JGi JAIN
DEEMED-TO-BE UNIVERSITY

Course: DISCRETE MATHEMATICS AND GRAPH THEORY
Subject Code: 21CIDS31

Course Objective: Objective of this course is to	
1.	Perform set operations and solve problems by logical reasoning to verify the correctness of the logical statements.
2.	Understand the concepts of relations and apply the properties to find the partially ordered sets and lattices.
3.	Analyze the techniques to solve recurrence relations.
4.	Analyze properties of graphs to study the Mathematical structures and techniques.

At the end of the course, Students will be able to:	
CO1	Discuss logical reasoning to verify the correctness of the logical statements and Perform set operations.
CO2	Illustrate the concepts of relations, partially ordered sets and lattices in data bases and data structures.
CO3	Employ generating function techniques to solve recurrence relations problems
CO4	Examine recurrence relations to solve problems involving an unknown sequence in engineering problems
CO5	Demonstrate the fundamental concepts in graph theory to learn network analysis.
CO6	Employ the concepts of graphs to understand Mathematical structures, trees, and shortest path techniques in computer applications.

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Pre-requisite: Sets, Relations and Functions

Module – I: Fundamentals of Logic (9 Hours)

Basic Connectives and Truth Tables, Logic Equivalence – The Laws of Logic, Logical Implication – Rules of Inference. Fundamentals of Logic contd. The Use of Quantifiers, Quantifiers.

Module – II: Principles of Counting and Inclusion-Exclusion (9 Hours)

Pigeonhole Principle, Relations and Digraphs, Equivalence relations, Partially ordered sets, Hasse diagram, lattices, Paths and related problems, transitive closure and Warshall's algorithm.
Principle of Inclusion Exclusion, Generalizations of the Principles, Derangements – Nothing is in its Right Place, Rook Polynomials.

Module – III: Generating Functions and Recurrence Relations (9 Hours)

Generating Functions, Definition & Examples, Calculational Techniques, Partitions of Integers, Exponential Generating Functions, and Summation Operator. Method of Generating Functions. Recurrence Relations: First - Order Linear Recurrence Relation, Second - Order Linear Homogeneous Recurrence Relations with Constant Coefficients, Non-Homogeneous Recurrence Relations.

Module – IV: Graph Theory (9 Hours)

Definitions and Examples, Sub graphs, Complements and Graph Isomorphism, Bipartite graphs, Planar graphs, Euler's formula. Vertex degree: Euler Trails and Circuits, Hamilton Paths and Cycles.Graph Colouring, Chromatic Polynomials.

Module-V: Trees (9 Hours) *Trees, Weighted Trees, Routed Trees, Sorting and Prefix Codes. Minimal Spanning Trees: Kruskal's and Prim's Algorithms, Dijkstra's Shortest Path Algorithm.*

Text Books:

1. *Ralph P Grimaldi, (2019), Discrete and Combinatorial Mathematics, 5th Edition, Pearson Education.*
2. *SC Gupta and VK Kapoor (2020) Fundamentals of mathematical statistics, 12th Edition, Sultan Chand & Sons Publication, New Delhi.*

References Books:

1. *Kenneth Rosen (2017), Discrete Mathematics and Its Applications with Combinatorics and Graph Theory (SIE),7th Edition, McGraw Hill Education.*
2. *Dr. S P Gupta, (2021) Statistical Methods, 46th edition, Sultan Chand & Sons publication.*
3. *Oscar Levin, (2016) Discrete Mathematics: An Open Introduction, 2nd edition Create space Independent Publication.*

Module – I: Fundamentals of Logic

- **Basic Connectives and Truth Tables**
- **Logic Equivalence – The Laws of Logic,**
- **Logical Implication – Rules of Inference.**
- **Fundamentals of Logic contd.**
- **The Use of Quantifiers, Quantifiers.**

Propositions- Proposition is a declarative statement which in a given context is either true or false but not both.

Propositions Examples-

- 7 + 4 = 10
- Apples are black.
- Narendra Modi is president of India.
- Two and two makes 5.
- 2016 will be the leap year.
- Delhi is in India.

All these statements are propositions. This is because they are either true or false but not both.

Truth Values

- **If proposition is True its value is '1'**
- **If proposition is False its value is '0'**

Types Of Propositions-

Types of Propositions

- Atomic Propositions
- Compound Propositions

Atomic Propositions-

Atomic propositions are those propositions that can not be divided further.

Small letters like p, q, r, s etc. are used to represent atomic propositions.

Examples-

p : Sun rises in the east.

q : Sun sets in the west.

r : Apples are red.

s : Grapes are green

Compound Propositions-

- Compound propositions are those propositions that are formed by combining one or more atomic propositions using connectives.
- In other words, compound propositions are those propositions that contain some connective.
- Capital letters like P, Q, R, S etc. are used to represent compound propositions.

Examples-

P : Sun rises in the east and Sun sets in the west.

Q : Apples are red and Grapes are green.

Statements That Are Not Propositions-

Following kinds of statements are not propositions-

1.Command
2.Question
3.Exclamation
4.Inconsistent
5.Predicate or Proposition Function

A predicate is **an expression of one or more variables defined on some specific domain**

Example:

Following statements are not propositions-

- Close the door. (Command)
- Do you speak French? (Question)
- What a beautiful picture! (Exclamation)
- I always tell lie. (Inconsistent)
- P(x) : x + 3 = 5 (Predicate)

PRACTICE PROBLEMS BASED ON PROPOSITIONS-

Identify which of the following statements are propositions-

1. France is a country.
2. 2020 will be a leap year.
3. Sun rises in the west.
4. P(x) : x + 6 = 7
5. P(5) : 5 + 6 = 2
6. Apples are oranges.
7. Grapes are black.
8. Two and two makes 4.
9. x > 10

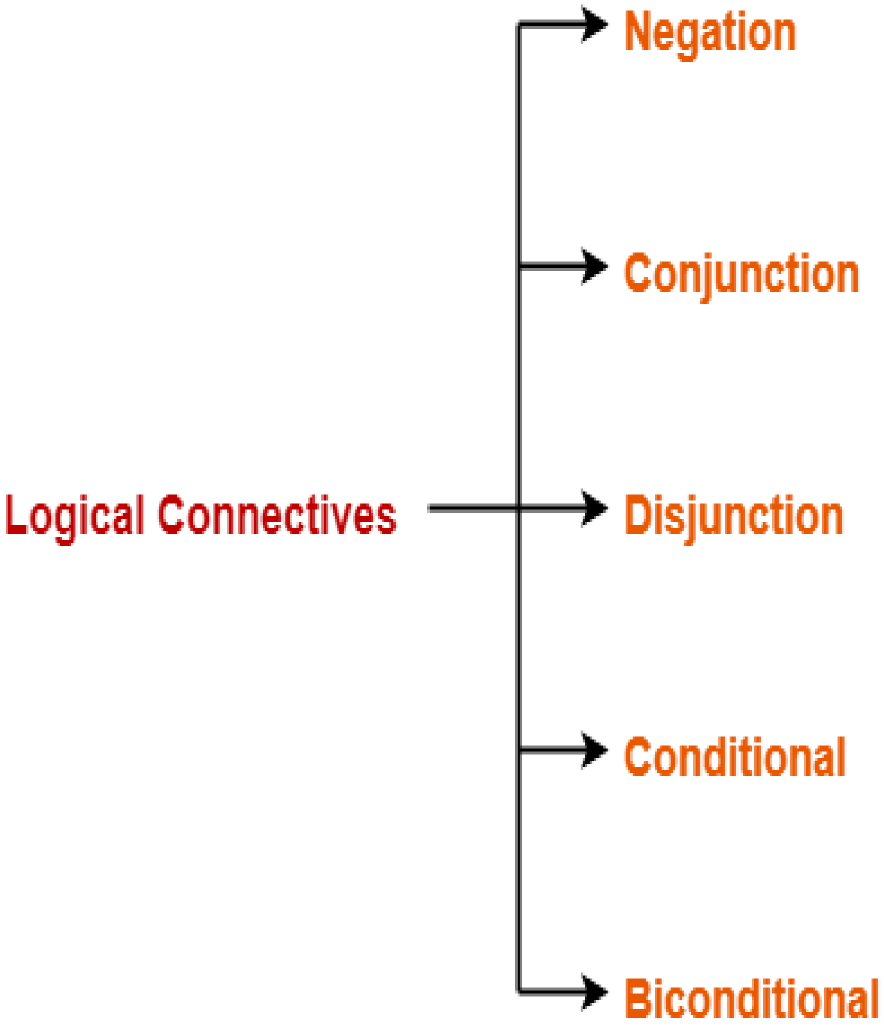
1. Proposition (True)
2. Proposition (True)
3. Proposition (False)
4. Not a proposition (Predicate)
5. Proposition (False)
6. Proposition (False)
7. Proposition (False)
8. Proposition (True)
9. Not a proposition (Predicate)

10. Open the door.
11. Are you tired?
12. What a bright sunny day!
13. Mumbai is in India.
14. I always tell truth.
15. I always tell lie.
16. Do not go there.
17. It will rain tomorrow.
18. Fan is rotating.

10. Not a proposition (Command)
11. Not a proposition (Question)
12. Not a proposition (Exclamation)
13. Proposition (True)
14. Proposition (True)
15. Not a proposition (Inconsistent)
16. Not a proposition (Command)
17. Proposition (Will be confirmed tomorrow whether true or false)
18. Proposition (True if fan is rotating otherwise false)

Logical Connectives-

Connectives are the operators that are used to combine one or more propositions.



Name of Connective	Connective Word	Symbol
Negation	Not	or ~ or ‘ or –
Conjunction	And	∧
Disjunction	Or	∨
Conditional	If-then	→
Biconditional	If and only if	↔

1. Negation-

If p is a proposition, then negation of p is a proposition which is-

- True when p is false
- False when p is true.

p	~p or
0	1
1	0

Example-

If p : It is raining outside.
Then, Negation of p is-
~p : It is not raining outside.

2. Conjunction-

If p and q are two propositions, then conjunction of p and q is a proposition which is-

- if both are true ----True
- else False

P	q	p ∧ q
0	0	0
0	1	0
1	0	0
1	1	1

Example- If p and q are two propositions where- p : 2 + 4 = 6
•q : It is raining outside.
Then, conjunction of p and q is-
p ∧ q : 2 + 4 = 6 and it is raining outside.
Then, conjunction of p and q is- p ∧ q : 2 + 4 = 6 and it is raining outside

3. Disjunction-

If p and q are two propositions, then disjunction of p and q is a proposition which is-

- If any one is true – True

p	q	p ∨ q
0	0	0
0	1	1
1	0	1
1	1	1

Example- If p and q are two propositions where-
•p : 2 + 4 = 6
•q : It is raining outside
Then, disjunction of p and q is-
p ∨ q : 2 + 4 = 6 or it is raining outside

4. Conditional- If p and q are two propositions, then-

- Proposition of the type "If p then q" is called a conditional or implication proposition.

Only in one case you get False

- 1st Is True 2nd is False--- False
- Else True

p	q	$p \to q$
0	0	1
0	1	1
1	0	0
1	1	1

Examples-

- If a = b and b = c then a = c.
- If I will go to Australia, then I will earn more money.

5. Biconditional- If p and q are two propositions, then-

- Proposition of the type "p if and only if q" is called a biconditional or bi-implication proposition.
- If Both Are Same ---- True

p	q	p q	q p	p q
0	0	1	1	1
0	1	1	0	0
1	0	0	1	0
1	1	1	1	1

Examples-

He goes to play a match if and only if it does not rain.

- Birds fly if and only if sky is clear.

Ex-1 let,

p : A circle is a conic,

q : $\sqrt{5}$ is a real number.

r : Exponential series is convergent.

Express the following compound propositions in words:

1. **p ^ (~q)** A circle is a conic and $\sqrt{5}$ is not a real number.
2. **(~p) ˅ q** A circle is not a conic or $\sqrt{5}$ is a real number.
3. **q → (~p)** If $\sqrt{5}$ is a real number, then a circle is not a conic.
4. **~p ↔ q** If a circle is not a conic then $\sqrt{5}$ is a real number and if $\sqrt{5}$ is a real number then a circle is not a conic

Ex-2: Construct the truth tables for the following compound propositions: **(i) p ^ (~q) (ii) (~p) ˅ q (iii) p → (~q)**

p	q	~p	~q	p ^ (~q)	(~p) ˅ q	p (~q)
0	0	1	1	0	1	1
0	1	1	0	0	1	1
1	0	0	1	1	0	1
1	1	0	0	0	1	0

Ex-3: Let p and q be primitive statements for which the conditional $p \rightarrow q$ is false. Determine the truth values of the following compound propositions:

(i) $p \wedge q$ (ii) $(\sim p) \vee q$ (iii) $q \rightarrow p$ (iv) $(\sim q) \rightarrow (\sim p)$

Solution:

Since $p \rightarrow q$ is given to be 0, **p has to be 1 and q has to be 0**. Consequently, $(\sim p)$ has to be 0 and $(\sim q)$ has to be 1, therefore:

i) Since p is 1 and q is 0

By conjunction: T T =T , else F

the truth value of $p \wedge q$ is 0 (False)

ii) Since $\sim p$ is 0 and q is 0, then

By Disjunction; F F = F

hence the truth value of $(\sim p) \vee q$ is 0.

iii) Since q is 0 , p is 1

Then by conditional, F T = T,

so the truth value of $q \rightarrow p$ is 1.

iv) Since $\sim q$ is 1, $\sim p$ is 0

By conditional, T F= F

the truth value of $(\sim q) \rightarrow (\sim p)$ is 0.

Conj:

$\wedge$ --- Both are T=T, Else F

Disjun:

$\vee$ -- Any one T=T, or Both False=F

Cond:

$\rightarrow$ --- T F – F else T

Bi Cond

$\leftrightarrow$ both same then T

Ex-4: Let p, q and r be propositions having truth values 0, 0 and 1 respectively. Find the truth values of the following compound propositions

(1) (p ˅ q) ˅ r (2) (p ^ q) ^ r (3) (p ^ q) → r

(4) p → (q ^ r) (5) p ^ (r → q) (6) p → (q → (~r))

Solution:

Since p, q and r having truth values 0, 0 and 1, ~r is 0

1) Since both p and q are 0,

By disjunction: "0 0 = 0"

=> (p ˅ q) is 0 Since r is 1,

By disjunction: 0 1 = 1

=> (p ˅ q) ˅ r is 1

2) Since both p and q are 0,

By Conjunction: "0 0= 0"

p ^ q is 0.

Since p ^ q is 0 and r is 1,

By Conjunction: 0 1= 0

Thus, the truth value of (p ^ q) ^ r is 0.

3) From (2) Since p ^ q is 0 and r is 1,

By Conditional: "1 0 = 0, else 1"

Hence (p ^ q) → r gives **"0 1 = 1" ,** (p ^ q) → r is 1.

Thus, the truth value of (p ^ q) → r is 1.

Conj:

Λ --- Both are T=T, Else F

Disjun:

V -- Any one T=T, or Both

False=F

Cond:

→ --- T F – F else T

Bi Cond

↔ both same then T

4) Since q is 0 and r is 1,
By conjunction: 1 1=1, else 0
Hence q ^ r gives **0 1 = 0**. , q ^ r is 0
Also, p is 0.
Therefore, p → (q ^ r) gives **0 0**
By conditional "1 0 = 0 else 1"
Thus, the truth value of p → (q ^ r) is 1.

5) Since r is 1 and q is 0
By conditional: "1 0 = 0", else 0
r → q gives "**1 0**" which is 0
Also, p is 0.
Hence, p ^ (r → q) gives 0 0
By conjunction: "0 0 = 0"
Thus, the truth value of p ^ (r → q) is 0.

6) Since r is 1, ~r is 0. Since q is 0,
q → (~r) gives 1 0
By conditional: 1 0 = 0
Therefore, q → (~r) is 0
Also, p is 0. Therefore, p → (q → (~r)) gives **0 0**
By conditional: 1 0 =0 else 1
Thus, the truth value of p → (q → (~r)) is 1.

Ex-5: Find the Possible truth values of p, q and r in the following cases:

(1) $p \rightarrow (q \vee r)$ is false

(2) $p \wedge (q \rightarrow r)$ is true

Sol

1) $p \rightarrow (q \vee r)$ is false

Means , $p \rightarrow (q \vee r)$ is 0

Which is only possible by **conditional when "1 0 = 0"**

i.e. p **is 1 and** $(q \vee r)$ is 0

⇒ $(q \vee r)$ is 0 when **by disjunction "0 0 = 0"**

⇒ **i.e. q is 0 and r is 0**

⇒ **hence the possible Truth values for** p, q and r is **1,0,0**

Sol

2) $p \wedge (q \rightarrow r)$ is true

(1) Means , $p \wedge (q \rightarrow r)$ **is 1**

Which is only possible by **conjunction when both are T = T**

i.e. "1 ∧ 1 = 1" ,

i.e. p is 1, and (q →r) is **1**

⇒ And **(q →r)** is **1** when **by conditional "0 → 1 = 1", "1 → 1=1", "0 → 0=1"**

⇒ **i.e. q is 0,1,0 and r is 1,1,0**

⇒ **hence the possible Truth values for** p, q and r is

p	q	r
1	0	1
1	1	1
1	0	0

Conj:

∧ --- Both are T=T, Else F

Disjun:

∨ -- Any one T=T, or Both

False=F

Cond:

→ --- T F – F , else T

Bi Cond

↔ both same then T

Ex-6: Construct the truth tables for the following compound propositions:

(i) $(p \vee q) \wedge r$ (ii) $p \vee (q \wedge r)$

p	**q**	**r**	**p ∨ q**	**(p ∨ q) ∧ r**	**q ∧ r**	**p ∨ (q ∧ r)**
0	0	0	0	0	0	0
0	0	1	0	0	0	0
0	1	0	1	0	0	0
0	1	1	1	1	1	1
1	0	0	1	0	0	1
1	0	1	1	1	0	1
1	1	0	1	0	0	1
1	1	1	1	1	1	1

Ex-7: Construct the truth tables for the following compound propositions: (i) $(p \wedge q)$ (ii) $q \wedge ((\sim r) \rightarrow p)$

p	q	r	~r	$(p \wedge q)$	$(p \wedge q) \rightarrow (\sim r)$	$(\sim r) \rightarrow p$	$q \wedge ((\sim r) \rightarrow p)$
0	0	0	1	0	1	0	0
0	0	1	0	0	1	1	0
0	1	0	1	0	1	0	0
0	1	1	0	0	1	1	1
1	0	0	1	0	1	1	0
1	0	1	0	0	1	1	0
1	1	0	1	1	1	1	1
1	1	1	0	1	0	1	1

Ex-7 If a proposition q has the truth value 1, determine all truth value assignments for the primitive propositions p, r and s for which the truth value of the following compound proposition is 1.

$$[q \rightarrow \{(\sim p \vee r) \wedge \sim s\}] \wedge \{\sim s \rightarrow (\sim r \wedge q)\}$$

Sol:

The given compound proposition is of the form $u \wedge v$, where

$u \equiv q \rightarrow \{(\sim p \vee r) \wedge \sim s\}$ and $v \equiv \sim s \rightarrow (\sim r \wedge q)$

Since the truth value of this compound proposition is 1,

the truth value of each of u and v is 1.

Since q has the truth value 1, and u has the truth value 1 it follows that the truth value of $(\sim p \vee r) \wedge \sim s$ must also be 1.

Consequently, $\sim p \vee r$ has the truth value 1 and $\sim s$ has the truth value 1.

Consequently, s hast the truth value 0.

Since $\sim s$ has the truth value 1 and v has the truth value 1,

the truth value of $(\sim r \wedge q)$ must be 1.

Since q has the truth value 1, it follows that $\sim r$ must also have the truth value 1;

that is r must have the truth value 0.

Since $(\sim p \vee r)$ has the truth value 1 and r has the truth value 0, it follows that $\sim p$ must have truth value 1; that is, p must have the truth value 0.

Thus, all of p, r, s must have the truth value 0.

Ex-8: Indicate how many rows are needed in the truth table for the compound proposition.

$(p \vee \sim q) \leftrightarrow \{(\sim r \wedge s) \rightarrow t\}$, Find the truth value of this proposition if p and r are true and q, s, t are false.

Sol : The given compound proposition contains five primitives (components) p, q, r, s, t. Therefore, the number of possible combinations of the truth values of these components which we have to consider $2^5 = 32$. Hence, 32 rows are needed in the truth table for the given compound proposition.

Next, suppose that p and r are true and q, s, r are false. Then ~q is true and ~r is false. Since p is true and ~q is true, $p \vee \sim q$ is true. On the other hand, since ~r is false and s is false, $\sim r \wedge s$ is false. Also, r is false. Hence $(\sim r \wedge s) \rightarrow t$ is true.

Since $(p \vee \sim q)$ is true and $(\sim r \wedge s) \rightarrow t$ is true, it follows that the truth value of the given proposition

$(p \vee \sim q) \leftrightarrow \{(\sim r \wedge s) \rightarrow t\}$ is 1.

Tautology and Contradiction

An expression involving logical variables that is true in all cases is a tautology. **We use the number 1 to symbolize a tautology.**

An expression involving logical variables that is false in all cases is a contradiction. **We use the number 0 to symbolize a contradiction.**

Ex-1: Prove that for any proposition p, the compound proposition $p \vee \neg p$ is a tautology and the compound proposition $p \wedge \neg p$ is contradiction

p	¬p	p ∨ ¬p	p ∧ ¬p
0	1	1	0
1	0	1	0

Conj: ∧ --- Both are T=T, Else F

Disjunct: ∨ --- Any one T=T, or Both False=F

Ex-2: Show that, for any propositions p and q, the compound proposition $p \rightarrow (p \vee q)$ is a tautology and the compound proposition $p \wedge (\neg p \wedge q)$ is a contradiction.

p	q	p ∨ q	p (p ∨q)	¬p	¬p ∧ q	p ∧ (¬p ∧ q)
0	0	0	**1**	1	0	**0**
0	1	1	**1**	1	1	**0**
1	0	1	**1**	0	0	**0**
1	1	1	**1**	0	0	**0**

Ex-3: Show that, the truth values of following compound proposition are independent of the truth values of their components: 1) $\{p \wedge (p \rightarrow q)\} \rightarrow q$ 2) $(p \rightarrow q) \leftrightarrow (\neg p \vee q)$

p	q	$p \rightarrow q$	$r = p \wedge (p \rightarrow q)$	$r \rightarrow q$
0	0	1	0	1
0	1	1	0	1
1	0	0	0	1
1	1	1	1	1

p	q	$u = p \rightarrow q$	$\neg p$	$r = \neg p \vee q$	$u \leftrightarrow r$
0	0	1	1	1	1
0	1	1	1	1	1
1	0	0	0	0	1
1	1	1	0	1	1

Ex-4: Prove that, for any proposition p, q, r the compound proposition $\{p \rightarrow (q \rightarrow r)\} \rightarrow \{(p \rightarrow q) \rightarrow (p \rightarrow r)\}$ is a tautology.

p	q	r	$p \rightarrow q$	$q \rightarrow r$	$p \rightarrow r$	$p \rightarrow (q \rightarrow r)$	$(p \rightarrow q) \rightarrow (p \rightarrow r)$	$p \rightarrow (q \rightarrow r)\} \rightarrow \{(p \rightarrow q) \rightarrow (p \rightarrow r)\}$
0	0	0	1	1	1	1	1	1
0	0	1	1	1	1	1	1	1
0	1	0	1	0	1	1	1	1
0	1	1	1	1	1	1	1	1
1	0	0	0	1	0	1	1	1
1	0	1	0	1	1	1	1	1
1	1	0	1	0	0	0	0	1
1	1	1	1	1	1	1	1	1

Cond:

$\rightarrow$ --- T F – F , else T

Homework problems:

1. **Prove that, following are tautology.**

i) $p \vee [(\neg p \wedge q)]$ ii) $(p \vee q) \vee \neg p$

2. Find the Possible truth values of p, q, r, s, t for which following are contradiction

i) $[(p \vee q) \wedge r] \rightarrow (s \vee t)$

Logical Equivalence and The Law of Logics

Definition : Two Propositions u and *v* are said to be **logically equivalent** or simply **equivalent** whenever u and v have same truth values, or equivalently, the biconditional **$u \leftrightarrow v$ is a tautology.**

Then we write $u \Leftrightarrow v$. Here symbol $\Leftrightarrow$ stands for "logically equivalent to"

When the proposition are not logically equivalent we write u<≠>v.

logically equivalent propositions are treated as identical propositions

Ex:1 let x be a specific positive integer. Consider the following propositions:

p: x is an odd integer **q: x is not divisible by 2**

Are p and q logically equivalent?

Sol: we note that p, q have the same truth values. As p and q are logically equivalent; i.e. $p \Leftrightarrow q$

Ex:2 For any two proposition p, q, prove that $(p \rightarrow q) \Leftrightarrow (\neg p) \vee q$

Observe that $p \rightarrow q$ and $(\neg p) \vee q$ have same truth values for all possible truth values of p, q. Therefore $p \rightarrow q \Leftrightarrow (\neg p) \vee q$

p	q	$p \rightarrow q$	$\neg p$	$(\neg p) \vee q$
0	0	1	1	1
0	1	1	1	1
1	0	0	0	0
1	1	1	0	1

<u>Ex:3</u> Prove that for any three propositions p, q and r: $[(p \vee q) \rightarrow r] \Leftrightarrow [(p \rightarrow r) \wedge (q \rightarrow r)]$

p	q	r	pVq	(pVq)→r	p →r	q →r	(p →r) ∧(q →r)
0	0	0	0	1	1	1	1
0	0	1	0	1	1	1	1
0	1	0	1	0	1	0	0
0	1	1	1	1	1	1	1
1	0	0	1	0	0	1	0
1	0	1	1	1	1	1	1
1	1	0	1	0	0	0	0
1	1	1	1	1	1	1	1

Properties of Logical Equivalence.

1. Law of double negation

(i) $\sim(\sim p) \equiv p$

2. Idempotent Laws

(i) $p \vee p \equiv p$

(ii) $p \wedge p \equiv p$

3. Identity Laws

(i) $p \vee F \equiv p$

(ii) $p \wedge T \equiv p$

4. Inverse Laws

(i) $p \vee \sim p \equiv T$

(ii) $p \wedge \sim p \equiv F$

5. Domination Laws

(i) $p \vee T \equiv T$

(ii) $p \wedge F \equiv F$

6. Commutative Laws

(i) $p \vee q \equiv q \vee p$

(ii) $p \wedge q \equiv q \wedge p$.

7. Absorption Laws

(i) $p \vee (p \wedge q) \equiv p$

(ii) $p \wedge (p \vee q) \equiv p$

8. de Morgan's Laws

(i) $\neg (p \vee q) \equiv \neg p \wedge \neg q$

(ii) $\neg (p \wedge q) \equiv \neg p \vee \neg q$

9. Associative Laws

(i) $p \vee (q \vee r) \equiv (p \vee q) \vee r$

(ii) $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$.

10. Distributive Laws

(i) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

(ii) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

11. Equivalence of conditional :

$(p \rightarrow q) \Leftrightarrow (\neg p \vee q)$

$\neg[p \rightarrow q] \Leftrightarrow p \wedge \neg q$

Try

$\neg p \wedge (\neg p \vee \neg q)$

The laws of double negation and idempotent law and commutative laws are trivially true, other laws can be verified with the aid of truth tables

Proof of 8. de Morgan's Laws

(i) $\neg(p \vee q) \equiv \neg p \wedge \neg q$

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
0	0	0	1	1	1	1
0	1	1	0	0	0	0
1	0	1	0	0	1	0
1	1	1	0	0	0	0

Proof of 10. Distributive Laws

(i) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
0	0	0	0	0	0	0	0
0	0	1	0	0	0	1	0
0	1	0	0	0	1	0	0
0	1	1	1	1	1	1	1
1	0	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	1	0	0	1	1	1	1
1	1	1	1	1	1	1	1

Remarks:

Logically equivalence propositions are treated as identical proposition. In view of laws of indicated above we have following results

1. $\neg (p \vee q) \equiv \neg p \wedge \neg q$
2. $\neg (p \wedge q) \equiv \neg p \vee \neg q$
3. $\neg (p \longrightarrow q) \equiv (p \wedge \neg q)$
4. $(p \longrightarrow q) \equiv \neg \neg (p \longrightarrow q)$
   $\equiv \neg (p \wedge \neg q)$
   $\equiv \neg p \vee q$

Table for Negation of compound proposition

Proposition	Negation
$\neg p$	p
$p \wedge q$	$\neg p \vee \neg q$
$p \vee q$	$\neg p \wedge \neg q$
$p \longrightarrow q$	$p \wedge \neg q$

Ex:1 **let x be specified number. Write down the negation of the following conditional**

" if x is an integer, then x is a rational number."

Sol: p: x is an integer;
q: x is a rational number
the given condition is $p \longrightarrow q$ where
Hence according to the result
$\neg (p \longrightarrow q) \equiv (p \wedge \neg q)$
Negation of given statement reads
x is an integer and x is not a rational number."

<u>Ex:2</u> **let x be specified number. Write down the negation of the following conditional**

" if x is not a real number, then it is not a rational number and not an irrational number."

Sol: p: x is not a real number

q: x is not a rational number

r : x is not a irrational number

The given proposition reads: $p \longrightarrow (q \wedge r)$

Therefore, negation of the proposition is

$\neg [p \longrightarrow (q \wedge r)]$	$\equiv \neg [\neg p \vee (q \wedge r)]$		(Equivalence of conditional) $p \longrightarrow q \equiv (\neg p \vee q)$
	$\equiv$	$[\neg\neg p \wedge \neg(q \wedge r)$	(De Morgan's law)
	$\equiv$	$[p \wedge (\neg q \vee \neg r)$	(Double negation and De Morgan's law)
	$\equiv$	$[p \wedge \neg q) \vee (P \wedge \neg r)$	(Distributive)

Hence the negation of the given statement is:

"x is not a real number but a rational number or x is not a real number but it is an irrational number"

Ex-1: Prove following logical equivalence without using Truth table.

1. $p \vee [\, p \wedge (p \vee q)] \Leftrightarrow p$
2. $[p \vee q \vee (\neg\, p \wedge \neg q \wedge r)] \Leftrightarrow (p \vee q \vee r)$

Solution:

1. $p \vee [\, p \wedge (p \vee q)] \Leftrightarrow p$

$\Rightarrow p \vee [\, p \wedge (p \vee q)] \equiv p \vee [(p \wedge p) \vee (p \wedge q)]$ -------- Distributive law

$\Rightarrow\ p \vee [p \vee (p \wedge q)]$ -------- Idempotent Law

$\Rightarrow\ p \vee p$ -------- Absorption Law

$\Rightarrow p$ -------- Idempotent Law

2. $[p \vee q \vee (\neg\, p \wedge \neg q \wedge r)] \quad \Leftrightarrow \quad (p \vee q \vee r)$

$\Rightarrow [p \vee q \vee (\neg\, p \wedge \neg q \wedge r)] \quad \equiv \quad \{p \vee q \vee [\neg\,(p \vee q) \wedge r\}$ de Morgan's Laws

$\Rightarrow \quad \equiv \quad \{(p \vee q) \vee [\neg\,(p \vee q) \wedge r\}$ Associative Laws

$\Rightarrow \quad \equiv \quad [(p \vee q) \vee \neg\,(p \vee q)] \wedge [(p \vee q) \vee r]$ Distributive law

$\Rightarrow \quad \equiv \quad [T \wedge [(p \vee q) \vee r]$

$\Rightarrow \quad \equiv \quad [(p \vee q) \vee r]$

Ex-2: Prove following logical equivalence without using Truth table.

1. $[(p \vee q) \wedge (p \vee \neg q)] \vee q \Leftrightarrow p \vee q$

Solution: $[(p \vee q) \wedge (p \vee \neg q)] \vee q$	$\equiv [p \vee (q \wedge p) \vee (q \wedge \neg q)] \vee q$	Distributive
$\Rightarrow$	$\equiv [p \vee (q \wedge \neg q)] \vee q$	Commutative
$\Rightarrow$	$\equiv [p \vee (F)] \vee q$	inverse law
$\Rightarrow$	$\equiv p \vee q$	Absorption Law and identity law (pVF)=p

Ex-3: Prove by using laws of logic. $(p \rightarrow q) \wedge [\neg q \wedge (r \vee \neg q)] \Leftrightarrow \neg(q \vee p)$

Solution: $(p \rightarrow q) \wedge [\neg q \wedge (r \vee \neg q)] \Leftrightarrow (p \rightarrow q) \wedge [\neg q \wedge (\neg q \vee r)$	(Commutative law)
$\Leftrightarrow (\neg p \vee q) \wedge \neg q$	(Absorption law)
$\Leftrightarrow \neg q \wedge (\neg p \vee q)$	(Commutative law)
$\Leftrightarrow (\neg q \wedge \neg p) \vee (\neg q \wedge q)$	(Distributive law)
$\Leftrightarrow (\neg q \wedge \neg p) \vee F$	($\because u \wedge \neg u \Leftrightarrow F$)
$\Leftrightarrow (\neg q \wedge \neg p)$	($\because u \vee F \Leftrightarrow u$)
$\Leftrightarrow \neg(q \vee p)$	(De Morgan's law)

Ex-4: Prove without using Truth table. $[\sim p \wedge (\sim q \wedge r)] \vee [(q \wedge r) \vee (p \wedge r)] \Leftrightarrow r$

Solution $[\sim p \wedge (\sim q \wedge r)] \vee [(q \wedge r) \vee (p \wedge r)]$	
$\Leftrightarrow [(\sim p \wedge \sim q) \wedge r)] \vee [(r \wedge q) \vee (r \wedge p)]$	(commutative)
$\Leftrightarrow [(\sim p \wedge \sim q) \wedge r)] \vee [(r \wedge (q \vee p)]$	(Distributive)
$\Leftrightarrow [(\sim (p \vee q) \wedge r)] \vee [(r \wedge (p \vee q)]$	(De Morgan's and commutative)
$\Leftrightarrow [r \wedge \sim (p \vee q)] \vee [r \wedge (p \vee q)]$	(commutative)
$\Leftrightarrow [r \wedge \{\sim (p \vee q) \vee (p \vee q)\}]$	(Distributive)
$\Leftrightarrow r \wedge T$	($\because \sim u \vee u \Leftrightarrow T$.
	Here u : $(p \vee q) \Leftrightarrow r$ ($\because v \wedge T \Leftrightarrow v$)

Ex-5: Prove that $(p \lor q) \land ((p \land (p \land q)) \Leftrightarrow (p \land q)$, by using rules of Logic.

<u>Solution</u> LHS $\Leftrightarrow (p \lor q) \land ((p \land p) \land q)$	(Associative rule)
$\Leftrightarrow (p \lor q) \land (p \land q)$	(Idempotent law)
$\Leftrightarrow (p \land q) \land (p \lor q)$	(commutative law)
$\Leftrightarrow p \land [q \land (p \lor q)]$	(Associative law)
$\Leftrightarrow p \land [q \land (q \lor p)]$	(commutative law)
$\Leftrightarrow p \land q$	(Absorption law)

Ex-6: Prove the following using laws of logic : $p \rightarrow (q \rightarrow r) \Leftrightarrow (p \land q) \rightarrow r$

<u>Solution</u> $p \rightarrow (q \rightarrow r) \Leftrightarrow p \rightarrow (\neg q \lor r)$	(Equivalence of Conditional)
$\Leftrightarrow \neg p \lor (\neg q \lor r)$	(Equivalence of Conditional)
$\Leftrightarrow (\neg p \lor \neg q) \lor r$	(Associative law)
$\Leftrightarrow \neg(p \land q) \lor r$	(De Morgan's law)
$\Leftrightarrow (p \land q) \rightarrow r$	(Equivalence of Conditional

Ex-7: Simplify $\neg [\neg\{(p \lor q) \land r\} \lor \neg q]$

<u>Solution</u> $\neg [\neg\{(p \lor q) \land r\} \lor \neg q]$	$\equiv [\neg\neg\{(p \lor q) \land r\} \lor \neg\neg q]$	using Dem organ Law
	$\equiv \{(p \lor q) \land r)\} \land q$,	using the Law of double negation
	$\equiv \{q \land (p \lor q)\} \land r$,	using Associative law
	$\equiv \{q \land (q \lor p)\} \land r$,	Commutative law
	$\equiv q \land r$,	using Absorption law

Ex-8: Prove that $(\neg p \wedge q) \vee \neg (p \vee q) \Leftrightarrow \neg p$.

Solution : $(\neg p \wedge q) \vee \neg(p \vee q) \Leftrightarrow (\neg p \wedge q) \vee (\neg p \wedge \neg q)$ (DeMorgan0 slaw)

$\Leftrightarrow \neg p \wedge (q \vee \neg q)$ (Distributive law)

$\Leftrightarrow \neg p \wedge T0$ (Inverse law)

$\Leftrightarrow \neg p$ (Identity law)

Ex-9: Prove that $[(\neg p \vee \neg q) \rightarrow (p \wedge q \wedge r) \Leftrightarrow p \wedge q$

Solution: $(\neg p \vee \neg q) \rightarrow (p \wedge q \wedge r) \Leftrightarrow \neg(\neg p \vee \neg q) \vee (p \wedge q \wedge r)$, (Equivalence of conditional)

$\Leftrightarrow (\neg\neg p \vee \neg\neg q) \vee (p \wedge q \wedge r)$, (De Morgans law)

$\Leftrightarrow (p \wedge q) \vee [(p \wedge q) \wedge r]$, Double Negation and Associative Law

$\Leftrightarrow p \wedge q$, by Absorption Law

Ex-10: Establish the following logical equivalence. $(p \vee q) \vee (\neg p \wedge \neg q \wedge r) \Leftrightarrow (p \vee q \vee r)$

Solution:

$(p \vee q) \vee (\neg p \wedge \neg q \wedge r) \Leftrightarrow (p \vee q) \vee (\neg p \wedge \neg q) \wedge r$, by Associative Law

$\Leftrightarrow (p \vee q) \vee [\neg(p \vee q) \wedge r]$ by De Morgan Law.

$\Leftrightarrow [(p \vee q) \vee \neg(p \vee q)] \wedge [(p \vee q) \vee r]$ (Distributive)

$\Leftrightarrow T0 \wedge [(p \vee q) \vee r]$ ($\because u \vee \neg u \Leftrightarrow T$, Here, $u : (p \vee q)$)

$\Leftrightarrow (p \vee q) \vee r$ ($\because T \wedge v \Leftrightarrow v$, Here $v : (p \vee q) \vee r$)

Ex-11: Prove that $(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

Solution:

$(p \rightarrow r) \vee (q \rightarrow r) \Leftrightarrow (\neg p \vee r) \vee (\neg q \vee r)$	(Equivalence of conditional)
$\Leftrightarrow (\neg p \vee \neg q) \vee (r \vee r)$	(Associative)
$\Leftrightarrow (\neg p \vee \neg q) \vee r$	($\because r \vee r \Leftrightarrow r$)
$\Leftrightarrow \neg(p \wedge q) \vee r$	(De Morgan's law)
$\Leftrightarrow (p \wedge q) \rightarrow r$	(Equivalence of conditional

Ex-12: Using Laws of Logic Prove that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent.

Solution: $\neg(p \vee (\neg p \wedge q)) \Leftrightarrow \neg p \wedge \neg(\neg p \wedge q)$,	(by the De Morgan law)
$\Leftrightarrow \neg p \wedge [\neg(\neg p) \vee \neg q]$	(De Morgan law)
$\Leftrightarrow \neg p \wedge (p \vee \neg q)$	(double negation law)
$\Leftrightarrow (\neg p \wedge p) \vee (\neg p \wedge \neg q)$	(distributive law)
$\Leftrightarrow F0 \vee (\neg p \wedge \neg q)$	($\because \neg p \wedge p \Leftrightarrow F0$) $\Leftrightarrow (\neg p \wedge \neg q)$ ($\because F0 \vee u \Leftrightarrow u$)

Converse, Inverse and Contrapositive; Logical implication

Consider a conditional $p \to q$ then:

- $q \to p$ is called the converse of $p \to q$
- $\neg p \to \neg q$ is called the inverse of $p \to q$
- $\neg q \to \neg p$ is called the contrapositive of $p \to q$.

For Ex, Let

p: 2 is an integer

q: 9 is a multiple of 3.

Then,

$p \to q$: if 2 is an integer, then 9 is a multiple of 3

Converse of this condition is

$q \to p$: If 9 is a multiple of 3, then 2 is an integer

inverse of this condition is

$\neg p \to \neg q$: if 2 is not an integer, then 9 is not a multiple of 3

contrapositive of this condition is

$\neg q \to \neg p$: If 9 is not a multiple of 3, then 2 is not an integer

Truth Table for Converse, Inverse and Contrapositive

p	q	$\neg p$	$\neg q$	$p \to q$	$q \to p$	$\neg p \to \neg q$	$\neg q \to \neg p$
0	0	1	1	1	1	1	1
0	1	1	0	1	0	0	1
1	0	0	1	0	1	1	0
1	1	0	0	1	1	1	1

$p \to q$ and $\neg q \to \neg p$ have same truth values

Also $q \to p$ and $\neg p \to \neg q$ have same truth values in all possible situation. And we have following two important results:

1. $p \to q \Leftrightarrow \neg q \to \neg p$ (Conditional and its contrapositive are logically equivalent)
2. $q \to p \Leftrightarrow \neg p \to \neg q$ (Converse and the inverse are logically equivalent)

Logical implication

let say, p: 6 is a multiple of 2 , q: 3 is a prime number, and we get
conditional $p \rightarrow q$: if 6 is a multiple of 2, then 3 is a prime number

p is true q is true

hence $p \rightarrow q$ is true But question is does this conditional $p \rightarrow q$ make any sense ?

The Answer is NO

Because there is no consistency in the statement $p \rightarrow q$
(all tough it is logically true!

Ex- consider the proposition p: 4 is a odd number, q: Bangalore is not in Karnataka

p is False q is False

But $p \rightarrow q$ is true: if 4 is odd no, then Bangalore is not in Karnataka is true

which is logically True but makes no sense!

We do not deal with conditional as stated above,
Our interest lies in conditional $p \rightarrow q$ where p and q are related in some way so that truth values of q depends upon truth values of p or vice-versa. Such Condition are called hypothetical (implicative) statements

When hypothetical statement $p \rightarrow q$ is such that q is true whenever p is true, we say that p (logically) implies q.
Symbolically written as $p \Rightarrow q$

When hypothetical statement p → q is such that q is not necessarily true whenever p is true, we say that p (logically) does not implies q. Symbolically written as p =/> q

Necessary and Sufficient condition

Necessary Condition:

Suppose that p and q are statements. We say that the statement p is necessary for the statement q if q cannot be true unless p is also true.

In other words, q *requires* p However it is possible for p to be true even if q is not true. We write $p \Leftarrow q$

For example, suppose p is the statement "you sit the exam" and q is the statement "you pass the exam".

You **cannot pass the exam without sitting the exam**: **sitting the exam is a *necessary* condition for passing the exam**. **However sitting the exam does not mean that you will necessarily pass the exam.**

Sufficient Condition

The statement **p is said to be a sufficient condition for the statement q** if knowing that p is true *guarantees* that q is also true.

However knowing that q is true does *not* guarantee that p is true. That is, q needn't be a sufficient condition for p.

We write $p \Rightarrow q$

For example, suppose p is the statement "you achieve an overall grade of over 70% in all of the modules that you have studied as part of your economics degree"

q is the statement "you get a first class degree in economics".

Achieving an overall grade of over 70% in all of the modules that you have studied as part of your economics degree means that you *will* get a first class in economics.

However, getting a first class degree in economics does *not* necessarily mean that you achieved a first in all of your economics modules.

Necessary and Sufficient Condition

We say that the statement p is a **necessary and sufficient** condition for the statement q when q is true *if and only if* p is also true.

That is, either p and q are both true, or they are both false.

Note that if p is necessary and sufficient for q, then q is necessary and sufficient for p. We write **$p \Leftrightarrow q$.**

For example, the statement “I am a male sibling” is necessary and sufficient for the truth of the statement “I am a brother”.

EX- let C denote some specific city, consider following propositions:

p: the city C is in Karnataka

q: the city C is in India

$\Rightarrow$ q is true does *not* guarantee that p is true,

$\Rightarrow$ That is, q needn't be a sufficient condition for p

Here p => q but q =/> p .

or we can say that p is sufficient but not necessary condition for q.

EX- consider a specific integer x and let :

p: the integer x is even

q: the integer x is divisible by 2

=> See that q is true *if and only if* p is also true

i.e. , either p and q are both true, or they are both false

We say that the statement A is a **necessary and sufficient** condition for the statement B

i.e., p => q and q => p or, $p \Leftrightarrow q$

Q-1- Write down the contrapositive of $[p \to (q \to r)]$ with

a) Only one occurrence of the connective $\to$

b) No occurrence of connective $\to$

Sol: contrapositive of $[p \to (q \to r)]$ is $[\sim(q \to r) \to (\sim p)]$

$$[\sim(q \to r) \to (\sim p)] \Leftrightarrow \sim [\sim(q \to r)] \vee \sim p$$

$$\Leftrightarrow (q \to r) \vee \sim p \text{ ------(a)}$$

$$\Leftrightarrow (\sim q \vee r) \vee \sim p \text{ -----(b)}$$

Q-2Write inverse, converse and contrapositive of "If you do your homework, you will not be punished

Sol: The inverse of the given statement is $(\neg p \to \neg q)$: "If you do not do your homework, you will be punished."

Converse $(q \to p)$: "If you will be punished, then you do your homework".

contrapositive $(\neg q \to \neg p)$: If you will not be punished, then you not do your homework"

Q-3 Replace the following statement with its contrapositive: "If x and y are rational, then x + y is rational."

Solution : p : x is rational,

q : y is rational,

r: x + y is rational.

This statement is in the form $(p \wedge q) \to r$.

Its contrapositive statement is $(\neg q \to \neg p)$: $\sim r \to \sim (p \wedge q)$

Using De Morgans law, this can be written as $\sim r \to (\sim p \vee \sim q)$

Hence contrapositive is the statement: " If x + y is irrational, then either x is irrational or y is irrational

Q-4: Write converse, inverse and contrapositive of

(1) If a quadrilateral is a parallelogram, then its diagonals bisect each other.

(2) If a real number x^2 is greater than zero, then x is not equal to zero.

(3) If a triangle is not isosceles, then it is not equilateral. (4) If two lines are parallel, then they are equidistant.

Solution :

(1)	converse:	If the diagonals of a quadrilateral bisect each other, then it is a parallelogram.
	inverse:	If a quadrilateral is not a parallelogram, then its diagonals do not bisect each other.
	contrapositive:	If the diagonals of a quadrilateral do not bisect each other, then it is not a parallelogram.
(2)	converse:	If a real number x is not equal to zero, then x^2 is greater than zero.
	inverse:	If a real number x^2 is not greater than zero, then x is equal to zero.
	contrapositive:	If a real number x is equal to zero, then x^2 is not greater than zero.
(3)	converse :	If a triangle is not equilateral, then it is not isosceles.
	inverse:	If a triangle is isosceles, then it is equilateral.
	contrapositive:	If a triangle is equilateral, then it is isosceles

Q-5: Prove the Following :

i. $[p \wedge (p \rightarrow q)] \Rightarrow q$

ii. $[(p \rightarrow q) \wedge \sim q] \Rightarrow \sim p$

iii. $[(p \vee q) \wedge \sim p] \Rightarrow q$

Note:

To Prove p=>q

1. Whenever p is true then q is true.
2. Proving p→q is a tautology

i. $[p \wedge (p \rightarrow q)] \Rightarrow q$

p	q	$p \rightarrow q$	$p \wedge (p\rightarrow q)$	$[p \wedge (p\rightarrow q)] \rightarrow q$
0	0	1	0	1
0	1	1	0	1
1	0	0	0	1
1	1	1	1	1

Note:

To Prove p=>q

1. Whenever p is true then q is true.
2. Proving $p\rightarrow q$ is a tautology

i. From table we find that both p and p->q is true then q is true $[p \wedge (p\rightarrow q)] \Rightarrow q$

ii. $[(p\rightarrow q) \wedge \sim q] \Rightarrow \sim p$

p	q	¬p	¬q	$p \rightarrow q$	$(p\rightarrow q) \wedge \neg q$	$[(p\rightarrow q) \wedge \sim q] \rightarrow \sim p$
0	0	1	1	1	1	1
0	1	1	0	1	1	1
1	0	0	1	0	0	1
1	1	0	0	1	0	1

ii. From table we find that both $p\rightarrow q$ and ~q is true then ~p is true $[(p\rightarrow q) \wedge \sim q] \Rightarrow \sim p$

iii.[(p V q) Λ~p]=> q

p	q	¬p	pvq	(p V q) Λ~p	[(p V q) Λ~p]--> q
0	0	1	0	0	1
0	1	1	0	1	1
1	0	0	1	0	1
1	1	0	1	0	1

iii. From table we find that both p V q and ~p is true then q is true

[(p V q) Λ~p]=> q

Q-6: Prove the Following :

i. $[p \wedge (p \rightarrow q) \wedge r] \Rightarrow [(p \vee q) \rightarrow r]$

ii. $\{[p \vee (q \vee r)] \wedge \sim q\} \Rightarrow p \vee r$

p	q	r	$p \rightarrow q$	$p \vee q$	$(p \vee q) \rightarrow r$	$p \wedge (p \rightarrow q)$	$p \wedge (p \rightarrow q) \wedge r$
0	0	0	1	0	1	0	0
0	0	1	1	0	1	0	0
0	1	0	1	1	0	0	0
0	1	1	1	1	1	0	0
1	0	0	0	1	0	0	0
1	0	1	0	1	1	0	0
1	1	0	1	1	0	1	0
1	1	1	1	1	1	1	1

i. From table we find that both p, p->q and r is true then $[(p \vee q) \rightarrow r]$

is true: $p \wedge (p \rightarrow q) \wedge r] \Rightarrow [(p \vee q) \rightarrow r]$

ii.{[p V (q V r)] Λ~q }=> p V r

p	q	r	p V (q V r)	~q	p V (q V r)] Λ~q	p V r
0	0	0	0	1	0	0
0	0	1	1	1	1	1
0	1	0	1	0	0	0
0	1	1	1	0	0	1
1	0	0	1	1	1	1
1	0	1	1	1	1	1
1	1	0	1	0	0	1
1	1	1	1	0	0	1

ii From table we find that {[p V (q V r)] Λ~q } is true then p V r
Is true

Rules of Inference for Propositional Logic

Rules of Inference are templates for constructing valid arguments

Deriving Conclusions from Evidences

Valid Argument Form

In the previous example, the argument belongs to the following form

$p \rightarrow q$
p

$\therefore q$

By definition, if a valid argument form consists – premises: p1, p2, … , pk – conclusion: q
then (p1 ∧ p2 ∧ … ∧ pk) → q is a tautology

Some simple valid argument forms, called rules of inference, are derived and can be used to construct complicated argument form

In mathematics, an argument is a sequence of propositions (called premises) followed by a proposition (called conclusion)

A valid argument is one that, if all its premises are true, then the conclusion is true

Ex: "If it rains, I drive to school."
"It rains."

∴ "I drive to school."

Types of Inference Rule

Rule	Tautology	Name
$p \to q$ p $\therefore q$	$((p \to q) \wedge p) \Rightarrow q$	Modus Ponens (Law of Detachment)
$p \to q$ $\neg q$ $\therefore \neg p$	$((p \to q) \wedge \neg q) \Rightarrow \boxed{\neg p}$	Modus Tollens
$p \to q$ $q \to r$ $\therefore p \to r$	$((p \to q) \wedge (q \to r)) \Rightarrow (p \to r)$	Hypothetical Syllogism (Transitivity)
$p \vee q$ $\neg p$ $\therefore q$	$((p \vee q) \wedge \neg p) \Rightarrow q$	Disjunctive Syllogism

p $\therefore p \vee q$	$p \Rightarrow p \vee q$	Disjunctive amplification Addition
$p \wedge q$ $\therefore p$	$(p \wedge q) \Rightarrow p$	Simplification
p q $\therefore p \wedge q$	$(p) \wedge (q) \Rightarrow (p \wedge q)$	Conjunction
$p \vee q$ $\neg p \vee r$ $\therefore q \vee r$	$((p \vee q) \wedge (\neg p \vee r)) \Rightarrow (q \vee r)$	Resolution

How to Build Arguments by using Rule of Inference

Ex-1 Test Whether following is a valid argument

Premises : If Sachin hits a century then he gets a free car

Premises : Sachin hits a century

Conclusion: sachin gets a free car

If Sachin hits a century then he gets a free car

Sachin hits a century

∴ Conclusion: sachin gets a free car

p: If Sachin hits a century then he gets a free car, q: Sachin hits a century

$$\begin{array}{l} p \rightarrow q \\ p \\ \hline \therefore q \end{array}$$

In view of modus Pones Rule this is valid argument

Q-1 Test Whether following is a valid argument

If Sachin hits a century then he gets a free car

sachin does not get a free car.

∴ Sachin has not hit century

p: Sachin hits a century, q: Sachin gets a free car

Then argument reads

$$\begin{array}{r} p \rightarrow q \\ \sim q \\ \hline \therefore \sim p \end{array}$$

In view of modus Tollens Rule this is valid argument

Q-2 Test Whether following is a valid argument
If Sachin hits a century he gets a free car
sachin get a free car.

$\therefore$ Sachin has hit a century

p: Sachin hits a century, q: Sachin gets a free car

Then argument reads

$$\begin{array}{r} p \rightarrow q \\ q \\ \hline \therefore\ p \end{array}$$

But p can be F when p → q are true

Thus [(p →q) ∧ q] → q is not a tautology, and given statement is not a valid argument

p	q	p →q	(p →q) ∧ q
0	1	1	1

Q-3 Test Whether following is a valid argument
If I drive to Work, Then I will arrive tired
I am not tired (when I arrive at work)

I do not drive to Work

p: I drive to Work q: I arrived tired

Then argument reads

$$\begin{array}{r} p \rightarrow q \\ \sim q \\ \hline \therefore\ \sim p \end{array}$$

In view of modus Tollens
Rule this is valid argument

Q-4 Test Whether following is a valid argument

If I Study, then I do not fail in the exam

If I do not fail in the exam, then my father gifts a car to me

---

if I study then my father gift me a car

p: I Study
q: I do not fail in the exam
r : my father gifts a car to me

Then argument reads

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow \mathbf{r} \\ \hline \therefore\ p \rightarrow \mathbf{r} \end{array}$$

In view of Rule of Syllogism this is valid argument

Q-5 Test Whether following is a valid argument

If Ravi goes out with friends, he will not study.

If Ravi does not study, his father becomes angry

His father is not angry

Ravi has not gone out with friends

p : Ravi goes out with friends
q: Ravi will not study
r : his father becomes angry

Then argument reads

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow \mathbf{r} \\ \sim r \\ \hline \therefore\ \sim p \end{array}$$

Here we have three premises, so we club two premises to simplify by using rues of inference to make conclusion

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow \mathbf{r} \\ \hline \therefore\ p \rightarrow \mathbf{r} \end{array}$$

By Rule of Syllogism

$$\begin{array}{l} p \rightarrow r \\ \sim \mathbf{r} \\ \hline \therefore\ \sim p \end{array}$$

By Rule of Modus Tollens

Q-6 Test Whether following is a valid argument

If I study, I will not fail in exam

If I do not watch TV in evenings, I will study

I failed in exam

I must have watched TV in the Evenings

p : I study

q: I fail in exam

r : I watch TV in evenings

Then argument reads

$$\begin{array}{l} p \rightarrow \sim q \\ \sim r \rightarrow p \\ \quad\; q \\ \hline \therefore \; r \end{array}$$

This argument is logically equivalent to

$$\begin{array}{l} q \rightarrow \sim p \\ \sim p \rightarrow r \\ \quad\; q \\ \hline \therefore \; r \end{array}$$

<u>As by contrapositive</u>

$p \rightarrow \sim q \Leftrightarrow (\sim\sim q \rightarrow \sim p)$

$\sim r \rightarrow p \Leftrightarrow (\sim p \rightarrow r)$

This argument is logically equivalent to

$$\begin{array}{l} q \rightarrow r \\ \quad\; q \\ \hline \therefore \; r \end{array}$$

By Modus Ponens

Q-7 Test Whether following is a valid argument

I will get grade A in this course or I will not graduate

if I do not graduate, I will join the army

I got grade A

I will not join Army

p : I will get grade A in this course

q: I will not graduate

r : I will join the army

Then argument reads

$$\begin{array}{l} p \vee q \\ q \rightarrow r \\ \quad\; p \\ \hline \therefore \; \sim r \end{array}$$

By contrapositive

$p \vee q \Leftrightarrow \sim p \rightarrow q \Leftrightarrow \sim q \rightarrow p$

$q \rightarrow r \Leftrightarrow \sim q \rightarrow \sim r$

This argument is logically equivalent to

$$\begin{array}{l} \sim q \rightarrow p \\ \sim q \rightarrow \sim r \\ \hline \therefore p \rightarrow \sim r \end{array}$$

This argument is logically equivalent to

$$\begin{array}{l} p \rightarrow \sim r \\ p \\ \hline \therefore \sim r \end{array}$$

By MP

given statement is not a valid argument

But ~r can be F when p→ ~r are true

Q-8: Test the validity of following arguments:

i.

$$\begin{array}{l} P \wedge q \\ p \rightarrow (q \rightarrow r) \\ \hline \therefore r \end{array}$$

ii.

$$\begin{array}{l} p \\ p \rightarrow \sim q \\ \sim q \rightarrow \sim r \\ \hline \therefore \sim r \end{array}$$

iii.

$$\begin{array}{l} P \rightarrow r \\ q \rightarrow r \\ \hline \therefore (p \vee q) \rightarrow r \end{array}$$

i. Since $P \wedge q$ is true, both p and q are True.
Since p is true and $p \rightarrow (q \rightarrow r)$ is true, $(q \rightarrow r)$ is true, and since q is true $(q \rightarrow r)$ is true, r has to be true hence given argument is valid

ii. The Premises $p \rightarrow \sim q$ and $\sim q \rightarrow \sim r$ together yields the premises $p \rightarrow \sim r$. (by Rule of syllogism)
Since p is true this premises $p \rightarrow \sim r$ establish that ~r is true. Hence given statement is true

iii. $(P \rightarrow r) \wedge (q \rightarrow r) \Leftrightarrow (\sim P \vee r) \wedge (\sim q \vee r)$

$\Leftrightarrow (r \vee \sim P) \wedge (r \vee \sim q)$ by Commutative

$\Leftrightarrow r \vee (\sim p \wedge \sim q)$ by distributive

$\Leftrightarrow r \vee \sim(p \vee q)$ By de Morgan's

$\Leftrightarrow \sim(p \vee q) \vee r$ by Commutative

$\Leftrightarrow (p \vee q) \rightarrow r$

This logical Equivalence shows that the given argument is valid

Q-9: Prove that the following are valid arguments:

i.

$$\begin{array}{l} P \to (q \to r) \\ \sim q \to \sim p \\ p \\ \hline \therefore\ r \end{array}$$

ii.

$$\begin{array}{l} \sim P \leftrightarrow q \\ q \to r \\ \sim r \\ \hline \therefore\ p \end{array}$$

Q-10: Test the validity for following arguments:

i.

$(\sim P \vee q) \rightarrow r)$

$r \rightarrow (s \vee t)$

$\sim s \wedge \sim u$

$\underline{\sim u \rightarrow \sim t}$

$\therefore p$

ii.

$P \rightarrow r$

$r \rightarrow s$

$t \vee \sim s$

$\sim t \vee u$

$\underline{\sim u}$

$\therefore \sim p$

Q-11 Establish the validity of the following Argument

$$\begin{array}{l} p \\ p \to q \\ s \vee r \\ r \to \sim q \\ \hline \therefore s \vee t \end{array}$$

$$\begin{array}{l} p \\ p \to q \\ \hline q \end{array}$$ Modus Ponens

$$\begin{array}{r} r \to \sim q \\ q \\ \hline \sim r \end{array}$$ Modus Tollens

$$\begin{array}{c} s \vee r \\ \sim r \\ \hline s \end{array}$$ Disjunctive syllogism

$$\begin{array}{c} s \\ \hline s \vee t \end{array}$$ Addition

Q-12: Establish the validity of the following Argument. $(\sim p \vee \sim q) \to (r \wedge s)$, $r \to t$, $\sim t$, $\therefore p$

$$\begin{array}{l} (\sim p \vee \sim q) \to (r \wedge s) \\ r \to t \\ \sim t \\ \hline \therefore p \end{array}$$

$$\begin{array}{l} r \to t \\ \sim t \\ \hline \therefore \sim r \end{array}$$, Modus Tollens Rule

$$\begin{array}{c} \sim r \\ \hline \therefore \sim r \vee \sim s \end{array}$$ disjunctive amplification

$\therefore \sim r \vee \sim s \Rightarrow \sim(r \wedge s)$ De Morgan's Law

$$\begin{array}{l} (\sim p \vee \sim q) \to (r \wedge s) \\ \sim(r \wedge s) \\ \hline \therefore \sim(\sim p \vee \sim q) \end{array}$$ Modus Tollens Rule

$\therefore (p \wedge q)$ De Morgan's Law

$\therefore (p \wedge q) \Rightarrow p$ Simplification

Open Statements

- In Mathematical discussions, declarative sentences such as those given below are encountered previously:

Ex: 1) $x+3=6$, 2) $x^2 < 10$ 3) x divides 4 4) $x = \sqrt{2}$, These sentences are not propositions unless symbol x is specified.

These sentences of these kind are called open sentences or open statements

And symbol x which is unspecified is called free variable

Let us consider open sentence (1) and set of Real number "R".
This sentence becomes a proposition if x is replaced by and=y element of R
For Ex: if x is replaced as 3,
The sentence
$x+3=6$ becomes true Proposition
And if x =5 it becomes False Proposition

Here we say R is Universe (universe of discourse) for variable x

- Open statement containing variable x are denoted by p(x), q(x) etc..
- If U is universe for variable x in an open statement p(x) and if a ∈ U, then proposition got by replacing a by a in p(x) is denoted by p(a)

p(x): x+3 where x ∈ set of integers
p(2): is false proposition " 2+3=6"

q(x) is open statement, q(x): $x^2 < 10$ where x ∈ set of real numbers as the universe for x then
p($\sqrt{2}$): is false proposition "2<10" which is true

Note: open statement p(x) becomes proposition only when x is replaced by chosen element of the universe

Note: Like compound proposition studied earlier, compound open statement p(x), q(x)... Can be formed by using logical connective's
Thus ~ p(x) is negation of open statement p(x)
Also for open statement p(x) and q(x)

i. $p(x) \wedge q(x)$ is conjunction
ii. $p(x) \vee q(x)$ is disjunction
iii. $p(x) \rightarrow q(x)$ is conditional
iv. $p(x) \leftrightarrow q(x)$ is bi conditional for given universe

Q-1: suppose the universe is consists of all integers. Consider the following open statements:

1. $p(x): x \leq 3$ 2. $q(x): x+1$ is odd 3. $r(x): x > 0$

Write down the truth values of the following:

1. p(2) 2. ~q(4) 3. $p(-1) \wedge q(1)$ 4. $\sim p(3) \vee r(0)$ 5. $p(0) \rightarrow q(0)$ 6. $p(1) \leftrightarrow \sim q(2)$ 7. $p(4) \vee ((q(1) \wedge r(2))$
8. $p(2) \wedge((q(0) \vee \sim r(2))$

1. p(2) is proposition " $2 \leq 3$" which is true
2. q(4) is proposition " 4+1" is odd which is true, therefore ~q(4) is false
3. p(-1) is proposition "$-1 \leq 3$" which is true, and q(1) is proposition "1+1" is odd which is false. Therefore $p(-1) \wedge q(1)$ is false
4. p(3) is true so that ~p(3) is false and r(0) is false. $\sim p(3) \vee r(0)$ is false.
5. p(0) is true and q(0) is true. Therefore $p(0) \rightarrow q(0)$ is true.
6. p(1) is true and q(2) is true. therefore $p(1) \leftrightarrow \sim q(2)$ is false.
7. p(4) is false, q(1) is false and r(2) is true. Hence $((q(1) \wedge r(2))$ is false, so that $p(4) \vee ((q(1) \wedge r(2))$ is false.
8. p(2) is true, q(0) is true and r(2) true. Therefore, $q(0) \vee \sim r(2)$ is true, so that $p(2) \wedge((q(0) \vee \sim r(2))$ is true.

Quantifiers

Consider the following propositions:

1. **All** squares are rectangles.
2. **For every** integer x, x^2 is non-negative integer.
3. **Some** determinants are equal to zero
4. **There exists** a real number whose square is equal to itself

In these propositions
"All" $\forall$
"For Every" $\forall$ Universal quantifiers
"Some" $\exists$
"there exists" $\exists$ Existential quantifiers
Are associated with idea of quantity. Such words are called quantifiers

The proposition (1)-(4) considered above may be re written in alternative forms as explained below.

Let S denote set of all squares. Then the proposition (1) may be rewritten as:

1. for **All** x $\in$ S , x is rectangles. Symbolically, $\forall x \in S$, p(x)

$\forall$ denotes for all, p(x) stands for open statement x is rectangles

2. $\forall x \in Z$, q(x) , Z is set of all integers

1. **D denotes set of all** determinants, $\exists\ x \in D$, p(x)
2. **R denotes a set of** a real number , $\exists\ x \in R$, q(x)

A proposition involving Universal or existential quantifiers is called quantified statement

Q-2 For the universe of all integers, let p(x), q(x), r(x), s(x) and t(x) denote the following open statements

$p(x)$: $x > 0$,

$q(x)$: x is even,

$r(x)$: x is a perfect square,

$s(x)$: x is divisible by 3,

$t(x)$: x is divisible by 7.

Write the following statements in symbolic form:

i) At least one integer is even.

ii) There exists a positive integer that is even.

iii) Some even integers are divisible by 3

iv) If x is even and a perfect square, then x is not divisible by 3.

v) If x is odd or is not divisible by 7, then x is divisible by 3.

Solution :

(i) $\exists x, q(x)$

(ii) $\exists x, [p(x) \wedge q(x)]$

(iii) $\exists x, [q(x) \wedge s(x)]$

(iv) $\forall x, [q(x) \wedge r(x)] \rightarrow \neg s(x)$

(v) $\forall x, [\sim q(x) \vee \sim t(x)] \rightarrow s(x)$

Truth values of quantified statement

The following Rules are employed for determining the truth value of a quantified statement
Rule-1: The statement '$\forall x \in S$, $p(x)$' is true only when $p(x)$ is true for each $x \in S$.
Rule-2: The statement '$\forall x \in S$, $p(x)$' is False only when $p(x)$ is False for every $x \in S$.

Accordingly,

- To infer that proposition of the form '$\forall x \in S$, $p(x)$' is false, it is enough to exhibit one element a of S such that p(a) is false. The element a is called counter example.
- To infer that proposition of the form '$\forall x \in S$, $p(x)$' is true, it is enough to exhibit one element a of S such that p(a) is true.

From these quantified statement

1. **All** squares are rectangles.
2. **For every** integer x, x^2 is non-negative integer.
3. **Some** determinants are equal to zero
4. **There exists** a real number whose square is equal to itself

All propositions are true propositions

But it is obvious that following propositions are false

1. **All** rectangles are squares
2. **For every** integer x, x^2 is positive integer.
3. The square of real numbers are negative

Two rules of Inference

As a consequence of rule 1 and 2 indicated above, we obtain the following rules of inference

Rule-3: If an open statement **p(x) is known to be true** for all x in a universe S and if $a \in S$, then p(a) is true [known as rule of universal specification)

Rule-4: If an open statement **p(x) is proved to be true** for any arbitrary x chosen from set S, then quantified statement "$\forall x \in S$, $p(x)$' is true [known as rule of universal generalization]

Logical Equivalence

1. ∀x, [p(x) ∧ q(x)] ⇔ (∀x, p(x)) ∧ (∀x, q(x))
2. ∃x, [p(x) V q(x)] ⇔ (∃x, p(x)) V (∃x, q(x))
3. ∃x, [p(x) → q(x)] ⇔ ∃x, [~p(x) V q(x)]
4. **∀x, ~p(x) ⇔ for no x, p(x)**

For ex: the statement " for every integer x, x^2 is non-negative" is ⇔ **statement " for no integer x, x^2 is negative**

Rules for Negation of a quantified statement

Rule 5:

~ [∀x, p(x)] ⇔ ∃x, [~p(x)]

~ [∃x, p(x)] ⇔ ∀ x, [~p(x)]

For ex: let us consider the quantified statement **" All equilateral triangle are isosceles"**

Symbolic form: **∀x** ∈ T, p(x)" T is set of all equilateral triangle, p(x) is open statement "x is isosceles"

As per rule of negation

~ [∀x, p(x)] ⇔∃x, [~p(x)]

i.e. ∃x ∈ T, [~p(x)]

" for some equilateral triangle x, x is not isosceles.

Or, " some equilateral triangle are not isosceles

Q-1 For the universe of all integers, let p(x), q(x), r(x), s(x) and t(x) denote the following open statements

$p(x) : x > 0$,

q(x): x is even,

r(x): x is a perfect square,

s(x) : x is divisible by 3,

t(x) : x is divisible by 7.

Write the following symbolic statements in words and indicate its truth value:

i) $\forall x, [r(x) \rightarrow p(x)]$ ii) $\exists x, [s(x) \wedge \sim q(x)]$ iii) $\forall x, \sim[r(x)]$ iv) $\forall x,[r(x) \vee t(x)]$

i)	$\forall x, [r(x) \rightarrow p(x)]$	: For any integer x, if x is a perfect square then x > 0	[false take x=0]
ii)	$\exists x, [s(x) \wedge \sim q(x)]$	: For some integer, x is divisible by 3 and x is not even	[true take x=9]
iii)	$\forall x, \sim[r(x)]$	: for any integer, x is not a perfect square	[false]
iv)	$\forall x,[r(x) \vee t(x)]$	: For any integer x, x is a perfect square or divisible by 7	[false take x=8]

Q-2 Consider the Following open statements with the set of all real numbers as universe

$p(x) : |x| > 3$, $q(x): x>3$.

Find the truth value of quantified statement : $\forall x, [p(x) \rightarrow q(x)]$

Also write the converse, inverse and contrapositive of this statement and find their truth values

We note that: $p(-4) \Leftrightarrow |-4|>3 \Leftrightarrow 4>3$ is True, $q(-4) \Leftrightarrow -4>3$ is false

Thus $[p(x) \rightarrow q(x)]$ is false for x=-4.

i. The converse statement of $\forall x, [p(x) \rightarrow q(x)]$ is $\forall x, [q(x) \rightarrow p(x)]$ which reads : *for Every real number x, if x>3 then |x|>3*

ii. The Inverse statement of $\forall x, [p(x) \rightarrow q(x)]$ is $\forall x, [\sim p(x) \rightarrow \sim q(x)]$ which reads : *for Every real number x, if |x|<=3, then x<=3.*

Since converse $\Leftrightarrow$ Inverse hence truth value of (ii) is true

iii. The Contrapositive is $\forall x, [\sim q(x) \rightarrow \sim p(x)]$ which reads : *for Every real number x, if x<=3 then |x|<=3. [False]*

Q-3Write the following sentences in symbolic form, and find its negation:
(i) If all triangles are right angled, then no triangle is equiangular. (ii) All integers are rational numbers and some rational numbers are not integers

Solution :
(i) Let T denote the set of all triangles.
Also, let $p(x)$: x is right-angled,
$q(x)$: x is equiangular.
Then the given proposition can be written in symbolic form as $\{\forall x \in T, p(x)\} \rightarrow \{\forall x \in T, \neg q(x)\}$
whose negation is $\{\forall x \in T, p(x)\} \wedge \{\exists x \in T, q(x)\}$
This reads as "All triangles are right-angled and some triangles are equiangular"
(ii) Let $p(x)$: x is a rational number.
$q(x)$: x is an integer. Z : Set of all integers. Q : Set of all rational numbers.
Then the given proposition can be written in symbolic form as $\{\forall x \in Z, p(x)\} \wedge \{\exists x \in Q, \neg q(x)\}$.
The negation of this is : $\neg\{\forall x \in Z, p(x)\} \vee \neg\{\exists x \in Q, \neg q(x)\}$ which is logically equivalent to $\{\exists x \in Z, \neg p(x)\} \vee \{\forall x \in Q, q(x)\}$
i.e. "Some integers are not rational numbers or every rational number is an integer".

Q-4: Write the negation of each of the following statements for (i) and (ii) the universe consists of all integers and for (iii) the universe consists of all real numbers.
(i) For all integers n, if n is not divisible by 2, then n is odd.
(ii) If k, m, n are any integers where $k–m$ and $m - n$ are odd, then $k - n$ is even.
(iii) For all real numbers x, if $|x - 3| < 7$, then $-4 < x < 10$.

(Let Z denote the set of all integers and R denote the set of all real numbers.

(i) The given statement can be written as $\forall n \in Z, \neg p(n) \rightarrow q(n)$ where

$p(n)$: n is divisible by 2,

$q(n)$: n is odd.

Its negation is : $\exists n \in Z, \neg p(n) \wedge \neg q(n)$ i.e. For some integer n, n is not divisible by 2 and n is not odd.

(ii) The given statement can be written as : $\forall k, m, n \in Z, [p(x) \wedge q(x)] \rightarrow r(x)$

Its negation is : $\exists k, m, n \in Z, [p(x) \wedge q(x)] \wedge \neg r(x)$

i.e. There exist integers k, m, n such that $k - m$, $m - n$ are odd and $k - n$ is not even.

(iii) The given statement can be written as : $\forall x \in R, p(x) \rightarrow q(x)$ where $p(x) : |x - 3| < 7$ and $q(x) : -4 < x < 10$

Its negation is : $\exists x \in R, \neg[p(x) \rightarrow q(x)]$ i.e. $\exists x, [p(x) \wedge \sim q(x)]$ i.e. For some real number x, $|x - 3| < 7$ and $x \notin (-4, 10)$

Q-5: Let the universe comprise of all integers i) Given $p(x)$: x is odd, $q(x)$: x 2 − 1 is even. Express the statement "If x is odd then x 2 − 1 is even" in symbolic form using quantifiers and negate it

Solution :

(i) The given statement can be written in symbolic form as "$\forall x \in Z, [p(x) \rightarrow q(x)]$"

where Z is the set of all integers. Its negation is $\exists x \in Z, [p(x) \wedge \neg q(x)]$.

i.e. For some integer x, x is odd and x 2 − 1 is not even

Statements with more than one variable

Consider the following statements:

(1) x-2y is a positive integer.

(2) x+y-z=0.

These are open statements which contain more than one free variable.

These become propositions if each variable is replaced by an element of a certain Universe.

For example, if ***W*** take the set of all integers as the Universe and replace x and y in the statement **(1)** by 5 and -3 respectively, then this statement becomes the proposition

"5-2(-3) is a positive integer" **(which is true).**

Similarly, if we take the set of all rational numbers as the Universe and replace x, y, z in the statement **(2)** by 1/2, 1/4, 1/4, then the statement becomes the proposition

"1/2+1/4-1/4= 0" **(which is false).**

Open statements containing two variables x and y are usually denoted by ***p(x, y), q(x,y)*** etc and those with three variables x, y, z are denoted by ***p(x, y, z). q(x, y, z),*** etc.

For an open statement with more than one variable, the Universe can be the same for all variables ***or*** *can be different for different variables.*

Example: in the case of the open statement "x - 2y is a positive integer“ the set of all integers can be the Universe for both x and y, ***or*** the set of ***all integers*** can be the Universe for x and the set of ***all positive integers*** can be the Universe for y .

If U is the universe for x and V is the Universe for y in an open statement p(x, y) and if a € U and b € V, then the proposition got by replacing x by a and y by b in p(x, y) is denoted by p(a, b).

Thus,

if p(x, y) is the open statement **"x-2y is a positive integer**" with Z as the Universe for both x and y,

then p(6, 4) is the proposition "**6-(2x4) is a positive integer**".

Similarly, p(-4, 2) is the proposition **"-4-(2 x 2) is a positive integer**".

Example 1: Let p(x, y) and q(x, y) denote the following open statements.

p(x, y): $x^2 \geq y$***,*** ***q(x, y): (x+2) < y***

If the universe for both of x, y is the set of all real numbers, determine the truth value of each of the following statements:

$(\boldsymbol{i})p(2,4)$ $(\boldsymbol{ii})q(1,\pi)$ $(\boldsymbol{iii})p(-3,8)\Lambda\, q(1,3)$

$(\boldsymbol{iv})\ p(1/2,1/3)\ V \sim q(-2,-3)$ $(\boldsymbol{v})\ p(2,2) \rightarrow\ q(1,1)$ $(\boldsymbol{vi})\ p(1,2)\ \leftrightarrow\ \sim q(3,8).$

(i) $p(2,4)\ =\ 22\ \geq\ 4, which\ is\ true.$

(ii) $q(1,л)\ =\ (1\ +\ 2)\ <\ л, which\ is\ true.$

(iii)$(p(-3,8)\ \wedge\ q(1,3))\ =\ [(-3)^2 \geq\ 8]\Lambda[(1\ +\ 2)\ <\ 3], which\ is\ false.$

(iv)$(p(1/2,1/3)\ V \neg\, q(-2,-3))\ =\ [(1/2)^2\ \geq\ (1/3)|\ V\ [(-2\ +\ 2)\ \geq\ -3]$ which is true

(v) $(p(2,2) \rightarrow\ q(1,1))\ =\ (2^2\ \geq\ 2) \rightarrow ((1+2)\ <\ 1), which\ is\ false.$

(vi)$(p(1,2) \leftrightarrow \sim q(3,8))\ =\ (1^2 \geq\ 2) \leftrightarrow (3+2\ \geq\ 8), which\ is\ true.$

Quantified Statements with more than one variable

When an open statement contains more than one free variable, quantification may be applied to each of the variables.

Thus, if p(x, y) is an open statement with variables x, y, we can have quantified statements of the following form:

(1) $\forall x, \forall y, p(x, y)$ **(2)** $\exists x, \exists y, p(x, y)$

(3) $\forall x, \exists y, p(x, y)$ **(4)** $\exists x, \forall y, p(x, y)$

In the above statements, x and y can have the same universe or different universes.

When x *and* y have the same universe, the statements (1) and (2) are respectively rewritten as

(1) $\forall x, y, p(x, y)$, **(2)** $\exists x, y, p(x, y)$.

From the meaning of the quantifiers, the following results are readily obtained:

$$\forall x, \forall y, p(x, y) \Leftrightarrow \forall y, \forall x, p(x, y).$$

$$\exists x, \exists y, p(x, y) \Leftrightarrow \exists y, \exists x, p(x, y).$$

Let us analyse the quantified statement (3) in some detail.

Let us consider the open statement $\boldsymbol{p(x,y)\colon\ x+y=1}$

with the set of all integers as the universe.

Then, the statement $\forall\ \boldsymbol{x}, \exists\ \boldsymbol{y}, \boldsymbol{p(x,y)}$ reads:

For every (each) integer x, there exists an integer y such that x+y=1".

This statement carries the same meaning as the statement:

"Given any integer, we can find a corresponding integer y such that x+y=1". This is a true statement;

because once we select any x, there does exist y = 1 - x which meets the requirement x+y=1.

On the other hand, the statement $\exists\, y, \forall\, x, p(x, y)$

Reads: ***"For some integer y and for all integers x, we have*** $\boldsymbol{x + y = 1}$***"***.This is a false statement;

because if this statement were to be true, then every integer would be equal to $1 - y$ for some (fixed) integer y.

The above example illustrates the following important result:

$$\forall\, \boldsymbol{x}, \exists\, \boldsymbol{y}, \boldsymbol{p}(\boldsymbol{x}, \boldsymbol{y}) \;<\neq>\; \exists\, \boldsymbol{y}. \forall\, \boldsymbol{x}, \boldsymbol{p}(\boldsymbol{x}, \boldsymbol{y}).$$

$$\exists\, \boldsymbol{x}, \forall\, \boldsymbol{y}, \boldsymbol{p}(\boldsymbol{x}, \boldsymbol{y}) \;<\neq>\; \forall\, \boldsymbol{y}, \exists\, \boldsymbol{x}, \boldsymbol{p}(\boldsymbol{x}, \boldsymbol{y}).$$

Similarly.

Quantified statements involving more than two variables can be analysed similarly.

All rules applicable to quantified statements with one variable can be extended in a natural way to those involving more than one variable.

Example 2: Let x and y denote integers. Consider the statement

$$\mathrm{p}(x,y):\ x + y \text{ is even}$$

Write down the following statements in words:

$$(i)\ \forall\, x, \exists\, y, p(x,y) \qquad (ii)\ \exists\, x, \forall\, y, p(x,y).$$

Solution: In words, the required statements are

(i) With every integer x, there exists an integer y such that x + y is even

(ii) There exists an integer x such that x+y is even for every integer y.

Example 3: Write down the following statements in symbolic form using quantifiers:

(1) Every real number has an additive inverse

(2) The set of real numbers has a multiplicative identity.

(3) The integer 58 is equal to the sum of two perfect squares.

(1) The statement
"Every real number has an additive inverse" is the same as:
"Given any real number x, there is a real number y such that x + y = y + x = 0".
In symbols, this reads $\forall x, \exists y, (x + y = y + x = 0]$.
Here, the set of all real numbers is the universe.

(2) The statement
"The set of real numbers has a multiplicative identity" is the same as:
"There exists a real number x such that xy= yxy for every y".
In symbols, this reads $\exists x, \forall y, \ [xy = yx = y]$
Here, the set of all real numbers is the universe.

(3) The given statement is the same as ***"There exist integers m and n such that*** $58 = m^2 + n^2$
"In symbols, this reads $\exists m, \exists n, 58 = m^2 + n^2$.
Here, the set of all integers is the universe.

Example 4 : Determine the truth value of each of the following quantified statements , the universe being the set of all non-zero integers.

(i) $\exists\, x, \exists\, y, [xy = 1]$

(ii) $\exists\, x, \forall\, y, [xy = 1]$

(iii) $\forall\, x, \exists\, y, [xy = 1]$

(iv) $\exists\, x, \exists\, y, [(2x + y = 5) \Lambda (x - 3y = -8)]$

(v) $\exists\, x, \exists\, y, [(3x - y = 17) \Lambda (2x + 4y = 3)]$

Solution:

(i) True. (Take x = 1,y= 1).

(ii) False. (For a specified x, xy= 1 for every y is not true).

(iii) False. (For x = 2, there is no integer y such that xy = 1).

(iv) True. (Take x = 1,y= 3).

(v) False. (Equations 3x - y = 7 and 2x+4y = 3 do not have a common integer solution).