Complex Analysis

HHH

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Part I

Basic Theory

Chapter I

Preliminaries to Complex Analysis

§1 Power series

§1.1 Formal Power Series

Definition 1.1. *A **formal power series** of a neutral letter T over $\mathbb{C}$ is an expansion of the form*

$$f(T) = \sum_{n=0}^{\infty} a_n T^n$$

where $a_n \in \mathbb{C}$.

Definition 1.2. *Suppose a power series is of the form*

$$f = a_r T^r + a_{r+1} T^{r+1} + \cdots$$

*and $a_r \neq 0$. Thus r is the smallest integer n such that $a_n \neq 0$. Then we call r the **order** of f, and write $r = \operatorname{ord} f$.*

Proposition 1.3. *Suppose that f and $g \in \mathbb{C}[[T]]$. Then $\operatorname{ord} fg = \operatorname{ord} f + \operatorname{ord} g$.*

Corollary 1.4. *A formal power series $f \in \mathbb{C}[[T]]$ has an inverse iff $\operatorname{ord} f = 0$.*

Theorem 1.5. *Given a power series $\sum\limits_{n=0}^{\infty} a_n z^n$, there exists $0 \leq R \leq \infty$ such that:*

(i) If $|z| < R$ the series converges absolutely.

(ii) If $|z| > R$ the series diverges. Moreover, R is given by Hadamard's formula

$$1/R = \limsup |a_n|^{1/n}$$

The number R is called the radius of convergence of the power series, and the region $B(0.R)$ the disc of convergence.

Theorem 1.6. *The power series $\sum\limits_{n=0}^{\infty} a_n z^n$ defines a holomorphic function in its disc of convergence. The derivative of f is also a power series obtained by differentiating term by term the series for f,* *that is,*

$$\sum_{n=0}^{\infty} na_n z^{n-1}$$

Moreover, f has the same radius of convergence as f.

Corollary 1.7. *A power series f is infinitely complex differentiable in its disc of convergence, and the higher derivatives are also power series obtained by termwise differentiation.*

Definition 1.8. *A function f defined on an open set Ω is said to be analytic (or have a power series expansion) at a point $z_0 \in \Omega$ if there exists a power series $\sum a_n(z-z_0)^n$ centered at z_0, with positive radius of convergence δ, such that*

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n, \quad z \in B(z_0, \delta) \subset \Omega$$

If f has a power series expansion at every point in Ω, we say that f is analytic on Ω.

§2 Differential Functions

The letter Ω will from now on denote a plane open set.

Definition 2.1. *Suppose f is a complex function defined in Ω. If $z_0 \in \Omega$ and if*

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

*exists, we denote this limit by $f'(z_0)$ and call it the **derivative** of f at z_0. If $f'(z_0)$ exists for every $z_0 \in \Omega$, we say that f is **holomorphic** (or analytic) in Ω. The class of all holomorphic functions in Ω will be denoted by $H(\Omega)$.*

Theorem 2.2. *If f is representable by power series in Ω, then $f \in H(\Omega)$ and f' is also representable by power series in Ω. In fact, if*

$$f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n$$

for $z \in D(a; r)$, then for these z we also have

$$f'(z) = \sum_{n=1}^{\infty} nc_n(z-a)^{n-1}$$

Theorem 2.3. *Suppose (X, μ) is a complex (finite) measure space, φ is a complex measurable function on X, Ω is an open set in the plane which does not intersect $\varphi(X)$, and*

$$f(z) = \int_X \frac{\mathrm{d}\mu(\zeta)}{\varphi(\zeta) - z} \quad (z \in \Omega)$$

Then f is representable by power series in Ω.

Proof. Suppose $D(a;r) \subset \Omega$. Since

$$\left|\frac{z-a}{\varphi(\zeta)-a}\right| \le \frac{|z-a|}{r} < 1$$

for every $z \in D(a;r)$ and every $\zeta \in X$, the geometric series

$$\sum_{n=0}^{\infty} \frac{(z-a)^n}{(\varphi(\zeta)-a)^{n+1}} = \frac{1}{\varphi(\zeta)-z}$$

converges uniformly on X, for every fixed $z \in D(a;r)$. Hence the series (3) may be substituted into (1), and $f(z)$ may be computed by interchanging summation and integration. It follows that

$$f(z) = \sum_{0}^{\infty} c_n (z-a)^n \quad (z \in D(a;r))$$

where

$$c_n = \int_X \frac{d\mu(\zeta)}{(\varphi(\zeta)-a)^{n+1}} \quad (n = 0, 1, 2, \dots)$$

Note: The convergence of the series (4) in $D(a;r)$ is a consequence of the proof. We can also derive it from (5), since (5) shows that

$$|c_n| \le \frac{|\mu|(X)}{r^{n+1}} \quad (n = 0, 1, 2, \dots).$$

□

§3 Integration along curve

Definition 3.1. *Let*

§4 Integration along curve

Definition 4.1. *Suppose f is a function on the open set Ω. A* ***primitive*** *for f on Ω is a function F that is holomorphic on Ω and such that $F'(z) = f(z)$ for all $z \in \Omega$.*

Theorem 4.2. *If a continuous function f has a primitive F in Ω, and γ is a curve in Ω that begins at w_1 and ends at w_2, then*

$$\int_\gamma f(z)\,\mathrm{d}z = F(w_2) - F(w_1)$$

Proof: If γ is smooth, let $z(t) : [a,b] \longrightarrow C$ is a parametrization for γ, then $z(a) = w_1$ and

$z(b) = w_2$, and we have

$$\begin{aligned}\int_\gamma f(z)\,\mathrm{d}z &= \int_a^b f(z(t))z'(t)\,\mathrm{d}t \\ &= \int_a^b F'(z(t))z'(t)\,\mathrm{d}t \\ &= F(z(b)) - F(z(a)) \\ &= F(w_2) - F(w_1)\end{aligned}$$

Corollary 4.3. *If γ is a closed curve in an open set Ω, and f is continuous and has a primitive in Ω, then*

$$\int_\gamma f(z)\,\mathrm{d}z = 0$$

Corollary 4.4. *If f is holomorphic in a region and $f' = 0$, then f is constant.*

Proof: Since Ω is connected, for any $w \in \Omega$, there exists a curve γ which joins w_0 to w. Since f is clearly a primitive for f', we have

$$f(w) - f(w_0) = \int_\gamma f'(z)\,\mathrm{d}z = 0$$

we conclude that $f(w) = f(w_0)$ foe any $w \in \Omega$ as desired.

§5 Cauchy Theorems

Theorem 5.1 (Goursat's). *If Ω is an open set in $\mathbb{C}$, and $T \subset \Omega$ a triangle whose interior is also contained in Ω, then*

$$\int_T f(z)\,\mathrm{d}z = 0$$

whenever f is holomorphic in Ω

Theorem 5.2. *A holomorphic function in an open disc has a primitive.*

Proof: After a translation, we may assume without loss of generality that the disc, say D, is centered at the origin.

Define

$$F(z) = \int_{\gamma_z} f(\zeta)\,\mathrm{d}\zeta$$

We contend that F is holomorphic in D and $F' = f$. To prove this, fix $z \in D$ and let $h \in \mathbb{C}$ be so *small that $z+h$ also belongs to the disc. Now consider the difference*

$$\begin{aligned}\left|\frac{F(z+h)-F(z)-f(z)h}{h}\right| &= \left|\frac{\int_{\gamma_{z+h}} f(\zeta)\,\mathrm{d}\zeta - \int_{\gamma_z} f(\zeta)\,\mathrm{d}\zeta - f(z)h}{h}\right| \\ &= \left|\frac{\int_{z\to z+h} f(\zeta+z)\,\mathrm{d}\zeta - f(z)h}{h}\right| \quad \textit{(Goursat)} \\ &= \left|\frac{\int_{z\to z+h} f(\zeta+z) - f(z)\,\mathrm{d}\zeta}{h}\right| \\ &\to 0\end{aligned}$$

as $h \to 0$, thereby proving that F is a primitive for f on the disc.

Theorem 5.3 (Cauchy's Theorem for a Disk)**.** *If f is holomorphic in a simply connected region, then*

$$\int_\gamma f(z)\,\mathrm{d}z = 0$$

for any closed curve γ in that disk.

§6 Cauchy'S Integral Formula

Theorem 6.1 (Cauchy integral formula)**.** *Suppose f is holomorphic in an open set that contains the closure of a disc D, If C denotes ∂D withthe positive orientation, then*

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta$$

for any point $z \in D$.

Proof: By Cauchy's theorem, we claim that

$$\frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta = \frac{1}{2\pi i}\int_{\partial B(z,\delta)} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta$$

for any small δ that $B(z,\delta)$ contained in D. Then

$$\begin{aligned}\frac{1}{2\pi i}\int_{\partial B} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta &= \frac{1}{2\pi i}\int_{\partial B} \frac{f(\zeta)-f(z)-f'(z)(\zeta-z)}{\zeta - z} + f'(z) + \frac{f(z)}{\zeta - z}\,\mathrm{d}\zeta \\ &= f(z) + \frac{1}{2\pi i}\int_{\partial B} \frac{f(\zeta)-f(z)-f'(z)(\zeta-z)}{\zeta - z}\,\mathrm{d}\zeta \\ &\to f(z)\end{aligned}$$

as $\delta \to 0$.

Corollary 6.2 (Regularity theorem)**.** *If f is holomorphic in an open set Ω, then f has infnitely many complex derivatives in Ω. Moreover, if $C \subset \Omega$ is a circle whose interior is also contained in*

Ω, then

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, \mathrm{d}\zeta$$

for all z in the interior of C.

§7 Application

Theorem 7.1 (Morera)**.** *Suppose f is a continuous function in the open disc D such that for any triangle T contained in D*

$$\int_T f(z) \, \mathrm{d}z = 0$$

then f is holomorphic in D

Proof: The function f has a primitive F in D that satisfies $F' = f$. By the regularity theorem, we know that F is indefinitely complex differentiable, and therefore f is holomorphic.

Theorem 7.2 (Taylor's Theorem)**.** *Suppose f is holomorphic in an open set Ω. If D is a disc centered at z_0 and $D \subset\subset \Omega$, then f has a power series expansion at z_0*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \textit{for all } z \in D$$

and the coefficients are given by

$$a_n = \frac{f^{(n)}(z_0)}{n!}$$

for all $n \geq 0$.

Proof: Fix $z \in D$. By the Cauchy integral formula, we have

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta \\
&= \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z_0} \frac{1}{1 - \left(\frac{z - z_0}{\zeta - z_0}\right)} \, \mathrm{d}\zeta \\
&= \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0}\right)^n \\
&= \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta\right) \cdot (z - z_0)^n
\end{aligned}$$

Since $\zeta \in C$ and $z \in D$ is fixed, there exists $0 < r < 1$ such that $\left|\frac{z-z_0}{\zeta-z_0}\right| < r < 1$, therefore $\sum_{n=0}^{\infty} \left(\frac{z-z_0}{\zeta-z_0}\right)^n$ converges uniformly. This allows us to interchange the infinite sum with the integral.

Corollary 7.3 (Cauchy inequalities)**.** *If f is holomorphic in an open set that contains $\overline{B(z_0, R)}$, then*

$$\left|f^{(n)}(z_0)\right| \leq \frac{n!}{R^n} \|f\|_{L^\infty(\partial B(z_0, R))}$$

Corollary 7.4 (Liouville's theorem). *If f is entire and bounded, then f is constant.*

Corollary 7.5. *Let non-constant polynomial $P(z)$ in $\mathbb{C}[z]$ of degreee $n \geq 1$.*
(i) $P(z)$ has a roots in $\mathbb{C}$.
(ii) $p(z)$ has precisely n roots in $\mathbb{C}$.
(ii) If these roots are denoted by $w_1, \ldots, w_n$, then P can be factored as

$$P(z) = a_n(z - w_1)(z - w_2) \cdots (z - w_n)$$

§7.1 On compact set

Theorem 7.6. *If f is holomorphic in a open set Ω, let*

$$\Omega_\delta = \{z \in \Omega : \overline{B(z, \delta)} \subset \Omega\} = \{d(z, \partial\Omega) > \delta\}$$

then

$$\sup_{z \in \Omega_\delta} |F'(z)| \leq \frac{1}{\delta} \sup_{\zeta \in \Omega} |F(\zeta)|$$

Proof: Since for every $z \in \Omega_\delta$, $\overline{B(z, \delta)}$ is contained in Ω. Then

$$F'(z) = \frac{1}{2\pi i} \int_{\partial B(z,\delta)} \frac{F(\zeta)}{(\zeta - z)^2} \,\mathrm{d}\zeta$$

Hence

$$\begin{aligned} |F'(z)| &\leq \frac{1}{2\pi} \int_{\partial B(z,\delta)} \frac{|F'(z)|}{|\zeta - z|^2} \,|d\zeta| \\ &\leq \frac{1}{2\pi} \sup_{\zeta \in \Omega} |F(z)| \,\frac{1}{\delta^2}\, 2\pi\delta \\ &= \frac{1}{\delta} \sup_{\zeta \in \Omega} |F(z)| \end{aligned}$$

as was to be shown.

Theorem 7.7 (Weierstrass's). *If $\{f_n\}_{n=1}^{\infty}$ is a sequence of holomorphic functions in Ω that converges uniformly to a function f in every compact subset of Ω, then f is holomorphic in Ω.*

Proof: Let D be any disc whose closure is contained in Ω and T any triangle in that disc. Then, since each f_n is holomorphic, Goursat's theorem implies

$$\int_T f_n(z) \,\mathrm{d}z = 0 \quad \textit{for all } n$$

By assumption $f_n \to f$ uniformly in the $\overline{D}$, so f is continuous and

$$\int_T f_n(z) \,\mathrm{d}z \to \int_T f(z) \,\mathrm{d}z$$

As a result, we find $\int_T f(z)\,\mathrm{d}z = 0$*, and by Morera's theorem, we conclude that* f *is holomorphic in* D *.*

Since this conclusion is true for every D *whose closure is contained in* Ω*, we find that* f *is holomorphic in all of* Ω*.*

Theorem 7.8. *If* $\{f_n\}_{n=1}^{\infty}$ *is a sequence of holomorphic functions in* Ω *that converges uniformly to a function* f *in every compact subset of* Ω*, then the sequence of* k*-th derivatives* $\{f_n^{(k)}\}$ *converges uniformly to* $f^{(k)}$ *on every compact set of* Ω *and* $f^{(k)}$ *is holomorphic in* Ω*.*

Proof: We only need to prove $k = 1$*.*

Suppose K *is any compact subset of* Ω*, then let* $\varepsilon = d(K, \partial\Omega) - \delta$ *that* $\varepsilon > 0$ *and* $\delta > 0$*. Since* K *is compact and* $K \subset \bigcup_{z\in K} B(z,\varepsilon) \subset \Omega$*, we can obtain an open set* $V = B(z_1,\varepsilon) \cup B(z_2,\varepsilon) \cdots \cup B(z_t,\varepsilon)$ *and* $V \subset \Omega_\delta$*. Then*

$$\sup_{z\in\Omega_\delta} |f_n' - f'| \leq \sup_{z\in\Omega} \frac{1}{\delta}\, |f_n - f|$$

Theorem 7.9. *Let* $F(z,s)$ *be defined for* $(z,s) \in \Omega \times [0,1]$ *where* Ω *is an open set in* $\mathbb{C}$*. Suppose* F *satisfies the following properties:*
(a) $F(z,s)$ *is holomorphic in* z *for each* s*.*
(b) F *is continuous on* $\Omega \times [0,1]$*.*
Then the function f *defined on* Ω *by*

$$f(z) = \int_0^1 F(z,s)\,\mathrm{d}s$$

is holomorphic.

Proof: To prove this result, it suffices to prove that f *is holomorphic in any disc* D *contained in* Ω*, and by Morera's theorem this could be achieved by showing that for any triangle* T *contained in* D *we have*

$$\int_T \int_0^1 F(z,s)\,\mathrm{d}s dz = 0$$

Interchanging the order of integration (Fubini theorem), and using property (a) would then yield the desired result.

Abother proof: For each $n \geq 1$*, we consider the Riemann sum*

$$f_n(z) = \frac{1}{n}\sum_{k=1}^{n} F(z, \frac{k}{n})$$

Then f_n *is holomorphic in all of* Ω*, and we claim that on any disc* D *whose closure is contained in* Ω*, the sequence* $\{f_n\}_{n=1}^{\infty}$ *converges uniformly to* $f(z) = \int_0^1 F(z,s)\,\mathrm{d}s$*. Then* $f(z)$ *is holomorphic in* D*. As a consequence,* f *is holomorphic in* Ω*, as was to be shown.*

Theorem 7.10 (Symmetry principle)**.** *If* f^+ *and* f^- *are holomorphicfunctions in* $\Omega^+ = \Omega \cap \{z : Re(z) > 0\}$ *and* $\Omega^- = \Omega \cap \{z : Re(z) < 0\}$ *respectively, that extend continuously to* $I = \Omega \cap \{z :$

$Re(z) = 0\}$ and

$$f^+(x) = f^-(x) \quad \textit{for all } x \in I$$

then the function f defined on Ω by

$$f = \begin{cases} f^+(z) & , z \in \Omega^+ \\ f^+(z) = f^-(z) & , z \in I \\ f^-(z) & , z \in \Omega^- \end{cases}$$

is holomorphic on all of Ω

Proof:

§8 Local Properties of Analytic Function

§8.1 Zeros and Poles

Theorem 8.1. *Suppose that f is holomorphic in a region Ω, and does not vanish identically in Ω. If $f(z_0) = 0$, then there exists a neighborhood $U \subset \Omega$ of z_0, a non-vanishing holomorphic functiong g on U and a unique positive integer n such that*

$$f(z) = (z - z_0)^n g(z)$$

*for all $z \in U$. We say that f has a **zero of order** n (or multiplicity n) at z_0. If a zero is of order 1, we say that it is **simple zero**.*

Proof: Since Ω is connected and f is not identically zero, we conclude that f is not identically zero in a neighborhood of z_0. (If not, then $f^{-1}(0)$ is open and closed)

In a small disc centered at z_0 the function f has a power series expansion by Taylor's theorem

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

Since f is not identically zero near z_0, there exists a smallest integer n such that $a_n \neq 0$. Then, we can write

$$f(z) = (z - z_0)^n [a_n + a_{n+1}(z - z_0) + \cdots] = (z - z_0)^n g(z) \ ,$$

where g is defned by the series in brackets, and hence is holomorphic in this small disk, and is nowhere vanishing for all z close to z_0.

To prove the uniqueness of the integer n, suppose that we can also write

$$f(z) = (z - z_0)^n g(z) = (z - z_0)^m h(z)$$

where $h(z_0) \neq 0$. If $m > n$, then we may divide by $(z-z_0)^n$ to see that

$$g(z) = (z-z_0)^{m-n} h(z)$$

and letting $z \to z_0$ yields $g(z_0) = 0$, a contradiction. If $m < n$ a similar argument gives $h(z_0) = 0$, which is also a contradiction. We conclude that $m = n$, thus $h = g$, and the theorem is proved.

Corollary 8.2. *We can see that the zeros of analytic funcion which does not vanish identically are isolated.*

Corollary 8.3 (Uniqueness)**.** *If f and g are analytic in Ω, and if $f(z) = g(z)$ on a set which has an convergence point in Ω, then f is identically equal to $g(z)$.*

§8.2 Pole

Definition 8.4. *We say that a function f defned in a deleted neighborhood of z_0 has a **pole** at z_0, if the function*

$$g(z) = \begin{cases} \frac{1}{f(z)} & , \quad z \neq z_0 \\ 0 & , \quad z = z_0 \end{cases}$$

is holomorphic in a full neighborhood of z_0.

Theorem 8.5. *If f has a pole at $z_0 \in \Omega$ then in a neighborhood of z_0 there exist a non-vanishing holomorphic function h and a unique positive integer n such that*

$$f(z) = (z-z_0)^{-n} h(z)$$

*The integer n is called the **order (or multiplicity) of the pole**. If the pole is of order 1, we say that it is **simple**.*

Proof: By the previous theorem we have $g(z) = (z-z_0)^n h_1(z)$, where h_1 is holomorphic and non-vanishing in a neighborhood U of z_0, so the result follows with $h(z) = 1/h_1(z)$ that is holomorphic in U.

Theorem 8.6. *If f has a pole of order n at z_0, then*

$$f(z) = \frac{a_{-n}}{(z-z_0)^n} + \frac{a_{-n+1}}{(z-z_0)^{n-1}} + \cdots + \frac{a_{-1}}{z-z_0} + G(z)$$

where G is a holomorphic function in a neighborhood of z_0

Proof: The proof follows from the multiplicative statement in the previous theorem. Indeed, the function h has a power series expansion in a neighbourhood of z_0 with $h(0) = A_0 \neq 0$

$$h(z) = A_0 + A_1(z-z_0) + \cdots$$

so that

$$\begin{aligned} f(z) &= (z-z_0)^{-n}(A_0 + A_1(z-z_0)+\cdots) \\ &= \frac{a_{-n}}{(z-z_0)^n} + \frac{a_{-n+1}}{(z-z_0)^{n-1}} + \cdots + \frac{a_{-1}}{(z-z_0)} + G(z). \end{aligned}$$

The sum

$$\frac{a_{-n}}{(z-z_0)^n} + \frac{a_{-n+1}}{(z-z_0)^{n-1}} + \cdots + \frac{a_{-1}}{(z-z_0)}$$

*is called the **principal part of** f **at the pole** z_0, and the coefficient a_{-1} is the **residue of** f **at that pole**. We write $Res_{z_0}f = a_{-1}$.*

Theorem 8.7. *If f has a pole of order n at z_0, then*

$$\operatorname{Res}_{z_0} f = \lim_{z\to z_0} \frac{1}{(n-1)!}\left(\frac{d}{dz}\right)^{n-1}(z-z_0)^n f(z)$$

§8.3 Singularities

Removable Singularity

Definition 8.8. *Let f be a function holomorphic in an open set Ω except possibly at one point z_0 in Ω. If we can define f at z_0 in such a way that f becomes holomorphic in all of Ω, we say that z_0 is a **removable singularity** for f*

Theorem 8.9 (Riemann's theorem on removable singularities)**.** *Suppose that f is holomorphic in an open set Ω except possibly at a point z_0 in Ω. If f is bounded on some deleted neighbourhood of z_0, then z_0 is a removable singularity.*

Proof: Since the problem is local we may consider a small disc D centered at z_0 and whose closure is contained in Ω. We shall prove that if $z \in D$ and $z \neq z_0$, then under the assumptions of the theorem we have

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta \quad z \in D - \{z_0\}$$

Since the right-hand side defines a holomorphic function on all of D that agrees with $f(z)$ when $z \neq z_0$, this give us a (unique) desired extension of f.

Fix $z \in D$ with $z \neq z_0$, we have

$$\frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta = \frac{1}{2\pi i}\int_{\partial B(z_0,\varepsilon)} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta + \frac{1}{2\pi i}\int_{\partial B(z_0,\varepsilon)} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta$$

By Cauchy integral formula. We find that

$$\frac{1}{2\pi i}\int_{\partial B(z,\varepsilon)} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta = f(z)$$

and

$$\left|\frac{1}{2\pi i}\int_{\partial B(z_0,\varepsilon)} \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta\right| < C\delta$$

Corollary 8.10. *Suppose that f has an isolated singularity at the point z_0. Then z_0 is a pole of f if and only if $|f(z)| \to \infty$ as $z \to z_0$.*

Proof: If z_0 is a pole, then we know that $1/f$ has a zero at z_0, and therefore $|f(z)| \to \infty$ as $z \to z_0$.

Conversely, suppose that this condition holds. Then $1/f$ is bounded near z_0. Therefore, $1/f$ has a removable singularity at z_0 and must vanish there. This proves the converse, namely that z_0 is a pole of f.

Essential Singularity

Definition 8.11. *Any singularity that is not removable or a pole is defined to be an **essential singularity**.*

Theorem 8.12 (Casorati-Weierstrass)**.** *Suppose f is holomorphic in the punctured disc $B(z_0, r) - \{z_0\}$ and has an essential singularity at z_0. Then, the image of $B(z_0, r) - \{z_0\}$ under f is dense in the complex plane.*

Proof: We argue by contradiction. Assume that the range of f is not dense, so that there exists $w \in \mathbb{C}$ and $\delta > 0$ such that

$$|f(z) - w| > \delta \quad \textit{for all } z \in D_r(z_0) - \{z_0\}$$

We may therefore define a new function on $D_r(z_0) - \{z_0\}$ by

$$g(z) = \frac{1}{f(z) - w}$$

which is holomorphic on the punctured disc and bounded by $1/\delta$. Hence g has a removable singularity at z_0.

If $g(z_0) \neq 0$, then $f(z) - w = 1/g(z)$ is holomorphic at z_0, which contradicts the assumption that z_0 is an essential singularity. In the case that $g(z_0) = 0$, then $f(z) - w$ has a pole at z_0 also contradicting the nature of the singularity at z_0. The proof is complete.

Theorem 8.13 (Picard)**.**

§8.4

Definition 8.14. *A function f on an open set Ω is **meromorphic** if there exists a sequence of points $\{z_0, z_1, z_2, \ldots\}$ that has no limit points in Ω, and such that*
(a) the function f is holomorphic in $\Omega - \{z_0, z_1, z_2, \ldots\}$
(b) f has poles at the points $\{z_0, z_1, z_2, \ldots\}$.

Definition 8.15. *If f is holomorphic in $B(\infty, R) = \{z : |z| \geq R\}$, we consider*

$$F(z) = f(1/z)$$

which is holomorphic in a deleted neighborhood of the origin. We say that f ***has a pole(essential singularity, removable singularity) at infinity*** *if F has a pole(essential singularity, removable singularity) at the origin.*

Definition 8.16. *A meromorphic function in the complex plane $\mathbb{C}$ that is either holomorphic at infnity or has a pole at infinity is said to be meromorphic in the extended complex plane $\overline{\mathbb{C}}$.*

Theorem 8.17. *The meromorphic functions in the extended complex plane are the rational functions.*

Proof: Suppose that f is meromorphic in the extended plane, so f can have only finitely many poles in the plane, say at $z_1, \cdots, z_n$. Near each pole $z_k \in \mathbb{C}$ we can write

$$f = f_k + g_k$$

where $f_k(z)$ is the principal part of f at z_k and g_k is holomorphic in a neighborhood of z_k. In particular, f_k is a polynomial in $1/(z - z_k)$. Similarly, we can write

$$f(1/z) = \tilde{f}_\infty(z) + \tilde{g}_\infty(z)$$

where $\tilde{g}_\infty$ is holomorphic in a neighborhood of the origin and $\tilde{f}_\infty$ is the principal part of $f(1/z)$ at 0, that is, a polynomial in $1/z$. Finally, let $f_\infty(z) = \tilde{f}_\infty(z)$.

We contend that the function $H = f - f_\infty - \sum f_k$ is entire and bounded. Indeed, near the pole z_k we subtracted the principal part of f so that the function H has a removable singularity there. Also, $H(1/z)$ is bounded for z near 0 since we subtracted the principal part of the pole at ∞. This proves our contention, and by Liouville's theorem we conclude that H is constant. From the definition of H, we find that f is a rational function, as was to be shown.

§9 The argument principle and applications

§9.1 Argument Principle

Theorem 9.1 (Argument Principle)**.** *Suppose f is meromorphic in Ω, then*

$$n\left(f(\gamma), w\right) = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z) - w}\,\mathrm{d}z = \sum_j n(\gamma, z_j(w)) - \sum_k n(\gamma, p_k)$$

for every closed curve γ which is homologous to 0 in Ω and does not pass through any of zeros of $f(z) - w$ an poles.

Corollary 9.2. *Suppoes f is analytic in a disk Δ, then the number of roots of equation $f(z) = a$ in Δ is*

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z) - a}\,\mathrm{d}z$$

Corollary 9.3. *Suppoes that $f(z)$ is analytic in sufficiently small $B(z_0, \varepsilon)$ such that*

(i) $f(z) - f(z_0)$ *has a only zero of order n at z_0*

(ii) $f'(z)$ *has a only zero at z_0*

Then there exists $\delta > 0$ such that for all $w \in B(f(z_0), \delta)$ the equation $f(z) - w$ has exactly n different simple roots in $B(z_0, \varepsilon)$

Corollary 9.4 (Open Maps)**.** *A nonconstant analytic funcion is a open map.*

Corollary 9.5. *If $f(z)$ is analytic at z_0 with $f'(z_0) \neq 0$, it maps a neighborhood of z_0 conformally and homeomorphic onto a region*

Theorem 9.6 (Rouche' theorem)**.** *Suppose that f and g are holomorphic in an open set containing a circle C and its interior. If*

$$|f(z) - g(z)| < |f(z)| \quad \textit{for all } z \in C$$

then f and g have the same number of zeros inside the circle C

Theorem 9.7 (A.Hurwitz)**.** *If f_n are analytic and $\neq 0$ in a region Ω, and if f_n converges to f uniformly on every compact subset of Ω. Then f is either identically zero or never equal to zero in Ω.*

Proof: Suppoes that f is not identically zero. For any $z_0 \in \Omega$ there is therefore a r such that f is defined $\neq 0$ for $0 < |z - z_0| \leq r$, thus $|f|$ has a positive minimum on circle $|z - z_0| = r$. It follows that $\frac{1}{f_n}$ converges unifformly to $\frac{1}{f}$ on C. And since f' converges unifformly to f' on C.

We may conclude that

$$\frac{1}{2\pi} \int_C \frac{f'(z)}{f(z)} \, \mathrm{d}z = \lim_{n \to \infty} \frac{1}{2\pi} \int_C \frac{f_n'(z)}{f_n(z)} \, \mathrm{d}z = 0$$

Consequently, $f(z_0) \neq 0$.

§10 The General Form of Cauchy's Theorem

Theorem 10.1. *If f is holomorphic in Ω , then*

$$\int_{\gamma_0} f(z) \, \mathrm{d}z = \int_{\gamma_1} f(z) \, \mathrm{d}z$$

whenever the two curves γ_0 and γ_1 are homotopic in Ω.

Theorem 10.2. *Any holomorphic function in a simply connected domain has a primitive.*

§11 The Calculus of Residues

Lemma 11.1 (Jordan)**.** *If f is continuous and any $\alpha > 0$*

$$\lim_{\substack{z\to\infty \\ \mathrm{Im}(z)>0}} f(z) = 0$$

Then

$$\lim_{R\to\infty} \int_{\gamma_R} e^{\alpha z} f(z)\,\mathrm{d}z = 0$$

where $\gamma_R = \left\{z : z = Re^{\theta}, 0 \leq \theta \leq \pi\right\}$

Theorem 11.2. *Suppose f is meromorphic in* $\mathrm{Im}\, z > 0$ *and For any $\alpha > 0$,*

$$\int_{-\infty}^{\infty} e^{\alpha x} f(x)\,\mathrm{d}x = 2\pi \sum \mathrm{res}\left(e^{\alpha x} f(x), a_k\right)$$

Chapter II

Entire Function

§1 Infinite Products

Definition 1.1. *An finite product of complex numbers*

$$p_1 p_2 \cdots p_n = \prod_{n=1}^{\infty} p_n$$

is evaluated by taking the limit of partial product $P_n = p_1 p_2 \cdots p_n$. *It said to converge to the* $P = \lim P_n$ *is this limit sxists and is different from zero.*

Theorem 1.2. *The infinite product* $\prod\limits_1^\infty (1 + a_n)$ *with* $1 + a_n \neq 0$ *converges simultaneously with the series* $\sum\limits_1^\infty \log(1 + a_n)$ *whose terms represent the values of the principal branch of the logarithm.*

Definition 1.3. *An infinite product* $\prod\limits_1^\infty (1 + a_n)$ *is said to be absolutely convergent if the corresponding series* $\sum\limits_1^\infty \log(1 + a_n)$ *converges absolutely.*

Theorem 1.4. *A necessary and sufficient condition for the absolute convergence of the product* $\prod\limits_1^\infty (1 + a_n)$ *is the convergence of the series* $\sum\limits_1^\infty |a_n|$ *(if and only if* $\sum \log(1 + |a_n|)$ *converge).*

Theorem 1.5. *The value of an absolutely convergent product does not change if the factors are reordered*

Theorem 1.6. *Suppose* $\{g_n = 1 + f_n\}$ *is a sequence of holomorphic functions on the open set* Ω. *If there exist constants* $c_n > 0$ *such that*

$$\sum c_n < \infty \quad \textit{and} \quad |f_n(z)| \leq c_n \quad \textit{for all } z \in \Omega$$

then:

(i) The product $\prod\limits_{n=1}^\infty (1 + f_n)$ *converges uniformly in* Ω *to a holomorphic function* $G(z)$.

(ii) If $g_n = 1 + f_n(z)$ does not vanish for any n, then

$$\frac{G'(z)}{G(z)} = \sum_{n=1}^{\infty} \frac{g_n'(z)}{g_n(z)}$$

Proof:

$$G_n(z) = \prod_{i=1}^{n} g_i(z) = e^{\sum^{n} \log(1+f_i(z))}$$

converges uniformly to a holomorphic function

To establish the second part of the theorem, suppose that K is a compact subset of Ω. We have just proved that $G_n \to F$ uniformly in Ω, so the sequence $\{G_n'\}$ converges uniformly to F' in K. Since G_N is uniformly bounded from below on K (it cannot be omitted), we conclude that $G_n'/G_n \to F'/F$ uniformly on K.

And because K is an arbitrary compact subset of Ω, the limit holds for every point of Ω. Moreover, as we saw

$$\frac{G_n'}{G_n} = \sum_{i=1}^{n} \frac{g_i'}{g_i}$$

so part (ii) of the theorem is also proved.

§2 Jensen's formula

Lemma 2.1. *Let $a \in \mathbb{C}$ and $|a| = 1$*

$$\int_0^{2\pi} \log\left|a - e^{i\theta}\right| \, \mathrm{d}\theta = \int_0^{2\pi} \log\left|1 - e^{i\theta}\right| \, \mathrm{d}\theta = 0$$

Theorem 2.2. *If f is an analytic function, then $\log|f(z)|$ is harmonic except the zeros f. Therefore, if $f(z)$ is analytic and free from zeros in $B(0,R)$ then*

$$\log|f(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log\left|f(Re^{i\theta})\right| \, \mathrm{d}\theta$$

and $\log|f(z)|$ can be expressed by Poission's formula.

Proof: If f has zeros the circle $\partial B(0,R)$. Denotes these zeros by $Re^{i\theta_k}$ ($k = 1, 2 \cdots m$), multiple zeros being repeated, then define

$$F(z) = f(z) \prod_{k=1}^{m} \frac{1}{z - Re^{i\theta_k}}$$

is analytic and has on zeros in $\partial B(0,R)$.

$$\log|f(0)| - m\log R = \frac{1}{2\pi} \int_0^{2\pi} \log\left|f(Re^{i\theta})\right| - \sum \log\left|Re^{i\theta} - Re^{i\theta_k}\right| \, \mathrm{d}\theta$$

Theorem 2.3 (Jensen's formula)**.** *Let Ω be an open set that contains closed disc $\overline{B(0,\rho)}$ the and suppose that f is holomorphic in* **Ω***, and vanish at $z_1, z_2, \cdots, z_n$ in $B(0,\rho)$, mutiple zeros being repeated, and assume that $f(0) \neq 0$. Then*

$$\log|f(0)| = -\sum_{k=1}^{n} \log\left(\frac{\rho}{|z_k|}\right) + \frac{1}{2\pi}\int_0^{2\pi} \log\left|f(\rho e^{i\theta})\right| \,\mathrm{d}\theta$$

Proof: 1. Let

$$F(z) = f(z)\prod_{i=1}^{n} \frac{\rho^2 - \overline{z_i}z}{\rho(z - z_i)}$$

is free fron zeros in the disk $B(0,\rho)$, and $|F| = |f|$ on $\partial B(0,\rho)$. Consequently we obtain

$$\log|F(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log\left|F(\rho e^{i\theta})\right| \,\mathrm{d}\theta$$

and, substituting the value

$$\log|f(0)| + \sum \log\left(\frac{\rho}{|z_i|}\right) = \frac{1}{2\pi}\int_0^{2\pi} \log\left|f(\rho e^{i\theta})\right| \,\mathrm{d}\theta$$

2. If $f(0) = 0$ we write $g(z) = f(z)/z^h$, then

$$\log|a_h| + \sum \log\left(\frac{\rho}{|z_i|}\right) = \frac{1}{2\pi}\int_0^{2\pi} \log\left|f(\rho e^{i\theta})\right| \,\mathrm{d}\theta - h\log\rho$$

Theorem 2.4 (Poission-Jensen formula)**.** *Let Ω be an open set that contains closed disc $\overline{B(0,\rho)}$ the and suppose that f is holomorphic in* **Ω***, and vanish at $z_1, z_2, \cdots, z_n$ in $B(0,\rho)$, mutiple zeros being repeated. Then*

Proof: If $|z_0| < \rho$ and $f(z_0) \neq 0$ we write

$$\varphi_{\rho,z_0}(z) = \frac{\rho^2(z - z_0)}{\overline{z_0}z - \rho^2}$$

then let

$$h(z) = f \circ \varphi_{\rho,z_0}(z)$$

that $h(0) = f(z_0)$ and h vanish at $\varphi_{\rho,z_0}(z_i)$, then

$$\begin{aligned}\log|f(z_0)| + \sum \log\left(\frac{\rho}{|\varphi_{\rho,z_0}(z_i)|}\right) &= \frac{1}{2\pi}\int_0^{2\pi} \log\left|f \circ \varphi_{R,z_0}(Re^{i\theta})\right| \,\mathrm{d}\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \log\left|f \circ \varphi_{R,z_0}(Re^{i\theta})\right| \,\mathrm{d}\theta\end{aligned}$$

We obtain

$$\log|f(z)| = -\sum \log\left(\frac{z_i z - \rho^2}{\rho(z - z_i)}\right) + \frac{1}{2\pi}\int_0^{2\pi} \frac{\rho^2 - z^2}{|\rho e^{i\theta} - z|^2} \log\left|f(\rho e^{i\theta})\right| \,\mathrm{d}\theta$$

Provided that $f(z) \neq 0$.

Lemma 2.5. *If f is a holomorphic function in Ω, we denote by $\mathfrak{n}(r)$ the number of zeros of f (counted with their multiplicities) inside the disc $B(0,r) \subset \Omega$. If $z_1, \ldots, z_N$ are the zeros of f inside the disc $B(0,\rho)$, then*

$$\int_0^\rho \mathfrak{n}(r)\,\frac{dr}{r} = \sum_{k=1}^N \log\left|\frac{\rho}{z_k}\right|.$$

§3 Weierstrass infnite product

Definition 3.1. *For each integer $k \geq 0$ we define* ***canonical factors*** *by*

$$E_0 = 1 - z \qquad E_k = (1 - z)e^{z + \frac{z^2}{2} + \cdots \frac{z^k}{k}}$$

the integer k is called the degreee of the canonical factors

Lemma 3.2. *If $|z| \leq 1/2$, then $|1 - E_k| \leq C\,|z|^{k+1}$ for some $C > 0$ (independent of k)*

Theorem 3.3 (Weierstrass)**.** *Given any sequence $\{a_n\}$ of complex numbers with $|a_n| \to \infty$ as $n \to \infty$. Every entire function with these and no other zeros can be written in the form*

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n} + \frac{1}{2}\left(\frac{z}{a_n}\right)^2 + \cdots + \frac{1}{m_n}\left(\frac{z}{a_n}\right)^{m_n}}$$

$a_n \neq 0$ where the product is taken over all $a_n \neq 0$, the m_n are certain integers, and $g(z)$ is an entire function.

Proof: *The product converges together with the series with the gengral term*

$$r_n(z) = \log\left(1 - \frac{z}{a_n}\right) + p_n(z)$$

where the branch of logarithm shall

For a given R we consider only the terms with $|a_n| > R$. In the disk $\overline{B(0,R)}$ the principal *branch of* $\log(1 - z/z_n)$ *can be developed in Taylor series*

$$\begin{aligned} r_n(z) &= \log E_{m_n}\left(\frac{z}{a_n}\right) \\ &= \log\left(1 - \frac{z}{a_n}\right) + p_n(z) \\ &= \sum_{n=1}^{\infty} -\frac{1}{n}\left(\frac{z}{a_n}\right)^n + \sum_{n=1}^{m_n} \frac{1}{n}\left(\frac{z}{a_n}\right)^n \\ &= \sum_{n=m_n+1}^{\infty} -\frac{1}{n}\left(\frac{z}{a_n}\right)^n \end{aligned}$$

and we obtain the estimate

$$|r_n(z)| \leq \frac{1}{m_n+1}\left(\frac{R}{a_n}\right)^{m_n+1}\left(1 - \frac{R}{|a_n|}\right)^{-1}$$

where $R/|a_n| \leq \delta < 1$ *for all* $n \in \{n : |a_n| > R\}$. *Suppoes now that the series*

$$\sum_{n=1}^{\infty} \frac{1}{m_n+1}\left(\frac{R}{a_n}\right)^{m_n+1}$$

converges, the comparison shows that the series $\sum r_n(z) = \sum\limits_{|a_n|>R} r_n + \sum\limits_{else} r_k$ *is absolutely and uniformly convergent for* $|z| \leqq R$, *and thus the product represents an analytic function in* $B(0, R)$. *It remains only to show that the series can be made convergent for all* R. *But this is obvious, for if we take* $m_n = n$.

Corollary 3.4. *Every meromorphic function in the whole plane* $\mathbb{C}$ *is the quotient of two entire function*

Corollary 3.5 (Interpolation)**.** *Suppose that* $a_n \to \infty$ *and that the* A_n *are arbitrary complex numbers. Then there exists an entire function* $f(z)$ *which satisfies* $f(a_n) = A_n$.

Proof: Let $g(z)$ *be a function with simple zeros at the* a_n. *We show that*

$$\sum_{n=1}^{\infty} \frac{g(z)A_n}{(z-a_n)g'(a_n)} \cdot e^{\gamma_n(z-a_n)}$$

converges for some choice of the numbers γ_n.

For any $R > 0$, *we consider*

Definition 3.6. *Assume that* h *is the smallest integer for which* $\sum \frac{1}{|a_n|^{h+1}}$ *converges; the expression*

$$\prod_{n=1}^{\infty} E_h\left(\frac{z}{a_n}\right)$$

is then called the ***canonical product associated with sequence*** $\{a_n\}$, *and* h *is the* ***genus of the***

canonical product

whenever possible we use the canonical product in the representation of f, which is thereby uniquely determined

$$f = z^m e^{g(z)} \prod_{n=1}^{\infty} E_h\left(\frac{z}{a_n}\right)$$

that

$$|r_n(z)| = \left|\log E_h\left(\frac{z}{a_n}\right)\right| \leq \frac{1}{h+1}\left(\frac{R}{a_n}\right)^{h+1}\left(1 - \frac{R}{|a_n|}\right)^{-1}$$
$$\leq \frac{C_R}{(h+1)\,|a_n|^{h+1}}$$

then product converges uniformly for $B(0,R)$.

*If in this representation $g(z)$ reduces to a polynomial, the function $f(z)$ is said to be of finite genus, and **genus of** $f(z)$ is by definition equal to* $\max\{h, \mathit{deg}\ g\}$.

Theorem 3.7.

$$\sin \pi z = z\pi \prod_{n\neq 0}\left(1 - \frac{z}{n}\right) e^{z/n} = \pi z \prod_{1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)$$

In order to determine $g(z)$ we form the logarithmic derivatives on both sides. We find

$$\pi \cot \pi z = \frac{1}{z} + g'(z) + \sum_{n\neq 0}\left(\frac{1}{z-n} + \frac{1}{n}\right)$$

where the procedure is easy to justify by uniform convergence on any compact set which does not contain the points $z = n$. By comparison with the previous formula (10) we conclude that $g'(z) = 0$. Hence $g(z)$ is a constant, and since $\lim_{z\to 0} \sin \pi z/z = \pi$ we must have $e^{g(z)} = \pi$.

§4 Hadamard's factorization theorem

Definition 4.1. *Let f be an entire function. If there exist a positive number ρ and constants $A, B > 0$ such that*

$$|f(z)| \leq Ae^{B|z|^{\rho}} \quad \textit{for all } z \in \mathbb{C}$$

*then we say that f has an order of growth $\leq \rho$. We define the **order of growth of** f as*

$$\lambda = \inf \rho$$

Denote by $M(r)$ the maximum of $|f(z)|$ on $|z| = r$. For any given $\varepsilon > 0$ as sonn as r is sufficiently large, we have

$$M(r) \leq e^{r^{\lambda+\varepsilon}}$$

then $M(r) = o(e^{r^{\lambda+\varepsilon}})$ *and* $\log M(r) = o(r^{\lambda+\varepsilon})$ *for large* r. *Actually,*

$$\lambda = \limsup \frac{\log\log M(r)}{\log r}$$

Lemma 4.2. *For all* $u \in C$ *and* $h \in N$

$$\log |E_h(u)| \leq (2h+1)\,|u|^{h+1}$$

Furthermore, $E_h(z)$ *has order of growth* h.

Proof: If $|u| < 1$ *we have power series development*

$$\log |E_h(u)| \leq \frac{|u|^{h+1}}{h+1} + \frac{|u|^{h+2}}{h+2} + \cdots \leq \frac{1}{h+1}\frac{|u|^{h+1}}{1-|u|}$$

and thus

$$(1-|u|)\log |E_h(u)| \leq |u|^{h+1}$$

For arbitrary u *and* $h \geq 1$ *it is also clear that*

$$\log |E_h(u)| \leq \log |E_{h-1}(u)| + |u|^h$$

since $E_h(u) = E_{h-1}(u)e^{u^h/h}$.

We assume that with $h-1$ *in the place of* h*, that is to say*

$$\log |E_{h-1}(u)| \leq (2h-1)\,|u|^h$$

If $|u| \geq 1$*, this imply*

$$\begin{aligned}\log |E_h(u)| &\leq \log |E_{h-1}(u)| + |u|^h\\ &\leq 2h\,|u|^h\\ &\leq (2h+1)\,|u|^{h+1}\end{aligned}$$

But if $|u| < 1$ *we can also obtain*

$$\begin{aligned}\log |E_h(u)| &= (1-|u|)\log |E_h(u)| + |u|\log |E_h(u)|\\ &\leq |u|^{h+1} + |u|\left((2h-1)\,|u|^h + |u|^h\right)\\ &= (2h+1)\,|u|^{h+1}\end{aligned}$$

Theorem 4.3. *If* f *is an entire function that has an order of growth* λ*, then for every* $\varepsilon > 0$

(i) $\mathfrak{n}(r) = o(r^{\lambda+\varepsilon})$

(ii) If $a_1, a_2, \ldots$ *denote the zeros of* f*, with* $a_k \neq 0$*, then we have*

$$\sum_{k=1}^{\infty} \frac{1}{|a_k|^{\lambda+\varepsilon}} < \infty$$

Proof: It suffices to prove the estimate for $\mathfrak{n}(r)$ *when* $f(0) \neq 0$. *Indeed, consider the function* $F(z) = f(z)/z^\ell$ *where* ℓ *is the order of the zero of* f *at the origin. Then* $\mathfrak{n}_f(r)$ *and* $\mathfrak{n}_F(r)$ *differ only by a constant, and* F *also has an of order of growth* $\leq \rho$.

1. If $f(0) \neq 0$, *then for* $R = 2r > 0$ *that* f *vanish nowhere on* $\partial B(0, R)$

$$\begin{aligned}\mathfrak{n}(r)\log 2 \leq \int_r^{2r} \frac{\mathfrak{n}(x)}{x}\,\mathrm{d}x &\leq \int_0^R \frac{\mathfrak{n}(x)}{x}\,\mathrm{d}x \\ &= \frac{1}{2\pi}\int_0^{2\pi} \log\left|f(Re^{i\theta})\right|\,\mathrm{d}\theta - \log|f(0)| \\ &\leq \log|M(2r)| - \log|f(0)| \\ &\leq (2r)^{\lambda+\varepsilon} - \log|f(0)|\end{aligned}$$

Consequently, $\lim \mathfrak{n}(r) r^{-\lambda-\varepsilon} = 0$

2. Since $\mathfrak{n}(x)$ *vanish near the* 0 *and* $\mathfrak{n}(x) = o(x^{\lambda+\frac{1}{2}\varepsilon})$, *we have*

$$\sum \frac{1}{|a_n|^{\lambda+\varepsilon}} = \frac{1}{\lambda+\varepsilon}\int_0^\infty \frac{\mathfrak{n}(x)}{x^{\lambda+\varepsilon+1}}\,\mathrm{d}x < \infty$$

Theorem 4.4 (Hadamard Theorem)**.** *The genus and the order of an entire function satisfiy the double inequality*

$$h \leq \lambda \leq h + 1$$

Proof: 1. Suppose that $f(z)$ *is of genus* h, *then the previous lemma gives the estimate*

$$\log|P(z)| = \sum_n \log\left|E_h\left(\frac{z}{a_n}\right)\right| \leq (2h+1)\,|z|^{h+1}\sum_n \frac{1}{|a_n|^{h+1}}$$

an it follows that $P(z)$ *is at most of order* $h + 1$

2. For the opposite inequality assume $f(z)$ *is of finite order* λ *and* $h_1 = [\lambda]$. *Then* $h_1 + 1 > \lambda$, *and we have to prove that* $\sum 1/|a_n|^{h_1+1}$ *converges. It is obvious by the previous lemma. It remains to prove that* $g(z)$ *is a polynomial of degreee* $\leq h_1 = [\lambda]$. *If the operation* $\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}$ *is applied to both sides of the Poission Jensen formula, we obtain*

Corollary 4.5. *An entire function of fractional ordered assemues every finite valus infinity many times*

Proof: It is clear that f *and* $f - a$ *have the same order for any constant* a. *Therefore we need only show that* f *has infinitely many zeros.*

If f *has only a finite number of zeros we can divide by a polynomial and obtain a function of the same order without zeros.*

$$\frac{f(z)}{\prod_{k=1}^{n} z - a_k} = e^{g(z)}$$

By the theorem $g(z)$ *must be a polynomial. But it is evident that the order of* $e^{g(z)}$ *is exactly the degree of* g*, and hence an integer. The contradiction proves the corollary.*

Chapter III

Special Function

§1 The Gamma Function

§1.1 The Gamma Function

Definition 1.1.
$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{\frac{z}{n}}$$

We observe that $\gamma(z)$ is meromorphic function with simole poles at $z = 0, -1 \cdots$ but without zeros

Proof:
$$G(z) = \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}}$$

We observe that $G(z-1)$ has the same zeros as $G(z)$, and in addition a zero at the origin. It is therefore clear that we can write

$$G(z-1) = ze^{\gamma(z)} G(z),$$

where $\gamma(z)$ is an entire function. In order to determine $\gamma(z)$ we take the logarithmic derivatives on both sides. This gives the equation

$$\sum_{n=1}^{\infty} \left(\frac{1}{z-1+n} - \frac{1}{n}\right) = \frac{1}{z} + \gamma'(z) + \sum_{n=1}^{\infty} \left(\frac{1}{z+n} - \frac{1}{n}\right)$$

In the series to the left we can replace n by $n+1$. By this change we obtain

$$\sum_{n=1}^{\infty}\left(\frac{1}{z-1+n}-\frac{1}{n}\right)=\frac{1}{z}-1+\sum_{n=1}^{\infty}\left(\frac{1}{z+n}-\frac{1}{n+1}\right)$$
$$=\frac{1}{z}-1+\sum_{n=1}^{\infty}\left(\frac{1}{z+n}-\frac{1}{n}\right)+\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)$$
$$=\frac{1}{z}+\sum_{n=1}^{\infty}\left(\frac{1}{z+n}-\frac{1}{n}\right)$$

Hence $\gamma'(z)=0$ and $\gamma(z)$ is a constant, which we denote by γ, and $G(z-1)=e^{\gamma}G(z)$. Taking $z=1$ we have

$$1=G(1)=e^{\gamma}G(1)$$

an hence

$$\gamma=\lim\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log n\right)$$

Let Euler's gamma function

$$\Gamma(z)=\frac{1}{ze^{\gamma z}G(z)}$$

satisfies

$$\Gamma(z+1)=z\Gamma(z)$$

Proposition 1.2. *The function Γ has the following properties:*

(i)

$$\Gamma(z+1)=z\Gamma(z)$$

(ii) $\frac{1}{\Gamma(s)}$ is an entire function of s with simple zeros at $s=0,-1,-2,\ldots$ and it vanishes nowhere else.

Theorem 1.3. $1/\Gamma(s)$ *has growth*

$$\left|\frac{1}{\Gamma(s)}\right|\leq c_1e^{c_2|s|\log|s|}$$

Therefore, $1/\Gamma$ is of order 1.

Proof. By the theorem we may write

$$\frac{1}{\Gamma(s)}=\Gamma(1-s)\frac{\sin\pi s}{\pi}$$

and therefore $1/\Gamma$ is entire with simple zeros at $s=0,-1,-2,-3,\ldots$.

To prove the estimate, we begin by showing that

$$\int_1^{\infty}e^{-t}t^{\sigma}\,\mathrm{d}t\leq e^{(\sigma+1)\log(\sigma+1)}$$

whenever $\sigma = Re(s)$ is positive. Choose n so that $\sigma \le n \le \sigma + 1$. Then

$$\begin{aligned}\int_1^\infty e^{-t}t^\sigma dt &\le \int_0^\infty e^{-t}t^n dt \\ &= n! \\ &\le n^n \\ &= e^{n\log n} \\ &\le e^{(\sigma+1)\log(\sigma+1)}\end{aligned}$$

Since the relation (3) holds on all of $\mathbb{C}$, we see from (5) that

$$\frac{1}{\Gamma(s)} = \left(\sum_{n=0}^\infty \frac{(-1)^n}{n!(n+1-s)}\right)\frac{\sin\pi s}{\pi} + \left(\int_1^\infty e^{-t}t^{-s}dt\right)\frac{\sin\pi s}{\pi}$$

However, from our previous observation,

$$\left|\int_1^\infty e^{-t}t^{-s}dt\right| \le e^{(|\sigma|+1)\log(|\sigma|+1)}$$

and because $|\sin\pi s| \le e^{\pi|s|}$ (by Euler's formula for the sine function) we find that the second term in the formula is dominated by $ce^{(|s|+1)\log(|s|+1)}e^{\pi|s|}$, which is itself majorized by $c_1e^{c_2|s|\log|s|}$. Next, we consider the term

$$\sum_{n=0}^\infty \frac{(-1)^n}{n!(n+1-s)}\frac{\sin\pi s}{\pi}$$

There are two cases: $|\operatorname{Im}(s)| > 1$ and $|\operatorname{Im}(s)| \le 1$. In the first case, this expression is dominated in absolute value by $ce^{\pi|s|}$. If $|\operatorname{Im}(s)| \le 1$, we choose k to be the integer so that $k - 1/2 \le \operatorname{Re}(s) < k + 1/2$. Then if $k \ge 1$,

$$\begin{aligned}\sum_{n=0}^\infty \frac{(-1)^n}{n!(n+1-s)}\frac{\sin\pi s}{\pi} &= (-1)^{k-1}\frac{\sin\pi s}{(k-1)!(k-s)\pi} + \\ &+ \sum_{n\ne k-1}(-1)^n\frac{\sin\pi s}{n!(n+1-s)\pi}\end{aligned}$$

Both terms on the right are bounded; the first because $\sin\pi s$ vanishes at $s = k$, and the second because the sum is majorized by $c\sum 1/n\,!$.

When $k \le 0$, then $\operatorname{Re}(s) < 1/2$ by our supposition, and $\sum_{n=0}^\infty \frac{(-1)^n}{n!(n+1-s)}$ is again bounded by $c\sum 1/n\,!$. This concludes the proof of the theorem.

The fact that $1/\Gamma$ satisfies the type of growth conditions discussed in Chapter 5 leads naturally to the product formula for the function $1/\Gamma$, which we treat next.

§1.2 Properties

Theorem 1.4. *The residue of Γ at $s = -n$ is $(-1)^n/n!$.*

Proof:

$$\Gamma(z) = \frac{\Gamma(z+n+1)}{(z+1)(z+2)\cdots(z+n+1)}$$

then

$$\begin{aligned}\mathrm{Res}_{z=-n}\Gamma(z) &= (z+n)\Gamma(z)|_{z=-n} \\ &= \frac{\Gamma(1)}{(-n+1)\cdots(-n+n-1)(-n+n+1)} \\ &= \frac{(-1)^n}{n!}\end{aligned}$$

Theorem 1.5 (Gauss)**.**

$$(2\pi)^{\frac{n-1}{2}}\,\Gamma(nz) = n^{nz-\frac{1}{2}}\Gamma(z)\Gamma\left(z+\frac{1}{n}\right)\cdots\Gamma\left(z+\frac{n-1}{n}\right)$$

Proof: Considering the second derivative of $\log\Gamma(z)$

$$\frac{d}{dz}\left(\frac{\Gamma'(z)}{\Gamma(z)}\right) = \sum_{k=0}^{\infty}\frac{1}{(z+k)^2}$$

then

$$\begin{aligned}\sum_{m=0}^{n-1}\frac{d}{dz}\left(\frac{\Gamma(z+\frac{m}{n})}{\Gamma'(z+\frac{m}{n})}\right) &= \sum_{m=0}^{n-1}\sum_{k=0}^{\infty}\frac{1}{(z+\frac{m}{n}+k)^2} \\ &= n^2\sum_{s=0}^{\infty}\frac{1}{(nz+s)^2} \\ &= n\frac{d}{dz}\left(\frac{\Gamma(nz)}{\Gamma'(nz)}\right)\end{aligned}$$

By integration we obtain

$$\Gamma(nz) = e^{az+b}\Gamma(z)\Gamma\left(z+\frac{1}{n}\right)\cdots\Gamma\left(z+\frac{n-1}{n}\right)$$

where the constants a *and* b *have yet to be determined. Considering the residues of two sides at the poles* $z=0$*, we have*

$$\frac{1}{n} = e^b\prod_{k=1}^{n-1}\Gamma\left(\frac{k}{n}\right) = e^b\sqrt{\prod_{k=1}^{n-1}\frac{\pi}{\sin\frac{\pi k}{n}}} = e^b\sqrt{\frac{\pi^{n-1}}{\frac{n}{2^{n-1}}}}$$

$e^b = \frac{1}{n^{1/2}(2\pi)^{\frac{n-1}{2}}}$. *Next, substituting* $z = 1$

$$\begin{aligned}(n-1)! &= e^{a+b}\prod_{k=1}^{n-1}\Gamma\left(1+\frac{k}{n}\right)\\ &= e^{a+b}\prod_{k=1}^{n-1}\frac{k}{n}\cdot\Gamma\left(\frac{k}{n}\right)\\ &= e^{a+b}\frac{(n-1)!}{n^(n-1)}\frac{1}{ne^b}\end{aligned}$$

hence $e^a = n^n$*. The final result is thus*

$$(2\pi)^{\frac{n-1}{2}}\ \Gamma(nz) = n^{nz-\frac{1}{2}}\Gamma(z)\Gamma\left(z+\frac{1}{n}\right)\cdots\Gamma\left(z+\frac{n-1}{n}\right)$$

Corollary 1.6 (Legendre's duplication formula)**.**

$$\sqrt{\pi}\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma(z+\frac{1}{2})$$

Theorem 1.7. *For all* $z \in \mathbb{C}$

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

§1.3 The Integral Form

Lemma 1.8. *The funcion*

$$F(s) = \int_0^\infty e^{-t}t^{s-1}\,\mathrm{d}t$$

is an analytic function in the half-plane $Re(s) > 0$*.*

Proof: We first observe that the integral is exist for every $\mathrm{Re}(s) > 0$*. It suffess to show that the integral defines a holomorphic function in every strip*

$$S_{\delta,M} = \{s : \delta < Re(s) < M\}$$

where $0 < \delta < M < \infty$*.*

For $\epsilon > 0$*, let*

$$F_\epsilon(s) = \int_\varepsilon^{1/\epsilon} e^{-t}t^{s-1}\,\mathrm{d}t$$

the function F_ϵ *is holomorphic in the strip* $S_{\delta,M}$*, it suffices to show the* F_ϵ *converges unifformly to* Γ *on that the strip* $S_{\delta,M}$*. To see this, we first*

$$\begin{aligned}|F(s)-F_\varepsilon(s)| &= \left|\left(\int_0^\epsilon + \int_{1/\epsilon}^\infty\right) e^{-t}t^{s-1}\,\mathrm{d}t\right| \\ &\le \int_0^\epsilon e^{-t}t^{\sigma-1}dt + \int_{1/\epsilon}^\infty e^{-t}t^{\sigma-1}\,\mathrm{d}t \\ &\le \int_0^\epsilon e^{-t}t^{\delta-1}dt + \int_{1/\epsilon}^\infty e^{-t}t^{M-1}\,\mathrm{d}t\end{aligned}$$

converges uniformly to 0 *in* $S_{\delta,M}$.

Lemma 1.9. *The function* F *initially defined for* $\mathrm{Re}(s) > 0$ *has an analytic continuation to a meromorphic function on* $\mathbb{C}$ *whose only singularities are simple poles at the negative integers* $s = 0, -1, \cdots$ *The residue of* F *at* $s = -n$

$$\mathrm{Res}_{z=-n}F(z) = \frac{(-1)^n}{n!}$$

Proof: It suffices to extend F *to each half-plane* $\mathrm{Re}(s) > -m$, *where* $m \ge 1$ *is an integer. For* $\mathrm{Re}(s) > -1$, *we define*

$$F_1(s) = \frac{F(s+1)}{s}$$

Since $\Gamma(s+1)$ *is holomorphic in* $\mathrm{Re}(s) > -1$, *we see that* F_1 *is meromorphic in that half-plane, with the only possible singularity a simple pole at* $s = 0$ *with residue 1. Moreover, if* $\mathrm{Re}(s) > 0$, *then*

$$F_1(s) = \frac{F(s+1)}{s} = F(s)$$

So F_1 *extends* F *to a meromorphic function on* $\mathrm{Re}(s) > -1$.

We can now continue in this fashion by defining a meromorphic F_m *for* $\mathrm{Re}(s) > -m$ *that agrees with* F *on* $\mathrm{Re}(s) > 0$. *For* $\mathrm{Re}(s) > -m$, *where* m *is an integer* ≥ 1, *define*

$$F_m(s) = \frac{\Gamma(s+m)}{(s+m-1)(s+m-2)\cdots s}.$$

The function F_m *is meromorphic in* $\mathrm{Re}(s) > -m$ *and has simple poles at* $s = 0, -1, -2, \ldots, -m+1$ *with residues*

$$\begin{aligned}res_{s=-n}F_m(s) &= \frac{\Gamma(-n+m)}{(m-1-n)!(-1)(-2)\cdots(-n)} \\ &= \frac{(m-n-1)!}{(m-1-n)!(-1)(-2)\cdots(-n)} \\ &= \frac{(-1)^n}{n!}\end{aligned}$$

Successive applications of the lemma show that $F_m(s) = F(s)$ *for* $\mathrm{Re}(s) > 0$. *By uniqueness, this also means that* $F_m = F_k$ *for* $1 \le k \le m$ *on the domain of definition of* F_k. *Therefore, we have obtained the desired continuation of* F.

Theorem 1.10. *For* $\mathrm{Re}(z) > 0$*, we have*

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$$

§2 The Zeta Function

§2.1 The Zeta Function

Theorem 2.1. *The Riemann's Zeta function*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

represents an analytic funcion of s in the half plane $\mathrm{Re}(s) > 1$.

Theorem 2.2. *For* $\sigma = \mathrm{Re}(s) > 1$*,*

$$\frac{1}{\zeta(s)} = \prod_{n=1}^{\infty} \left(1 - p_n^{-s}\right)$$

§2.2 Extension to the Whole Plane

Theorem 2.3. *For* $\sigma > 1$

$$\zeta(s) = -\frac{\Gamma(1-s)}{2\pi i} \int_C \frac{(-z)^{s-1}}{e^z - 1} \, \mathrm{d}z$$

where $(-z)^{s-1}$ *is defined on the* $\mathbf{C} - (0, \infty)$ *as* $e^{(s-1)\log(-z)}$ *with* $-\pi < \mathrm{Im}\left(\log(-z)\right) < \pi$

Proof: The integral is obviously convergent. By Cauchy's theorem its value does not depend on the shape of C as long as C does not enclose any limit we are left with an integral back and forth along the positive real axis.

On the upper edge $(-z)^{s-1} = e^{(s-1)(\log z - \pi i)} = x^{s-1} e^{-(s-1)\pi i}$ *and on the lower edge* $(-z)^{s-1} = e^{(s-1)(\log z + \pi i)}\ x^{s-1} e^{(s-1)\pi i}$*. We obtain*

$$\begin{aligned} \int_C \frac{(-z)^{s-1}}{e^z - 1} dz &= -\int_0^\infty \frac{x^{s-1} e^{-(s-1)\pi i}}{e^x - 1} dx + \int_0^\infty \frac{x^{s-1} e^{(s-1)\pi i}}{e^x - 1} dx \\ &= 2i \sin((s-1)\pi) \zeta(s) \Gamma(s) \\ &= -2\pi i \frac{\zeta(s)}{\Gamma(1-s)} \end{aligned}$$

Theorem 2.4. *The* $\zeta(s)$ *can be extended to a meromorphic funcion in the whole plan whose only pole is a simple pole at* $s = 1$ *with the residue 1.*

Proof: It is indeed quite obvious that the integral $\int_C \frac{(-z)^{s-1}}{e^z - 1} \, \mathrm{d}z$ *is an entire function of s, while* $\Gamma(1-s)$ *is meromorphic with poles at* $s = 1, 2, \ldots$*. Because* $\zeta(s)$ *is already known to be analytic for* $\sigma > 1$*, the poles at the integers* $n \geq 2$ *must cancel against zeros of the integral.*

At $s = 1, -\Gamma(1-s)$ *has a simple pole with the residue 1. On the other hand,*

$$\frac{1}{2\pi i}\int_C \frac{dz}{e^z - 1} = 1$$

by residues, so $\zeta(s)$ *has the residue 1.*

Theorem 2.5. *The values* $\zeta(-n)$ *at the negative integers and zero can be evaluated explicitly. Recall the expansion*

$$\frac{1}{e^z-1} = \frac{1}{z} - \frac{1}{2} + \sum_{k=1}^{\infty}(-1)^{k-1}\frac{B_k}{(2k)!}z^{2k-1}$$

From (59)

$$\zeta(-n) = (-1)^n\frac{n!}{2\pi i}\int_C \frac{z^{-n-1}}{e^z-1}dz$$

Hence $\zeta(-n)$ *is equal to* $(-1)^n n!$ *times the coefficient of* z^n *in (60), and*

§2.3 Functional Equation

Theorem 2.6. *We have*

$$\zeta(s) = 2^s\pi^{s-1}\sin\frac{\pi s}{2}\Gamma(1-s)\zeta(1-s)$$

It is equivalent that

$$\zeta(1-s) = 2^{1-s}\pi^{-s}\cos\frac{\pi s}{2}\Gamma(s)\zeta(s)$$

Corollary 2.7. *The function*

$$\xi(s) = \frac{1}{2}s(1-s)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

is entire and satisfies

$$\xi(s) = \xi(1-s)$$

Chapter IV

Conformal Mappings

§1 Conformal Mappings

Definition 1.1. *A bijective holomorphic function $f : U \longrightarrow V$ is called a* ***conformal map*** *or biholomorphism. Given such a mapping f, we say that U and V are* ***conformally equivalent*** *or simply biholomorphic.*

Definition 1.2. *A conformal map from an open set Ω to itself is called an automorphism of Ω. The set of all automorphisms of Ω is denoted by* $\mathrm{Aut}(\Omega)$,*and carries the structure of a group.*

Theorem 1.3. *If $f : U \longrightarrow V$ is holomorphic and injective, then $f' \neq 0$ for all $z \in U$. In particular, the inverse of f defined on its range is holomorphic, and thus the inverse of a conformal map is also holomorphic.*

In particular, the inverse of f defined on its range is holomorphic, and thus the inverse of a conformal map is also holomorphic (conformal).

Proof: We argue by contradiction, and suppose that $f'(z_0) = 0$ for some $z_0 \in U$. Then

$$f(z) - f(z_0) = \frac{f^k(z_0)}{k!}(z - z_0)^k + G(z) \quad \textit{for all } z \text{ near } z_0$$

with $f^k(z_0) \neq 0, k \geq 2$ and G vanishing to order $k+1$ at z_0. For sufficiently small $|w| < \left|\frac{f^k(z_0)}{k!}\right|^{\frac{1}{k}}$, we write

$$f(z) - f(z_0) - w = F(z) + G(z)$$

where $F(z) = \frac{f^k(z_0)}{k!}(z - z_0)^k - w$. Let $M\,|w|^{1/k} > r > \left|\frac{wk!}{f^{(k)}(z_0)}\right|^{1/k}$, then $|G(z)| < \varepsilon_1 < |w| - \varepsilon_2 < |F(z)|$ on circle $\partial B(z_0, r)$ (For sufficiently small $|w|$), and F has at least two zeros in $B(z_0, r)$. Rouche's theorem implies that $f(z) - f(z_0) - w = F + G$ has at least two zeros in $B(z_0, r)$. Since $f'(z) \neq 0$ for all $z \in B(z_0, r) - \{z_0\}$, it follows that the roots of $f(z) - f(z_0) - w$ are distinct, hence f is not injective, a contradiction.

Now let $g = f^{-1}$ denote the inverse of f on its range $f(U)$. Suppose $f(z_0) = w_0 \in f(U)$ and

$f(z) = w$ is close to w_0. If $w \neq w_0$, we have

$$\frac{g(w)-g(w_0)}{w-w_0} = \frac{1}{\frac{w-w_0}{g(w)-g(w_0)}} = \frac{1}{\frac{f(z)-f(z_0)}{z-z_0}}$$

Since $f'(z_0) \neq 0$, we may let $w \to w_0$ that implies $z \to z_0$ and conclude that g is holomorphic at w_0 with $g'(w_0) = 1/f'(g(w_0))$.

§2 The Schwarz Lemma

Theorem 2.1 (Schwarz lemma)**.** *Let $f : B(0,1) \longrightarrow B(0,1)$ be holomorphic with $f(0) = 0$. Then*

$$|g(z)| = \left|\frac{f(z)}{z}\right| \leq 1 \quad \text{for all } z \in \mathbb{D}$$

If for some $z_0 \in \mathbb{D}$ we have $|f(z_0)| = |z_0|$, then f is a rotation.

Theorem 2.2 (Aut $(\mathbb{D})$)**.** *If f is an automorphism of the unit disc, then there exist $\theta \in \mathbb{R}$ and $\alpha \in \mathbb{D}$ such that*

$$f(z) = e^{\theta}\frac{z-\alpha}{\overline{\alpha}z-1}$$

Theorem 2.3 (Aut $(\mathbb{H})$)**.** *Every automorphism of upper half-plane $\mathbb{H}$ takes the form*

$$f_M(z) = \frac{az+b}{cz+d}$$

for some

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(R)$$

Conversely, every map of this form is an automorphism of H.

Proof: Step 1. If $M \in SL_2(R)$, then f_M maps H to itself. This is clear from the observation that

Step 2. If M and M' are two matrices in G, then $f_M \circ f_{M'} = f_{MM'}$. As a consequence, we can prove the first half of the theorem. Each f_M is an automorphism because it has a holomorphic inverse $f_{M^{-1}}$.

Step 3. Given any two points w_1 and w_2 in $\mathbb{H}$, there exists $M \in SL_2(R)$ such that $f_M(w_1) = w_2$, and therefore G acts transitively on $\mathbb{H}$. (Indeed, let $f(z) = \frac{z+Re(w_2)-Re(w_1)}{Im(w_1)/Im(w_2)}$)

Step 4.

Step 5. We can now complete the proof of the theorem. We suppose f is an automorphism of $\mathbb{H}$ with $f(\beta) = i$, and consider a matrix $N \in G$ such that $f_N(i) = \beta$. Then $g = f \circ f_N$ satisfies $g(i) = i$, and therefore $F \circ g \circ F^{-1}$ is an automorphism of the disc that fixes the origin. So $F \circ g \circ F^{-1}$ is a rotation, and by Step 4 there exists R such that

Therefore, if we identify the two matrices M and $-M$, then we obtain a new group $PSL_2(R)$ calledthe projective special linear group; this group is isomorphic with $Aut(\mathbb{H})$.

§3 The Riemann mapping theorem

Theorem 3.1. *Given any simply connected region Ω which is not the whole palne, and a point $z_0 \in \Omega$, there exists a unique analytic funcion $f(z)$ in Ω, normalized by the conditions $f(z_0) = 0$, $f'(z_0) > 0$, such that f defines a one-to-one mapping of Ω to the disk $|w| < 1$*

§4 The Schwarz-Christoffel Formula

§4.1 Mapping unit disk to polygons

Theorem 4.1. *The funcion $w = f(z)$ which maps $|z| < 1$ conformally onto polygons with angles $\alpha_k\pi$ are of the form*

$$f(z) = C\int_0^z \prod_{k=1}^{n} (z - z_k)^{\alpha_k - 1}\, dz + C'$$

where z_k are points on the $|z| = 1$, and C, C' are complex constants.

§4.2 Mapping Upper-half Plane to Polygons

Theorem 4.2. *If funcion $w = f(z)$ which maps $\mathbb{H} = \{z : \mathrm{Im} z > 0\}$ conformally onto the inside of a polygons G with vertices $w_1, w_2, \ldots, w_n$ and interior angles $\alpha_k\pi$ of vertex w_k. Suppoes that $z_k \in \mathbb{R}$ correspond to w_k that; $f(z_k) = w_k$, and $-\infty < z_1 < z_2 < \ldots < z_n \leq \infty$. Then f is of the form*

(1)

$$f(z) = C\int_0^z \prod_{k=1}^{n} (\zeta - z_k)^{\alpha_k - 1}\, d\zeta + C'$$

if $|z_k| < \infty$.

(2) Otherwise, if $z_n = \infty$, the form of f is

$$f(z) = C\int_0^z \prod_{k=1}^{n-1} (\zeta - z_k)^{\alpha_k - 1}\, d\zeta + C'$$

Theorem 4.3 (Uniqueness in a way)**.** *Let z_1, z_2, z_3 belong to $\mathbb{R}$ and polygons with vertexs w_k (regardless of the order). Then there exists a unique $f \in C(\overline{\mathbb{H}}) \bigcap H(\mathbb{H})$ that maps $\mathbb{H}$ conformally onto the inside of a polygons with $f(z_i) = w_i$ ($i = 1, 2, 3$).*

§5

Theorem 5.1.

$$\varphi_a(z) = \frac{z - a}{\bar{a}z - 1}$$

which maps a to 0, 0 to a and $\varphi \circ \varphi(z) = z$.

Theorem 5.2. *Map the complement of a line segment onto the inside (of outside) of a circle.*

$$\overline{\mathbb{C}} - (a, b) \longrightarrow \overline{\mathbb{C}} - (-\infty, 0)$$

where $a, b \in \mathbb{R}$ *by*

$$f(z) = \frac{z - a}{b - z}$$

Theorem 5.3. *Let*

$$f(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

maps $z = \rho e^{i\theta}$ *to*

$$x = \frac{1}{2}\left(\rho + \frac{1}{\rho}\right)\cos\theta$$
$$y = \frac{1}{2}\left(\rho - \frac{1}{\rho}\right)\sin\theta$$

Elimination of θ *yields*

$$\frac{x^2}{\left[\frac{1}{2}(\rho + \rho^{-1})\right]^2} + \frac{y^2}{\left[\frac{1}{2}(\rho - \rho^{-1})\right]^2} = 1$$

Chapter V

Elliptic Functions

§1 Elliptic functions

Definition 1.1. *There are two non-zero complex numbers ω_1 and ω_2 such that*

$$f(z+w_1) = f(z) \quad \textit{and} \quad f(z+w_2) = f(z)$$

for all $z \in \mathbb{C}$. A function with two periods is said to be doubly periodic.

If the periods w_1 and w_2 are linearly independent over $\mathbb{R}$, we now describe a normalization. Let $\tau = \omega_2/\omega_1$ and assume (after possibly interchanging the roles of ω_1 and ω_2) that $Im(\tau) > 0$.

It is therefore natural to consider the lattice in $\mathbb{C}$ defined by

$$\Lambda = \{n\omega_1 + m\omega_2 : \ n, m \in \mathbb{Z}\}$$

We say that 1 and τ generate $Lambda$.

Associated to the lattice Λ is the fundamental parallelogram defined by

$$P_0 = \{z \in \mathbb{C} : z = a + b\tau \textit{ where } 0 \leq a < 1 \textit{ and } 0 \leq b < 1\}$$

A period parallelogram P is any translate of the fundamental parallelogram, $P = P_0 + h$ with $h \in \mathbb{C}$

Two complex numbers z and w are congruent modulo Λ if $z - w \in \Lambda$, and we write $z \sim w$

Proposition 1.2. *Suppose f is a meromorphic function with two periods 1 and τ which generate the lattice Λ. Then:*

(i) Every point in $\mathbb{C}$ is congruent to a unique point in any given period parallelogram (fundamental parallelogram).

(ii) The lattice Λ provides a disjoint covering of the complex plane, in the sense of

$$\mathbb{C} = \bigcup_{h \in \Lambda} P_0 + h$$

(iii) The function f is completely determined by its values in any period parallelogram.

(iv) The number of poles of f is same in all period parallelograms.

Theorem 1.3. *An entire doubly periodic function is constant.*

Definition 1.4. *A non-constant doubly periodic meromorphic function is called an* ***elliptic funnc-tion****.*

Theorem 1.5. *The total number of poles of an elliptic function in P_0 is always ≥ 2.*

Proof: Suppose first that f has no poles on the boundary ∂P_0. By the residue theorem and period of f we have

$$2\pi i \sum \mathrm{res} f = \int_{\partial P_0} f(z)\,\mathrm{d}z = 0$$

Therefore f must have at least two poles in P_0.

If f has a pole on ∂P_0 choose a small $h \in \mathbb{C}$ so that if $P = h + P_0$, then f has no poles on ∂P. Arguing as before, we find that f must have at least two poles in P, and therefore the same conclusion holds for P_0.

Theorem 1.6. *The total number of poles (counted according to their multiplicities) of an elliptic function in (P_0) is called its order.*

Theorem 1.7. *Every elliptic function of order m has m zeros in P_0.*

Proof: Assuming first that f has no zeros or poles on the boundary ∂P of P, we know by the argument principle and periodicity of f that

$$0 = \int_{\partial P_0} \frac{f'(z)}{f(z)}\,\mathrm{d}z = 2\pi i(N_z - N_p)$$

In the case when a pole or zero of f lies on ∂P_0 it suffices to apply the argument to a translate of P.

Corollary 1.8. *If f is elliptic then the equation $f(z) = c$ has as many solutions as the order of f for every $c \in \mathbb{C}$.*

§2 The Weierstrass $\wp$ function

Definition 2.1. *Let Λ^* denote the lattice $\Lambda = \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}$ minus the origin. The* ***Weierstrass*** *$\wp(z;\Lambda)$* ***function****, which is given by the series*

$$\wp(z;\Lambda) = \frac{1}{z^2} + \sum_{\omega \in \Lambda^*} \left[\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right]$$

Theorem 2.2. *The function $\wp$ is an elliptic function that has periods ω_1 and ω_1, and double poles at the lattice points.*

Proof: Step 1. To see this, suppose that $|z| < R$, and write

$$\wp(z;\Lambda) = \frac{1}{z^2} + \sum_{|\omega| \leq 2R} \left[\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right] + \sum_{|\omega| > 2R} \left[\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right]$$

The term in the second sum is $O(1/|\omega|^3)$ uniformly for $|z| < R$, so by Lemma 1.5 this second sum defines a holomorphic function in $B(0,R)$. Finally, note that the first sum exhibits double poles at the lattice points in the disc $B(0,R)$.

Step 2. To prove that $\wp$ is periodic with the correct periods, note that the derivative is given by differentiating the series for $\wp$ termwise so

$$\wp'(z) = -2 \sum \frac{1}{(z-\omega)^3}$$

This accomplishes two things for us. First, the differentiated series converges absolutely whenever z is not a lattice point, by the case r = 3 of Lemma 1.5. Second, the differentiation also eliminates the subtraction term $1/\omega^2$; therefore the series for $\wp'$ is clearly periodic with periods 1 and τ, that is, $\wp'(z+w_1) = \wp'(z)$ and $\wp'(z+w_2) = \wp'(z)$. Hence, there are two constants a and b such that

$$\wp(z+w_1) = \wp(z) + a \quad \textit{and} \quad \wp(z+w_2) = \wp(z) + b$$

It is clear from the definition, however, that $\wp$ is even, since the sum over $\omega \in \Lambda$ can be replaced by the sum over $-\omega \in \Lambda$. Therefore $\wp(-w_1/2) = \wp(w_1/2)$ and $\wp(-w_2/2) = \wp(w_2/2)$, respectively, in the two expressions above proves that $a = b = 0$.

§2.1

Theorem 2.3 (Legendre's relation)**.** *Since $\wp$ has zero residues, it is the derivative of a single-valued function denote $-\zeta$*

$$\zeta(z) = \frac{1}{z} + \sum_{\omega \neq 0} \frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2}$$

It is clear that ζ satisfies conditions

$$\zeta(z+\omega_1) = \zeta(z) + \eta_1, \ \zeta(z+\omega_2) = \zeta(z) + \eta_2$$

since $\zeta' = -\wp$. We choose any $a \neq 0$ and observe that

$$\frac{1}{2\pi i} \int_{\partial P_a} \zeta(z) \, \mathrm{d}z = 1$$

by residue theorem, and obtain

$$\eta_1 \omega_2 - \eta_2 \omega_1 = 2\pi i$$

*known as **Legendre's relation**.*

Theorem 2.4. *The canonical product associated with* Λ

$$\sigma(z) = z \prod_{\omega \neq 0} \left(1 - \frac{z}{\omega}\right) e^{\frac{z}{\omega} + \frac{1}{2}\left(\frac{z}{\omega}\right)^2}$$

converges and represent an entire function which satisfies

$$\frac{\sigma'(z)}{\sigma(z)} = \frac{d \log \sigma(z)}{dz} = \zeta(z)$$

and

$$\sigma(z+\omega_1) = -\sigma(z) e^{\eta_1 (z + \frac{\omega_1}{2})}, \ \sigma(z+\omega_2) = -\sigma(z) e^{\eta_2 (z + \frac{\omega_2}{2})}$$

Proof: Then we have

$$\frac{\sigma'(z+\omega_1)}{\sigma(z+\omega_1)} = \frac{\sigma'(z)}{\sigma(z)} + \eta_1$$

it is follows at once that

$$\sigma(z+\omega_1) = C\sigma(z) e^{\eta_1 z}$$

On setting $z = -\omega_1/2$ *the value of* C *can be determined, and we find that*

$$\sigma(z+\omega_1) = -\sigma(z) e^{\eta_1 (z + \frac{\omega_1}{2})}$$

Similarly, it is also that

$$\sigma(z+\omega_2) = -\sigma(z) e^{\eta_2 (z + \frac{\omega_2}{2})}$$

§2.2

Proposition 2.5. *Let* $\wp(z; \Lambda)$

(1) $\wp$ *is even and* $\wp'$ *is old.*

(2) $\wp'$ *vanish at* $\frac{1}{2}\Lambda$ *and*

(3)

$$\wp\left(\frac{1}{2}\Lambda\right) = \left\{\wp\left(\frac{\omega_1}{2}\right), \wp\left(\frac{\omega_2}{2}\right), \wp\left(\frac{\omega_1+\omega_2}{2}\right)\right\}$$

(4) If we define

$$\wp\left(\frac{\omega_1}{2}\right) = e_1, \quad \wp\left(\frac{\omega_2}{2}\right) = e_2 \quad \textit{and} \quad \wp\left(\frac{\omega_1+\omega_2}{2}\right) = e_3$$

we conclude that the equation $\wp(z) = e_i$ *has a double root* $(\wp'(e_i) = 0)$. *Since* $\wp$ *has order* 2, *there are no other solutions to the equation* $\wp(z) = e_i$ *in the fundamental parallelogram. In particular, the three numbers* e_1, e_2 *and* e_3 *are distinct.*

Theorem 2.6. *The function* $(\wp')^2$ *is the cubic polynomial in* $\wp$

$$(\wp')^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3)$$

§3 The Representation of Elliptic Function

Theorem 3.1. *Any elliptic function with period ω_1, ω_2 can be written as*

$$C\prod_{k=1}^{n}\frac{\sigma(z-a_k)}{\sigma(z-b_k)}$$

where n is the order and a_k are congruence class of all zeros, b_k are congruence class of all poles that satisfies $\sum a_k = \sum b_k$.

Lemma 3.2. *Every even elliptic function F with periods ω_1 and ω_1 is a rational funcion of $\wp$.*

Proof: If F has a zero or pole at the origin it must be of even order, since F is an even function. As a consequence, there exists an integer m so that $F\wp^m$ has no zero or pole at the lattice points. We may therefore assume that F itself has no zero or pole on Λ.

If a is a zero of F, then so is $-a$, since F is even. If the points $a_1, -a_1 \cdots, a_m, -a_m$ counted with multiplicities (modulo Λ) describe all the zeros of F has precisely the same roots as F. A similar argument, where $b_1, -b_1, \cdots, b_m, -b_m$ (with multiplicities) describe all the poles of F, then shows that

$$G(z) = \frac{[\wp(z)-\wp(a_1)]\cdots[\wp(z)-\wp(a_m)]}{[\wp(z)-\wp(b_1)]\cdots[\wp(z)-\wp(b_m)]}$$

is periodic and has the same zeros and poles as F. Therefore, F/G is holomorphic and doubly-periodic, hence constant. This concludes the proof of the lemma.

Theorem 3.3. *Every elliptic function f with periods ω_1 and ω_2 is a rational function of $\wp$ and $\wp'$.*

Proof: We first recall that $\wp$ is even while $\wp'$ odd. We then write f as a sum of an even and an odd function

$$f = f_{even} + f_{ood}$$

Then, since $f_{ood}/\wp'$ is even, it is clear from the lemma applied to f_{even} and $f_{ood}/\wp'$ that f is a rational function of $\wp$ and $\wp'$.

Theorem 3.4.

$$\wp(z)-\wp(u) = -\frac{\sigma(z-u)\sigma(z+u)}{\sigma(z)^2\sigma(u)^2}$$

Taking logarithmic derivatives, then

$$\frac{\wp'(z)}{\wp(z)-\wp(u)} = \zeta(z-u)+\zeta(z+u)-\zeta(2z)$$

If we change z and u and add them

$$\zeta(z+u) = \zeta(z)+\zeta(z)+\frac{1}{2}\frac{\wp'(z)-\wp'(zu)}{\wp(z)-\wp(u)}$$

Theorem 3.5 (Second derivative)**.**

$$\wp'' = 6\wp^2-\frac{1}{2}g_2$$

Theorem 3.6 (Addition theorem for $\wp$)**.**

$$\wp(z+u) = -\wp(z) - \wp(u) + \frac{1}{4}\left(\frac{\wp'(z) - \wp'(zu)}{\wp(z) - \wp(u)}\right)^2$$

Proof: We recall that

$$\zeta(z+u) = \zeta(z) + \zeta(z) + \frac{1}{2}\frac{\wp'(z) - \wp'(zu)}{\wp(z) - \wp(u)}$$

Taking derivative

$$-\wp(z+u) = -\wp(z) + \frac{1}{2}\frac{\wp''(z)(\wp(z) - \wp(u)) - \wp'(z)(\wp'(z) - \wp'(u))}{(\wp(z) - \wp(u))^2}$$

then change z and u, we

$$\begin{aligned}
&-2\wp(z+u) \\
&= -\wp(z) - \wp(u) + \frac{1}{2}\frac{(\wp''(z) - \wp''(u))(\wp(z) - \wp(u)) - (\wp'(z) - \wp'(u))^2}{(\wp(z) - \wp(u))^2} \\
&= 2(\wp(z) + \wp(u)) + \frac{1}{2}\left(\frac{\wp'(z) - \wp'(u)}{\wp(z) - \wp(u)}\right)^2
\end{aligned}$$

by $\wp'' = 6\wp^2 - \frac{1}{2}g_2$.

Corollary 3.7.

$$\wp(2z) = -2\wp(z) + \frac{1}{4}\left(\frac{\wp''(z)^2}{\wp'(z)^2}\right)^2$$

§4 The Differential Equation

Definition 4.1. *The **Eisenstein series** of order k is defined by*

$$E_k(\tau) = \sum_{\omega \neq 0} \frac{1}{\omega^{2k}}$$

whenever k is an integer ≥ 2.

Theorem 4.2. *Eisenstein series have the following properties:*

(i) The series $E_k(\tau)$ converges if $k \geq 2$, and is holomorphic in the upper half-plane.

(ii) $E_k(\tau)$ satisfies the following transformation relations:

$$E_K(\tau+1) = E_K(\tau) \quad \textit{and} \quad E_K(\tau) = \tau^{-k}E_K(-1/\tau)$$

The last property is sometimes referred to as the modular character of the Eisenstein series.

Theorem 4.3. *For z near 0, we have*

$$\begin{aligned}\wp(z) &= \frac{1}{z^2} + 3E_2 z^2 + 5E_3 z^4 + \cdots \\ &= \frac{1}{z^2} + \sum_{k=2}^{\infty}(2k-1)E_k z^{2k-2}\end{aligned}$$

Proof: From the definition of $\wp$, if we note that we may replace ω by $-\omega$ without changing the sum, we have

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega\in\Lambda^*}\left[\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2}\right] = \frac{1}{z^2} + \sum_{\omega\in\Lambda^*}\left[\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right]$$

The identity

$$\frac{1}{(1-w)^2} = \sum_{\ell=0}^{\infty}(\ell+1)w^{\ell}, \quad \textit{for } |w|<1$$

implies that for all small z

$$\frac{1}{(z-\omega)^2} = \frac{1}{\omega^2}\sum_{\ell=0}^{\infty}(\ell+1)\left(\frac{z}{\omega}\right)^{\ell} = \frac{1}{\omega^2} + \frac{1}{\omega^2}\sum_{\ell=1}^{\infty}(\ell+1)\left(\frac{z}{\omega}\right)^{\ell}$$

Therefore

$$\begin{aligned}\wp(z) &= \frac{1}{z^2} + \sum_{\omega\in\Lambda^*}\sum_{\ell=1}^{\infty}(\ell+1)\frac{z^{\ell}}{\omega^{\ell+2}} \\ &= \frac{1}{z^2} + \sum_{\ell=1}^{\infty}(\ell+1)\left(\sum_{\omega\in\Lambda^*}\frac{1}{\omega^{\ell+2}}\right)z^{\ell} \\ &= \frac{1}{z^2} + \sum_{k=2}^{\infty}(2k-1)E_k z^{2k-2}\end{aligned}$$

Corollary 4.4 (Differential Equation)**.** *If $g_2 = 60E_2$ and $g_3 = 140E_3$, then*

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3$$

Proof: From the previous theorem, we obtain the following three expansions for z near 0

$$\begin{aligned}\wp'(z) &= \frac{-2}{z^3} + 6E_4 z + 20E_6 z^3 + \cdots, \\ (\wp'(z))^2 &= \frac{4}{z^6} - \frac{24E_4}{z^2} - 80E_6 + \cdots, \\ (\wp(z))^3 &= \frac{1}{z^6} + \frac{9E_4}{z^2} + 15E_6 + \cdots\end{aligned}$$

From these, one sees that the difference $(\wp'(z))^2 - 4(\wp(z))^3 + 60E_4\wp(z) + 140E_6$ is holomorphic near 0, and in fact equal to 0 at the origin. Since this difference is also doubly periodic, we conclude that it is constant, and hence identically 0.

§5 Modular Function

Definition 5.1. *Notes that* $e_1 = \wp\left(\frac{\omega_1}{2}\right), e_2 = \wp\left(\frac{\omega_2}{2}\right)$ *and* $e_3 = \wp\left(\frac{\omega_1+\omega_2}{2}\right)$ *are homogeneous of order* -2 *in* ω_1, ω_2. *We conclude taht the quantity*

$$\lambda(\tau) = \frac{e_3 - e_2}{e_1 - e_2}$$

depends on the $\tau = \frac{\omega_2}{\omega_1}$ *is analytic in the half plan* $\operatorname{Im}\tau > 0$.

Theorem 5.2.

$$\lambda\left(\frac{a\tau + b}{c\tau + d}\right) = \lambda(\tau)$$

for all

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod 2$$

Theorem 5.3. *Under a modular transformation*

(1)

$$\begin{pmatrix} \omega_2' \\ \omega_1' \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \omega_2 \\ \omega_1 \end{pmatrix}$$

we have $e_1' = e_1$, $e_2' = e_3$, $e_3' = e_2$ *and* $\tau' = \tau + 1$

$$\lambda(\tau + 1) = \lambda(\tau') = \frac{e_3' - e_2'}{e_1' - e_2'} = \frac{e_2 - e_3}{e_1 - e_3} = \frac{\lambda(\tau)}{\lambda(\tau) - 1}$$

(2)

$$\begin{pmatrix} \omega_2' \\ \omega_1' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \omega_2 \\ \omega_1 \end{pmatrix}$$

we have

$$\lambda\left(-\frac{1}{\tau}\right) = 1 - \lambda(\tau)$$

Chapter VI

Global Analytic Function

§1 Picard's Theorem

Theorem 1.1 (Picard's Theorem)**.** *An entire funcion with more than one finite lacunary value reduces to a conatant.*

Proof: Suppoes that f has two finite lacunary values 0 and 1. Considering the modular funcion

$$\tau : \mathbb{H} \to \mathbb{C} - \{0, 1\}$$

is a covering map and holomorphic, where $\mathbb{H} = \{z : \operatorname{Im}(z) > 0\}$. By lifting theorem (since domain $\mathbb{C}$ is simply connected), there is holomorphic funcion

$$\tilde{f} : \mathbb{C} \to \mathbb{H}$$

that

$$\tau \circ \tilde{f} = f$$

$$\begin{tikzcd} \mathbb{C} \arrow[r, "\tilde{f}"] \arrow[dr, "f"'] & \mathbb{H} \arrow[d, "\tau"] \\ & \mathbb{C}\backslash\{0,1\} \end{tikzcd}$$

Thus

$$\phi \circ \tilde{f} : \mathbb{C} \to \mathbb{D}$$

is a bounded entire funcion, where

$$\phi(z) = \frac{z - i}{z + i} : \mathbb{H} \to \mathbb{D}$$

Therefore, $\tilde{f}$ must be a constant and $f = \tau \circ \tilde{f}$