

Abstract Algebra

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Chapter I

Topological Space

§1 Topological Space

§1.1 Basic Definition

Definition 1.1. A *topology* on a set X is a collection \mathcal{T} of subsets of X having the following properties:

- (i) \emptyset and X are in \mathcal{T} .
- (ii) The union of the elements of any subcollection of \mathcal{T} is in \mathcal{T} .
- (iii) The intersection of the elements of any finite subcollection of \mathcal{T} is in \mathcal{T} .

A set X for which a topology \mathcal{T} has been specified is called a **topological space**.

Proposition 1.2. Let X be a topological space. Then the following conditions hold:

- (1) \emptyset and X are closed.
- (2) Arbitrary intersections of closed sets are closed.
- (3) Finite unions of closed sets are closed.

Definition 1.3. Suppose that \mathcal{T} and \mathcal{T}' are two topologies on a given set X . If $\mathcal{T}' \supset \mathcal{T}$, we say that \mathcal{T}' is **finer** than \mathcal{T} ; if \mathcal{T}' properly contains \mathcal{T} , we say that \mathcal{T}' is **strictly finer** than \mathcal{T} .

We also say that \mathcal{T} is **coarser** than \mathcal{T}' , or **strictly coarser**, in these two respective situations.

We say \mathcal{T} is **comparable** with \mathcal{T}' if either $\mathcal{T}' \supset \mathcal{T}$ or $\mathcal{T} \supset \mathcal{T}'$.

§1.2 Limit Points, Closure and Interior

Definition 1.4. Given a subset A of a topological space (X, \mathcal{T}) , the **interior** of A is defined as the union of all open sets contained in A , denoted by $\text{Int } A$. And the **closure** of A is defined as the intersection of all closed sets containing A , denoted by \bar{A} .

Obviously $\text{Int } A$ is an open set and \bar{A} is a closed set; furthermore,

$$\text{Int } A \subset A \subset \bar{A}$$

Definition 1.5. If A is a subset of the topological space X and if x is a point of X , we say that x is a **limit point** of A if every neighborhood of x intersects A in some point other than x itself.

The sets A' consisting of all limit points of A is called **derived set** of A .

Theorem 1.6. Let A be a subset of the topological space X .

- (1) Then $x \in \bar{A}$ if and only if every open set U containing x intersects A .
- (2) x is a limit point of A if and only if it belongs to the closure of $A - \{x\}$.
- (3) $\bar{A} = A \cup A'$

Corollary 1.7. A subset of a topological space is closed if and only if it contains all its limit points.

§1.3 Basis For a Topological Space

Definition 1.8. If X is a set,

(1) A **basis** for a topology on X is a collection \mathcal{B} of subsets of X (called basis elements) such that

(i) For each $x \in X$, there is at least one basis element B containing x .

(ii) If x belongs to the intersection of two basis elements B_1 and B_2 , then there is a basis element B_3 containing x such that $B_3 \subset B_1 \cap B_2$.

(2) If \mathcal{B} satisfies these two conditions, then we define the topology \mathcal{T} **generated by basis \mathcal{B}** as follows: A subset U of X is said to be open in X if for each $x \in U$, there is a basis element $B \in \mathcal{B}$ such that $x \in B$ and $B \subset U$.

Proposition 1.9. Let X be a set; let \mathcal{B} be a basis for a topology \mathcal{T} on X . Then \mathcal{T} equals the collection of all unions of elements of \mathcal{B} .

Definition 1.10. A **subbasis** \mathcal{S} for a topology on X is a collection of subsets of X whose union equals X . The topology generated by the subbasis \mathcal{S} is defined to be the collection \mathcal{T} of all unions of finite intersections of elements of \mathcal{S} .

§1.4 Subspace Topology

Definition 1.11. Let X be a topological space with topology \mathcal{T} . If Y is a subset of X , the collection

$$\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}\}$$

is a topology on Y , called the **subspace topology**. With this topology, Y is called a subspace of X ; its open sets consist of all intersections of open sets of X with Y .

Theorem 1.12. Let X be a topological space and Y a subset of X

- (1) If \mathcal{B} is a basis for the topology of X then the collection

$$\mathcal{B}_Y = \{B \cap Y \mid B \in \mathcal{B}\}$$

is a basis for the subspace topology on Y .

(2) Then a set A is closed in Y if and only if it equals the intersection of a closed set of X with Y .

(3) If A is closed in Y and Y is closed in X , then A is closed in X .

Proof. (1)

(2) Assume that $A = C \cap Y$, where C is closed in X . Then $X - C$ is open in X , so that the complement of A in Y , $Y - A = (X - C) \cap Y$, is open in Y , then so A is closed in Y .

Conversely, assume that A is closed in Y . Then $Y - A$ is open in Y , so that it equals the intersection of an open set U of X with Y . The set $X - U$ is closed in X , and $A = Y \cap (X - U)$, so that A equals the intersection of a closed set of X with Y , as desired. \square

Chapter II

Countability

§1 The Countability Axioms

Definition 1.1. A space X is said to have a countable basis at x if there is a countable collection \mathcal{B} of neighborhoods of x such that each neighborhood of x contains at least one of the elements of \mathcal{B} .

A space that has a countable basis at each of its points is said to satisfy the first countability axiom, or to be **first-countable**.

Theorem 1.2. Let X be a topological space.

(1) Let A be a subset of X . If there is a sequence of points of A converging to x , then $x \in \bar{A}$; the converse holds if X is first-countable.

(2) Let $f : X \rightarrow Y$. If f is continuous, then for every convergent sequence $x_n \rightarrow x$ in X , the sequence $f(x_n)$ converges to $f(x)$. The converse holds if X is first countable.

Definition 1.3. If a space X has a countable basis for its topology, then X is said to satisfy the second countability axiom, or to be **second-countable**.

Definition 1.4. A subset A of a space X is said to be **dense** in X if $\bar{A} = X$. A space having a countable dense subset is often said to be **separable**.

Theorem 1.5 (Hereditarity and Countable Multiplicativity). A subspace of a first-countable space is first-countable, and a countable product of first-countable spaces is first-countable.

A subspace of a secondcountable space is second-countable, and a countable product of second-countable spaces is second-countable.

Definition 1.6. A space for which every open covering contains a countable subcovering is called a **Lindelöf space**.

Theorem 1.7. Suppose that X has a countable basis. Then:

- (1) X is a Lindelöf space.
- (2) X is separable.

§2 The Separation Axioms

Definition 2.1. A space X is said to be **Frechét** (or T_1) if for each pair of distinct point x and y , there exist neighborhood of x which not contains y .

A space X is said to be **Hausdorff** (or T_2) if for each pair consisting of distinct point x and y , there exist disjoint open sets containing x and y , respectively.

A space X is said to be **regular** (or T_3) if for each pair consisting of a point x and a closed set B disjoint from x , there exist disjoint open sets containing x and B , respectively.

The space X is said to be **normal** (or T_4) if for each pair A, B of disjoint closed sets of X , there exist disjoint open sets containing A and B , respectively.

Theorem 2.2. Let (X, τ) be topology space

- (1) X is T_1 if and only if single point set $\{x\}$ is closed.
- (2) X is T_2 if and only if $\Delta = \{(x, x) : x \in X\}$ is closed in $X \times X$
- (3) X is T_3 if and only if for every $x \in X$ and open set U containing x , there exists open V that $x \in V \subset \bar{V} \subset U$.
- (4) X is T_4 if and only if for every closed $A \subset X$ and open set U containing A , there exists open V that $A \subset V \subset \bar{V} \subset U$.

Proposition 2.3. Let X be a space. Then

- (1) $T_2 \Rightarrow T_1$
- (2) Compact + $T_2 \Rightarrow T_3$
- (3) Compact + $T_3 \Rightarrow T_4$; thus Compact + $T_2 \Rightarrow T_4$
- (4) Lindelof + $T_3 \Rightarrow T_4$; Thus $A_2 + T_3 \Rightarrow T_4$ and σ -compact + $T_3 \Rightarrow T_4$

§3 Urysohn Lemma

Theorem 3.1 (Urysohn lemma). Let X be a normal (T_4) space, let A and B be disjoint closed subsets of X . Let $[a, b]$ be a closed interval in the real line. Then there exists a continuous map

$$f : X \longrightarrow [a, b]$$

such that $f(A) = a$ and $f(B) = b$.

Proof. Put $r_1 = 0, r_2 = 1$, and let r_3, r_4, r_5, \dots be an enumeration of the rationals in $(0, 1)$. By Theorem 2.7, we can find open sets V_0 and then V_1 such that \bar{V}_0 is compact and

$$A \subset V_1 \subset \bar{V}_1 \subset V_0 \subset \bar{V}_0 \subset B^c$$

Suppose $n \geq 2$ and V_{r_1}, \dots, V_{r_n} have been chosen in such a manner that $r_i < r_j$ implies $\bar{V}_{r_j} \subset V_{r_i}$. Then one of the numbers r_1, \dots, r_n , say r_i , will be the largest one which is smaller than r_{n+1} , and

another, say r_j , will be the smallest one larger than r_{n+1} . Using Theorem 2.7 again, we can find $V_{r_{n+1}}$ so that

$$\overline{V}_{r_j} \subset V_{r_{n+1}} \subset \overline{V}_{r_{n+1}} \subset V_{r_j}$$

Continuing, we obtain a collection $\{V_r\}$ of open sets, one for every rational $r \in [0, 1]$, with the following properties: $A \subset V_1$, $\overline{V}_0 \subset B^c$, each \overline{V}_r , and

$$s > r \text{ implies } \overline{V}_s \subset V_r$$

Define

$$f_r(x) = r\chi_{V_r} \quad g_s(x) = s + (1 - s)\chi_{\overline{V}_s}$$

and

$$f = \sup_r f_r, \quad g = \inf_s g_s$$

then f is lower semicontinuous and that g is upper semicontinuous. It is clear that $0 \leq f \leq 1$, that $f(A) = 1$ and that $f(B) = 0$. The proof will be completed by showing that $f = g$.

The inequality $f_r(x) > g_s(x)$ is possible only if $r > s$, $x \in V_r$, and $x \notin \overline{V}_s$. But $r > s$ implies $V_r \subset V_s$. Hence $f_r \leq g$, for all r and s , so $f \leq g$. Suppose $f(x) < g(x)$ for some x . Then there are rationals r and s such that $f(x) < r < s < g(x)$. Since $f(x) < r$, we have $x \notin V_r$; since $g(x) > s$, we have $x \in \overline{V}_s$. By (3), this is a contradiction. Hence $f = g$. \square

Corollary 3.2. *Supposes that X is a T_4 space, then there exists $f \in C(X)$ that $f^{-1}(0) = A$ if and only if A is a closed G_δ 's.*

Proof. 1.

$$A = f^{-1}(0) = \bigcup_n f^{-1}\left(\left(-\frac{1}{n}, \frac{1}{n}\right)\right)$$

2. Assume that A is a closed G_δ 's, there exists open sets U_i that $A = \bigcap U_i$. By Urysohn lemma, there is $g_n : X \rightarrow [0, 1]$ that

$$A \subset g_n^{-1}(0), \quad U_n^c \subset g_n^{-1}(1)$$

Define

$$f = \sum_{n=1}^{\infty} \frac{g_n(x)}{2^n}$$

thus $f^{-1}(0) = A$

Theorem 3.3 (Strong Urysohn lemma). *Let X be a normal (T_4) space, let A and B be disjoint closed G_δ 's subsets of X . Let $[a, b]$ be a closed interval in the real line. Then there exists a continuous map $f : X \rightarrow [a, b]$ such that*

$$f^{-1}(a) = A, \quad f^{-1}(b) = B$$

§4 Tietze Extension Theorem

Theorem 4.1 (Tietze extension theorem). *Let X be a normal T_4 space; let A be a closed subspace of X . Then any continuous map $f : A \rightarrow \mathbb{R}$ may be extended to a continuous map $F : X \rightarrow \mathbb{R}$ that $F|_A = f$*

§5 Urysohn Metrization Theorem

Proposition 5.1. *The topology space (X, τ) is metrizable if and only if X can be topologically embedded into a metric space.*

Theorem 5.2. *If X is T_1, T_4 and A_2 (thus T_i, A_j), then X can be imbedding into $[0, 1]^\omega$ or \mathbb{R}^ω ; thus metrizable.*

Chapter III

Connectness

§1 Connected Space

Definition 1.1. Let X be a topological space. A **separation** of X is a pair U, V of disjoint nonempty open subsets of X whose union is X . The space X is said to be **connected** if there does not exist a separation of X .

Theorem 1.2. If Y is a subspace of X , a separation of Y is a pair of disjoint nonempty sets A and B whose union is Y , neither of which contains a limit point of the other. The space Y is connected if there exists no separation of Y .

Proof. Suppose first that A and B form a separation of Y . Then A is both open and closed in Y . The closure of A in Y is the set $\bar{A} \cap Y$ (where \bar{A} as usual denotes the closure of A in X). Since A is closed in Y , $A = \bar{A} \cap Y$; or to say the same thing, $\bar{A} \cap B = \emptyset$. Since \bar{A} is the union of A and its limit points, B contains no limit points of A . A similar argument shows that A contains no limit points of B .

Conversely, suppose that A and B are disjoint nonempty sets whose union is Y , neither of which contains a limit point of the other. Then $\bar{A} \cap B = \emptyset$ and $A \cap \bar{B} = \emptyset$; therefore, we conclude that $\bar{A} \cap Y = A$ and $\bar{B} \cap Y = B$. Thus both A and B are closed in Y , and since $A = Y - B$ and $B = Y - A$, they are open in Y as well. \square

Theorem 1.3. Let (X, τ) be a topology space.

(1) If the sets C and D form a separation of X , and if Y is a connected subspace of X , then Y lies entirely within either C or D .

(2) The union of a collection of connected subspaces of X that have a point in common is connected.

(3) Let A be a connected subspace of X . If $A \subset B \subset \bar{A}$, then B is also connected.

Proof. We only prove (3). Let A be connected and let $A \subset B \subset \bar{A}$. Suppose that $B = C \cup D$ is a separation of B , then A must lie entirely in C or in D ; suppose that $A \subset C$. Then $\bar{A} \subset \bar{C}$; since \bar{C} and D are disjoint, B cannot intersect D . This contradicts the fact that D is a nonempty subset of B . \square

Theorem 1.4. *A finite cartesian product of connected spaces is connected.*

Proof. We prove the theorem first for the product of two connected spaces X and Y . Choose a "base point" $a \times b$ in the product $X \times Y$. Note that the "horizontal slice" $X \times b$ is connected, and each "vertical slice" $x \times Y$ is connected. As a result, each "T-shaped" space

$$T_x = (X \times b) \cup (x \times Y)$$

is connected, being the union of two connected spaces that have the point $x \times b$ in common. Now the union

$$\bigcup_{x \in X} T_x = X \times Y$$

is connected because it is the union of a collection of connected spaces that have the point $a \times b$ in common.

The proof for any finite product of connected spaces follows by induction, using the fact that $X_1 \times \cdots \times X_n$ is homeomorphic with $(X_1 \times \cdots \times X_{n-1}) \times X_n$. \square

Theorem 1.5. *The image of a connected space under a continuous map is connected.*

§2

Lemma 2.1. *Given X , define an equivalence relation on X by setting $x \sim y$ if there is a connected subspace of X containing both x and y . The equivalence classes are called the **connected components** of X .*

Theorem 2.2. *The components of X are connected disjoint subspaces of X whose union is X , such that each nonempty connected subspace of X intersects only one of them.*

Definition 2.3. *We define another equivalence relation on the space X by defining $x \sim y$ if there is a path in X from x to y . The equivalence classes are called the **path components** of X .*

Theorem 2.4. *The path components of X are path-connected disjoint subspaces of X whose union is X , such that each nonempty path-connected subspace of X intersects only one of them.*

Definition 2.5. *A space X is said to be **locally connected at** x if for every neighborhood U of x , there is a connected neighborhood V of x contained in U . If X is locally connected at each of its points, it is said simply to be **locally connected**.*

Similarly, a space X is said to be **locally path connected at** x if for every neighborhood U of x , there is a path-connected neighborhood V of x contained in U . If X is locally path connected at each of its points, then it is said to be **locally path connected**.

Theorem 2.6. *A space X is locally connected if and only if for every open set U of X , each connected component of U is open in X .*

Proof. Suppose that X is locally connected; let U be an open set in X ; let C be a component of U . If x is a point of C , we can choose a connected neighborhood V of x such that $V \subset U$. Since V is connected, it must lie entirely in the component C of U . Therefore, C is open in X .

Conversely, suppose that components of open sets in X are open. Given a point x of X and a neighborhood U of x , let C be the component of U containing x . Now C is connected; since it is open in X by hypothesis, X is locally connected at x . \square

Theorem 2.7. *A space X is locally path connected if and only if for every open set U of X , each path connected component of U is open in X .*

Theorem 2.8. *Let (X, τ) be a topology space.*

- (1) If X is path connected, then X is connected.*
- (2) Each path component of X lies in a component of X*
- (3) If X is locally path connected, then the components and the path components of X are the same.*

Chapter IV

Compactness

§1 Basic Definition

Definition 1.1. A collection \mathcal{A} of subsets of a space X is said to cover X , or to be a covering of X , if the union of the elements of \mathcal{A} is equal to X . It is called an open covering of X if its elements are open subsets of X .

Definition 1.2. A space X is said to be **compact** if every open covering \mathcal{A} of X contains a finite subcollection that also covers X .

A space X is said to be **sequentially compact** if every sequence of X contains a convergent subsequence.

Theorem 1.3 (Criterion for compact sets). Let Y be a subspace of X . Then Y is compact if and only if every covering of Y by sets open in X contains a finite subcollection covering Y .

Proof. Suppose that Y is compact and $\mathcal{A} = \{A_\alpha\}_{\alpha \in J}$ is a covering of Y by sets open in X . Then the collection

$$\{A_\alpha \cap Y \mid \alpha \in J\}$$

is a covering of Y by sets open in Y ; hence a finite subcollection

$$\{A_{\alpha_1} \cap Y, \dots, A_{\alpha_n} \cap Y\}$$

covers Y . Then $\{A_{\alpha_1}, \dots, A_{\alpha_n}\}$ is a subcollection of \mathcal{A} that covers Y .

Conversely, suppose the given condition holds; we wish to prove Y compact. Let $\mathcal{A}' = \{A'_\alpha\}$ be a covering of Y by sets open in Y . For each α , choose a set A_α open in X such that

$$A'_\alpha = A_\alpha \cap Y.$$

The collection $\mathcal{A} = \{A_\alpha\}$ is a covering of Y by sets open in X . By hypothesis, some finite subcollection $\{A_{\alpha_1}, \dots, A_{\alpha_n}\}$ covers Y . Then $\{A'_{\alpha_1}, \dots, A'_{\alpha_n}\}$ is a subcollection of \mathcal{A}' that covers Y . □

Theorem 1.4. *Let X be a topology space.*

- (1) *If X is compact, then closed set of X is compact.*
- (2) *Every compact subspace of a Hausdorff space is closed.*
- (3) *If Y is a compact subspace of the T_2 space X and $x_0 \notin Y$, then there exist disjoint open sets U and V of X containing x_0 and Y , respectively.*
- (4) *The image of a compact space under a continuous map is compact.*
- (5) *The product of finitely many compact spaces is compact.*

Lemma 1.5 (The tube lemma). *Consider the product space $X \times Y$, where Y is compact. If N is an open set of $X \times Y$ containing the slice $x_0 \times Y$, then N contains some tube $W \times Y$, where W is a neighborhood of x_0 in X .*

Definition 1.6. *A collection \mathcal{C} of subsets of X is said to have the **finite intersection property** if for every finite subcollection*

$$\{C_1, \dots, C_n\}$$

of \mathcal{C} , the intersection $C_1 \cap \dots \cap C_n$ is nonempty.

Theorem 1.7. *Let X be a topological space. Then X is compact if and only if for every collection \mathcal{C} of closed sets in X having the finite intersection property, the intersection $\bigcap_{C \in \mathcal{C}} C$ is nonempty.*

§2 Tychonoff Theorem

Theorem 2.1 (Tychonoff theorem). *An arbitrary product of compact spaces is compact in the product topology*

Proof. Let

$$X = \prod_{\alpha \in J} X_\alpha$$

where each space X_α is compact. Let \mathcal{A} be a collection of subsets of X having the finite intersection property. We prove that the intersection is nonempty.

Compactness of X follows. Applying Lemma 37.1, choose a collection \mathcal{D} of subsets of X such that $\mathcal{D} \supset \mathcal{A}$ and \mathcal{D} is maximal with respect to the finite intersection property. It will suffice to show that the intersection $\bigcap_{D \in \mathcal{D}} \bar{D}$ is nonempty.

Given $\alpha \in J$, let $\pi_\alpha : X \rightarrow X_\alpha$ be the projection map, as usual. Consider the collection

$$\{\pi_\alpha(D) \mid D \in \mathcal{D}\}$$

§37 The Tychonoff Theorem 235 of subsets of X_α . This collection has the finite intersection property because \mathcal{D} does. By compactness of X_α , we can for each α choose a point x_α of X_α such that

$$x_\alpha \in \bigcap_{D \in \mathcal{D}} \overline{\pi_\alpha(D)}$$

Let \mathbf{x} be the point $(x_\alpha)_{\alpha \in J}$ of X . We shall show that $\mathbf{x} \in \bar{D}$ for every $D \in \mathcal{D}$; then our proof will be finished.

First we show that if $\pi_\beta^{-1}(U_\beta)$ is any subbasis element (for the product topology on X) containing \mathbf{x} , then $\pi_\beta^{-1}(U_\beta)$ intersects every element of \mathcal{D} . The set U_β is a neighborhood of x_β in X_β . Since $x_\beta \in \overline{\pi_\beta(D)}$ by definition, U_β intersects $\pi_\beta(D)$ in some point $\pi_\beta(\mathbf{y})$, where $\mathbf{y} \in D$. Then it follows that $\mathbf{y} \in \pi_\beta^{-1}(U_\beta) \cap D$.

It follows from (b) of Lemma 37.2 that every subbasis element containing \mathbf{x} belongs to \mathcal{D} . And then it follows from (a) of the same lemma that every basis element containing \mathbf{x} belongs to \mathcal{D} . Since \mathcal{D} has the finite intersection property, this means that every basis element containing \mathbf{x} intersects every element of \mathcal{D} ; hence $\mathbf{x} \in \bar{D}$ for every $D \in \mathcal{D}$ as desired. \square

§3 Compactness in Metric Space

Theorem 3.1. *If X is a metric space, then the following*

1. *X is compact*
2. *X is sequentially compact*
3. *X is complete and totally bounded.*
4. *Bolzano-Weierstrass property. Every infinite subset $A \subset X$ has a limit point at least.*

Chapter V

The Fundamental Group

§1 Homotopy of Paths

Definition 1.1. If f and f' are continuous maps of the space X into the space Y , we say that f is homotopic to f' if there is a continuous map $F : X \times I \rightarrow Y$ such that

$$F(x, 0) = f(x) \quad \text{and} \quad F(x, 1) = f'(x)$$

for each x . (Here $I = [0, 1]$.) The map F is called a **homotopy** between f and f' . If f is homotopic to f' , we write $f \simeq f'$. If $f \simeq f'$ and f' is a constant map, we say that f is **nulhomotopic**.

Two paths f and f' , mapping the interval $I = [0, 1]$ into X , are said to be **path homotopic** if they have the same initial point x_0 and the same final point x_1 , and if there is a continuous map $F : I \times I \rightarrow X$ such that

$$\begin{aligned} F(s, 0) &= f(s) & \text{and} & & F(s, 1) &= f'(s) \\ F(0, t) &= x_0 & \text{and} & & F(1, t) &= x_1 \end{aligned}$$

for each $s \in I$ and each $t \in I$. We call F a **path homotopy** between f and f' . If f is path homotopic to f' , we write $f \simeq_p f'$.

Lemma 1.2. The relations \simeq and \simeq_p are equivalence relations.

Definition 1.3. If f is a path in X from x_0 to x_1 , and if g is a path in X from x_1 to x_2 , we define the product $f * g$ of f and g to be the path h given by the equations

$$h(s) = \begin{cases} f(2s) & \text{for } s \in [0, \frac{1}{2}] \\ g(2s - 1) & \text{for } s \in [\frac{1}{2}, 1] \end{cases}$$

The function h is well-defined and continuous, by the pasting lemma; it is a path in X from x_0 to x_2 .

Theorem 1.4. The operation $*$ has the following properties:

(1) The product operation on paths induces a well-defined operation on path-homotopy classes, defined by the equation

$$[f] * [g] = [f * g]$$

(2) Associativity. If $[f] * ([g] * [h])$ is defined, so is $([f] * [g]) * [h]$, and they are equal.

(3) Right and left identities. Given $x \in X$, let e_x denote the constant path $e_x : I \rightarrow X$ carrying all of I to the point x . If f is a path in X from x_0 to x_1 , then

$$[f] * [e_{x_1}] = [f] \quad \text{and} \quad [e_{x_0}] * [f] = [f]$$

(4) Inverse. Given the path f in X from x_0 to x_1 , let \bar{f} be the path defined by $\bar{f}(s) = f(1 - s)$. It is called the reverse of f . Then

$$[f] * [\bar{f}] = [e_{x_0}] \quad \text{and} \quad [\bar{f}] * [f] = [e_{x_1}]$$

Definition 1.5. Let X be a space; let x_0 be a point of X . A path in X that begins and ends at x_0 is called a **loop based at x_0** . The set of path homotopy classes of loops based at x_0 , with the operation $*$, is called the **fundamental group of X relative to the base point x_0** . It is denoted by $\pi_1(X, x_0)$.

Theorem 1.6. Let α be a path in X from x_0 to x_1 . Then α induces a group isomorphism by

$$\hat{\alpha} : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1)$$

by the equation

$$\hat{\alpha}([f]) = [\bar{\alpha}] * [f] * [\alpha] = [\bar{\alpha}]^{-1} * [f] * [\alpha]$$

Proposition 1.7. Let α be a path in X from x_0 to x_1 ; let β be a path in X from x_1 to x_2 . Show that if $v = \alpha * \beta$, then $\hat{v} = \hat{\beta} \circ \hat{\alpha}$.

Corollary 1.8. If X is path connected and x_0 and x_1 are two points of X , then $\pi_1(X, x_0)$ is isomorphic to $\pi_1(X, x_1)$.

Definition 1.9. A space X is said to be **simply connected** if it is a path-connected space and if $\pi_1(X, x_0)$ is the trivial (one-element) group for some $x_0 \in X$, and hence for every $x_0 \in X$. We often express the fact that $\pi_1(X, x_0)$ is the trivial group by writing $\pi_1(X, x_0) = 0$.

Definition 1.10. Let $h : (X, x_0) \rightarrow (Y, y_0)$ be a continuous map. Define

$$h_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$$

by the equation

$$h_*([f]) = [h \circ f]$$

The map h_* is called the **homomorphism induced by h** , relative to the base point x_0 .

Theorem 1.11. *If $h : (X, x_0) \rightarrow (Y, y_0)$ and $k : (Y, y_0) \rightarrow (Z, z_0)$ are continuous, then $(k \circ h)_* = k_* \circ h_*$. If $i : (X, x_0) \rightarrow (X, x_0)$ is the identity map, then i_* is the identity homomorphism.*

Corollary 1.12. *If $h : (X, x_0) \rightarrow (Y, y_0)$ is a homeomorphism of X with Y , then h_* is an isomorphism of $\pi_1(X, x_0)$ with $\pi_1(Y, y_0)$.*

§2 Covering Space

Definition 2.1. *Let $p : E \rightarrow B$ be a continuous surjective map. The open set U of B is said to be **evenly covered by p** if the inverse image $p^{-1}(U)$ can be written as the union of disjoint open sets V_α in E such that for each α , the restriction of p to V_α is a homeomorphism of V_α onto U . The collection $\{V_\alpha\}$ will be called a **partition of $p^{-1}(U)$ into slices***

*If every point b of B has a neighborhood U that is evenly covered by p , then p is called a **covering map**, and E is said to be a **covering space** of B*

Theorem 2.2 (Hereditarity). *Let $p : E \rightarrow B$ be a covering map. If B_0 is a subspace of B , and if $E_0 = p^{-1}(B_0)$, then the map $p_0 : E_0 \rightarrow B_0$ obtained by restricting p is a covering map.*

Proof. Given $b_0 \in B_0$, let U be an open set in B containing b_0 that is evenly covered by p ; let $\{V_\alpha\}$ be a partition of $p^{-1}(U)$ into slices. Then $U \cap B_0$ is a neighborhood of b_0 in B_0 , and the sets $V_\alpha \cap E_0$ are disjoint open sets in E_0 whose union is $p^{-1}(U \cap B_0)$, and each is mapped homeomorphically onto $U \cap B_0$ by p . \square

Theorem 2.3 (Finite Multiplicativity). *If $p : E \rightarrow B$ and $p' : E' \rightarrow B'$ are covering maps, then*

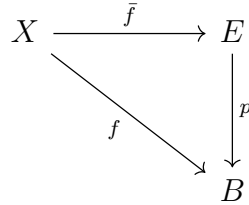
$$p \times p' : E \times E' \rightarrow B \times B'$$

is a covering map.

Proof. Given $b \in B$ and $b' \in B'$, let U and U' be neighborhoods of b and b' , respectively, that are evenly covered by p and p' , respectively. Let $\{V_\alpha\}$ and $\{V'_\beta\}$ be partitions of $p^{-1}(U)$ and $(p')^{-1}(U')$, respectively, into slices. Then the inverse image under $p \times p'$ of the open set $U \times U'$ is the union of all the sets $V_\alpha \times V'_\beta$. These are disjoint open sets of $E \times E'$, and each is mapped homeomorphically onto $U \times U'$ by $p \times p'$. \square

§3 Lifting Lemma

Definition 3.1. Let $p : E \rightarrow B$ be a map. If f is a continuous mapping of some space X into B , a **lifting** of f is a map $\bar{f} : X \rightarrow E$ such that $p \circ \bar{f} = f$.



Theorem 3.2 (Path-lifting lemma). Let $p : (E, e_0) \rightarrow (B, b_0)$ be a covering map. Any path $f : [0, 1] \rightarrow B$ beginning at b_0 has a unique lifting to a path \bar{f} in E beginning at e_0 .

Theorem 3.3. Let $p : (E, e_0) \rightarrow (B, b_0)$ be a covering map. Let the map $F : I \times I \rightarrow B$ be continuous, with $F(0, 0) = b_0$. There is a unique lifting of F to a continuous map

$$\tilde{F} : I \times I \rightarrow E$$

such that $\tilde{F}(0, 0) = e_0$. If F is a path homotopy, then \tilde{F} is a path homotopy.

Definition 3.4. Let $p : (E, e_0) \rightarrow (B, b_0)$ be a covering map. Given an element $[f]$ of $\pi_1(B, b_0)$, let \tilde{f} be the lifting of f to a path in E that begins at e_0 . Let

$$\phi([f]) = \tilde{f}(1)$$

Then ϕ is a well-defined set map

$$\phi : \pi_1(B, b_0) \rightarrow p^{-1}(b_0)$$

We call ϕ the **lifting correspondence derived from the covering map** p .

§4

Definition 4.1. If $A \subset X$, a **retraction** of X onto A is a continuous map $r : X \rightarrow A$ such that $r|_A$ is the identity map of A . If such a map r exists, we say that A is a **retract** of X .

Proposition 4.2. If A is a retract of X , then the homomorphism of fundamental groups induced by inclusion $j : A \rightarrow X$ is injective.

Proof: If $r : X \rightarrow A$ is a retraction, then the composite map $r \circ j$ equals the identity map of A . It follows that $r_* \circ j_*$ is the identity map of $\pi_1(A, a)$, so that j_* must be injective.

Theorem 4.3 (No-retraction theorem). There is no retraction of B^2 onto S^1 .

Proof: If S^1 were a retract of B^2 , then the homomorphism induced by inclusion $j : S^1 \rightarrow B^2$ would be injective. But the fundamental group of S^1 is nontrivial and the fundamental group of B^2 is trivial.

Lemma 4.4. Let $h : S^1 \rightarrow X$ be a continuous map. Then the following conditions are equivalent:

- (1) h is nullhomotopic.
- (2) h extends to a continuous map $k : B^2 \rightarrow X$.
- (3) h_* is the trivial homomorphism of fundamental groups.

Proof: (1) \Rightarrow (2). Let $H : S^1 \times I \rightarrow X$ be a homotopy between h and a constant map. Let $\pi : S^1 \times I \rightarrow B^2$ be the map

$$\pi(x, t) = (1 - t)x$$

Then π is continuous, closed and surjective, so it is a quotient map; it collapses $S^1 \times 1$ to the point 0 and is otherwise injective. Because H is constant on $S^1 \times 1$, it induces, via the quotient map π , a continuous map $k : B^2 \rightarrow X$ that is an extension of h .

(2) \Rightarrow (3). If $j : S^1 \rightarrow B^2$ is the inclusion map, then h equals the composite $k \circ j$. Hence $h_* = k_* \circ j_*$. But

$$j_* : \pi_1(S^1, b_0) \rightarrow \pi_1(B^2, b_0)$$

is trivial because the fundamental group of B^2 is trivial. Therefore h_* is trivial. (3) \Rightarrow (1). Let $p : \mathbb{R} \rightarrow S^1$ be the standard covering map, and let $p_0 : I \rightarrow S^1$ be its restriction to the unit interval. Then $[p_0]$ generates $\pi_1(S^1, b_0)$ because p_0 is a loop in S^1 whose lift to \mathbb{R} begins at 0 and ends at 1.

Let $x_0 = h(b_0)$. Because h_* is trivial, the loop $f = h \circ p_0$ represents the identity element of $\pi_1(X, x_0)$. Therefore, there is a path homotopy F in X between f and the constant path at x_0 . The map $p_0 \times id : I \times I \rightarrow S^1 \times I$ is a quotient map, being continuous, closed, and surjective; it maps $0 \times t$ and $1 \times t$ to $b_0 \times t$ for each t , but is otherwise injective. The path homotopy F maps $0 \times I$ and $1 \times I$ and $I \times 1$ to the point x_0 of X , so it induces a continuous map $H : S^1 \times I \rightarrow X$ that is a homotopy between h and a constant map. See Figure 55.2.