DISCRETE MATHEMATICS AND GRAPH THEORY (21CIDS31)

MODULE – II: PRINCIPLES OF COUNTING AND INCLUSION-EXCLUSION

- ❖ Pigeonhole Principle
- ❖ Relations and Digraphs
- ❖ Equivalence Relations
- ❖ Partially Ordered Set
- ❖ Hasse Diagram
- ❖ Lattices
- ❖ Paths and Related Problems
- ❖ Transitive Closure and Wars hall's Algorithm
- ❖ Principle Of Inclusion-Exclusion
- ❖ Generalizations of the Principles
- ❖ Derangements – Nothing is in its Right Place
- ❖ Rook Polynomials

PRINCIPLES OF COUNTING

The Rules of Sum and Product

In many situations of computational work, we employ two basic rules of counting, called the Sum Rule and the Product Rule. These rules are restated and illustrated in the following paragraphs.

The Sum Rule

Suppose two tasks T_1 and T_2 are to be performed. If the task T_1 can be performed in m different ways and the task T_2 can be performed in n different ways and if these two tasks cannot be performed simultaneously, then one of the two tasks (T_1 and T_2) can be performed in $m + n$ ways.

More generally, if T_1, T_2, T_3.......T_k are k tasks such that no two of these tasks can be performed at the same time and if the task T_i can be performed in n_i different ways, then one of the k tasks (namely T_1 or T_2 or T_3.......or T_k) can be performed in $n_1 + n_2 + n_3 +$.......$+ n_k$ different ways.

PRINCIPLES OF COUNTING

FACULTY OF ENGINEERING AND TECHNOLOGY

Example 1: Suppose there are 16 boys and 18 girls in a class and we wish to select one of these students (either a boy or a girl) as the class representative.

The number of ways of selecting a boy is 16 and the number of ways of selecting a girl is 18. Therefore, the number of ways of selecting a student (boy or girl) is 16 + 18 = 34.

Example 2: Suppose a Hostel library has 12 books on Mathematics, 10 books on Physics, 16 books on Computer Science and 11 books on Electronics. Suppose a student wishes to choose one of these books for study.

The number of ways in which he can choose a book is 12 + 10 + 16 + 11 = 49.

PRINCIPLES OF COUNTING

The Product Rule

Suppose that two tasks T_1 and T_2 are to be performed one after the other. If T_1 can be performed in n_1 different ways, and for each of these ways T_2 can be performed in n_2 different ways, then both of the tasks can be performed in n_1 n_2 different ways.

More generally, suppose that k tasks T_1, T_2, T_3…….T_k are to be performed in a sequence. If T_1 can be performed in n_1 different ways and for each of these ways T_2can be performed in n_2 different ways, and for each of n_2 different ways of performing T_1 and T_2 in that order, T_3 can be performed in n_3 different ways, and so on, then the sequence of tasks T_1, T_2, T_3…….T_k can be performed in $n_1 n_2 n_3$…….n_k different ways.

PRINCIPLES OF COUNTING

Example 4. Suppose a person has 8 shirts and 5 ties. How many different ways of choosing a shirt and a tie?

Then he has $8 \times 5 = 40$ different ways of choosing a shirt and a tie.

Example 5. Suppose we wish to construct sequences of four symbols in which the first 2 are English letters and the next 2 are single digit numbers.

If no letter or digit can be repeated then the number of different sequences that we can construct is

$26 \times 25 \times 10 \times 9 = 58500$.

If repetition of letters and digits are allowed then the number of different sequences that we can construct is

$26 \times 26 \times 10 \times 10 = 67600$.

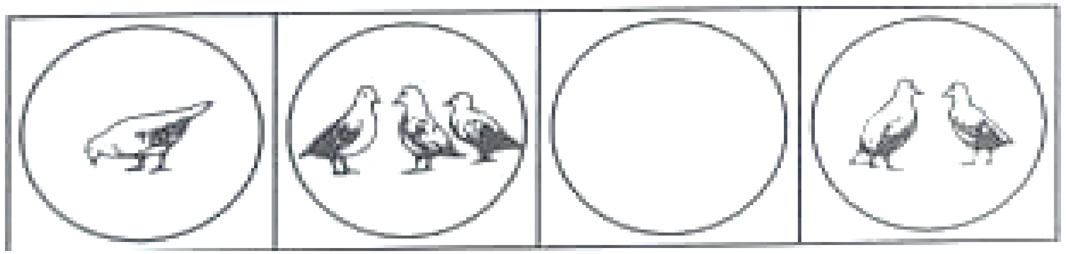
PIGEONHOLE PRINCIPLE

Statement: - If n pigeons occupy m pigeonholes and if $m < n$ then at least one pigeonhole contains two or more pigeons

The use of pigeonhole principle is to

- Identify the pigeons (objects)
- Identify the pigeonhole (categories the desired characteristics)

A simple Illustration of above principle is that, If 6 pigeons occupy 4 pigeon holes, then at least one pigeonhole must contain two or more pigeons in it.



EXTENDED PIGEONHOLE PRINCIPLE: -

Statement: - If n pigeon are assigned to m pigeonholes then one of the pigeonholes must contains at least $\left\lfloor\frac{n-1}{m}\right\rfloor + 1$ pigeons.

PROBLEMS

1. Show that if you pick any five numbers from the integers 1-8, then two of them must add up to 9.

Solution: - Let us first write all the numbers from 1 to 8 as (1,2,3,4,5,6,7,8)

Now let's take any 5 numbers from 1 to 8 such as (1,3,4,7,8)

As it given that any two of the numbers out of the 5 numbers we have chosen should be equal to sum 9.

Let's add every two numbers so that we can get one such pair of numbers whose sum would be 9.

Case 1>. 1 + 3 = 4

Case 2>. 3 + 4 = 7

Case 3>. 4 + 7 = 11

Case 4>. 7 + 8 = 15

Case 5>. 8 + 1 = [9]

PROBLEMS

Hence in Case 5 we get a pair of numbers 8 and 1 whose sum is equal to 9 , so we present them together in a same set as {8,1}.So according to Pigeonhole Principle ,We can take any 5 numbers and there will always exist one pair whose sum is equal to 9.

PROBLEMS

2. **ABC is an equilateral triangle whose sides are of the length 1cm each. If we select 5 points inside the triangles, Prove that at least two of these points are such that the distance is less than ½ cm.**

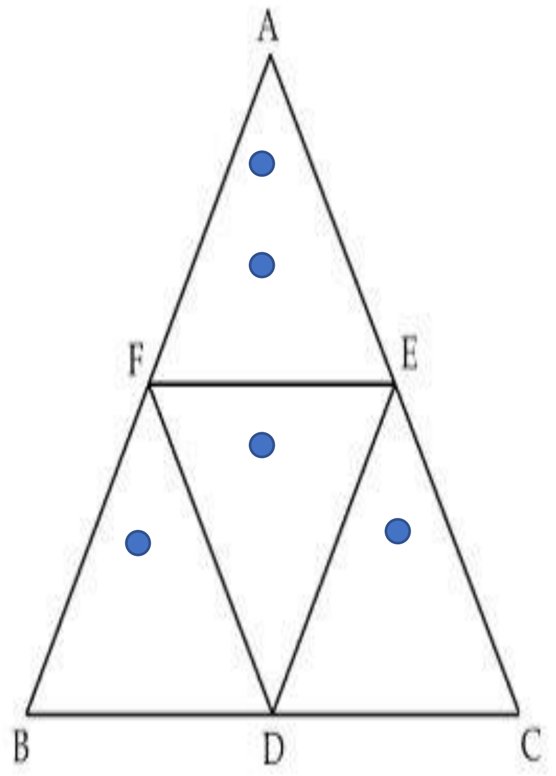
Solution: -

Consider a Δ DEF formed by midpoint of the sides BC, CA, AB divide ABC into four small equilateral triangle(position) each of which has sides equal to ½ cm.

Treating each of the triangle as pigeonhole and 5 points chosen inside the triangle as a pigeon.

We find by using pigeonhole principle that at least one portion must contain two or more points.

Evidently the distance between such points is less than ½ cm



PROBLEMS

3. Show that if 30 dictionaries in a library contain a total of 61,327 pages, then one of the dictionaries must have at least 2045 pages.

Solution: - Given that pigeon(n) = 61327 and pigeonhole(m) = 30

We need to prove $p + 1 = 2045$

Therefore $$p + 1 = \left\lfloor \frac{n-1}{m} \right\rfloor + 1$$

$$= \left\lfloor \frac{61327 - 1}{30} \right\rfloor + 1$$

$$= \lfloor 2044.2 \rfloor + 1$$

$$= 2044 + 1$$

$$p + 1 = 2045$$

PROBLEMS

4. Show that if seven colors are used to paint 50 bicycles, at least eight bicycles will be the same color.

Solution: - Given that pigeon(n) = 50 and pigeonhole(m) = 7

We need to prove $p + 1 = 8$

Therefore $$p + 1 = \left\lfloor \frac{n-1}{m} \right\rfloor + 1$$

$$= \left\lfloor \frac{50-1}{7} \right\rfloor + 1$$

$$= 7 + 1$$

$$p + 1 = 8$$

PROBLEMS

5. If 13 people are assembled in a room, show that at least two of them must have their birthdays in the same month.

Solution: - Given that pigeon(n) = 13 and pigeonhole(m) = 12

We need to prove $p + 1 = 2$

Therefore $$p + 1 = \left\lfloor \frac{n-1}{m} \right\rfloor + 1$$

$$= \left\lfloor \frac{13-1}{12} \right\rfloor + 1$$

$$= 1 + 1$$

$$p + 1 = 2$$

PROBLEMS

6. How many friends must you have to guarantee that at least 5 of them will have birthdays in the same month?

Solution: - Given that pigeon(n) = ? and pigeonhole(m) = 12, $p + 1 = 5$

We have to find n = ?

Therefore $p + 1 = \left\lfloor \frac{n-1}{m} \right\rfloor + 1$

$$5 = \left\lfloor \frac{n-1}{12} \right\rfloor + 1$$

$$5 - 1 = \left\lfloor \frac{n-1}{12} \right\rfloor$$

$$4 = \left\lfloor \frac{n-1}{12} \right\rfloor$$

$$4 \times 12 = n - 1$$

$$48 = n - 1$$

$$n = 48 + 1$$

$$n = 49$$

PROBLEMS

7. Six books each of physics, chemistry, mathematics and four books of biology totally contains 12225 pages. Find the number of pages contained in a book. [Homework].

8.Find how many of a sample size of 1000 people,

a. **Are born in the same month.**
b. **Born on a particular day are born in the same hour. [Homework].**

Introduction

Relationships between elements of sets occur in many contexts. Every day we deal with relationships such as those between a business and its telephone number, an employee and his or her salary, a person and a relative, and so on. Relationships such as that between a program and a variable it uses, and that between a computer language and a valid statement in this language often arise in computer science

Relationships between elements of sets are represented using the structure called a relation, which is just a subset of the Cartesian product of the sets. Relations can be used to solve problems such as determining which pairs of cities are linked by airline flights in a network, finding a viable order for the different phases of a complicated project, or producing a useful way to store information in computer databases.

RELATIONS AND DIAGRAPHS

Relations:

Let A and B be two sets. Then a subset of $A \times B$ is called a binary relation or just a relation.

The Cartesian product (or) cross product of A and B denoted by $A \times B$ is set of all ordered pairs (a, b) where "a" belongs to A and "b" belongs to B.

$\therefore A \times B = \{(a, b)/a \in A, b \in B\}$

Thus, a relation R from a non-empty set A to a non-empty set B is a subset of a Cartesian product $A \times B$.

i.e., If $(a, b) \in R$, we say that a is related b and we write aRb

Note:

$|A| = m$, $|B| = n$ then total number of relations from A to B formed is given by 2^{mn}

When a relation is defined on A i.e., $(A \; to \; A)$ then the relation is called binary relation.

PROBLEMS

Example 1: Let $\mathbf{A = \{2,4,6,8\}}$ **and** $\mathbf{B = \{1,2,3\}}$ **relations** $\mathbf{R_1, R_2, R_3, R_4}$ **from A to B be defined as follows**

i. $\mathbf{aR_1b}$ **if** $\mathbf{a \le b}$

ii. $\mathbf{aR_2b}$ **if** $\mathbf{a > b}$

iii. $\mathbf{aR_3b}$ **if a divides b**

iv. $\mathbf{aR_4b}$ **if b divides a**

Solution: Given that $A = \{2,4,6,8\}$ and $B = \{1,2,3\}$

$$\therefore A \times B = \{(2,1), (2,2), (2,3), (4,1), (4,2), (4,3), (6,1), (6,2), (6,3), (8,1), (8,2), (8,3)\}$$

i) $R_1 = \{(2,2), (2,3)\}$

ii) $R_2 = \{(2,1), (4,1), (4,2), (4,3), (6,1), (6,2), (6,3), (8,1), (8,2), (8,3)\}$

iii) $R_3 = \{(2,2)\}$

iv) $R_4 = \{(2,1), (2,2), (4,1), (4,2), (6,1), (6,2), (6,3), (8,1), (8,2)\}$

Example 2: Let A and B be finite sets with $|B| = 3$. If there are 4096 relations from A to B then what is $|A|$?

Solution

We know that if $|A| = m$, $|B| = n$ then total number of relations from A to B formed is given by 2^{mn}

It is given that $|B| = 3 = n$

Thus, we have

$$2^{3m} = 4096$$

$$\log 2^{3m} = \log 4096$$

$$3m \log 2 = \log 4096$$

$$m = \frac{\log 4096}{3 \log 2}$$

$$m = 4 = |A|$$

Or

$$2^{3m} = 4096 = 2^{12} = 2^{3\times 4}$$

Hence $m = 4$

PROBLEMS

3. **Let $A = \{1, 2, 3, 4\}$ and R_1, R_2, R_3 are relations on A defined as follows**

i) aR_1b **if** $a \leq b$

ii) aR_2b **if** $a > b$

iii) aR_3b **if a is odd b and b is even [Homework].**

Representation of relations:

Matrix of a relation (Zero-one matrix): Let R be a binary relation defined on a set A then a matrix $[m_{ij}]$ with

$$m_{ij} = \begin{cases} 1 \; if \; (a_i, b_j) \in R \;\; or \; a_i R b_j \\ 0 \; if \; (a_i, b_j) \notin R \;\; or \; a_i \not R b_j \end{cases}$$

The matrix $[m_{ij}]$ is called matrix of a relation. $[m_{ij}]$ is also called zero-one matrix.

Example : Let $A = \{1,2,3,4\}$ and $R = \{(1,1)(2,3),(3,4),(4,1)\}$ then $M_R = M(R) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

RELATIONS AND DIAGRAPHS

Digraph or Directed graph of a relation:

Let A relation R on set A $[R \subseteq A \times A]$ can be represented pictorially as follows:

Step 1: Draw small circles representing the element of A. This circle is called **vertex or node.**

Step 2: Draw an arrow from vertex a_1 to vertex a_2 iff a_1 is related to a_2. This arrow is called **edge.**

Note: If a_1 is related to itself then, we write the arrow which starts from vertex a_1 and ends at vertex a_1 and it is called **loop.**

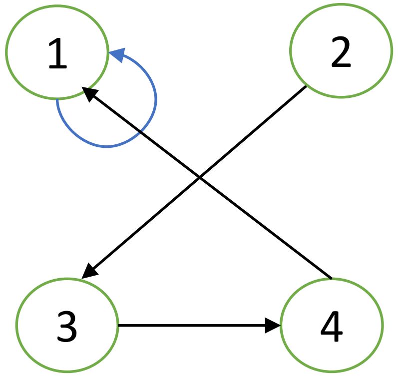
The resulting pictorial representation is known as a directed graph of a relation.

Note: In a digraph the number of edges leaving a vertex is known as **out-degree** and number of edges which enters a vertex is known as **in degree** of a vertex.

PROBLEMS

1.Let $A = \{1, 2, 3, 4\}$ and $R = \{(1, 1)(2, 3), (3, 4), (4, 1)\}$

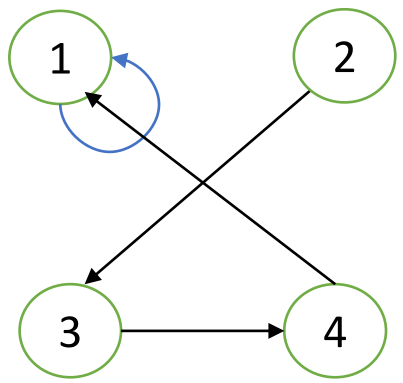
Vertex	In degree	Out degree
1		
2		
3		
4		



PROBLEMS

1.Let $A = \{1, 2, 3, 4\}$ and $R = \{(1, 1)(2, 3), (3, 4), (4, 1)\}$

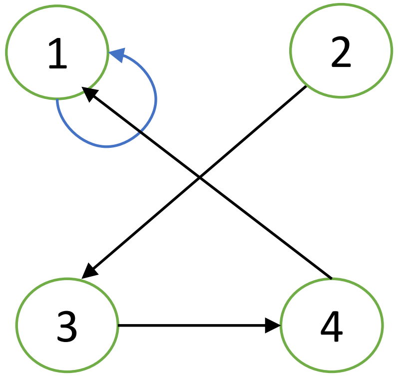
Vertex	In degree	Out degree
1	2	1
2		
3		
4		



PROBLEMS

1.Let $A = \{1, 2, 3, 4\}$ and $R = \{(1, 1)(2, 3), (3, 4), (4, 1)\}$

Vertex	In degree	Out degree
1	2	1
2	0	1
3		
4		



PROBLEMS

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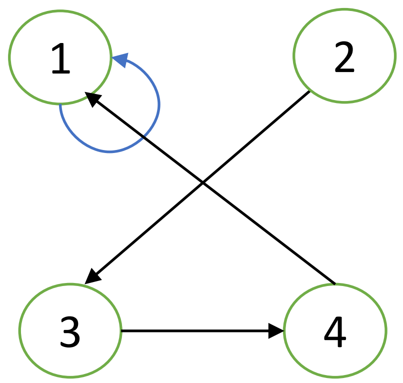
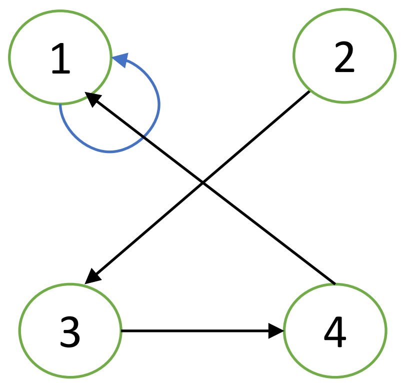


Vertex	In degree	Out degree
1	2	1
2	0	1
3	1	1
4		

PROBLEMS

1.Let $A = \{1, 2, 3, 4\}$ and $R = \{(1, 1)(2, 3), (3, 4), (4, 1)\}$



Vertex	In degree	Out degree
1	2	1
2	0	1
3	1	1
4	1	1

PROBLEMS

2. Let $A = \{1, 2, 3, 4\}$ let R be a relation defined by xRy iff x divides y [divisibility relation].

(a)Write R as ordered pairs.

(b)Write Matrix of relation.

(c)Draw digraph.

(d)Find indegree & outdegree

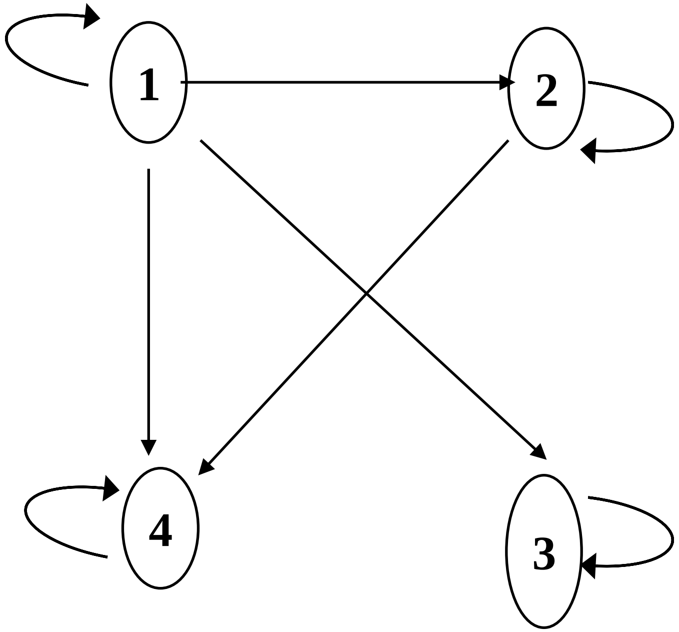
Solution:

(a) $R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$

(b) $M(R) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

PROBLEMS

Diagraph:



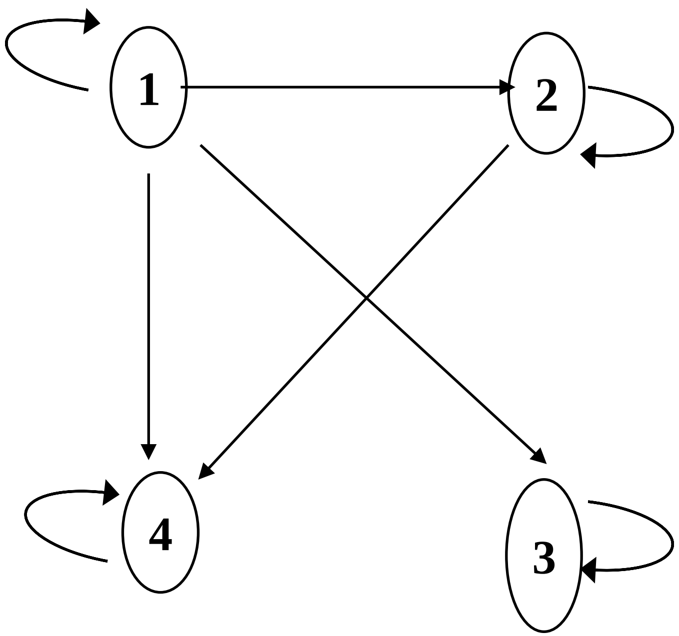
(a) R = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)}

Indegree and Outdegree:

Vertex	In degree	Out degree
1		
2		
3		
4		

PROBLEMS

Diagraph:



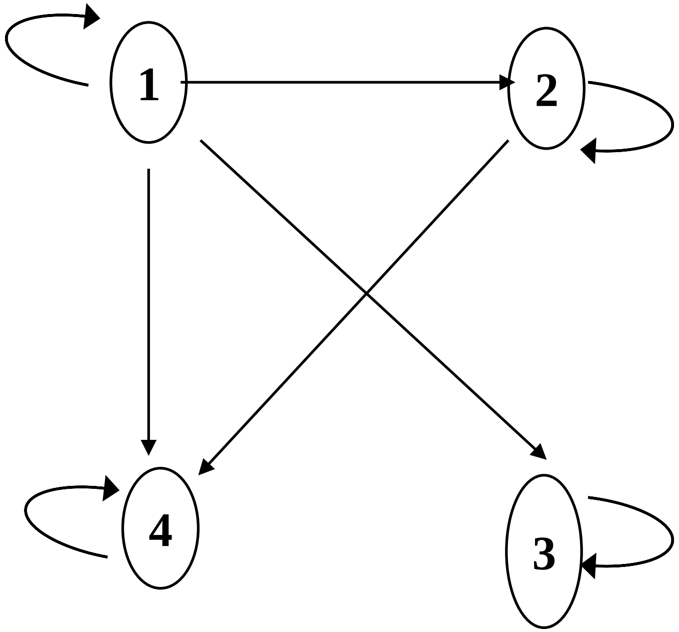
(a) R = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)}

Indegree and Outdegree:

Vertex	In degree	Out degree
1		
2		
3		
4		

PROBLEMS

Diagraph:



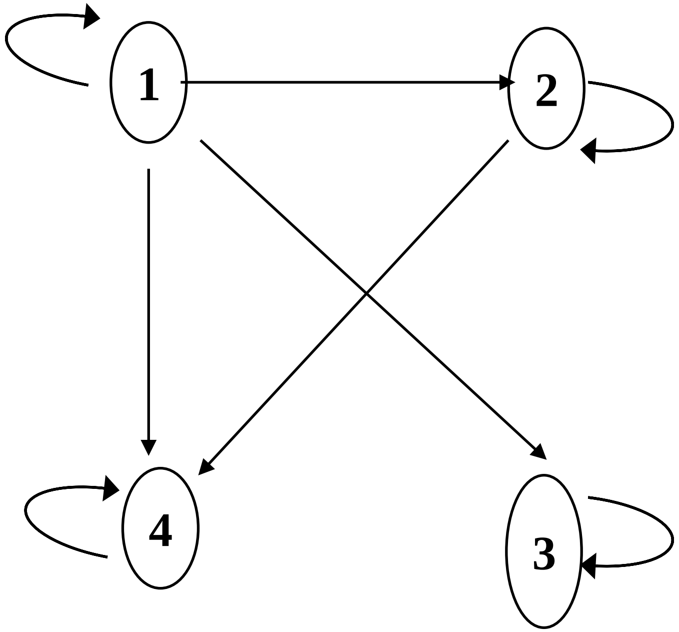
(a) R = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)}

Indegree and Outdegree:

Vertex	In degree	Out degree
1	1	4
2		
3		
4		

PROBLEMS

Diagraph:



(a) R = {(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)}

Indegree and Outdegree:

Vertex	In degree	Out degree
1	1	4
2	2	2
3	2	1
4	3	1

PROBLEMS

3.. Let $A = \{1, 2, 3, 4, 6\}$ Let R be a relation defined by xRy iff x is a multiple of y

a) **Write R as ordered pair.**

b) **Write $M(R)$.**

c) **Draw directed graph of relation.**

d) **Find indegree and out degree.**

Solution

(a) $R = \{(1,1), (2,1), (3,1), (4,1), (6,1), (2,2), (4,2), (6,2), (3,3), (6,3), (4,4), (6,6)\}.$

$$(b)\, M(R) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

PROBLEMS

(a) $R =$

$\{(1,1), (2,1), (3,1), (4,1), (6,1), (2,2), (4,2), (6,2), (3,3), (6,3), (4,4), (6,6)\}.$

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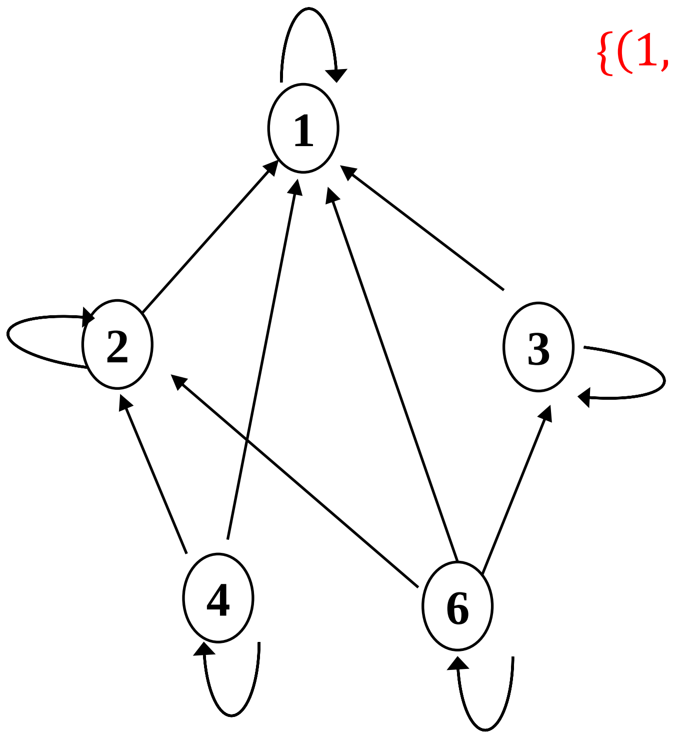
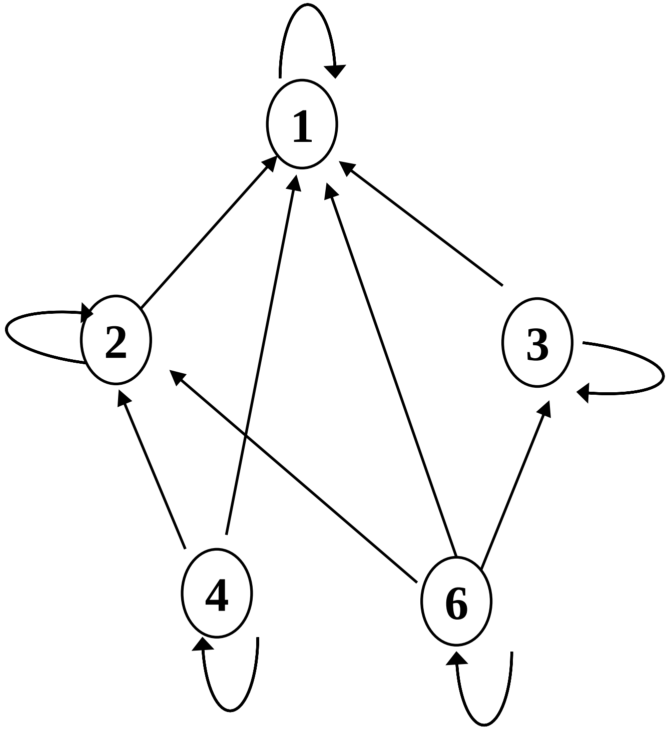


Indegree and Outdegree:

Vertex	In degree	Out degree
1		
2		
3		
4		
6		

PROBLEMS

Diagraph:



(a) R =

{(1,1), (2,1), (3,1), (4,1), (6,1), (2,2), (4,2), (6,2), (3,3), (6,3), (4,4), (6,6)}

.

Indegree and Outdegree:

Vertex	In degree	Out degree
1	5	1
2		
3		
4		
6		

PROBLEMS

(a) R =

{(1,1), (2,1), (3,1), (4,1), (6,1), (2,2), (4,2), (6,2), (3,3), (6,3), (4,4), (6,6)}

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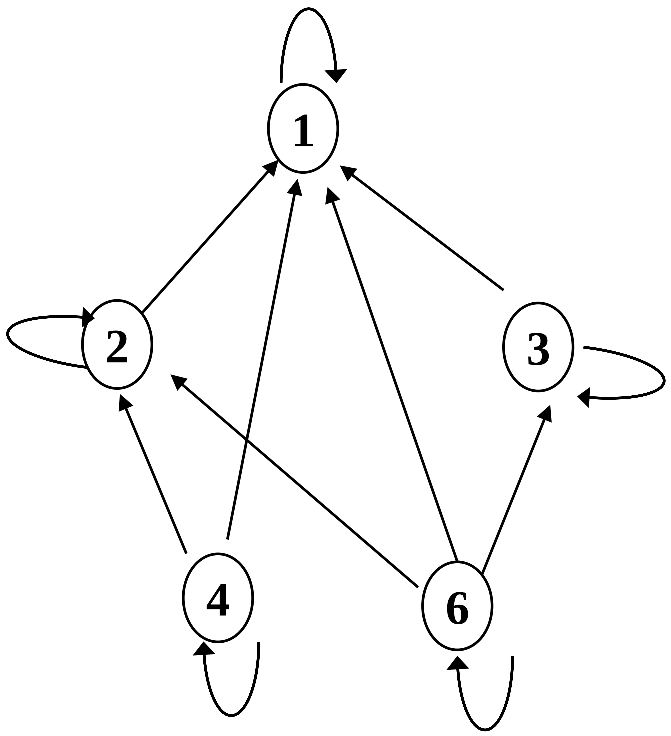

Indegree and Outdegree:

Vertex	In degree	Out degree
1	5	1
2	3	2
3		
4		
6		

PROBLEMS

(a) $R =$

$\{(1,1), (2,1), (3,1), (4,1), (6,1), (2,2), (4,2), (6,2), (3,3), (6,3), (4,4), (6,6)\}$

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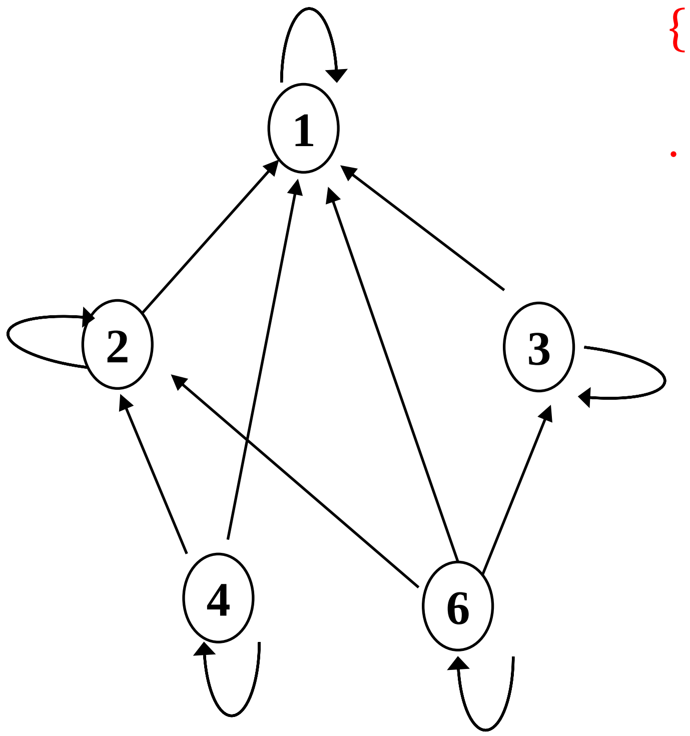

Indegree and Outdegree:

Vertex	In degree	Out degree
1	5	1
2	3	2
3	2	2
4		
6		

PROBLEMS

(a) $R =$

$\{(1,1),(2,1),(3,1),(4,1),(6,1),(2,2),(4,2),(6,2),(3,3),(6,3),(4,4),(6,6)\}$

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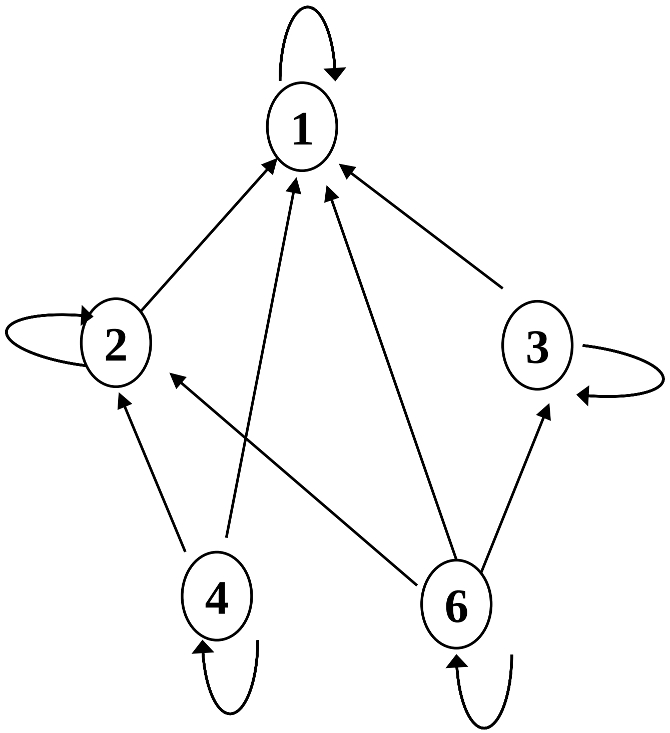

Indegree and Outdegree:

Vertex	In degree	Out degree
1	5	1
2	3	2
3	2	2
4	1	3
6		

PROBLEMS

(a) R =

{(1,1), (2,1), (3,1), (4,1), (6,1), (2,2), (4,2), (6,2), (3,3), (6,3), (4,4), (6,6)}

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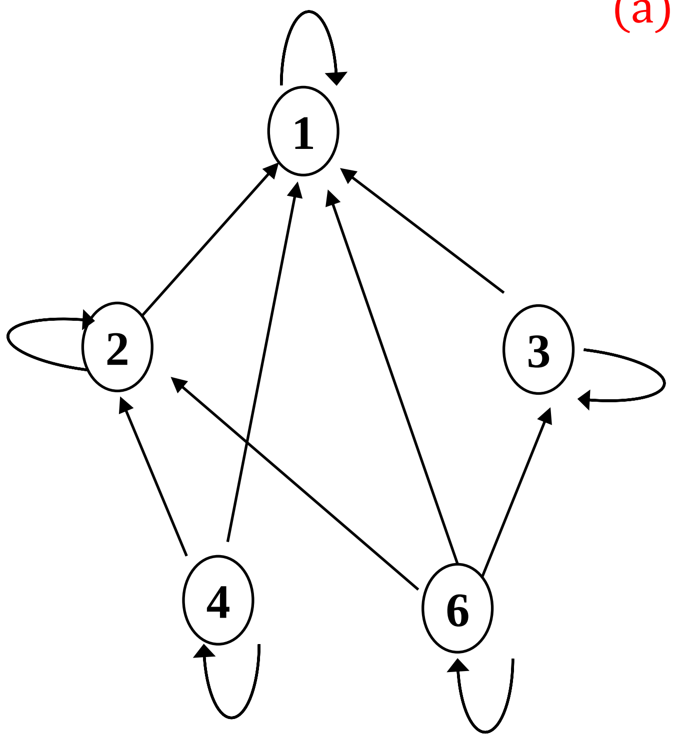
Diagraph:



Indegree and Outdegree:

Vertex	In degree	Out degree
1	5	1
2	3	2
3	2	2
4	1	3
6	1	4

PROBLEMS

4.Let $A = \{u, v, x, y, z\}$, and R be a relation on A whose matrix is as given below :

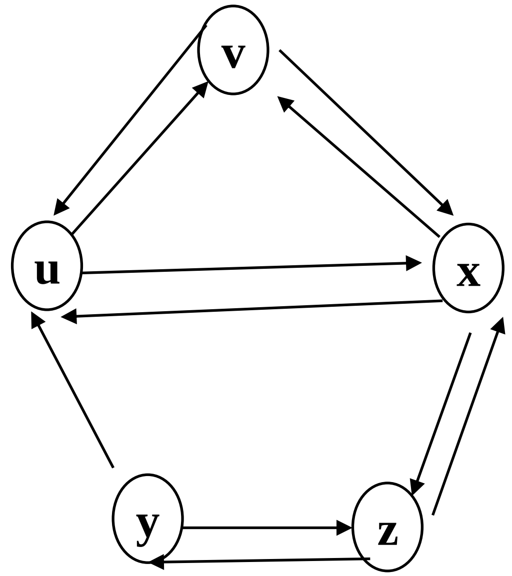
$$M(R) = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

- **a) Write R as ordered pair.**
- **b) Draw directed graph of relation.**

Solution:

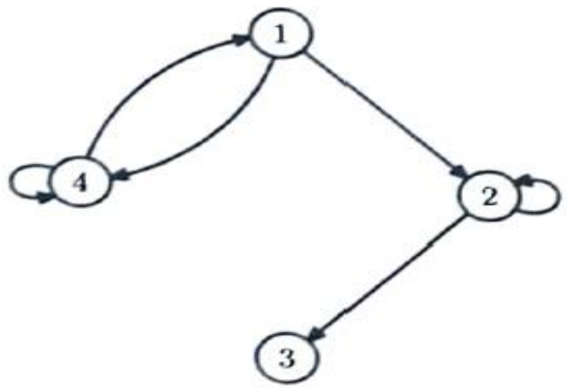
(a) $R = \{(u, v), (u, x), (v, u), (v, x), (x, u), (x, v), (x, z), (y, u), (y, z), (z, x), (z, y)\}.$

(b) Diagraph



PROBLEMS

5.Find the relation represented by the digraph given below. Also, write down its matrix.



Solution:

Let A $= \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}\}$

$$R = \{(1,2), (1,4), (2,2), (2,3), (4,1), (4,4)\}$$

Matrix R

$$M(R) = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

PROBLEMS

6.Let $\mathrm{A} = \{\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}\}$, and R be the relation on A that has the matrix:

$$\mathrm{M(R)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

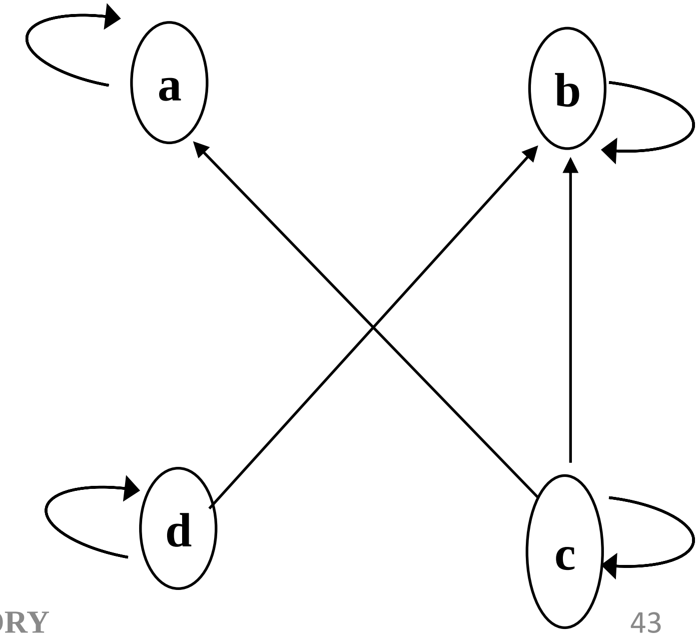
Construct the diagraph of R and list the in-degree and out-degree:

Solution:

Let $\boldsymbol{A} = \{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}\}$,

$$\mathrm{R} = \{(\mathrm{a}, \mathrm{a}), (\mathrm{b}, b), (\mathrm{c}, a), (\mathrm{c}, b), (\mathrm{c}, c), (\mathrm{d}, b), (\mathrm{d}, d)\}$$

Diagraph:



Indegree and Outdegree:

Vertex	In degree	Out degree
a		
b		
c		
d		

PROBLEMS

Indegree and Outdegree:

Vertex	In degree	Out degree
a	2	1
b		
c		
d		

PROBLEMS

Indegree and Outdegree:

Vertex	In degree	Out degree
a	2	1
b	3	1
c		
d		

PROBLEMS

Indegree and Outdegree:

Vertex	In degree	Out degree
a	2	1
b	3	1
c	1	3
d		

PROBLEMS

Indegree and Outdegree:

Vertex	In degree	Out degree
a	2	1
b	3	1
c	1	3
d	1	2

PROBLEMS

7.Let $\mathbf{A} = \{1, 2, 3, 4\}$, **R is relation xRy iff** $\mathbf{x \leq y}$

a) **Write R as ordered pair.**

b) **Write** $\mathbf{M(R)}$.

c) **Draw directed graph of relation.**

d) **Find indegree and out degree.** **[Home Work]**

OPERATIONS ON RELATIONS:

FACULTY OF ENGINEERING AND TECHNOLOGY

Union:

Let R and S be two relations then $R \cup S = \{(x, y)/(x, y) \in R \text{ } or \text{ } (x, y) \in S\}$

Intersection:

Let R and S be two relations then $R \cap S = \{(x, y)/(x, y) \in R \text{ } and \text{ } (x, y) \in S\}$

Complement:

Given a relation R from set A to B, the complement of R is denoted by $\bar{R}$ is defined as a relation from $A \text{ } to \text{ } B$ with the property that $(x, y) \in \bar{R} \text{ } iff \text{ } (x, y) \notin R.$

i.e., $\bar{R}$ is the complement of the set R in the universal set $A \times B$.

Symbolically $\overline{R} = \{(x, y)/(x, y) \notin R\}$

OPERATIONS ON RELATIONS:

Converse:

Given a relation R from set A to B, the converse of R is denoted by R^c is defined as a relation from $A\ to\ B$ with the property that $(x, y) \in R^c\ iff\ (y, x) \in R$

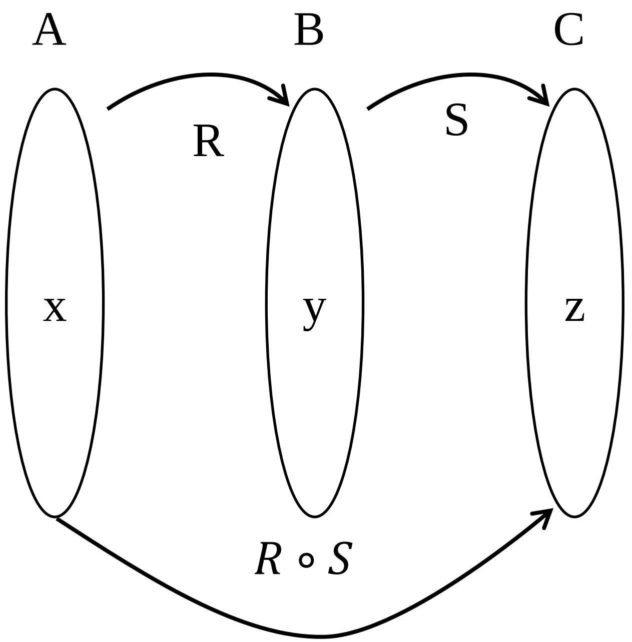
Symbolically $R^c = \{(x, y)/(y, x) \in R\}$

Composition of Relation:

Let $A, B\ and\ C$ be non-empty set, let R be relation from A to B and S be relation from B to C

The composition of R and S is denoted by $R \circ S$ and defined as

$R \circ S = \{(x, z)/(x, y) \in R\ and (y, z) \in S\}$



OPERATIONS ON RELATIONS:

Connectivity Relation

Given a relation R on a set A we define a relation R^{∞} on A as follows, for any $x, y \in A$, $(x, y) \in R^{\infty}$ iff there is some path in R from x *to* y if $(x, y) \in R^{\infty}$ we write xRy^{∞}.

i.e., R^{∞} is list of all ordered pairs if vertices for which there is a path of any length from one vertex to another.

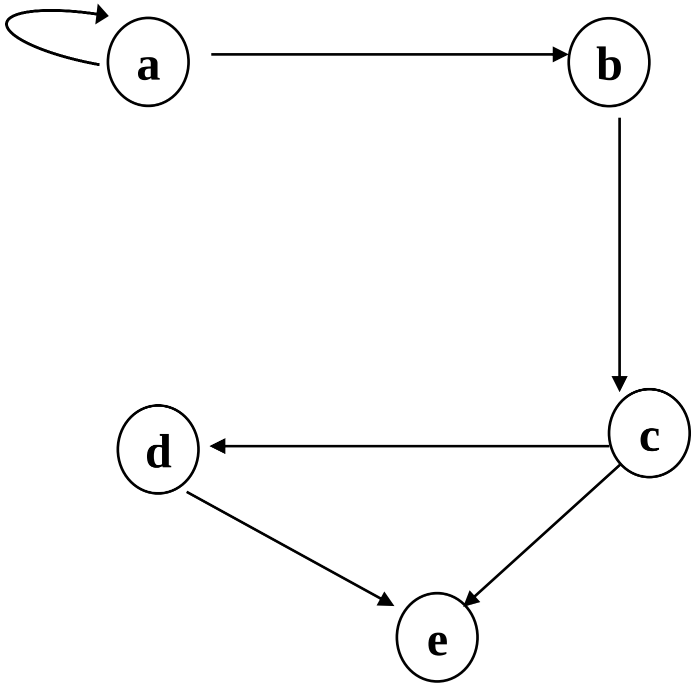
Note:

- R^{n} represents the relation of path length n from one vertex to another.
- A directed graph of relation R is said to be strongly connected if $(x, y) \in R^{\infty}$ for all distinct vertices x & y.

PROBLEMS:

1.Let $A=\{a,b,c,d,e\}$ & $R=\{(a,a),(a,b),(b,c),(c,e),(c,d),(d,e)\}$ find R^2 & R^∞ also write the digraph of R

Solution: The digraph of R is given by



aR^2a, since aRa & aRa.

aR^2c, since aRb & bRc.

aR^2b, since aRa & aRb.

bR^2e, since bRc & cRe.

bR^2d, since bRc & cRd.

cR^2e, since cRd & dRe.

$\therefore R^2=\{(a,a),(a,b),(a,c),(b,e),(b,d),(c,e)\}$.

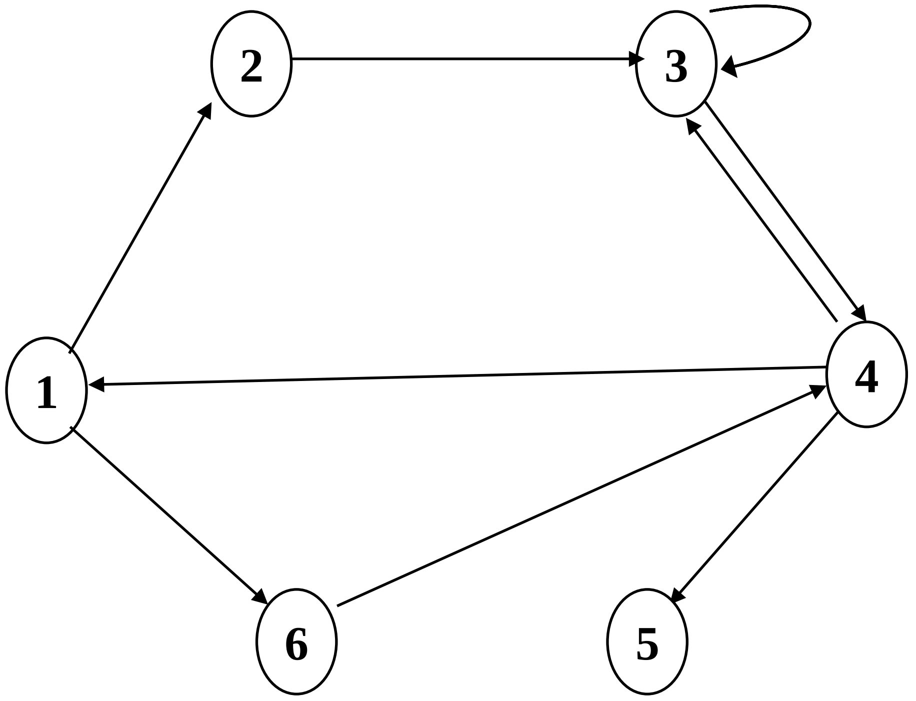
w.k.t R^∞ is list of all ordered pairs if vertices for which there is a path of any length from one vertex to another.

Thus, $R^\infty=\{(a,a),(a,b),(a,c),(a,d),(a,e),(b,c),(b,e),(b,d),(c,d),(c,e),(d,e)\}$

PROBLEMS:

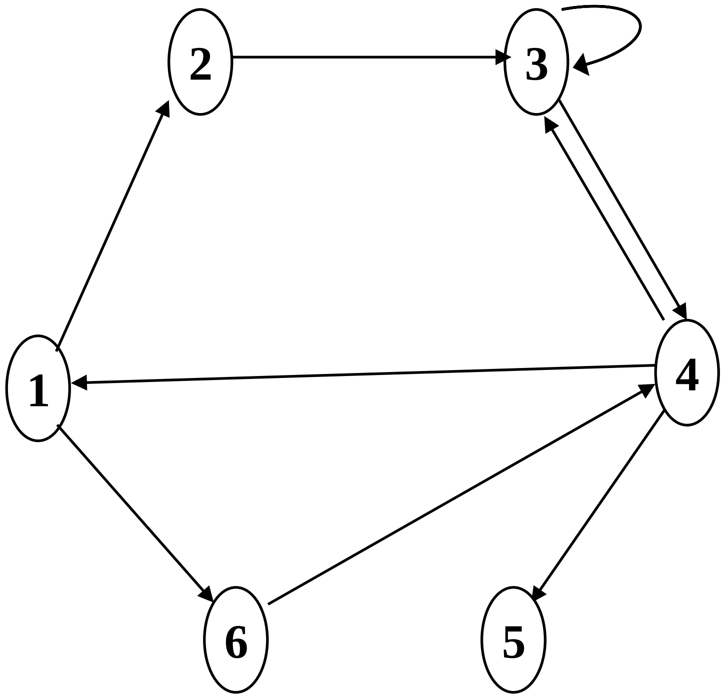
2.Let R be the relation its digraph is given below

a) List all paths of length 1 and write in the form of R
b) Find $R^2$by Listing all path of length 2 starting from vertex 2
c) List all paths of length 2
d) List all paths of length 3 starting from vertex 3[homework]
e) List all path of length 3 [homework]
f) Find a cycle starting at vertex 6 and vertex 2
g) Draw the digraph of R^2
h) Find $M(R^2)$
i) Find $M(R^\infty)$



PROBLEMS:

Solution:



a)$R = \{(1,2),(1,6),(2,3),(3,3),(3,4),(4,3),(4,5),(6,4),(4,1)\}$

b) R^2 from vertex 2 is given by

$2R3$ since $2R3$ & $3R3$, $2R4$ since $2R3$ & $3R4$

Thus, R^2 from vertex 2= {(2,3),(2,4)}.

c)1,2,3; 1,6,4; 2,3,3; 2,3,4; 3,3,3; 3,3,4; 3,4,3; 3,4,5; 3,4,1; 4,3,4;

4,3,3; 4,1,6; 4,1,2; 6,4,3; 6,4,1; 6,4,5;

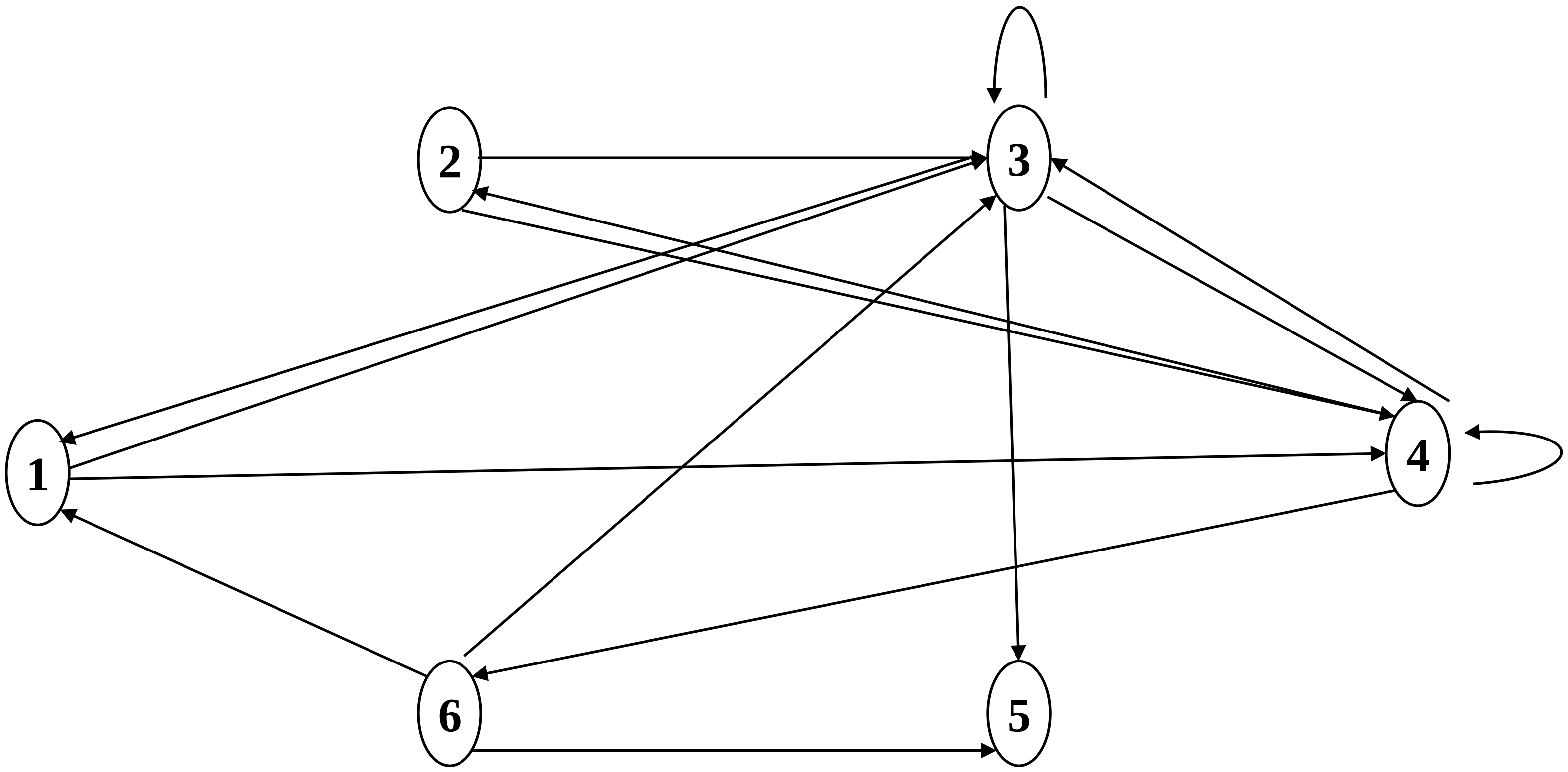
f) Cycle from vertex 6 is 6,4,1,6

Cycle from vertex 2 is 2,3,3,4,1,2

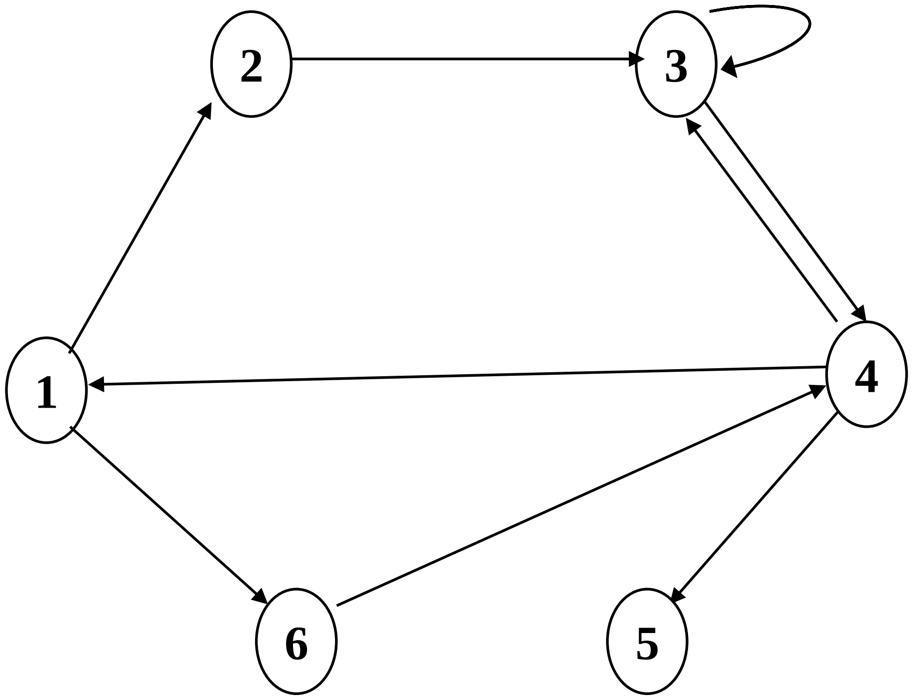
PROBLEMS:

g) To draw R^2 we need $R^2 = \{(1,3), (1,4), (2,3), (2,4), (3,4), (3,3), (3,1), (3,5), (4,4),$

$(4,3), (4,6), (4,2), (6,3), (6,1), (6,5)\}$



PROBLEMS:



h) $M(R^2) = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$

$i)\ R^{\infty}$

$= \{(1,2), (1,3), (1,4), (1,5), (1,6), (1,1), (2,3), (2,4), (2,5), (2,1), (2,6),$

$(2,2), (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6),$

$(6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}$

$M(R^{\infty}) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$

PROPERITIES OF RELATIONS

Reflexive relation:

A relation R defined on set A is called reflexive relation if $aRa,\ \forall\ a \in A$

Example :

Let $A = \{1,2,3,4\}$ then

$R_1 = \{(1,1), (2,2), (3,3), (4,4)\}$ →Reflexive

$R_2 = \{(1,1), (2,2), (3,2), (4,4)\}$ → Non-reflexive

$R_3 = \{(1,2), (2,3), (3,4), (4,1)\}$ → Irreflexive

Note:

- A relation in which no element is related to itself is called Irreflexive relation.
- In the relation if at least one element is not related itself then it is called Non-reflexive.

PROPERITIES OF RELATIONS

Symmetric relation:

Let R be relation defined on set A, R is called symmetric relation $If\ aRb \Rightarrow bRa$

Example : Let $A = \{1,2,3,4\}$ then

$R_1 = \{(1,2),(2,1),(3,2),(2,3),(4,4)\} \rightarrow$ Symmetric.

$R_2 = \{(1,2),(3,2),(2,3),(4,4)\} \rightarrow$ Not symmetric (Asymmetric).

Note: A relation which is not symmetric is called asymmetric relation.

Antisymmetric relation:

Let R be set defined on a set A. Then R is an antisymmetric relation $If\ aRb\ and\ bRa \Rightarrow a = b\ OR$

$$aRb\ and\ (b,a) \notin R \Rightarrow a \neq b$$

PROPERITIES OF RELATIONS

Example :

i. $a \leq b \text{ and } b \leq a \Rightarrow a = b.$

ii. $a \geq b \text{ and } b \geq a \Rightarrow a = b.$

iii. $A \subseteq B \text{ and } B \subseteq A \Rightarrow A = B.$

Transitive relation:

Let R is a relation defined on a set A. Then R is called transitive relation

$$If\ aRb\ and\ bRc \Rightarrow aRc$$

Example :Let $A = \{1,2,3,4,5\}$ then

$R_1 = \{(1,2), (2,3), (1,3), (1,4), (2,4), (2,5), (3,4), (4,5), (3,5), (1,5)\}$ → Transitive.

$R_2 = \{(1,2), (2,3)\}$ → Not Transitive.

PROBLEMS

1.Let A={1,2,3} . Determine the nature of the following relations on A:

i. $R_1 = \{(1,2),(2,1),(1,3),(3,1)\}$.

ii. $R_2 = \{(1,1),(2,2),(3,3),(2,3)\}$.

iii. $R_3 = \{(1,1),(2,2),(3,3)\}$.

iv. $R_4 = \{(1,1),(2,2),(3,3),(2,3),(3,2)\}$.

v. $R_5 = \{(1,1),(2,3),(3,3)\}$.

vi. $R_6 = \{(2,3),(3,4),(2,4)\}$.

vii. $R_7 = \{(1,3),(3,2)\}$.

PROBLEMS

Solution: By examining all ordered pairs present in the relations given, we find that:

R_1 is symmetric and irreflexive , but neither reflexive nor transitive.

R_2 is reflexive and transitive, but not symmetric.

$R_3 \; and \; R_4$ are both reflexive and symmetric.

R_5 is neither reflexive nor symmetric.

R_6 is transitive and irreflexive, but not symmetric.

R_7 is irreflexive , but neither transitive not symmetric

PROBLEMS

Solution: By examining all ordered pairs present in the relations given, we find that:

R_1 is symmetric and irreflexive , but neither reflexive nor transitive.

R_2 is reflexive and transitive, but not symmetric.

R_3 *and* R_4 are both reflexive and symmetric.

R_5 is neither reflexive nor symmetric.

R_6 is transitive and irreflexive, but not symmetric.

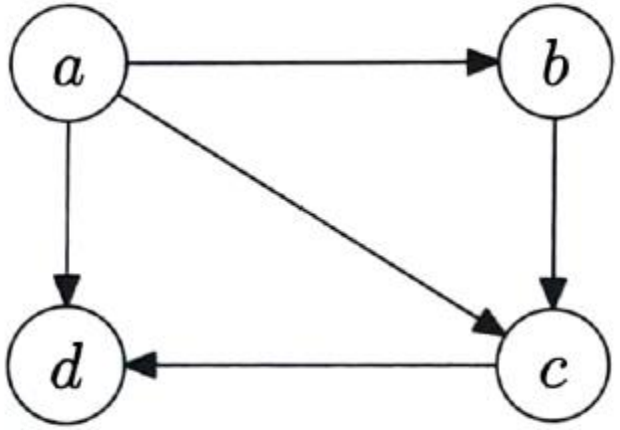
R_7 is irreflexive , but neither transitive not symmetric

PROBLEMS

2.Let A={1,2,3,4} . Determine the nature of the following relations on A:

i. $\boldsymbol{R_1 = \{(1,1),(1,2),(2,1),(2,2),(3,3),(3,4),(4,3),(4,4)\}}$ **.**

ii. $\boldsymbol{R_2}$ **represented by the following diagraph:**



Solution: By examining all ordered pairs present in the relations given, we find that:

R_1 is reflexive, symmetric and transitive.

R_2 is both asymmetric and antisymmetric.

PROBLEMS

3.Find the nature of the relations represented by the following matrices on A:

$$(a)\begin{bmatrix}0&1&1&0\\1&1&0&0\\1&0&1&1\\0&0&1&1\end{bmatrix}\qquad (b)\begin{bmatrix}1&0&1&0\\0&1&0&1\\1&0&1&0\\0&1&0&1\end{bmatrix}\qquad (c)\begin{bmatrix}0&0&1&1\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{bmatrix}$$

Solution: By examining the given matrix, we find that:

a) Relation is Symmetric. Since the matrix is symmetric.

b) Relation is Reflexive and Symmetric.

c) Relation is not symmetric and not antisymmetric.

PROBLEMS

4.Let A={a,b,c,d,e} and R= $\{(\boldsymbol{a},\boldsymbol{d}),(\boldsymbol{d},\boldsymbol{a}),(\boldsymbol{c},\boldsymbol{b}),(\boldsymbol{b},\boldsymbol{c}),(\boldsymbol{c},\boldsymbol{e}),(\boldsymbol{e},\boldsymbol{c}),(\boldsymbol{b},\boldsymbol{e}),(\boldsymbol{e},\boldsymbol{b}),(\boldsymbol{e},\boldsymbol{e})\}$

be a symmetric relation on A. Draw the graph of R. [HOMEWORK]

EQUIVALENCE RELATION

Equivalence relation:

A relation R is defined on a set A. then R is called an equivalence relation. If it is reflexive, symmetric, and transitive.

Problems 1.Let A={1,2,3,4} and $R = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,3),(3,3),(4,4)\}$ **be a relation on A. Verify that**

R is Equivalence relation.

Solution: We have to show that R is reflexive, symmetric, and transitive.

First we note that all of $(1,1),(1,2),(2,1),(2,2),(3,3),(4,4)$ belong to R.That is $aRa,\ \forall\ a \in A$.Therefore, R is a reflexive relation.

Next we note the following: $(1,2),(2,1) \in R$ and $(3,4),(4,3) \in R$

That is, if $aRb \Rightarrow bRa$ **for** $a,b \in A$. Therefore, R is a symmetric relation**.**

PROBLEMS

Lastly we note that

$(1,2),(2,1),(1,1) \in R$, $(2,1),(1,2),(2,2) \in R$, $(4,3),(3,4),(4,4) \in R$ That is, If $aRb \text{ and } bRc \Rightarrow aRc$ for $a, b, c \in A$

Therefore, R is a transitive relation.

PROBLEMS

2.Show that the following relation R defined on the set of all integers Z is an equivalence relation. $\mathbf{R = \{(x, y):\ x, y \in Z \text{ and } (x - y) \text{is an even number}\}}$ **.**

Solution: It is given that xRy iff $x - y[= 2m]$ is an even number

We shall show that R is reflexive, symmetric, transitive.

Reflexive relation:

Let us consider xRx i.e., $x - x = 0 = 2(0)$ is an even number. Hence R is Reflexive

Symmetric Relation:

Let $xRy \Rightarrow x - y = 2m$ is even integer $m \in Z$

$\therefore yRx \Rightarrow y - x = -(x - y) = -2m = 2(-m)$ is also an even integer

That is $xRy \Rightarrow yRx$. Hence R is Symmetric

PROBLEMS

Transitive Relation:

Let $xRy \Rightarrow (x-y)$ is even. That is $x-y=2m$(say), $m \in Z$.

$yRz = (y-z)$ is even. That is $y-z=2n$(say), $n \in Z$.

Now $(x-y)+(y-z)=2m+2n$

$(x-z)=2(m+n)=2k$(say) $k=m+n \in Z$ is also even $\Rightarrow xRz$

That is $xRy, yRz \Rightarrow xRz$. Hence R is transitive.

Thus, we conclude that the relation R is equivalence relation.

3.Show that the following relation R defined on the set of all integers Z is an equivalence relation. $\mathbf{R = \{(x, y) : x, y \in Z}$ and $\mathbf{(x-y)}$ is a multiple of 5$\}$ [homework].

4.Show that the relation congruence modulo m, $a \equiv b(\text{mod } m)$ on the set of all positive integers Z is an equivalence relation.

Solution :By the definition of congruence modulo m, $a \equiv b(\text{mod } m) \Rightarrow m \text{ divides } (a - b) \Rightarrow (a - b) = km$ where m is fixed and $a, b, k \in Z$

We shall show that relation $R = \{(a, b): (a - b) = km\}$ is an equivalence relation

Reflexive relation: aRa is $a - a = 0$ and m divides 0. Hence R is reflexive.

Symmetric Relation:

$$aRb \Rightarrow (a - b) = km$$

$$\therefore bRa = (b - a) = -(a - b) = -km \Rightarrow m \text{ divides } (b - a)$$

That is $aRb \Rightarrow bRa$. Hence R is Symmetric

PROBLEMS

Transitive Relation:

$aRb \Rightarrow (a - b) = km$ and $bRc = (b - c) = lm$ where $k, l \in Z$.

Now $a - c = (a - b) + (b - c) = km + lm = (k + l)m, \quad (k + l) \in Z$

$\therefore a - c = (k + l)m \Rightarrow m$ divides $(a - c)$ or $a \equiv c(\text{mod } m)$

That is $aRb, bRc \Rightarrow aRc$. Hence R is transitive.

Thus, we conclude that the relation R is equivalence relation.

PROBLEMS

5.The matrix of relation R is given. Is R an equivalence relation? Also write digraph of R . Where $\mathbf{M(R)} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{1} \end{bmatrix}$**.**

Solution:$M(R)$ is a 3×3 matrix and hence let us take the set $A = \{a, b, c\}$ to write the relation R explicitly as follows with reference to the given $M(R)$.

Thus, $R = \{(a, a), (b, b), (b, c), (c, b), (c, c)\}$

By observation we can see all elements are related itself. i.e., $(a, a), (b, b), (c, c) \in R$

Hence R is reflexive.

We observe that $(b, c) \in R$ and also $(c, b) \in R$.

Hence R is symmetric

PROBLEMS

Further we observe the following

$$(b,b),(b,c) \in R; \quad (b,c) \in R$$

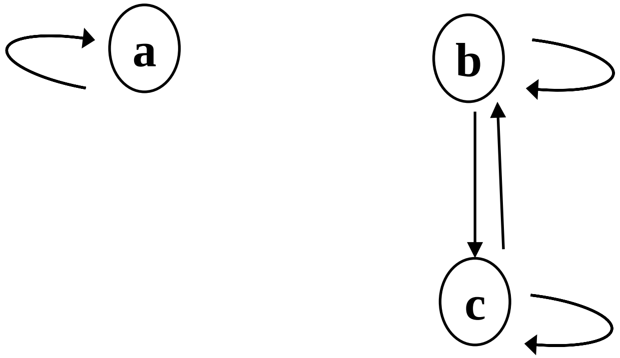
$$(c,b),(b,b) \in R; \quad (c,b) \in R$$

$$(b,c),(c,c) \in R; \quad (b,c) \in R$$

$$(c,c),(c,b) \in R; \quad (c,b) \in R$$

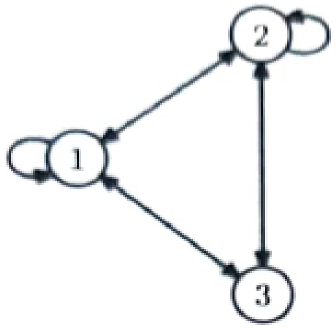
Hence R is transitive.

Thus, we conclude that R is an equivalence relation. Digraph of R is as follows



PROBLEMS

6.The diagraph of a relation R on A={1,2,3,4} as given below. Determine whether R is an Equivalence relation.



Solution: By examining the diagraph, we note that the given relation is symmetric and transitive but not reflexive.

Here (3,3) $\notin$ R. Thus, we conclude that R is not an equivalence relation.

PROBLEMS

7.Let $A = \{1, 2, 3, 4\}$ and let R be the relation defined as follows. $R = \{(x, y): x, y \in A \text{ and } x < y\}$. Determine whether R is reflexive, symmetric, transitive. [Homework].

PARTIAL ORDER SET AND HASSE DIAGRAM:

Partial order: A relation R is defined on a set A is called a partial order. If it is reflexive, antisymmetric and transitive.

(R, A) is called partially ordered or POSET.

HASSE Diagram (or) POSET Diagram (or) Digraph of a partial order:

While drawing the directed graph of a partial order we use the following conventional conditions

1) As R is reflexive every element is related to itself, therefore self-loop need not be shown.

2) As R is transitive aRb and bRc $\Rightarrow$ aRc **we need not draw** an edge from a to c.

3) We represent element of a set A by dots.

4) The digraph drawn in such a way that there edges always upwards. We need not show direction.

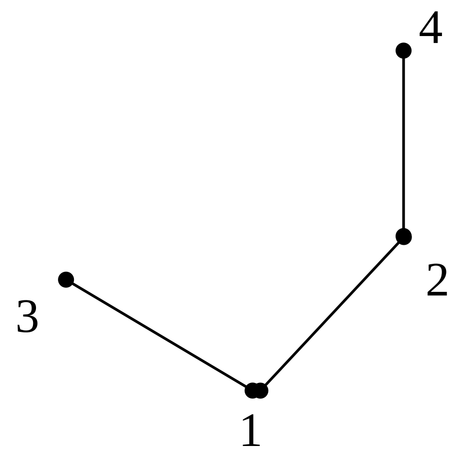
The directed graph obtained using the above is known as POSET diagram or Hasse diagram.

PROBLEMS

Example for Hasse Diagram

Let $A = \{1,2,3,4\}$ then

$R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$



PROBLEMS

1.Let R be a relation defined on set $A = \{1, 2, 3, 4\}$ "xRy iff x divides y" show that R is a partial order and draw the Hasse diagram/Poset Diagram.

Solution:

Reflexive: We know every number divides itself

$\Rightarrow$ x divides x

$\Rightarrow$ xRx

$\Rightarrow$ R is reflexive

Antisymmetric: wkt if x divides y & y divides x then $x = y$

$\Rightarrow$ xRy & yRx $\Rightarrow x = y$

$\Rightarrow$ R is antisymmetric

PROBLEMS

Transitive:

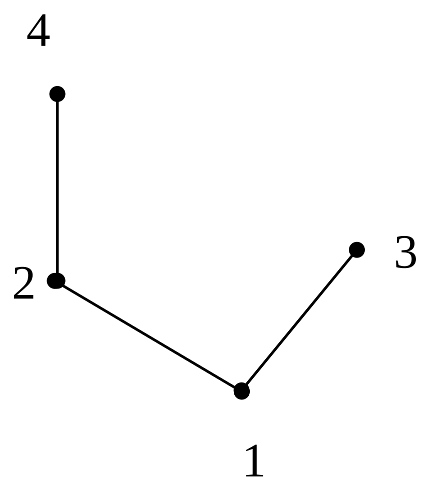
wkt if $x|y$ & $y|z$ then $x|z$

i.e., if xRy & yRz then xRz

$\Rightarrow R\ is\ Transitive$

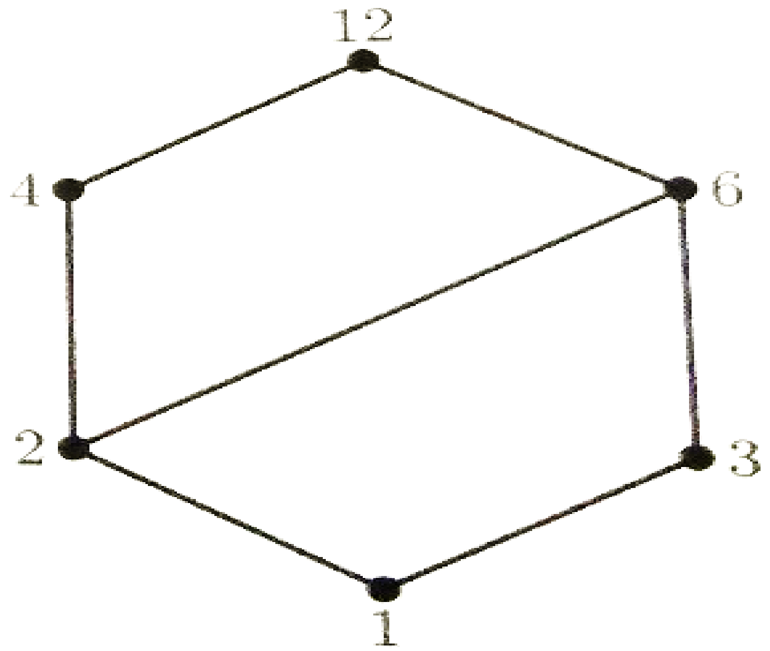
$\therefore$ R is partial order

$R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$



PROBLEMS

2.Let A $= \{1, 2, 3, 4, 6, 12\}$ and on A define the relation R by aRb if "a divides b" prove that R is a partial order on A and Hasse Diagram of this relation.



Solution:$R = \{(1,1), (1,2), (1,3), (1,4), (1,6), (1,12), (2,2), (2,4), (2,6), (2,12), (3,3), (3,6), (3,12), (4,4), (4,12), (6,6), (6,12), (12,12)\}$

We observe the following

$(x, x) \in R\ \forall x \in A \Rightarrow R\ is\ reflexive$

When $(x, y) \in R$ and$(y, x) \in R, y = x, \Rightarrow R\ is\ antisymmetric$

Also, we can see that when $(x, y)\ and\ (y, z) \in R$, then $(x, z) \in R$ which ensures that the transitive property. Hence, we conclude that R is a partial order on A

PROBLEMS

3.Draw the Hasse diagram representing the positive divisors of 36

Solution :

The set of all positive divisors of 36 is $D_{36} = \{1,2,3,4,6,9,12,18,36\}$

The relation R is a divisibility (that aRb if and only if $a\ divides\ b$) is a partial order on this set.

The Hasse diagram for this partial order is required here

We note that, under R

1 is related all elements of D_{36}

2 is related to 2, 4, 6, 12, 18, 36

3 is related to $3, 6, 9, 12, 18, 36$

PROBLEMS

4 is related to 4, 12, , 36

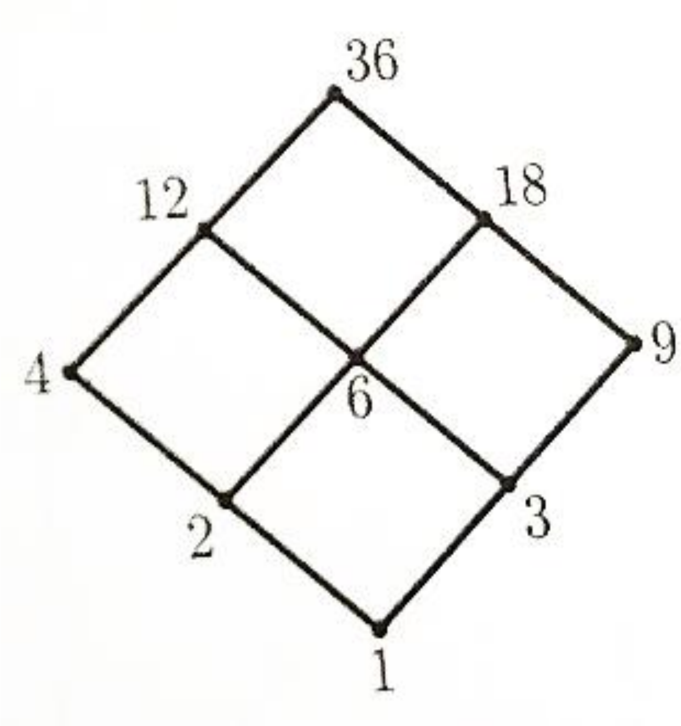
6 is related to 6, 12, 18, 36

9 is related to 9, 18, 36

12 is related to 12, 36

18 is related to 18, 36

36 is related 36



Note: Let R be a partial order on set A. Then R is called a total order on A if for all $x, y \in A$ either xRy or yRx. In this case the poset (A, R) is called a totally ordered set.

PROBLEMS

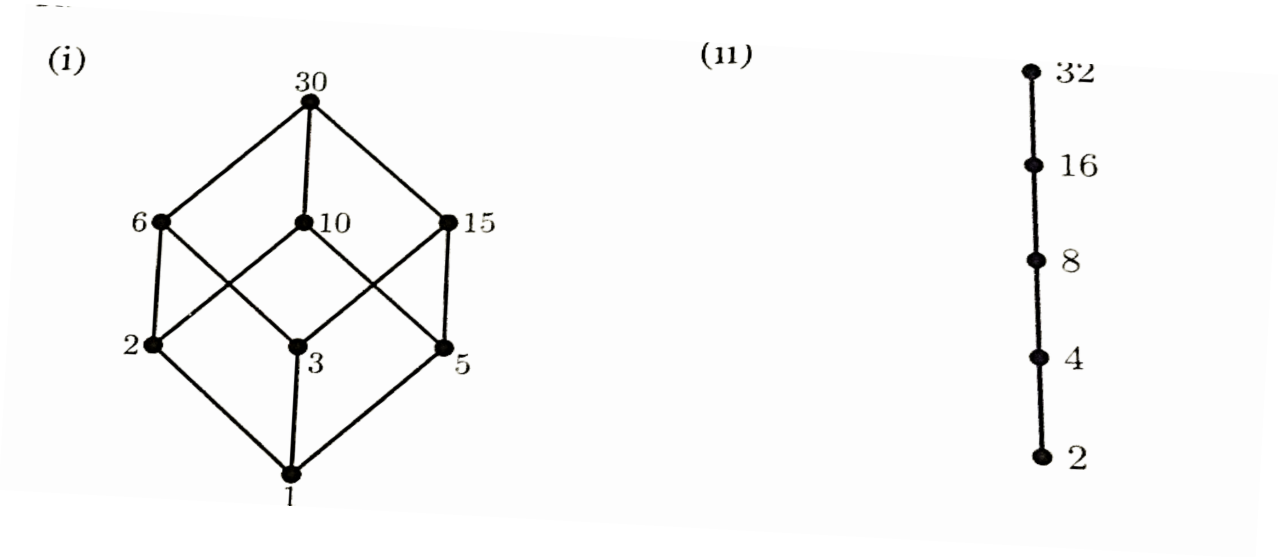
4.In the following cases consider the partial order of divisibility on the set A. Draw the Hasse diagram of poset and determine whether the poset is totally ordered or not

i) $\mathbf{A = \{1, 2, 3, 5, 6, 10, 15, 30\}}$ **[positive divisors of 30]**

ii)$\mathbf{A = \{2, 4, 8, 16, 32\}}$

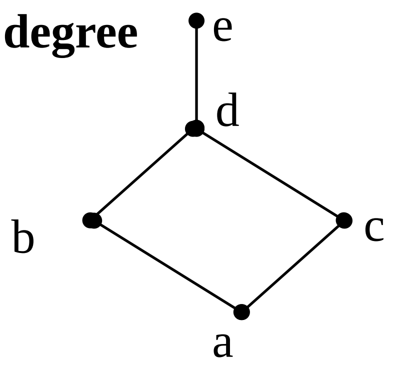
Total order set :Let R be a partial order on a set A.Then R is called a Total order on A if for all x,y belongs to A, either xRy or yRx. In this case the POSET(A,R) is called a totally ordered set.

By examining the above Hasse diagrams we find that the given relation is totally ordered in case (ii) as it is evident that every member of A dividies each one of the succeeding members of A. But is not totally ordered in case (i)



PROBLEMS

5.For $\mathbf{A = \{a, b, c, d, e\}}$ the Hasse diagram for the poset $\mathbf{(A, R)}$ is as show below
Determine the relation matrix for R and construct the digraph for R also find indegree and out degree



Solution:

Relation:

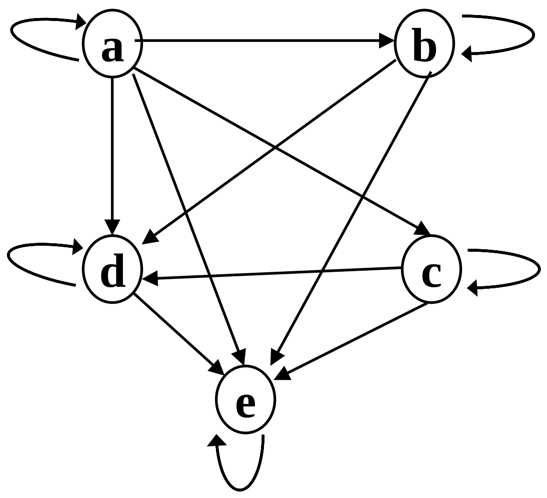
$$R = \{(a,a),(b,b),(c,c),(d,d),(e,e),(a,b),(a,c),(a,d),(a,e),(b,d),(b,e),(c,d),(c,e),(d,e)\}$$

Matrix of relation R:

$$M(R) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

PROBLEMS

Diagraph:



Indegree and Outdegree:

Vertex	Indegree	Outdegree
a	1	5
b	2	3
c	2	3
d	4	2
e	5	1

PROBLEMS

6.Let A $= \{1, 2, 3, 4, 6, 8, 12\}$ and on A define the relation R by aRb if "a divides b" prove that R is a partial order on A and Hasse Diagram of this relation. [homework]

Draw the Hasse diagram representing the positive divisors of 72 [homework]

EXTERNAL ELEMENT:

Maximal element: An element a is maximal element of A iff in the Hasse diagram of R no edge **starts** at a.

Minimal element: An element a is minimal element of A iff in the Hasse diagram of R no edge **terminates** at a.

Greatest element: An element $a \in A$ is called the greatest element if all elements are related to a.

Least element: An element $a \in A$ is called the least element if a related to all element of A.

Upper bound: Let B be a subset of A, an element $a \in A$ is called an upper bound of B if all the element of B are related to a.

EXTERNAL ELEMENT:

Least upper bound (Supremum): An element $a \in A$ is called as least upper bound of a subset B, if the following two conditions holds good

1) If a is an upper bound.

2) If a related to all upper bound.

Greatest lower bound (Infimum): An element $a \in A$ is called as greatest lower bound of a subset B, if the following two conditions holds good

i) If a is lower bound.

ii) All lower bounds are related to a.

PROBLEM

1.Find the maximal, minimal, greatest, and least element (if any) in the given Hasse diagram

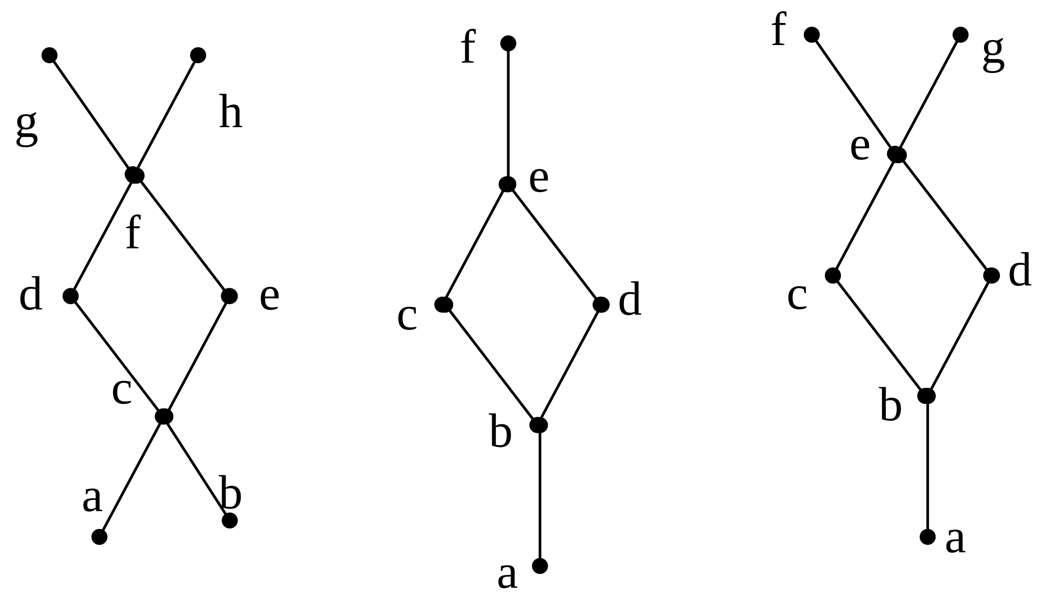


Fig 1 Fig 2 Fig 3

	Fig		
Element	1	2	3
Maximal			
Minimal			
Greatest			
Least			

PROBLEM

1.Find the maximal, minimal, greatest, and least element (if any) in the given Hasse diagram

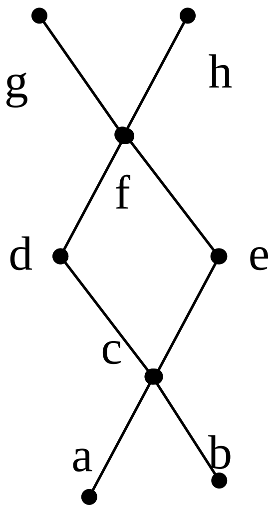


Fig 1

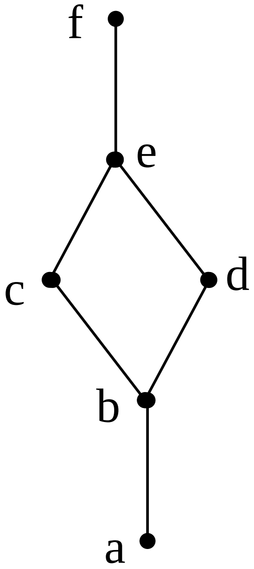


Fig 2

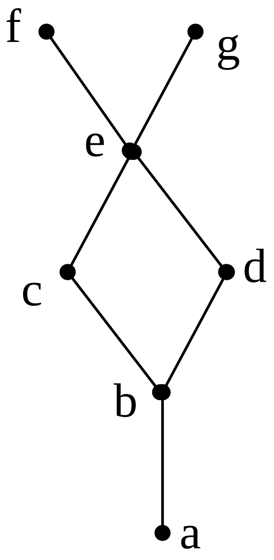


Fig 3

	Fig		
Element	1	2	3
Maximal	g & h	f	f & g
Minimal			
Greatest			
Least			

PROBLEM

1.Find the maximal, minimal, greatest, and least element (if any) in the given Hasse diagram

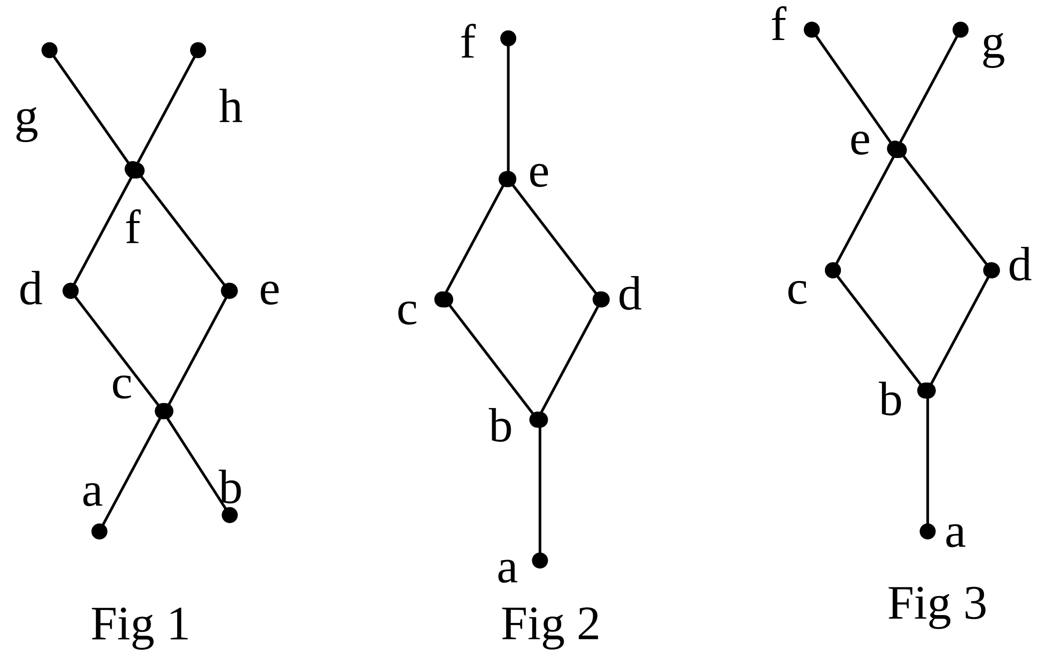


	Fig		
Element	1	2	3
Maximal	g & h	f	f & g
Minimal	a & b	a	a
Greatest			
Least			

PROBLEM

1.Find the maximal, minimal, greatest, and least element (if any) in the given Hasse diagram

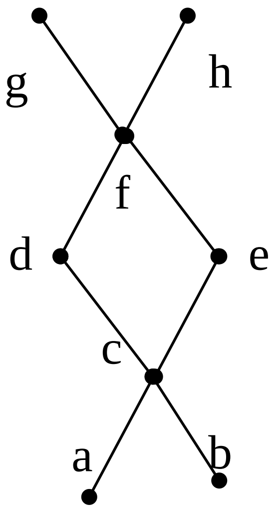


Fig 1

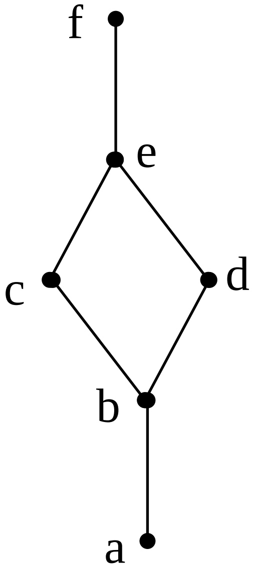


Fig 2

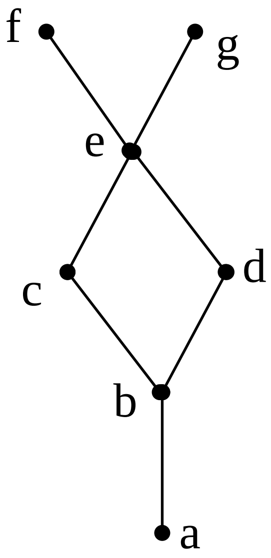


Fig 3

	Fig		
Element	1	2	3
Maximal	g & h	f	f & g
Minimal	a & b	a	a
Greatest	Nil	f	Nil
Least			

PROBLEM

1.Find the maximal, minimal, greatest, and least element (if any) in the given Hasse diagram

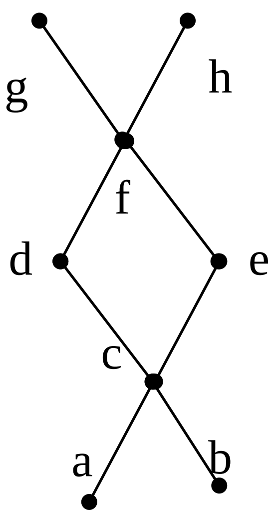


Fig 1

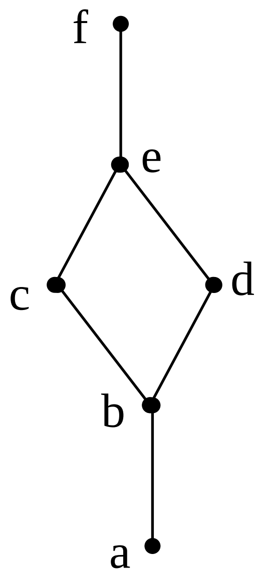


Fig 2

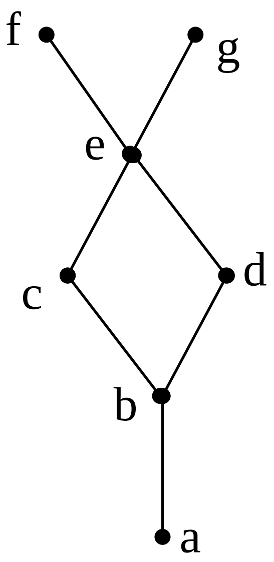
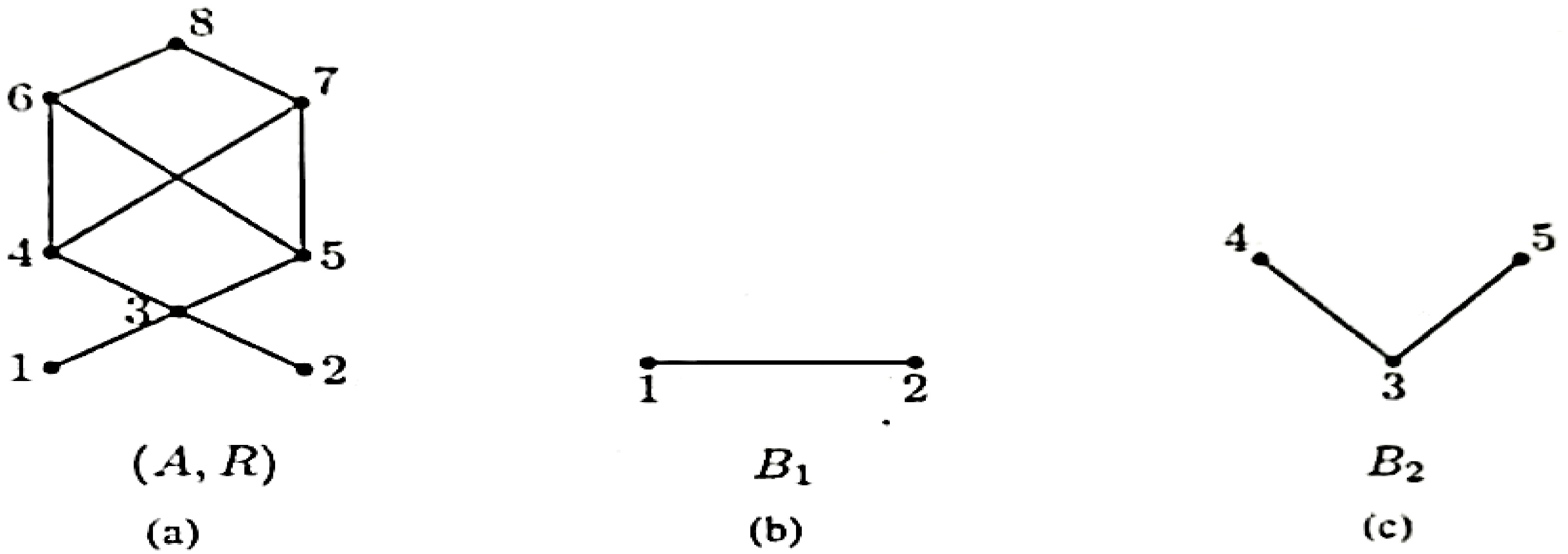


Fig 3

	Fig		
Element	1	2	3
Maximal	g & h	f	f & g
Minimal	a & b	a	a
Greatest	Nil	f	Nil
Least	Nil	a	a

PROBLEM

2. Consider the set A={ 1,2,3,4,5,6,7,8} and a partial order on A whose Hasse diagram as shown below figure 01(a).Consider the subset B1={1,2} and B2={3,4,5} of A shown in the figure 1(b) and 1(c)



Find

- Upper bound of B1
- Least upper bound of B1
- Lower bound of B1
- Greatest lower bound of B1
- Upper bound of B2
- Least upper bound of B2
- Lower bound of B2
- Greatest lower bound of B2

Solution: By examining the Hasse diagram, we make the following observations:

(1) $1R3$, $2R3$. Therefore, 3 is an upper bound of B_1. For a similar reason, 4,5,6,7,8 are also upper bounds of B_1.

(2) The upper bound 3 of B_1 is such that $3Rx$ for all upper bounds x of B_1. Therefore, 3 is a least upper bound (LUB) of B_1; we write this as LUB $(B_1) = 3$

PROBLEM

(3) In A, there is no element x such that $xR1$ and $xR2$. Therefore, B_1 has no lower bounds.

(4) Since B_1 has no lower bounds, it has no greatest lower bound.

(5) For each $x \in B_2$, we have $xR6$. Therefore, 6 is an upper bound of B_2. For a similar reason, 7 and 8 are also upper bounds of B_2.

(6) Although 6 is the least of the upper bound of B_2, 6 is not related to the upper bound 7. Therefore, B_2 has no least upper bound.

(7) For each $x \in B_2$, we have $1Rx$. Therefore, 1 is a lower bound for B_2. For a similar reason, 2 and 3 are also lower bounds of B_2.

(8) Among the lower bounds 1,2,3 of B_2, 3 is such that $1R3$, $2R3$ and $3R3$. Therefore, 3 is the greatest lower bound (GLB) of B_2; we write this as $\text{GLB}(B_2) = 3$.

The following theorems contain some results on extremal elements.

LATTICES

Lattice:

A POSET in which every pair of elements has both a least upper bound and greatest lower bound.

In other words, it is a structure with two binary operations

1.Join: The join of two elements is their least upper bound. It is denoted by $\vee$, not to be confused with disjunction.

2.Meet: The meet of two elements is their greatest lower bound. It is denoted by $\wedge$, not to be confused with a conjunction

LATTICES

In mathematically,

A lattice is a partially ordered set $(L, \leq)$ in which every subset $\{a, b\}$ consisting of two elements has a least upper bound and a greatest lower bound.

We denote LUB($\{a, b\}$) by $a \vee b$ and call it join or sum of a and b. Similarly, we denote GLB ($\{a, b\}$) by $a \wedge b$ and call it meet or product of a and b.

Other symbols used are

LUB: $\oplus$, $+$, $\cup$,

GLB: $*$, $\cdot$, $\cap$.

Thus Lattice is a mathematical structure with two binary operations, join and meet.

A totally ordered set is obviously a lattice but not all partially ordered sets are lattices.

EXAMPLE

1.Let A = {1,2,3,6} where a is related to be by divisibility, meaning “a divides b.” Let’s prove that the relation is a partial order, construct a Hasse diagram, and determine its maximal and minimal elements.

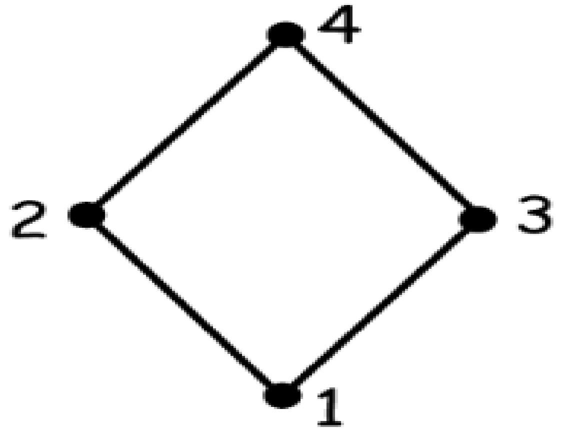
Solution:

$(\{1,2,3,6\},|)$ then $R=\{(1,1),(1,2),(1,3),(1,6),(2,2),(2,6),(3,3),(3,6),(6,6)\}$

Reflexive: $a\,|\,a$ for every integer (true)

Antisymmetric: if $a\,|\,b$ then $b\,|\,a$ only if $a=b$ (true)

Transitive: if $a\,|\,b$ and $b\,|\,c$ then $a\,|\,c$ (true)



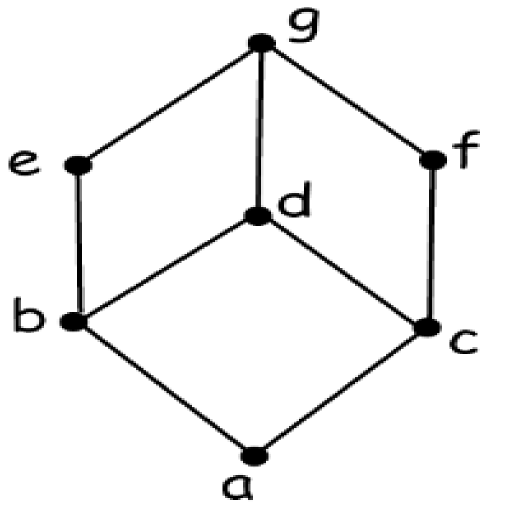
Maximal Element(s): 6
Greatest Element: 6
Minimal Element(s): 1
Least Element: 1

EXAMPLE

And sometimes, we wish to find the upper and lower bounds of a subset of a partial order. An easy way to think of this is to look for downward and upward paths. If a vertex is an “upper bound,” then it has a downward path to all vertices in the subset. And a vertex is a “lower bound” if it has an upward path to all vertices in the subset.

EXAMPLE

2.Let’s determine if the following posets are lattice using a Hasse diagram

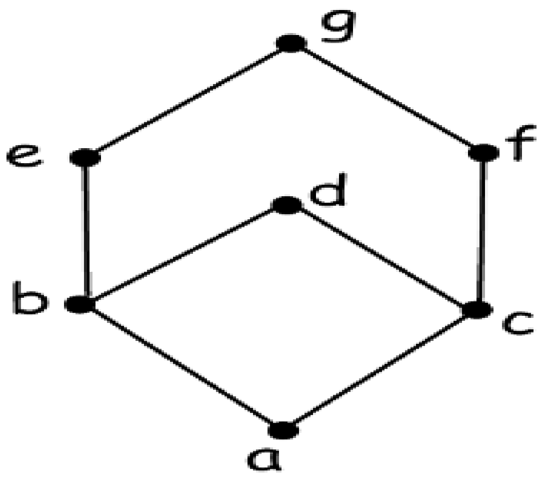


Lattice

Solution:

The partial ordering on the left indicates a lattice because each pair of elements has both a least upper bound and greatest lower bound. In other words, each pair of elements is comparable.

However, the partial ordering on the right is not a lattice because elements b and c are incomparable. Notice that while the upper bound for b and c is {d,e,f,g}, we can’t identify which one of these vertices is the least upper bound (LUB) — therefore, this poset is not a lattice.



Not a Lattice

TRANSITIVE CLOSURE, WARSHALL'S ALGORITHM

Closure and Transitive closure:

If a & b are in a set, then $aR^{\infty}b$ iff there is a path in R from a to b.

Now R^{∞} is certainly transitive since if $aR^{\infty}b$ and $bR^{\infty}c$ then the composition of the paths from a to b and b to c forms a path from a to c in R then $aR^{\infty}c$

Warshall's Algorithm:

It is an efficient method of finding the adjacency matrix of the transitive closure of relation R on a finite set S from the adjacency matrix of R. It uses properties of the diagraph D in particular ,walks of various lengths in D

TRANSITIVE CLOSURE, WARSHALL'S ALGORITHM

Warshall's Algorithm

Step 1: First transfer to W_k all as in W_{k-1}

Step 2: List the locations $p_1, p_2, \ldots,$ in column k of W_{k-1}, where entry is 1 & locations

$q_1, q_2, \ldots$ in row k of W_{k-1} where the entry is 1

Step 3: Put 1 in all the positions p_i, q_j of W_k [if there are not already there].

Example

1. Let $A = \{1,2,3,4\}$ and let $R = \{(1,2), (2,3), (3,4), (2,1)\}$. Compute the transitive closure R^∞ by using Warshall's Algorithm.

PROBLEMS

Solution:

$$W_0 = \mathrm{M(R)} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Column	Row	Transitive
2R1	1R2	2R2

Step1: First we find W_1 so that $k = 1$

$$W_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Column	Row	Transitive
1R2	2R1	1R1
2R2	2R2	1R2
	2R3	1R3

Step2: Second, we find W_2 so that $k = 2$

Column	Row	Transitive
1R3	3R4	1R4
2R3		2R4

$$W_2 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

PROBLEMS

Step3: Next, we find W_3 so that $k = 3$

$$W_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Column	Row	Transitive
1R4		
2R4		
3R4		

Step4: Next, we find W_4 so that $k = 4$

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here W_4 is same as W_3. i.e., $W_3 = W_4$

$$\mathbf{R}^{\infty} = \{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3)(2,4), (3,4)\}$$

PROBLEMS

2. Let $A = \{1, 2, 3, 4\}$ for the relation R whose matrix is given find transitive closure by Warshall's algorithm

Where $W = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Solution:

$$W_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Column	Row	Transitive
1R1		
2R1		

Step1: First we find W_1 so that $k = 1$

$$W_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Column	Row	Transitive
2R2	2R1	2R1
	2R2	2R2

PROBLEMS

Step2: First we find W_2 so that $k = 2$

$$W_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Column	Row	Transitive
3R3	3R3	3R3

Step3: First we find W_2 so that $k = 3$

$$W_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Column	Row	Transitive
4R4	4R4	4R4

Step4: First we find W_2 so that $k = 4$

$$W_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Here W_4 is same as W_3. i.e., $W_3 = W_4$

Hence $R^{\infty} = \{(1,1), (2,1), (2,2), (3,3), (4,4)\}$

PROBLEMS

3. Let $A = \{a,b,c,d,e\}$ and let $M(R) = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$ Compute the W_1, W_2 and W_3 .

Solution

Let $W_0 = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$

Column	Row	Transitive
aRa	aRa	aRa
dRa	aRd	dRd

$W_1 = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$

Column	Row	Transitive
bRb	bRb	bRb
eRb		

PROBLEMS

$$W_2 = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Column	Row	Transitive
	cRd	
	cRe	

$$W_3 = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

PROBLEMS

4. Let $A = \{1,2,3,4\}$ and let $\mathrm{M(R)} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ Compute the transitive closure R^∞ by using Warshall's Algorithm.

Solution: Same as First Problem

5. Let $A = \{1,2,3,4\}$ and let $\mathrm{M(R)} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$ Compute the transitive closure R^∞ by using Warshall's Algorithm.[Home work]

PRINCIPLE OF INCLUSION EXCLUSION

If S is a finite set, then the number of elements in S is called cardinality of S and is denoted by $|S|$. If A and B are subset of S, then the cardinality of $A \cup B$ is given by the formula.

$$|A \cup B| = |A| + |B| - |A \cap B| --- (1)$$

Thus, to determine the elements of $A \cup B$, we include all elements of A and B, but exclude all elements common to A and B.

$$\overline{A} \cap \overline{B} = (\overline{A \cup B}) \text{ and}$$

$$(\overline{A \cup B}) = |S| - |A \cup B|$$

$$|\overline{A} \cap \overline{B}| = (\overline{A \cup B}) = |S| - |A| - |B| + |A \cap B| -- (2)$$

The formula (1) and (2) are equivalent to one another, and either of these is referred to as Addition Principle or the Principle of inclusion-exclusion.

PRINCIPLE OF INCLUSION EXCLUSION

Principle of Inclusion-Exclusion for n sets

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$$

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = \Sigma |A_i| - \Sigma |A_i \cap A_j| + \Sigma |A_i \cap A_j \cap A_k| - \ldots\ldots\ldots + (-1)^{n-1} \Sigma |A_1 \cap A_2 \cap \ldots \cap A_n| \text{-----}(3)$$

By De'morgan law, we have

$$(\overline{A_1 \cup A_2 \cup \ldots \cup A_n}) = \overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \ldots\ldots\ldots\ldots\ldots\ldots \cap \overline{A_n}.$$

Since $|\bar{A}| = |S| - |A|$ for any subset $A\ of\ S$, this yields

$$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \ldots . \cap \overline{A_n}| = |\overline{A_1 \cup A_2 \cup \ldots \cup A_n}|$$

$$= |S| - |(A_1 \cup A_2 \cup \ldots \cup A_n)|$$

Using equation (3), this becomes

$$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \ldots . \cap \overline{A_n}| = |S| - \Sigma |A_i| + \Sigma |A_i \cap A_j| - \Sigma |A_i \cap A_j \cap A_k| - \ldots + (-1)^n \Sigma |A_1 \cap A_2 \cap \ldots \cap A_n| \text{-----}(4)$$

PRINCIPLE OF INCLUSION EXCLUSION

Equation (4) can also be written as

$$\bar{N} = S_1 - S_2 + S_3 + \cdots \ldots\ldots + (-1)^{n-1} S_n \text{-----}(5)$$

$$\bar{N} = S_0 - S_1 + S_2 - S_3 + \cdots \ldots\ldots + (-1)^{n} S_n \text{-----}(6)$$

Where $S_0 = |S|$

$$S_1 = \Sigma \mid A_i \mid$$

$$S_2 = \Sigma \mid A_i \cap A_j \mid$$

$$S_3 = \Sigma \mid A_i \cap A_j \cap A_k \mid$$

$$S_n = \Sigma \mid A_1 \cap A_2 \cap \ldots \cap A_n \mid$$

GENERALIZATION

The principle of inclusion-exclusion as given by equation (6) gives the number of elements in s that satisfy none of the conditions.

The following expression determines the number of elements in S that satisfy exactly m of n conditions $(0 \leq m \leq n)$.

$$E_m = S_m - \binom{m+1}{1} S_{m+1} + \binom{m+2}{2} S_{m+2} - \cdots \ldots \ldots + (-1)^{n-m} \binom{n}{n-m} S_n$$

For $m = 0$, this expression reduces to equation (6).

Further, the following expression determines the number of elements in S that satisfy at least m of n conditions $(1 \leq m \leq n)$.

$$L_m = S_m - \binom{m}{m-1} S_{m+1} + \binom{m+1}{m-1} S_{m+2} - \cdots \ldots \ldots + (-1)^{n-m} \binom{n-1}{m-1} S_n$$

for $m = 1$, this expression in reduces to eq (5).

PROBLEMS

EXAMPLE 01 *Among the students in a hostel, 12 students study Mathematics (A), 20 study Physics (B), 20 study Chemistry (C), and 8 study Biology (D). There are 5 students for A and B, 7 students for A and C, 4 students for A and D, 16 students for B and C, 4 students for B and D, and 3 students for C and D. There are 3 students for A, B and C, 2 for A, B, and D, 2 for B, C and D, 3 for A, C and D. Finally, there are 2 who study all of these subjects. Furthermore, there are 71 students who do not study any of these subjects. Find the total number of students in the hostel.*

Solution:

From, what is given, we have

$$|A| = 12, \quad |B| = 20, \quad |C| = 20, \quad |D| = 8,$$
$$|A \cap B| = 5, \quad |A \cap C| = 7, \quad |A \cap D| = 4,$$
$$|B \cap C| = 16, \quad |B \cap D| = 4, \quad |C \cap D| = 3,$$

PROBLEMS

$$|A \cap B \cap C| = 3, \quad |A \cap B \cap D| = 2, \quad |B \cap C \cap D| = 2,$$

$$|A \cap C \cap D| = 3, \quad |A \cap B \cap C \cap D| = 2, \quad |\overline{A} \cap \overline{B} \cap \overline{C} \cap \overline{D}| = 71,$$

We are required to find $|S|$ where S is the set of all students in the hostel.

The Principle of inclusion-exclusion

$$71 = |S| - (12 + 20 + 20 + 8) + (5 + 7 + 4 + 16 + 4 + 3) - (3 + 2 + 2 + 3) + 2 = |S| - 29$$

This gives $|S| = 71 + 29 = 100.$

Thus, the total number of students in the hostel is 100.

PROBLEMS

EXAMPLE 02 *Out of 30 students in a hostel, 15 study History, 8 study Economics, and 6 study Geography. It is known that 3 students study all these subjects. Show that 7 or more students study none of these subjects.*

Solution:

► Let S denote the set of all students in the hostel, and A_1, A_2, A_3 denote the sets of students who study History, Economics and Geography, respectively. Then, from what is given, we have

$$S_1 = \Sigma|A_i| = 15 + 8 + 6 = 29, \quad \text{and} \quad S_3 = |A_1 \cap A_2 \cap A_3| = 3.$$

The number of students who do not study any of the three subjects is $|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}|$.

PROBLEMS

$$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}| = |S| - \Sigma|A_i| + \Sigma|A_i \cap A_j| - \Sigma|A_1 \cap A_2 \cap A_3|$$

$$= |S| - S_1 + S_2 - S_3$$

$$= 30 - 29 + S_2 - 3 = S_2 - 2 \qquad ((i))$$

where $S_2 = \Sigma|A_i \cap A_j|$.

We note that $(A_1 \cap A_2 \cap A_3)$ is a subset of $(A_i \cap A_j)$ for $i, j = 1, 2, 3$. Therefore, *each* of $|A_i \cap A_j|$, which are 3 in number, is greater than or equal to $|A_1 \cap A_2 \cap A_3|$. Hence

$$S_2 = \Sigma|A_i \cap A_j| \geq 3|A_1 \cap A_2 \cap A_3| = 9.$$

Using this in (i), we find that

$$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}| \geq 9 - 2 = 7.$$

This proves the required result

PROBLEMS

EXAMPLE 03 *Determine the number of positive integers n such that $1 \le n \le 100$ and n is not divisible by 2, 3, or 5.*

Solution:

▶ Let $S = \{1, 2, 3, \ldots, 100\}$. Then $|S| = 100$. Let A_1, A_2, A_3 be the subsets of S whose elements are divisible by 2, 3 and 5 respectively. Then, we have to find $|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}|$.

$$|A_1| = \text{No. of elements in } S \text{ that are divisible by } 2$$
$$= \lfloor 100/2 \rfloor = \lfloor 50 \rfloor = 50,$$

$$|A_2| = \text{No. of elements in } S \text{ that are divisible by } 3$$
$$= \lfloor 100/3 \rfloor = \lfloor 33.333 \rfloor = 33,$$

PROBLEMS

$$|A_3| = \text{No. of elements in } S \text{ that are divisible by 5}$$
$$= \lfloor 100/5 \rfloor = \lfloor 20 \rfloor = 20,$$

$$|A_1 \cap A_2| = \text{No. of elements in } S \text{ that are divisible by 2 and 3}$$
$$= \lfloor 100/6 \rfloor = \lfloor 16.666 \rfloor = 16,$$

$$|A_1 \cap A_3| = \text{No. of elements in } S \text{ that are divisible by 2 and 5}$$
$$= \lfloor 100/10 \rfloor = \lfloor 10 \rfloor = 10,$$

$$|A_2 \cap A_3| = \text{No. of elements in } S \text{ that are divisible by 3 and 5}$$
$$= \lfloor 100/15 \rfloor = \lfloor 6.666 \rfloor = 6,$$

$$|A_1 \cap A_2 \cap A_3| = \text{No. of elements in } S \text{ that are divisible by 2, 3 and 5}$$
$$= \lfloor 100/30 \rfloor = \lfloor 3.333 \rfloor = 3.$$

PROBLEMS

Now, the Principle of inclusion-exclusion gives

$$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}| = |S| - \{|A_1| + |A_2| + |A_3|\} + \{|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|\} - |A_1 \cap A_2 \cap A_3|$$

$$= 100 - (50 + 33 + 20) + (16 + 10 + 6) - 3 = 26.$$

Thus, the required number is 26.

PROBLEMS

EXAMPLE 04 *How many integers between 1 and 300 (inclusive) are*

(i) *divisible by at least one of 5, 6, 8?*

(ii) *divisible by none of 5, 6, 8?*

Solution:

► Let $S = \{1, 2, \ldots, 300\}$ so that $|S| = 300$. Also, let A_1, A_2, A_3 be subsets of S whose elements are divisible by 5, 6, 8 respectively. Then:

(i) The number of elements of S that are divisible by at least one of 5, 6, 8 is $|A_1 \cup A_2 \cup A_3|$. This is given by (see expression (3))

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - \{|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|\} + |A_1 \cap A_2 \cap A_3| \quad \text{(i)}$$

PROBLEMS

We note that

$$|A_1| = \lfloor 300/5 \rfloor = 60, \quad |A_2| = \lfloor 300/6 \rfloor = 50, \quad |A_3| = \lfloor 300/8 \rfloor = 37,$$

$$|A_1 \cap A_2| = \lfloor 300/30 \rfloor = 10, \quad |A_1 \cap A_3| = \lfloor 300/40 \rfloor = 7,$$

$$|A_2 \cap A_3| = \lfloor 300/24 \rfloor = 12 \quad \text{(Note that the l.c.m. of 6 and 8 is 24)},$$

$$|A_1 \cap A_2 \cap A_3| = \lfloor 300/120 \rfloor = 2 \text{ (Note that the l.c.m. of 5, 6, 8 is 120).}$$

Using these in (i), we get

$$|A_1 \cup A_2 \cup A_3| = (60 + 50 + 37) - (10 + 7 + 12) + 2 = 120.$$

Thus, 120 elements of S are divisible by at least one of 5, 6, 8.

PROBLEMS

(ii) The number of elements of S that are divisible by none of 5, 6, 8 is

$$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}| = |S| - |A_1 \cup A_2 \cup A_3| = 300 - 120 = 180.$$

PROBLEMS

EXAMPLE 05 *In how many ways 5 number of a's, 4 number of b's and 3 number of c's can be arranged so that all the identical letters are not in a single block?*

Solution:

► The given letters are $5 + 4 + 3 = 12$ in number, of which 5 are a's 4 are b's and 3 are c's. If S is the set of all permutations (arrangements) of these letters, we have

$$|S| = \frac{12!}{5!\,4!\,3!}.$$

Let A_1 be the set of arrangements of the letters where the 5 a's are in a single block. The number of such arrangements is

$$|A_1| = \frac{8!}{4!\,3!}.$$

(Because in such an arrangement all the a's taken together can be regarded as a single letter and the remaining letters consist of 4 b's and 3 c's).

PROBLEMS

$$|A_1| = \frac{8!}{4!\,3!}.$$

(Because in such an arrangement all the a's taken together can be regarded as a single letter and the remaining letters consist of 4 b's and 3 c's).

Similarly, if A_2 is the set of arrangements where the 4 b's are in a single block, and A_3 is the set of arrangements where the 3 c's are in a single block, we have

$$|A_2| = \frac{9!}{5!\,3!} \quad \text{and} \quad |A_3| = \frac{10!}{5!\,4!}.$$

Likewise,

$$|A_1 \cap A_2| = \frac{5!}{3!}, \quad |A_1 \cap A_2| = \frac{6!}{4!}, \quad |A_2 \cap A_3| = \frac{7!}{5!}, \quad |A_1 \cap A_2 \cap A_3| = 3!.$$

PROBLEMS

Accordingly, the required number of arrangements is

$$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}| = |S| - \{|A_1| + |A_2| + |A_3|\} + \{|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|\} - |A_1 \cap A_2 \cap A_3|$$

$$= \frac{12!}{5!\,4!\,3!} - \left\{\frac{8!}{4!\,3!} + \frac{9!}{5!\,3!} + \frac{10!}{5!\,4!}\right\} + \left\{\frac{5!}{3!} + \frac{6!}{4!} + \frac{7!}{5!}\right\} - 3!$$

$$= 27720 - (280 + 504 + 1260) + (20 + 30 + 42) - 6$$

$$= 25762. \quad \blacksquare$$

PROBLEMS

EXAMPLE 06 *In how many ways can the 26 letters of the English alphabet be permuted so that none of the patterns* CAR, DOG, PUN *or* BYTE *occurs?*

Solution:

- Let S denote the set of all permutations of the 26 letters. Then $|S| = 26!$.

Let A_1 be the set of all permutations in which CAR appears. This word, CAR, consists of three letters which form a single block. The set A_1 therefore consists of all permutations which contain this single block and the 23 remaining letters. Therefore, $|A_1| = 24!$

Similarly, if A_2, A_3, A_4 are the sets of all permutations which contain DOG, PUN and BYTE respectively, we have

$$|A_2| = 24!, \quad |A_3| = 24!, \quad |A_4| = 23!.$$

PROBLEMS

Likewise, we find that*

$$|A_1 \cap A_2| = |A_1 \cap A_3| = |A_2 \cap A_3| = (26 - 6 + 2)! = 22!,$$

$$|A_1 \cap A_4| = |A_2 \cap A_4| = |A_3 \cap A_4| = (26 - 7 + 2) = 21!,$$

$$|A_1 \cap A_2 \cap A_3| = (26 - 9 + 3)! = 20!,$$

$$|A_1 \cap A_2 \cap A_4| = |A_1 \cap A_3 \cap A_4| = |A_2 \cap A_3 \cap A_4| = (26 - 10 + 3)! = 19!,$$

$$|A_1 \cap A_2 \cap A_3 \cap A_4| = (26 - 13 + 4)! = 17!.$$

Therefore, the required number of permutations is given by

$$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4}| = |S| - \Sigma|A_i| + \Sigma|A_i \cap A_j| - \Sigma|A_i \cap A_j \cap A_k| + |A_1 \cap A_2 \cap A_3 \cap A_4|$$

$$= 26! - (3 \times 24! + 23!) + (3 \times 22! + 3 \times 21!) - (20! + 3 \times 19!) + 17!$$

PROBLEMS

EXAMPLE 07 *Determine the number of integers between 1 and 300 (inclusive) which are (i) divisible by exactly two of 5, 6, 8, and (ii) divisible by at least two of 5, 6, 8.*

Solution:

► Let $S = \{1, 2, \ldots, 300\}$ so that $|S| = 300$. Also, let A_1, A_2, A_3 be subsets of S whose elements are divisible by 5, 6, 8 respectively. Then:

$$S_0 = |S| = 300, \quad S_1 = \Sigma|A_i| = 60 + 50 + 37 = 147,$$

$$S_2 = \Sigma|A_i \cap A_j| = 10 + 7 + 12 = 29, \quad S_3 = |A_1 \cap A_2 \cap A_3| = 2.$$

PROBLEMS

Therefore:

(i) the number of integers between 1 and 300 which are divisible by exactly two of 5, 6, 8

$$E_2 = S_2 - \binom{3}{1} S_3 = 29 - \binom{3}{1} \times 2 = 29 - (3 \times 2) = 23,$$

(ii) the number of integers between 1 and 300 which are divisible by at least two of 2, 3, 5 is

$$L_2 = S_2 - \binom{2}{1} S_3 = 29 - (2 \times 2) = 25.$$

PROBLEMS

EXAMPLE 08 *Find the number of permutations of the English letters which contain (i) exactly two, (ii) at least two, (iii) exactly three, and (iv) at least three, of the patterns* CAR, DOG, PUN *and* BYTE.

Solution:

- Let S denote the set of all permutations of the 26 letters. Then $|S| = 26!$.

Let A_1 be the set of all permutations in which CAR appears. This word, CAR, consists of three letters which form a single block. The set A_1 therefore consists of all permutations which contain this single block and the 23 remaining letters. Therefore, $|A_1| = 24!$

Similarly, if A_2, A_3, A_4 are the sets of all permutations which contain DOG, PUN and BYTE respectively, we have

$$|A_2| = 24!, \quad |A_3| = 24!, \quad |A_4| = 23!.$$

PROBLEMS

Likewise, we find that*

$$|A_1 \cap A_2| = |A_1 \cap A_3| = |A_2 \cap A_3| = (26 - 6 + 2)! = 22!,$$

$$|A_1 \cap A_4| = |A_2 \cap A_4| = |A_3 \cap A_4| = (26 - 7 + 2) = 21!,$$

$$|A_1 \cap A_2 \cap A_3| = (26 - 9 + 3)! = 20!,$$

$$|A_1 \cap A_2 \cap A_4| = |A_1 \cap A_3 \cap A_4| = |A_2 \cap A_3 \cap A_4| = (26 - 10 + 3)! = 19!,$$

$$|A_1 \cap A_2 \cap A_3 \cap A_4| = (26 - 13 + 4)! = 17!.$$

$$S_0 = |S| = 26!$$

$$S_1 = \Sigma|A_i| = (3 \times 24!) + 23!,$$

PROBLEMS

$$S_2 = \Sigma|A_i \cap A_j| = (3 \times 22!) + (3 \times 21!),$$

$$S_3 = \Sigma|A_i \cap A_j \cap A_k| = 20! + (3 \times 19!),$$

$$S_4 = |A_1 \cap A_2 \cap A_3 \cap A_4| = 17!.$$

(i) $E_2 = S_2 - \binom{3}{1}S_3 + \binom{4}{2}S_4 = 3 \times (22! + 21!) - 3 \times \{20! + (3 \times 19!)\} + 6 \times 17!.$

(ii) $L_2 = S_2 - \binom{2}{1}S_3 + \binom{3}{1}S_4 = 3 \times (22! + 21!) - 2 \times \{20! + (3 \times 19!)\} + 3 \times 17!.$

(iii) $E_3 = S_3 - \binom{4}{1}S_4 = \{20! + (3 \times 19!)\} - 4 \times 17!.$

(iv) $L_3 = S_3 - \binom{3}{2}S_4 = \{20! + (3 \times 19!)\} - (3 \times 17!).$

PROBLEMS (Home Work)

EXAMPLE 09 *Find the number of permutations of the letters* $a, b, c, \ldots, x, y, z$ *in which none of the patterns* spin, game, path *or* net *occurs.*

EXAMPLE 10 *Consider the students referred to in Example 1. Find, among the subjects indicated, how many study* (i) *exactly 1 subject,* (ii) *exactly 2 subjects,* (iii) *exactly 3 subjects,* (iv) *at least 1 subject,* (v) *at least 2 subjects,* (vi) *at least 3 subjects.*

EXAMPLE 11 *In how many ways can one arrange the letters in the word* CORRESPONDENTS *so that*

(i) *there is no pair of consecutive identical letters?*

(ii) *there are exactly two pairs of consecutive identical letters?*

(iii) *there are at least three pairs of consecutive identical letters?*

DERANGEMENTS – NOTHING IS IN ITS RIGHT PLACE

Derangements:The word ‘arrangement’ refers as things in order. The word derangement is opposite of arrangements which refers as nothing is in its right place.

A permutation of n distinct objects in which none of the objects is in its original place is called **derangement**. The number of possible derangement of n distinct things is denoted by $\mathrm{D_n}$ or $\mathrm{d_n}$.

If there are n integers 1 2 3 $\cdots n$, then the derangement is that 1 should not be in first place, 2 should not be second place, 3 should not be in third place, and so on.

Examples:

i. If there is only one thing, evidently $d_1 = 0$.

ii. Suppose we have 2 things, the arrangement is 1 2 and the only possible derangement is 2 1. Hence $D_2 = 1$.

iii. Suppose we have 3 things, the arrangement is 1 2 3 and the only possible derangements are 3 1 2 and 2 3 1. Hence $D_3 = 2$.

DERANGEMENTS – NOTHING IS IN ITS RIGHT PLACE

Formula for d_n for $n \geq 1$ is given by

$$d_n = n!\left\{1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} \pm \cdots + \frac{(-1)^n}{n!}\right\}$$

$$d_n = n! \times \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

If n is large that is $n \geq 7$ $d_n = n! \times e^{-1}$.

PROBLEMS

1. Compute D_4 and verify the result by actually listing all the derangements of 1234.

Solution

Here there ae 4 objects. Therefore, the number arrangements is,

$$d_n = n!\left\{1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} \pm \cdots + \frac{(-1)^n}{n!}\right\}$$

$$d_4 = 4!\left\{1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right\}$$

$$= 24 \times \left\{1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24}\right\}$$

$$= 12 - 4 + 1 = 9$$

Now we check that the nine derangements of 1 2 3 4 are

2143, 2341, 2413, 3142, 3412, 3421, 4123, 4312, 4321 totally 9 arrangement.

Thus, $D_4 = 9$ and the same is verified by listing all derangements.

PROBLEMS

2. Find the number of ways of the alphabets A, B, C, D, E, F, G are arranged such that A is not in first position, B is not in second position,…, G is not in seventh position.

Solution

Here we need to find d_7 as there are 7 objects

$$d_n = n! \times e^{-1}$$

$$d_7 = 7! \times e^{-1} = 5040 \times 0.3679 = 1854.11 \approx 1854$$

3. Seven books are distributed among 7 students for reading. The books are recollected and redistributed. In how many ways will each student get to read two different books.

Solution

Given 7 books are distributed to 7 students in ${}^{7}P_{7} = 7!$ Ways.

As books are recollected and redistributed so that each student get to read two different books means that we need to find the derangement of 7 books multiplied total number of arrangements $= 7! \times 7! \times e^{-1} = 9342708.48 \approx 9342709$

PROBLEMS

Note:

From the set of all permutation of n distinct objects, one permutation is chosen at random. What id the probability that it is not a derangement?

- The number of permutation of n distinct object is $n!$.
- The number of derangement of these objects is d_n.
- Therefore , the probability that a permutation chosen is not a derangement is

$$p = 1 - \frac{d_n}{n!} = 1 - \frac{1}{1!} + \frac{1}{2!} - \cdots \ldots . \frac{(-1)^n}{n!}$$

For $n \geq 7$, this is approximated to

$$p = 1 - e^{-1} = 1 - 0.3679 = 0.6321$$

PROBLEMS

4.How many permutations of 1, 2, 3, 4, 5, 6, 7, 8 are not derangements?

Solution

The number of permutation of 1, 2, …,8 are 8!

The number of derangements 1, 2, …,8 are d_8

$$d_8 = 8! \times e^{-1} = 40320 \times 0.3679 = 14832.89 \approx 14833$$

Thus required number of permutations that are not derangements is denoted by $\overline{d_8}$

$$\overline{d_8} = 8! - d_8 = 40320 - 14833 = 25487$$

ROOK POLYNOMIALS

Introduction

Chess is an indoor game played between to opponents on opposite side using board which contain 64 of alternating colours. Each player has 16 pieces of pawns in opposite colours white and black with various names placed on each square of keeping top 2 rows and bottom 2 rows on the board having 8 rows and 8 columns consist 16 squares.

Rook is the name of the pawn placed on square of the chess board. Two of the pawns placed on any board having squares greater than or equal to two are said to be attack each other if they are in the same row or column.

Rook Polynomial:

Consider a board C, that assemble a chess board or part of a chess board, consisting of n number of a square pawns are to be placed in the squares such that no two pawns occupy the same squares. Two pawns on the board having more than two squares are set to be captured each other. If they(pawns) are in the same row or same column of the same board.

ROOK POLYNOMIALS

For $2 \leq k \leq r$, let r_k denote the number of ways in which k pawns can be placed on a board C such that no two pawns can captured each other, then the polynomial denoted by

$$R_k(x) = r(C, x) = 1 + r_1 x + r_2 x^2 + r_3 x^3 + \ \dots \dots \dots + r_n x^n$$

is the rook polynomial for each board C. Here r_1 always represents the number of squares on the board.

Product Formula:

Suppose a board C is made of two parts C_1 and C_2 where C_1 and C_2 have no square in same row or column of C such parts of C are called disjoint sub board of C then the rook polynomial of C can be written as $r(C, x) = r(C_1, x)\, r(C_2, x)$

In generally,

If C is made up of pair wise disjoint sub board $c_1, c_2, c_3 \dots \dots . . c_n$ are of

$$r(C, x) = r(c_1, x)\, r(c_2, x) \dots \dots \dots (c_n, x)$$

ROOK POLYNOMIALS

Expansion Formula:

Let C be the given board we choose particular square $\circledast$. Let D be the board obtained from C by deleting the row and column containing the $\circledast$ and let E be the board obtained from C by deleting only the square $\circledast$.

The rook polynomial for board C can be written has $r(C,x) = xr(D,x) + r(E,x)$

Note: Shaded portion of board is called forbidden position.

PROBLEMS

1. **Find the rook polynomial for the following 1×1 board**

1

Solution

w.k.t $r(C, k) = 1 + r_1 x + r_2 x^2 + r_3 x^3 + \ \ldots \ldots \ldots + r_n x^n$

Here $r_1 = 1$ as there is only one square. And all other $r_2 = r_3 = \cdots = r_n = 0$

Thus rook polynomial is given by

$r(C, x) = 1 + x$

PROBLEMS

2. Find the rook polynomial for the following 2×2 board

1	2
3	4

Solution

w.k.t $r(C,x) = 1 + r_1x + r_2x^2 + r_3x^3 + \ldots\ldots\ldots + r_nx^n$.

For this board $r_1 = 4$

The number of ways in which two rooks can be place on this board such that no two pawns capture each other is $r_2 = 2$

Because the two possible such positions are (1,4)& (2,3)

Three rooks cannot be place in this board such that no two pawns capture each other. Thus $r_3 = 0$. Similarly, $r_4 = 0$ and so on.

Accordingly, the rook polynomial for the board is

$$r(C,x) = 1 + 4x + 2x^2$$

PROBLEMS

3. Find the rook polynomial for the following 3×3 board

1	**2**	**3**
4	**5**	**6**
7	**8**	**9**

Solution

w.k.t $r(C, x) = 1 + r_1 x + r_2 x^2 + r_3 x^3 + \; \ldots \ldots \ldots + r_n x^n$.

For this board $r_1 = 9$

We note that 2 non-capturing rooks can be placed on the board in the following positions

(1,5), (1,6), (1,8), (1,9), (2,4), (2,7), (2,6), (2,9), (3,4), (3,5), (3,7), (3,8), (4,8), (4,9),

(5,7), (5,9), (6,7), (6,8)

Thus $r_2 = 18$

PROBLEMS

1	2	3
4	5	6
7	8	9

Next, we look out for the positions of placing 3 mutually non attacking rooks. The positions are as follows.

(1,5,9), (1,6,8), (2,4,9), (2,6,7), (3,4,8), (3,5,7)

Thus, $r_3 = 6$

Later we find that four or more mutually non-capturing rooks can not be placed on board. Thus $r_4 = 0$. Similarly, $r_5 = 0$ and so on.

Accordingly, the rook polynomial for the board is $r(C, x) = 1 + 9x + 18x^2 + 6x^3$

PROBLEMS

Note:

- Rook polynomial for 4×4 board is given by

$$r(C, x) = 1 + 16x + 72x^2 + 96x^3 + 24x^2$$

- Rook polynomial for $n \times n$ board can be obtained by the following expansion

$$r(C, x) = 1 + (1!)({}^nC_1)^2\, x + (2!)({}^nC_2)^2\, x^2 + (3!)({}^nC_3)^2\, x^3 + \cdots + (n!)({}^nC_n)^2\, x^n$$

PROBLEMS

4. Find the rook polynomial for the following board

	1	**2**
	3	**4**
5	**6**	**7**

Solution:w.k.t $r(C,x) = 1 + r_1 x + r_2 x^2 + r_3 x^3 + \ \dots\dots\dots + r_n x^n$.

For this board $r_1 = 7$

The positions for 2 non-capturing rooks are (1,4), (1,5), (1,7), (2,3), (2,5), (2,6), (3,5), (3,7), (4,5), (4,6). Thus $r_2 = 10$

The positions for 3 non-capturing rooks are (1,4,5), (2,3,5). Thus $r_3 = 2$

The board has no positions for four or more mutually non-capturing rooks cannot be placed on board. Thus $r_4 = 0$. Similarly, $r_5 = 0$ and so on.

Accordingly, the rook polynomial for the board is

$$r(C,x) = 1 + 7x + 10x^2 + 2x^3$$

PROBLEMS

5. Find the rook polynomial using product formulae for the board shown below (shaded part)

1	2			
3	4			
			5	6
			7	8
		9	10	11

Solution

We note that the given board C is made up of two disjoint sub-boards C_1 and C_2, where C_1 is the 2×2 board with squares numbered 1 to 4 and C_2 is the board with squares numbered 5 to 11

PROBLEMS

Since C_1 is the 2×2 board we have [by problem 2]

$$r(C_1, x) = 1 + 4x + 2x^2$$

For the board C_2 the rook polynomial is given by [by problem 4]

$$r(C_2, x) = 1 + 7x + 10x^2 + 2x^3$$

Therefore, the product formula yields the rook polynomial for the given board as

$$r(C, x) = r(C_1, x)\, r(C_2, x)$$

$$r(C, x) = [1 + 4x + 2x^2][1 + 7x + 10x^2 + 2x^3]$$

$$r(C, x) = 1 + 11x + 40x^2 + 56x^3 + 28x^4 + 4x^5$$

PROBLEMS

6. Find the rook polynomial for the 3×3 board by using expansion formula

Solution

w.k.t $r(C,x) = 1 + r_1x + r_2x^2 + r_3x^3 + \ \ldots\ldots\ldots + r_nx^n$.

The 3×3 board is given below

1	**2**	**3**
4	**5**	**6**
7	**8**	**9**

Let us mark the square which is at the centre of the board as $\circledast$. Then the boards D and E as shown bellow (the shaded parts are the deleted parts)

PROBLEMS

D

1	2	3
4	5	6
7	8	9

E

1	2	3
4	5	6
7	8	9

For the board D we find that $r_1 = 4$

The positions for 2 non-capturing rooks are $(1,9), (3,7)$. Thus $r_2 = 2$

The board D has no positions for three or more mutually non-capturing rooks cannot be placed on

board. Thus $r_3 = 0$. Similarly, $r_4 = 0$ and so on.

Accordingly, the rook polynomial for the board is

$$r(D, x) = 1 + 4x + 2x^2$$

PROBLEMS

E

1	**2**	**3**
4	**5**	**6**
7	**8**	**9**

For the board E we find that $r_1 = 8$

The positions for 2 non-capturing rooks are

(1,6), (1,8), (1,9), (2,4), (2,7), (2,9), (2,6), (3,4), (3,7), (3,8), (4,8), (4,9), (6,7), (6,8)

Thus $r_2 = 14$

PROBLEMS

The positions for 3 non-capturing rooks are$(1,6,8), (2,4,9), (2,6,7), (3,4,8)$. Thus $r_3 = 4$

The board has no positions for four or more mutually non-capturing rooks cannot be placed on board. Thus $r_4 = 0$.

Similarly, $r_5 = 0$ and so on.

Accordingly, the rook polynomial for the board is $r(E,x) = 1 + 8x + 14x^2 + 4x^3$

Thus, by expansion formula, the rook polynomial for board C can be written has

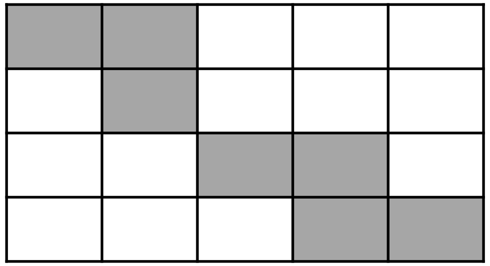
$$r(C,x) = xr(D,x) + r(E,x)$$

$$r(C,x) = x[1 + 4x + 2x^2] + [1 + 8x + 14x^2 + 4x^3]$$

$r(C,x) = 1 + 9x + 18x^2 + 6x^3$.

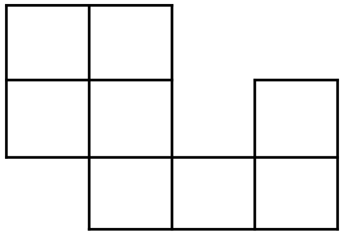
PROBLEMS

7. Find the rook polynomial for the shaded region [HOMEWORK]



PROBLEMS

8. Find the rook polynomial for the following board [homework]



Arrangement with forbidden position

Suppose m objects are to be arranged in n places, where $n \geq m$. Suppose there are constraints under which some objects cannot occupy certain places - such places are called the ***forbidden positions*** for the said objects. The number of ways of carrying out this task is gives by the following rule:

$$\overline{N} = S_0 - S_1 + S_2 - S_3 + \cdots + (-1)^n S_n$$

$$\text{where} \quad S_0 = n!$$

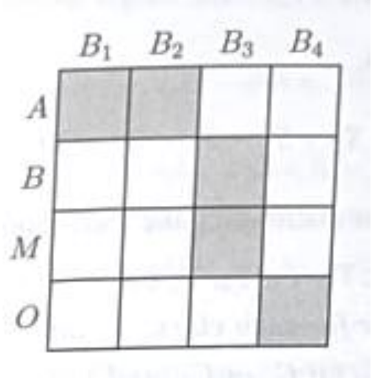
$$\text{and} \qquad S_k = (n-k)! \times r_k. \quad \text{for } k = 1, 2, 3, \ldots, n$$

Her r_k is the coefficient of x^k in the rook polynomial of the board of m rows and n columns ***whose squares represent the forbidden places*** (under the specified conditions).

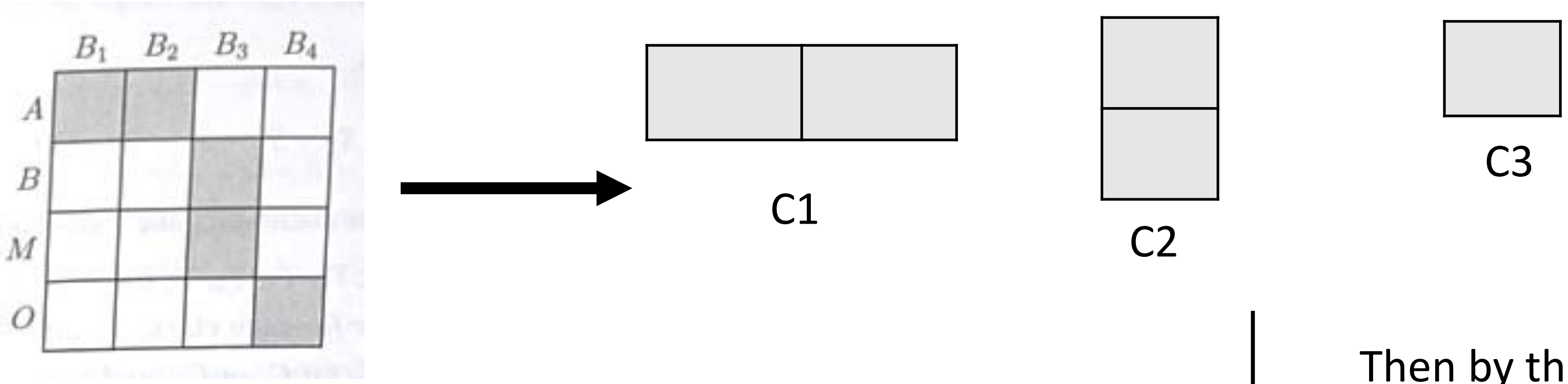
PROBLEMS

1. An apple, a banana, a mango and an orange are to be distributed to four boys B1, B2, B3, B4 The boys B1 and B2 do not wish to have apple, the boy B3 does not want banana or mango, and B4 refuses orange. In how many ways the distribution can be made so that no boy is displeased?

Solution: The situation can be described by the board shown below in which the rows respectively represent apple, banana, mango and orange, and the columns represent the boys B1, B2, B3, B4, respectively. Also, the shaded squares together represent the forbidden places in the distribution.



Let us consider the board C consisting of the shaded squares in above Figure. We note that C is formed by the mutually disjoint boards C1, C2, C3 shown in below figure



As such, the rook polynomial for C is (by the product formula)

$$r(C,x) = r(C_1,x) \times r(C_2,x) \times r(C_3,x)$$

By inspection, we find that

$$r(C_1,x) = 1+2x, \quad r(C_2,x) = 1+2x, \quad r(C_3,x) = 1+x$$

Accordingly, we have

$$r(C,x) = (1+2x)^2(1+x) = 1+5x+8x^2+4x^3$$

Thus we have r1 =5, r2 = 8, r3 = 4.

Then by the formula

$$S_0 = 4! = 24, \quad S_1 = (4-1)! \times r_1 = 30,$$

$$S_2 = (4-2)! \times r_2 = 16, \qquad S_3 = (4-3)! \times r_3 = 4$$

Therefore

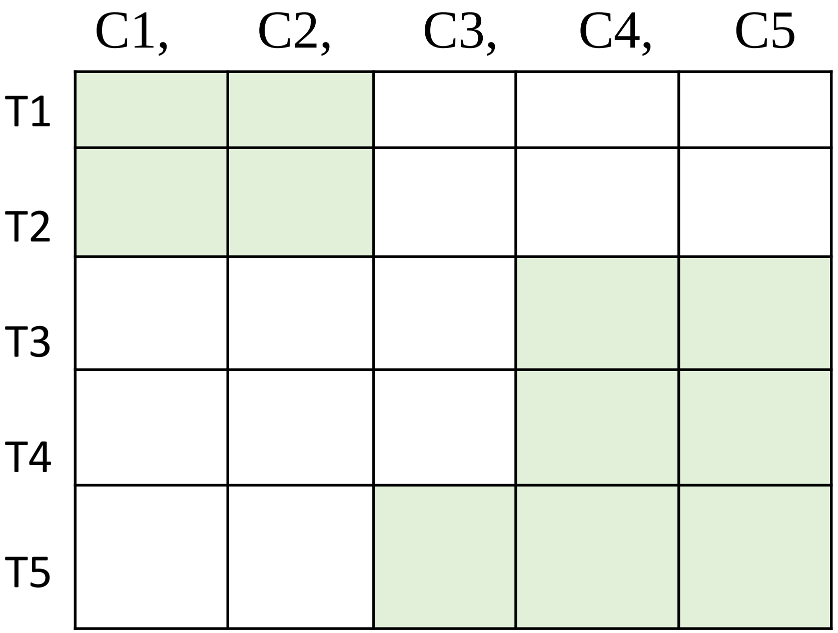
$$\bar{N} = S_0 - S_1 + S_2 - S_3 = 24 - 30 + 16 - 4 = 6$$

This is the number of ways the distribution can be made so that no boy is displeased.

PROBLEMS

2. Five teachers T1, T2, T3, T4, T5 are to be made class teachers for five class C1, C2, C3, C4, C5, one teacher for each class. T1 and T2 do not wish to become the class teachers for C1 or C2, T3 and T4 for C4 or C5, and T5 for C3 or C4 or C5. In how many ways can the teachers be assigned the work (without displeasing any teacher)?

Solution: The situation can be represented by the board



By example 7, we know that

$$r(C,x) = 1 + 11x + 40x^2 + 56x^3 + 28x^4 + 4x^5$$

$$r_1 = 11,\ r_2 = 40,\ r_3 = 56,\ \ r_4 = 28,\ \ r_5 = 4$$

$$So = 5! = 120, S1 = (5-1)!.r1 = 264$$

$$S_2 = (5-2)! \times r_2 = 240, \quad S_3 = (5-3)! \times r_3 = 112,$$
$$S_4 = (5-4)! \times r_4 = 28, \quad S_5 = (5-5)! \times r_5 = 4.$$

therefore

$$S_0 - S_1 + S_2 - S_3 + S_4 - S_5 = 120 - 264 + 240 - 112 + 28 - 4 = 8$$

3. Four persons P1, P2, P3, P4 who arrive late for a dinner party find that only one chair at each of five tables T1, T2, T3, T4 and Ts is vacant. P1 will not sit at T1 or T2, P2 will not sit at T2, P3 will not sit at T3 or T4, and P4 will not sit at T4 or T5. Find the number of ways they can occupy the vacant chairs.**[home work]**

THANK YOU