Representation of Groups

HHH

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Chapter I

Introduction

§1 Basic Definitions

Definition 1.1. *Suppose now G is a finite group and k is a field.*

1. *A k-**linear representation** of G is a group homomorphism*

$$\rho : G \to \mathrm{GL}(V)$$

   *where V is a k-vector space. The space V is called the **representation space** of ρ and $\dim_k V$ (if finite) is called the **degree** of the representation.*

2. *Let ρ and ρ' be two representations of the same group G in vector spaces V and V'. The **transforms** of V and V' is a k-linear map from V to V' such that*

$$\tau \circ \rho(s) = \rho'(s) \circ \tau \quad \text{for all } s \in G.$$

   *These representations are said to be **similar** (or isomorphic) if τ is an isomorphism.*

Remark. *A k-linear representation $\rho : G \to \mathrm{GL}(V)$ is equivalent to a left $k[G]$-module V. In this case, the transforms between two representations are exactly the $k[G]$-module homomorphisms.*

Definition 1.2. *Let $\rho : G \to \mathrm{GL}(V)$ be a k-linear representation of a finite group G in the vector space V. A subspace W of V is called a **subrepresentation** if it is a $k[G]$-submodule of V.*

*A representation is called **irreducible** if it has no proper non-zero subrepresentation.*

Definition 1.3. *Let $\rho_i : G \to \mathrm{GL}(V_i)$ be k-linear representation of a finite group G in the vector space V_i or* $\operatorname{char} k = 0$.

1. *The **sum representation** $\rho_1 \oplus \rho_2$ is defined on the vector space $V_1 \oplus V_2$ by*

$$(\rho_1 \oplus \rho_2)(s)(v_1, v_2) = (\rho_1(s)v_1, \rho_2(s)v_2)$$

   for all $s \in G, v_1 \in V_1, v_2 \in V_2$. (the direct sum of $k[G]$-modules $V_1 \oplus V_2$).

2. *The **tensor product representation** $\rho_1 \otimes \rho_2$ is defined on the vector space $V_1 \otimes V_2$ by*

$$\left(\rho_1 \otimes \rho_2\right)(s)\left(v_1 \otimes v_2\right)=\left(\rho_1(s) v_1\right) \otimes\left(\rho_2(s) v_2\right)$$

for all $s \in G, v_1 \in V_1, v_2 \in V_2$.

§2 Maschke's Theorem

In this section, we always assume that k be a field whose characteristic does not divide the order of the finite group G

Lemma 2.1. *Let V_i be $k[G]$-module for $i=1,2$ and $f: V_1 \rightarrow V_2$ be a k-linear map, then*

$$F(x):=\frac{1}{|G|} \sum_{g \in G} g \cdot f\left(g^{-1} x\right)$$

is a $k[G]$-module homomorphism from V_1 to V_2.

Remark. *Thus every k-linear map $h: V_1 \rightarrow V_2$ can be "averaged" to a transform.*

$$h^0=\frac{1}{|G|} \sum_{g \in G}\left(\rho_g^2\right)^{-1} h \rho_g^1 .$$

Theorem 2.2 (Maschke)**.** *Then $k[G]$ is semisimple.*

Proof. Let V be a $k[G]$-module, W be a submodule of V. And let $W^{\prime}$ be a k-complement of W in V, and let p be the corresponding projection of V onto V. Define a map $P: V \rightarrow W$ by

$$P(v):=\frac{1}{|G|} \sum_{g \in G} g \cdot p\left(g^{-1} v\right)$$

where $1/|G|$ is the inverse of the order of G in the field k (or say $\mathbb{F}_p$). Then P is a $k[G]$-projection from V to W (i.e. P is a $k[G]$-module homomorphism with $P^2=P$ and $W=\operatorname{Im}(P)$), so $\ker(P)$ is a $k[G]$-complement of W in V. □

Corollary 2.3. *Every representation of G over k is a direct sum of irreducible representations and has finite number of irreducible components up to isomorphism.*

Remark. *Noted that every irreducible representation is finite degree. (Since it is a simple module over the finite dimensional algebra $k[G]$)*

§2.1

Corollary 2.4. *For a finite dimension $k[G]$, we have the following decomposition of the group algebra $k[G]$:*

$$k[G] \cong \bigoplus_{i=1}^r M_{n_i}(k)$$

and each $M_{n_i}(k)$ has only one simple module $V_i = k^{n_i}$ up to isomorphism.

Corollary 2.5. *Consider the regular representation of G over k. Then each irreducible representation V_i appears in it with multiplicity equal to its degree $n_i = \dim V_i$.*

$$ {}_{k[G]}k[G] \cong \bigoplus_{i=1}^{h} V_i^{\oplus n_i}. $$

§3

Proposition 3.1. *Let $z = \sum a_g g \in k[G]$ with coefficients a_g in k. Then $z \in Z\left(k[G]\right)$ if and only if $a_{hgh^{-1}} = a_g$ for all $g, t \in G$.*

Corollary 3.2. *For each conjugacy class C of G, The elements*

$$ z_C := \sum_{g \in C} g $$

where C runs through the conjugacy classes of G, form a k-basis of $Z\left(k[G]\right)$. Thus $\dim_k Z\left(k[G]\right)$ *is equal to the number of conjugacy classes of G.*

Chapter II

Character Theory

§1 Characters of Representations

Definition 1.1. *Let V be a $k[G]$-module of finite dimension.*

1. *The **character** χ of the representation is defined by*

$$\chi(s) := \operatorname{Tr}(\rho_s).$$

2. *If W be a submodule of V, the **subcharacter** χ_W of the is defined by*

$$\chi_W(s) := \operatorname{Tr}(\rho_s|_W).$$

3. *A character χ is called **irreducible** if it is the character of an irreducible representation.*

Proposition 1.2. *If χ is the character of a representation ρ of degree n, we have:*

1. $\chi(1) = n$,
2. $\chi(s^{-1}) = \overline{\chi(s)}$ *for* $s \in G$,
3. $\chi(tst^{-1}) = \chi(s)$ *for* $s, t \in G$. $\chi(ab) = \chi(ba)$

Proposition 1.3. *Let $\rho^1 : G \to \mathrm{GL}(V_1)$ and $\rho^2 : G \to \mathrm{GL}(V_2)$ be two linear representations of G. Then:*

1. $\chi_{\rho^1 \oplus \rho^2} = \chi_{\rho^1} + \chi_{\rho^2}$.
2. $\chi_{\rho^1 \otimes \rho^2} = \chi_{\rho^1} \cdot \chi_{\rho^2}$.

§2 Schur's lemma

We first give the general version of Schur's lemma in module theory.

Lemma 2.1. *Let A be a ring and M_1, M_2 be simple left A-modules. Then*

$$\operatorname{Hom}_A(M_1, M_2) = \begin{cases} 0 & , M_1 \ncong M_2 \\ \textit{division ring} & , M_1 \cong M_2 \end{cases}$$

Lemma 2.2. *Let k be an algebraically closed field and A be k-algebra. If V_i are simple left A-module of finite dimension and V_1 is A-isomorphic to V_2 Then*

$$\dim_k \operatorname{Hom}_A(V_1, V_2) = 1$$

Indeed, $\operatorname{Hom}_A(V_1, V_2) = k \cdot \phi$ *where ϕ is the A-isomorphism of V_1 and V_2.*

Remark. *Especially, for a simple left A-module V of finite dimension, we have*

$$\operatorname{End}_A(V) = k \cdot \mathrm{id}$$

Proposition 2.3 (Schur's lemma)**.** *Let k be a algebraically closed field and G be a finite group. Let $\rho^1 : G \to \mathrm{GL}(V_1)$ and $\rho^2 : G \to \mathrm{GL}(V_2)$ be two irreducible representations of G, and let $f : V_1 \to V_2$ be a transform. Then:*

1. *If ρ^1 and ρ^2 are not isomorphic, we have $f = 0$.*
2. *If $\rho^1 \cong \rho^2$ ($\rho^1 = \rho^2$ and $V_1 = V_2$) then f is a homothety (i.e., a scalar multiple of the identity).*

Corollary 2.4. *Let h be a linear mapping of V_1 into V_2, and put:*

$$h^0 = \frac{1}{|G|} \sum_{g \in G} \left(\rho_g^2\right)^{-1} h \rho_g^1.$$

Then:

1. *If ρ^1 and ρ^2 are not isomorphic, we have $h^0 = 0$.*
2. *If $V_1 = V_2$ and $\rho^1 = \rho^2, h^0$ is a homothety of ratio $(1/n)\operatorname{Tr}(h)$, with $n = \dim(V_1)$.*

Corollary 2.5.

§3 Main

In this section, we will derive the character theory of finite degree representations over a algebraically closed field.

Definition 3.1. *Let G be a finite group and φ, ϕ be complex valued functions on G. The **inner product** of φ and ϕ is defined by*

$$(\varphi, \phi) := \frac{1}{|G|} \sum_{g \in G} \varphi(g) \overline{\phi(g)}.$$

Definition 3.2. *For any $z \in Z(k[G])$ acting on an irreducible representation V, by Schur's lemma, we know that z acts as a homothety on V. Thus one can define the k-algebra homomorphism*

$$\omega_V : Z(k[G]) \to k$$

called ***central character*** *of irreducible representation V over k*

Proposition 3.3. *Let V be an irreducible representation of G over k and C be a conjugacy class of G. And let $z_C := \sum_{g \in C} g \in Z(k[G])$*

$$\omega_V(z_C) = \frac{\#C}{\dim V} \chi_V(c)$$