^+(v_5) = 1, d^+(v_6) = 1.$

**3.** (i) $d^+(A) = 0, d^+(B) = 1, d^+(C) = 4, \quad d^-(A) = 3, d^-(B) = 2, d^-(C) = 0.$

(ii) $d^+(v_1) = 3, d^+(v_2) = 1, d^+(v_3) = 2, d^+(v_4) = 1, \quad d^-(v_1) = 0, d^-(v_2) = 2, d^-(v_3) = 3, d^-(v_4) = 2.$

**4.**

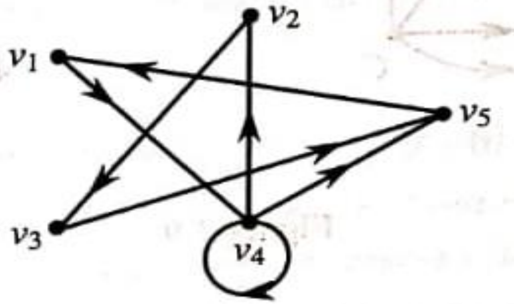



**Figure 1.12**

| Vertices | $v_1$ | $v_2$ | $v_3$ | $v_4$ | $v_5$ |
|---|---|---|---|---|---|
| $d^+$ | 1 | 1 | 1 | 3 | 1 |
| $d^-$ | 1 | 1 | 1 | 2 | 2 |

# Graphs

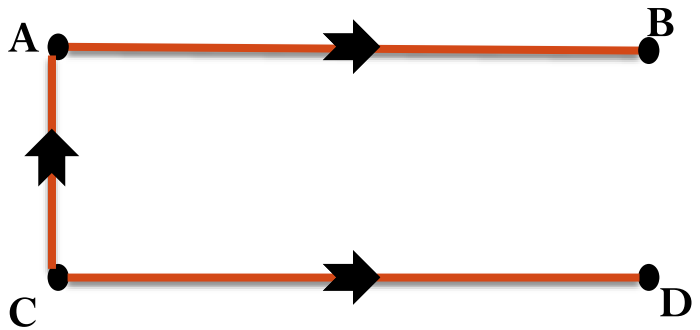



Fig 1.1


Fig 2.1

*Fig 1.1 is a directed graph* and *Fig 2.1 is a undirected Graph (Graph).*

The edge AB is determined by the vertices A and B and represented by unordered pair $\{AB\} = \{BA\}$.

***Definition:*** ***A graph is pair*** $(\boldsymbol{V}, \boldsymbol{E})$***, where V is a nonempty set and E is a nonempty set of unordered pairs of elements taken from the set V.***

***The graph is denoted by*** $\boldsymbol{G = (V, E), G = G(V, E), G.}$

# Graph

According to the definition of a graph/diagraph, the vertex set in a graph/diagraph has to be nonempty set but the *edge set can be empty.*

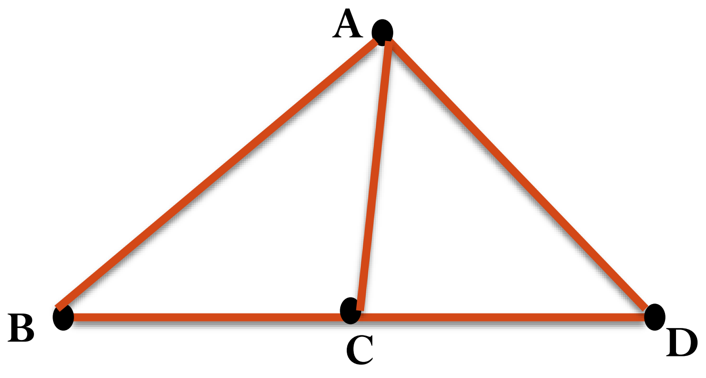

(a) (b)

**Fig 2.2**

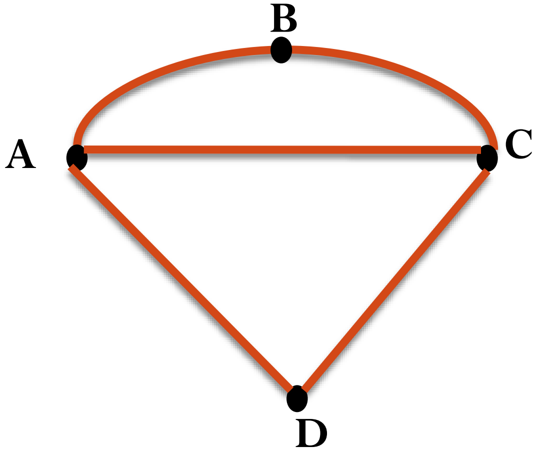
*A graph containing no edges is called a* ***null graph (Fig 2.2 (a)).*** *A null graph with only one vertex is called a* ***trivial graph (Fig 2.2 (b)).***

# Graphs

**Draw the graphs with four vertices A, B, C, D with AB, AC, AD, BC and CD.**




(a)




(b)

**Fig 2.3**

The way one draws a diagram of a graph is basically immaterial. There can be more than one diagram for the same graph. Yet they represent the same graph since each convey the same information.

A graph with a finite number of vertices and finite number of edges is called **finite graph** otherwise is called **infinite graph.**

# Order and size of graph

The *number of vertices* in a graph is called **order of the graph** and the *number of edges* in the graph are called its **size.**

**Cardinality of V, $|V|$ is the order.**

**Cardinality of E, $|E|$ is the size.**

A graph of order $n$ and size $m$ is called a $\boldsymbol{(n, m)}$ **graph**.

A null graph with $n$ vertices is a $(n, 0)$ graph.

# End vertices, loop, multiple edges

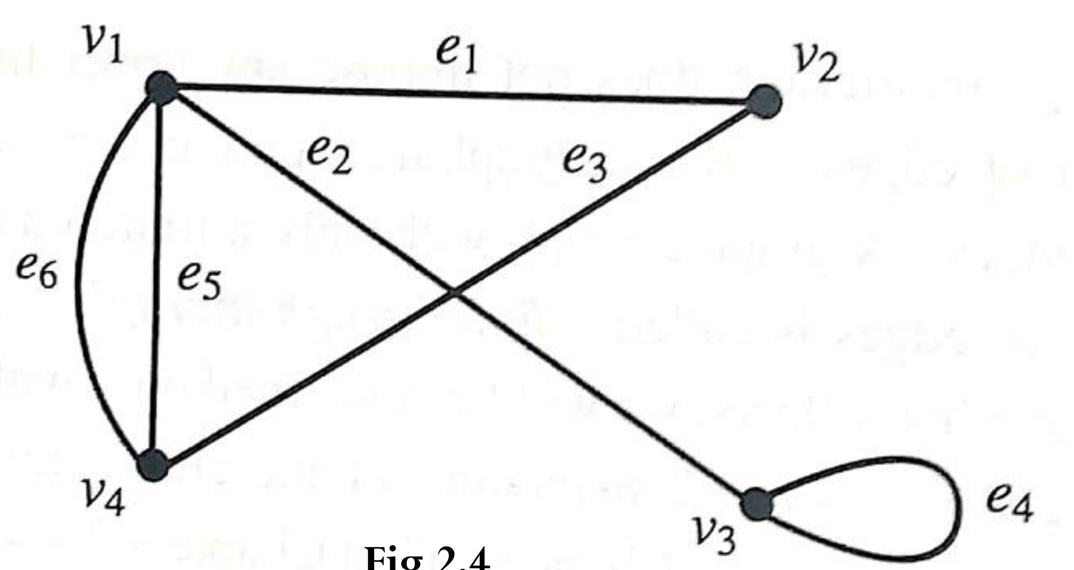



**Fig 2.4**

The fig 2.4 has 4 vertices $v_1, v_2, v_3, v_4$ and six edges $e_1, e_2, e_3, e_4, e_5, e_6$.

$v_1, and\ v_2$ are **end vertices** of the edge $e_1$. Written as $e_1 = \{v_1, v_2\}$, Similarly other end vertices can be defined.

An edge such as $e_4$ is called **loop,** $e_4 = \{v_3, v_3\}$.

The both edges $e_5, e_6$ have same end vertices $v_1, v_4$ such edges are called **parallel edges.** $e_5 = \{v_1, v_4\}, e_6 = \{v_1, v_4\}$.

**If in a graph there are two or more edges with the same end vertices, the edges are called multiple edges.**

# Simple graph, Multiple graph, General graph

**Simple Graph:** A graph does not contain *loops and multiple edges* is called a **simple graph.** A graph which does not contains a loops is called a **loop-free** graph.

**Multigraph:** A graph which *contains multiple edges but no loops* is called **multigraphs**.

**General Graph:** A graph which *contains multiple edges or loops (or both)* is called **general graphs**.

Fig 2.1 is a simple graph, Fig 2.4 is a general graph, Fig 2.5 is Multigraph graph


Fig 2.1

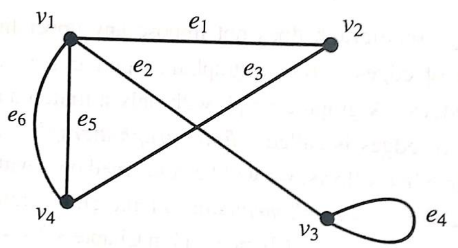



Fig 2.4

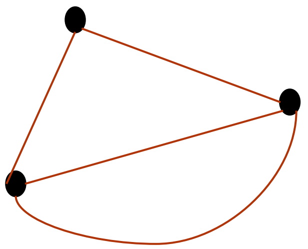


Fig 2.5

# Adjacent edges and vertices

The two *non parallel edges* are said to be ***adjacent edges*** if they are *incident on a common vertex.*

*Two vertices are said to be **adjacent vertices** if there is an edge joining them.*

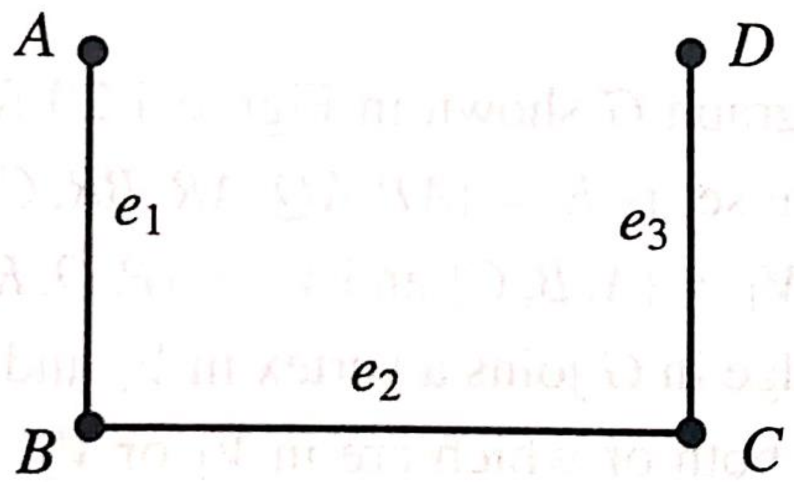



**Fig 2.6**

In Fig 2.6, *A and B are adjacent vertices* and $e_1$ *and* $e_2$ *are adjacent edges.*

*A and C are not adjacent vertices* and $e_1$ *and* $e_3$ *are not adjacent edges.*

# Complete Graph

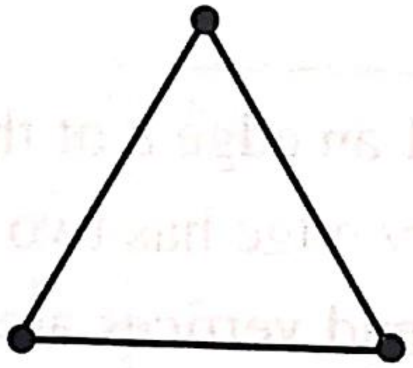
**Def:** A simple graph of *order* $\geq 2$ in which there is an edge between every pair of vertices is called a **complete graph**. **OR**

A *complete graph* is a *simple graph* in which every pair of distinct vertices are adjacent.

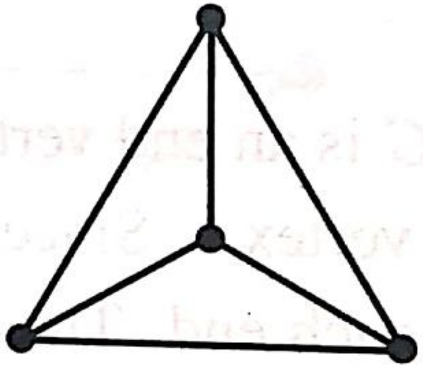
A complete graph with $\boldsymbol{n} \geq \mathbf{2}$ vertices is denoted by $\boldsymbol{k_n}$.

Following are the complete graph with 2, 3,, 4 and 5 vertices are shown below

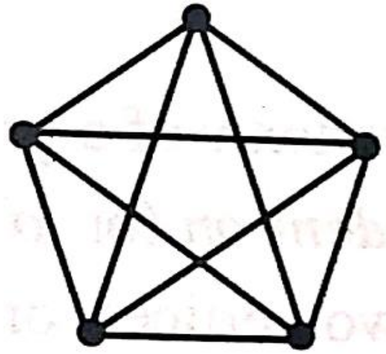
(a) $K_2$



(b) $K_3$



(c) $K_4$



(d) $K_5$

$K_5$ is called the Kuratowski's first graph

**Fig 2.7**

# Bipartite graph

**Def:** Suppose a simple graph G is such that its vertex set V is the union of two mutually disjoint nonempty sets $V_1 and\ V_2$ which are such that each edges in G joins a vertex in $V_1$ and a vertex $V_2$. Then G is called a **bipartite graph.**

If E is the edge set of this graph, the graph is denoted by $G = (V_1, V_2; E)$, or $G = G(V_1, V_2; E)$.

The sets $V_1 and\ V_2$ are called **bipartites** of the vertex set V.

$\boldsymbol{V = \{A, B, C, P, Q, R, S\}}$, $\boldsymbol{V_1 = \{A, B, C\}, V_2 = \{P, Q, R, S\}}$

$\boldsymbol{E = \{AP, AQ, AR, BR, CQ, CS\}}$.

*(i)* $V_1 and\ V_2$ **are disjoint.**

**(ii)** **Every edge in G is a join of a vertex in $V_1$ and a vertex in $V_2$.**

**(iii)** **G contains no edge that joins two vertices both of which are in $V_1 and\ V_2$ .**

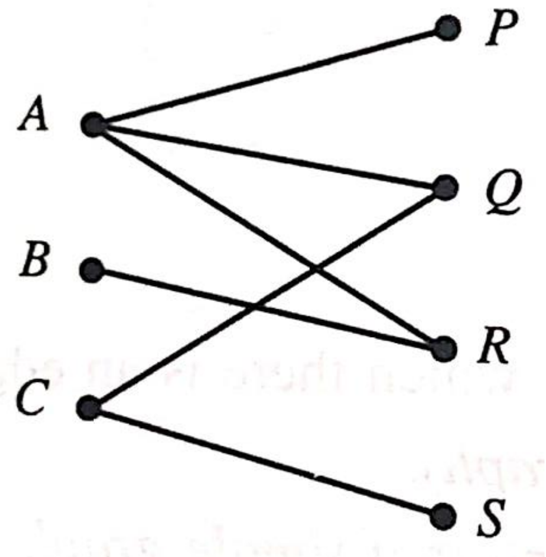



**Fig 2.8**

# Complete Bipartite graph

**Def:** A bipartite graph $G = (V_1, V_2; E)$ is called a *complete bipartite graph* if there is an edge between every vertex in $V_1$ and every vertex in $V_2$.

A complete bipartite graph $G = (V_1, V_2; E)$ in which the bipartites $V_1 \, and \, V_2$ contain $r$ and $s$ vertices respectively, with $r \leq s$, is denoted by $K_{r,s}$.

$K_{r,s}$ has $r + s$ vertices and $rs$ edges; that is $K_{r,s}$ is of order $r + s$ and size $rs$; it is therefore a $(r + s, rs)$ graph.

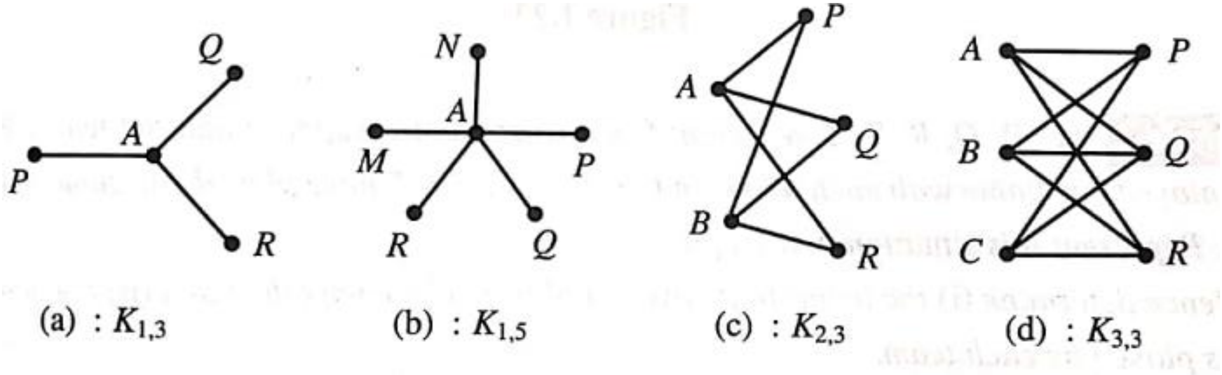



**Fig 2.9**

**$K_{3,3}$ is called the Kuratowski's second graph**

# Example

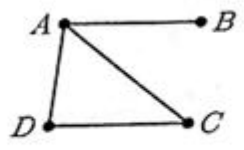
1. **Draw a diagram of the graph $G = G(V, E)$ in each of the following cases.**

*(i)* $V = \{A, B, C, D\}, E = \{(A, B), (A, C), (A, D), (C, D)\}$

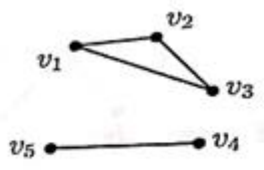
*(ii)* $V = \{v_1, v_2, v_3, v_4, v_5\}, E = \{(v_1, v_2), (v_1, v_3), (v_2, v_3), (v_4, v_5)\}$

*(iii)* $V = \{P, Q, R, S, T\}, E = \{(P, S), (Q, R), (Q, S)\}$

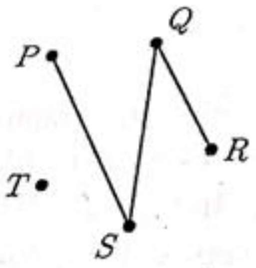
*(iv)* $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}, E = \{(v_1, v_4), (v_1, v_6), (v_3, v_2), (v_4, v_6), (v_3, v_5), (v_2, v_5)\}$



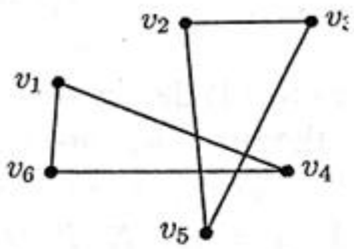

(i)




(ii)




(iii)



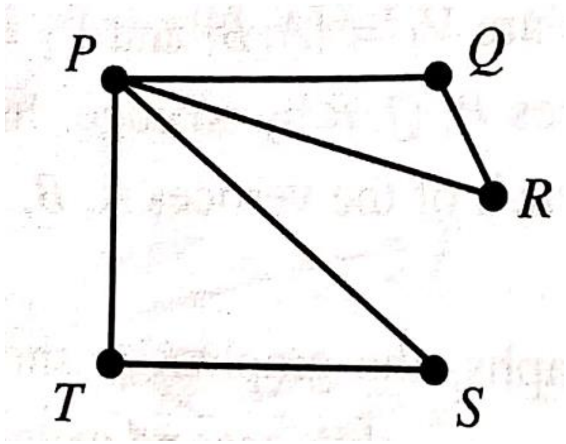

(iv)

# Example

**2. Let P, Q, R, S, T represent five cricket teams. Suppose that the teams P, Q, R have played one game with each other, and the teams P, S, T have played one game with each other. Represent this in a graph.**

**Hence determine (i) the teams that have not played with each other (ii) the number of games played by each team.**

Solution: Let the team represents vertices and edge represents the playing. Then the graph represents the given situation is as shown below.




There is no edge between Q and S, Q and T, R and S, R and T.

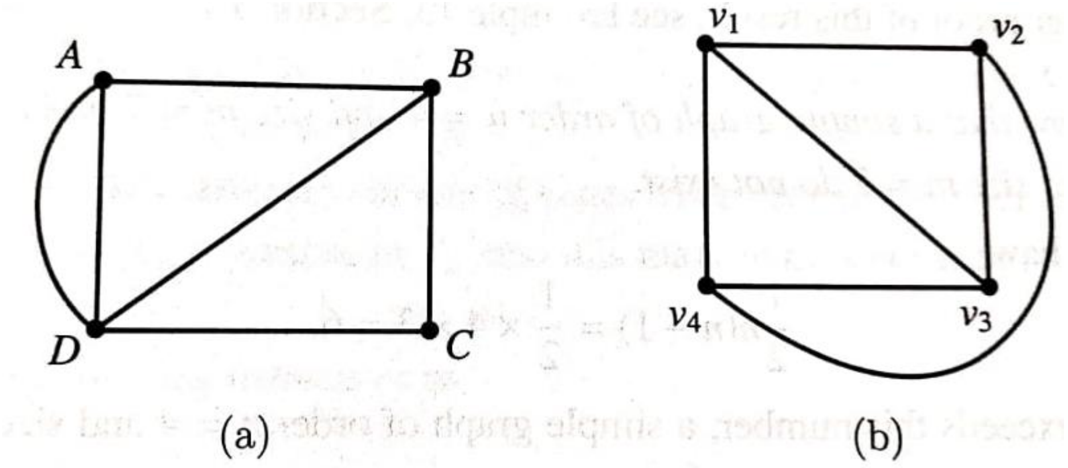
Therefore the teams Q and S, Q and T, R and S, R and T have not played together.

The degree of Q, R, S, T is 2 and degree of P is 4.

Therefore Q, R, S, T played two games each and P played 4 games

## Example

**3. Which of the following is a complete graph?**




Solution: (a) is not complete graph- it is not a simple graph, no edge between A and C

(b) Is complete graph- there is an edge between every pair of vertices

**4. If $\boldsymbol{G = G(V, E)}$ is a simple graph, prove that $\boldsymbol{2|E| = |V|^2 - |V|}$.**

**Solution:**

- Each edge of a graph is determined by pair of vertices.
- In simple graph there are no multiple edges
- The number of edges cannot exceed the number of vertices.
- The number of pairs of vertices that can be choosen from $n$ vertices is $^{n}C_2 = \frac{1}{2}n(n-1)$
- Thus, for a simple graph with $n$ $(\geq 2)$ vertices, the number of edges cannot exceed $\frac{1}{2}n(n-1)$.

Accordingly if the simple graph G has $n$ vertices and $m$ edges, then $m \leq 1/2n(n-1)$

Therefore $2m \leq n^2 - n$ , that is $\boldsymbol{2|E| = |V|^2 - |V|}$.

**5. Show that a complete graph with $n$ vertices, namely $K_n$, has $\frac{1}{2}n(n-1)$ edges.**

**Solution:**

In a complete graph, there exist one edge between every pair of vertices.

Therefore

Number of of edges = Number of pair of vertices

If $n$ is number of vertices, then number of pair of vertices is $\frac{1}{2}n(n-1)$

**Thus the number of edges is a complete graph is with $n$ vertices is $\frac{1}{2}n(n-1)$**

6. show that a simple graph of order $n = 4$ and size $m = 7$ and a complete graph of order $n = 4$ and size $m = 5$ do not exist.

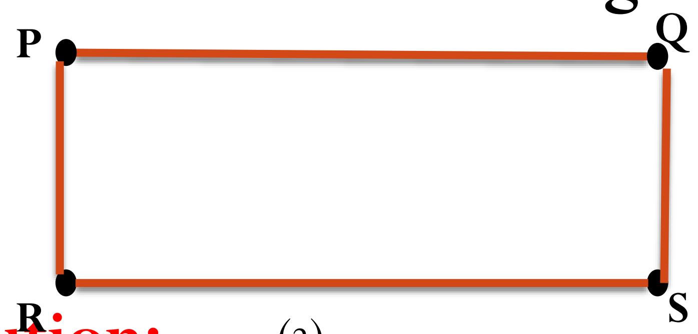
**Solution:**

For $n = 4$, m $= \frac{1}{2}n(n-1) = 6$ .

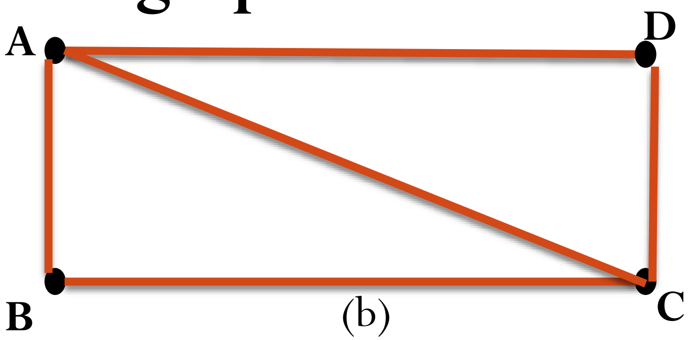
Since $m = 7$ exceeds this number, a simple graph of order 4 and size 7 do not exist.

Similarly, since $m = 5$ is not equal to $\frac{1}{2}n(n-1) = 6$, a complete graph od order 4 and size m=5 does not exist.

7. Which of the following is a bipartite graph?



(a)



(b)

**Solution:**

(a) Is a bipartite graph $V = \{P, Q, R, S\}, V_1 = \{P, S\}, V_2 = \{Q, R\}$

8. How many vertices and how many edges are there in the complete bipartite graph $K_{4,7}$ and $K_{7,11}$.?

**Solution:**

11 vertices and 28 edges

18 vertices and 77 edges.

**9. Show that a simple graph of order $n = 4$ and size $m = 5$ cannot be a bipartite graph.**

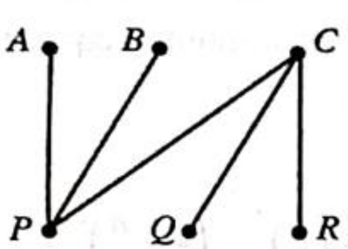
**Solution:**

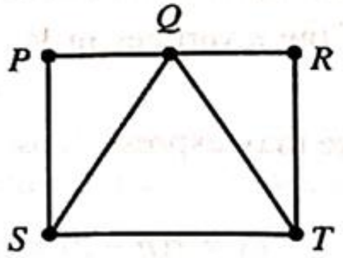
A simple graph to be bipartite graph $4m \leq n^2$

$4m = 20$ and $n^2 = 16$.

So that $4m > n^2$, therefore the given simple graph cannot be bipartite graph.

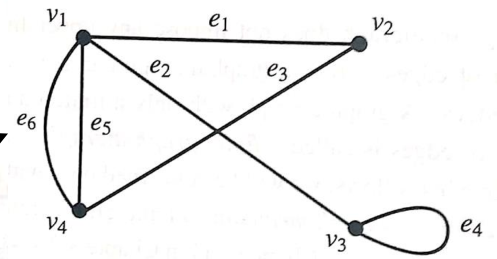
1. Indicate the order and size of each of the graphs shown below.



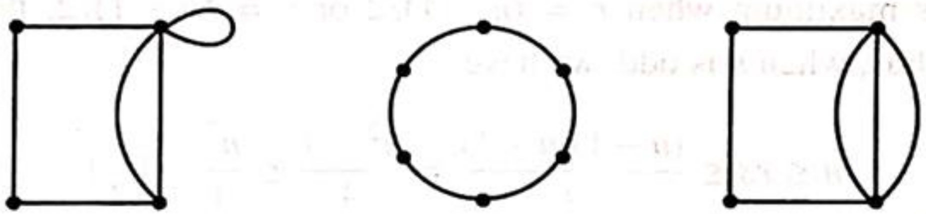




2. Identify the adjacent vertices in the graphs of the preceding exercise.

3. Identify the adjacent vertices and adjacent edges in the graph shown in Figure

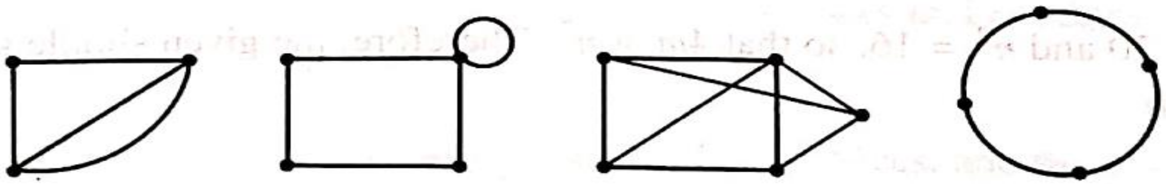



4. Which of the following graphs is a simple graph? a multigraph? a general graph?



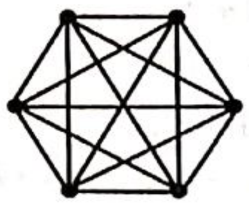
(i) (ii) (iii)

5. Which of the following are complete graphs?

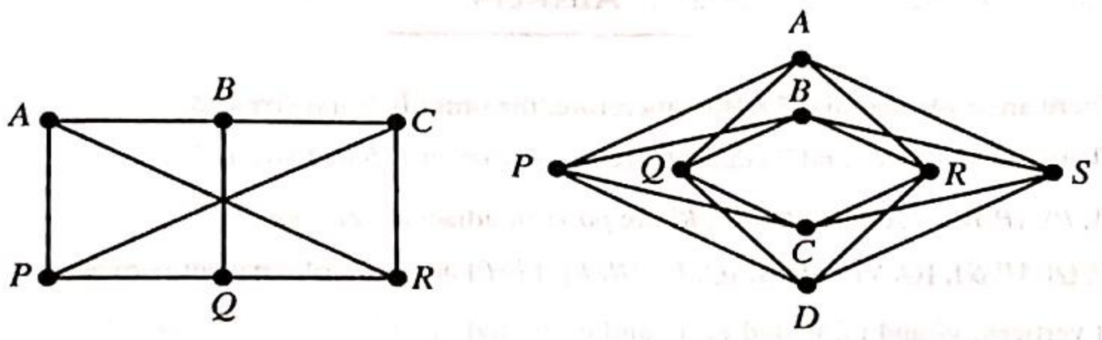


(i) (ii) (iii) (iv)

6. Identify the graph shown below:

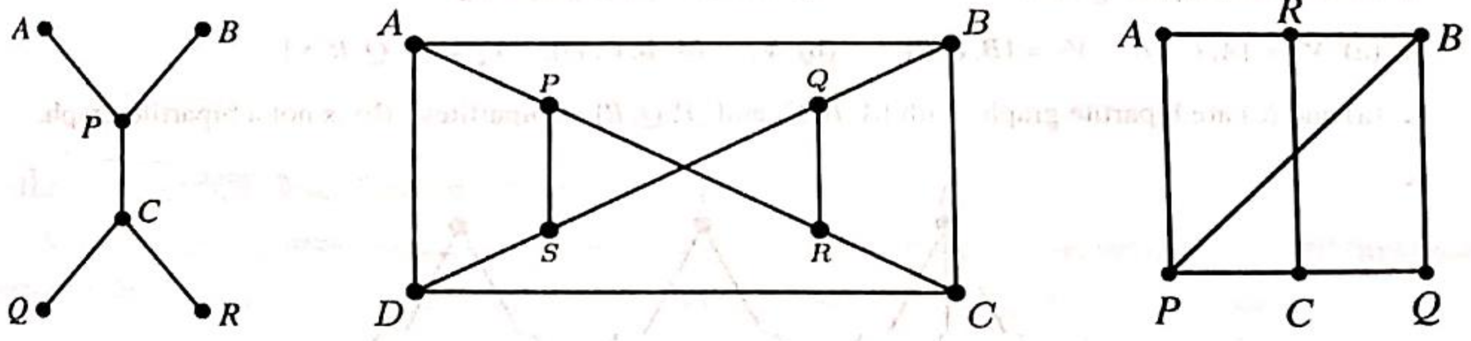


7. Verify that the following are bipartite graphs. What are their bipartites?




(a) (b)

8. Which of the graphs shown below are bipartite graphs?




(a) (b) (c)

9. Company $X$ has offices in cities $B$, $D$ and $K$; company $Y$ in cities $B$ and $M$; company $Z$ in cities $C$ and $M$. Represent this situation by a bipartite graph. Is this a complete bipartite graph?

10. State whether the following graphs can exist or cannot exist.

(1) Simple graph of order 3 and size 2.

(2) Simple graph of order 5 and size 12.

(3) Complete graph of order 5 and size 10.

(4) Bipartite graph of order 4 and size 3.

(5) Bipartite graph of order 3 and size 4.

(6) Complete bipartite graph of order 4 and size 4.

**1.** (i) There are 6 vertices and 5 edges; therefore, the order is 6 and size is 5.

(ii) There are 5 vertices and 7 edges; therefore, the order is 5 and size is 7.

**2.** (i) $\{A, P\}, \{P, B\}, \{P, C\}, \{C, Q\}, \{C, R\}$ are *pairs* of adjacent vertices.

(ii) $\{P, Q\}, \{P, S\}, \{Q, S\}, \{Q, R\}, \{Q, T\}, \{R, T\}, \{S, T\}$ are *pairs* of adjacent vertices.

**3.** Adjacent vertices: $v_1$ and $v_2$, $v_1$ and $v_3$, $v_1$ and $v_4$, $v_2$ and $v_4$.

Adjacent edges: $e_1$ and $e_2$, $e_1$ and $e_3$, $e_1$ and $e_5$, $e_1$ and $e_6$,

$e_2$ and $e_4$, $e_2$ and $e_5$, $e_2$ and $e_6$, $e_3$ and $e_5$,

$e_3$ and $e_6$.

**4.** (i) general graph (ii) simple graph, (iii) multigraph.

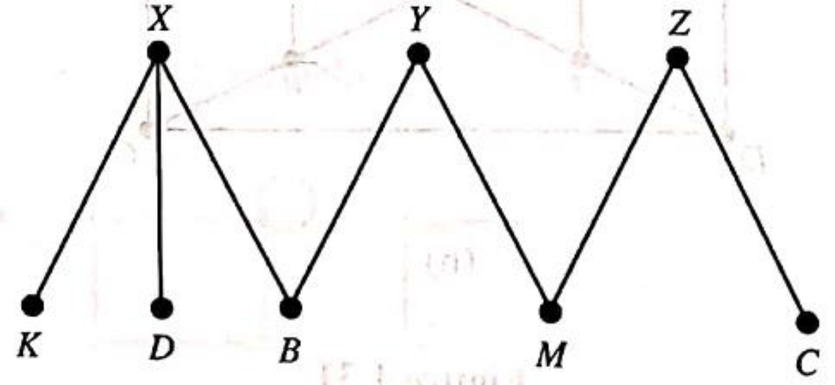
**5.** None is a complete graph.

**6.** The complete graph $K_6$.

**7.** (a) $V_1 = \{A, C, Q\}$, $V_2 = \{B, P, R\}$ (b) $V_1 = \{A, B, C, D\}$, $V_2 = \{P, Q, R, S\}$

**8.** (a) and (c) are bipartite graphs with $\{A, B, C\}$ and $\{P, Q, R\}$ as bipartites. (b) is not a bipartite graph

**9.**



# Vertex Degree and Handshaking Property

Let $G = (V, E)$ be a graph and $v$ be a vertex of G. Then, the number of edges of G that are incident on $v$ (that is, the number of edges that join $v$ to other vertices of G) with the *loops counted twice* is called the **degree** of the vertex $v$ and is denoted by $\deg(v)$, or $d(v)$.

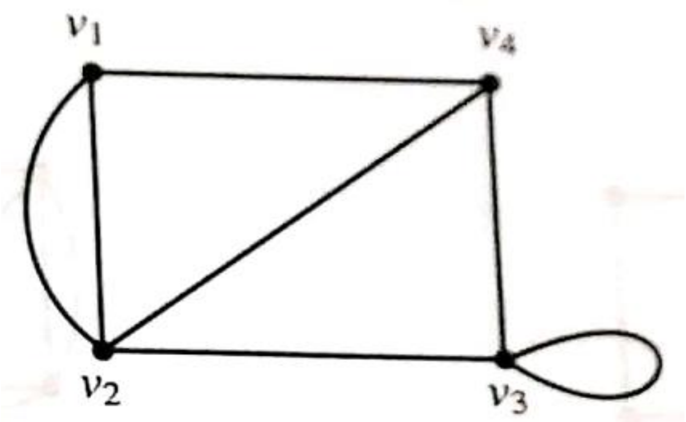


Fig 4.1

$$d(v_1) = 3, d(v_2) = 4, d(v_3) = 4, d(v_3)$$

Degree sequence of a graph is 3, 3, 4, 4. (non-decreasing order).

# Isolated vertex, pendant Vertex

A vertex is a graph which is not an end vertex of any edge of the graph is called an ***isolated vertex***.

A vertex of degree 1 is called a *pendent vertex.* An edge incident on a pendant vertex is called a *pendant edge.*

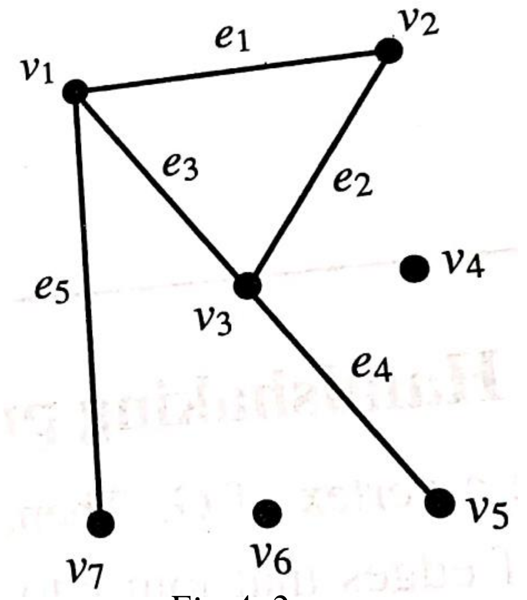


Fig 4.2

*In Fig 4.2.* $v_4, v_6$ *are isolated vertex.* $v_5, v_7$ *are pendent vertices. And edge* $e_4, e_5$ *are pendant edges.*

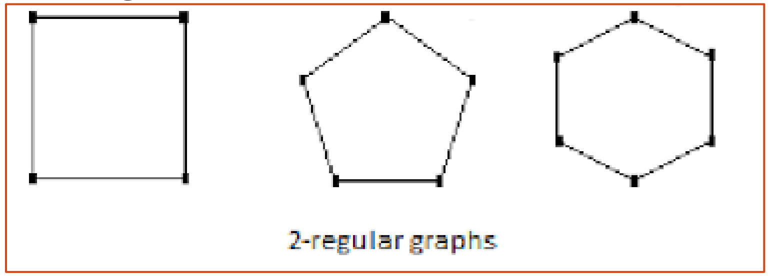
*In Null graph every vertex is a isolated vertex .*

# Regular Graph

A graph in which all the vertices are of the same *degree* $k$ is called a *regular graph of degree* $k$. *Or* $k - regular\ graph$.

*Draw 2-regular, and 4-regular graph.*

*2-regular*




*3-regular*

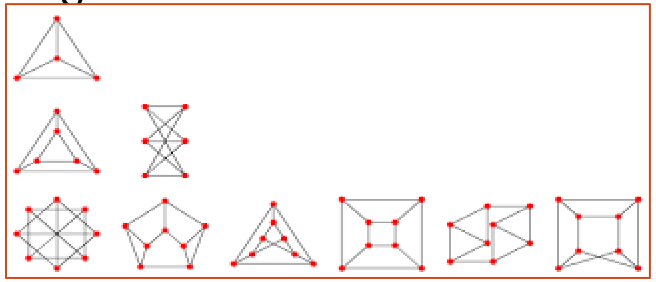
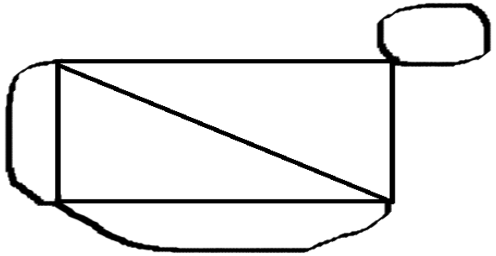


Fig 4.3

*4-regular*



This 3-regular graph is called **Petersen graph. Which has 10 vertices and 15 edges.**

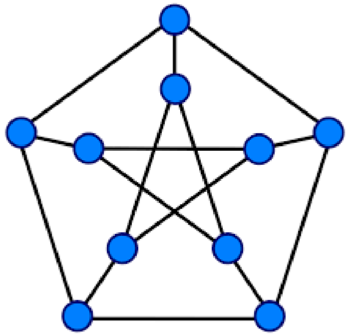


Fig 4.4

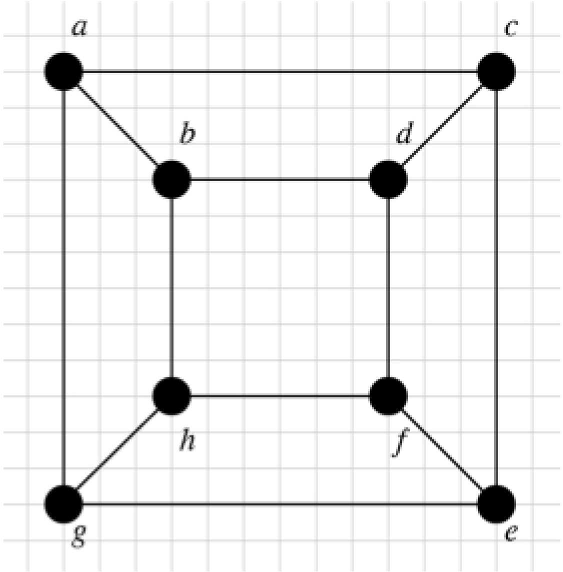



Fig 4.4

The graph in Fig 4.4 is a cubic graph with $8 = 2^3$ vertices. This particular graph is called the *three-dimensional hypercube and is denoted by* $Q_3$.

*In general, for any positive integer k, a loop-free k-regular graph with* $2^k$ *vertices is called the k-dimensional hypercube (or k-cube) and is denoted by* $Q_k$.

***Handshaking property:** The sum of the degrees of all the vertices in a graph is an even number; and this number is equal to twice the number of edges in the graph. (First theorem of graph theory)*

***For a Graph G=G(V, E).***

$$\sum_{v\in V} \mathbf{deg}(v) = 2|E|$$

***If several people shake hands, the total number of hands shake must be even.***

**Theorem: Prove that the sum of the degrees of all the vertices in a graph is an even number and this number is equal to twice the number of edges in the graph.**

$$\sum_{v \in V} deg(v) = 2|E|$$

**Proof:** Since the degree of a vertex is the number of edges incident with that vertex,

the sum of degree counts the total number of times an edge is incident with a vertex.

Since every edge is incident with exactly two vertices, each edge counted twice, once at each end.

Therefore, the sum of the degrees is equal twice the number of edges.

$$\sum_{v \in V} deg(v) = 2|E|$$

**Note: this proof applies if multiple edges and loops are present.**

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# Theorem: In every graph, the number of vertices of odd degrees is even.

**Proof:** If a graph has a $n$ vertices we know by hand shaking property that

$$\sum_{i=1}^{n} \deg(v_i) = 2e \text{ where } e \text{ is the number of edges.}$$

Now let $\sum_{i=1}^{n} \deg(v_i) = \sum_{i=1}^{k} \deg(v_i) + \sum_{i=k+1}^{n} \deg(v_i) ---1$

$[i = 1,2 \ldots k \text{ are vertices with even degree and}, k, k+1, \ldots n \text{ are vertices with odd degree}]$

The sum on the left hand side of (1) is twice the number of edges. As such, this sum is even.

$$2e = \sum_{i=1}^{k} \deg(v_i) + \sum_{i=k+1}^{n} \deg(v_i)$$

The first sum on the right hand side is the sum of the degrees of vertices with even degree. As such, this sum is also even.

Therefore, the second sum in the right hand side must also be even, that is

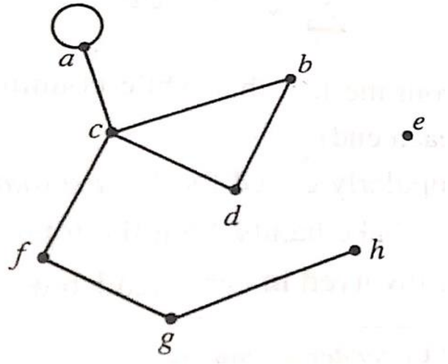
$$\sum_{i=k+1}^{n} \boldsymbol{deg(v_i)} = \textbf{even}.$$

12 = 8 + ?

1+3=4,
1+3+5=9

**But each $\boldsymbol{deg(v_{k+1})}.\boldsymbol{deg(v_{k+2})}$ …….n is odd. Therefore number of terms must be even.**

1. For a graph shown below indicate the degree of each vertex and verify handshaking property.




Solution: $\deg(a) = 3, \deg(b) = 2, \deg(c) = 4, \deg(d) = 2$

$\deg(e) = 0, \deg(f) = 2, \deg(g) = 2, \deg(h) = 1$

Note that, $e$ is a isolated vertex and $h$ is a pendent vertex.

Sum of the degree of vertices =16

Number of edges = 8

Thus, $16 = 2 \times 8$

**2. For a graph $G = G(V, E)$, what is the largest possible value for $|V|$ if $|E| = 19$ and $\deg(v) \geq 4$ for all $v \in V$?**

**Solution**: We know by hand shaking property

$$\sum_{i=1}^{n} \deg(v_i) = 2e$$

Given that, In this graph $e = |E| = 19$. and degree of all vertices are greater than or equal to 4.

Let $4 + 4 + \cdots \ldots . \leq 2e$

$$4|V| \leq 2 \times 19$$

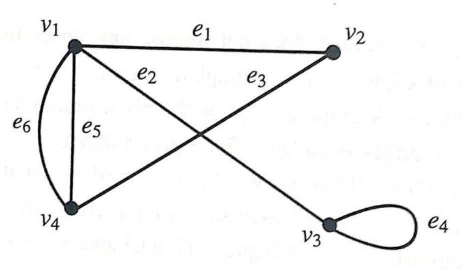
$$|V| \leq 38/4$$

$$|V| \leq 9.5$$

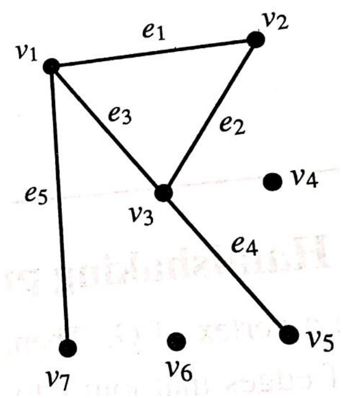
Thus, the largest possible value of $|V|$ is 9

**Problems:**

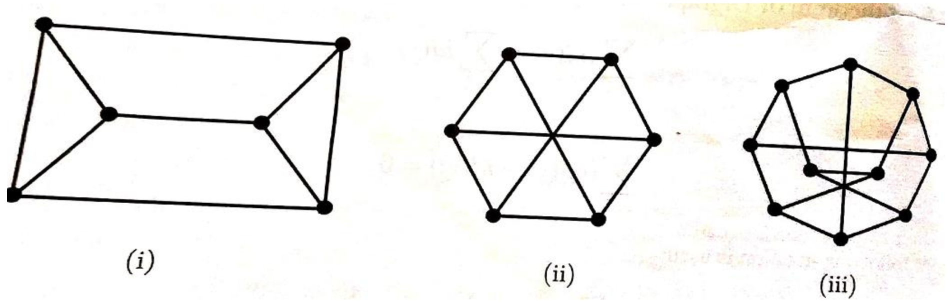
1. **Find the degree of all the vertices of the graph shown below, verify handshaking property.**




2. **Verify the handshaking property for the graph shown below**




3. **Are the following graphs regular?**




4. **Draw the diagram of a graph where the degrees of the vertices are 1, 1, 1, 2, 3, 5, 5, 7**
5. **For a graph $G = G(V, E)$, what is the largest possible value for $|V|$ if $|E| = 35$ and $\deg(v) \geq 3$ for all $v \in V$?**

# Isomorphism

Consider a graph $G = (V, E)$ and $G' = (V', E')$. Suppose there exists a function $f: V \to V'$ such that

*(i)* $f$ is one-to-one correspondence

(ii) For all vertices A, B of G, $\{A, B\}$ is an edge of G if and only if $\{f(A), f(B)\}$ is an edge of $G'$.

*Then $f$ is called an isomorphism between $G$ and $G'$*. And we say that $G$ and $G'$ *are isomorphic graphs*.

In other words, two graphs $G$ and $G'$ are said to be isomorphic (to each other) if there is one-to-one correspondence between their vertices and between their edges such that adjacency of vertices is preserved.

Such graph will have same structure; they differ only in the way their vertices and edges are labelled or only in the way they represented geometrically.

**When $G$ and $G'$ *are isomorphic, we write* $G \cong G'$.**

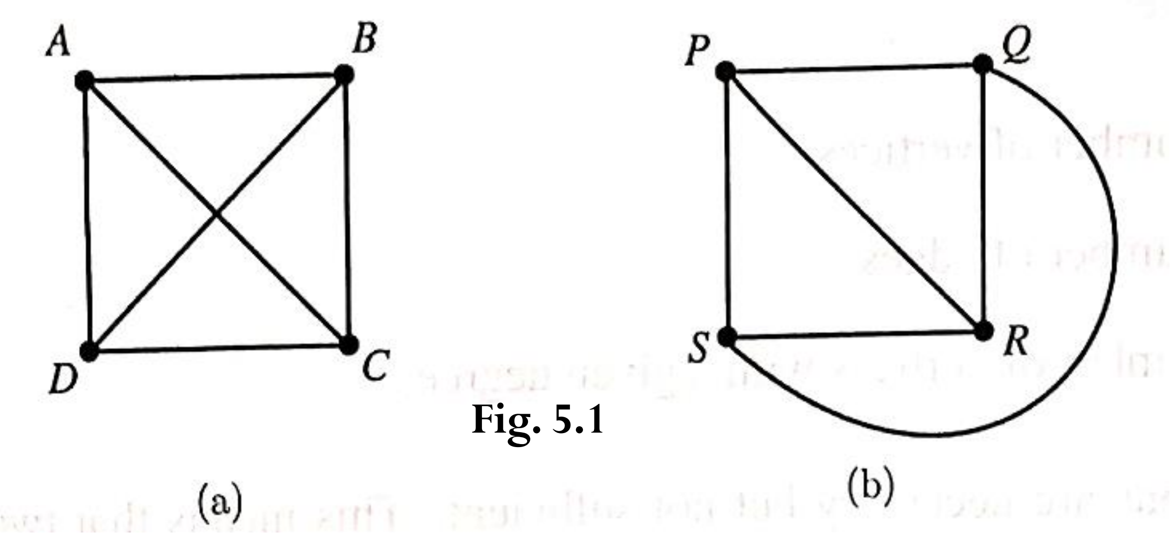



Fig. 5.1

In graphs shown above Fig 5.1.Consider the following *one-to-one correspondence between the vertices of these two graphs*

$$A \leftrightarrow P, B \leftrightarrow Q, \mathrm{C} \leftrightarrow R, \mathrm{D} \leftrightarrow S$$

Under this correspondence, the edge in two graphs correspond with each other, as indicated below:

$$\{A,B\} \leftrightarrow \{P,Q\}, \{A,C\} \leftrightarrow \{P,R\}, \{A,D\} \leftrightarrow \{P,S\}$$
$$\{B,C\} \leftrightarrow \{Q,R\}, \{B,D\} \leftrightarrow \{Q,S\}, \{C,D\} \leftrightarrow \{R,S\}$$

Therefore two graphs are isomorphic.

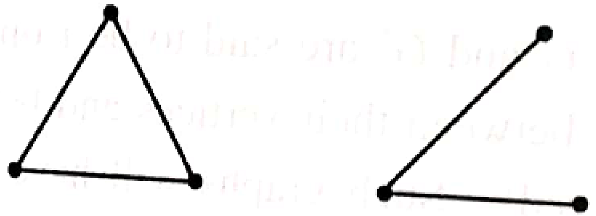


**Fig. 5.2**

We observe that both of these two graphs have same *number of vertices but different number of edges*. Therefore, although there can exist *one-to-one correspondence between vertices, there cannot be one-to-one correspondence between edges*. Two graphs are not isomorphic.

From the definition of isomorphism of graphs, it follows that if two graphs are isomorphic then they must have:

1. The same number of vertices
2. The same number of edges
3. An equal number of vertices with a given degree

These conditions are necessary but not sufficient.

In particular, two graphs of the same order and the same size need not be isomorphic. Consider the two graphs shown below **fig5.3**

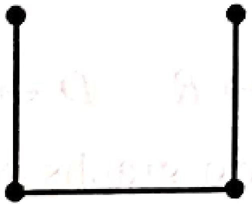
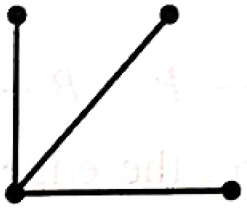


**Fig. 5.3**



Both of these graphs are order 4 and size 3. But the two graphs are isomorphic.

**Note:**

1. Every two complete graphs, with the same number of vertices are isomorphic.
2. Any two complete bipartite graphs with bipartite containing $r$ $and$ $s$ vertices are isomorphic.

1. **Prove that the graphs shown below are isomorphic.**


Sol$^n$:

No. of vertices $= 4$ ✓

No. of edges $= 4$ ✓

One-to-one correspondence btw vertices.

$u_1 \leftrightarrow v_1$, $u_2 \leftrightarrow v_4$, $u_3 \leftrightarrow v_3$, $u_4 \leftrightarrow v_2$ ✓

One-to-one correspondence btw edges.

$\{u_1, u_2\} \leftrightarrow \{v_1, v_4\}$, $\{u_1, u_3\} \leftrightarrow \{v_1, v_3\}$

$\{u_2, u_4\} \leftrightarrow \{v_4, v_2\}$, $\{u_3, u_4\} \leftrightarrow \{v_3, v_2\}$

$\therefore G \cong G'$

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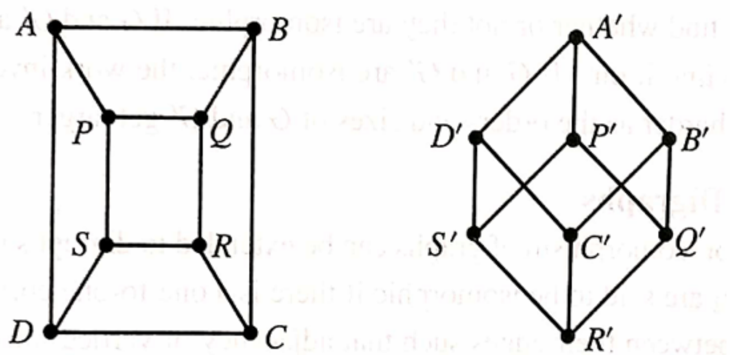
## 2. Verify that the two graphs shown below are isomorphic.




(a) $G$ (b) $G'$

Sol$^n$: No. of edges = 12

No. of vertices = 8

One-to-one correspondence of vertices.

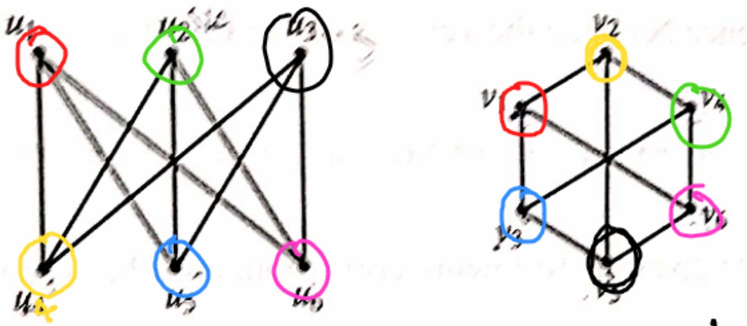
$A, B, C, D, P, Q, R, S \longleftrightarrow A', B', C', D', P', Q', R', S'$

One-to-one correspondence of edges

$\{A, B\} \longleftrightarrow \{A', B'\}$ .....

$\therefore G \cong G'$

# 3. Show that the following two graphs are isomorphic.




(a) G (b) G'

Sol: The No. of vertices in G & G' = 6

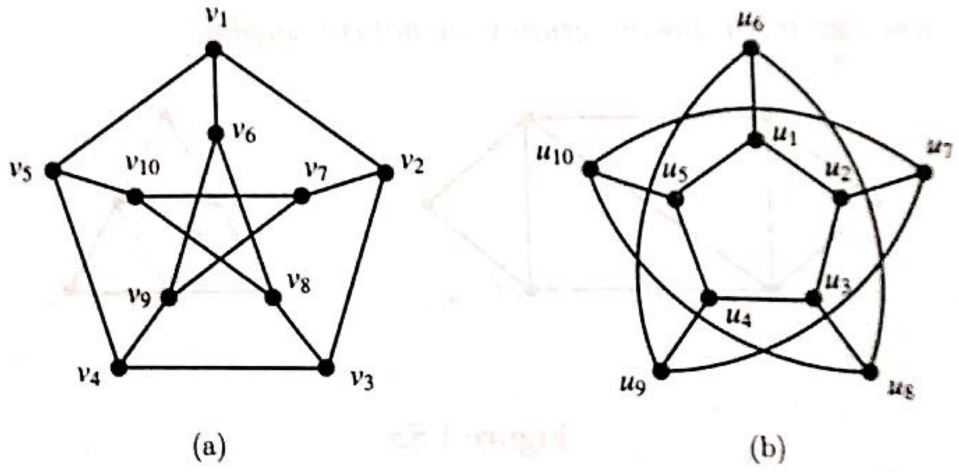
The No. of edges in G & G' = 9

Each vertex degree is 3,

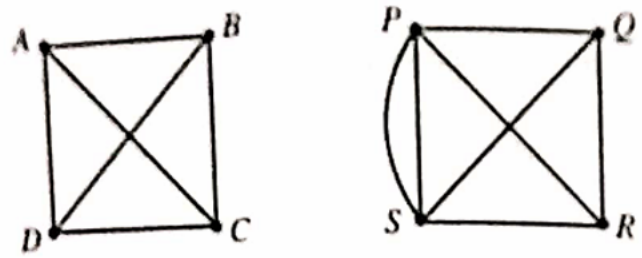
$u_1 \leftrightarrow v_1$, $u_2 \leftrightarrow v_4$, $u_3 \leftrightarrow v_5$, $u_4 \leftrightarrow v_2$

$u_5 \leftrightarrow v_3$, $u_6 \leftrightarrow v_6$.

4. **Show that the following two graphs are isomorphic.**



5. Show that the following two graphs are not isomorphic.



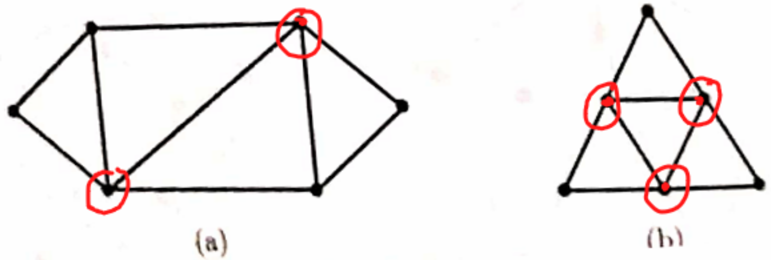

(a) (b)

Sol?: We observe that first graph has 4 vertices and 6 edges and second graph has 4 vertices and 7 edges.

∴ one-to-one correspondance b/w edges are not possible

∴ two graphs are not isomorphic

# 6. Show that the following two graphs are not isomorphic.



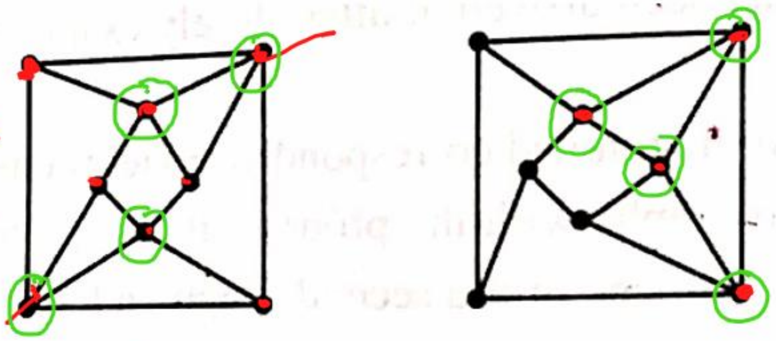

Sol$^n$: We note that, both the graphs has 6 vertices & 9 edges.

But the 1st graph has 2 vertices with degree 4, and 2nd graph has 3 vertices with degree 4.

$\therefore$ Hence two graphs are not isomorphic

7. Show that the following two graphs are not isomorphic.

4 → 4 degree
4 → 3 degree



4 — 4 degree
4 → 3 degree

(a) (b)

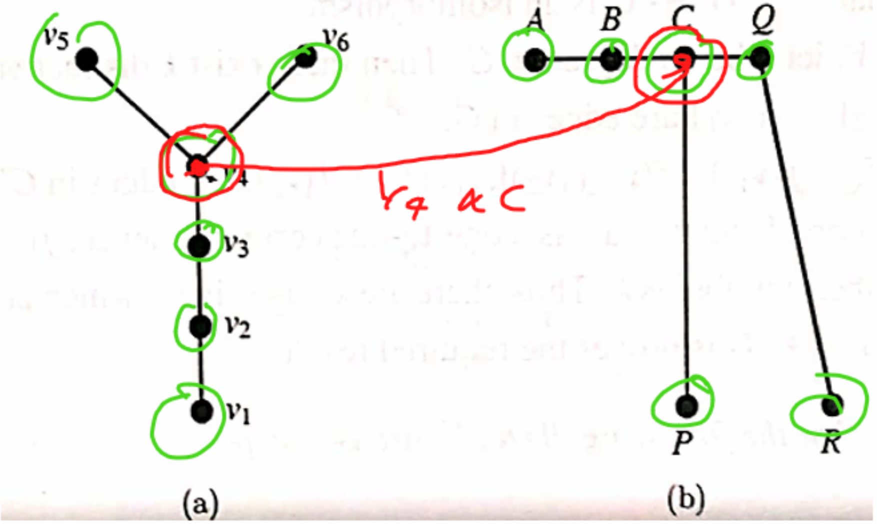
Solⁿ:

No. of vertices of both graph = 8
No. of edges of " " = 12

we note that first graph has a pair of vertices of degree 4 which are not adjacent, whereas the second graph has a pair of vertices of degree 4 which are adjacent.

∴ Two graphs are not isomorphic.

**8. Show that the two graphs need not to be isomorphic even if they have same number of vertices, the same number of edges and equal number of vertices with the same degree.**




(a) (b)

Sol$^n$:

No. of vertices = 6 —

No. of edn. = 5 —

3 vertices with degree 1, 2 vertices of degree 2, 1 vertex degree 3,

But, two are not isomorphic. because there are two pendant edges from $v_4$ in first graph and only one pendant edge from C in 2$^{nd}$ graph

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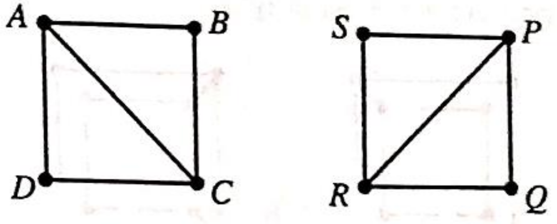
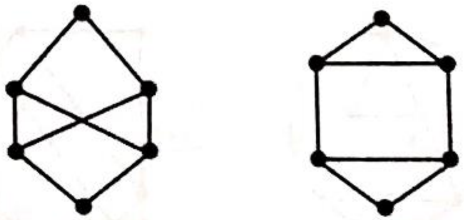
1 Show that the following graphs are isomorphic.

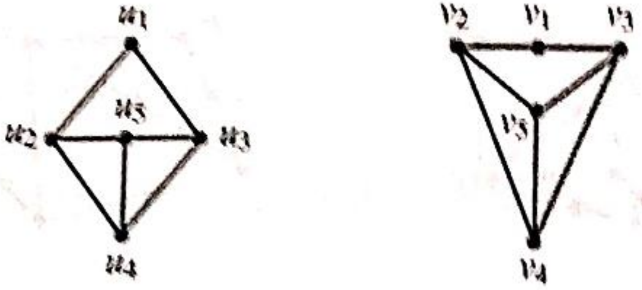



(a) (b)

2 Show that the following graphs are isomorphic.

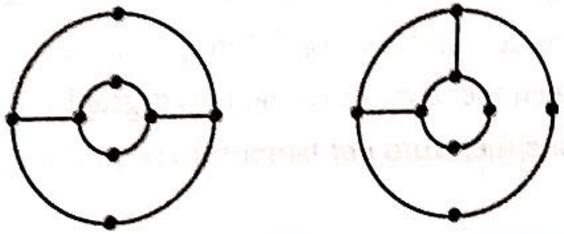


3 Show that the following graphs are isomorphic.



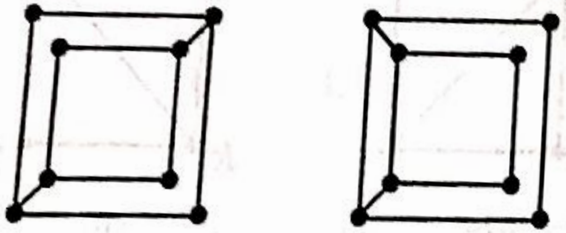

(a) (b)

4 Show that the following graphs are not isomorphic.



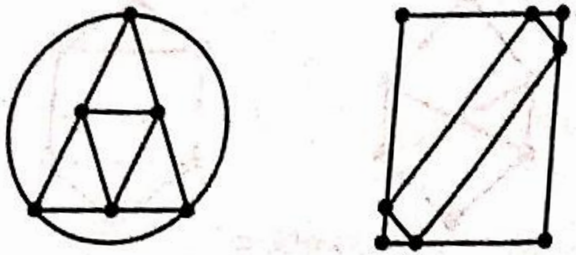
(a) (b)

5 Show that the following graphs are not isomorphic.



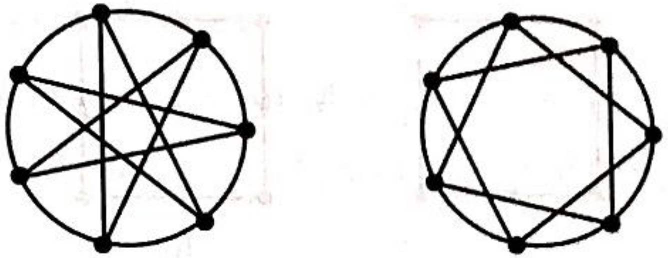
(a) (b)

6 Show that the following graphs are not isomorphic.



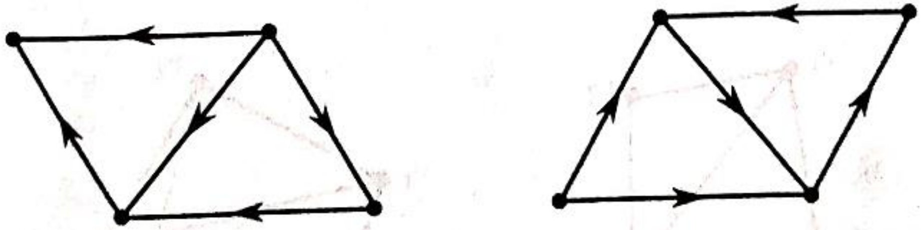
(a) (b)

7 Show that the following graphs are isomorphic.



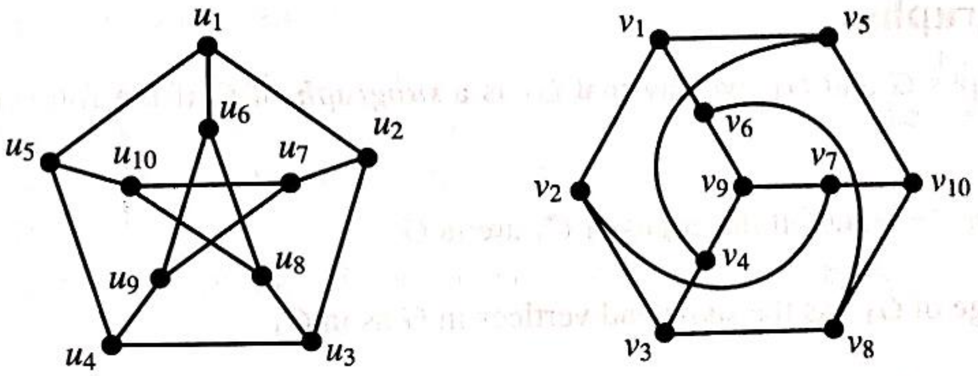
(a) (b)

9 Show that the following digraphs are isomorphic.



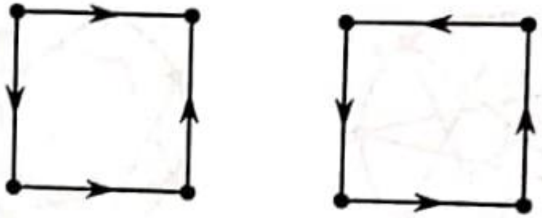
(a) (b)

8 Verify that the following graphs are isomorphic.




(a) (b)

10 Show that the following digraphs are not isomorphic.



(a) (b)

# Subgraphs

Given two graphs $G$ and $G_1$, we say that $G_1$ is a **subgraph** of G if the following condition holds.

1. All the vertices and all the edges of $G_1$ are in G.
2. Each edge of $G_1$ has the same end vertices in G as in $G_1$.

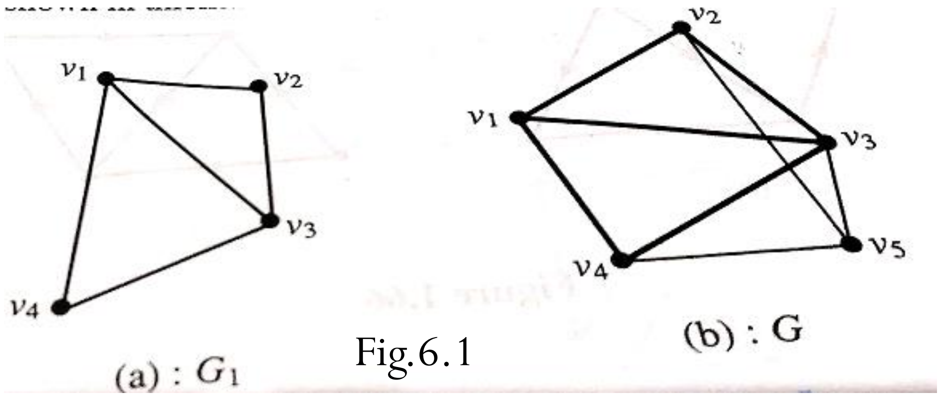



Fig.6.1

Consider two graph in Fig.6.1 (a) and (b), we observe that

*All vertices and all edges of the graph $G_1$ are in the graph G.*

*Every edge in $G_1$ has the same end vertices in G as in $G_1$.*

Therefore, $G_1$ is a subgraph of G.

# Spanning subgraphs

Given a graph $G = (V, E)$, if there is a subgraph $G_1 = (V_1, E_1)$ of G such that $V_1 = V$, then $G_1$ is called a *spanning subgraph of G.*

*In other words, a subgraph* $G_1$ of a graph $G$ *is a spanning subgraph of* G *whenever* $G_1$ *contains all vertices of G. Thus, a graph and all its spanning subgraphs have the same vertex set.*

*Every graph is its own spanning subgraph.*

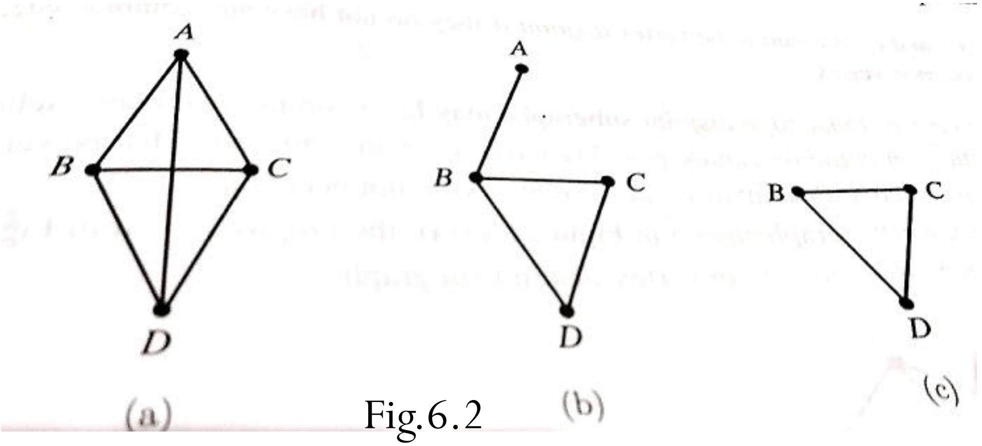



Fig.6.2

*Fig 6.2*

*(b) Is spanning subgraph of (a)*

*(c) Is not spanning subgraph of (a)*

# Induced subgraphs

Given a graph $G = (V, E)$, *suppose there is a subgraph* $G_1 = (V_1, E_1)$ *of* $G$ *such that every edge of* $\{A, B\}$ *of* $G$, *where* $A, B \in V_1$ *is an edge of* $G_1$ *also*. Then $G_1$ is called an induced subgraph of $G$ (induced by $V_1$) and is denoted by $< A >$.

*subgraph* $G_1 = (V_1, E_1)$ *of* $G = (V, E)$ *is not an induced subgraph of* $G$ *if for some* $A, B \in V_1$, *there is an edge of* $\{A, B\}$ *which is in* $G$ **but not in** $G_1$.

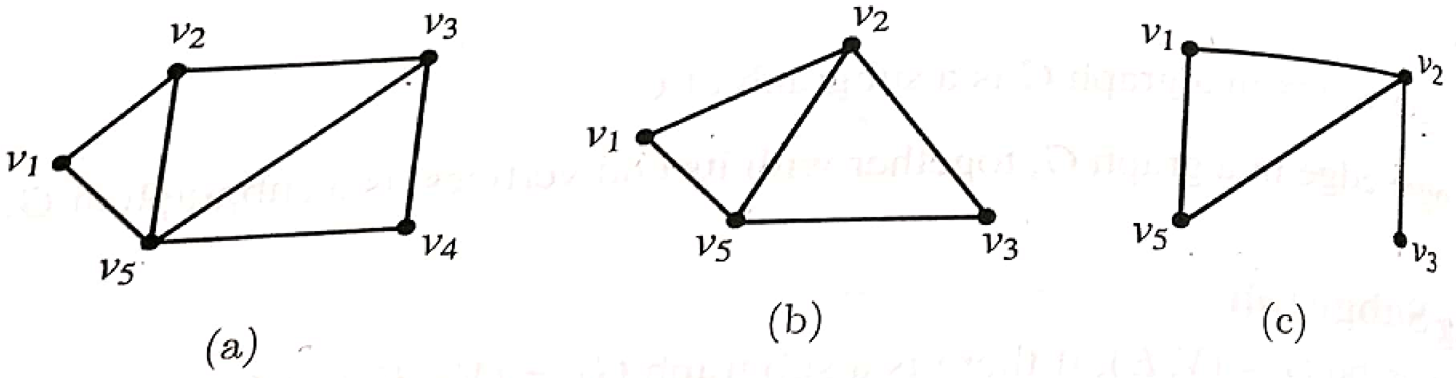



*Fig 6.3*

*(b) Is induced subgraph of (a)*

*(c) Is not induced subgraph of (a)*

Fig.6.3

# Edge disjoint and Vertex-disjoint Subgraphs

Let $G$ be a graph and $G_1$ and $G_2$ be two subgraphs of $G$. Then:

1. $G_1$ and $G_2$ are said to be edge-disjoint if they do not have any edge in common.(may have common vertices).
2. $G_1$ and $G_2$ are said to be Vertex-disjoint if they do not have any common edge and common vertex.

*Subgraphs that have no vertices in common cannot possibly have edges in common. That is, vertex disjoint subgraphs must be edge disjoint also but converse is not true.*

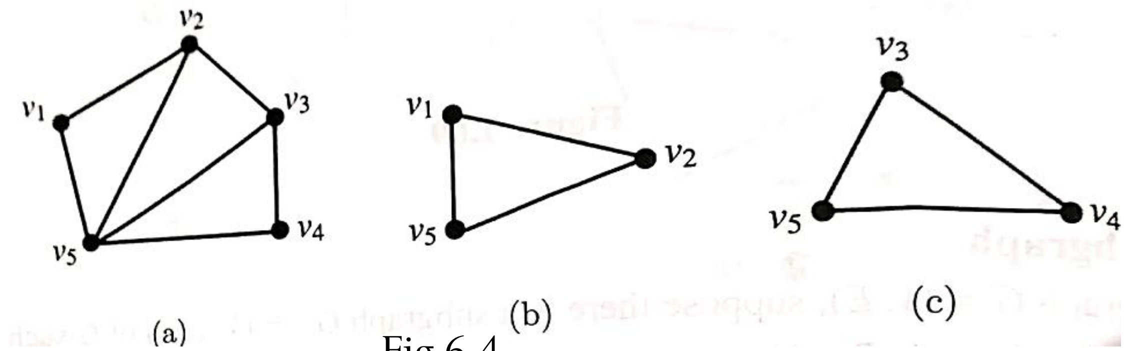
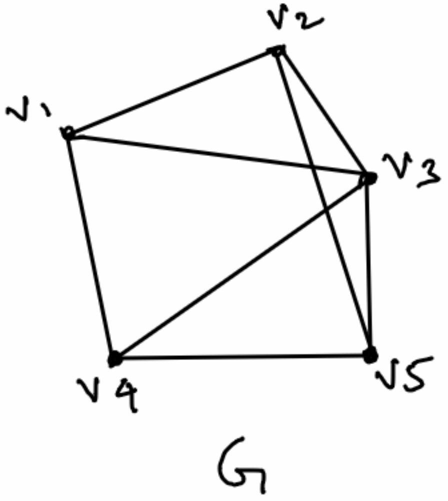
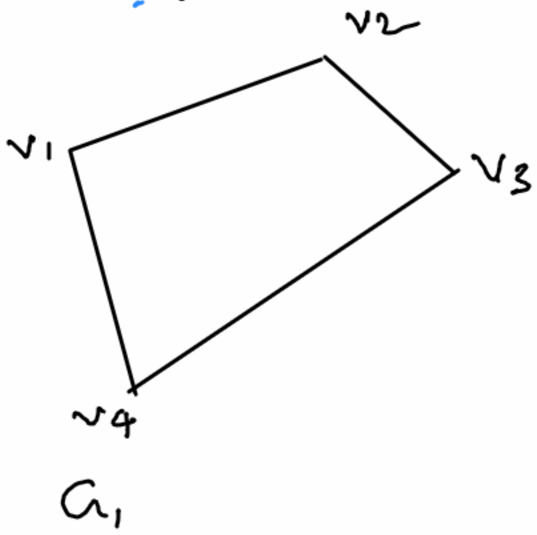


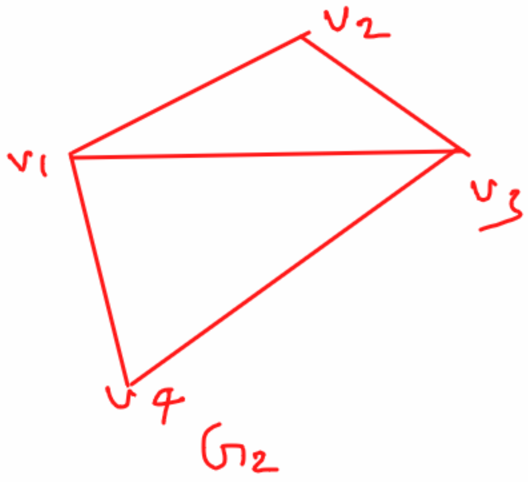

Fig.6.4

*Fig 6.4. (b) and (c) are edge disjoint but not vertex disjoint subgraphs.*

# Subgraphs, Spanning Subgraphs & Induced Subgraphs




$G_1$ is a subgraph ✓

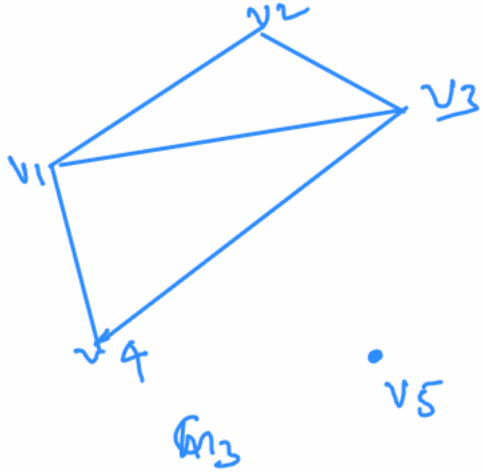
$G_1$ is not spanning

$G_1$ is not induced

$G_2$ is a subgraph

$G_2$ is not spanning

$G_2$ is induced subgraph




$G_3$ is subgraph

$G_3$ is spanning

$G_3$ is not induced

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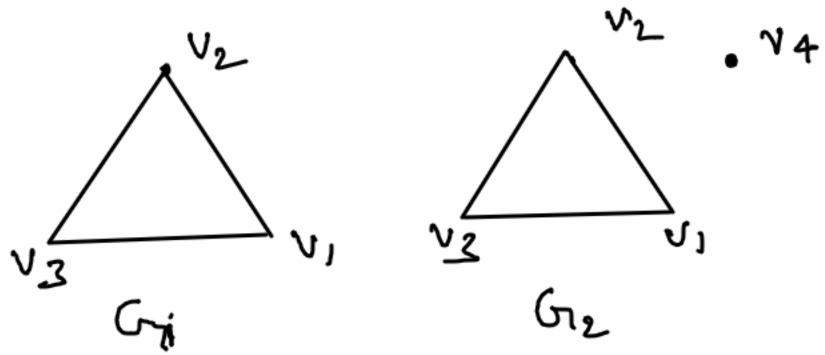
(1) Given a graph $G_1$, can there exist a graph $G_2$ such that $G_1$ is subgraph of $G_2$ but not a spanning subgraph of $G_2$ and yet $G_1$ & $G_2$ have the same size?

Sol$^n$: Yes.




$G_1$ is subgraph of $G_2$ but not spanning $(V_1 \neq V_2)$

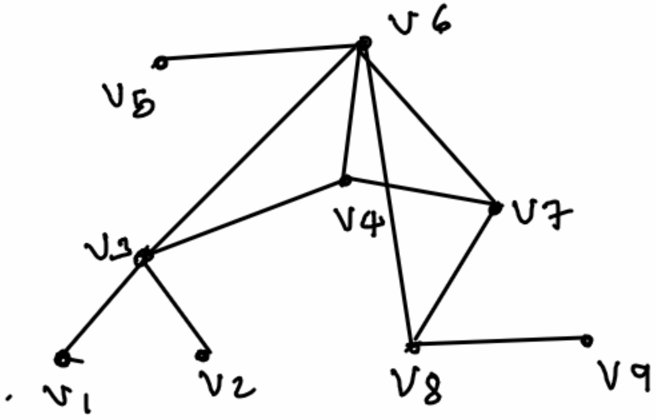
and size $= 3$

② Consider the graph G shown in (a)

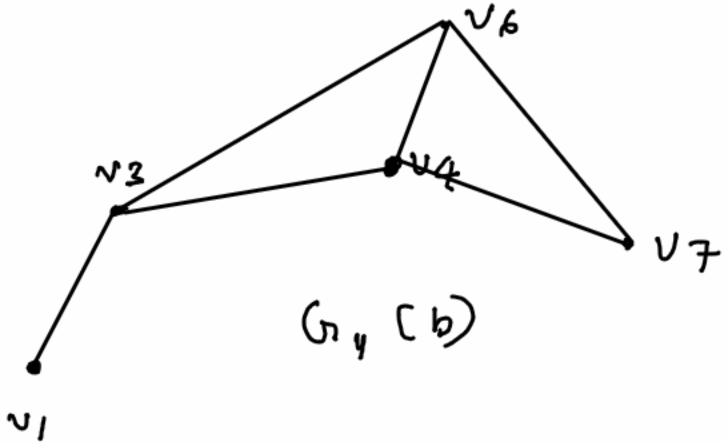
(a) Verify that graph $G_1$ (b) is an induced subgraph of G. I[...] a spanning subgraph of G.

(b) Draw the subgraph $G_2$ of G induced by the set $V_2 = \{V_3, V_4 \ldots$

Sol: Given:




G (a)



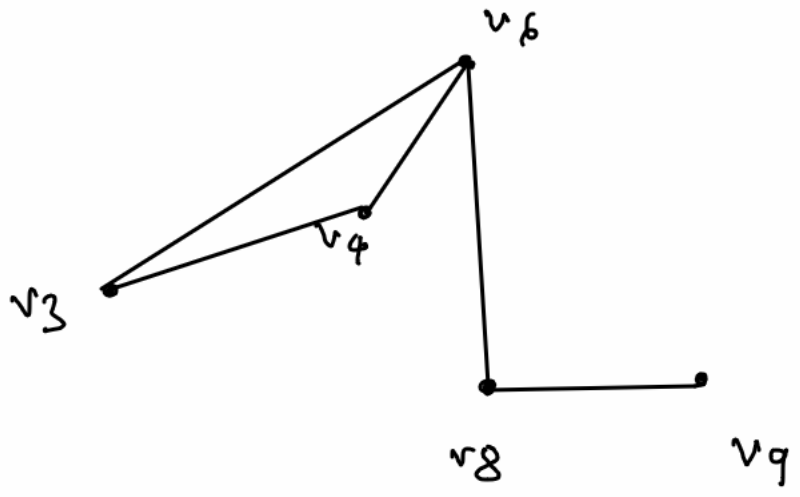

$G_1$ (b)

a) The vertex $V_1 = \{v_1, v_3, v_4, v_6, v_7\}$ is a subset of vertex set $V = \{v_1, \ldots v_9\}$ of G.

Also all the edges of $G_1$ are in G. ∴ $G_1$ is a subgraph ∴ $G_1$ is induced subgraph

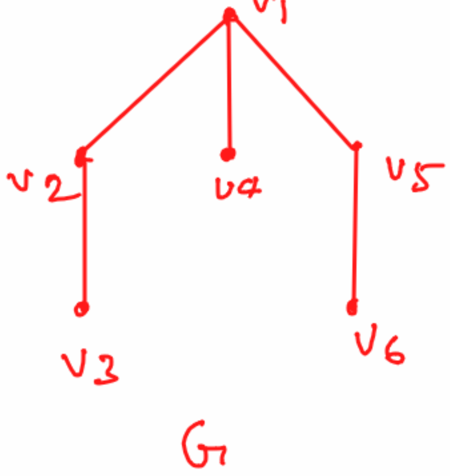
further every edge $\{v_i, v_j\}$ of G where $v_i, v_j \in V$ is an edge of $G_1$.

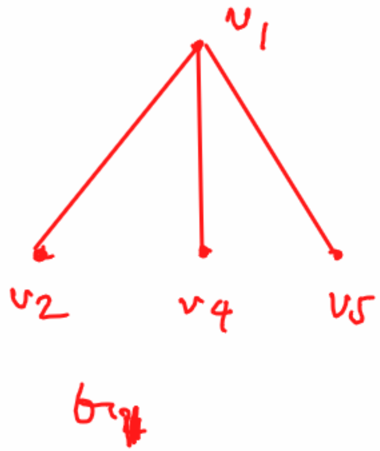
(b) $V_2 = \{v_3, v_4, v_6, v_8, v_9\}$

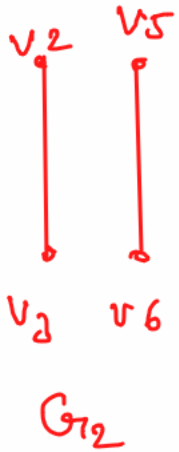


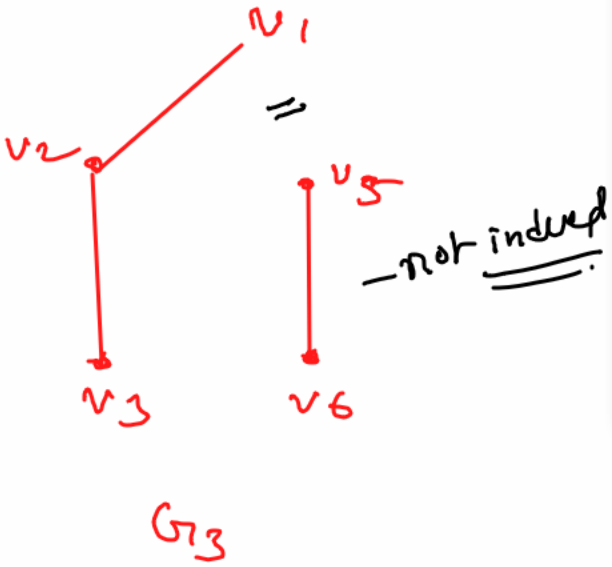

$G_2$ induced subgraph of $G$.

③ Consider a graph $G$. Verify that the graphs $G_1$ & $G_2$ are induced subgraphs of $G$ whereas the graph $G_3$ is not an induced subgraph of $G$.




Soln: We note that the vertex set of $G_1$, $G_2$, $G_3$ are subset of vertex set $V$ of $G$.

All edges in each $G_1$, $G_2$, & $G_3$ have same end vertices as in $G$.

$\therefore$ $G_1$, $G_2$ & $G_3$ are subgraphs.

Every edge $\{v_i, v_j\} \in G$ where $v_i, v_j \in V_1, V_2$ is an edge of $G_1$ & $G_2$

$\therefore$ $G_1$ & $G_2$ are induced subgraphs. $G_3$ is not induced. ($v_1 v_5$ is not edge)

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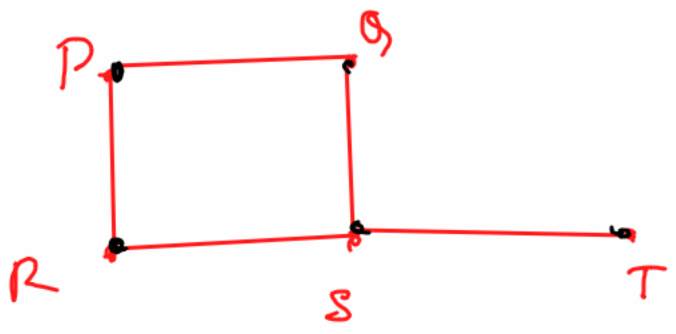
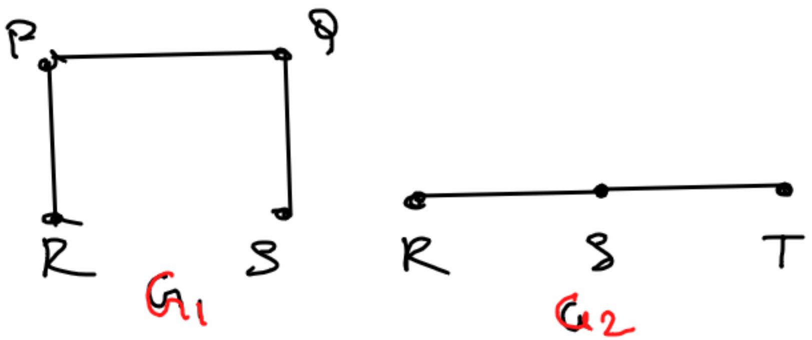
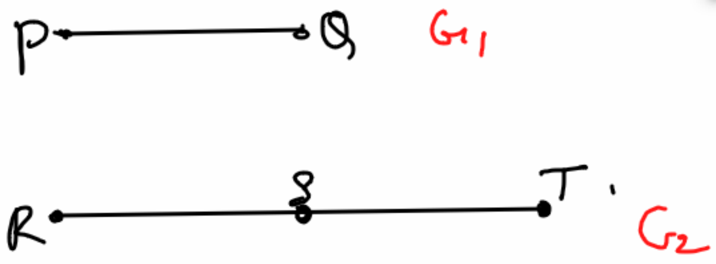
(4) For the graph shown in fig find two edge-disjoint subgraphs and two vertex disjoint subgraphs.




Sol:




Edge-disjoint




Vertex-disjoint

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# Complement of a subgraph

Given a *graph G and a subgraph $G_1$ of G*, the subgraph of G obtained by *deleting from G all the edges that belong to $G_1$* is called the ***complement of $G_1$ in G***; it is denoted by $G - G_1$, or $\overline{G_1}$.

If $E_1$ is the set of all edges of $G_1$, then the compliment of $G_1$ in G is given by $\overline{G_1} = \text{G} - E_1$.

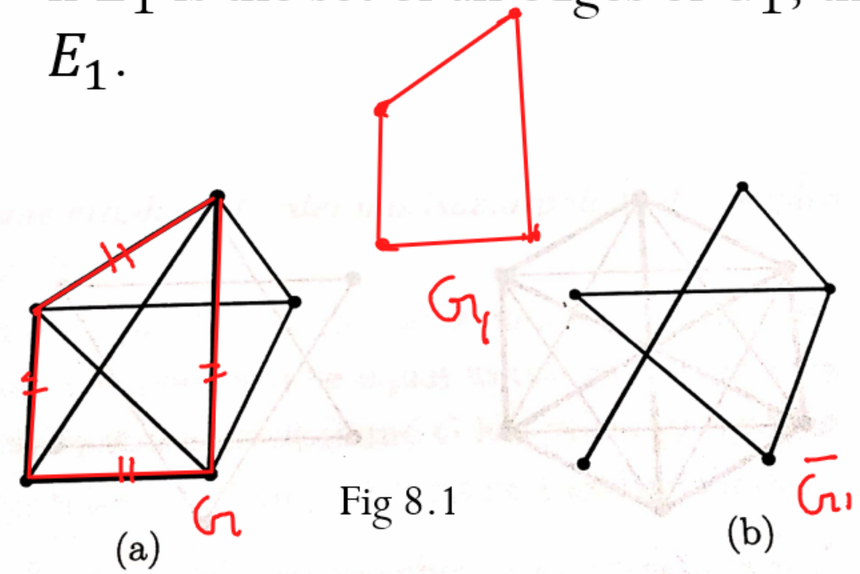



Fig 8.1

In fig 8.1 (a). Let $G_1$ be the subgraph of G shown by thick lines in this figure. The compliment of $G_1$ in G, namely $\overline{G_1}$, is as shown in Fig. 8.1 (b).

# Complement of a simple graph

We know that, *every simple graph of order $n$ is a subgraph of the complete graph $K_n$*. If G is a simple graph of order $n$, then the *complement of G in $K_n$ is called the complement of G*; it is denoted by $\overline{G}$ .

Thus, the complement $\overline{G}$ of a simple graph G with $n$ vertices is that graph which is obtained by deleting those in $K_n$ which belong to G. Thus $\overline{G} = K_n - G.$

Evidently $K_n$ , G and $\overline{G}$ have the same vertex set, and two vertices are adjacent in G if and only if they are not adjacent in $\overline{G}$ . Obviously, $\overline{G}$ is also a simple graph and the complement of $\overline{G}$ is G;

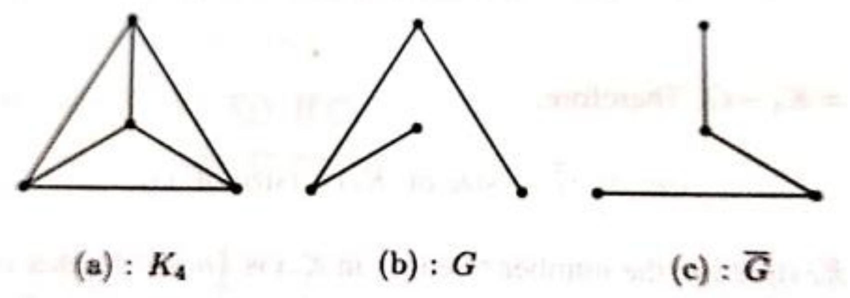
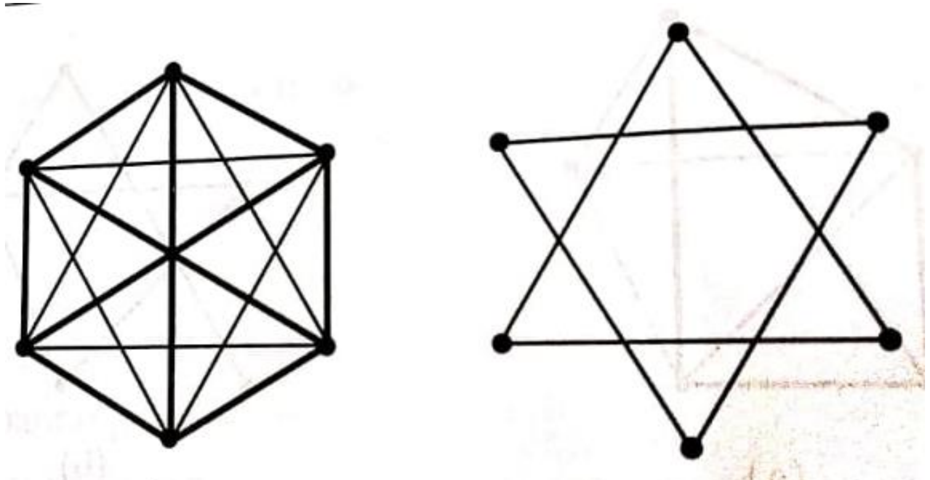



Fig 8.2

Fig 8.2 (a) is complete graph $K_4$. A simple graph of order 4 is shown in (b), the complement $\overline{G}$ of G is shown in (c).

Fig 8.3 (a), a graph of order 6 is shown as a subgraph of $K_6$, the edges of G is being shown in thick lines. Its complement $\bar{G}$ is shown in Fig 8.3 (b). This graph is known as **David graph.**



(a) Fig 8.3 (b)

1. *Show that the complement of a bipartite graph need not to be a bipartite graph.*

***Solution:*** *Fig 8.4 (a) shows a bipartite graph which is of order 5. The complement of this shown in Fig 8.4 (b), this is not bipartite.*

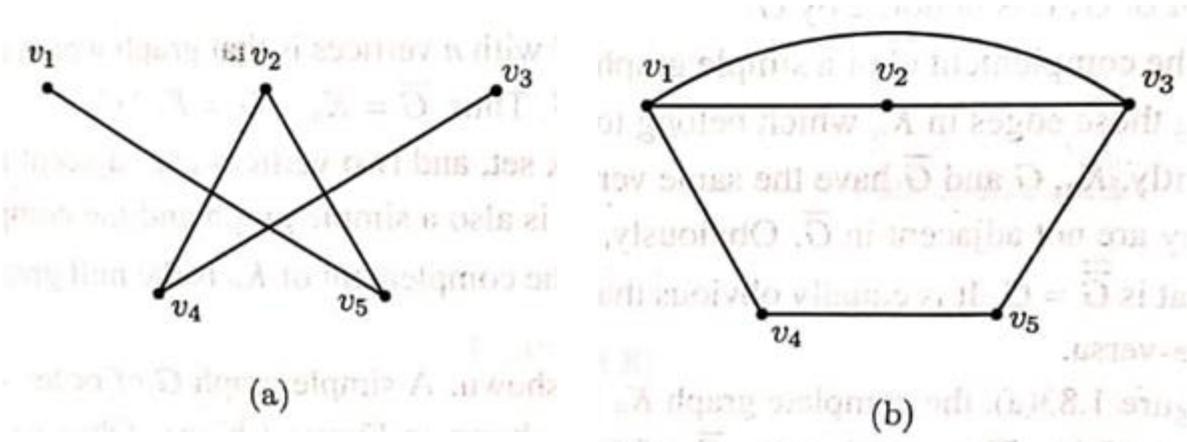



Fig 8.4

2. *Let G be a simple graph of order* $\boldsymbol{n}$*. If the size of G is 56 and the size of* $\overline{\mathbf{G}}$ *is 80, what is* $\boldsymbol{n}$*.*

**Solution:** we know that $\overline{G} = K_n - G.$

$$size\ of\ \overline{G} = size\ of\ K_n - size\ of\ G.$$

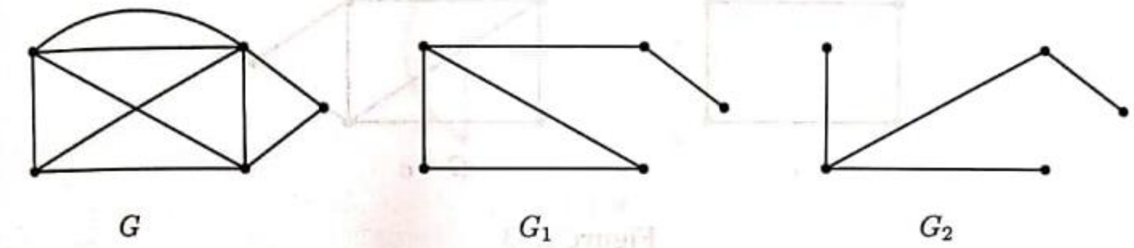
Size of $K_n$ (number of edges is $K_n$) is $\frac{1}{2}\,n(n-1)$

$$80 = \frac{1}{2}\,n(n-1) - 56$$

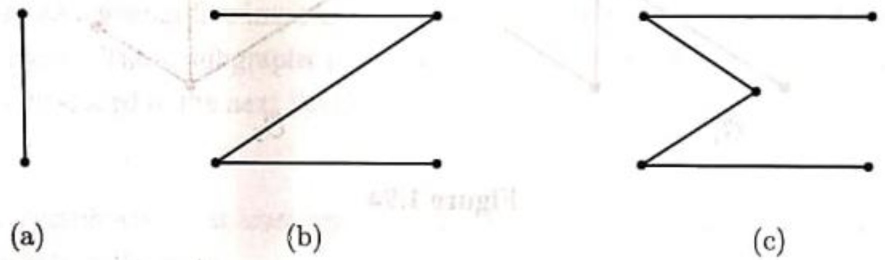
$$n = 17$$

# Problems

1. For the graph $G$ and its subgraphs $G_1$ and $G_2$ shown below, find $\overline{G}_1$ and $\overline{G}_2$.




2 Find the complement of each of the following simple graphs.




3 Draw diagrams of a self complementary graph $G$ with five vertices and its complement $\overline{G}$.

4 Find the complement of the complete bipartite graph $K_{3,3}$.

# Walks and their classification

In this section, we consider five important subgraphs of a graph, called a ***walk, a trail, a circuit, a path and a cycle.*** These subgraphs play a major role in studies concerned with connected graphs to be introduced in the next section.

***Walk***

*Let G be a graph having at least one edge. In G, consider a finite, alternating sequence of the vertices and edges of the form*

$$v_i e_j \; v_{i+1} e_{j+1} \, v_{i+2} e_{j+2} \cdots \cdots . \, e_k v_m .$$

*Which begins and ends with vertices and which is such that each edge in the sequence is incident in the vertices preceding and following it in the sequence. Such a sequence is called a* ***walk*** in G.

- *In a walk, a vertex or an edge (or both) can appear more than once.*
- *The number of edges present in a walk is called its* ***length.***

*For example, consider the graph shown below*

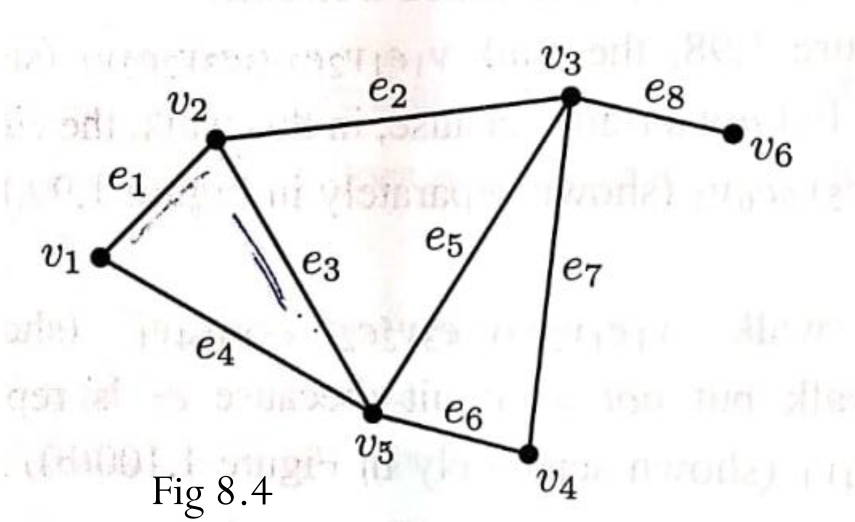


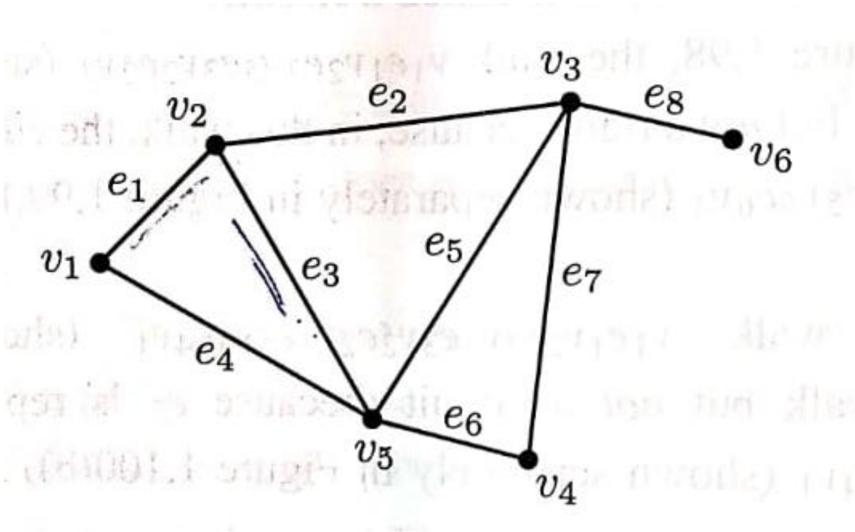

Fig 8.4

*In this graph:*

1. *The sequence* $\boldsymbol{v_1 e_1 v_2 e_2 v_3 e_8 v_6}$ *is a walk of length 3 (because, this walk contains 3 edges:* $\boldsymbol{e_1 e_2 e_8}$*). In this walk, no vertex and no edge is repeated.*
2. *The sequence* $\boldsymbol{v_1 e_4 v_5 e_3 v_2 e_2 v_3 e_5 v_5 e_6 v_4}$ *is a walk of length 5. In this walk the vertex* $\boldsymbol{v_5}$ *is repeated, but no edge is repeated*
3. *The sequence* $\boldsymbol{v_1 e_1 v_2 e_3 v_5 e_3 v_2 e_2 v_3}$ *is a walk of length 4. In this walk the edge* $\boldsymbol{e_3}$ *is repeated and the vertex* $\boldsymbol{v_2}$ *is repeated.*

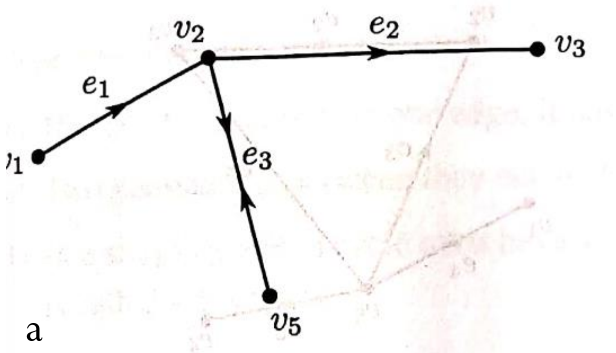
- *The vertex with which a walk begins is called the initial vertex (or the origin) of the walk*
- *and the vertex with which a walk ends is called the final vertex (or the terminus) of the walk.*
- *The initial vertex and the final vertex of a walk are together called its terminal vertices.*
- *The terminal vertices of a walk need not be distinct. Nonterminal vertices of a walk are called its internal vertices.*
- *A walk having u as the initial vertex and $\nu$ as the final vertex is called a walk from u to $\nu$, or briefly a u- $\nu$ walk.*
- *A walk that begins and ends at the same vertex is called a closed walk. In other words, a closed walk is a walk in which the terminal vertices are coincident. A walk which is not closed is called an open walk. In other words, an open walk is a walk that begins and ends at two different vertices.*

For example, in the graph shown in Fig 8.4 , $v_1 e_1 v_2 e_3 v_5 e_4 v_1$ is closed walk $v_1 e_1 v_2 e_2 v_3 e_5 v_5$ is an open walk.



# Trail and circuit

*As mentioned before, in a walk, vertices and/or edges may appear more than once. If in an open walk no edge appears more than once, then the walk is called a* ***trail****. A closed walk in which no edge appears more than once is called a* ***circuit.***




a

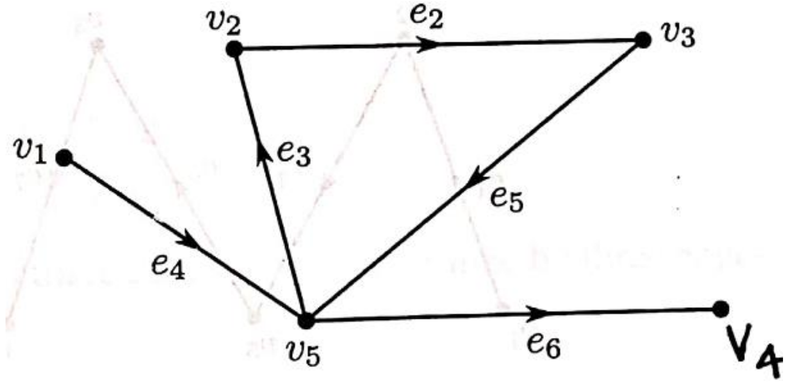
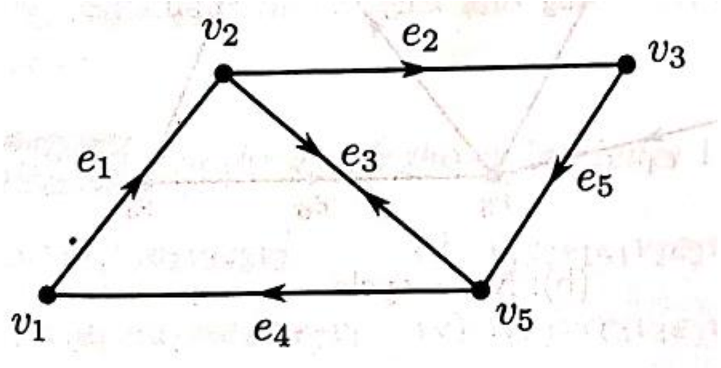



Fig 8.5

b

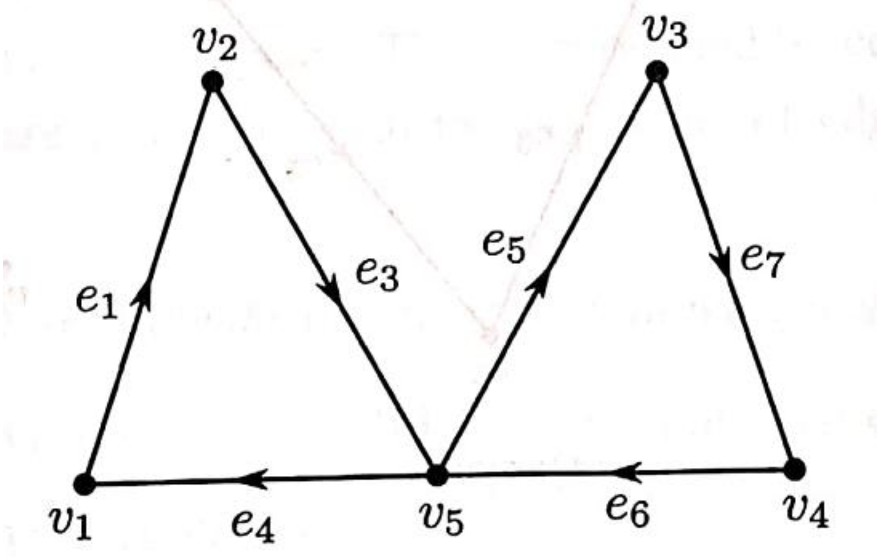
*For example, in Fig 8.4 , the walk* $v_1e_1v_2e_3v_5e_3v_2e_2v_3$*(shown separately in figure 8.5 (a) is an open walk but not a trail (because, in this walk, the edge* $e_3$ *is repeated)*

*The walk* $v_1e_4v_5e_3v_2e_2v_3e_5v_5e_6v_4$ *(shown separately in figure 8.5 (b) is an open walk which is a trail.*




c

*In Figure 8.4, the walk* $v_1e_1v_2e_3v_5e_3v_2e_2v_3e_5v_5e_4v_1$ *(shown separately in figure (c) is a closed walk* but not a circuit (because $e_3$ is repeated)



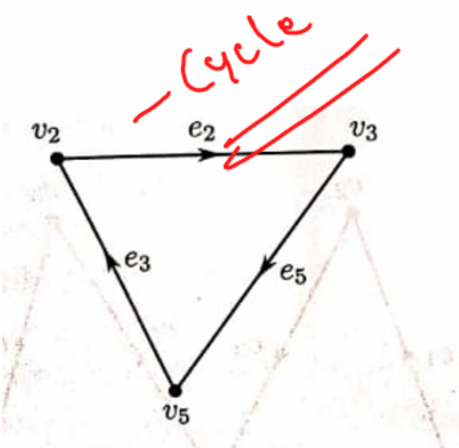

d

The walk $v_1e_1v_2e_3v_5e_3v_2e_7v_4e_6v_5e_4v_1$ (shown separately in figure (d) is a closed walk which is a circuit.

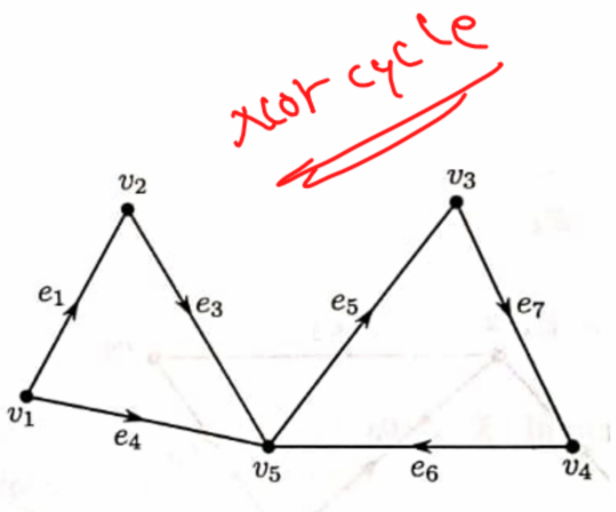
# Path and cycle

*A trail in which no vertex appears more than once is called a path.*

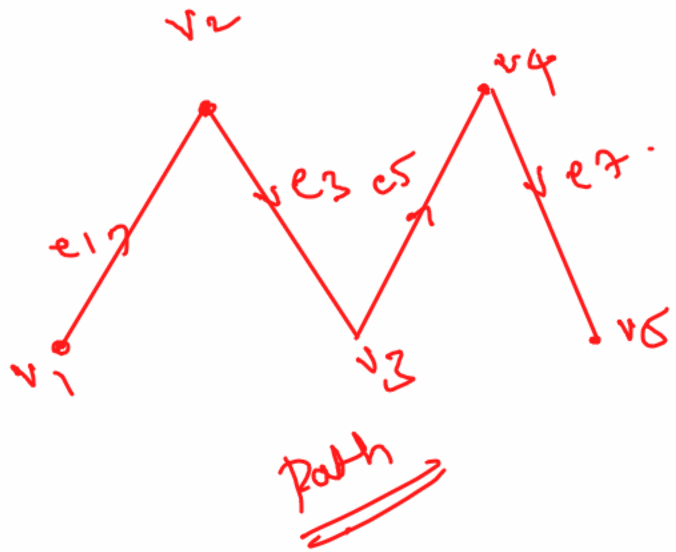
*A circuit in which the terminal vertex does not appear as an internal vertex and no internal vertex is repeated is called a cycle.*

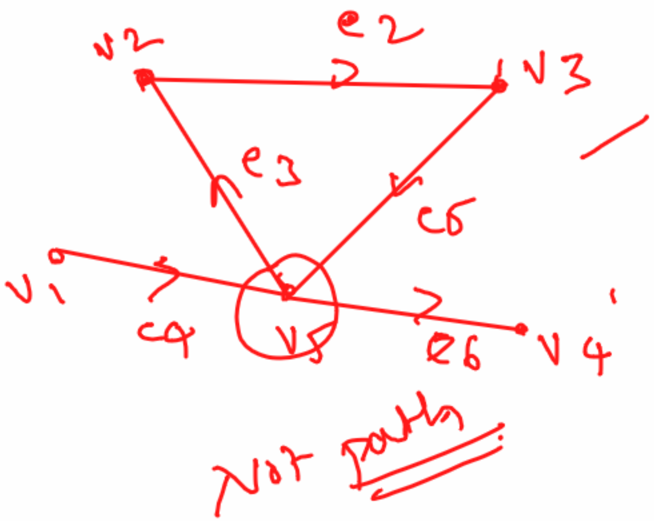

Cycle
$v_2$
$e_2$
$v_3$
$e_3$
$e_5$
$v_5$

(a))


Not cycle
$v_2$
$v_3$
$e_1$
$e_3$
$e_5$
$e_7$
$v_1$
$e_4$
$v_5$
$e_6$
$v_4$

(b))


v2
v4
e1
e3
e5
e7
v1
v3
v5
Path


v2
e2
v3
e3
e5
v1
e4
v5
e6
v4
Not path

*The following facts are to be emphasized.*

1. A *walk can be open or closed. In a walk (closed or open), a vertex and/or an edge can appear more than once.*
2. *A trail is an open walk in which a vertex can appear more than once but an edge cannot appear more than once.*
3. *A circuit is closed walk in which a vertex can appear more than once but an edge cannot appear more than once.*
4. *A path is an open walk in which neither a vertex nor an edge can appear more than once. Every path is a trail, but a trail need not be a path.*
5. *A cycle is closed walk in which neither a vertex nor an edge can appear more than once. Every cycle is circuit; but, a circuit need not be a cycle.*

The following results are obvious:

- If cycle contain only one edge, it has to be a loop.
- Two parallel edges (when they occur) form a cycle.
- In a simple graph, a cycle must have at least three edges. (A cycle formed by three edges is called a **triangle**)

**1.** **For the graph given below, indicate the nature of the following walks.**

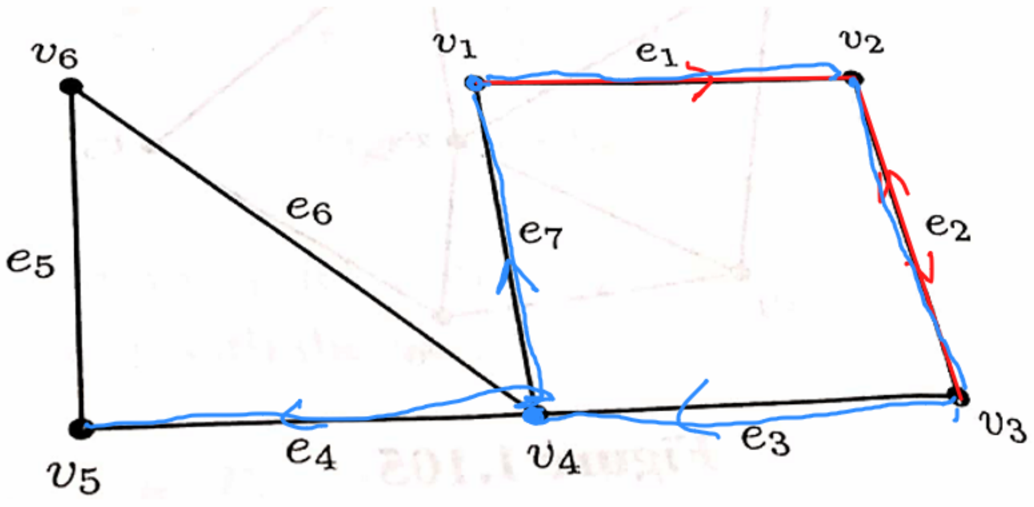
*(i)* $v_1 e_1\, v_2\, e_2\, v_3\, e_2\, v_2$

(ii) $v_4 e_7\, v_1\, e_1\, v_2\, e_2\, v_3\, e_3\, v_4\, e_4\, v_5$

*(iii)* $v_1 e_1\, v_2\, e_2\, v_3\, e_3 v_4\, e_4\, v_5$

*(iv)* $v_1 e_1\, v_2\, e_2\, v_3\, e_3 v_4 e_7 v_1$

*(v)* $v_6 e_5\, v_5\, e_4\, v_4\, e_3\, v_3 e_2 v_2\, e_1\, v_1\, e_7 v_4 e_6 v_6$




Sol^n: (i) open walk but not a trail ($e_2$ repeated)

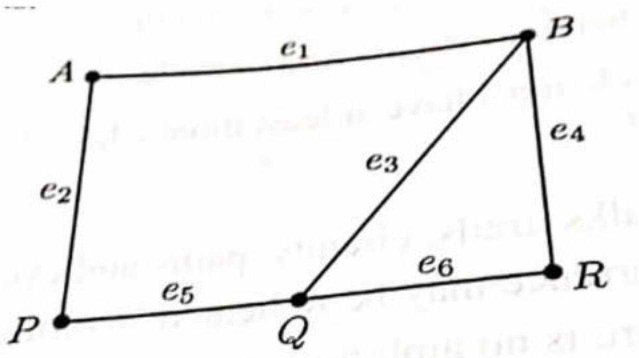
(ii) walk → open → trail → ~~path~~? ($v_4$ repeated)

(iii) Trail which is path.

(iv) Circuit which is cycle

(v) closed walk which is circuit

2. Consider the graph shown below. Find all paths from vertex A to vertex R. also, indicate their lengths.






Sol$^n$:

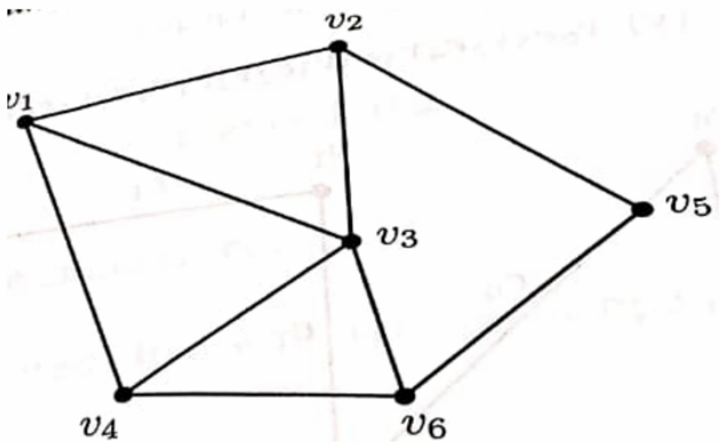
1) $A e_1 B e_3 Q e_6 R \rightarrow 3$

2) $A e_1 B e_4 R \rightarrow 2$

3) $A e_2 P e_5 Q e_6 R \rightarrow 3$

4) $A e_2 P e_5 Q e_3 B e_4 R \rightarrow 4$

3. Determine the number of different paths of length 2 in the graph shown below.




Sol$^n$: The no. of paths of length 2 that pass through the vertex $v_1$ is the no. of pairs of edges incident on $v_1$.

$$^nC_r = \frac{n!}{(n-r)!\, r!}$$

Since 3 edges are incident of $v_1$ $\therefore$ $^3C_2 = 3$

|||ly for $v_2, v_3, v_4, v_5, v_6$

$^3C_2 = 3$, $^4C_2 = 6$, $^3C_2 = 3$, $^2C_2 = 1$, $^3C_2 = 3$

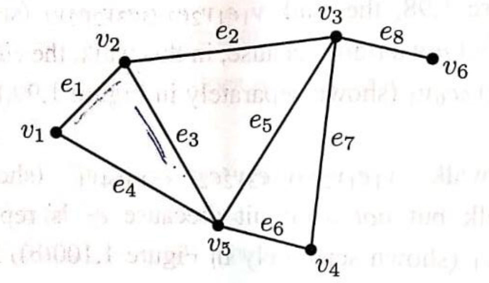
$\therefore$ The required no. is $= 19$

8/16/2021

# Problems (9)

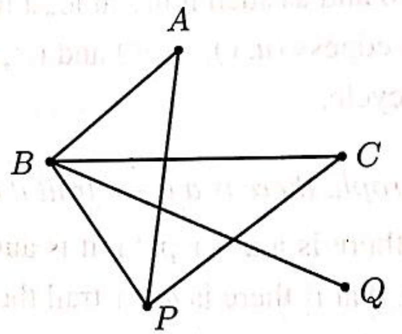
1. For the graph shown in Figure 1.98, find the nature of the following walks (the edges in-between the vertices are understood):

(i) $v_1v_2v_5v_3v_4v_5v_1$.  (ii) $v_1v_2v_3v_5v_1$.




2. For the graph shown in Figure 1.108, find the nature of the following walks.

(i) *BAPCB* (ii) *PABQ* (iii) *CBAPBQ*



3. For the graph shown in Figure 1.109, find the nature of the following walks:
(i) $ABEFDACDB$ (ii) $ABEFDCA$ (iii) $ACDFEBDA$ (iv) $ABDFEBDC$

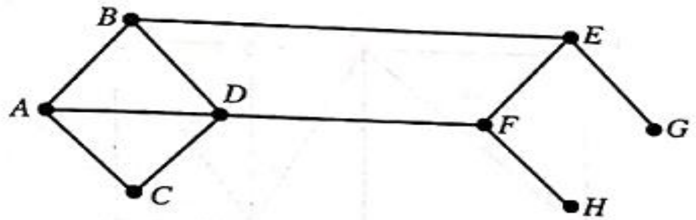



**Figure 1.109**

4. In the graph shown in Figure 1.110, verify that
   (i) $v_1v_2v_3v_4v_1$ is a cycle.
   (ii) $v_1v_2v_5v_3v_4v_5v_1$ is a circuit which is not a cycle.
   (iii) $v_1v_2v_5v_1$ is a triangle.

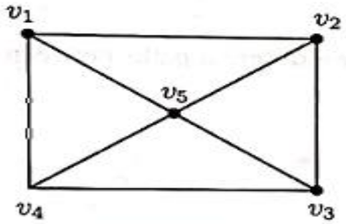
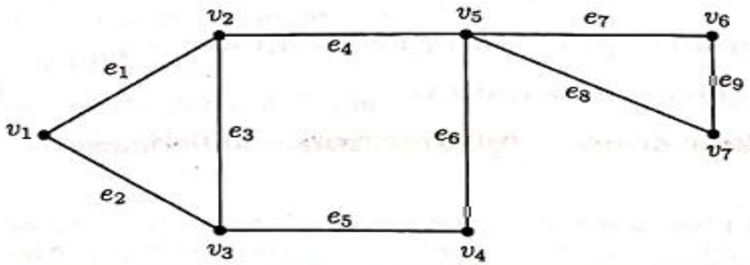



**Figure 1.110**

5. For the graph shown in Figure 1.111, determine (i) a walk from $v_2$ to $v_4$ which is not a trail (ii) a $v_2 - v_4$ trail which is not a path (iii) a path from $v_2$ to $v_4$ (iv) a closed walk from $v_2$ to $v_2$ which is not a circuit (v) a circuit from $v_2$ to $v_2$ which is not a cycle (vi) a cycle from $v_2$ to $v_2$, and (vii) the number of paths from $v_2$ to $v_6$.



# Connected and Disconnected Graphs

Consider a Graph G of order greater than or equal to two. Two vertices in G are said to be **connected** if there is at least one ***path*** from one vertex to the other.

*We say that a graph G is a **connected graph** if every pair if every pair of distinct vertices in G are connected. Otherwise, G is called a **disconnected graph**.*

*In other words, a graph G is said to be (i) connected if there is a at least one path between every two distinct vertices in G, and (ii) disconnected if G has at least one pair of distinct vertices between which there is no path.*

*A graph G is connected if we can reach any vertex of G from any other vertex of G by travelling along the edges, and disconnected otherwise.*

Example: Graph shown in Fig 9.1 (a) is connected whereas the graph shown in fig 9.1 (b) is disconnected.

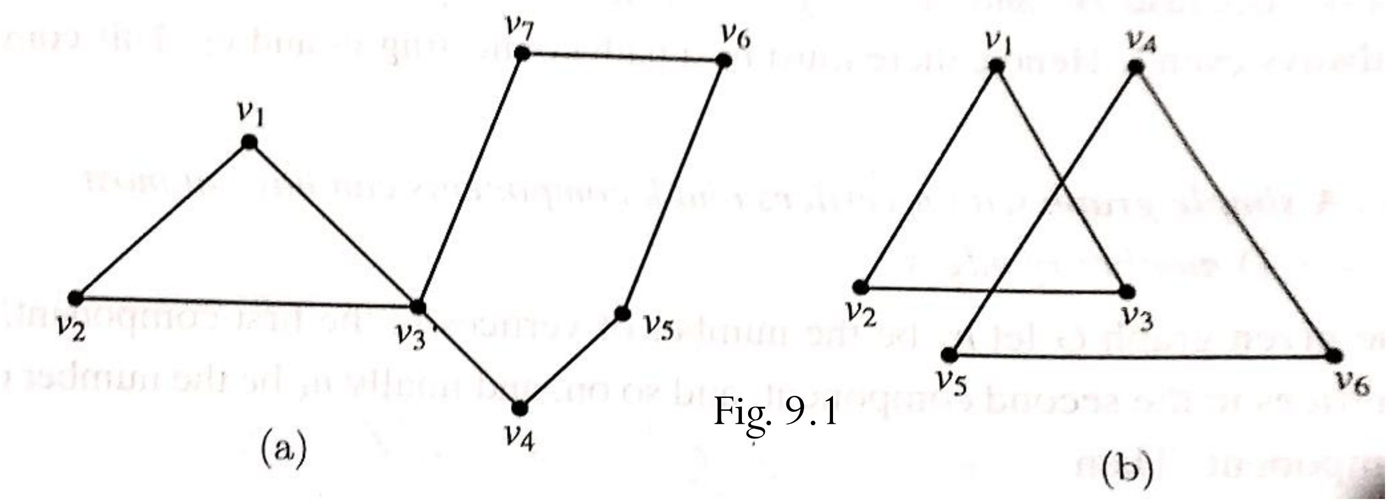


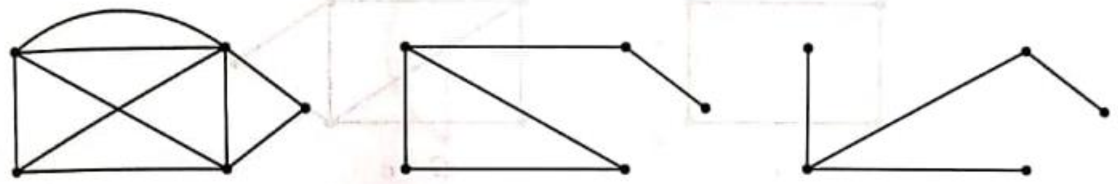

Fig. 9.1

**NOTE:**

1. *It is obvious that in a graph G all walks and, therefore, all trails, all circuits, all paths and all cycles ( when they exist) are connected subgraphs of G.*
2. *It is evident that every (nontrivial) graph G consists of one or more connected graphs. Each connected graph is a subgraph of G and is called a component of G.*
3. *Obviously, a connected graph has only one component and a disconnected graph has two or more components. The number of components of a graph G is denoted by $k(G)$.*

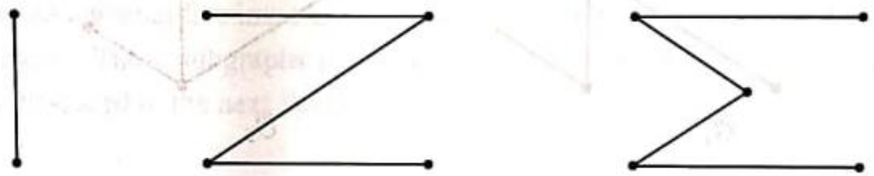
# Answer for the previous problems

1. For the graph $G$ and its subgraphs $G_1$ and $G_2$ shown below, find $\overline{G}_1$ and $\overline{G}_2$.



$G$ $G_1$ $G_2$

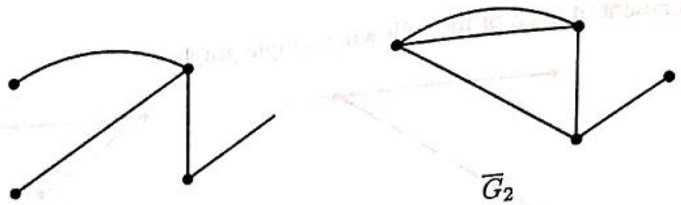
2 Find the complement of each of the following simple graphs.



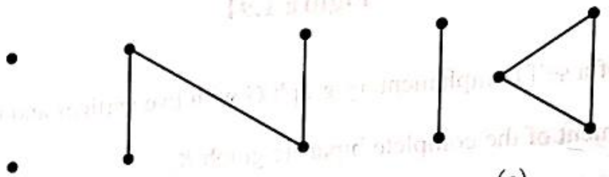
(a) (b) (c)

3 Draw diagrams of a self complementary graph $G$ with five vertices and its complement $\overline{G}$.

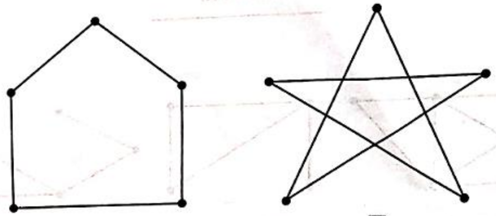
4 Find the complement of the complete bipartite graph $K_{3,3}$.

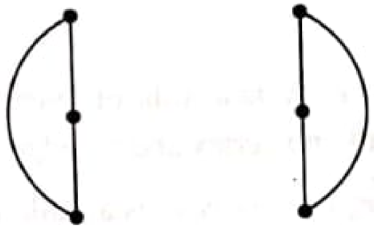


$\overline{G}_1$ $\overline{G}_2$



(a) (b) (c)





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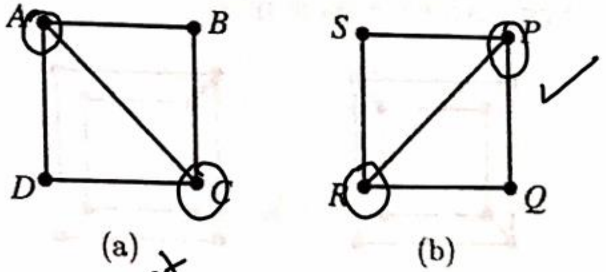
# MODULE-1
# INTRODUCTION TO GRAPH THEORY-I

CLASS-10

1 Show that the following graphs




Sol$^n$: The graph (a) & (b) both have same no. of vertices & edges [4 V, 5 Edge].

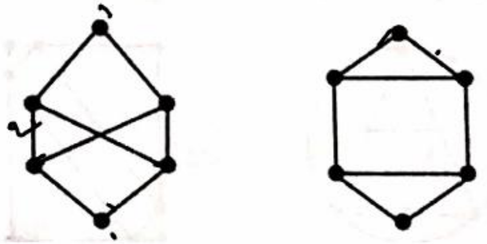
In Both graph two vertices with 2 degree & 2 vertices with 3 degree

A ⇔ P D ⇔ S ∴ Both are isomorphic

C ⇔ R

B ⇔ Q

2 Show that the following graphs are not isomorphic.



Sol$^n$: E = 8

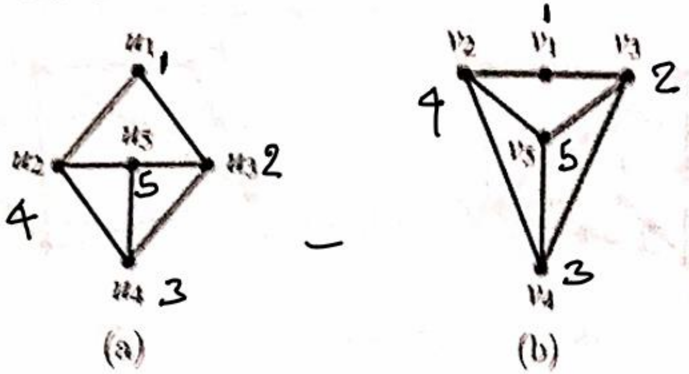
v = 6

2 v → 2 deg

4 v → 3 deg.

X/o adjancency btw vertices // Not Isomo

3 Show that the following graphs are isomorphic.



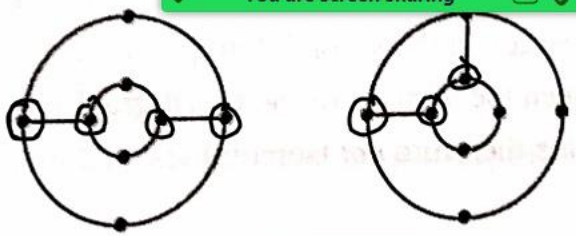

E = 7

V = 5

1 v = 2 degree

4 = 3 degree

∴ Both are Isomorphic



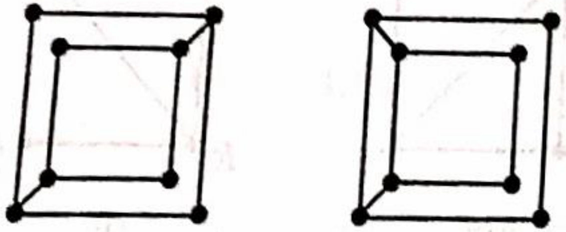
(a) (b)

Sol^n:
$E = 10$
$V = 8$
$4V = 3$ degrees
$4V = 2$ degrees

The pair of degree are not in (b)

$\therefore$ Both are not isomorphic

5 Show that the following graphs are not isomorphic.



(a) (b)

Sol^n:
$E = 10$
$v = 8$
$4V = 3$ degree
$4V = 2$ degree

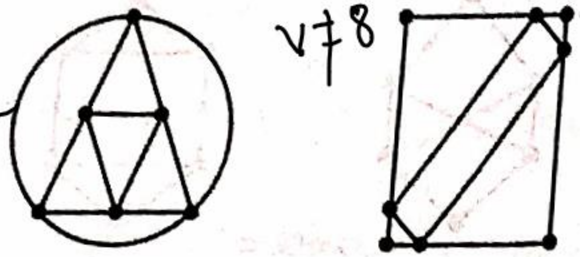
//

$\therefore$ Both a

6 Show that the following graphs are not isomorphic.

$V = 6$
$E = 12$

$V \neq 8$

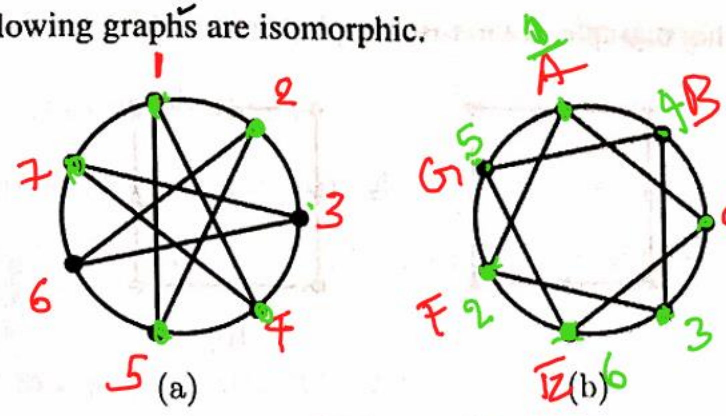


(a) (b)

Sol^n: No. of vertices in (a) $\neq$ No. of vertices (b)

$\therefore$ Both are not isomorphic

7 Show that the following graphs are isomorphic.




Sol^n:

$V = 7$

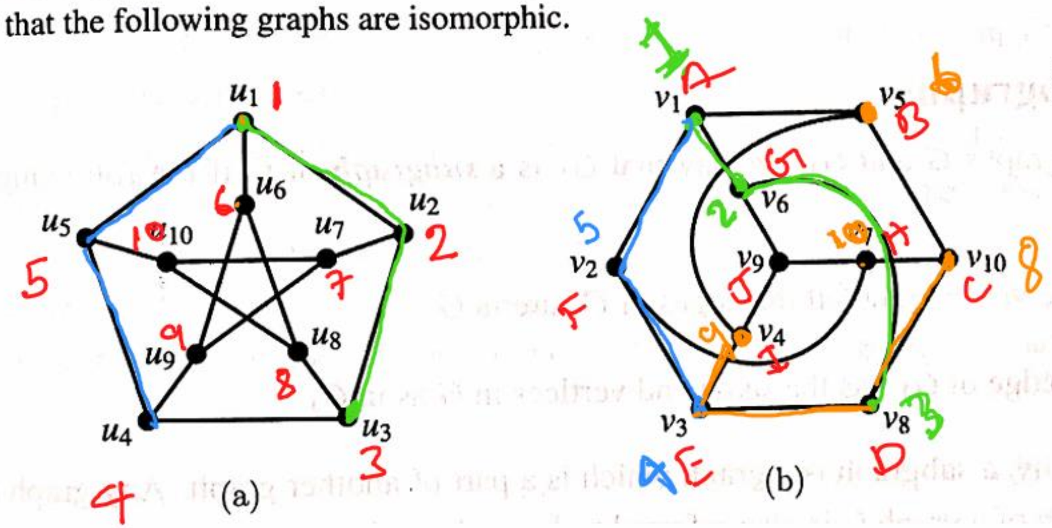
$E = 14$

$7\ V = 4$ degree

The adjacency of vertices are preserved

$\therefore$ Both are Isomorphic

8 Verify that the following graphs are isomorphic.




Sol^n

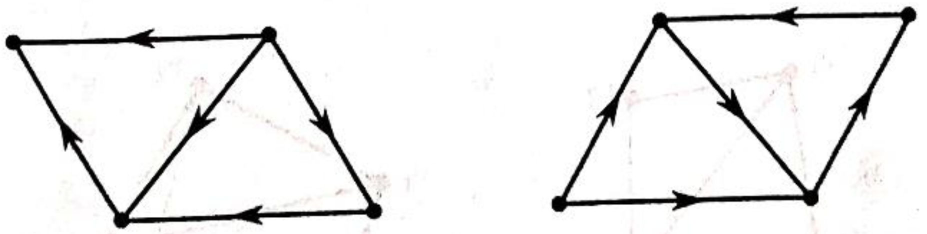
$V = 10$

$E = 15$

$10\ V = 3$ degree

$\therefore$ Both are Isomorphic

9 Show that the following digraphs are isomorphic.

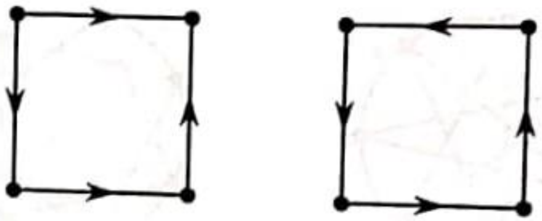


(a) (b)

Sol^n: The grap(a) has a vertex with 3 outdegree whereas in (b) we dont have
∴ Both are not isomorphic

(b)

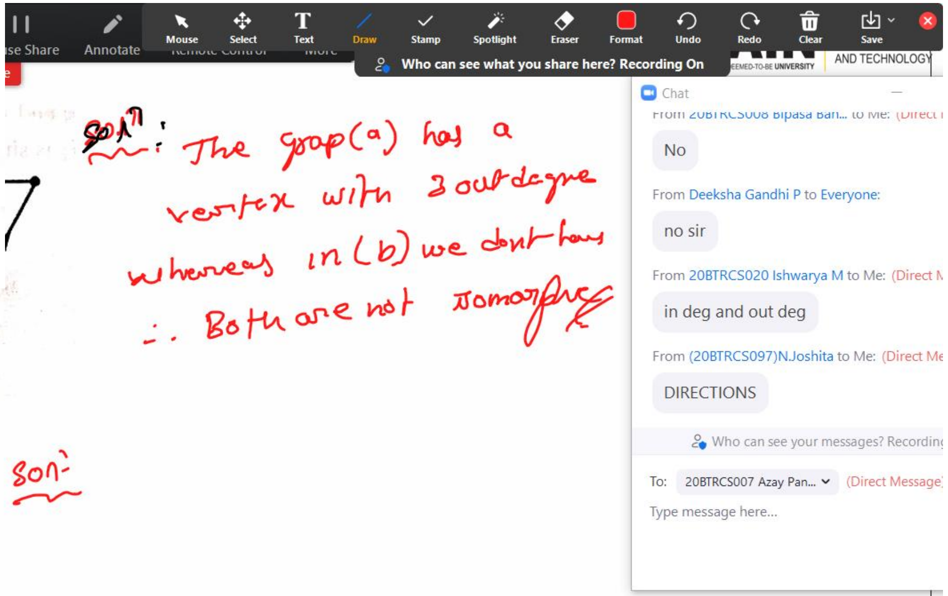
10 Show that the following digraphs are not isomorphic.



(a) (b)

Son:

b) somorphic




8/17/2021

# MODULE-1
# INTRODUCTION TO GRAPH THEORY-I

CLASS-11

# Euler circuits and Euler trails

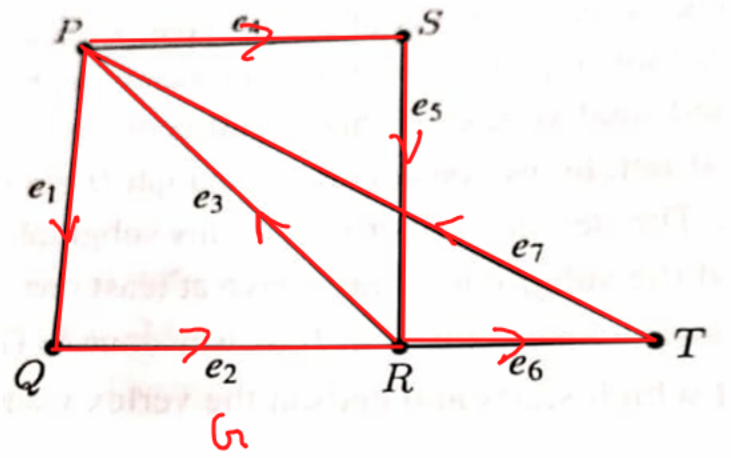
Consider a connected graph G. if there is circuit in G that contains *all the edges of G,* then that circuit is called an ***Euler circuit*** (or Euclerian line, or Euler tour) in G.

If there is a trail in G that contains all the edges of G, then that trail is called an ***Euler trail in G.***

*A connected graph that contains an Euler circuit is called an* ***Euler Graph (or Euclearian graph).***

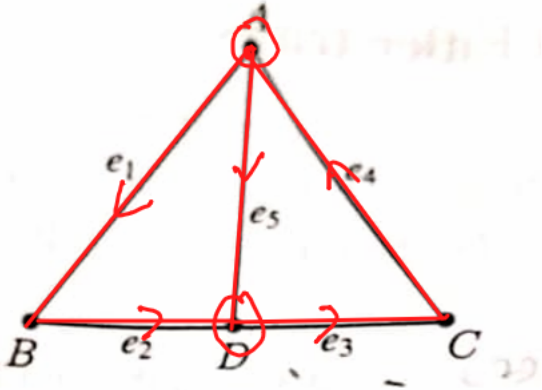
*A connected graph that contains an Euler trail is called an* ***semi-Euler Graph (or Euclearian graph).***

Example:




Closed walk $Pe_1Qe_2Re_3Pe_4Se_5Te_7P$

It is a Euler circuit. Therefore the graph is E




Open walk $Ae_1Be_2De_3Ce_4Ae_5D$

It is an Euler trail, it is an semi Euler graph.

**Theorem: A connected graph G has an Euler circuit (that is, G is an Euler graph) if and only if all vertices of G are of even degree.**

Proof:

Suppose that G has an Euler circuit. While tracing this circuit we observe that every time the circuit meets a vertex $v$ it goes through two edges incident on $v$.

This is true for all vertices that belong to the circuit,

Since the circuit contains all edges,

It meets all the vertices at least once.

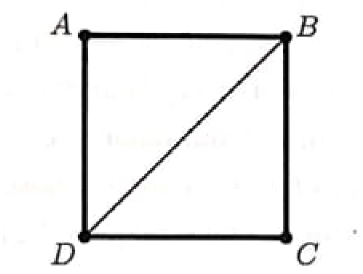
Therefore, the degree of every vertex is a multiple of two

That is every vertex is of even degree.

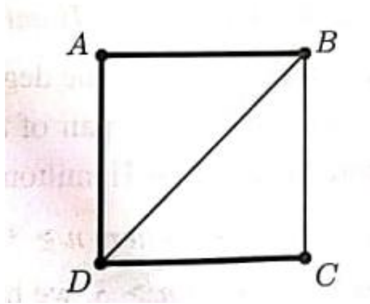
# Hamilton cycles and Hamilton paths

- *Let G be a connected graph. If there is a cycle in G that contains all the vertices of G, then that cycle is called a Hamilton cycle in G.*
- *A Hamilton cycle (when it exists) in a graph of n vertices consists of exactly n edges. Because, a cycle with n vertices has n edges.*
- *By definition, a Hamilton cycle (when it exists) in a graph G must include all vertices is G. This does not mean that it should include all edges of G.*

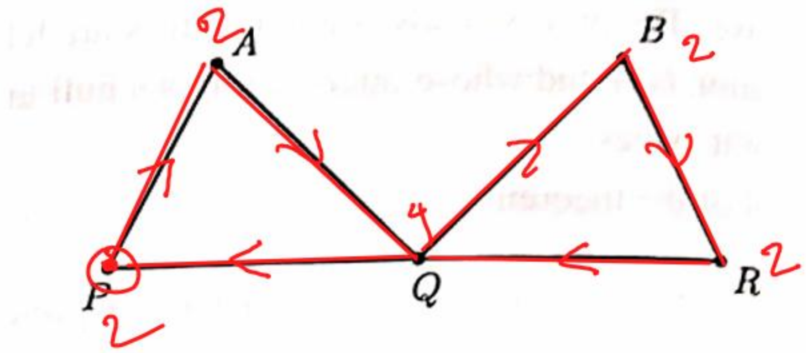
A graph that contains a Hamilton cycle is called Hamilton graph (or Hamiltonian graph). For example, in the graph shown in figure 10.1 the cycle shows in thick lines is a Hamilton cycle. (Observe that this cycle does not include the edge BD). The graph is therefore a Hamilton graph.




*A path (if any) in a connected graph which includes every vertex (but not necessarily every edge) of the graph is called a Hamilton Path (Hamiltonian path) in the graph. For example, in the graph shown in figure 10.2, the path shown in thick lines is a Hamilton path.*



1. Show that the graph shown below is an Euler graph

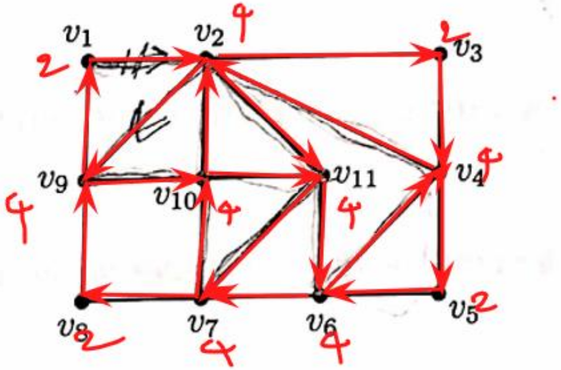



Sol^n:

P A Q B R Q P

The graph contains Euler circuit

2. Find an Euler circuit in the graph shown below

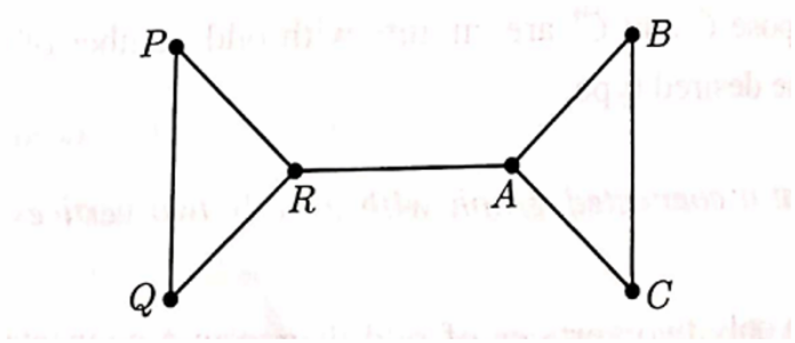



Sol^n:

Euler circuit of a graph

is $v_1 v_2 v_9 v_{10} v_2 v_{11} v_7 v_{10} v_{11} v_6 v_4 v_2 v_3$
$v_4 v_5 v_6 v_7 v_8 v_9 v_1$

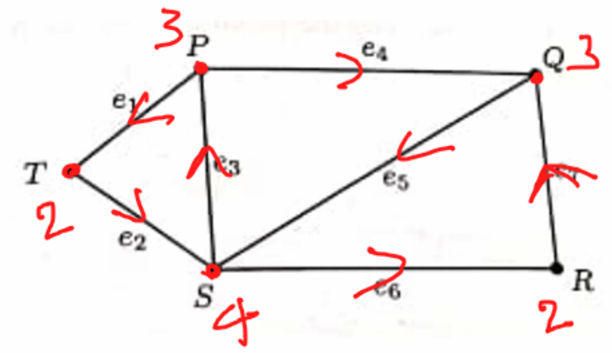
3. Show that the following graph does not contain an Euler circuit.




Sol: The vertices R & A having odd degrees

∴ The graph is not Euler circuit

4. Show that the following graph contains an Euler trail




Sol:

$P\, e_1\, T\, e_2\, S\, e_3\, P\, e_4\, Q\, e_5\, S\, e_6\, R\, e_7\, Q$

is an Euler trail

**(Dirac's Theorem) 1:** If in a simple connected graph with $n$ vertices (where $n \geq 3$) the sum of the degrees of every pair of non-adjacent vertices is greater than equal to $n$, then the graph is Hamiltonian.

**Theorem 2:** If in a simple connected graph with $n$ vertices (where $n \geq 3$) the degree of every vertex is greater than equal to $n/2$, then the graph is Hamiltonian.

Note:

Complete graph $K_n$, where $n \geq 3$, is a Hamiltonian graph.

Every simple k-regular graph with $2k - 1$ vertices is a Hamiltonian graph

① Draw the following

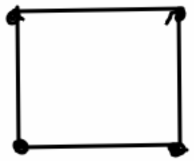
i) A graph has both an Euler circuit & a Hamilton Cycle

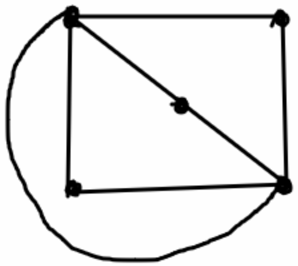
(ii) A graph which has an Euler Circuit but no Hamilton cycle

(iii) A graph which has Hamilton cycle but no Euler circuit

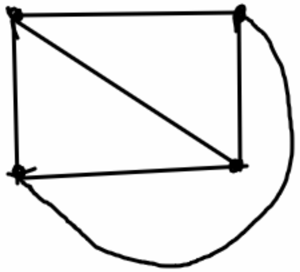
(iv) A graph which has niether Hamilton cycle nor Euler circuit

Find the gr ?

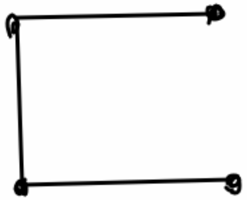
1


2


Euler circuit but not a Hamilton cycle.

3


Hamilton cycle but not Euler circuit

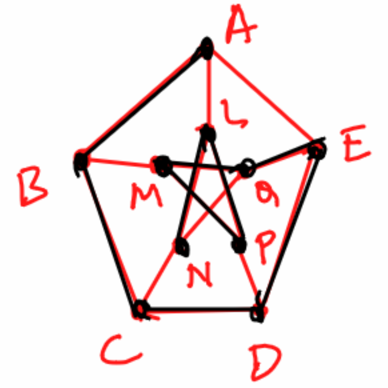
4


Neither Euler circuit nor Hamilton Cycle.

Mute Start Video Security Participants Chat Polls New Share Pause Share Annotate
Mouse Select Text Draw Stamp Spotlight Eraser Format Undo Redo Clear Save
You are screen sharing Stop Share
Who can see what you share here? Recording On

(1) Show that the peterson graph has no Hamilton cycle in it but it has Hamilton path.

Sol: The peterson graph is a 3-regular graph, 10 vertices & 15 edges.



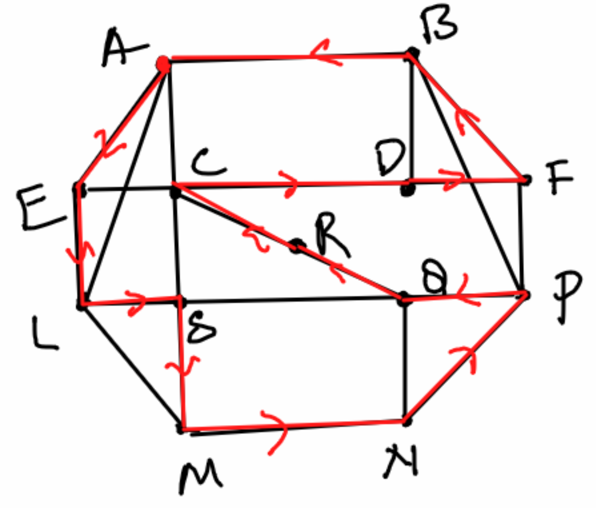

We cannot draw a Hamilton cycle without repeating a vertex.

Thus, Peterson graph does not contain a Hamilton cycle.

The edges AB, BC, CD, DE, EQ, QM, MP, PL, LN form a path & this path includes all the vertices.

$\therefore$ This path is called Hamilton path.

② S.T the graph shown below is a Hamilton graph.




Soln:

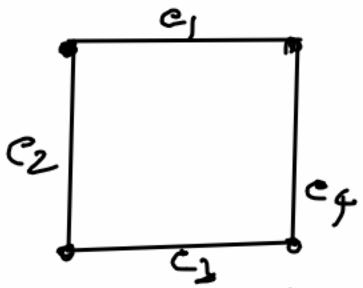
In the graph, there is a cycle AELSMNPQRCDFBA which contains all the vertices of the graph.

∴ This cycle is a Hamilton cycle.

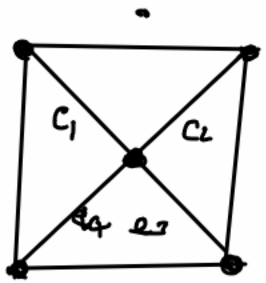
⇒ Hamilton path.

*On the other hand, a graph which cannot be represented by a plane drawing in which the edges meet only at the vertices is called a* ***non-planar graph****.*

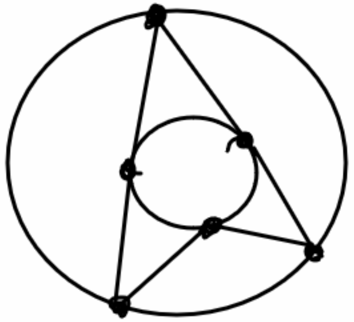
*In other words, a non-planar graph whose every possible plane drawing contains at least two edges which intersect each other at points other than vertices.*




Planar graph

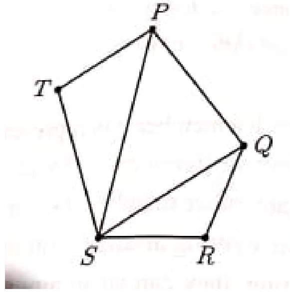



planar graph

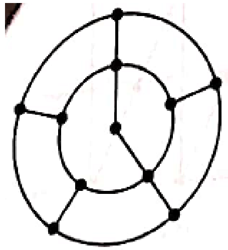


planar graph.

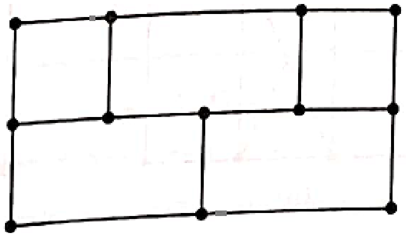
1. Identify five different Hamilton cycles in the following graph



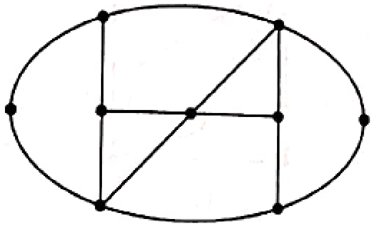

2. Show that the following graphs are Hamiltonian graphs.



(a)

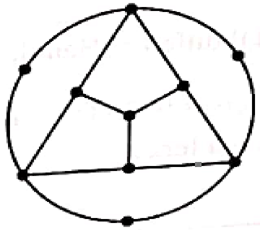


(b)

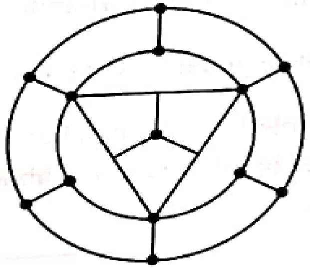


(c)

3. Show that the following graphs do not have Hamiltonian paths and Hamiltonian cycles.

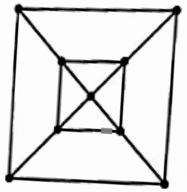


(a)

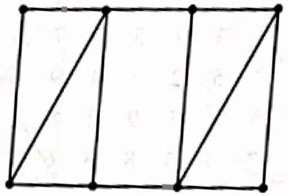


(b)

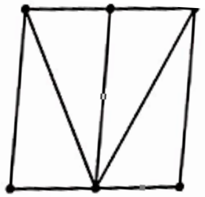
4. Show that the following graphs are Hamiltonian but not Eulerian.



(a)

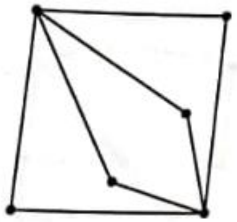


(b)

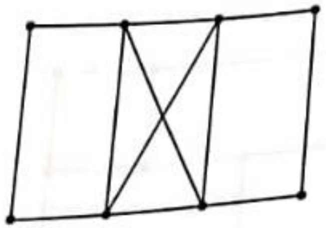


(c)

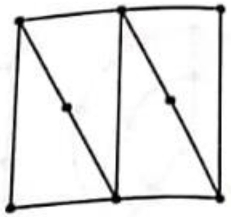
5. Which of the following are Euler graphs? Hamilton graphs?



(a)



(b)



(c)

# Planar graphs and coloring

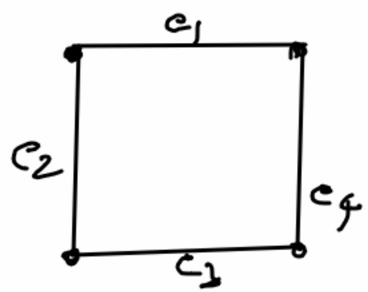
*In this chapter, we present an elementary discussion on planar and non-planar graphs. The so-called dual graphs associated with planar graphs are also defined and illustrated. The problem of graph coloring is briefly dealt-with.*

*A graph can be represented by more than one geometrical drawing. In some drawings representing graphs, the edges intersect (cross over) at points which are not vertices of the graph and in some others the edges meet only at the vertices.*

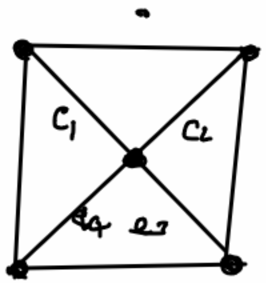
*A graph which can be represented by at least one plane drawing (drawing done on a plane surface) in which the edges meet only at the vertices is called a **planar graph**.*

*On the other hand, a graph which cannot be represented by a plane drawing in which the edges meet only at the vertices is called a* ***non-planar graph****.*

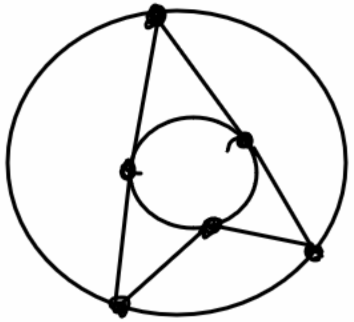
*In other words, a non-planar graph whose every possible plane drawing contains at least two edges which intersect each other at points other than vertices.*




Planar graph



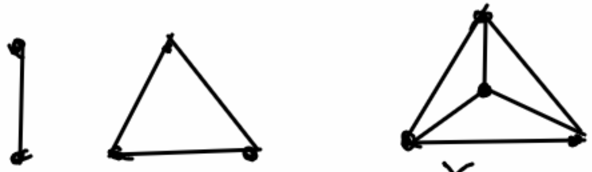

planar graph



planar graph.

Example

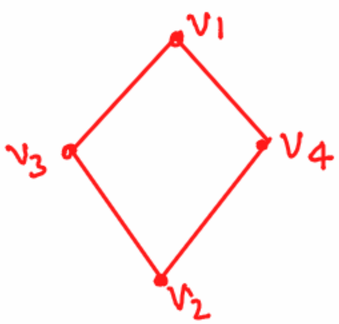
$K_2$, $K_3$ $K_4$ are planar graph?



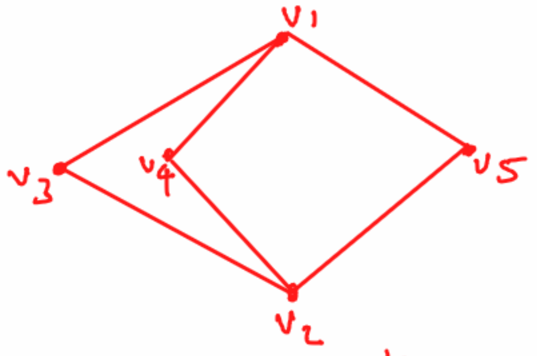
$K_2$ $K_3$ $K_4$

none of the edges meet at points other than vertices, ① S.T the Bipartite graphs $K_{2,2}$ & $K_{2,3}$ are planar graphs.

Sol^n: Given: $K_{2,2}$ and $K_{2,3}$.




(a) $K_{2,2}$




(b) $K_{2,3}$

In $K_{2,2}$ the vertex set made up of two bipartites

$V_1 = \{v_1, v_2\}$ & $V_2 = \{v_3, v_4\}$

The edges meets only at vertices

$\therefore$ $K_{2,2}$ is planar graph.

In $K_{2,3}$ the vertex set is made up of two bipartites $V_1 = \{v_1, v_2\}$ $V_2 = \{v_3, v_4, v_5\}$

The edges meet only at vertices

$\therefore$ $K_{2,3}$ is planar graph.



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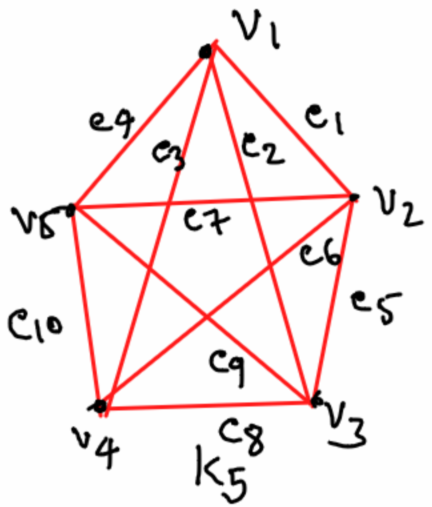
(2) P.T the complete graph $K_5$ is non planar graph. [kurtowskis first graph]

Sol$^n$: $K_5$ having 5 vertices & edge between every pair of vertices (10 edge).

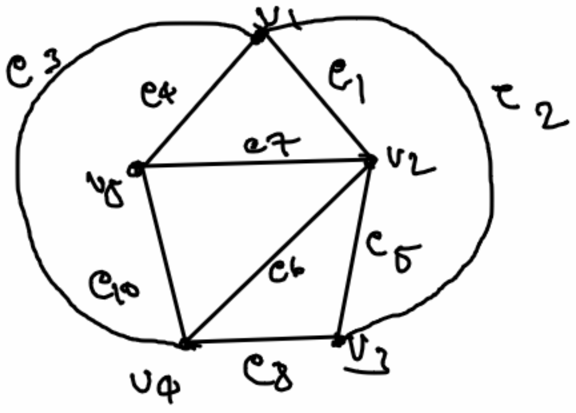
$V = \{v_1 \cdots v_5\}$ $E = \{e_1 \cdots e_{10}\}$




The edges $e_1, e_5, e_8, e_{10}, e_4$ form pentagonal cycle

remaining edges $e_2$ $e_3$ $e_6$ $e_7$ $e_9$ are all inside the cycle & intersect at points other than the vertices.

Let us try to draw $K_5$ in which edges meets only at vertices.




the edge $e_9 = \{v_3 v_5\}$ If we draw this edge outside the pentagon, it intersect $e_3$, & if we draw inside, it will intersect $e_6$.

$\therefore$ $K_5$ is non planar

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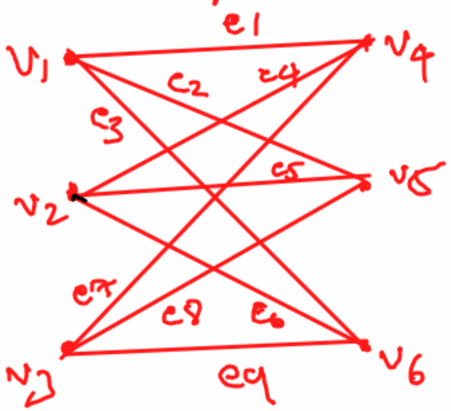
AND TECHNOLOGY

③ P.T complete bipartite graph $K_{3,3}$ is a non-planar graph. [Kuratowski's 2nd graph]

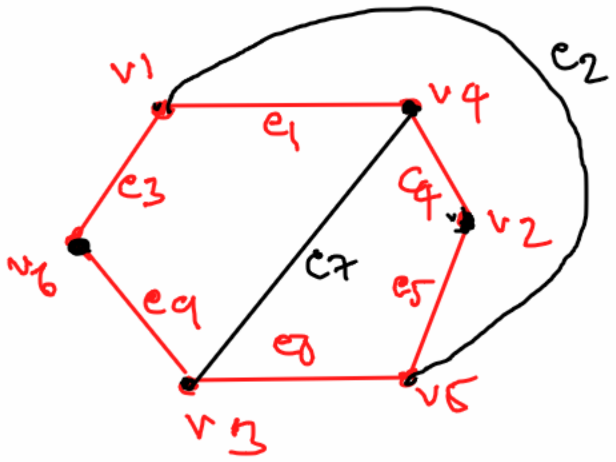
Sol:- $K_{33}$ - 6 vertices with vertex set. $V_1 = \{v_1 v_2 v_3\}$ $V_2 = \{v_4 v_5 v_6\}$

& 9 edges $E = \{e_1 \ldots e_9\}$.




The edges $e_1 = \{v_1 v_4\}$, $e_4 = \{v_4 v_2\}$ $e_5 = \{v_2 v_5\}$ $e_8 = \{v_5 v_3\}$ $e_9 = \{v_3 v_6\}$ $e_3 = \{v_6, v_1\}$.

forms an hexagonal.




The edge $e_6 = \{v_2, v_6\}$. If we draw from inside it will intersect $e_7$, outside it will intersect $e_2$.

$\therefore$ $K_{33}$ is not planar,

Chat

9

From 20BTRCS012 Dhairyajee... to Me:

e1, e4, e5,e8,e9

From 20BTRCS020 Ishwarya M to Me:

no doubts sir

Who can see your messages?

To: 20BTRCS007 Azay Pan... (Direct

Type message here...

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AND TECHNOLOGY

③ Suppose there are three houses & three utility points (ele, water, sewerage.) which are such that each utility point is joined to each house. Can the lines of joining be such that no two lines cross each other.

H.W

Chat

its similar to k33

yeah sir

From Deeksha Gandhi P to Me: (Direct Message)

it's just like a complete bipartite graph K3, 3 sir. V1 is houses and V2 is the points

Who can see your messages? Recording On

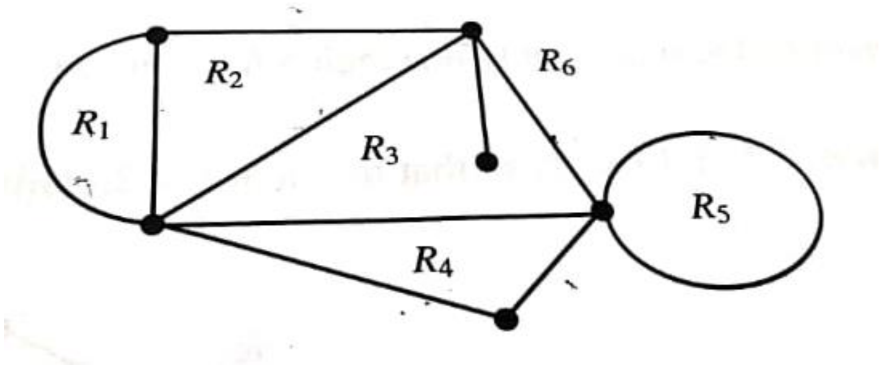
To: 20BTRCS007 Azay Pan... (Direct Message)

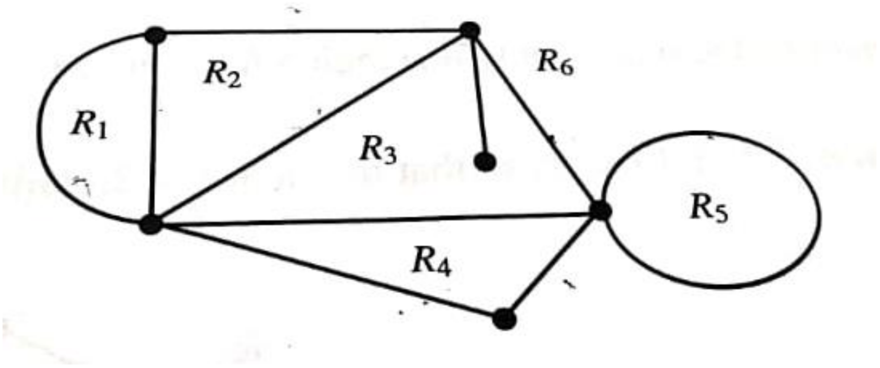
Type message here...

# Euler's formula

*If G is a planar graph, then G can be represented by a diagram in a plane in which the edges meet only at the vertices. Such a diagram divides the plane into a number of parts, called regions (or faces), of which exactly one part is unbounded. The number of edges that form the boundary of a region is called the degree of that region.*

*For example in the diagram of a planar graph shown in below Figure, the diagram divides the plane into 6 regions $R_1$, $R_2$, $R_3$, $R_4$, $R_5$, $R_6$. We observe that each of the regions R1 to R5 is bounded and the region R6 is unbounded. We say that the regions R1 to R5 are in the interior of the graph and the region R6 is in the exterior.*






$d(R_1) = 2$

$d(R_2) = 3$

$d(R_3) = 5$

$d(R_4) = 3$

$d(R_5) = 1$

$d(R_6) = 6$ and

$d(R_1) + d(R_2) + d(R_3) + d(R_4) + d(R_5) + d(R_6) = 20$

The twice the number of edges in the graph. This property is analogous to the handshaking property and true for all planar graphs.

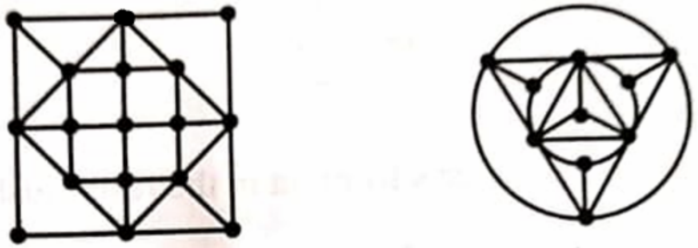
***Theorem:*** *A connected planar graph G with* $n$ *vertices and* $m$ *edges has exactly* $m - n + 2$ *regions in all of its diagram.*

That is $r = m - n + 2$ or $n - m + r = 2$

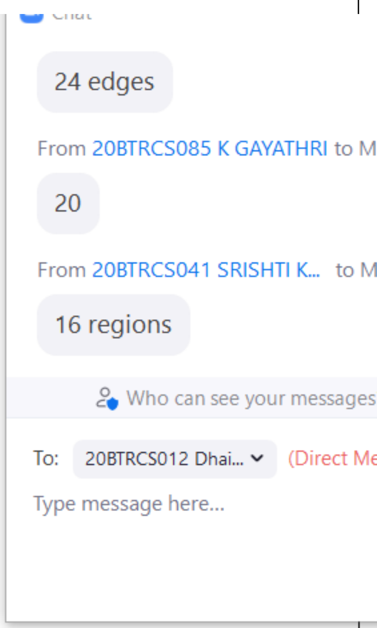
NOTE:

1. If G is a connected, simple planar graph with $n\ (\geq 3)$ vertices, $m (> 2)$ edges and $r$ regions then, $i. m \geq \frac{3}{2} r$, $ii. m \leq 3n - 6$
2. Kuratowski's first graph, is non planar
3. Kuratowski's second graph, is non planar
4. Every connected simple planar graph G contains a vertex of degree less than 6.

1. For the diagram of a graph shown below verify Euler's formula.



(a) (b)




Sol:

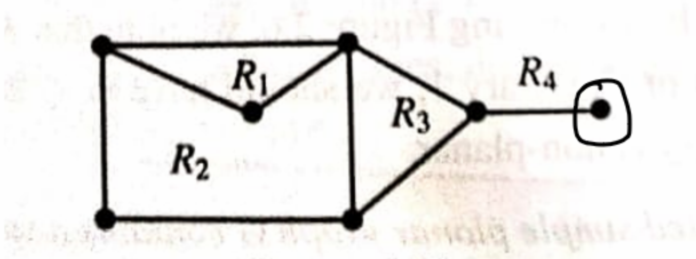
→ in graph (a) $n = 17$, $m = 34$ $r = 19$

$n - m + r = 2$ ∴ Euler's formula verified

→ in graph (b) = $n = 10$, $m = 24$ $r = 16$

$n - m + r = 2$ ∴ Euler's formula Verified

2. Verify Euler's formula for the planar graphs shown below.




Sol: In graph $n = 7$, $m = 9$, $r = 4$

Now $n - m + r = 2$ $\quad 7 - 9 + 4 = 2$

$\therefore$ Euler's formula verified.

Now $\deg(R_1) = 3$, $\deg(R_2) = 5$, $\deg(R_3) = 3$, $\deg(R_4) = 7$.

Sum of the degrees $= 2m$

$18 = 18$

3. A connected planar graph G has 9 vertices with degrees 2, 2, 3, 3, 3, 4, 5, 6, 6. Find the number of regions of G.

Sol$^n$: Given: $n = 9$ vertices, & degrees are

$$2 + 2 + 3 + 3 + 3 + 3 + 4 + 5 + 6 + 6 = 2m.$$

$$34 = 2m \Rightarrow m = 17$$

$\therefore$ By Euler's formula

$$n - m + r = 2$$

$$\Rightarrow r = 10$$

8/26/2021

# Detection of planarity

*Given a graph G, the determination of its planarity or otherwise is an important problem. This problem can be tackled by employing what is known as Elementary Reduction. The steps involved in this reduction are as explained below:*

**Elementary Reduction**

**Step 1.** Given a graph G, determine the set

$$A = \{G_1, G_2, \dots \dots \dots \dots .. G_k\}$$

Where $G_1, G_2, \dots \dots \dots \dots .. G_k$ are subgraphs of G every pair of which has exactly one vertex in common (-such subgraphs are called blocks).

**Step 2.** Remove all loops from all of $G_i s$.

**Step 3.** Remove all but one edge between every pair of vertices joined by multiple edges (if any).

**Step 4.** Eliminate all vertices of degree 2 by merging the edges incident on these vertices.

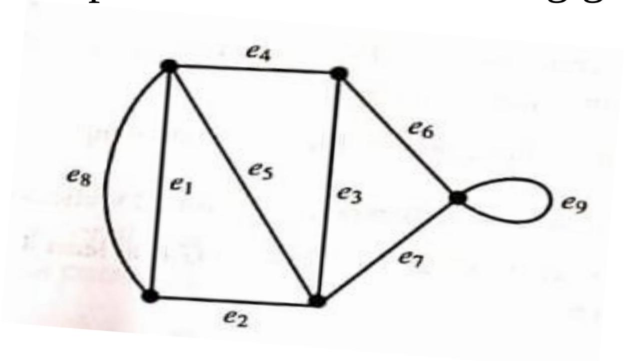
**Step 5.** Repeat the steps 3 and 4 repeatedly until each block $G_i$ is reduced to a new graph $H_i$ which will be one of the following.

(1) A graph with a single edge.

(2) A complete graph of order four.

If $H_i$ is in the first or second of the above possible forms, we conclude that $H_i$ is a planar graph. Consequently, each $G_i$ with which we started is planar and therefore $G$ is planar.

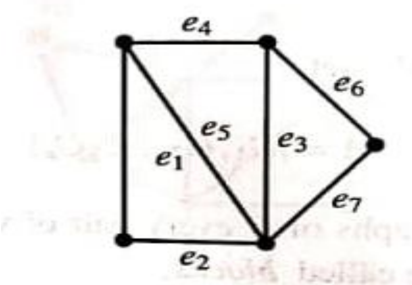
**Example 1:** Carry out the elementary reduction process for the following graph shown in below Figure.



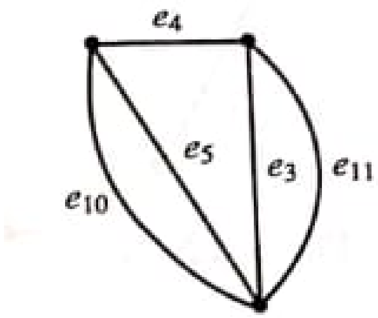

**Step 1 :** The given graph $G$ is a single block. Therefore, the set $A$ of step 1 contains only $G$.

**Step 2 :** we have to remove the loops. In the graph, there is one loop consisting of the edge $e_9$. Let us remove it.

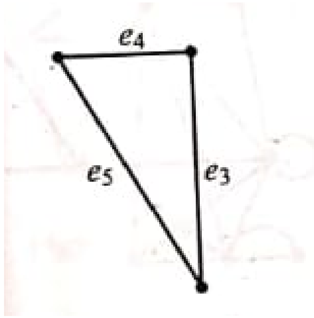
**Step 3:** we have to remove one of the two parallel edges from each vertex pair having such edges. In the given graph, $e_1$, $e_8$ are parallel edges. Let us remove $e_8$ from the graph. The graph left-out after the first three steps is as shown below:



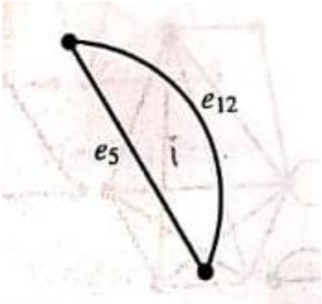
**Step 4:** we have to eliminate the vertices of degree 2 by merging the edges incident on these vertices. Thus, we merge (i) the edges, $e_1$ and $e_2$ into an edge, $e_{10}$(say) and (ii) the edges , $e_6$ and $e_7$ into an edge $e_{11}$ (say). The resulting graph will be shown below.



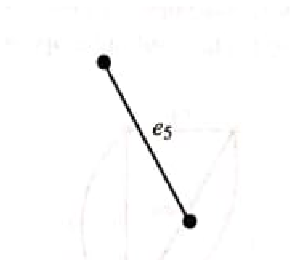

*As per Step 3, let us remove one of the parallel edges $e_5$ and $e_{10}$ and one of the parallel edges $e_3$ and $e_{11}$. The graph got be removing $e_{10}$ and $e_{11}$ will be as shown below:*



*As per step 4, we merge the edges $e_3$ and $e_4$ into an edge $e_{12}$ (say) to get the following graph.*

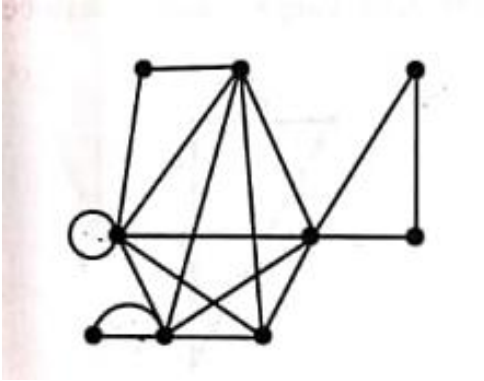



*As per step 3, we remove one of the two parallel edges, say $e_{12}$. Thus, we get the following graph.*



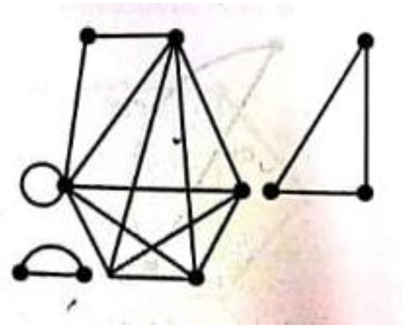

*This graph is final graph obtained by the process of elementary reduction applied to the given graph*

**Example 2: Check the planarity (or otherwise) of the following graph by the method of elementary reduction.**

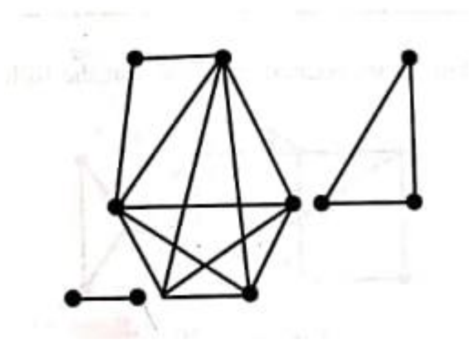


Solution: The elementary reduction of the given graph $G$ consists of the following steps:

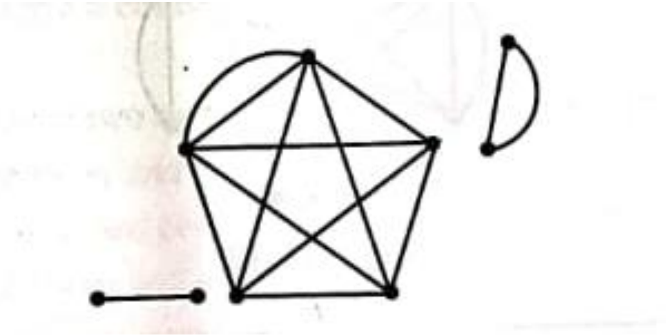
**Step 1:** splitting $G$ into blocks. This splitting is shown below:



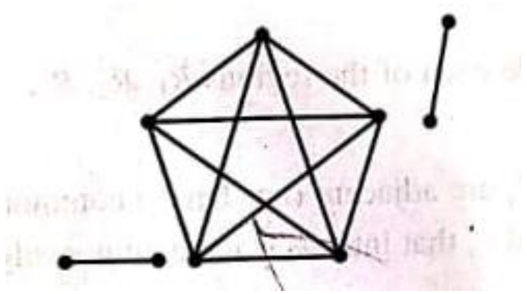
**Step 2:** Removing loops and eliminating multiple edges. The resulting graph is as shown below:



**Step 3:** Merging the edges incident on vertices of degree 2. The resulting graph is as shown below:



**Step 4:** Eliminating parallel edges. The resulting graph is as shown below:



The reduction is now complete. The final reduced graph (shown in figure) has three blocks, of which the first and the third (which are single edges) are obviously planar. The second one is the complete graph, $K_5$, which is non-planar. Thus, the given graph contains $K_5$ as a subgraph and is therefore non-planar.

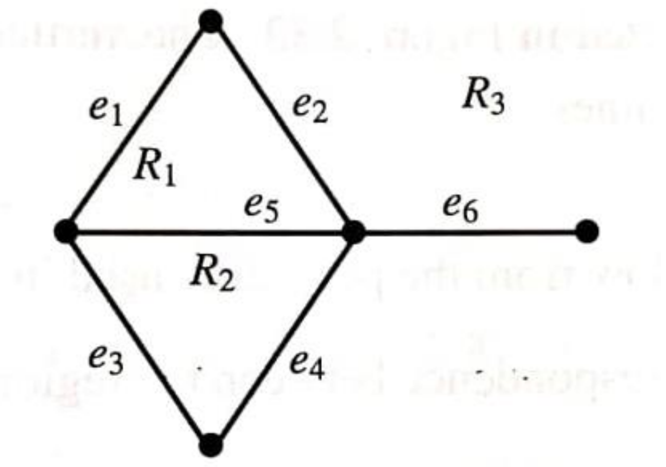
## Dual of a planar graph

- Consider a connected planar graph $G$ and a plane drawing thereof. Suppose $R_1, R_2, R_3, etc.$, are the regions (including the exterior region) in this drawing. Let us now construct a graph $G^*$ using the procedure given below:

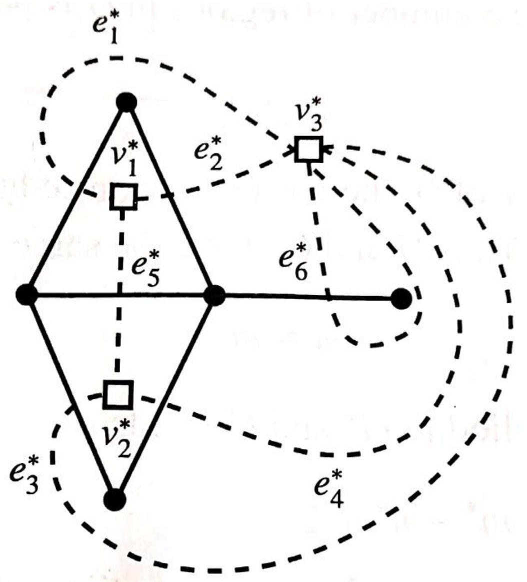
1. Choose one point inside each of the regions $R_1, R_2, R_3$ ,…… Denote these points by $v_1^*, v_2^*, v_3^*, \dots$, respectively.
2. If two regions $R_i$ and $R_j$ are adjacent (i.e. have a common edge, say $e_k$), draw a line $e_k^*$ joining the points $v_i^*$ and $v_j^*$ that intersects the common edge $e_k$ exactly once.

1. If there is more than one edge common to $R_i$ and $R_j$, draw one line $e_p^*$ between the points $v_i^*$ and $v_j^*$ for each common edge $e_p$, interesting $e_p$ exactly once.
2. For an edge $e_i$, lying entirely in one region, say $R_i$, draw a loop $e_i^*$ at the point $v_i^*$ intersecting $e_i$ exactly once.

The graph $G^*$ for which $v_i^*$ are vertices and $e_i^*$ are edges constructed as explained above is called the geometric dual or just the dual of $G$.



*For example, consider the connected planar graph $G$ a plane drawing of which is shown in Figure 2.31. we observe that the drawing divides the plane into regions $R_1, R_2$, and $R_3$, Of which $R_3$ is unbounded. We construct the dual $G^*$ of $G$ by using the afore- stated procedure. The step-by-step description of the construction is given below.*




We choose three points $v_1^*, v_2^*, v_3^*$ inside the regions $R_1, R_2, R_3$, respectively.

The regions $R_1, and\ R_2$ have a common edge $e_5$. We draw a line $e_5^*$ joining $v_1^*, and\ v_2^*$ that cross $e_5$ exactly once.

The regions $R_1\ and\ R_3$ have two common edges $e_1, e_2$. We draw two lines $e_1^*, e_2^*$ between $v_1^*, and\ v_3^*$, with $e_1^*$ crossing only $e_1$ and $e_2^*$ crossing only $e_2$.

The regions $R_2\ and\ R_3$ have two common edges $e_3, e_4$. We draw two lines $e_3^*, e_4^*$ between $v_2^*, and\ v_3^*$, with $e_3^*$ crossing only $e_3$ and $e_4^*$ crossing only $e_4$.

The edge $e_6$ is completely contained in $R_3$. We draw a loop $e_6^*$ at $v_3^*$ interesting $e_6$ exactly once.

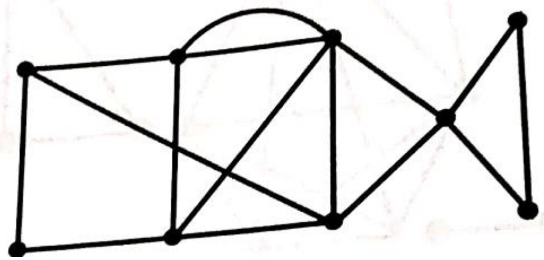
The construction of $G^*$ is now complete. For $G^*$, the points $v_1^*, v_2^*, v_3^*$, are the vertices and these lines $e_1^*, e_2^*, e_3^*, e_4^*, e_5^*, and\ e_6^*$ are the edges.

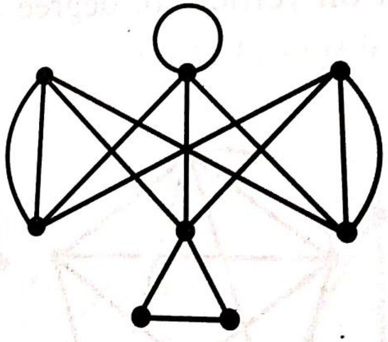
Some results

The results listed below follow from the procedure used in the construction of $G^*$:

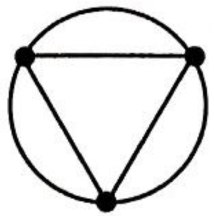
1. There is a one-to-one correspondence between the regions of $G$ and the vertices of $G^*$.
2. There is a one-to-one correspondence between the edges of $G$ and the edges of $G^*$. (if $e^*$is the edge of $G^*$that corresponds to the edge
3. The pendant edge in G yields a loop in $G^*$.
4. A loop in G yields a pendant edge in $G^*$.
5. Edges that are in series in G yield parallel edges in $G^*$.
6. Parallel edges in G yield edges in series in $G^*$
7. The number of edges which form a boundary of a Region $R_i$ in G is equal to the degree of the corresponding vertex $v_i^*$ in $G^*$, and vice-versa.
8. Like G, $G^*$is a connected planar graph.

1. Check the planarity of the following graph by elementary reduction

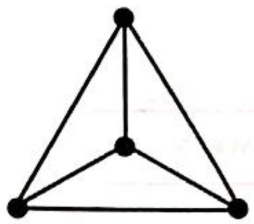




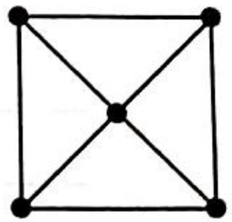
2. Construct the duals of the following planar graphs:



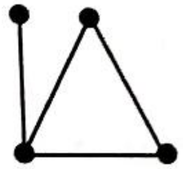
(a)



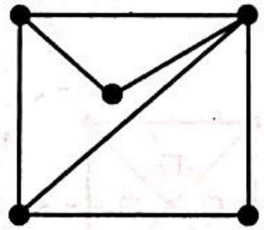
(b)



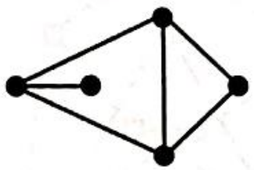
(c)



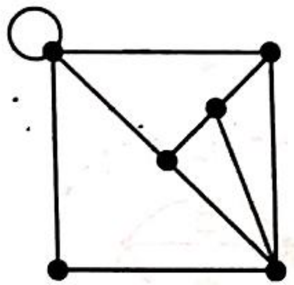
(d)



(e)



(f)



(g)

3. Verify that the two planar graphs shown in Figure 2.34 are the duals of each other.