Assignment 2

SFWRENG 2CO3: Data Structures and Algorithms–Winter 2023

Deadline: February 12, 2023

Department of Computing and Software
McMaster University

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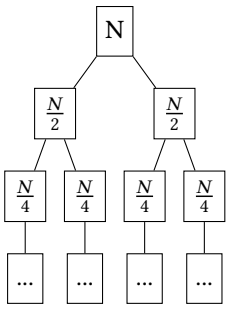
Problem 1. Assume we have a computer with infinite processor cores that can all operate at the same time. In an attempt to sort *as fast as possible*, we make a *parallelized* version of MergeSort that uses multiple processor cores at the same time. To achieve this, we change the top-down merge-sort algorithm as follows:

Algorithm ParallelMergeSort($L[start \ldots end]$)**:**

1: **if** $start + 1 < end$ **then**
2: $mid := (start + end)$ div 2.
3: Start ParallelMergeSort($L[start, mid]$) on a fresh processor core C.
4: L_2 := ParallelMergeSort($L[mid \ldots end]$).
5: Wait until C finished ParallelMergeSort($L[start, mid]$), resulting in L_1.
6: **return** Merge(L_1, L_2).
7: **else**
8: **return** L.
9: **end if**

Let $n = |L|$ be the length of the list being sorted by ParallelMergeSort.

P1.1. Give a recurrence $T(n)$ for the runtime complexity of ParallelMergeSort and solve the recurrence $T(n)$ by proving that $T(n) \sim e(n)$ for some expression e that uses n.
Answer:



Number	Cost	Total
1	N	$1 * N$
1	$\frac{N}{2}$	$1 * \frac{N}{2}$
1	$\frac{N}{4}$	$1 * \frac{N}{4}$
...	...	...
1	$\frac{N}{2^i}$	$1 * \frac{N}{2^i}$
—	—	—
		$2N$

As you can see, the tree shows that each step has $\frac{N}{2^i}$ cost. There is only 1 counted as they all occur in parallel, so unlike the cost complexity, the time complexity is as if the step is just run once. When you go to sum up all the steps, you can see that the steps will converge to 2N which is ~ N. Note: $N + \frac{N}{2} + \frac{N}{4} + \frac{N}{8} + ... = 2N$. Given N is not always a power of 2, we know that $O(N)$ is upper bounded by 2N and it must be lower bounded by N since the first step will always have a cost of N. This is enough to prove that the algorithm is $\sim N$.

P1.2. Provide a strict bound on the number of processor cores that ParallelMergeSort uses (hence, how many different processor cores are used by ParallelMergeSort at the same time).
Answer:
The number of processors is $2^{\log_2 N} = N$. This is because the recurrance tree proves to be $\log_2 N$ height and each step has double the processors of the last step. On the last reccurance level, it will terminate as there is only 1 element in the List, so it will hit the else statement (rather than the if condition) and return the List. This is an assumption that N is a power of 2. Given N is not always a power of 2, we know the number of processors is in the range of $2^{\log_2 N-1} <= processors <= N$, which contains the min and max number of processors needed per level on the reccurance tree.

Problem 2. Consider the following recursive sorting algorithm.

Algorithm WeirdSort(L, *start*, *end*)**:**

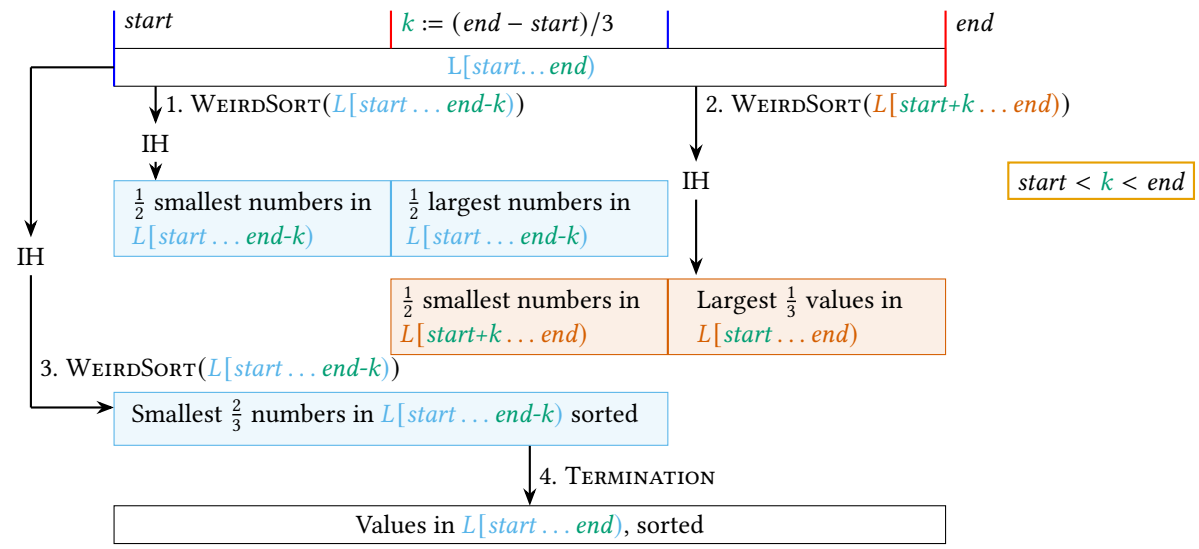
```
1:  if end − start = 2 then
2:      if L[start] ≥ L[start + 1] then
3:          Exchange L[start] and L[start + 1].
4:      end if
5:  else if end − start > 2 then
6:      k := (end − start) div 3.
7:      WeirdSort(L, start, end − k).
8:      WeirdSort(L, start + k, end).
9:      WeirdSort(L, start, end − k).
10: end if
```

Let $n = |L|$ be the length of the list being sorted by WeirdSort.

P2.1. Is WeirdSort a *stable* sort algorithm? If yes, explain why. If no, show why not and indicate whether the algorithm can be made stable.
Answer:
This is not a stable sort algorithm as in line 2, if the earlier element is equal to the next element, they get swapped. To make this a stable algorithm, change ≥ to >.

P2.2. Prove via induction that WeirdSort will sort list L.

Answer:



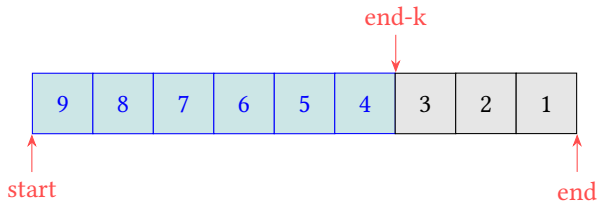
Base case 1: $0 <= end - start = 2$ values
In this case, it will hit the first if statement and sort the 2 values based on which is largest, then return. There is no recursion in this case.

Base case 2: $0 <= end - start = 3$ values
This is the lowest amount of values where there will be recursion. This will result in $k = (3-0)/3 = 1$
Induction Hypothesis: WeirdSort sorts $0 <= end - start < n$ values correct.
Induction Step: Prove WeirdSort sorts $end - start = n$ values

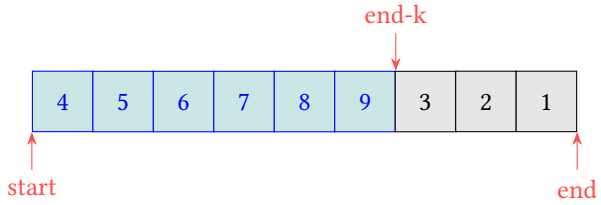
To start, assume $|L| = 3n, n \in \mathbb{N}, n > 0$ ($|L|$ is a multiple of 3). k will always give us $\frac{1}{3}|L|$, so each WeirdSort call will be $\frac{2}{3}$ of L. Since our base case 2 shows that $k >= 1$ always, this means the lists drawn for the 3 WeirdSort calls will be at least 1 smaller than the input array. Given this, we can see that, assuming WeirdSort can sort $< n$ values via IH, WeirdSort call 1 will sort the largest $\frac{1}{3}$ values of L into the middle third of L (second half of L in WeirdSort call 1). WeirdSort call 2 will then take those largest numbers sorted in the first call and sort then with the remaining $\frac{1}{3}$ values. By the end of this call, the largest $\frac{1}{3}$ values in L will be sorted at the end of the list. WeirdSort call 3 will then sort the last (and smallest) $\frac{2}{3}$ values.

The following is looking at the case where the largest values are at the start of the list and the smallest at the end (assume reverse ordered list). You can see that there are enough of the largest values moved to the end of the first WeirdSort call, so that they can be sorted to the end of the list. This is because as seen by the final WeirdSort call, you only need to move the $\frac{1}{3}$ largest values to the last $\frac{2}{3}$ of the list, in order for them to be sorted to the end of the list. The final WeirdSort call will handle the rest.

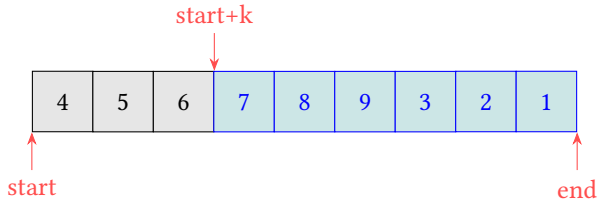
Start of WeirdSort call 1



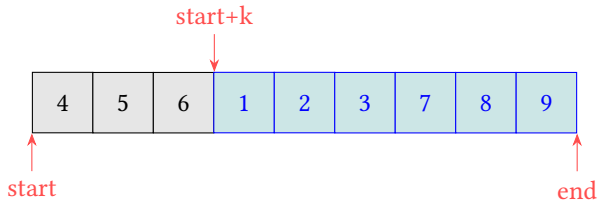
End of WeirdSort call 1



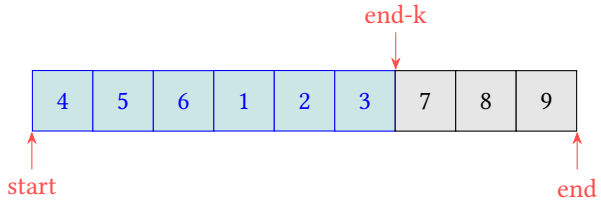
Start of WeirdSort call 2



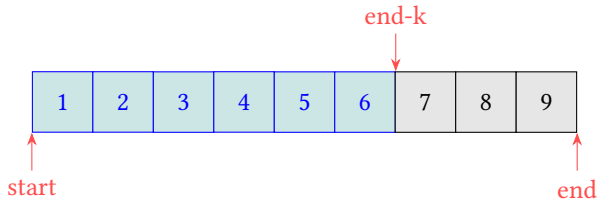
End of WeirdSort call 2



Start of WeirdSort call 3



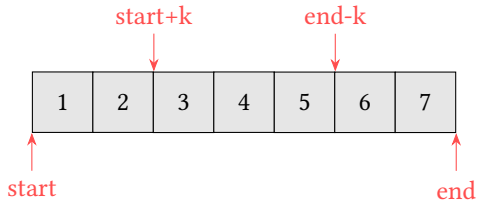
End of WeirdSort call 3



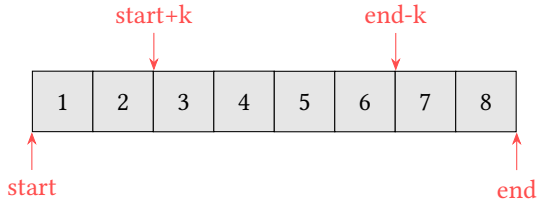
Now given that $|L|$ is not a multiple of 3, there are two cases: $|L| = 3n + 1$ and $|L| = 3n + 2$. For both of these cases, there will be more values pushed to the end of the list in WeirdSort call 1, for WerirdSort call 2 to move to the end of the list. This is because k always rounds down, so the subsequent lists from WeirdSort calls will be bigger. More formally, for this to occur $|L[start+k, end-k)| >= L[end-k, end)$

An example can be seen below given

$|L| = 3n + 1$



$|L| = 3n + 2$



P2.3. Give a recurrence $T(n)$ for the runtime complexity of WeirdSort and solve the recurrence $T(n)$ by proving that $T(n) \sim e(n)$ for some expression e that uses n.
Answer:

$$T(N) = \left\{ \begin{array}{ll} 0, & \text{if } N = 1 \\ 1, & \text{if } N = 2 \\ T(\frac{2N}{3}) + T(\frac{2N}{3}) + T(\frac{2N}{3}) + 3, & \text{if } N > 2 \end{array} \right\} \tag{1}$$

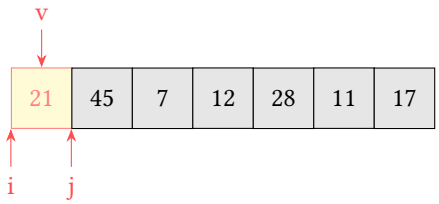
Using Master Theorem, we have $a = 3$, $b = 1.5$, $f(n) = 3 \sim 1 = N^{\log_{1.5}(1.5)} = O(N^{\log_{1.5}(3-\epsilon)})$ where $\epsilon = 1.5 > 0$, which is case 1 in Master Theorem. This means $T(N) \sim N^{\log_{1.5}(3)} \approx N^{2.71}$.

Problem 3. Consider the following Partition algorithm used by QuickSort (this version of Partition is based on the algorithm from the slides with the for-loop replaced by a while-loop).

Algorithm Partition(L, *start*, *end*)**:**

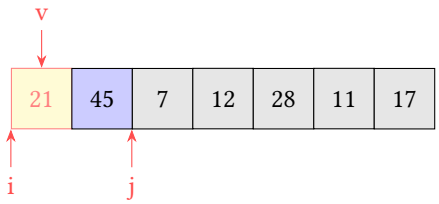
1: $v, i, j := L[start], start, start + 1.$
2: **while** $j \neq end$ **do**
3: **if** $L[j] \leq v$ **then**
4: $i := i + 1.$
5: Exchange $L[i]$ and $L[j]$.
6: **end if**
7: $j := j + 1.$
8: **end while**
9: Exchange $L[i]$ and $L[start]$
10: **return** L.

P3.1. Illustrate the operations performed by Partition on the array $A = [21, 45, 7, 12, 28, 11, 17]$. Show the content of A after each execution of the loop body.
Answer:
Start of function

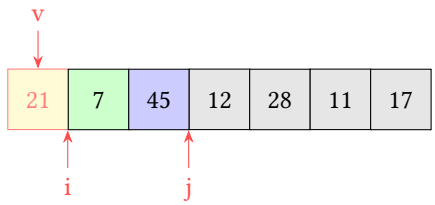


Start of loop

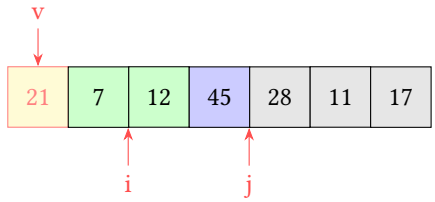
Did not enter if statement
End Loop 1



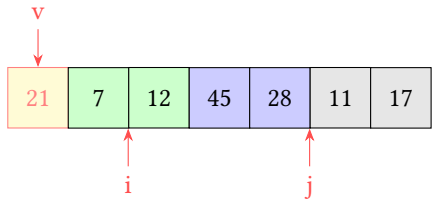
Entered if statement
End Loop 2



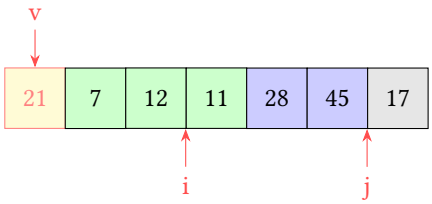
Entered if statement
End Loop 3



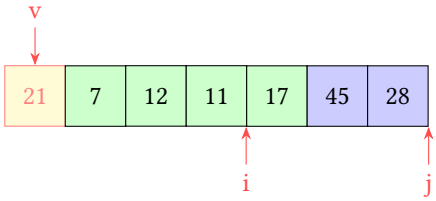
Did not enter if statement
End Loop 4



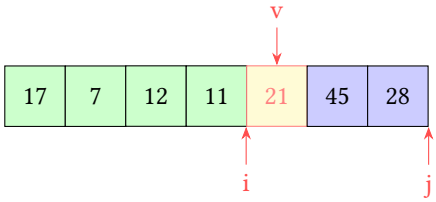
Entered if statement
End Loop 5



Entered if statement
End Loop 6



End of loop
Exchange L[i] and L[start]



P3.2. Provide pre-conditions and post-conditions for PARTITION and provide an invariant and bound function for the while-loop at Line 2. Prove the correctness of PARTITION.
Pre-condition:

- $|L| > 0$, else L[start] would error

Post-condition:

- All elements to the left of v are smaller than or equal to v, all elements to the right of v are larger than v, v is in the middle of the smaller and larger elements.
- The input array is the same length as the output array and contains the same values (not necessarily in the same order)

Loop Invariant:

- All entries in A[$start + 1 ... i$] are <= pivot.
- All entries in A[$i + 1 ... j$) are > pivot.
- A[start] = pivot.

Bound function:

- $end - j$, this is because when $end = j$, this means $end - j = 0$ and the loop will terminate

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Algorithm PARTITION(L, $start$, end)**:**

1: $v, i, j := L[start], start, start + 1.$
Base case: v=L[start] (pivot) and i=0 and j=1 implies invariant (L1 and L2 are both empty because their range is out of bounds)

Given the invariant is:
All entries in L1 = L[$start$ + 1...i] are <= pivot.
All entries in L2 = L[i + 1...j) are > pivot.
Induction hypothesis: The invariant holds at every step in the loop.
Induction step: Prove that the invariant holds at every step in the loop.

2: **while** $j \neq end$ **do**
3: Known: $j \neq end$
4: **if** $L[j] \leq v$ **then**
5: $i := i + 1$.
6: Known: $L[j] \leq pivot$, L[i_{new}] is ≥ pivot because i_{old} is always 1 cell behind the first element in L1
7: Exchange $L[i]$ and $L[j]$.
8: Known: j isn't at end of list yet. L[$start$ + 1...i] are all smaller than or equal to pivot and L[i + 1...j − 1) are all larger than pivot
9: **end if**
10: $j := j + 1$.
11: $j_{new} = j_{old} + 1$ meaning L[i + 1...j_{new}) are all larger than pivot, which is our invariant.
12: **end while**
13: Known: We know that L[$start$ + 1...i] are all smaller or equal to pivot and L[i + 1...j) are all larger than pivot. We know that the i^{th} value is the last value included in L1 and everything past the i^{th} value is part of L2, so we can swap it with the start value (pivot), making the new list have all smaller or equal values to the left of the pivot and all larger to the right.
14: Exchange $L[i]$ and $L[start]$
Known: L is a list where L[start...i) values are smaller than or equal to the pivot, L[i] is the pivot value, L[i+1...end) are all values larger than pivot.
15: **return** L.

P3.3. Argue how Partition can be adjusted to run on singly linked lists L, while keeping a running time of $\sim|L|$.

Instead of holding the index i and j, we need to hold the nodes right before position i and j. Holding the node before is since for a singly linked list, you can't go back, so you need to always have a reference (node.next) to the node you want to compare and swap. We also need a function to swap two nodes in the linked list. This function is constant time as it is just a few assignments. As exchanges only happen between nodes i and j, it is sufficient to keep track of only these nodes, while updating them by calling node.next whenever we need to do i++ or j++. This way, we can traverse the list once while using constant time functions in each iteration, giving us a complexity of $\sim L$.

The while loop condition would now need to be when j.next.next = null as that is when we've reached the second last node in the list. This is when we want to terminate since we are always holding the node one before the one we want to compare (index j-1).

Note: You don't need to hold the node at v as we only need the value for comparisons during the loop, and you can quickly access the head of a linked list when it comes to getting the value of the head at the start and exchanging it at the end of the function.

Problem 4. Consider pairs (x_i, y_i) such that x_i is the time at which person $i = 0, 1, 2, \ldots$ enters the museum and y_i is the time at which person i leaves the museum. You may assume that consecutive people enter the museum in order of increasing time ($x_0 \leq x_1 \leq \ldots$).

P4.1. Provide an algorithm MaxVisitors that takes as input $L = [(x_0, y_0), \ldots, (x_{N-1}, y_{N-1})]$ and computes in $\sim N \log N$ the maximum number of visitors in the museum at any time.

P4.2. Argue why your algorithm MaxVisitors is correct and has a runtime complexity of $\sim N \log N$.

P4.3. Assume that the museum has a maximum capacity of M. Provide a datastructure with an operation PersonEnters(x_i, y_i) that computes in at-most $\sim \log M$ the number of visitors in the museum when person i enters the museum (for any number of persons).

P4.4. Argue why your algorithm PersonEnters is correct and has a runtime complexity of $\sim \log M$.

LOOK INTO MAX HEAP

Assignment Details

Write a report in which you solve each of the above problems. Your submission:

1. must be a PDF file;
2. must have clearly labeled solutions to each of the stated problems;
3. must be clearly presented;
4. must *not* be hand-written: prepare your report in LaTeX or in a word processor such as Microsoft Word (that can print or exported to PDF).

Submissions that do not follow the above requirements will get a grade of zero.

Grading

Each problem counts equally toward the final grade of this assignment.