Differential Manifolds

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Contents

Chapter 1

Manifolds

1.1 Manifolds

1.1.1 Topological Manifolds

Definition 1.1.1. *Suppose M is a topological space. We say that M is a* ***topological manifold*** *of dimension n or a topological n-manifold if it has the following properties:*

(i) M is a Hausdorff space

(ii) M is second-countable

(iii) M is locally Euclidean of dimension n: each point p of M has a neighborhood U that is homeomorphic to an open subset $\widehat{U}$ of $\mathbb{R}^n$. (U, φ) called ***coordinate chart****.*

Given a chart (U, φ), we call the set U a ***coordinate domain****, and φ is called a* ***local coordinate map****.*

1.1.2

Proposition 1.1.2. *Suppose M is a topological manifold, then*
(1) M is local compact
(2) M is connected
(4) M is path-connected
(5) M is Lindelof's : any open

Definition 1.1.3. *Let M be a topological space. A collection $\mathcal{X}$ of subsets of M is said to be* ***locally finite*** *if each point of M has a neighborhood that intersects at most finitely many of the sets in $\mathcal{X}$.*

Given a cover $\mathcal{U}$ of M, another cover $\mathcal{V}$ is called a ***refinement*** *of $\mathcal{U}$ if for each $V \in \mathcal{V}$ there exists some $U \in \mathcal{U}$ such that $V \subseteq U$.*

We say that M is ***paracompact*** *if every open cover of M admits an open, locally finite refinement.*

Theorem 1.1.4 (σ-compact)**.** *A second-countable, locally compact Hausdorff space (thus a manifold) admits an exhaustion by compact sets.*

Proof: Let X be such a space. Because X is a locally compact Hausdorff space, it has a basis of precompact open subsets; since it is second-countable, it is covered by countably many such sets. Let $(U_i)_{i=1}^{\infty}$ be such a countable cover. Beginning with $K_1 = \overline{U}_1$, assume by induction that we have constructed compact sets $K_1, \ldots, K_k$ satisfying $U_j \subseteq K_j$ for each j and $K_{j-1} \subseteq \operatorname{Int} K_j$ for $j \geq 2$. Because K_k is compact, there is some m_k such that $K_k \subseteq U_1 \cup \cdots \cup U_{m_k}$. If we let $K_{k+1} = \overline{U}_1 \cup \cdots \cup \overline{U}_{m_k}$, then K_{k+1} is a compact set whose interior contains K_k. Moreover, by increasing m_k if necessary, we may assume that $m_k \geq k+1$, so that $U_{k+1} \subseteq K_{k+1}$. By induction, we obtain the required exhaustion.

Theorem 1.1.5 (Paracompact)**.** *Every topological manifold is paracompact. In fact, given a topological manifold M, an open cover $\mathcal{X}$ of M, and any basis $\mathfrak{B}$ for the topology of M, there exists a countable, locally finite open refinement of $\mathcal{X}$ consisting of elements of $\mathcal{B}$.*

Proof: Given $M, \mathcal{X}$, and $\mathcal{B}$ as in the hypothesis of the theorem, let $(K_j)_{j=1}^{\infty}$ be an exhaustion of M by compact sets (Proposition A.60). For each j, let $V_j = K_{j+1} \setminus \operatorname{Int} K_j$ and $W_j = \operatorname{Int} K_{j+2} \setminus K_{j-1}$ (where we interpret K_j as $\varnothing$ if $j < 1$). Then V_j is a compact set contained in the open subset W_j. For each $x \in V_j$, there is some $X_x \in \mathcal{X}$ containing x, and because $\mathcal{B}$ is a basis, there exists $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq X_x \cap W_j$. The collection of all such sets B_x as x ranges over V_j is an open cover of V_j, and thus has a finite subcover. The union of all such finite subcovers as j ranges over the positive integers is a countable open cover of M that refines $\mathcal{X}$. Because the finite subcover of V_j consists of sets contained in W_j, and $W_j \cap W_{j'} = \varnothing$ except when $j-2 \leq j' \leq j+2$, the resulting cover is locally finite.

1.1.3 Smooth Structures and Smooth Manifolds

Definition 1.1.6. *Let M be a topological n-manifold.*

1. *Two charts (U, φ) and (V, ψ) are said to be **smoothly compatible** if either $U \cap V = \varnothing$ or the two **transition map***

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V), \quad \varphi \circ \psi^{-1} : \psi(U \cap V) \to \varphi(U \cap V)$$

   are C^{∞}.

2. *We define an **smooth atlas** $\mathcal{A}$ for M to be a collection of charts $\{(U_\alpha, \varphi_\alpha)\}$ whose domains cover M and any two charts in $\mathcal{A}$ are smoothly compatible with each other.*

3. *A smooth atlas $\mathcal{A}$ on M is **maximal** if it is not properly contained in any larger smooth atlas. If M is a topological manifold, a **smooth structure** on M is a maximal smooth atlas.*

   *A C^{∞} manifold is a pair $(M, \mathcal{A})$, where M is a topological manifold and $\mathcal{A}$ is a **smooth structure** on M.*

Proposition 1.1.7. *Let M be a topological manifold.*

*(1) Every smooth atlas $\mathcal{A}$ for M is contained in a unique maximal smooth atlas $\overline{\mathcal{A}}$, called the **smooth structure determined by** $\mathcal{A}$. Indeed, $\overline{\mathcal{A}}$ denote the set of all charts that are smoothly compatible with every chart in $\mathcal{A}$.*

(2) Two smooth atlases for M determine the same smooth structure if and only if their union is a smooth atlas.

Proof. We only prove (1). Let $\mathcal{A}$ be a smooth atlas for M, and let $\overline{\mathcal{A}}$ denote the set of all charts that are smoothly compatible with every chart in $\mathcal{A}$.

Step 1. To show that $\overline{\mathcal{A}}$ is a smooth atlas. For any $(U, \varphi), (V, \psi) \in \overline{\mathcal{A}}$, let $x = \varphi(p) \in \varphi(U \cap V)$ be arbitrary, then there is some chart $(W, \theta) \in \mathcal{A}$ such that $p \in W$. Since every chart in $\overline{\mathcal{A}}$ is smoothly compatible with (W, θ), both of the maps $\theta \circ \varphi^{-1}$ and $\psi \circ \theta^{-1}$ are smooth where they are defined. Since $p \in U \cap V \cap W$, it follows that $\psi \circ \varphi^{-1} = (\psi \circ \theta^{-1}) \circ (\theta \circ \varphi^{-1})$ is smooth on a neighborhood of x. Thus, $\psi \circ \varphi^{-1}$ is smooth in a neighborhood of each point in $\varphi(U \cap V)$. Therefore, $\overline{\mathcal{A}}$ is a smooth atlas.

Step 2. To check that it is maximal, just note that any chart that is smoothly compatible with every chart in $\overline{\mathcal{A}}$ must in particular be smoothly compatible with every chart in $\mathcal{A}$, so it is already in $\overline{\mathcal{A}}$. This proves the existence of a maximal smooth atlas containing $\mathcal{A}$.

Step 3. Uniqueness. If $\mathcal{B}$ is any other maximal smooth atlas containing $\mathcal{A}$, each of its charts is smoothly compatible with each chart in $\mathcal{A}$, so $\mathcal{B} \subseteq \overline{\mathcal{A}}$. By maximality of $\mathcal{B}, \mathcal{B} = \overline{\mathcal{A}}$. $\square$

Definition 1.1.8. *A subset S of a C^∞ manifold M of dimension n is a **regular submanifold of dimension** s if for every $p \in S$ there is a coordinate neighborhood (U, φ) of p such that $U \cap S$ is defined by the vanishing of $n - s$ of the coordinate functions. By renumbering the coordinates, we may assume that these $n - k$ coordinate functions are $x^{s+1}, \dots, x^n$.*

$$\varphi(U \cap S) = \varphi(U) \cap (\mathbb{R}^s \times 0)$$

Let

$$\varphi_S : U \cap S \to \mathbb{R}^s$$

be the restriction of the first k components of φ to $U \cap S$. Then $(U \cap S, \varphi_S)$ is a chart for S in the subspace topology.

1.1.4 Manifold with Boundary

1.1.5 Topological Manifold with Boundary

Definition 1.1.9. *An **n-dimensional topological manifold with boundary** is a second-countable Hausdorff space M in which every point p has a neighborhood U homeomorphic φ to an open subset of $\mathbb{R}^n_+$. The open subset $U \subseteq M$ together with a map $\varphi : U \to \mathbb{R}^n_+$ that is a homeomorphism onto an open subset of $\mathbb{R}^n_+$ will be called a **chart** for M.*

*We will call (U, φ) an **interior chart** if $\varphi(U)$ is an open subset of $\mathbb{R}^n$, and a **boundary chart** if $\varphi(U)$ is an open subset of $\mathbb{R}^n_+$ such that $\varphi(U) \cap \partial\mathbb{R}^n_+ \neq \varnothing$.*

*A point $p \in M$ is called an **interior point** of M if it is in the domain of some interior chart. It is a **boundary point** of M if it is in the domain of a boundary chart that sends p to $\partial\mathbb{R}^n_+$.*

The boundary of M (the set of all its boundary points) is denoted by ∂M; similarly, its interior, the set of all its interior points, is denoted by $\operatorname{Int} M$.

Remark *∂M and $\operatorname{Int} M$ are disjoint sets whose union is M*

Proposition 1.1.10. *Let M be a topological n-manifold with boundary.*

(1) $\operatorname{Int} M$ is an open subset of M and a topological n-manifold without boundary.

(2) ∂M is a closed subset of M and a topological $(n-1)$-manifold without boundary.

(3) M is a topological manifold if and only if $\partial M = \varnothing$.

(4) If $n = 0$, then $\partial M = \varnothing$ and M is a 0 -manifold.

Proposition 1.1.11. *Let M be a topological manifold with boundary.*

(1) M has a countable basis of precompact coordinate balls and half-balls.

(2) M is locally compact.

(3) M is paracompact.

(4) M is locally path-connected.

(5) M has countably many components, each of which is an open subset of M and a connected topological manifold with boundary.

(6) The fundamental group of M is countable.

1.1.6 Smooth Manifold with Boundary

Definition 1.1.12. *Let M be a topological manifold with boundary. A **smooth structure** for M is defined to be a maximal smooth atlas $\mathcal{A}$, a collection of charts whose domains cover M and whose transition maps (and their inverses) are smooth. With such a structure, $(M, \mathcal{A})$ is called a **smooth manifold with boundary**.*

Just as for smooth manifolds, if M is a smooth manifold with boundary, any chart in the given smooth atlas is called a smooth chart for M.

Recall that a map from an arbitrary subset $A \subseteq \mathbb{R}^n$ to $\mathbb{R}^k$ is said to be smooth if in a neighborhood of each point of A it admits an extension to a smooth map defined on an open subset of $\mathbb{R}^n$.

1.2 Smooth Maps on a Manifold

1.2.1 Smooth Functions on a Manifold

Definition 1.2.1. *Suppose M is a smooth n-manifold, k is a nonnegative integer, and $f : M \to \mathbb{R}^k$ is any function. We say that f is a **smooth function** if for every $p \in M$, there exists a smooth chart*

(U, φ) for M whose domain contains p and such that the composite function

$$f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^k$$

is smooth on the open subset $\varphi(U) \subseteq \mathbb{R}^n$. The function $f \circ \varphi^{-1}(x)$ is called the ***local coordinate representation*** *of f.*

Proposition 1.2.2. *Let M be a manifold of dimension n, and $f : M \to \mathbb{R}$ a real-valued function on M. The following are equivalent:*

1. *The function $f : M \to \mathbb{R}$ is C^∞.*
2. *The manifold M has an atlas such that for every chart (U, ϕ) in the atlas, $f \circ \phi^{-1} : \mathbb{R}^n \supset \phi(U) \to \mathbb{R}$ is C^∞.*
3. *For every chart (V, ψ) on M, the function $f \circ \psi^{-1} : \mathbb{R}^n \supset \psi(V) \to \mathbb{R}$ is C^∞.*

1.2.2 Smooth Maps Between Manifolds

Definition 1.2.3. *Let M, N be smooth manifolds, and let $f : M \to N$ be a continuous map. We say that f is a* ***smooth map*** *if for every $p \in M$, there exist smooth charts (U, φ) containing p and (V, ψ) containing $f(p)$ (assume that $f(U) \subseteq V$ without generality) and the composite map*

$$\tilde{f} = \psi \circ f \circ \varphi^{-1} : \varphi(f^{-1}(V) \cap U) \to \psi(V)$$

is smooth. We call $\tilde{f} = \psi \circ f \circ \varphi^{-1}$ the ***coordinate representation of f with respect to the given coordinates****.*

Proposition 1.2.4. *Let N and M be smooth manifolds, and $F : N \to M$ a continuous map. The following are equivalent:*

1. *The map $F : N \to M$ is C^∞.*
2. *There are atlases $\mathfrak{U}$ for N and $\mathfrak{V}$ for M such that for every chart (U, ϕ) in $\mathfrak{U}$ and (V, ψ) in $\mathfrak{V}$, the map*
$$\psi \circ F \circ \phi^{-1} : \phi\left(U \cap F^{-1}(V)\right) \to \mathbb{R}^m$$
is C^∞.
3. *For every chart (U, ϕ) on N and (V, ψ) on M, the map*
$$\psi \circ F \circ \phi^{-1} : \phi\left(U \cap F^{-1}(V)\right) \to \mathbb{R}^m$$
is C^∞.

Proposition 1.2.5. *Let M, N, and P be smooth manifolds with or without boundary.*

(1) Every constant map c:$M \to N$ is smooth.

(2) The identity map of M is smooth.

(3) If $U \subseteq M$ is an open submanifold with or without boundary, then the inclusion map $U \hookrightarrow M$ is smooth.

(4) If $F : M \to N$ and $G : N \to P$ are smooth, then so is $G \circ F : M \to P$.

1.2.3 Diffeomorphisms

Definition 1.2.6. *If M and N are smooth manifolds with or without boundary, a **diffeomorphism** from M to N is a smooth bijective map $F : M \to N$ that has a smooth inverse. We say that M and N are **diffeomorphic** if there exists a diffeomorphism between them. Sometimes this is symbolized by $M \approx N$.*

Proposition 1.2.7. *(1) Every composition of diffeomorphisms is a diffeomorphism.*

(2) Every finite product of diffeomorphisms between smooth manifolds is a diffeomorphism.

(3) Every diffeomorphism is a homeomorphism and an open map.

(4) The restriction of a diffeomorphism to an open submanifold with or without boundary is a diffeomorphism onto its image.

(5) "Diffeomorphic" is an equivalence relation on the class of all smooth manifolds with or without boundary.

Definition 1.2.8. *Let $f : M \to N$ be a smooth map, and let (U, φ) and (V, ψ) be charts on M and N respectively such that $f(U) \subset V$. Denote by*

$$\tilde{f} = \psi \circ f \circ \varphi^{-1}$$

Then the matrix

$$\left(\frac{\partial \tilde{f}^i}{\partial x^j}\right)$$

*is called the **Jacobian matrix of f relative to the charts (U, φ) and (V, ψ)**.*

We define the rank of f at p

$$\operatorname{rank}_p f := \operatorname{rank}\left(\frac{\partial \tilde{f}^i}{\partial x^j}\right)_{\varphi(p)}$$

Theorem 1.2.9 (Constant rank theorem)**.** *Let M and N be manifolds of dimensions m and n respectively. Suppose $f : M \to N$ has constant rank k in a neighborhood of a point p in M. Then there are charts (U, φ) centered at p in N and (V, ψ) centered at $f(p)$ in M such that for $(x^1, \ldots, x^m)$ in $\varphi(U)$,*

$$\psi \circ f \circ \varphi^{-1} : \left(x^1, \ldots, x^m\right) \mapsto \left(x^1, \ldots, x^k, 0, \ldots, 0\right)$$

Proof. Choose a chart $(\overline{U}, \overline{\varphi})$ about p in M and $(\overline{V}, \overline{\psi})$ about $f(p)$ in M. Then $\overline{\psi} \circ f \circ \overline{\varphi}^{-1}$ is a map between open subsets of Euclidean spaces. Because $\overline{\varphi}$ and $\overline{\psi}$ are diffeomorphisms, $\overline{\psi} \circ f \circ \overline{\varphi}^{-1}$ has the same constant rank k as f in a neighborhood of $\overline{\varphi}(p)$ in $\mathbb{R}^n$.

By the constant rank theorem for Euclidean spaces there are a diffeomorphism G of a neighborhood of $\overline{\varphi}(p)$ in $\mathbb{R}^m$ and a diffeomorphism F of a neighborhood of $(\overline{\psi} \circ f)(p)$ in $\mathbb{R}^m$ such that

$$F \circ \overline{\psi} \circ f \circ \overline{\varphi}^{-1} \circ G^{-1} \left(x^1, \ldots, x^m\right) = \left(x^1, \ldots, x^k, 0, \ldots, 0\right).$$

Set $\phi = G \circ \overline{\varphi}$ and $\psi = F \circ \overline{\psi}$.

1.3 Partition of unity

1.3.1 Topological Preliminaries

Definition 1.3.1. *Let X be a topological space.*

*(1) A collection $\mathcal{A}$ of subsets of X is said to be **locally finite** in X if every point of X has a neighborhood that intersects only finitely many elements of $\mathcal{A}$.*

*(2) A collection $\mathcal{B}$ of subsets of X is said to be **countably locally finite** if $\mathcal{B}$ can be written as the countable union of collections $\mathcal{B}_n$, each of which is locally finite.*

Definition 1.3.2. *Let X be a topological space, if there exists X_j such that*

(i) For each j, the closure $\overline{X}_j$ is compact.

(ii) For each j, $\overline{X}_j \subset X_{j+1}$.

(iii) $M = \bigcup_j X_j$.

*The subsets $\{X_j\}$ described is called an **exhaustion** of M.*

Definition 1.3.3. *A real-valued continuous function f on X is called an **exhausion function** for X if for any $c \in \mathbb{R}$, the sublevel set $f^{-1}((\infty, c])$ is compact.*

Lemma 1.3.4 (Lindelof's)**.** *Let X be a second countable space, then every open cover $\mathcal{A}$ has a countable subcover.*

Theorem 1.3.5. *If X is a second countable, locally compact and Hausdorff space (thus a manifold), then there exists a exhaustion of X.*

Proof. First, there exists a open cover $\mathcal{A}$ that every element of $\mathcal{A}$ has compact closure, then there exists countable many

$$A_1, A_2, \ldots$$

such that $\bigcup A_i = X$ and $\overline{A_i}$ is compact.

We let $X_1 = A_1$. Since A_i is an open cover of $\overline{X}_1$ which is compact, there exists finitely many open sets $A_{i_1}, \cdots, A_{i_k}$ so that $\overline{X}_1 \subset A_{i_1} \cup \cdots \cup A_{i_k}$. Let $X_2 = A_{i_1} \cup \cdots \cup A_{i_k} \cup A_2$. Obviously

$\overline{X}_2$ is compact. Repeat this procedure again and again, we could get a desired sequence of open sets $X_1, X_2, X_3, \cdots$.

Definition 1.3.6. *Let $\mathcal{A}$ be a open cover of the space X, then a open cover $\mathcal{B}$ of X is said to be a **refinement** of $\mathcal{A}$ (or is said to refine $\mathcal{A}$) if for each element B of $\mathcal{B}$, there is an element A of $\mathcal{A}$ containing B.*

Definition 1.3.7. *A space X is **paracompact** if every open cover $\mathcal{A}$ of X has a locally finite open refinement $\mathcal{B}$ of $\mathcal{A}$.*

Theorem 1.3.8. *Let M be any topological manifold. For any open cover $\mathcal{U} = \{U_\alpha\}$ of M, one can find two countable family of open covers $\mathcal{V} = \{V_j\}$ and $\mathcal{W} = \{W_j\}$ of M so that*

(i) $\mathcal{W}$ is a locally finite open refinement of $\mathcal{U}$.

(ii) For each j, $\overline{V}_j$ is compact and $\overline{V}_j \subset W_j$.

Proof. *Let $\{X_j\}$ be a exhaustion of M. For each $p \in M$, there is an j and an $\alpha(p)$ so that $p \in \overline{X}_{j+1} \backslash X_j$ and $p \in U_{\alpha(p)}$. Since M is locally Euclidean, one can always choose open neighborhoods V_p, W_p of p so that $\overline{V}_p$ is compact and*

$$p \in V_p \subset \overline{V}_p \subset W_p \subset U_{\alpha(p)} \cap \left(X_{j+2} \backslash \overline{X}_{j-1}\right)$$

Now for each j, since the "stripe" $\overline{X}_{j+1} \backslash X_j$ is compact, one can choose finitely many points $p_1^j, \cdots, p_{k_j}^j$ so that $V_{p_1^j}, \cdots, V_{p_{k_j}^j}$ is an open cover of $\overline{X}_{j+1} \backslash X_j$. Denote all these $V_{p_k^j}$'s by $V_1, V_2, \cdots$, and the corresponding $W_{p_k^j}$'s by $W_1, W_2, \cdots$. Then $\mathcal{V} = \{V_k\}$ and $\mathcal{W} = \{W_k\}$ are open covers of M that satisfies all the conditions.

Corollary 1.3.9. *Manifolds is paracompact, σ-compact*

1.3.2 Existence

Lemma 1.3.10. *There exists f_1, f_2, f_3 such that*

Theorem 1.3.11 (Bump function on manifold)**.** *Let M be a smooth manifold, $K \subset M$ is a compact subset, and $U \subset M$ an open subset that contains K. Then there is a $\varphi \in C^\infty(M)$ so that $K \prec f \prec U$ ($0 \leq \varphi \leq 1, \varphi \equiv 1$ on K and $\operatorname{supp}(\varphi) \subset U$).*

Proof. *For each $q \in K$, there is a chart (φ_q, U_q, V_q) near q so that $U_q \subset U$ and V_q contains the open ball $B_3(0)$ in $\mathbb{R}^n$. Let $U_q' = \varphi_q^{-1}(B_1(0))$, and let*

$$f_q(p) = \begin{cases} f_3(\varphi_q(p)) & , p \in U_q \\ 0 & , p \notin U_q \end{cases}$$

Then $f_q \in C^\infty(M), \operatorname{supp}(f_q) \subset U_q \subset U$ and $f_1 \equiv 1$ on U_q'.

Now the family of open sets $\{U'_q\}$ *is an open cover of* K. *Since* K *is compact, there is a finite sub-cover* $\{U'_{q_1}, \cdots, U'_{q_n}\}$. *Let*

$$\psi = \sum_{i=1}^{n} f_{q_i}$$

Then ψ *is a smooth and compactly supported function on* M *so that* $\psi \geq 1$ *on* K *and* $\operatorname{supp}(\psi) \subset U$. *It follows that the function* $\varphi(p) = f_2(\psi(p))$ *satisfies all the conditions we required.*

Definition 1.3.12. *Suppose* M *is a topological space, and let* $\mathcal{A} = \{A_\alpha\}$ *be an arbitrary open cover of* M. *A* ***partition of unity subordinate to*** $\mathcal{A}$ *is an indexed family*

$$\{\rho_\alpha : \rho_\alpha : M \to \mathbb{R} \text{ is continue}\}$$

with the following properties:

(i) $\rho_\alpha \prec A_\alpha$ $(\operatorname{supp} \rho_\alpha \subset A_\alpha)$ *for all* α

(ii) *The family of supports* $\{\operatorname{supp} \rho_\alpha\}$ *is locally finite.*

(iii) $\sum_\alpha \rho_\alpha(x) \equiv 1$ *on* M

Theorem 1.3.13. *Suppose* M *is a smooth manifold with or without boundary, and* $\{U_\alpha\}_{\alpha \in A}$ *is any indexed open cover of* M. *Then there exists a smooth partition of unity subordinate to* $\{U_\alpha\}_{\alpha \in A}$.

Proof. *We can find nonnegative functions* $\varphi_j \in C^\infty(M)$ *so that* $\overline{V}_j \prec \varphi_j \prec W_j$ *on* $\overline{V}_j$ *and* $\operatorname{supp}(\varphi_j) \subset W_j$. *Since* $\mathcal{W}$ *is a locally finite cover,* $\varphi = \sum \varphi_j$ *is a well-defined smooth function on* M. *Since each* φ_j *is nonnegative, and* $\mathcal{V}$ *is a cover of* M, φ *is strictly positive on* M. *It follows that the functions* $\psi_j = \frac{\varphi_j}{\varphi}$ *are smooth and satisfy* $0 \leq \psi_j \leq 1$ *and* $\sum_j \psi_j = 1$.

Let

$$\rho_\alpha = \sum_{W_j \subset U_\alpha} \psi_j$$

Note that the right hand side is a finite sum near each point, so it does define a smooth function. Clearly the family $\{\rho_\alpha\}$ *is a partition of unity subordinate to* $\{U_\alpha\}$.

1.3.3 Application

Theorem 1.3.14 (Smooth Urysohn lemma)**.** *Let* M *be a smooth manifold,* $F \subset M$ *is a closed subset, and* $U \subset M$ *an open subset that contains* F. *Then there is a "bump" function* $\varphi \in C^\infty(M)$ *so that* $0 \leq \varphi \leq 1, \varphi \equiv 1$ *on* F *and* $\operatorname{supp}(\varphi) \subset U$.

Proof. *Let* $\{\rho_1, \rho_2\}$ *be a partition of unity subordinate to the open cover* $\{U, M \backslash F\}$. *Then* $\varphi = \rho_1$ *is what we need:* $\rho_1 = 1$ *on* F *since* $\rho_2 = 0$ *on* F.

Theorem 1.3.15 (Whitney Approximation Theorem)**.** *Let* M *be a smooth manifold,* $F \subset M$ *a closed subset and* k *be a positive integer. Then for any continuous function* $f : M \to \mathbb{R}^k$ *which is smooth on* F *and any positive continuous function* $\delta : M \to \mathbb{R}_{>0}$, *there exists* $f \in C^\infty(M)$ *so that*

$$f(p) = g(p), \quad \forall p \in F$$

and

$$|f(p) - g(p)| < \delta(p), \quad \forall p \in M$$

Proof. By definition, there exists an open set $U \supset A$ and a smooth function f_0 defined on U so that $f_0 = f$ on F. Let

$$U_0 = \{p \in U : |f_0(p) - f(p)| < \delta(p)\}$$

Then U_0 is open in M and $U_0 \supset F$.

Next we construct a open cover of $M \backslash F$. For any $q \in M \backslash F$, we let

$$U_q = \{p \in M \backslash A : |f(p) - f(q)| < \delta(p)\}$$

Then $\{U_q \mid q \in M \backslash F\}$ is an open covering of $M \backslash F$. Now let $\{\rho_0, \rho_q : q \in M\}$ be P.O.U. subordinate to the open cover $\{U_0, U_q : q \in M\}$ of M, and define a function on M via

$$g(p) = \rho_0(p) f_0(p) + \sum_{q \in M} \rho_q(p) f(q).$$

Since the summation is locally finite, g is smooth. Also by definition, $g = f_0 = f$ on F. Moreover, for any $q \in M$ one has

$$\begin{aligned}
|g(p) - f(p)| &= \left| \rho_0(p) f_0(p) + \sum_q \rho_q(p) f(q) - \rho_0(p) f(p) - \sum_q \rho_q(p) f(p) \right| \\
&\leq \rho_0(p) \, |f_0(p) - f(p)| + \sum_q \rho_q(p) |f(q) - f(p)| \\
&< \rho_0(p) \delta(p) + \sum_q \rho_q(p) \delta(p) \\
&= \delta(p)
\end{aligned}$$

Corollary 1.3.16 (Tietze)**.** *Let M be a smooth manifold and closed set $F \subset$. If f is smooth on F, then there exists $g \in C^\infty(M)$ that $f = g$ on F.*

Theorem 1.3.17 (Existence of Smooth Exhaustion Function)**.** *Every smooth manifold with or without boundary admits a smooth positive exhaustion function.*

Proof. Let M be a smooth manifold with or without boundary, let $\{V_j\}_{j=1}^\infty$ be any countable open cover of M by precompact open subsets, and let $\{\psi_j\}$ be a smooth partition of unity subordinate to this cover. Define $f \in C^\infty(M)$ by

$$f(p) = \sum_{j=1}^{\infty} j \psi_j(p)$$

Then f is smooth because only finitely many terms are nonzero in a neighborhood of any point, and positive because $f(p) \geq \sum_j \psi_j(p) = 1$.

To see that f is an exhaustion function, let $c \in \mathbb{R}$ be arbitrary, and choose a positive integer $N > c$. If $p \notin \bigcup_{j=1}^{N} \bar{V}_j$, then $\psi_j(p) = 0$ for $1 \leq j \leq N$, so

$$f(p) = \sum_{j=N+1}^{\infty} j\psi_j(p) \geq \sum_{j=N+1}^{\infty} N\psi_j(p) = N\sum_{j=1}^{\infty} \psi_j(p) = N > c.$$

Equivalently, if $f(p) \leq c$, then $p \in \bigcup_{j=1}^{N} \bar{V}_j$. Thus $f^{-1}((-\infty, c])$ is a closed subset of the compact set $\bigcup_{j=1}^{N} \bar{V}_j$ and is therefore compact. □

Theorem 1.3.18 (Level Sets of Smooth Functions)**.** *Let M be a smooth manifold. If K is any closed subset of M, there is a smooth nonnegative function $f : M \to \mathbb{R}$ such that $f^{-1}(0) = K$.*

Proof. We begin with the special case in which $M = \mathbb{R}^n$ and $K \subseteq \mathbb{R}^n$ is a closed subset. For each $x \in M \backslash K$, there is a positive number $r \leq 1$ such that $B_r(x) \subseteq M \backslash K$. By Proposition A.16, $M \backslash K$ is the union of countably many such balls $\{B_{r_i}(x_i)\}_{i=1}^{\infty}$.

Let $h : \mathbb{R}^n \to \mathbb{R}$ be a smooth bump function that is equal to 1 on $\bar{B}_{1/2}(0)$ and supported in $B_1(0)$. For each positive integer i, let $C_i \geq 1$ be a constant that bounds the absolute values of h and all of its partial derivatives up through order i. Define $f : \mathbb{R}^n \to \mathbb{R}$ by

$$f(x) = \sum_{i=1}^{\infty} \frac{(r_i)^i}{2^i C_i} h\left(\frac{x - x_i}{r_i}\right)$$

The terms of the series are bounded in absolute value by those of the convergent series $\sum_i 1/2^i$, so the entire series converges uniformly to a continuous function by the Weierstrass M-test. Because the i th term is positive exactly when $x \in B_{r_i}(x_i)$, it follows that f is zero in K and positive elsewhere.

It remains only to show that f is smooth. We have already shown that it is continuous, so suppose $k \geq 1$ and assume by induction that all partial derivatives of f of order less than k exist and are continuous. By the chain rule and induction, every k th partial derivative of the i th term in the series can be written in the form

$$\frac{(r_i)^{i-k}}{2^i C_i} D_k h\left(\frac{x - x_i}{r_i}\right),$$

where $D_k h$ is some k th partial derivative of h. By our choices of r_i and C_i, as soon as $i \geq k$, each of these terms is bounded in absolute value by $1/2^i$, so the differentiated series also converges uniformly to a continuous function. It then follows from Theorem C. 31 that the k th partial derivatives of f exist and are continuous. This completes the induction, and shows that f is smooth.

Now let M be an arbitrary smooth manifold, and $K \subseteq M$ be any closed subset. Let $\{B_\alpha\}$ be an open cover of M by smooth coordinate balls, and let $\{\psi_\alpha\}$ be a subordinate partition of unity. Since each B_α is diffeomorphic to $\mathbb{R}^n$, the preceding argument shows that for each α there is a smooth nonnegative function $f_\alpha : B_\alpha \to \mathbb{R}$ such that $f_\alpha^{-1}(0) = B_\alpha \cap K$. The function $f = \sum_\alpha \psi_\alpha f_\alpha$ does the trick. □

1.4 In paracompact Hausdorff space

Lemma 1.4.1 (Shrinking lemma)**.** *Let X be a paracompact Hausdorff space; let $\{U_\alpha\}_{\alpha\in J}$ be open cover of X. Then there exists a locally finite open cover $\{V_\alpha\}_{\alpha\in J}$ of X such that $\overline{V}_\alpha \subset U_\alpha$ for each α.*

Proof. Let $\mathcal{A}$ be the collection of all open sets A such that $\overline{A}$ is contained in some element of the collection $\{U_\alpha\}$. Regularity of X implies that $\mathcal{A}$ covers X. Since X is paracompact, we can find a locally finite collection $\mathcal{B} = \{B_\beta\}_{\beta\in K}$ of open sets covering X that refines $\mathcal{A}$

Let us index $\mathcal{B}$ bijectively with some index set K, then the general element of $\mathscr{B}$ can be denoted B_β, for $\beta \in K$, and $\{B_\beta\}_{\beta\in K}$ is a locally finite indexed famuly. Since $\mathscr{B}$ refines $\mathscr{A}$, we can define a function $f : K \to J$ by choosing, for each β in K, an element $f(\beta) \in J$ such that

$$\overline{B}_\beta \subset U_{f(\beta)}$$

Then for each $\alpha \in J$, we define V_α to be the union of many B_β

$$V_\alpha = \bigcup_{f(\beta)=\alpha} B_\beta$$

Because the collection $\mathcal{B}_\alpha$ is locally finite, $\overline{V}_\alpha = \bigcup \overline{B}_\beta$, so that $\overline{V}_\alpha \subset U_\alpha$.

Finally, we check local finiteness. Given $x \in X$, choose a neighborhood W of x such that W intersects B_β for only finitely many values of β, say $\beta = \beta_1, \ldots, \beta_K$. Then W can intersect V_α only if α is one of the indices $f(\beta_1), \ldots, f(\beta_K)$.

Theorem 1.4.2. *Let X be a paracompact Hausdorff space; let $\{U_\alpha\}_{\alpha\in J}$ be an indexed open covering of X. Then there exists a partition of unity on X subordinate to $\{U_\alpha\}$.*

Proof. We begin by applying the shrinking lemma twice, to find locally finite indexed famles of open sets $\{W_\alpha\}$ and $\{V_\alpha\}$ covering X, such that

$$W_\alpha \subset \overline{W}_\alpha \subset V_\alpha \subset \overline{V}_\alpha \subset U_\alpha$$

for each α Since X is normal, we may choose, for each α, a continuous function $\varphi_\alpha : X \to [0, 1]$ such that $\varphi_\alpha\left(\overline{W}_\alpha\right) = \{1\}$ and $\varphi_\alpha(X - V_\alpha) = \{0\}$. Since φ_α is nonzero only at points of V_α, we have

$$\operatorname{supp} \varphi_\alpha \subset \overline{V}_\alpha \subset U_\alpha$$

Furthermore, the indexed family $\{\overline{V}_\alpha\}$ is locally finite (since an open set intersects $\overline{V}_\alpha$ only if it intersects V_α); hence the indexed family $\operatorname{supp} \varphi_\alpha$ is also locally finite. Note that because $\{W_\alpha\}$ covers X, for any given x at least one of the functions ψ_α is positive at x.

We can now make sense of the formally infinite sum

$$\varphi(x) = \sum_\alpha \varphi_\alpha(x)$$

and define

$$\rho_\alpha(x) = \frac{\varphi_\alpha(x)}{\varphi(x)}$$

to obtain our desired partition of unity.

Chapter 2

Tangent Vectors

Contents

2.1 The Tangent Space at a Point

Definition 2.1.1. *We define a* ***germ*** *of a C^∞ function at p in M to be an equivalence class of C^∞ functions defined in a neighborhood of p in M, two such functions being equivalent if they agree on some, possibly smaller, neighborhood of p. The set of germs of C^∞ real-valued functions at p in M is denoted by $C_p^\infty(M)$. The addition and multiplication of functions make $C_p^\infty(M)$ into a ring; with scalar multiplication by real numbers, $C_p^\infty(M)$ becomes an algebra over $\mathbb{R}$.*

Definition 2.1.2. *We define a* ***derivation*** *at a point in a manifold M, or a point-derivation of $C_p^\infty(M)$, to be a linear map $D : C_p^\infty(M) \to \mathbb{R}$ such that*

$$D(fg) = (Df)g(p) + f(p)Dg$$

A ***tangent vector*** *at a point p in a manifold M is a derivation at p, the tangent vectors at p form a vector space $T_p(M)$, called the* ***tangent space*** *of M at p. We also write T_pM instead of $T_p(M)$.*

2.2 The Differential of a Smooth Map

Definition 2.2.1. *If M and N are smooth manifolds with or without boundary and $F : M \to N$ is a smooth map, we define a map*

$$dF_p : T_pM \to T_{F(p)}N$$

*called the **differential** of F at p, as follows. Given $v \in T_pM$, we let $dF_p(v)$ be the derivation at $F(p)$ that acts on $C^\infty_{F(p)}(N)$ by the rule*

$$\langle dF_p(v), f\rangle = \langle v, f \circ F\rangle \quad \text{for all } f \in C^\infty_{F(p)}(N)$$

Proposition 2.2.2. *Let M, N, and P be smooth manifolds with or without boundary, let $F : M \to N$ and $G : N \to P$ be smooth maps, and let $p \in M$.*

1. *$dF_p : T_pM \to T_{F(p)}N$ is linear.*
2. *The chain rule. $d(G \circ F)_p = dG_{F(p)} \circ dF_p : T_pM \to T_{G\circ F(p)}P$.*
3. *$d\left(\mathrm{Id}_M\right)_p = \mathrm{Id}_{T_pM} : T_pM \to T_pM$.*
4. *If F is a diffeomorphism, then $dF_p : T_pM \to T_{F(p)}N$ is an isomorphism of vector spaces, and $(dF_p)^{-1} = d\left(F^{-1}\right)_{F(p)}$.*

2.3 Computations in Coordinates

Proposition 2.3.1. *Let $(U, \phi) = (U, x^1, \ldots, x^n)$ be a chart about a point p in a manifold M. Then*

1.
$$d\phi\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) = \left.\frac{\partial}{\partial r^i}\right|_{\phi(p)}$$

2. *If $(U, \phi) = (U, x^1, \ldots, x^n)$ is a chart containing p, then the tangent space T_pM has basis*
$$\left.\frac{\partial}{\partial x^1}\right|_p, \ldots, \left.\frac{\partial}{\partial x^n}\right|_p$$

3. *(Transition matrix for coordinate vectors). Suppose $(U, x^1, \ldots, x^n)$ and $(V, y^1, \ldots, y^n)$ are two coordinate charts on a manifold M. Then*
$$\frac{\partial}{\partial x^j} = \sum_i \frac{\partial y^i}{\partial x^j}\frac{\partial}{\partial y^i}$$
on $U \cap V$.

2.3.1 The Differential in Coordinates

2.3.2 Change of Coordinates

2.4 The Tangent Bundle

Definition 2.4.1. *Given a smooth manifold M with or without boundary, we define the* ***tangent bundle*** *of M, denoted by TM, to be the disjoint union of the tangent spaces at all points of M :*

$$TM = \coprod_{p \in M} T_p M$$

We usually write an element of this disjoint union as an ordered pair (p, v) of v_p, with $p \in M$ and $v \in T_p M$. The tangent bundle comes equipped with a ***natural projection map*** *$\pi : TM \to M$, which sends each vector in $T_p M$ to the point p at which it is tangent: $\pi(p, v) = p$.*

Proposition 2.4.2. *For any smooth n-manifold M, the tangent bundle TM has a natural topology and smooth structure that make it into a $2n$-dimensional smooth manifold. With respect to this structure, the projection $\pi : TM \to M$, $(x^i, v^i) \mapsto (x^i)$ is smooth.*

Proof. We begin by defining the maps that will become our smooth charts. Given any smooth chart $(U, \varphi, x^1, \ldots, x^n)$ for M and define a injective map $\tilde{\varphi} : \pi^{-1}(U) \to \mathbb{R}^{2n}$ by

$$\tilde{\varphi}\left(v^i \frac{\partial}{\partial x^i}\bigg|_p\right) = \left(x^1(p), \ldots, x^n(p), v^1, \ldots, v^n\right)$$

Now suppose we are given two smooth charts (U, φ) and (V, ψ), and let $(\pi^{-1}(U), \tilde{\varphi})$, $(\pi^{-1}(V), \tilde{\psi})$ be the corresponding charts on TM. The sets

$$\begin{aligned} \tilde{\varphi}\left(\pi^{-1}(U) \cap \pi^{-1}(V)\right) &= \varphi(U \cap V) \times \mathbb{R}^n \quad \text{and} \\ \tilde{\psi}\left(\pi^{-1}(U) \cap \pi^{-1}(V)\right) &= \psi(U \cap V) \times \mathbb{R}^n \end{aligned}$$

are open in $\mathbb{R}^{2n}$, and the transition map $\tilde{\psi} \circ \widetilde{\varphi}^{-1} : \varphi(U \cap V) \times \mathbb{R}^n \to \psi(U \cap V) \times \mathbb{R}^n$ can be written explicitly as

$$\begin{aligned} &\tilde{\psi} \circ \tilde{\varphi}^{-1}\left(x^1, \ldots, x^n, v^1, \ldots, v^n\right) \\ &= \left(\tilde{x}^1(x), \ldots, \tilde{x}^n(x), \frac{\partial \tilde{x}^1}{\partial x^j}(x) v^j, \ldots, \frac{\partial \tilde{x}^n}{\partial x^j}(x) v^j\right). \end{aligned}$$

This is clearly smooth.

Choosing a countable cover $\{U_i\}$ of M by smooth coordinate domains, we obtain a countable cover of TM by coordinate domains $\{\pi^{-1}(U_i)\}$ satisfying conditions (i)-(iv) of the smooth manifold chart lemma (Lemma 1.35). To check the Hausdorff condition (v), just note that any two points in the same fiber of π lie in one chart, while if (p, v) and (q, w) lie in different fibers, there exist disjoint smooth coordinate domains U, V for M such that $p \in U$ and $q \in V$, and then $\pi^{-1}(U)$ and $\pi^{-1}(V)$ are disjoint coordinate neighborhoods containing (p, v) and (q, w), respectively.

To see that π is smooth, note that with respect to charts (U, φ) for M and $(\pi^{-1}(U), \tilde{\varphi})$ for TM, its coordinate representation is $\pi(x, v) = x$.

The coordinates (x^i, v^i) are called **natural coordinates** on TM. $\square$

2.5 Curves in a Manifold and Velocity Vectors

Definition 2.5.1. *If M is a manifold with or without boundary, we define a **curve** in M to be a continuous map $\gamma : J \to M$; where $J \subset \mathbb{R}$ is an interval.*

Definition 2.5.2. *Now let M be a smooth manifold with or without boundary. Our definition of tangent spaces leads to a natural interpretation of velocity vectors: given a smooth curve $\gamma : J \to M$ and $t_0 \in J$, we define the **velocity** of γ at t_0, denoted by $\gamma'(t_0)$, to be the vector*

$$\gamma'(t_0) = d\gamma\left(\left.\frac{d}{dt}\right|_{t_0}\right) \in T_{\gamma(t_0)}M$$

where $\left.d/dt\right|_{t_0}$ is the standard coordinate basis vector in $T_{t_0}\mathbb{R}$.

This tangent vector acts on functions by

$$\gamma'(t_0) f = d\gamma\left(\left.\frac{d}{dt}\right|_{t_0}\right) f = \left.\frac{d}{dt}\right|_{t_0} (f \circ \gamma) = (f \circ \gamma)'(t_0)$$

In other words, $\gamma'(t_0)$ is the derivation at $\gamma(t_0)$ obtained by taking the derivative of a function along γ. (If t_0 is an endpoint of J, this still holds, provided that we interpret the derivative with respect to t as a one-sided derivative, or equivalently as the derivative of any smooth extension of $f \circ \gamma$ to an open subset of $\mathbb{R}$.)

Now let (U, φ, x^i) be a smooth chart. If $\gamma(t_0) \in U$, we can write the coordinate representation of γ as $\gamma(t) = (\gamma^1(t), \ldots, \gamma^n(t))$ for t sufficiently close to t_0, and then the coordinate formula for the differential yields

$$\gamma'(t_0) = \frac{d\gamma^i}{dt}(t_0) \left.\frac{\partial}{\partial x^i}\right|_{\gamma(t_0)}$$

Proposition 2.5.3 (The Velocity of a Composite Curve)**.** *Let $F : M \to N$ be a smooth map, and let $\gamma : J \to M$ be a smooth curve. For any $t_0 \in J$, the velocity at $t = t_0$ of the composite curve $F \circ \gamma : J \to N$ is given by*

$$(F \circ \gamma)'(t_0) = dF(\gamma'(t_0))$$

Proof. Just go back to the definition of the velocity of a curve:

$$(F \circ \gamma)'(t_0) = d(F \circ \gamma)\left(\left.\frac{d}{dt}\right|_{t_0}\right) = dF \circ d\gamma\left(\left.\frac{d}{dt}\right|_{t_0}\right) = dF(\gamma'(t_0))$$

□

Corollary 2.5.4 (Computing the Differential Using a Velocity Vector)**.** *Suppose* $F : M \to N$ *is a smooth map,* $p \in M$*, and* $v \in T_pM$*. Then*

$$dF_p(v) = (F \circ \gamma)'(0)$$

for any smooth curve $\gamma : J \to M$ *such that* $0 \in J, \gamma(0) = p$*, and* $\gamma'(0) = v$*.*

Chapter 3

Submersions, Immersions, and Embeddings

3.1 Maps of Constant Rank

Definition 3.1.1. *Suppose M and N are smooth manifolds with or without boundary. Given a smooth map $F : M \to N$ and a point $p \in M$, we define the rank of **F** at **p** to be the rank of the linear map $dF_p : T_pM \to T_{F(p)}N$.*

*If the rank of dF_p is equal to this upper bound, we say that **F** has full rank at **p**, and if F has full rank everywhere, we say F has full rank.*

*A smooth map $F : M \to N$ is called a smooth **submersion** if its differential is surjective at each point (or equivalently, if* rank $F =$ dim N *). It is called a smooth **inmersion** if its differential is injective at each point (equivalently,* rank $F =$ dim M *).*

Chapter 4

Sard's Theorem

4.1 Sets of Measure Zero

Definition 4.1.1. *If M is a smooth m-manifold with or without boundary, we say that a subset $A \subseteq M$* ***has measure zero in*** *M if for every smooth chart (U, φ) for M, the subset $\varphi(A \cap U) \subseteq \mathbb{R}^n$ has n-dimensional measure zero.*

Lemma 4.1.2. *Let M be a smooth n-manifold with or without boundary and $A \subseteq M$. Suppose that for some collection $\{(U_\alpha, \varphi_\alpha)\}$ of smooth charts whose domains cover $A, \varphi_\alpha(A \cap U_\alpha)$ has measure zero in $\mathbb{R}^n$ for each α. Then A has measure zero in M.*

Proof: Let (V, ψ) be an arbitrary smooth chart. We need to show that $\psi(A \cap V)$ has measure zero. Some countable collection of the U_α 's covers $A \cap V$. For each such U_α, we have

$$\psi(A \cap V \cap U_\alpha) = \left(\psi \circ \varphi_\alpha^{-1}\right) \circ \varphi_\alpha(A \cap V \cap U_\alpha)$$

Now, $\varphi_\alpha(A \cap V \cap U_\alpha)$ is a subset of $\varphi_\alpha(A \cap U_\alpha)$, which has measure zero in $\mathbb{R}^n$ by hypothesis. By Proposition 6.5 applied to $\psi \circ \varphi_\alpha^{-1}$, therefore, $\psi(A \cap V \cap U_\alpha)$ has measure zero. Since $\psi(A \cap V)$ is the union of countably many such sets, it too has measure zero.

Theorem 4.1.3. *Suppose M and N are differential m-manifolds with or without boundary, $F : M \to N$ is a C^1 map, and $A \subseteq M$ is a subset of measure zero. Then $F(A)$ has measure zero in N.*

Corollary 4.1.4. *Suppose M and N are differential manifolds with or without boundary,* $\dim M \leq \dim N$*, and $F \in C^1(M, N)$. If $A \subset M$ is a subset of measure zero, then $F(A)$ has measure zero in N.*

4.2 Sard's Theorem

Definition 4.2.1. *If $f : M \to N$ is a smooth map,*

(1) A point $p \in M$ is said to be a ***regular point of*** *f if $df_p : T_pM \to T_{f(p)}N$ is surjective* $(\mathrm{rank}_p f = \dim N)$*. A point $c \in N$ is said to be a* ***regular value*** *of f if every point of the level set $f^{-1}(c)$ is a regular point.*

(2) A point $q \in M$ is said to be a ***critical point of*** *f if $df_p : T_pM \to T_{f(p)}N$ is not surjective* $(\mathrm{rank}_p f < \dim N)$. *A point $d \in N$ is said to be a* ***critical value*** *of f if there exists a critical point in level set $f^{-1}(d)$.*

Theorem 4.2.2 (Sard's Theorem)**.** *Suppose M and N are smooth manifolds with or without boundary and $f : M \to N$ is a smooth map. Then the set of critical values of f has measure zero in N.*

Corollary 4.2.3. *Suppose M and N are smooth manifolds with or without boundary, and $F : M \to N$ is a smooth map. If* $\dim M < \dim N$*, then $F(M)$ has measure zero in N.*

Corollary 4.2.4. *Suppose M is a smooth manifold with or without boundary, and $S \subseteq M$ is an immersed submanifold with or without boundary. If* $\dim S < \dim M$*, then S has measure zero in M.*

4.3 The Whitney Embedding Theorem

Theorem 4.3.1 (Whitney Embedding Theorem)**.** *Every smooth n-manifold with or without boundary admits a proper smooth embedding into $\mathbb{R}^{2n+1}$.*

Corollary 4.3.2. *Every smooth n-dimensional manifold with or without boundary is diffeomorphic to a properly embedded submanifold (with or without boundary) of $\mathbb{R}^{2n+1}$.*

Corollary 4.3.3. *Suppose M is a compact smooth n-manifold with or without boundary. If $N \geq 2n+1$, then every smooth map from M to $\mathbb{R}^N$ can be uniformly approximated by embeddings.*

Theorem 4.3.4 (Whitney Immersion Theorem)**.** *Every smooth n-manifold with or without boundary admits a smooth immersion into $\mathbb{R}^{2n}$.*

Theorem 4.3.5 (Strong Whitney Immersion Theorem)**.** *If $n > 1$, every smooth n-manifold admits a smooth immersion into $\mathbb{R}^{2n-1}$.*

Chapter 5

123

5.1

Definition 5.1.1. *Let X, B, and F be Hausdorff spaces and $p : X \to B$ a map. Then p is called a* ***bundle projection with fiber*** *F, if each point of B has a neighborhood U such that there is a homeomorphism*

$$\phi : U \times F \to p^{-1}(U), \quad \textit{that } p(\phi\langle b, y\rangle) = b$$

for all $b \in U$ and $y \in F$. Such a map ϕ is called a ***trivialization of the bundle over*** *U.*

Definition 5.1.2. *Let K be a topological group acting effectively on the Hausdorff space F as a group of homeomorphisms. Let X and B be Hausdorff spaces. By a fiber bundle over the base space B with total space X, fiber F, and structure group K, we mean a bundle projection $p : X \to B$ together with a collection $\mathcal{A}$ of trivializations $\phi : U \times F \to p^{-1}(U)$, of p over U, called charts over U, such that:*

(i) each point of B has a neighborhood over which there is a chart in Φ;

(ii) if $\phi : U \times F \to p^{-1}(U)$ is in $\mathcal{A}$ and $V \subset U$ then the restriction of ϕ to $V \times F$ is in $\mathcal{A}$

(iii) if $\phi, \psi \in \mathcal{A}$ are charts over U then there is a map $\theta : U \to K$ such that $\psi\langle u, y\rangle = \phi\langle u, \theta(u)(y)\rangle$; and

(iv) the set $\mathcal{A}$ is maximal among collections satisfying (a), (b), and (c).

The bundle is called smooth if all these spaces are manifolds and all maps involved are smooth.

Definition 5.1.3. *A* ***vector bundle*** *is a fiber bundle in which the fiber is a euclidean space and the structure group is the general linear group of this euclidean space or some subgroup of that group.*

A vector bundle is usually denoted by a Greek letter such as ξ and its total space by $E(\xi)$ and base space by $B(\xi)$. Its fiber projection is denoted by π_ξ or just by π. The following definition, given only for vector bundles, has a fairly obvious generalization to general fiber bundles, but we need it only for vector bundles.

Definition 5.1.4. *If ξ and η are vector bundles then a **bundle map** $\xi \to \eta$ is a map $g : E(\xi) \to E(\eta)$ carrying each fiber of ξ onto some fiber of η isomorphically. A bundle map g is a **bundle isomorphism** or a bundle equivalence if it is a homeomorphism. (In particular, the fibers have the same dimension and there is an induced map $B(\xi) \to B(\eta)$.)*

5.2

Definition 5.2.1. *Let M, X, Y be smooth manifolds and let $f : X \to M$ and $g : Y \to M$ be smooth maps with g an embedding (Y is a submanifold of M). Then f is said to be transverse :o g (denoted by $f \pitchfork g$) if, whenever $f(x) = g(y)$, the images of the differentials $f_* : T_x(X) \to T_{f(x)}(M)$ and $g_* : T_y(Y) \to T_{g(y)}(M) = T_{f(x)}(M)$ span $T_{f(x)}(M)$.*

Definition 5.2.2. *Let $f : M \to N$ be a smooth map, and $X \subset N$ be a smooth submanifold. We say f **intersect** X **transversally**, and denote by $f \pitchfork g$, if*

$$\operatorname{Im}(df_p) + T_{f(p)}X = T_{f(p)}N, \quad \forall p \in f^{-1}(X)$$

Definition 5.2.3. *We say two smooth submanifolds X_1 and X_2 in M **intersect transversally** if for any $p \in X_1 \cap X_2$,*

$$T_pX_1 + T_pX_2 = T_pM$$

In this case we write $X_1 \pitchfork X_2$.