Name: Ramu Barai
USN: 21BTRCD059
Subject: Discrete Mathematics and Graph theory
Subcode: 21CIDS31

Assignment : 02

1) State the Pigeonhole principle. Also, prove that if 5 colours are used to paint 26 doors, then atleast 6 doors will have the same colour.

Ans

Pigeonhole principle

Statement: If n pigeons occupy m pigeonholes and if $m < n$ then at least one pigeonhole contains two or more pigeons.

ATQ

we have

Pigeonhole $(m) = 5$

Pigeon $(n) = 26$

To show, $P+1 = 6$

Now

we know that

$$P+1 = \left\lfloor \frac{n-1}{m} \right\rfloor + 1$$

$$= \left\lfloor \frac{26-1}{5} \right\rfloor + 1$$

$$= 5+1$$

$$= 6$$

$\therefore P+1 = 6$

Hence Proved

2) Find the least number of ways of choosing three different numbers from 1 to 10 so that all choices have the same sum.

Soln

Let us first write all the numbers from 1 to 10 as (1, 2, 3, 4, 5, 6, 7, 8, 9, 10)

Now,

Let's take randomly three number in each steps so that the sum is same in all case.

i.e. We define the following as

Case 1, $1+2+10$ or

Case 2, $1+4+8$ or

Case 3, $1+5+7$ or

Case 4, $1+3+9$ or

Case 5, $2+3+8$ or

Case 6, $2+4+7$ or

Case 7 $2+5+6$ or

Case 8, $3+4+6$.

Hence, In all case, we get three different numbers whose sum is same.

Hence, The required number of ways of choosing is 8.

3) Let $A = \{2, 8, 14, 18\}$ Let R be a relation on A defined by xRy if and only if $x - y > 5$.

a) Write down R as a set of ordered pairs.

b) Write $M(R)$.

c) Draw a directed graph of the relation.

d) Determine the indegree and outdegree of the vertices in the diagraph.

Soln

Given

$$A = \{2, 8, 14, 18\}$$

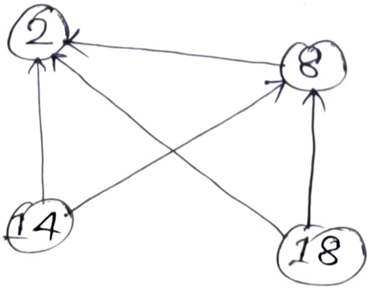
Let R be a relation on A defined by xRy if $x - y > 5$.

$\therefore$ (a) $R = \{(8,2), (14,2), (18,2), (14,8), (18,8)\}$

(b) $MR =$

$$\begin{array}{c|cccc} & 2 & 8 & 14 & 18 \\ 2 & 0 & 0 & 0 & 0 \\ 8 & 1 & 0 & 0 & 0 \\ 14 & 1 & 1 & 0 & 0 \\ 18 & 1 & 1 & 0 & 0 \end{array}$$

(c)



d) Indegree and outdegree

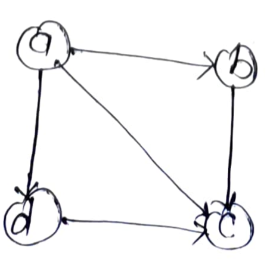
Vertex	Indegree	outdegree
2	3	0
8	2	1
14	0	2
18	0	2

4) Let $A = \{1, 2, 3, 4\}$. Determine the nature of the following relations on A.

i) $R_1 = \{(1,1), (2,2), (3,3), (4,4), (1,2), (2,1), (3,4), (4,3), (4,4)\}$

ii) $R_2 = \{(1,2), (1,3), (3,1), (1,1), (3,3), (3,2), (1,4), (4,2), (3,4)\}$

iii) R_3 represented by the following diagraph.



Soln

i) $R_1 = \{(1,1), (2,2), (3,3), (4,4), (1,2), (2,1), (3,4), (4,3), (4,4)\}$

a) $\forall\, a \in A, (a,a) \in R$

$1 \in A, (1,1) \in R$

$2 \in A, (2,2) \in R$

$3 \in A, (3,3) \in R$

$4 \in A, (4,4) \in R$

$\therefore R_1$ is Reflexive.

b) $\forall\, a,b \in A, (a,b) \in R = (b,a) \in R$

$\forall\, (1,1) \in A, (1,1) = (1,1) \in R$

$\forall\, (2,2) \in A, (2,2) = (2,2) \in R$

$\forall\, (3,3) \in A, (3,3) = (3,3) \in R$

$\forall\, (4,4) \in A, (4,4) = (4,4) \in R$

$\forall\, (1,2) \in A, (1,2) = (2,1) \in R$

$\forall\, (3,4) \in A\ (3,4) = (4,3) \in R$

$\therefore R_1$ is symmetric.

c) $\forall a,b,c \in A, (a,b) \in R$ & $(b,c) \in R$ then $(a,c) \in R$

$\forall (1,1) \in R$ & $(1,2) \in R$, then $(1,2) \in R$

$\forall (1,2) \in R$ & $(2,1) \in R$, then $(1,1) \in R$

$\forall (1,2) \in R$ & $(2,2) \in R$, then $(1,2) \in R$

$\forall (2,1) \in R$ & $(1,2) \in R$, then $(2,2) \in R$

$\forall (3,4) \in R$ & $(4,3) \in R$, then $(3,3) \in R$

$\forall (4,3) \in R$ & $(3,4) \in R$, then $(4,4) \in R$

$\therefore R_1$ is transitive.

Finally, R_1 is reflexive, symmetric and transitive.

ii) $R_2 = \{(1,2), (1,3), (3,1), (1,1), (3,3), (3,2),$
$(1,4), (4,2), (3,4)\}$

(a) $\forall a \in A, (a,a) \in R$

$1 \in A, (1,1) \in R$

$2 \in A, (2,2) \notin R$

$\therefore R_2$ is not reflexive.

(b) $\forall\ a, b \in A.\ (a,b) \in R = (b,a) \in R$

$\forall\ (1,3) \in R = (3,1) \in R$

$\forall\ (1,2) \in R = (2,1) \notin R$

$\therefore$ R_2 is not symmetric.

(c) $\forall\ a, b, c \in A.$ If $(a,b) \in R$ & $(b,c) \in R$ then $(a,c) \in R$

$\forall\ (1,3) \in R$ & $(3,1) \in R$, then $(1,1) \in R$

$\forall\ (3,1) \in R$ & $(1,1) \in R$, then $(3,1) \in R$

$\forall\ (1,1) \in R$ & $(1,3) \in R$, then $(1,3) \in R$

$\forall\ (1,3) \in R$ & $(3,2) \in R$, then $(1,2) \in R$

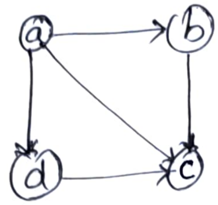
$\forall\ (1,4) \in R$ & $(4,2) \in R$, then $(1,2) \in R$

$\forall\ (3,1) \in R$ & $(1,4) \in R$, then $(3,4) \in R$

$\therefore$ R_2 is Transitive.

Finally, R_2 is Transitive but neither reflexive nor symmetric.

(iii)



from diagram, we have following relation

$R_3 = \{(a,b), (a,c), (a,d), (b,c), (d,c)\}$

(a) $\forall a \in A, (a,a) \in R$

$a \in A, (a,a) \notin R$

$b \in A, (b,b) \notin R$

$c \in A, (c,c) \notin R$

$d \in A, (d,d) \notin R$

$\therefore R_3$ is not reflexive.

(b) $\forall a,b \in A, (a,b) \in R = (b,a) \in R$

$\forall (a,b) \in R, (b,a) \notin R$

$\forall (a,c) \in R, (c,a) \notin R$

$\forall (a,d) \in R, (d,a) \notin R$

$\therefore R_3$ is not symmetric.

(c) $\forall a,b,c$ If $(a,b) \in R$, & $(b,c) \in R$, then $(a,c) \in R$.

$\forall (a,b) \in R$ & $(b,c) \in R$, then $(a,c) \in R$

$\forall (a,d) \in R$ & $(d,c) \in R$, then $(a,c) \in R$

$\therefore R_3$ is transitive.

Finally, R_3 is neither reflexive nor symmetric but is transitive.

5) Define equivalence relation. On set of all integer Z the relation R is defined by $(a,b) \in R \Leftrightarrow a^2-b^2$ is an even integer. Show that R is an equivalence relation.

Ans Equivalence relation :-

A relation R is defined on a set A. then R is called an equivalence relation. If it is reflexive, symmetric, and transitive.

Soln

It is given that aRb if $a^2-b^2 [= 2m]$ is an even number.

we shall show that, R is reflexive, symmetric and transitive.

① Reflexive,

$\forall a \in A, \; aRa$

$\forall a \in Z, \; aRa = a^2-a^2 = 0 = 2(0)$. is an even number.

Hence R is Reflexive.

② Symmetric

$\forall a, b \in A$ such that $aRb \Rightarrow bRa$

$\therefore \forall a, b \in Z$ such that $aRb = a^2-b^2 = 2m$

$\Rightarrow -b^2+a^2 = 2m$

$\Rightarrow -(b^2-a^2) = 2m$

$\Rightarrow b^2-a^2 = 2(-m)$

$\therefore bRa = 2(-m)$

$= 2(-m)$ is also an even integer.

i.e. $aRb \Rightarrow bRa$. hence R is symmetric.

(3) <u>Transitive</u>

$\forall\, a, b, c \in A,\ aRb \,\&\, bRc \Rightarrow aRc$

$\forall\, a, b, c \in Z,\ aRb : a^2 - b^2 = 2m$ (say), $m \in Z$.

$\therefore\ bRc : b^2 - c^2 = 2n$ (say), $n \in Z$.

<u>Now</u>,

$$(a^2 - b^2) + (b^2 - c^2) = 2m + 2n$$

$$\Rightarrow a^2 - b^2 + b^2 - c^2 = 2(m+n)$$

$$\Rightarrow a^2 - c^2 = 2(m+n)$$

$$\Rightarrow aRc = 2k \text{ (say) } k = m+n \in Z$$

i.e. $aRb,\ bRc \Rightarrow aRc$. Hence R is transitive.

Thus, we conclude that the relation R is equivalence relation.

6) Define POSET. Verify that R_1, R_2 is a partial order set on A. Also draw Hasse Diagram.

i) let $A = \{1, 2, 3, 6, 8\}$ and
$R_1 = \{(1,1), (1,2), (1,3), (1,6), (1,8), (2,2), (2,6), (3,3)$
$(2,8), (3,6), (6,6), (8,8)\}$

ii) Let $A = \{1, 2, 3, 4, 5\}$ and
$R_2 = \{(1,1), (1,3), (1,4), (1,5), (2,2), (2,3), (2,4), (2,5),$
$(3,3), (3,4), (3,5), (4,4), (5,5)\}$

Ans A relation R is defined on a set A is called a partial order. If it is reflexive, antisymmetric and transitive. (R, A) is called POSET.

i)

$R_1 = \{(1,1), (1,2), (1,3), (1,6), (1,8), (2,2), (2,6), (3,3)$
$(2,8), (3,6), (6,6), (8,8)\}$

→ Reflexive

$\forall a \in A, aRa \in R$
$\forall 1 \in A, (1,1) \in R$
$\forall 2 \in A, (2,2) \in R$
$\forall 3 \in A, (3,3) \in R$
$\forall 6 \in A, (6,6) \in R$
$\forall 8 \in A, (8,8) \in R$

$\therefore R_1$ is reflexive.

Symmetric

$\forall\ a,b \in A,\ (a,b) \in R\ ;\ (b,a) \in R$ Then $a = b$

$\forall\ (1,1) \in R,\ (1,1) \in R$ Then $1 = 1$

$\forall\ (2,2) \in R,\ (2,2) \in R$ Then $2 = 2$

$\forall\ (3,3) \in R,\ (3,3) \in R$ Then $3 = 3$

Hence, R is antisymmetric.

Transitive

$\forall\ a, b, c \in A,$

If $(a,b) \in R$ & $(b,c) \in R$, then $(a,c) \in R$

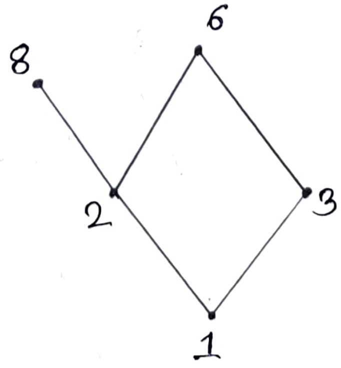
$\forall\ (1,1) \in R, (1,3) \in R,$ then $(1,3) \in R$

$\forall\ (1,2) \in R, (2,2) \in R,$ then $(1,2) \in R$

$\forall\ (1,4) \in R\ (4,4) \in R,$ then $(1,4) \in R$

From above observation, we can see that when (a,b) and $(b,c) \in R$ then $(a,c) \in R$ which ensures that the transitive property.

Hence, we conclude that R_1 is a partial order on A.



ii) Let $A = \{1, 2, 3, 4, 5\}$ and

$R_2 = \{(1,1), (1,3), (1,4), (1,5), (2,2), (2,3), (2,4),$
$(2,5), (3,3), (3,4), (3,5), (4,4), (5,5)\}$

Soln

→ Reflexive

$\forall\ a \in A,\ aRa \in R$

$\forall\ 1 \in A,\ (1,1) \in R$

$\forall\ 2 \in A,\ (2,2) \in R$

$\forall\ 3 \in A,\ (3,3) \in R$

$\forall\ 4 \in A,\ (4,4) \in R$

$\forall\ 5 \in A,\ (5,5) \in R$

$\therefore R_1$ is reflexive.

Symmetric.

$\forall\ a, b \in A.\ (a,b) \in R, (b,a) \in R$, Then $a = b$

$\forall$, 1, 2, 3, 4 and 5 A, $(1,1) \in R$, Then $1 = 1$

$(2,2) \in R$, Then $2 = 2$

$(3,3) \in R$, Then $3 = 3$

$(4,4) \in R$, Then $4 = 4$

$(5,5) \in R$, Then $5 = 5$

$\therefore$ R is antisymmetric.

Transitive.

$\forall\ a, b, c \in A.$

If $(a, b) \in R$, & $(b, c) \in R$, then $(a, c) \in R$.

$\forall\ (1, 1) \in R$ & $(1, 2) \in R$, then $(1, 2) \in R$

$\forall\ (1, 3) \in R$ & $(3, 6) \in R$, then $(1, 6) \in R$

$\forall\ (1, 3) \in R$ & $(3, 4) \in R$, then $(1, 4) \in R$

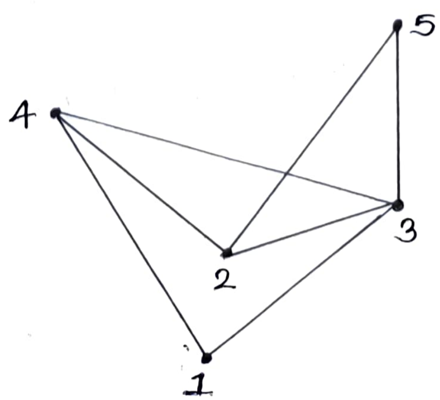
$\forall\ (1, 4) \in R$ & $(4, 4) \in R$, then $(1, 4) \in R$

$\forall\ (1, 5) \in R$ & $(5, 5) \in R$, then $(1, 5) \in R$

$\forall\ (2, 2) \in R$ & $(2, 5) \in R$, then $(2, 5) \in R$

From the above observation, we conclude that when (x, y) and $(y, z) \in R$, then $(x, z) \in R$ which ensures that the transitive property.

Hence, we conclude that R_2 is a partial order on A.



7) Show that divisibility relation of D_{30} is a partial order and draw its Hasse diagram.

Soln,

Let A be the set of all divisors of 30, then
$A = \{1, 2, 3, 5, 6, 10, 15, 30\}$

The relation R is a divisibility (that aRb if and only if a divides b) is a partial order on this set.

$\therefore R = \{(1,1), (1,2), (1,3), (1,5), (1,6), (1,10), (1,15), (1,30), (2,6), (2,10), (2,30), (3,6), (3,15), (3,10), (5,10), (5,15), (5,30), (6,30), (10,30), (15,30), (30,30), (2,2), (3,3), (5,5), (6,6), (10,10), (15,15)\}$

We observe the following

$(x,x) \in R \ \forall x \in A \Rightarrow R$ is reflexive.

when $(x,y) \in R$ and $(y,x) \in R$, $y = x$. $\Rightarrow R$ is antisymmetric.

Also, we can see that (x,y) and $(y,z) \in R$, then $(x,z) \in R$ which ensures that the transitive property. He we conclude that R is a partial order on A.

The Hasse diagram for this partial order is required here.

we note that, under R

1 is related to all elements of D_{30}.

2 is related to 2, 6, 10, 30.

3 is related to 3, 6, 15, 30.

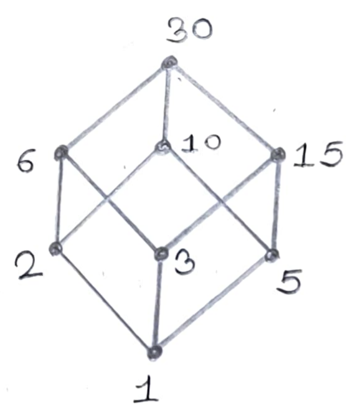
5 is related to 5, 10, 15, 30.

6 is related to 6, 30.

10 is related to 10, 30.

15 is related to 15, 30.

30 is related to 30.



8) Consider the Hasse diagram of POSET (A, R) given below. If $A = \{c, d, e\}$ in Fig 01, $B = \{3, 4, 5\}$ in Fig 02, $C = \{2, 3, 6\}$ in Fig 03. Find if any

i) All upper bounds

ii) All lower bounds

iii) The least upper bounds

iv) Greatest lower bound.

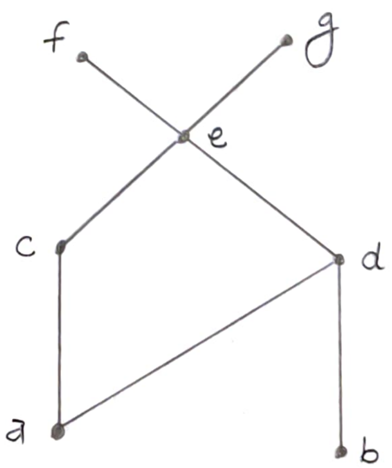


Fig 1

All upper bounds - e, f, g

All Lower bounds - a, b

The least upper bound - e

The Greatest lower bound - a

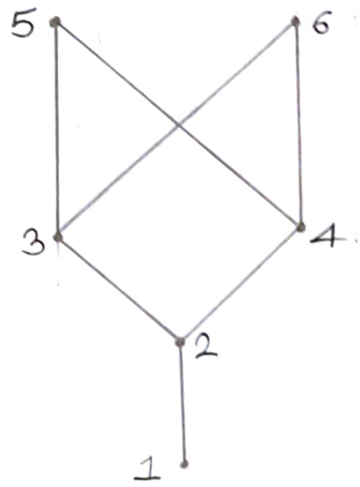


Fig. 02

All upper bounds - 5, 6

All lower bounds - 1, 2

The least upper bound - None

Greatest lower bound - 2

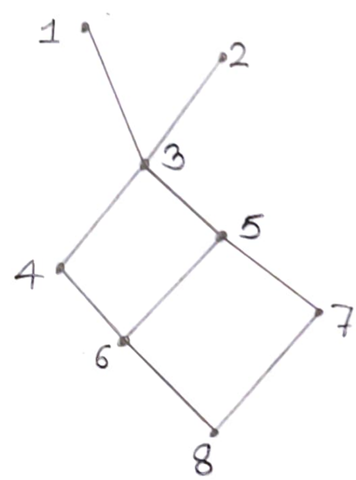


Fig. 03

All upper bounds - 2

All lower bounds - 6, 8

The least upper bound - 2

The greatest lower bound - 6

9) Compute the transitive closure R^{∞} by using warshall's Algorithm.

Let $A = \{1, 2, 3, 4\}$

i) $M_r = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Soln

Let

$$W_0 = M_r = \begin{array}{c|c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

	Col	Row	Transitive
	aRb	bRc	aRc
1.	1R1	1R1	1R1
	2R1		2R1

Step 1.

$$W_1 = \begin{array}{c|c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

	Col	Row	TC
2.	$2R2$	$2R1$	$2R1$
		$2R2$	$2R2$

Step 2

$$W_2 = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

	Col	Row	TC
3.	$3R3$	$3R3$	$3R3$

Step 3

$$W_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

	Col	Row	TC
4.	$4R4$	$4R4$	$4R4$

Step 4.

$$W_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\therefore R^{\infty} = \{(1,1),(2,1),(2,2),(3,3),(4,4)\}$

ii) $M(R) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

Soln

Let

$W_0 = M(R) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

Col	Row	Transitive
1R1	1R1	1R1
4R1	1R4	1R4
		4R1
		4R4

Step 1.

$W_1 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

Col	Row	Transitive
2R2	2R2	2R2
3R2	2R3	2R3
		3R2
		3R3

Step. 2

$$W_2 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Col	Row	TC
2R3	3R2	2R2
3R3	3R3	2R3
		3R2
		3R3

Step 3.

$$W_3 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Col	Row	TC
1R4	4R1	1R1
4R4	4R4	1R4
		4R1
		4R4

Step.4

$$W_4 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$\therefore R^{\infty} = \{(1,1),(1,4),(2,2),(2,3),(3,2),(3,3),(4,1),(4,4)\}$

10.) Among 100 students, 32 study mathematics, 20 study physics, 45 study biology, 15 study mathematics and biology, 7 study mathematics and physics, 10 study biology and physics and 30 donot study any of the subjects.

a) Find the number of students studying all three subjects?

b) Find the number of students studying exactly one of three subjects?

Soln

Here,

Let M, P and B denotes Mathematics, Physics and Biology respectively.

ATQ

$|S| = 100$

$|M| = 32$

$|P| = 20$

$|B| = 45$

$|M \cap B| = 15$

$|M \cap P| = 7$

$|B \cap P| = 10$

$|\overline{M \cup B \cup P}| = 30$

$(M \cap B \cap P) = ?$

Now

$$|S| = |M \cup B \cup P| + |\overline{M \cup B \cup P}|$$

or, $$|S| = |M| + |B| + |P| - |M \cap B| - |M \cap P| - |B \cap P| + |M \cap B \cap P| + |\overline{M \cup B \cup P}|$$

or, $$100 = 32 + 45 + 20 - 15 - 7 - 10 + |M \cap B \cap P| + 30$$

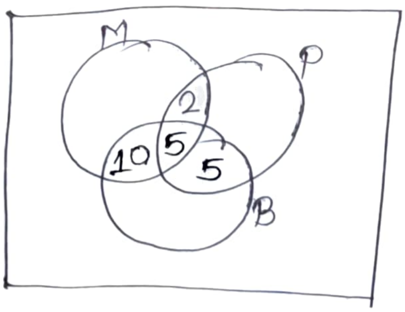
or, $$100 = 127 - 32 + |M \cap B \cap P|$$

or, $$|M \cap B \cap P| = 100 - 127 + 32$$

$$\therefore |M \cap B \cap P| = 5$$

b)

In order to find the number of student studying exactly one of three subjects, we need to draw venn diagram.



$\therefore |M_0| = 32 - (10+5+2) = 15$ study math only

$\therefore |P_0| = 20 - (2+5+5) = 8$ study Physics only

$\therefore |B_0| = 45 - (10+5+5) = 25$ study Biology only

Hence, Number of students studying exactly one of three subjects $= (15+8+25) = 48$.

11) How many integers between 1 and 300 (inclusive) are

a) Divisible by at least two of 5, 6, 8 ?

b) Divisible by exactly two of 5, 6, 8 ?

Soln

Let $S = \{1, 2, 3, \ldots, 300\}$

$\therefore |S| = 300$

Also Let A_1, A_2 and A_3 be subsets of S whose elements are divisible by 5, 6 and 8. Then

(a) The number of elements of S that are divisible by at least two of 5, 6, 8 is

$$L_2 = S_2 - \binom{2}{1} S_3.$$

we have

$S_0 |S| = 300.$

$S_1 = |A_1 + A_2 + A_3|$

$S_2 = |A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|$

$S_3 = |A_1 \cap A_2 \cap A_3|$

Now

$\therefore |A_1| =$ No. of Integer divisible by 5 $= \left\lfloor \frac{300}{5} \right\rfloor = 60$

$\therefore |A_2| =$ No. of Integer divisible by 6 $= \left\lfloor \frac{300}{6} \right\rfloor = 50$

$\therefore |A_3| =$ No. of Integer divisible by 8 $= \left\lfloor \frac{300}{8} \right\rfloor = 37$

$\therefore |A_1 \cap A_2| =$ No. of Integer divisible by 5 and 6

$$= \left\lfloor \frac{300}{5 \times 6} \right\rfloor = 10$$

$\therefore |A_1 \cap A_3| =$ No. of Integer divisible by 5 and 8

$$= \left\lfloor \frac{300}{5 \times 8} \right\rfloor = 7$$

$\therefore |A_2 \cap A_3| =$ No. of Integer divisible by 6 and 8

$$= \left\lfloor \frac{300}{24} \right\rfloor = 12$$ (LCM of 6 and 8 is 24)

$\therefore |A_1 \cap A_2 \cap A_3| =$ No. of Integer divisible by 5, 6 and 8

$$= \left\lfloor \frac{300}{120} \right\rfloor = 2$$ (LCM of 5, 6 and 8 is 120)

$\therefore S_1 = \Sigma|A_i| = 60 + 50 + 37 = 147$

$\therefore S_2 = \Sigma|A_i \cap A_j| = 10 + 7 + 12 = 29$

$\therefore S_3 = |A_1 \cap A_2 \cap A_3| = 2$

Finally,

$$L_2 = 29 - \binom{2}{1}2$$

$$= 29 - 2 \times 2$$

$$= 25$$

b) The number of integers between 1 and 300 which are divisible by exactly two of 5, 6, 8

$$E_2 = S_2 - \binom{3}{1} S_3$$

$$= 29 - (3 \times 2)$$

$$= 23$$

#

13) For the positive integers 1, 2, 3, ..., n there are 11660 de-rangements where 1, 2, 3, 4, 5 appear in the first five positions. What is the value of 'n'?

Soln

Here

The number of permutation of 1,2,3,4,5 are 5!

The number of de-rangements of 1,2,3,4,5 are d_5.

$$\therefore d_5 = 5! \times e^{-1} = 120 \times 0.3679 = 44.14$$
$$\approx 44.$$

The integer 1, 2, 3, 4 and 5 can be dearanged in the first five place in d_5 ways; the last $n-5$ integers in d_{n-5}. Hence, The total no. of derangements is $d_5 \times d_{n-5}$. This is given as 11660.

Thus, we have

$$d_5 \times d_{n-5} = 11660$$

$$\text{or, } d_{n-5} = \frac{11660}{d_5}$$

$$\text{or, } d_{n-5} = 265$$

[But we know $d_6 = 264.87 \approx 265$]

Hence, $d_{n-5} = d_6$

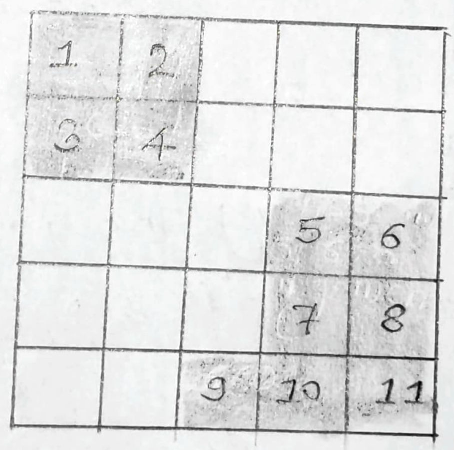
$\therefore \quad n-5 = 6$

$\Rightarrow \quad n = 6+5$

$\therefore \quad n = 11$

Hence, The required value of n is 11.

14) Find the rook polynomial for the board shown below (shaded part) using product formulae.



Solution

We have

$$r(C,x) = 1 + r_1 x + r_2 x^2 + r_3 x^3 + \dots + r_n x^n$$

$r_1 = 4$ (No. of square box)

$C = \{(1,4), (2,3)\}$

$\therefore r_2 = 2$

Three rooks cannot be place in this board such that no two pawn capture each other.

Thus $r_3 = 0$ and $r_4 = 0$. and so on.

Acc. to rook polynomial for the board is,

$$r(C, x) = 1 + 4x + 2x^2$$

For the second board

$r_1 = 7$ (No. of square boxes)

$C = \{(5,8), (5,9), (5,11), (6,7), (6,9), (6,10),$
$(7,9), (7,11), (8,9), (8,10)\}$

$\therefore r_2 = 10$

Similarly, The position for 3 non-capturing rooks are
$\{(5,8,9), (6,7,9)\}$

$\therefore r_3 = 2$.

The board has no position for four or more mutually non capturing rooks cannot be placed on board.
Thus $r_4 = 0$. similarly, $r_5 = 0$

Acc, the rook polynomial for the board is

$$r(C, x) = 1 + 7x + 10x^2 + 2x^3.$$

∴ The product formula yields the rook polynomial for the given board as

$$r(C,x) = r(C_1,x)\, r(C_2,x)$$

$$\Rightarrow r(C,x) = (1+4x+2x^2)(1+7x+10x^2+2x^3)$$

$$\therefore r(C,x) = 1+11x+40x^2+56x^3+28x^4+4x^5.$$

15) Find the Rook polynomial for the 3×3 board by using the expansion formula.

Soln

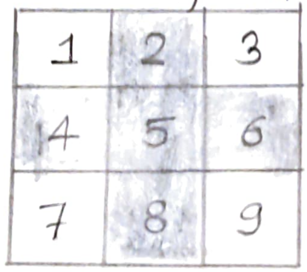
We have

$$r(C,x) = 1+r_1x+r_2x^2+r_3x^3+ \dots + r_nx^n$$

The 3×3 board is given below

1	2	3
4	5	6
7	8	9

Let us mark the square which is at the centre of the board as ⊛. Then the boards D and E as shown below (the shaded parts are the deleted parts)



D

E

For the board D,

$r_1 = 4$ (No. of square box)

$C = \{(1,9), (3,7)\}$

$\therefore r_2 = 2$

and $r_3 = 0$, $r_4 = 0$.

The rook polynomial for the board D is given by

$$r(D,x) = 1 + 4x + 2x^2$$

For the board E,

$r_1 = 8$ (No. of square box)

The positions for 2 non-capturing rook are $C = \{(1,6), (1,8), (1,9), (2,4), (2,6), (2,7), (2,9), (3,4), (3,7), (3,8), (4,8), (4,9), (6,7), (6,8)\}$

$\therefore r_2 = 14$

The position of 3 non-capturing rooks are
$C = \{(1,6,8), (2,4,9), (3,4,8), (2,6,7)\}$

$\therefore r_3 = 4$

The board has no positions for four or more mutually non-capturing rooks cannot be placed on board.

Thus $r_4 = 0$ and $r_5 = 0$.

acc, the rook polynomial for the board is

$$r(E,x) = 1 + 8x + 14x^2 + 4x^3$$

Finally, by expansion formula, the rook polynomial for board C can be written as

$$r(C,x) = x\,r(D,x) + r(E,x)$$

$$= x(1+4x+2x^2) + (1+8x+14x^2+4x^3)$$

$$= (x+4x^2+2x^3) + (1+8x+14x^2+4x^3)$$

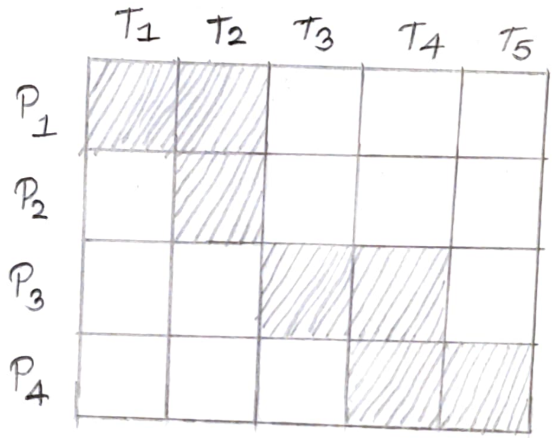
$$\therefore r(C,x) = 1+9x+18x^2+6x^3.$$

#

16) four persons $P_1, P_2, P_3 \cdot P_4$ who arrive late for a dinner Party. Find that only one chair at each of five tables $T1, T2, T3, T_4$ and T_5 is vacant. P_1 will not sit at T_1 or $T2$, P_2 will not sit at T_2, P_3 will not sit at T_3 or T_4 and P_4 will not sit at T_4 or T_5. Find the number of ways they can occupy the vacant chairs.

Soln

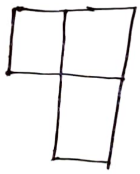
Here



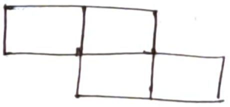
The situation can be represented by above board.

This board is made up of 7 sequence & It is made up of 2 disjoint subboards. ie.

C_1



C_2.



for C_1, $r_1 = n = 3$ & $r_1 = 1$; $r_3 = 0$

So, $r(C_1, x) = 1 + 3x + x^2$

for C_2, $r_1 = n = 4$; $r_2 = 3$; $r_3 = r_4 = 0$

So $r(C_2, x) = 1 + 4x + 3x^2$

$\therefore$ Rook polynomial expansion formula is

$$r(C, x) = r(C_1, x) \times r(C_2, x)$$

$$= (1 + 3x + x^2) \times (1 + 4x + 3x^2)$$

$$= 1 + 7x + 16x^2 + 13x^3 + 3x^4$$

So from the equation, we have

$\Rightarrow$ $r_1 = 7$, $r_2 = 16$, $r_3 = 13$ and $r_4 = 3$

Then by the formula,

$S_0 = 5! = 120$,

$S_1 = (5-1)! \times r_1 = 4! \times 7 = 168$

$S_2 = (5-2)! \times r_2 = 3! \times 16 = 96$

$S_3 = (5-3)! \times r_3 = 2! \times 13 = 26$

$S_4 = (5-4)! \times r_4 = 1! \times 3 = 3$

Finally,

No. of way in which 4 person can occupy the vacant chairs is

$$\bar{N} = S_0 - S_1 + S_2 - S_3 + S_4$$

$$= 120 - 168 + 96 - 26 + 3$$

$$= 25 \quad \#$$

12) While at race track, a person bets on each of the ten horses in a race to come in accordance to how they have favored. In how many way can they reach the finish line so that he losses all his bets?

Soln

Here

We have to find the number of ways of arranging the horses 1, 2, 3, . . ., 10 so that 1 is not in its favored place, 2 is not in its favored placed, . . . and 10 is not in its Favored placed. Thus the required number of ways in the number of derangements of 10 objects, namely.

$$d_{10} = e^{-1} \times 10! \approx 0.3679 \times (10!)$$

$$= 1335035$$