Functional Analysis

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Contents

Chapter I

Topological Linear Space

§1 Vector Spaces

Definition 1.1. *If X is a vector space, $A \subset X, B \subset X, x \in X$, and $\lambda \in \mathbb{F}$, the following notations will be used:*

$$x + A = \{x + a : a \in A\}$$
$$x - A = \{x - a : a \in A\}$$
$$A + B = \{a + b : a \in A, b \in B\}$$
$$\lambda A = \{\lambda a : a \in A\}$$

*A set $Y \subset X$ is called a **subspace** of X if Y is itself a vector space (with respect to the same operations, of course).*

*A set $C \subset X$ is said to be **convex** if*

$$tC + (1-t)C \subset C \quad (0 \le t \le 1)$$

*A set $B \subset X$ is said to be **balanced** if $\alpha B \subset B$ for every $\alpha \in \mathbb{F}$ with $|\alpha| \le 1$*

Definition 1.2. *Suppose τ is a topology on a vector space X such that*

(a) every point of X is a closed set.

(b) the vector space operations

$$+ : X \times X \to X$$

and

$$\cdot : \mathbb{F} \times X \to X$$

*are continuous with respect to τ. Under these conditions, τ is said to be a vector topology on X, and X is a **topological vector space**.*

*A subset E of a topological vector space is said to be **bounded** if to every neighborhood V of 0 in X corresponds a number $s > 0$ such that $E \subset tV$ for every $t > s$*

Proposition 1.3. *$T_a = a + x$ and $M_\lambda = \lambda x$ are homeomorphisms of X onto X.*

Definition 1.4. *Types of topological vector spaces In the following definitions, X always denotes a topological vector space, with topology τ.*

(a) X is ***locally convex*** *if there is a local base $\mathscr{B}$ whose members are convex.*

(b) X is ***locally bounded*** *if 0 has a bounded neighborhood.*

(c) X is ***locally compact*** *if 0 has a neighborhood whose closure is compact.*

(d) X is metrizable if τ is compatible with some metric d.

(e) X is an F-space if its topology τ is induced by a complete invariant metric d. (Compare Section 1.25.)

(f) X is a Fréchet space if X is a locally convex F-space.

(i) X has the ***Heine-Borel property*** *if every closed and bounded subset of X is compact.*

§2 Linear Span

Definition 2.1. *The intersection*

$$\bigcap Y_\sigma$$

of all linear subspaces Y_σ containing the set S is called the ***linear span of the set*** *S.*

Theorem 2.2. *Let X be linear space and $S \subset X$.*

(1) The linear span of a set S is the smallest linear subspace containing S.

(2) The linear span of S consists of all elements x of the form

$$x = \sum_1^n a_i x_i, \quad x_i \in S, a_i \in \mathbb{F}, n \text{ any natural number}$$

(3)

$$\operatorname{span} S = \bigcup_F \operatorname{span}\{x_1, x_2, \ldots, x_n\}$$

§3 Convex Set

Definition 3.1. *X is a linear space over $\mathbb{R}$; a subset K of X is called* ***convex*** *if, whenever x and y belong to K, the whole segment with endpoints x, y, meaning all points of the form*

$$ax + (1-a)y, \quad 0 \le a \le 1,$$

also belong to K.

Theorem 3.2. *Let K be a convex subset of a linear space X over $\mathbb{R}$. Suppose that $x_1, \ldots, x_n$ belong to K; then so does every x of the form*

$$x = \sum_{j=1}^n a_j x_j$$

where $a_j \geq 0, \sum_1^n a_j = 1$.*The form is called* ***convex combinations*** *of* $x_1, x_2, \ldots x_n$.

Theorem 3.3. *Let X be a linear space over $\mathbb{R}$.*

(1) The empty set is convex.

(2) A subset consisting of a single point is convex.

(3) Every linear subspace of X is convex.

(4) The sum of two convex subsets is convex.

(5) If K is convex, so is $-K$.

(6) The intersection of an arbitrary collection of convex sets is convex.

(7) Let $\{K_\alpha\}$ be a collection of convex subsets that is totally ordered by inclusion. Then their union $\bigcup K_\alpha$ is convex.

(8) The image of a convex set under a linear map is convex.

(9) The inverse image of a convex set under a linear map is convex.

Definition 3.4. *Let S be any subset of a linear space X over $\mathbb{R}$. The* ***convex hull*** *of S is defined as the intersection of all convex sets containing S. The hull is denoted as $\widehat{S}$.*

Definition 3.5. *The* ***closed convex hull*** *of a subset M of a normed linear space X is the smallest closed convex set containing M, that is the intersection of all closed convex sets containing M. We denote this set as $\breve{M}$.*

Indeed, the closed convex hull of M is the closure of the convex hull of M.

Theorem 3.6. *(i) The convex hull of S is the smallest convex set containing S.*

(2) The convex hull of S consists of all convex combinations of points of S.

Definition 3.7. *A subset E of a convex set K is called an* ***extreme subset*** *of K if:*

(a) E is convex and nonempty.

(b) whenever a point x of E is expressed as

$$x = \frac{y+z}{2}, \quad y, z \text{ in } K$$

then both y and z belong to E. An extreme subset consisting of a single point is called an ***extreme point*** *of K.*

§4 Linear Maps

Theorem 4.1. *The subspaces $N_j = N(T^j)$ defined have these properties:*

$$N_j \subset N_{j+1} \quad \text{for all } j$$

and

$$\dim\left(\frac{N_j}{N_{j-1}}\right) \geq \dim\left(\frac{N_{j+1}}{N_j}\right) \quad \text{for all } j$$

Proof: We claim that T maps

$$N_{j+1}/N_j \to N_j/N_{j-1}$$

in a injective function. To see this, note that a nonzero element of N_{j+1}/N_j is represented by $z + N_j$ where z does not lie in N_j but in N_{j+1}. Clearly, Tz lies in N_j but not in N_{j-1} ; this shows the one-to-oneness.

Corollary 4.2. *Suppose that for some i*

$$N_i = N_{i+1}$$

then

$$N_i = N_k \quad \textit{for all } k > i$$

Definition 4.3. *A subspace Y of X is called an **invariant subspace** of a linear map $T : X \to X$ if $T(Y) \subset Y$.*

Theorem 4.4. *Suppose that Y is an invariant subspace of X for a mapping $T : X \to X$. Then*

(1) there is a natural interpretation of T' as a mapping $X/Y \to X/Y$.

(2) if both maps

$$T : Y \longrightarrow Y \textit{ and } T' : X/Y \longrightarrow X/Y$$

are invertible, so is $T : X \to X$.

Theorem 4.5. *Let T be a linear map: $X \to X$.*

(1) For any y in X, the set $\{p(T)y\}$, where p represents any polynomial, is an invariant subspace of M.

(2) Let A be a linear map: $X \to X$ that commutes with T. Then the nullspace of T is an invariant subspace of A.

Chapter II

The Hahn-Banach Theorem

§1 The Hahn-Banach Theorem

Definition 1.1. *A* ***seminorm*** *on a vector space X is a real-valued function p on X such that*

(i) Subadditivity: $p(x+y) \leq p(x) + p(y)$.

(ii) Homogeneity: $p(\alpha x) = |\alpha| p(x)$ *for all x and y in X and all scalars $\alpha \in \mathbb{F}$.*

Therefore a seminorm also satisfies that (c) Positive semi-definite : $p(x) \geq 0$ *for all $x \in X$ and* $p(0) = 0$

Definition 1.2. *A* ***sublinear functional*** *on a vector space X is a real-valued function p on X such that*

(i) Subadditivity

(ii) Positive Homogeneity

Theorem 1.3 (Hahn-Banach theorem for real linear space)**.** *Let X be a linear space over $\mathbb{R}$, and p a sublinear functional. And Y denotes a linear subspace of X on which a linear functional ℓ is defined that is dominated by p :*

$$\ell(y) \leq p(y) \quad \textit{for all } y \textit{ in } Y$$

Then, ℓ can be extended to all of X as a linear functional dominated by p :

$$\ell(x) \leq p(x) \quad \textit{for all } x \textit{ in } X$$

Proof: Step 1. Suppose that Y is not all of X; then there is some z in X that is not in Y. Denote by $Z = \operatorname{span}\{Y, z\}$. *Our aim is to extend ℓ as a linear functional to Z, that is,*

$$\ell(y + az) = \ell(y) + a\ell(z) \leq p(y + az)$$

holds for all y in Y and all real a. The inequality holds for $a=0$. Since p is positive homogeneous, it suffices to verify it for $a=\pm 1$:

$$\ell(y)+\ell(z) \leq p(y+z), \quad \ell\left(y^{\prime}\right)-\ell(z) \leq p\left(y^{\prime}-z\right)$$

Thus for all $y, y^{\prime}$ in Y,

$$\ell\left(y^{\prime}\right)-p\left(y^{\prime}-z\right) \leq \ell(z) \leq p(y+z)-\ell(y)$$

must hold. Such an $\ell(z)$ exists iff for all pairs $y, y^{\prime}$,

$$\ell\left(y^{\prime}\right)-p\left(y^{\prime}-z\right) \leq p(y+z)-\ell(y)$$

This is the same as

$$\ell\left(y^{\prime}\right)+\ell(y)=\ell\left(y^{\prime}+y\right) \leq p(y+z)+p\left(y^{\prime}-z\right)$$

We prove the possibility of extending ℓ from Y to Z by the Subadditivity of p.

Step 2. Consider all extensions of ℓ to linear spaces Z containing Y on which domination condition continues to hold. We order these extensions by defining

$$(Z, \ell) \leq\left(Z^{\prime}, \ell^{\prime}\right)$$

to mean that $Z^{\prime}$ contains Z, and that $\ell^{\prime}$ agrees with ℓ on Z. Let $\left\{\left(Z_{\nu}, \ell_{\nu}\right)\right\}$ be a totally ordered collection of extensions of ℓ. Then we can define ℓ on the union $Z=\cup Z_{\nu}$ as being ℓ_{ν} on Z_{ν}. Clearly, ℓ on Z satisfies ($3^{\prime}$); equally clearly, $\left(Z_{\nu}, \ell_{\nu}\right) \leq(Z, \ell)$ for all ν. This shows that every totally ordered collection of extensions of ℓ has an upper bound. So the hypothesis of Zorn's lemma is satisfied, and we conclude that there exists a maximal extension. But according to the foregoing, a maximal extension must be to the whole space X.

Theorem 1.4 (Hahn-Banach Theorem for complex linear space)**.** *Let X be a linear space over $\mathbb{C}$, and p a semi-norm. Let Y be a linear subspace of X over $\mathbb{C}$, and let ℓ be a complex linear functional on Y that satisfies*

$$|\ell(y)| \leq p(y) \quad \text{for } y \text{ in } Y$$

Then ℓ can be extended to all of X so that

$$|\ell(x)| \leq p(x) \quad \text{for } x \text{ in } X$$

Proof: Step 1. Split ℓ into its real and imaginary part:

$$\ell(y)=\ell_{1}(y)+i \ell_{2}(y)$$

where
$$\ell_1 = \frac{\ell + \overline{\ell}}{2} \text{ and } \ell_2 = \frac{\ell - \overline{\ell}}{2i}$$
Clearly, ℓ_1 and ℓ_2 are linear over $\mathbb{R}$, and are related by

$$\ell_1(iy) = -\ell_2(y)$$

Conversely, if ℓ_1 is a linear functional over $\mathbb{R}$,

$$\ell(x) = \ell_1(x) - i\ell_1(ix)$$

is linear over $\mathbb{C}$.

Step 2. We turn now to the task of extending ℓ by extending ℓ_1. It follows that

$$\ell_1(y) \le p(y)$$

Therefore by the real H-B theorem, ℓ_1 can be extended to all of X. We define ℓ on X

$$\ell(x) = \ell_1(x) - i\ell_1(ix)$$

Clearly, ℓ is linear over $\mathbb{C}$ and we claim that (29) holds. To see this, write

$$\ell(x) = r\alpha, \quad r \text{ is real }, \quad |\alpha| = 1$$

Then
$$|\ell(x)| = r = \alpha^{-1}\ell(x) = \ell\left(\alpha^{-1}x\right) = \ell_1\left(\alpha^{-1}x\right) \le p\left(\alpha^{-1}x\right) = p(x)$$
This completes the proof of the complex H-B theorem.

§2 Geometric Hahn-Banach Theorem

§2.1 Minkowski functional

Definition 2.1. *Let X be topological linear space and $A, B \subset X$.*

(1) Set A is said to be ***absorbing****, if every $x \in X$ lies in tA for some $t = t(x) > 0$, i.e.*

$$X = \bigcup_{n=1}^{\infty} nA$$

Noted that $0 \in A$ if A is absorbing.

(2) A set $B \subset X$ is said to be ***balanced*** *if $\alpha B \subset B$ for every $\alpha \in \mathbb{F}$ with $|\alpha| \le 1$.*

Proposition 2.2. *There exists balanced and absorbing neigbourdhood basis of 0.*

Definition 2.3. *Let K be a convex set that $0 \in K$. We denote the **gauge (Minkowski functional)** p_K of K with respect to the origin as follows:*

$$p_K(x) = \inf\left\{a : a > 0, \frac{x}{a} \in K\right\}$$

Proposition 2.4. *Let K be a convex set that that $0 \in K$. Then*

(1) $p_K(x) \in [0, \infty]$, $p_K(0) = 0$

(2) Positive Homogeneity

(3) Subadditivity

Proposition 2.5. *Let K be a convex set in topological linear space X over $\mathbb{F}$, which we take to be the origin and the gauge p_K of K*

(1) $0 \leq p_K < \infty$ if and only if K is absorbing sets.

(2) If K is balanced, then p_K is homogeneity over $\mathbb{F}$

Corollary 2.6. *If K is a balanced, absorbing convex set, then p_k is a seminorm on X.*

Theorem 2.7. *For any convex set K,*

(1) If $x \in K$, then $p_K(x) \leq 1$

(2) $p_K(x) < 1$ iff x is an interior point of K.

Theorem 2.8. *Let p denote a sublinear functional defined on a linear space X over $\mathbb{R}$.*

(1) The set of points x satisfying

$$p(x) < 1$$

is a convex subset of X, and 0 is an interior point of it.

(2) The set of points x satisfying

$$p(x) \leq 1$$

is a convex subset of X.

§2.2 Separation Theorem

Definition 2.9. *Suppose that ℓ is a linear functional not $\equiv 0$; for any real c, all points of X belong to one, and only one, of the following three sets:*

$$\ell(x) < c, \quad \ell(x) = c, \quad \ell(x) > c.$$

The set of x that satisfies

$$\ell(x) = c$$

*is called a **hyperplane**; the sets where $\ell(x) < c$, respectively $\ell(x) > c$ are called **open halfspaces**. The sets where*

$$\ell(x) \geq c, \quad \text{or} \quad \ell(x) \leq c$$

are called ***closed halfspaces****.*

Theorem 2.10 (Hyperplane Separation Theorem)**.** *Let K be a nonempty convex subset of a linear space X over $\mathbb{R}$; suppose that K has an interior point. Then any point y not in K can be separated from K by a hyperplane $\ell_y(x) = c$; that is, there is a linear functional ℓ, depending on y, such that*

$$\ell(x) \leq c \quad \textit{for all } x \textit{ in } K; \quad \ell(y) = c.$$

Proof: Assume that 0 is interior point of K, and denote by p_K the gauge of K. It follows that p_K is a sublinear functional and $p_K(x) \leq 1$ for every x in K. We set

$$\ell(y) = 1.$$

Then ℓ is defined on $\operatorname{span}\{y\}$ *and we observe that*

$$\ell(z) \leq p_K(z)$$

for all $z = ay \in \operatorname{span}\{y\}$. Then we conclude from the real H-B theorem that ℓ can be so extended to all of X and

$$\ell(x) \leq p_K(x) \leq 1$$

for all $x \in K$.

Theorem 2.11 (Extended Hyperplane Separation)**.** *Let X is a linear space over $\mathbb{R}, H$, and M disjoint convex subsets of X, at least one of which has an interior point. Then H and M can be separated by a hyperplane $\ell(x) = c$; that is, there is a nonzero linear functional ℓ, and a number c, such that*

$$\ell(u) \leq c \leq \ell(v)$$

for all u in H, all v in M.

Proof: First, we note that the difference set $H - M = K$ is convex; since either H or M contains an interior point, so does K.

Since H and M are disjoint, $0 \notin K$; there is a linear functional ℓ such that

$$\ell(x) \leq \ell(0) = 0 \textit{ for all } x \textit{ in } K$$

It is equivalent that

$$\ell(u) \leq \ell(v)$$

for all $u \in H$, $v \in M$, with $c = \sup_{u \in H} \ell(u)$.

Chapter III

Normed Space

§1 Norms

Definition 1.1. *Let X denote a linear space over $\mathbb{R}$ or $\mathbb{C}$. A norm in X is a real-valued function: $X \to \mathbb{R}$, denoted as $|x|$, with the following properties:*

(i) Positivity,

$$|x| > 0 \quad \textit{for } x \neq 0; |0| = 0$$

(ii) Subadditivity,

$$|x + y| \leq |x| + |y|$$

(iii) Homogeneity. For all scalars $a \in \mathbb{R}$ of $\mathbb{C}$,

$$|ax| = |a||x|$$

With the aid of a norm we can introduce a metric in X, by defining the distance of two points to be

$$d(x, y) = |x - y|$$

It is easy to verify that this has all properties of a metric. Conversely, it is easy to show that every metric in a linear space that is translation invariant and homogeneous:

$$d(x + z, y + z) = d(x, y), \quad d(ax, ay) = |a|d(x, y)$$

Definition 1.2. *Two different norms, $|x|_1$ and $|x|_2$, defined on the same space X are called **equivalent** if there is a constant c such that*

$$c|x|_1 \leq |x|_2 \leq c^{-1}|x|_1$$

for all x in X.

The significance of this notion is that equivalent norms induce the same topology.

Proposition 1.3. *Let X ba a normed space over $\mathbb{F}$.*

(1) A subspace Y of X is again a normed linear space.

(2) Given two linear spaces Z and U, their Cartesian product, denoted as a direct sum $Z \oplus U$, consists of all ordered pairs $(z,u), z \in Z, u \in U$. When Z and U are normed, $Z \oplus U$ can be normed, such as by setting

$$|(z,u)| = |z| + |u|, \quad |(z,u)|' = \max\{|z|, |u|\}, \text{ or } \quad |(z,u)|'' = \left(|z|^2 + |u|^2\right)^{1/2}$$

Definition 1.4. *Let Y be a closed subspace of a normed linear space X. Let $\overline{x}$ be an equivalence class of elements of X* mod *Y. We define norm on X/Y*

$$\|\overline{x}\| = \inf_{x \in \overline{x}} |x|$$

Definition 1.5. *A **Banach space** is a normed linear space that is complete.*

Theorem 1.6. *The completion $\overline{X}$ of a normed linear space X under the metric has a natural linear structure that makes $\overline{X}$ a complete normed linear space.*

Proof. *Recall that the points of the completion of a metric space are equivalence classes of Cauchy sequences. The term-by-term sum of two Cauchy sequences is again a Cauchy sequence, and sums of equivalent Cauchy sequences are equivalent.*

Proposition 1.7. *If X is Banach space, Y a closed subspace of X, the quotient space X/Y is complete.*

Definition 1.8. *A normed linear space is called **separable** if it contains a countable set of points that is dense, namely, whose closure is the whole space.*

§2

Lemma 2.1 (Riesz's Lemma)**.** *Let Y be a closed, proper subspace of the normed linear space X. Then for all $\alpha < 1$, there is a united vector z in X,*

$$|z| = 1$$

and that satisfies

$$d(z,Y) > \alpha$$

Proof: Since Y is a proper subspace of X, some point x of X does not belong to Y. Since Y is closed, x has a positive distance to Y :

$$\inf_{y \in Y} |x - y| = d > 0$$

There is then a y_0 in Y such that

$$d \leq |x - y_0| < d + \varepsilon$$

Denote $z' = x - y_0$ we can then write

$$|z'| < d + \varepsilon$$

It follows that

$$|z' - y| \geq d(x, Y) \geq d$$

for all $y \in Y$. We set

$$z = \frac{z'}{|z'|}$$

Then

$$d(z, Y) = \frac{1}{|z'|} d(z', Y) \geq \frac{d}{d + \varepsilon} > \alpha$$

for sufficiently small ε.

Corollary 2.2. *Let X be an infinite-dimensional normed linear space; then the unit ball B_1 defined in X is not compact.*

Proposition 2.3. *(1) All norms are equivalent on finite-dimensional spaces.*

(2) Every finite-dimensional subspace of a normed linear space is closed. (Hint: Use the fact that)

Definition 2.4. *A norm is called* ***strictly subadditive*** *if in (2) strict inequality holds except when x or y is a nonnegative multiple of the other.*

Definition 2.5. *If there is an increasing function $\epsilon(r)$ defined for positive r,*

$$\epsilon(r) > 0, \quad \lim_{r \to 0} \epsilon(r) = 0$$

such that for all x, y in the unit ball $|x| \leq 1, |y| \leq 1$, the inequality

$$\left| \frac{x + y}{2} \right| \leq 1 - \epsilon(|x - y|)$$

holds. A normed linear space whose norm satisfies this for all united vectors x, y, where $\epsilon(r)$ is some function satisfying, is called ***uniformly convex****.*

Theorem 2.6. *Let X be a uniformly convex Banach space. Let K be a closed, convex subset of X, z any point of X. Then there is a unique point k of K which is closer to z than any other point of K.*

$$d(z, k) = d(z, K)$$

Proof: We may take $z = 0$, provided that we assume that 0 does not lie in K. Denote by s the distance of 0 to K, that is,

$$s = \inf |k|, \quad k \text{ in } K$$

Since 0 *does not lie in* K*, and since* K *is closed,* $s > 0$*. Let* $\{y_n\}$ *be a minimizing sequence for, that is,*

$$y_n \text{ in } K, \quad |y_n| = s_n \to s$$

Define the unit vectors x_n *as*

$$x_n = \frac{k_n}{s_n}$$

we can write

$$\begin{aligned} \frac{x_n + x_m}{2} &= \frac{1}{2s_n} y_n + \frac{1}{2s_m} y_m \\ &= \left(\frac{1}{2s_n} + \frac{1}{2s_m} \right) (c_n y_n + c_m y_m) \end{aligned}$$

Clearly, c_n *and* c_m *are positive, and* $c_n + c_m = 1$*. Since* K *is convex, it follows that* $c_n y_n + c_m y_m$ *belongs to* K*. Therefore,*

$$|c_n k_n + c_m k_m| \geq s$$

And we get that

$$\left| \frac{x_n + x_m}{2} \right| \geq \frac{s}{2s_n} + \frac{s}{2s_m}$$

Since $\{k_n\}$ *is a minimizing sequence,* $s_n \to s$*; therefore the right side tends to* 1*. So it follows from uniformly convexity that* $\lim_{n,m\to\infty} |x_n - x_m| = 0$*. It follows that also*

$$\lim_{n,m\to\infty} |k_n - k_m| = 0$$

meaning that the minimizing sequence $\{k_n\}$ *is a Cauchy sequence. Since* X *is complete and* K *is closed, the sequence* $\{k_n\}$ *converges to an element* k *of* K*. Clearly,* $|k| = s$*.*

§3 Isometries

Definition 3.1. *We turn now to* ***isometries*** *of a Banach space* X *onto itself, meaning mappings* T *of* X *onto* X *which preserve the distance of any pair of points:*

$$|T(x) - T(y)| = |x - y| \quad \text{for all } x, y \text{ in } X$$

Theorem 3.2. *Let* X *be a linear space over* $\mathbb{R}$ *with a strictly subadditive norm. Let* T *be an isometric mapping of* X *into itself that maps the origin into itself. Then* T *is linear.*

Proof: Denote for simplicity $T(x)$ *by* x'*. Take any pair of points* x *and* y *and define*

$$z = \frac{x + y}{2}$$

We have

$$|x' - z'| = |x - z| = \frac{|x - y|}{2}$$
$$|z' - y'| = |z - y| = \frac{|x - y|}{2}$$

and

$$|x' - y'| = |x - y|$$

These imply that

$$|x' - y'| = |x' - z' + z' - y'| = |x' - z'| + |z' - y'|$$

Since the norm is strictly subadditive, $x' - z'$ and $z' - y'$ must be positive multiples of each other. Since they have the same norm, they must be equal: $x' - z' = z' - y'$. Hence

$$2z' = x' + y'$$

Then we can conclude that T is linear.

Theorem 3.3 (Mazur and Ulam)**.** *Let X and X' be two normed linear spaces over $\mathbb{R}$, T an isometric mapping of X onto X' that carries 0 into 0. Then T is linear.*

Proof: 1. There may be other points u also halfway between x and y :

$$|x - u| = |y - u| = \frac{|x - y|}{2}.$$

We denote the set of all such u by A. We claim that this set A is symmetric with respect to the midpoint z. That is, that if u belongs to A, then so does

$$v = 2z - u$$

We define the diameter d_A of A as the greatest distance between pairs of points of A :

$$d_A = \sup_{u,w \in A} |u - w|$$

Since A is symmetric with respect to z, for all u in A,

$$|u - z| \le \frac{1}{2} d_A$$

2. Of course, there may be other points p in A with this property:

$$|u - p| \le \frac{1}{2} d_A \quad \textit{for all } u \textit{ in } A$$

We denote the set of all such p by A_1. We claim that A_1 is symmetric with respect to the midpoint z. That is, if p belongs to A_1, so does

$$q = 2z - p$$

It follows that the diameter of A_1 does not exceed half the diameter of A :

$$d_{A_1} \leq \frac{1}{2} d_A$$

3. We now repeat this construction, obtaining a nested sequence of sets $A \supset A_1 \supset A_2 \cdots$, each containing the midpoint z, each symmetric with respect to z, and their diameters satisfying

$$d_{A_{n+1}} \leq \frac{1}{2} d_{A_n}$$

Clearly, d_{A_n} tends to zero; it follows that the intersection of all the sets A_n consist of the single point z. This characterizes the midpoint z of x, y purely in terms of the metric structure of X.

4. It follows that M maps every point of A_n into A'_n bijectively and thus

$$T\left(\frac{x+y}{2}\right) = T\left(\bigcap A_n\right) = \bigcap T\left(A_n\right) = \frac{T\left(x\right) + T\left(y\right)}{2}$$

Chapter IV

Bounded Linear Maps

§1 Baire Space

Definition 1.1. *Let topological space X and A, B, C be subsets of X.*

(1) If $\overline{A} = X$, A is called ***dense****.*

(2) If $B^\circ = \varnothing$, we say that B has ***empty interior****.*

(2) If $\overline{C}^\circ = \varnothing$, then C is called ***nonwhere dense****.*

Remark *It is following from $\overline{U^c} = (U^\circ)^c$ that U has empty interior if and only if U^c is dense in X*

Definition 1.2. *Let topological space X and A be a subset of X. A is called* ***first category set*** *if there is a sequence of nowwhere dense sets $\{E_n\}_{n=1}^\infty$ that*

$$A = \bigcup E_n$$

If $B \subset X$ is not a first category set, then B is called ***second category set****.*

Definition 1.3. *A topological space X is called* ***Baire Space*** *if every nonempty open sets are the second category sets.*

Proposition 1.4 (Equivalent condition)**.** *Let topological space X. Then the following propositions are equivalent.*

(1) X is Baire space.

(2) Let $\{E_n\}$ be a denumerable sequence of dense open sets, then $\bigcap E_n$ is also dense.

(3) Let $\{F_n\}$ be a denumerable sequence of closed sets with empty interior, then $\bigcup F_n$ has also empty interior.

(4) Every nonempty open set is the second category set.

(5) If A is the first category set, then A has empty interior (Equivalently, A^c is dense in X).

Theorem 1.5 (Baire category theorem)**.** *If X is a complete metric space or a locally compact Hausdorff space, then X is Baire space.*

§2 Basic Theory

Definition 2.1. *Let X and Y are a pair of normed spaces. A linear map*

$$T : X \to Y$$

is called continuous if it maps convergent sequences into convergent ones, that is, if

$$x_n \longrightarrow x \quad \textit{implies} \quad Tx_n \longrightarrow Tx$$

Definition 2.2. *A linear map $T : X \to U$ of one normed space X into another Y is called* ***bounded*** *if there is a constant c such that for all x in X*

$$\|Tx\| \leq c|x|$$

Its norm, denoted as $|T|$, is defined by

$$\|T\| = \sup_{x \neq 0} \frac{|Tx|}{|x|}$$

The set of all bounded maps of one Banach space X into another U is denoted by

$$\mathcal{L}(X, U)$$

Theorem 2.3. *Let X and U denote normed linear spaces, $T : X \to U$ a bounded linear map.*

(1) $N(T)$ is a closed linear subspace of X. Then

(2) T, when regarded as a map

$$T_0 : X/N_T \longrightarrow U$$

is a injective, bounded linear map with $|T_0| = |T|$. And $R(T_0) = R(T)$.

Proof: N_T is the inverse image in X of $\{0\}$ in U. Since $\{0\}$ is a closed set, and T is continuous, N_T is closed.

Let $x_1 \equiv x_2 \bmod N$ if $x_1 - x_2 \in N_T$. By definition, and linearity, $Tx_1 = Tx_2$; therefore the mapping T_0 is unequivocally defined.

Using the definition of the norm of a map, and some obvious manipulations, we have

$$|T| = \sup_{x \neq 0} \frac{|Tx|}{|x|} = \sup_{\overline{x} \neq 0} \sup_{y \equiv x} \frac{|Tx|}{|y|} = \sup_{\overline{x} \neq 0} \frac{|Tx|}{\inf\limits_{y \in \overline{x}} |y|} = \sup_{\overline{x} \neq 0} \frac{|T\overline{x}|}{|\overline{x}|} = |T_0|$$

Theorem 2.4. *Suppose X and U are normed space.*

(1) $\mathcal{L}(X, U)$ is a normed space

(2) If U is Banach space then so $\mathcal{L}(X, U)$.

Definition 2.5. *Let X, U be normed space, and linear operator*

$$T : X \to U$$

Let ℓ be a point of U', then composite $\ell(Tx) \in X'$

$$\ell(Tx) = \xi(x)$$

The linear functional $\xi \in X'$ clearly depends linearly on $\ell : \xi = T'\ell$

$$T' : U' \to X'$$

*is called the **transpose** of T*

$$\langle \ell, Tx \rangle_U = \langle T'\ell, x \rangle_X$$

Theorem 2.6. *Let X, U be normed space*

(1) The transpose T' of a bounded linear map T is bounded, and

$$|T'| = |T|$$

(ii) The nullspace of T' is the annihilator of the range of T,

$$N(T') = R(T)^{\perp}$$

(iii) The nullspace of T is the annihilator of the range of T',

$$N(T) = R(T')^{\perp}$$

(iv) $(T + N)' = T' + N'$.

§3 Uniformly Bounded Theorem

Theorem 3.1 (Banach-Steinhauss theorem)**.** *Let X be Banach space, Y normed space and a collection of bounded linear operator $\{T_\alpha\} \subset \mathcal{L}(X, Y)$. If*

$$\sup_\alpha \|T_\alpha x\| < \infty$$

for each $x \in X$. Then

$$\sup_\alpha \|T_\alpha\| < \infty$$

Proof: Let

$$f(x) = \sup_\alpha \|T_\alpha x\|$$

is well defined and lower-continues. Then we define

$$M_n = \{x \in X : f(x) = \sup_\alpha \|T_\alpha x\| < n\}$$

is closed and $X = \bigcup M_n$. *Therefore, there exist a* n_0 *such that* M_{n_0} *has nonempty interior i.e.*

$$B(x_0, r) \subset M_{n_0}$$

For all $\|y\| \leq 1$, *we write* $x = x_0 + ry \in B(x_0, r) \subset M_{n_0}$. *Thus*

$$\|T_\alpha y\| = \frac{1}{r}\|T_\alpha x - T_\alpha x_0\| \leq \frac{n + f(x_0)}{r} \leq M$$

for every α, *then*

$$\sup_\alpha \|T_\alpha\| \leq M$$

Corollary 3.2. *A weakly convergent sequence of maps of Banach space* X *into normed space* Y *is uniformly bounded.*

§4 The Openning Map Theorem

Theorem 4.1. *Suppose* X *and* Y *are Banach spaces, and* $T : X \to U$ *a bounded linear mapping of* X *onto* U. *Then there is a* $d > 0$ *such that*

$$TB_1(0) \supset B_d(0)$$

Proof: We have

$$Y = \bigcup_{n=1}^{\infty} T(B_X(0, n))$$

thus

$$Y = \bigcup_{n=1}^{\infty} \overline{T(B_X(0, n))}$$

then there exist a n_0 *that* $\overline{T(B_X(0, n))}$ *has nonempty interior in* Y *i.e.*

$$B_Y(y_0, r) \subset \overline{T(B_X(0, n))}$$

We can conclude that

$$B_Y(0, r) \subset \overline{T(B_X(0, n))}$$

and

$$B_Y(0, \varepsilon) \subset \overline{T(B_X(0, 1))}$$

where ε is any positive number $< \min\{\frac{r}{n_0}, 1\}$. Furthermore,

$$B_Y(0, \varepsilon^{i+1}) \subset \overline{T(B_X(0, \varepsilon^i))}$$

Then we prove that $B_Y(0, \varepsilon^2) \subset T(B_X(0, 1))$ for some ε .

For every $y \in B_Y(0, \varepsilon^2)$, there exists $x_1 \in B_X(0, \varepsilon)$

$$\|y - Tx_1\| \le \varepsilon^3$$

and $x_2 \in B_X(0, \varepsilon^2)$

$$\|y - Tx_1 - Tx_2\| \le \varepsilon^4$$

$$\dots$$

We have a sequence $\{x_k\} \in X$ with $\|x_k\| \le \varepsilon^k$ and

$$T(x_1 + \cdots + x_n) \to y$$

as $n \to \infty$. Noted $\{x_k\}$ is a Cauchy sequence in X thus there exist x with $\|x\| \le \frac{\varepsilon}{1-\varepsilon} < 1$ (if we chooes $\varepsilon < \frac{1}{2}$) with $Tx = y$.

Corollary 4.2 (Openning Map Theorem)**.** *X and U are Banach spaces, $T : X \to U$ a bounded linear map onto U. Then T is a open map.*

Corollary 4.3 (Banach inverse operator theorem)**.** *Suppoes that X and Y are Banach spaces, $T : X \to Y$ a bounded linear map that carries X to Y surjectively. Then*

$$T^{-1} \in \mathcal{L}(Y, X)$$

Corollary 4.4 (Equivalent Norms Theorem)**.** *Let X be Banach space with norms $\|\cdot\|_1$, $\|\cdot\|_2$. If $\|\cdot\|_2$ is stronger than $\|\cdot\|_1$, then the two norms are equivalent.*

§5 Closed Graph Theorem

Definition 5.1. *Let X, U be normed space and T be linear map with domain $D\,(T)$ a subspace of X. A map $M : D(T) \to U$ is called **closed** if whenever $\{x_n\}$ is a sequence in $D(T)$ such that*

$$x_n \longrightarrow x \quad \text{and} \quad Tx_n \to u$$

then

$$Tx = u$$

It equivalent that the graph of T

$$G(T) = \{(x, Tx) : x \in X\}$$

is closed in $X \times U$.

Proposition 5.2. *Let* X *be normed space,* Y *be Banach space.*

(1) If linear operator

$$T : D(T) \to Y$$

be continues. Then Y *can be extended to* $\overline{D(T)}$ *that*

$$T_1 \mid_{D(T)} = T$$

and $\|T_1\| = \|T\|$.

(2) If the domain of continues linear operator T *is closed, then* T *is a closed linear operator.*

(3) Thus every continues linear operator can be seen as closed linear operator.

Theorem 5.3 (Closed Graph Theorem)**.** *Let* X *and* Y *be Banach spaces,* $T : X \to Y$ *is a closed linear map. Then* T *is continuous.*

Proof: Define the linear space G to consist of all pairs g *of form*

$$g = (x, Tx)\,, \quad x \text{ in } X$$

We define the following norm for g *in G:*

$$|g| = |x| + |Tx|$$

Clearly, G is Banach space.

Define the mapping $P : G \to X$ *to be the projection By definition of* $|g|, |Pg| \leq |g|$*, meaning that* P *is a bounded operator,* $|P| \leq 1$*. Clearly,* P *is linear and maps G one-to-one* X*. Therefore, the inverse of* P *is bounded; that is, there is a constant* c *such that*

$$c|Pg| \geq |g|$$

It follows that $(c-1)|x| \geq |Tx|$*, meaning that* T *is bounded.*

Remark *The condition can be weakened to* $D(T)$ *is closed in Banach space* X*, then* $T : D(T) \to U$ *is continues.*

Theorem 5.4. *Suppoes that* X *is a linear space equipped with two norms* $|x|_1$ *and* $|x|_2$ *that are compatible in the following sense:*

(a) If a sequence $\{x_n\}$ *converges in both norms, the two limits are equal.*

(b) X *is complete with respect to both norms*

then the two norms are equivalent.

Proof: Denote by X_1*, resp.* X_2 *the space* X *under the 1-, resp. 2-norm. By hypothesis, both* X_1 *and* X_2 *are complete. Compatibility clearly means that the identity map between* X_1 *and* X_2 *is closed. Therefore, by the closed graph theorem, it is bounded in both directions.*

Theorem 5.5. *X and U are Banach spaces, $T : X \to U$ a bounded linear map. Assume that the range $R(T)$ is a finite-codimensional subspace of U; then $R(T)$ is closed.*

Exercise 11. Prove theorem 14. (Hint: Extend T to $X \oplus Z$ so that its range is all of U.)

Exercise 12. Show that for every infinite-dimensional Banach space there are linear subspaces of finite codimension that are not closed. (Hint: Use Zorn's lemma.)

Theorem 5.6. *Let X be a Banach space, Y and Z closed subspaces of X that complement each other:*

$$X = Y \oplus Z$$

Denote the two components of $x = y + z$ by

$$y = P_Y x, \quad z = P_Z x$$

(1) P_Y and P_Z are linear maps on Y and Z, respectively.

(2) $P_Y^2 = P_Y, P_Z^2 = P_Z, P_Y P_Z = 0$.

(3) P_Y and P_Z are continuous.

Proof: Parts (1) and (2) are obvious. To prove part (3) we observe that since Y and Z are closed, and the decomposition is unique, it follows that the graphs of P_Y and P_Z are closed. The closed graph theorem does the rest.

Theorem 5.7. *X and U are Banach spaces, M: $X \to U$ a bounded linear map whose range $R(M)$ is a subset of U of second category. Then the range of T is all of U.*

Chapter V

Duals of Normed Linear Space

§1 Bounded Linear Functional

Definition 1.1. *The collection of all continuous linear functionals on normed space X is called the* ***dual*** *of X. It is denoted by X^*.*

Theorem 1.2. *A linear functional ℓ on X is continuous if and only if it is bounded.*

Theorem 1.3. *The nullspace of a bounded linear functional ℓ on a normed linear space is a closed linear subspace. For ℓ nontrivial, meaning $\not\equiv 0$, the nullspace has codimension 1 .*

Theorem 1.4. *The dual X^* of normed linear space X over $\mathbb{F}$ is a Banach space over $\mathbb{F}$.*

§2 Extension of Bounded Linear functional

Theorem 2.1. *Let X be a normed linear space over $\mathbb{F}$, Y a subspace, and ℓ a linear functional defined on Y and bounded there:*

$$|\ell(y)| \leq c|y|, \quad y \text{ in } Y$$

Then ℓ can be extended as a bounded linear functional to all of X so that its bound on X equals its bound on Y.

Theorem 2.2. *Say that $y_1, \ldots, y_N$ are N linearly independent vectors in a normed linear space X over $\mathbb{F}$, $a_1, \ldots, a_N$ arbitrary numbers. Then there exists a bounded linear functional ℓ such that*

$$\ell\left(y_j\right) = a_j, \quad j = 1, \ldots, N$$

Corollary 2.3. *Every finite-dimensional subspace Y of a normed linear space X has a closed complement Z such that*

$$X = Y \oplus Z$$

Proof. Choose a basis $y_1, \dots, y_N$ *in* Y*, there exist* N *bounded linear functionals* $\ell_j, j = 1, \dots, N$*, such that*

$$\ell_j\left(y_k\right) = \delta_{jk}$$

and the nullspace Z_j *of* ℓ_j *is closed. So then is their intersection*

$$Z = Z_1 \cap \ldots \cap Z_N = \{\ell_k\}^\perp$$

It is easy to check that Z *and* Y *are complementary, namely that* $X = Y \oplus Z$*.*

Theorem 2.4. *Let* X *be a normed linear space over* $\mathbb{F}, Y$ *a linear subspace of* X*. For any* z *in* X

$$d(z, Y) = \max_{\substack{|\ell| \leq 1 \\ \ell \in Y^\perp}} |\ell(z)|$$

Proof: Step 1. Since the functionals $\ell \in Y^\perp$*, and since* $|\ell| \leq 1, |\ell(z)| = |\ell(z-y)| \leq |z-y|$ *holds for all* y *in* Y*; therefore*

$$|\ell(z)| \leq d(z, Y)$$

Step 2. To show equality, we look at the linear space $Y_0 = \text{span}\{Y, z\}$*, and define on* Y_0 *the linear functional* ℓ_0 *:*

$$\ell_0(y + az) = a d(z, Y)$$

It follows that ℓ_0 *is bounded on* Y_0 *by* 1*; so it can be extended to all of* X *so that* $|\ell_0| = 1$ *and*

$$\ell_0(z) = d(z, Y)$$

Corollary 2.5. *For every* y *in a normed linear space* X *over* $\mathbb{F}$*, then*

$$|y| = \max_{|\ell|=1} |\ell(y)|$$

§3

Definition 3.1. *Let* X *be normed space. The set of linear functionals* ℓ *that vanish on a subset* S *of* X *is called the* ***annihilator*** *of* S*, and is denoted by* $S^\perp$*.*

Definition 3.2. *The* ***closed linear span*** *of a subset* $\{y_\alpha\}$ *of a normed linear space is the smallest closed linear space containing all* y_α*, that is, the intersection of all closed linear spaces containing all* y_j*.*

Indeed, the closed linear span of $\{y_\alpha\}$ *is the closure of the linear span* Y *of* $\{y_j\}$*, consisting of all finite linear combinations of the* y_α *:*

$$\text{span} F = \left\{ \sum_F \lambda_\alpha y_\alpha \right\}$$

Proposition 3.3. *Suppose X be a normed linear space over $\mathbb{F}$ and $S \subset X$.*

(1) $S^\perp$ is a closed linear subspace of X^*.

(2) Let Y be a closed subspace of a normed linear space X, then $(X/Y)'$ is isomorphic with $Y^\perp$.

(3) Spanning Criterion.

$$\overline{\text{span}S} = (S^\perp)^\perp$$

Theorem 3.4. *X is a normed linear space over $\mathbb{F}, Y$ a subspace of X. For any ℓ in X', define*

$$\|\ell\|_Y = |\ell_{|_Y}| = \sup_{\substack{y\in Y\\|y|=1}} |\ell(y)|$$

Then

$$|\ell|_Y = \min_{m\in Y^\perp} |\ell - m|$$

Proof: For any m in $Y^\perp$, and any y in Y with $|y| = 1$,

$$|\ell(y)| = |(\ell - m)(y)| \leq |\ell - m|.$$

It follows that

$$|\ell|_Y \leq |\ell - m|$$

for all $m \in Y^\perp$.

Then the restriction of ℓ to Y has an extension to X, call it ℓ_0, whose norm on X equals its norm on Y :

$$|\ell_0| = |\ell|_Y$$

Since ℓ_0 and ℓ are equal on $Y, \ell - \ell_0 = m_0$ belongs to $Y^\perp$ thus

$$|\ell - m| = |\ell_0| = |\ell|_Y$$

§4 Reflexive Spaces

Definition 4.1. *A Banach space is called **reflexive** if $X^{**} = X$, that is, if X is all of X^{**}.*

Theorem 4.2. *Every Hilbert space is reflexive.*

Theorem 4.3 (Milman)**.** *A uniformly convex Banach space is reflexive.*

Theorem 4.4. *A closed linear subspace Y of a reflexive Banach space X is reflexive.*

Proof: Every bounded linear functional ℓ on X, when restricted to Y, becomes a bounded linear functional $\ell|_Y$ on Y; we denote this functional by ℓ_0. Since by Hahn Banach every bounded linear functional on Y can be extended to X, this restriction map $\ell \to \ell_0$,

$$X^* \longrightarrow Y^*$$

maps X^* onto Y^*. The restriction map induces the following mapping from Y'' to X'' : For any η in Y'' we define ζ in X'' by setting, for any ℓ in X',

$$\zeta(\ell) = \eta(\ell_0)$$

where ℓ_0 is the restriction of ℓ to Y. Since X is reflexive, ζ can be identified with an element z of X :

$$\zeta(\ell) = \ell(z)$$

setting this into (29) gives

$$\ell(z) = \eta(\ell_0)$$

We claim that z belongs to Y. To show this, we note that if ℓ belongs to $Y^{\perp}$, meaning it vanishes on Y, then $\ell_0 = 0$, and so by $(29')$, $\ell(z) = 0$. We appeal now to theorem 8 to conclude that z belongs to the closure of Y. But since Y is closed, z belongs to Y. So we can rewrite ($29'$) as

$$\ell_0(z) = \eta(\ell_0)$$

Since every functional in Y' occurs as ℓ_0, (30) shows that every η in Y'' can be identified with some z in Y.

§5 Support Function

Definition 5.1. *For any bounded subset M of a normed linear space X over $\mathbb{R}$, we define the **support function** of M, $S_M : X' \to \mathbb{R}$*

$$S_M(\ell) = \sup_{y \in M} \ell(y)$$

Theorem 5.2. *Support functions S_M, S_N have the following properties:*

(i) Subadditivity

(ii) $S_M(0) = 0$.

(iii) Positive homogeneity

(iv) Monotonicity, for $M \subset N$, $\quad S_M(\ell) \leq S_N(\ell)$.

(v) Additivity, $S_{M+N} = S_M + S_N$.

(vi) $S_{-M}(\ell) = S_M(-\ell)$.

(vii) $S_{\overline{M}} = S_M$

(viii) $S_{\widehat{M}} = S_M$.

Theorem 5.3. *Let X be a normed linear space over $\mathbb{R}$, M a bounded subset of X. Then a point $z \in \breve{M}$ if and only if*

$$\ell(z) \leq S_M(\ell) \quad \text{for all } \ell \in X'$$

Chapter VI

Weak Converge

§1 Topological Preliminaries

Definition 1.1. *Let τ_1 and τ_2 be two topologies on a set X, and assume $\tau_1 \subset \tau_2$; that is, every τ_1-open set is also τ_2-open. Then we say that τ_1 is* ***weaker*** *than τ_2, or that τ_2 is* ***stronger*** *than τ_1.*

In this situation, the identity mapping on X is continuous from (X, τ_2) to (X, τ_1) and is an open mapping from (X, τ_1) to (X, τ_2).

Definition 1.2. *Suppose that X is a set and $\mathscr{F}$ is a nonempty family of mappings $f_\alpha : X \to Y_{f_\alpha}$, where each Y_{f_α} is a topological space. (In many important cases, Y_f is the same for all $f_\alpha \in \mathscr{F}$). Let τ be topology generated by the subbasis*

$$\left\{ f_\alpha^{-1}(V) : f_\alpha \in \mathscr{F}, V \text{ is open in } Y_{f_\alpha} \right\}$$

It is in fact the weakest topology on X that makes every $f \in \mathscr{F}$ continuous. This τ is called the ***weak topology on** X **induced by** $\mathscr{F}$, or, more succinctly,* ***the** $\mathscr{F}$**-topology of** X.*

§2 The Weak Topology

Definition 2.1. *Suppose X is a topological vector space with topology τ, whose dual X^* separates points on X. The X^*-topology of X is called the* ***weak topology of*** *X, denoted by $\sigma(X, X^*)$*

Proposition 2.2. *Suppoes X is a normed sapce.*

(1) $\sigma(X, X^*)$ is weaker than normed topology; that is, every weak open sets is open.

(2) $(X, \sigma(X, X^*))$ is Hausdorff.

(3) Thus the weak limit, if it exists, is unique.

§2.1 Weak convergence

Theorem 2.3. *Let X be a normed space and $\{x_n\} \to x$ weakly in X. Then*

(1) $\{\|x_n\|\}$ is bounded.

(2)

$$\|x\| \leq \liminf \|x_n\|$$

Proof: It follows that uniformly bounded theorem that (1) *holds. Then, there is a* $\ell \in X^*$ *such that*

$$|x| = |\ell(x)|, \quad |\ell| = 1$$

Since weak convergence means that

$$\ell(x) = \lim \ell(x_n)$$

and since

$$|\ell(x_n)| \leq |\ell| \, |x_n| = |x_n|$$

Theorem 2.4. *Let X be a normed space, $\{x_n\} \subset X$ and $x \in X$. Then $x_n \to x$ weakly if and only if*

1. *$\|x_n\|$ is bounded*
2. *there exists a dense subset M of X^* such that*

   $$\lim_{n\to\infty} \langle f, x_n\rangle = \langle f, x\rangle$$

   holds for all $f \in M$

§2.2 Weakly Closedness

Proposition 2.5. *Let X be a normed space and $A \subset X$. Then*
(1) If A is weakly compact, then A is closed weakly.
(2) If A is weakly sequentially compact, then A is closed weakly.

Theorem 2.6 (Mazur)**.** *Suppose X be a normed space and $\{x_n\} \to x$ weakly in X. Then x belongs to the closed convex hull of $\{x_n\}$; that is, for any $\varepsilon > 0$, there exists $\lambda_1, \lambda_2, \ldots, \lambda_n \geq 0$ with $\sum \lambda_i = 1$, such that*

$$\left\| x_0 - \sum_{i=1}^{n} \lambda_i x_i \right\| < \varepsilon$$

Corollary 2.7 (Mazur)**.** *Let K be a closed, convex subset of a normed linear space X. Then K is closed weakly.*

Another proof: Let S_K be the support function of K. It follows from that definition that for any ℓ in X^*

$$\ell(x_n) \leq S_K(\ell)$$

Since $\ell(x_n)$ tends to $\ell(x)$, it follows that also

$$\ell(x) \leq S_K(\ell)$$

this guarantees that x also belongs to K.

§2.3 Weakly compact and weakly sequentially compact

Definition 2.8. *A subset C of a Banach space X is called **weakly sequentially compact** if any sequence of points in C has a subsequence weakly convergent to a point of C.*

Proposition 2.9. *Let X be a Banach space. Then*

(1) A weakly sequentially compact set is bounded in norm and closed weakly.

(2) A weakly compact set is bounded in norm and closed weakly.

Theorem 2.10. *Assume that X is Banach space. Then every bounded sequence $\{x_n\}$ has a weakly convergence subsequence if and only if X is reflexive.*

Proof: Step 1. Assume that X is reflexive. Let $\{x_n\}$ be any sequence of points in the unit ball of X, that is, $|x_n| \leq 1$. Denote

$$X_0 = \overline{\text{span}\{x_n\}}$$

Since X is assumed reflexive, it follows that closed subspace X_0 is reflexive. Since $X_0^{**} = X_0$ is separable as well, it follows that X_0^* also is separable. Meaning that it contains a dense, denumerable subset $\{f_j\}$ of $\{X_0^*\}$

Step 2. Using the classical diagonal process, we can select a subsequence $\{z_n\}$ of $\{x_n\}$ such that

$$\lim_{n\to\infty} \langle f_j, z_n \rangle$$

exists for every f_j. Since all z_n satisfy $|z_n| \leq 1$, and since the $\{f_j\}$ are dense, it follows that for all f in X_0^*, $\langle f, z_n \rangle$ tends to a limit as $n \to \infty$. This limit is a bounded linear functional $\langle \cdot, Z \rangle$ on X_0^* :

$$\lim_{n\to\infty} \langle f, z_n \rangle = \langle f, z \rangle$$

Since $|m(z_n)| \leq |m| \, |z_n| \leq |m|$, it follows from ($16'$) that the linear functional $y(m)$ has norm ≤ 1.

Since Y is reflexive, there is a y in Y such that $y(m) = m(y), |y| \leq 1$, and so (16) says that for all m in Y', $m(z_n)$ tends to $m(y)$ as $n \to \infty$. Since the restriction of any ℓ in X' to Y is an m in Y', this proves that z_n converges weakly to a point y in the unit ball.

Corollary 2.11. *In a reflexive Banach space X, subset A of X is weakly sequentially compact if and only if A is bounded in norm and closed weakly.*

Thus the closed unit ball in reflexive Banach space is weakly sequentially compact.

Theorem 2.12 (Eberlein-Smulian)**.** *Assume that X is Banach space and $A \subset X$, then the following conditions are equivalent*

(1) A is weakly compact.

(2) A is weakly sequentially compact.

(3) A is weakly countably compact

§3 The Weak*-Topology

Definition 3.1. *Let X be a topological vector space whose dual is X^*. And X separates points on X^*. The X-topology of X^* is called the* ***weak*-topology of*** *X^*, denoted by $\sigma(X^*, X)$*

§3.1 Weak* convergence and weak* sequentially compact

Theorem 3.2. *Let X be a Banach space and $\{f_n\} \to f$ weakly* in X^*. Then*

$$\|f\| \leq \liminf \|f_n\|$$

Theorem 3.3 (Helly)**.** *Let X be a separable Banach space. Then*

(1) Every bounded sequence $\{f_n\}$ has a weak convergence subsequence.*

(2) Thus the closed unit ball in X^* is weak sequentially compact.*

Proof: Step 1. Given a sequence $\{f_n\}$ in closed united ball of X^*,

$$|f_n| \leq 1,$$

and take a denumerable set $\{x_k\}$ that is dense in X. Since

$$\{\langle f_n, x_k \rangle\}_{n=1}^{\infty}$$

is bounded for all k, we can, by the diagonal process, select a subsequence $\{g_n\}$ of $\{f_n\}$ such that

$$F(x) = \lim_{n\to\infty} \langle g_n, x_k \rangle$$

exists for all x_k, It follows that $\langle g_n, x \rangle$ tends to a limit for all x that lie in the closure of the set $\{x_k\} = X$.

Step 2. It is easy to see that this limit F is a linear function of x, and that it is bounded by 1.

§3.2 Weak* compact

Theorem 3.4 (Banach-Alaoglu)**.** *Let X be normed space. The closed unit ball*

$$B = \{f \in X^* : \|f\| \leq 1\}$$

in dual space X^* is weak compact.*

Corollary 3.5. *Let X be normed space and subset S is weak* closed in X^*. Then S is weak* compact if and only if S is bounded in norm.*

§4 Strong and Weak Topology on $\mathcal{L}(X, Y)$

Definition 4.1. *Let X, Y be normed space.*

*(1) The norm of linear maps $X \to Y$ defines a metric topology in $\mathcal{L}(X, Y)$ that is sometimes called the **uniform topology**. $\{T_n\}$ is called uniformly convergent if*

$$\|T_n - T\| \to 0$$

as $n \to \infty$.

*(2) The **strong topology** in $\mathcal{L}(X, Y)$ is the X-topology in which all functions $\mathcal{L} \to U$ of the form*

$$x : T \mapsto Tx$$

*are continuous, x being any point of X. $\{T_n\}$ is called **strongly convergent** if*

$$s - \lim_{n\to\infty} T_n x$$

exists for every x in X.

*(3) The **weak topology** in $\mathcal{L}(X, Y)$ is the weakest topology in which all linear functionals of the form*

$$T \longrightarrow \langle \ell, Tx \rangle_Y$$

*are continuous, $x \in X$ and $\ell \in Y^*$. $\{T_n\}$ is called **weakly convergent** if $\{T_n x\}$ converge weakly in Y*

$$w - \lim_{n\to\infty} T_n x$$

exists for all x in X.

Theorem 4.2. *Let X, Y be Banach spaces, T_n a sequence of linear maps uniformly bounded in norm. Suppose further that*

$$s - \lim T_n x$$

exists for a dense set of x in X. Then $\{T_n\}$ converges strongly.

§4.1 Application

Theorem 4.3. *Let X be a reflexive Banach space, K a closed, convex subset of X, z any point of X. Then there is a unique point y such that*

$$d(z, y) = d(z, K)$$

Proof. *We may take $z = 0$, and assume that $0 \notin K$. Denote by s the distance of 0 to K, that is,*

$$s = \inf |y|, \quad y \text{ in } K$$

Let $\{y_n\}$ *be a minimizing sequence. We may assume that each* y_n *lies in* $K \cap B(0, 2s)$*. This is a bounded, closed, convex set, therefore, a subsequence* $\{z_n\}$ *of* $\{y_n\}$ *converges weakly to some point* z *of* K*. And*

$$|z| \leq \liminf |z_n| = s$$

since $\{z_n\}$ *is the subsequence of a minimizing sequence. We have that* $|z| = s$ *and* z *is a point of* K *closest to* 0*.*

Chapter VII

Compact operator

§1 Topology

Proposition 1.1. *In a complete metric space X. The following propositions are equivalent*

(1) S is precompact : $\overline{S}$ is compact.

(2) Every sequence of points of S contains a convergent subsequence.

(3) S is completely bounded: for every $\epsilon > 0$ it can be covered by a finite number of balls of radius ϵ.

Proposition 1.2. *(c) If C_1 and C_2 are precompact subsets of a Banach space X, then $C_1 + C_2$ is precompact.*

(d) If C is a precompact set in a Banach space, so is its convex hull.

(e) If C is a precompact subset of a Banach space X, T a linear, bounded map of X into another Banach space U, then TC is a precompact subset of U.

§2 Basic Theory

Definition 2.1. *Suppose X and Y denote Banach spaces. A linear map $T : X \to Y$ is called* ***compact*** *if the image $T(B)$ of the unit ball B in X is precompact in Y.*

Theorem 2.2. *Let X, Y be Banach space ,$A, B \in \mathfrak{C}(X, Y)$, and $\alpha, \beta \in \mathbb{F}$. Then*

(1)

$$\alpha A + \beta B \in \mathfrak{C}(X, Y)$$

(2) If $M \in \mathcal{L}(W, X)$, Then

$$MA \in \mathfrak{C}(W, Y)$$

(3) If $N \in \mathcal{L}(Y, Z)$, then

$$AN \in \mathfrak{C}(X, Z)$$

(4) Let $C_n \in \mathfrak{C}(X,Y)$ be a sequence of compact maps that converge uniformly to C,

$$\lim_{n\to\infty} \|C_n - C\| = 0$$

Then C is compact.

***Remark** Let Banach space $X = Y$, the compact operator form a closed two-side ideal in $\mathcal{L}(X)$*

Theorem 2.3. *X and Y are Banach spaces, $C : X \to Y$ a compact linear map. Let X_1 be a closed subspace of X, and Y_2 the closure in Y of CX_1.*

(1) The restriction of C to $X_1 \to Y_1$ is a compact map.

(2) Suppose that $Y = X$, and the closed subspace Y is invariant under C, namely is mapped into itself by C. Then $C : X/Y \to X/Y$ is compact.

(3) A degenerate bounded linear map D $(\dim R_D < \infty)$ is compact.

Theorem 2.4 (Completely continuous)**.** *Let X, Y be Banach space and $C \in \mathfrak{C}(X,Y)$. If $\{x_n\}$ converge to x weakly in X, then $\{Cx_n\}$ converges to Cx in norms.*

Proof: Assume that there $\{x_{n_i}\}$ and $\varepsilon > 0$ that $\|Cx_{n_i} - Cx\| \geq \varepsilon$. Since $\{x_n\}$ converges weakly, then $\{x_n\}$ is bounded by uniformly bounded theorem. It follows from the compactness of C then that there exist subsequence $\left\{x_{n_j}\right\}$ that

$$s - \lim Cx_{n_j} = y$$

and $\|y - Cx\| \geq \varepsilon$.

On the another hand, for any $y^* \in Y^*$

$$\langle y^*, Cx_n\rangle = \langle C^*y^*, x_n\rangle \to \langle C^*y^*, x\rangle = \langle y^*, Cx\rangle$$

so $w - \lim Cx_n = Cx$, thus $w - \lim Cx_{n_j} = Cx$ in Y, it follows that $y = Cx$. This contradicts.

Theorem 2.5. *Let X, Y be Banach space. Then $T \in \mathfrak{C}(X,Y)$ if and only if $T^* \in \mathfrak{C}(X^*, Y^*)$*

Proof: Denote $\{y_\alpha^*\} = B_{Y^*}$. Let

$$\varphi_\alpha(y) = \langle y_\alpha^*, y\rangle, \quad y \in \overline{T(B_X)}$$

It is obvious that φ_α is equi-continuous and uniformly bounded on compact set $\overline{T(B_X)}$. It follows from Arzela-Ascoli theorem that $\{\varphi_\alpha\}$ is precompact in $C\left(\overline{T(B_X)}\right)$. Thus $\{\varphi_\alpha\}$ is completely bounded, there exists finite $y_1^*, y_2^*, \ldots, y_k^* \in B_{Y^*}$, that for any y_α^* there is a i

$$\sup_{y\in\overline{T(B_X)}} |\langle y_i^*, y\rangle - \langle y_\alpha^*, y\rangle| < \varepsilon$$

for all $\varepsilon > 0$.

$$\begin{aligned}\|T^*y_i^* - T^*y_\alpha^*\| &= \sup_{x\in B_X} |\langle T^*y_i^*, x\rangle - \langle T^*y_\alpha^*, x\rangle| \\ &= \sup_{x\in B_X} |\langle y_i^*, Tx\rangle - \langle y_\alpha^*, Tx\rangle| \\ &= \sup_{y\in T(B_X)} |\langle y_i^*, y\rangle - \langle y_\alpha^*, y\rangle| \\ &< \varepsilon\end{aligned}$$

Therefore $T^*(B_{Y^*})$ *is completely bounded, thus* T^* *is compact.*

If T^* *is compact, it follows from that* $T^{**} \in \mathfrak{C}(X^{**}, Y^{**})$ *is compact, then* $T = T^{**}\,|_X\colon X \to Y$ *is compact.*

§3 Riesz-Fredholm Theory

§3.1 Closed range

Theorem 3.1 (Closed range)**.** *Let* X *be Banach space and* $A \in \mathfrak{C}(X)$*, set*

$$T = I - A$$

Then $R(T)$ *is closed.*

Proof: Consider

$$T_1 : X/N(T) \to X$$

it follows that T_1 *is bounded linear, injective and* $R(T_1) = R(T)$.

We prove that T^{-1} *is continuous. Assume that there exists* $\{\overline{x}_n\}$ *that* $\|\overline{x}\| > \delta$ *and* $T_1\overline{x}_n \to 0$. *Let* $\overline{w}_n = \frac{\overline{x}_n}{\|\overline{x}\|}$*, then*

$$\|\overline{w}_n\| = 1, \quad T_1\overline{w}_n \to 0$$

We have $w_n \in \overline{w}_n$ *that*

$$\|w_n\| \leq 2, \quad (I - A)w_n \to 0$$

Since A *is compact, there is* $\{w_{n_k}\}$ *that* $Aw_{n_k} \to w$. *It follows that*

$$w_{n_k} = Aw_{n_k} + (I - A)w_{n_k} \to w$$

then $Tw = 0$, $\|\overline{w}\| = 0$. *It contradicts with* $\|\overline{w}\| = 0$

Corollary 3.2. *It follows from span criterion that*

$$R(T) = \overline{R(T)} = N(T^*)^\perp$$

§3.2 Null Space

Theorem 3.3. *Let X be Banach space and $A \in \mathfrak{C}(X)$, set*

$$T = I - A$$

Then

(1) $N(T)$ is finite-dimensional.

(1') $N(T^*)$ is finite-dimensional.

(2) There is an integer i such that

$$N(T^k) = N(T^i) \quad \text{for } k > i$$

(It is equivalent that $N(T^{i+1}) = N(T^i)$)

Proof: (1) follows from the Riesz's Lemma.

(2) Assume, that is, that N_{i-1} is a proper subset of N_i for all i. There would be for every i a vector y_i such that

$$y_i \text{ in } N_i, \quad |y_i| = 1, \quad d\left(y_i, N_{i-1}\right) > \frac{1}{2}$$

Take $m < n$; by definition of T,

$$Ay_n - Ay_m = y_n - Ty_n - y_m + Ty_m$$

The last three terms on the right belong to N_{n-1}, so, their sum differs from y_n by $\frac{1}{2}$ at least. This proves that $|Ay_n - Ay_m| > \frac{1}{2}$ which contradicts compactness of A and $\|y_k\| = 1$.

Corollary 3.4. *It follows form $N(T) = R\left(T^*\right)^{\perp}$ that*

$$\dim N(T) = \text{Codim}\, R(T^*)$$

Lemma 3.5. *In infinite dimension Banach space X,*

(1) Let $x_1, x_2, \ldots x_n \in X$, then there exists closed subspace X_1 such that

$$X = \text{span}\left\{x_1, x_2, \ldots, x_n\right\} \oplus X_1$$

Indeed, $X_1 = \left\{x_1^*, x_2^*, \ldots, x_n^*\right\}^{\perp}$

(2) Let $f_1, f_2, \ldots, f_m \in X^*$, then there exists $y_1, y_2, \ldots, y_m$ such that

$$f_i(y_j) = \delta_{ij}$$

Theorem 3.6. *Let C be a compact map of a Banach space $X \to X$. Then $T = I - C$ satisfies*

$$\text{ind}\, T = \dim N\left(T\right) - \text{codim}\, R(T) = 0$$

Proof: Step 1. We start with the special case that $N(T)$ is trivial. We show that then $\operatorname{Codim} R(T) = 0$. *Now suppose, on the contrary, that $R(T) = X_1$ is a proper closed subspace of X. Then, since by assumption, T is one-to-one, it follows from $C|_{X_1}$ is compact that $TX_1 = X_2$ is a proper closed subspace of X_1. Define X_k as $T^k X$. We deduce similarly that $X \supset X_1 \supset X_2 \supset \cdots$, and that all inclusions are proper, X_k is closed.*

We appeal now to Riesz's lemma; we can choose x_k in X_k so that

$$|x_k| = 1, \quad \operatorname{dist}(x_k, X_{k+1}) > \frac{1}{2}$$

Let m and n be two distinct indices, $m < n$. Then

$$Ax_m - Ax_n = x_m - Tx_m - x_n + Tx_n$$

The last three terms on the right all belong to X_{m+1}; therefore,

$$\|Ax_m - Ax_n\| > \frac{1}{2}$$

This contradicts the assumption that C maps the unit ball into a precompact set.

Step 2. Denote $N(T) = \operatorname{span}\{x_1, x_2, \ldots, x_n\}$, if $\operatorname{Codim} R(T) = \dim N(T^*) > n$ *then we define*

$$T_1 : \operatorname{span}\{x_1, x_2, \ldots, x_n\} \oplus X_1 \to \operatorname{span}\{y_1, y_2, \ldots, y_n\} \oplus R(T)$$

$$T_1\left(\sum_1^n c_i x_i + y\right) = \sum_1^n c_i y_i + Ty$$

It obvious that T_1 is also compact, $N(T_1) = 0$ thus T_1 is surjectively by Step 1. But it contradicts.

Corollary 3.7.

$$\dim N(T) = \dim N(T^*)$$

§3.3 Riesz-Fredholm

Theorem 3.8. *Let X be Banach space and $C \in \mathfrak{C}(X)$, $T = I - C$. Then*

(1) $R(T) = N(T^*)^\perp$, $R(T^*) = N(T)^\perp$

(2) $\operatorname{ind} T = \dim N(T) - \operatorname{Codim} R(T) = 0$, $\dim N(T^*) = \operatorname{Codim} R(T) = N(T)$

§4 Spectral Theorey of Compact Maps

§4.1

Theorem 4.1 (Riesz-Schauder)**.** *Let X be a Banach space of infinite dimension and $C \in \mathfrak{C}(X)$.*

(1) $0 \in \sigma(C)$

(2) $\sigma(C)\setminus\{0\}\subset\sigma_p(C)$; and for each nonzero eigenvalue $\lambda\in\sigma_p$, $N(\lambda I-C)$ is finite dimension.

(3) σ is at most denumerable sets that accumulate only at 0

Proof: (1),(2). To prove (3), suppose that there is $\{\lambda_n\}\subset\sigma(C)\setminus\{0\}$ that $\lambda_n\neq\lambda_m$ for all $n\neq m$ and $\lambda_n\to\lambda\neq 0$. Then there exists nonzero

$$x_n\in N(\lambda_n I-C)$$

It follows that $\{x_n\}$ is linear independent. Let $E_n=\text{span}\{x_1,x_2,\ldots,x_n\}$, then there $y\in E_{n+1}$ that

$$\|y_{n+1}\|=1,\quad d(y_{n+1,E_n})>\frac{1}{2}$$

Then for all $n,m\in\mathbb{N}$ with $n>m$

$$M\,\|Cy_n-Cy_m\|>\left\|\frac{Cy_n}{\lambda_n}-\frac{Cy_m}{\lambda_m}\right\|=\left\|y_n-\left(y_n-\frac{Cy_n}{\lambda_n}+\frac{Cy_m}{\lambda_n}\right)\right\|>\frac{1}{2}$$

since $y_n-\frac{Cy_n}{\lambda_n}+\frac{Cy_m}{\lambda_m}\in E_{n-1}$. It contradicts with the compactness of C.

§5 Hilbert-Schmidt

Definition 5.1. *An operator A mapping a Hilbert space H into itself is called **symmetric** if $A=A^*$*

Theorem 5.2. *A symmetric operator A as above is closed, thus bounded.*

Theorem 5.3. *Let A be a symmetric operator on H:*

(1) The (hermitean) quadratic form (Ax,x) is real for all x in H.

(2) The quadratic form is not identically zero unless the operator $A\equiv 0$.

Definition 5.4. *A symmetric operator K mapping a Hilbert space H into itself is called **positive definite** if the associated quadratic form (Kx,x) is nonnegative for every x in H. This is denoted as $0\le K$.*

Let A and B denote two symmetric operators mapping a Hilbert space H into itself. The inequality $A\le B$ means that $0\le B-A$.

§5.1 Fundmental Theorem

Theorem 5.5. *Let H be Hilbert space, A is symmetric and compact operator on H, Then there exists unit element x_0 such that*

$$|(Ax_0,x_0)|=\sup_{\|x\|=1}|(Ax,x)|$$

and

$$Ax_0=\lambda x_0$$

where $\lambda = \sup_{\|x\|=1} |(Ax, x)|$.

Proof: Since A *is a bounded operator,* $|(Ax, x)|$ *does not exceed* $\|A\|$ *on the unit sphere. Suppose that :*

$$\sup_{\|x\|=1} (Ax, x) = \lambda$$

Let $\{x_n\}$ *be a maximizing sequence on the unit sphere. Since the unit ball in a Hilbert space is weakly sequentially compact, a subsequence, also denoted as* $\{x_n\}$*, converges weakly to a limit we denote as* z*. It follows that* Ax_n *converges strongly to* Az *and*

$$(Ax_n, x_n) = (Az, x_n) + (Ax_n - Az, x_n) \to (Az, z)$$

Therefore

$$(Az, z) = \lambda$$

And it obvious $|z| = 1$ *by maximality*

The homogeneous function

$$R_A(x) = \frac{(Ax, x)}{\|x\|^2}$$

is called the Rayleigh quotient. Clearly, the vector z *maximizes* $R_A(z)$ *among all nonzero vectors, not just unit vectors. Let* w *be any vector in* H*,* t *any real number. The function* $R(z + tw)$ *as function of* t *achieves its maximum at* $t = 0$*; therefore*

$$R'(z + tw) = \frac{(Aw, z) + (Az, w)}{\|z\|^2} - (Az, z)\frac{(w, z) + (z, w)}{\|z\|^4} = 0$$

from which, using the symmetry of A *and (* $4'$ *), we get*

$$\mathrm{Re}(Az - mz, w) = 0$$

thus $Az = \lambda z$

Corollary 5.6. *A symmetric compact operator* A *mapping a Hilbert space* H *into itself has real eigenvalues. If* $A \neq 0$*,* A *has nonzero eigenvalues.*

Theorem 5.7 (Spectral Theorem)**.** *A denotes a compact symmetric operator mapping Hilbert space* H *into itself. Then there is an at most denumerable eigenvalues* $\{\lambda_n\} \subset \mathbb{R}$ *that accumulate only at* 0*, and an orthonormal base* $\{e_\alpha\}$ *for* H *consisting of eigenvector of* A *:*

$$Ae_\alpha = \lambda_{n,\alpha} e_\alpha$$

Proof : For any $\lambda \in \sigma_p(A) \setminus \{0\}$*, suppose the orthonormal base of* $N(\lambda I - A)$ *is*

$$\left\{e_i^{(\lambda)}\right\}_{i=1}^{m(\lambda)}$$

where $m(\lambda) = \dim N(\lambda I - A)$ *is the geometric multiplicity. If* $0 \in \sigma_p(A)$*, denote the orthonormal base of* $N(A)$ *by*

$$\left\{e_\alpha^{(0)}\right\}$$

Let

$$M = \{e_\alpha\}$$

be the union of base as above. We prove that $\overline{\operatorname{span} M} = H$*. If not, then* $M^\perp \neq \{0\}$*. Let*

$$A_1 = A \mid_{M^\perp}$$

it follows that A_1 *has no eigenvalues and is compact, symmetric. It contradicts.*

§5.2

Theorem 5.8. *Let* A *be a compact symmetric operator; denote its positive eigenvalues, indexed in decreasing order, by* $\lambda_k \leq \lambda_{k+1}, k = 1, 2, \ldots$*. Denote by* $R_A(x)$ *its Rayleigh quotient.*

(1) Fischer's principle:

$$\lambda_N = \max_{S_N} \min_{x \in S_N} R_A(x)$$

where S_N *is any linear subspace of* H *of dimension* N*.*

(2) Courant's principle:

$$\lambda_N = \min_{S_{N-1}} \max_{x \perp S_{N-1}} R_A(x)$$

§5.3 Normal Operator

Definition 5.9. *An operator* N *mapping a Hilbert space* H *into itself is called* ***normal*** *if* N *and its adjoint commute:*

$$N^* N = N N^*$$

Theorem 5.10. *Every compact normal operator on has a complete set of orthonormal eigenvectors.*

Proof: Decompose N *into the sum of its symmetric and antisymmetric parts:*

$$N = R + J, \quad \text{where } R = \frac{N + N^*}{2}, J = \frac{N - N^*}{2}$$

Clearly, R *is symmetric,* J *antisymmetric, and* $N^* = R - J$*. Since* N *and* N^* *commute, so do* R *and* J*. And the adjoint* N^* *of the compact operator* N *is compact; therefore so are* J *and* R*.*