Partial Differential Equatons

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§1.1 Preliminaries and notation

Throughout the text X denotes a subset of $\mathbb{R}^n$ and $\Omega \subset \mathbb{R}^n$ an open set. We use $C(X)$ for the space of real-valued continuous functions on X (when X is equipped with the subspace topology of $\mathbb{R}^n$). For a multi-index $\alpha \in \mathbb{N}^n$ we write $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $\partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$.

§1.2 The space C^m

Lemma 1.1. *Let X be subset of $\mathbb{R}^n$.*

1. *Then*
$$C_b(X) = \left\{ f \in C(X) : \sup_{x \in X} |f(x)| < \infty \right\}$$
is a Banach space with the norm $\|f\|_{C_b(X)} := \sup_{x \in X} |f(x)|$.

   Remark. *Noted that if X is compact, then $C_b(X) = C(X)$.*

2. *One can view $C\left(\overline{X}\right)$ as the set*
$$\left\{ f \in C(X) : \exists \text{ continuous } \tilde{f} : \overline{X} \to \mathbb{R} \text{ s.t. } \tilde{f}\Big|_X = f \right\}$$

   Remark. *The extension $\tilde{f}$ is unique if it exists thus we identify f with $\tilde{f}$ if f can be extended to $\overline{X}$.*

Definition 1.2. *Let Ω be a open set in $\mathbb{R}^n$, $m \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ and $0 < \gamma \leq 1$*

1. *The space*
$$C^m(\Omega) := \{u : \Omega \to \mathbb{R} \mid \partial^\alpha u \text{ exists and is continuous for all } |\alpha| \leq m\}$$

2. *The space*
$$C_b^m(\Omega) := \{u \in C^m(\Omega) \mid \partial^\alpha u \text{ bounded for all } |\alpha| \leq m\}$$
is a Banach space with the norm $\|u\|_{C_b^m(\Omega)} := \sum_{|\alpha| \leq m} \sup_{x \in \Omega} |\partial^\alpha u(x)|$

3. *The space*
$$C^m(\bar{\Omega}) := \left\{u \in C^m(\Omega) \mid \partial^\alpha u \text{ admits a continuous extension to } \bar{\Omega} \text{ for all } |\alpha| \leq m\right\}$$

4. *The space*
$$C_b^m\left(\bar{\Omega}\right) := C^m(\bar{\Omega}) \cap C_b^m(\Omega)$$

Remark. *Sometimes, we write $\left(C^m(\Omega), \|\cdot\|_{C^m(\Omega)}\right)$ to denote $\left(C_b^m(\Omega), \|\cdot\|_{C_b^m(\Omega)}\right)$, and $\left(C^m(\bar{\Omega}), \|\cdot\|_{C^m(\bar{\Omega})}\right)$ to denote $\left(C_b^m(\bar{\Omega}), \|\cdot\|_{C_b^m(\bar{\Omega})}\right)$ for simplicity.*

§1.3 Hölder spaces $C^{k,\gamma}\left(\bar{\Omega}\right)$

Definition 1.3. *Let Ω be a open set in $\mathbb{R}^n$, $m \in \mathbb{Z}_{\geq 0}$ and $0 < \gamma \leq 1$. If $u : \Omega \to \mathbb{R}$, we define*

1. *The* γ^{th} ***-Hölder seminorm*** *of* u *is*

$$[u]_{C^{0,\gamma}(\Omega)} := \sup_{\substack{x,y\in\Omega\\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^{\gamma}}$$

2. *The* ***Hölder space***

$$C^{k,\gamma}(\Omega)$$

consists of all functions $u \in C^k(\bar{\Omega})$ *for which the norm*

$$\|u\|_{C^{k,\gamma}(\Omega)} := \sum_{|\alpha|\leq k} \|D^{\alpha}u\|_{C(\Omega)} + \sum_{|\alpha|=k} [D^{\alpha}u]_{C^{0,\gamma}(\Omega)}$$

is finite.

Remark. *Noted that* $C^{k,\gamma}(\Omega) \subset C^k\left(\bar{\Omega}\right)$ *since* u^{β} *is uniformly continuous for all* $|\beta| = k$*. And if* $u \in C^{k,\gamma}(\Omega)$ *is extended to* $\bar{\Omega}$*, then the extension is also in* $C^{k,\gamma}\left(\bar{\Omega}\right)$*. Thus*

$$C^{k,\gamma}(\Omega) = C^{k,\gamma}\left(\bar{\Omega}\right) := \left\{u \in C^k\left(\bar{\Omega}\right) : \|D^{\alpha}u\|_{C(\bar{\Omega})}, [u]_{C^{0,\gamma}\left(\bar{\Omega}\right)} < \infty\right\}$$

and the norms are equal. We use $C^{k,\gamma}(\Omega)$ *and* $C^{k,\gamma}\left(\bar{\Omega}\right)$ *interchangeably, but always mean the latter.*

Theorem 1.4. *The space* $C^{k,\gamma}(\bar{\Omega})$ *is a Banach space*

§2 Positive Sobolev Spaces $W^{m,p}$

Definition 2.1. *Let* Ω *be an open set in* $\mathbb{R}^n$*,* $1 \leq p \leq \infty$ *and* $k \in \mathbb{Z}_{\geq 0}$*.*

1. *The* ***Sobolev space***

$$W^{m,p}(\Omega) = \{f \in \mathscr{D}'(\Omega) : D^{\alpha}f \in L^p(\Omega), 0 \leq |\alpha| \leq k\}$$

with the norm

$$\|u\|_{W^{m,p}(\Omega)} := \begin{cases} \left(\sum_{|\alpha|\leq k} \int_{\Omega} |D^{\alpha}u|^p\, dx\right)^{1/p} & (1 \leq p < \infty) \\ \sum_{|\alpha|\leq k} \operatorname*{ess\,sup}_{\Omega} |D^{\alpha}u| & (p = \infty) \end{cases}$$

2. $H^{m,p}(\Omega) \equiv$ *the completion of* $\left\{u \in C^m(\Omega) : \|u\|_{m,p} < \infty\right\}$ *with respect to the norm* $\|\cdot\|_{m,p}$
3. $W_0^{m,p}(\Omega) \equiv$ *the closure of* $C_0^{\infty}(\Omega)$ *in the space* $W^{m,p}(\Omega)$*.*

Theorem 2.2. *Let* $k \in \mathbb{Z}_{\geq 0}$ *and* $p \in [1, \infty]$*, then*

1. *$W^{m,p}(\Omega)$ is a Banach space.*

2. *$W^{m,p}(\Omega)$ is separable if $1 \leq p < \infty$.*

3. *$W^{m,p}(\Omega)$ is uniformly convex and reflexive if $1 < p < \infty$.*

4. *In particular, $W^{m,2}(\Omega)$ is a separable Hilbert space with inner product*

$$(u, v)_{W^{m,2}(\Omega)} = \sum_{0 \leq |\alpha| \leq m} (D^\alpha u, D^\alpha v)_{L^2} \tag{1}$$

Proof. Step 1. Let $\{f_n\}$ be a Cauchy sequence in $W^{m,p}(\Omega)$, then $\{D^\alpha f_n\}$ are Cauchy sequence in $L^p(\Omega)$ for all index $|\alpha| \leq k$. By the completeness of L^p, there exist f_α such that

$$D^\alpha f_n \to f_\alpha \quad \text{in } L^p(\Omega)$$

Considering in $\mathscr{D}(\Omega)'$, $f_n \to f$ in $\mathscr{D}(\Omega)'$, thus $D^\alpha f_n \to D^\alpha f$ in $\mathscr{D}(\Omega)'$. Then we have $D^\alpha f = f_\alpha$ and

$$f_n \to f \quad \text{in } W^{m,p}(\Omega)$$

Step 2. Suppose that $p < \infty$. Let $N = \sum_{|\alpha|<k} 1$ be the number of all distinct index $|\alpha| \leq k$, define the map

$$u \mapsto \{D^\alpha u\}_{|\alpha| \leq m}$$

from $W^{m,p}(\Omega)$ to $(L^p(\Omega))^N$. Thus $W^{m,p}(\Omega)$ can be viewed as a closed subspace of $(L^p(\Omega))^N$ which is reflexive, then so $W^{m,p}(\Omega)$. □

§3 Negative Sobolev Spaces $W^{-m,p'}(\Omega)$

Definition 3.1. *Let Ω be an open region in $\mathbb{R}^n$, $1 \leq p < \infty$ and $m \in \mathbb{Z}_{\geq 0}$. We define the dual space of $W_0^{m,p}(\Omega)$,*

$$W^{-m,p'}(\Omega) := (W_0^{m,p}(\Omega))^*$$

where $\frac{1}{p'} + \frac{1}{p} = 1$. Noted that $p' \in (1, \infty]$

Theorem 3.2. *Let Ω an open set, $k \in \mathbb{Z}_{\geq 0}$, $p' \in (1, \infty]$, we have*

$$W^{-m,p'}(\Omega) = \left\{ f = \sum_{|\alpha| \leq k} (-1)^{|\alpha|} D^\alpha f_\alpha : f_\alpha \in L^{p'}(\Omega) \right\}$$

where the functional f acts on $\varphi \in \mathscr{D}(\Omega)$ by

$$\langle f, \varphi \rangle = \sum_{|\alpha| \leq m} \int_\Omega f_\alpha D^\alpha \varphi \, \mathrm{d}x$$

and extend to $W_0^{m,p'}(\Omega)$ by density. Moreover, the norm of f in $W^{-m,p'}(\Omega)$ is equivalent to

$$\inf\left\{\left(\sum_{|\alpha|\leq m}\|g_\alpha\|_{L^{p'}(\Omega)}^{p'}\right)^{1/p'} : f = g \text{ in } W^{-m,p'}(\Omega)\right\}$$

§4 Extensions

Theorem 4.1 (Extension Theorem). *Let Ω be a bounded Lipschitz open set; $1 \leq p \leq \infty$. Select a bounded open set V such that $\Omega \subset\subset V$. Then there exists a bounded linear operator*

$$E : W^{k,p}(\Omega) \longrightarrow W^{k,p}(\mathbb{R}^n)$$

such that for each $u \in W^{k,p}(\Omega)$:

(i) $Eu|_\Omega = u$ *in* $W^{k,p}(\Omega)$,

(ii) Eu *has support within* V,

(iii) E *is bounded:* $\|Eu\|_{W^{k,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{k,p}(\Omega)}$ *where the constant* $C = C(n,p,\Omega,V)$ *depends only on* p, Ω, *and* V.

*We call Eu an **extension** of u to $\mathbb{R}^n$.*

Proof. **Step 1: Local flattening of the boundary.** Since Ω is Lipschitz, for each point $x_i^0 \in \partial\Omega$, there exists a radius $r_i > 0$ and a Lipschitz function $\gamma_i : \mathbb{R}^{n-1} \to \mathbb{R}$ such that, upon a suitable rotation of coordinates,

$$\Omega \cap B(x_i^0, r_i) = \{x = (x', x_n) \in B(x_i^0, r_i) : x_n > \gamma_i(x')\}.$$

By compactness of $\partial\Omega$, finitely many such neighborhoods $B(x_i^0, r_i)$ cover $\partial\Omega$.

Step 2: Local extensions. On each neighborhood $B(x_i^0, r_i)$, define a local extension $\tilde{u}_i$ of u to the whole $B(x_i^0, r_i)$ (or a slightly larger set) by reflection across the Lipschitz boundary:

$$\tilde{u}_i(x', x_n) := \begin{cases} u(x', x_n), & x_n > \gamma_i(x'), \\ u(x', 2\gamma_i(x') - x_n), & x_n \leq \gamma_i(x'). \end{cases}$$

Standard estimates show that

$$\|\tilde{u}_i\|_{W^{k,p}(B(x_i^0,r_i))} \leq C\|u\|_{W^{k,p}(\Omega\cap B(x_i^0,r_i))}.$$

Step 3: Partition of unity. Take smooth functions $\{\phi_i\}_{i=0}^N \subset C_c^\infty(\mathbb{R}^n)$ forming a partition of unity subordinate to $\{B(x_i^0, r_i)\}_{i=1}^N$ and an interior set $B_0 \subset \Omega$, such that

$$\phi_0 + \sum_{i=1}^{N} \phi_i = 1 \quad \text{on } \overline{\Omega}.$$

Step 4: Global extension. Define

$$Eu := \phi_0 u + \sum_{i=1}^{N} \phi_i \tilde{u}_i.$$

By construction:

(a) $(Eu)|_\Omega = u$, since on $\Omega \cap B(x_i^0, r_i)$ we have $\tilde{u}_i = u$,

(b) $\operatorname{supp}(Eu) \subset V$, by choosing $B(x_i^0, r_i) \subset V$ and $\operatorname{supp} \phi_i \subset B(x_i^0, r_i)$,

(c) $\|Eu\|_{W^{k,p}(\mathbb{R}^n)} \leq C \, \|u\|_{W^{k,p}(\Omega)}$, by Leibniz formula and finite overlap of supports.

Therefore, E is a bounded linear extension operator satisfying all required properties. □

§5 Density Theorem

§5.1 Approximation by Functions in $C^\infty(\Omega)$

Lemma 5.1 (Local approximation by smooth functions)**.** *Let Ω be an open set, $1 \leq p < \infty$. Then for every $u \in W^{k,p}(\Omega)$ there exist local smooth function u_n such that*

$$u_m \to u \quad \textit{in } W^{k,p}_{\text{loc}}(\Omega)$$

Proof. Define the

$$u^\varepsilon = \eta_\varepsilon * u \quad \text{in } \Omega_\varepsilon$$

Then we have $u^\varepsilon \in C^\infty(\Omega_\varepsilon)$ and $D^\alpha(\eta_\varepsilon * u) = \eta_\varepsilon * D^\alpha u$ in Ω_ε for all $|\alpha| \leq k$ and $\varepsilon > 0$. By the the consequence of mollifier,

$$D^\alpha u^\varepsilon = (D^\alpha u)^\varepsilon \to D^\alpha u \quad \text{in } L^p_{\text{loc}}(\Omega)$$

if we choose any open set $V \subset\subset \Omega$, then

$$\|u^\varepsilon - u\|^p_{W^{k,p}(V)} = \sum_{|\alpha| \leq k} \|D^\alpha(u^\varepsilon - u)\|^p_{L^p(V)} = \sum_{|\alpha| \leq k} \|(D^\alpha u)^\varepsilon - D^\alpha u\|^p_{L^p(V)} \to 0$$

as $\varepsilon \to 0$. This proves that $u_m \to u \quad$ in $W^{k,p}_{\text{loc}}(\Omega)$. □

Theorem 5.2 ($H = W$)**.** *Let Ω be an open set and $1 \leq p < \infty$. Then for every $u \in W^{m,p}(\Omega)$ there exist $u_m \in H^{m,p} = C^\infty(\Omega) \cap W^{m,p}(\Omega)$ such that*

$$u_m \to u \quad \textit{in } W^{m,p}(\Omega)$$

Proof. Step 1. Let $\{V_i\}_{i=1}^\infty$ be an exhaustion by open sets of Ω and $\{\zeta_i\}_{i=1}^\infty$ be a smooth partition of unity subordinate to $\{V_i\}_{i=1}^\infty$. Next, choose any function $u \in W^{m,p}(\Omega)$. We have $\operatorname{supp}(\zeta_i u) \subset \operatorname{supp} \zeta_i \subset V_i$ is compact, $\zeta_i u \in W^{m,p}(\Omega)$, and

$$u = \sum_{i=1}^\infty u^i$$

where $u^i := \zeta_i u$

Step 2. Fix $\delta > 0$. For every i, choose $\varepsilon_i > 0$ small that $\operatorname{supp} u^i \subset V_i^{\varepsilon_i}$ (dist $(\operatorname{supp} u^i, \partial V_i) > 0$) and v^i such that

$$\left\|v^i - u^i\right\|_{W^{m,p}(\Omega)} = \left\|v^i - u^i\right\|_{W^{m,p}(V_i)} \leq \frac{\delta}{2^{i+1}}$$

Write $v := \sum_{i=1}^\infty v^i$. This function is well defined and belongs to $C^\infty(\Omega)$ since $\operatorname{supp} u^i \subset \operatorname{supp} \zeta_i$ and $\{\operatorname{supp} u^i\}$ is locally finite. Then for any compact set $\Omega' \subset \Omega$, we have $\Omega' \subset \bigcup_{i=1}^N V_i$ for some N depending on Ω' and

$$\begin{aligned}
\|v - u\|_{W^{k,p}(\Omega')} &= \left\| \sum_{i=1}^N v^i - \sum_{i=1}^N u^i \right\|_{W^{k,p}(\Omega')} \\
&\leq \sum_{i=1}^\infty \left\|v^i - u^i\right\|_{W^{k,p}(\Omega)} \\
&\leq \delta \sum_{i=1}^\infty \frac{1}{2^i} \\
&= \delta
\end{aligned}$$

Take the supremum over compact sets $\Omega' \subset \Omega$, to conclude $\|v - u\|_{W^{k,p}(\Omega)} \leq \delta$ by Fatou's lemma. □

§5.2 Approximation by functions in $C^\infty(\bar{\Omega})$

Theorem 5.3. *Let Ω be a bounded Lipschitz open set; $1 \leq p < \infty$. Then for erery $u \in W^{k,p}(\Omega)$, then there exist $u_m \in C^\infty(\bar{\Omega})$ such that*

$$u_m \to u \quad \textit{in } W^{k,p}(\Omega)$$

Proof. **Step 1: Extension to the whole space.** Since Ω is Lipschitz, there exists a bounded linear extension operator

$$E : W^{k,p}(\Omega) \to W^{k,p}(\mathbb{R}^n)$$

such that $(Eu)|_\Omega = u$ and $\|Eu\|_{W^{k,p}(\mathbb{R}^n)} \leq C\,\|u\|_{W^{k,p}(\Omega)}$. Set $v := Eu$.

Step 2: Mollification in $\mathbb{R}^n$. By theorem 5.2 , there exist $v_\varepsilon \in C^\infty(\mathbb{R}^n)$ such that

$$\|v_\varepsilon - v\|_{W^{k,p}(\mathbb{R}^n)} \longrightarrow 0$$

Step 3: Restriction back to Ω. Set $u_\varepsilon := v_\varepsilon|_\Omega$. Since v_ε is C^∞ on $\mathbb{R}^n$, we have $u_\varepsilon \in C^\infty(\overline{\Omega})$. Moreover,

$$\|u_\varepsilon - u\|_{W^{k,p}(\Omega)} = \|v_\varepsilon - v\|_{W^{k,p}(\Omega)} \leq \|v_\varepsilon - v\|_{W^{k,p}(\mathbb{R}^n)} \longrightarrow 0.$$

Choosing a sequence $\varepsilon_m \downarrow 0$ and writing $u_m := u_{\varepsilon_m}$ yields the desired approximating sequence.

□

Proof. 1. Fix any point $x^0 \in \partial\Omega$. As $\partial\Omega$ is C^1, there exist, a radius $r > 0$ and a C^1 function $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ such that-upon relabeling the coordinate axes if necessary, we have

$$\Omega \cap B\left(x^0, r\right) = \left\{x \in B\left(x^0, r\right) \mid x_n > \gamma\left(x_1, \ldots, x_{n-1}\right)\right\}.$$

Set $V := \Omega \cap B\left(x^0, r/2\right)$.

2. Define the shifted point

$$x^\varepsilon := x + \varepsilon\lambda e_n \quad (x \in V, \varepsilon > 0),$$

and observe that for some fixed, sufficiently large number $\lambda > 0$ the ball $B\left(x^\varepsilon, \varepsilon\right) \subset \Omega \cap B\left(x^0, r\right)$ for all $x \in V$ and all small $\varepsilon > 0$.

Now define $u_\varepsilon(x) := u\left(x^\varepsilon\right) = u(x + \varepsilon\lambda e_n) \quad x \in V$. Next write $v^\varepsilon = \eta_\varepsilon * u_\varepsilon$. Clearly $v^\varepsilon \in C^\infty(\bar{V})$.

3. We now claim

$$v^\varepsilon \to u \quad \text{in } W^{k,p}(V)$$

To confirm this, take α to be any multiindex with $|\alpha| \leq k$. Then

$$\begin{aligned}
\|D^\alpha v^\varepsilon - D^\alpha u\|_{L^p(V)} &= \|\eta_\varepsilon * (D^\alpha u_\varepsilon - D^\alpha u) + \eta_\varepsilon * D^\alpha u - D^\alpha u\|_{L^p(V)} \\
&\leq \|\eta_\varepsilon * (D^\alpha u_\varepsilon - D^\alpha u)\|_{L^p(V)} + \|\eta_\varepsilon * D^\alpha u - D^\alpha u\|_{L^p(V)} \\
&\leq \|\eta_\varepsilon\| \cdot \|(D^\alpha u)_\varepsilon - D^\alpha u\| + \|\eta_\varepsilon * D^\alpha u - D^\alpha u\|
\end{aligned}$$

4. Select $\delta > 0$. Since $\partial\Omega$ is compact, we can find finitely many points $x_i^0 \in \partial\Omega$, radius $r_i > 0$, corresponding sets $V_i = \Omega \cap B^0\left(x_i^0, \frac{r_i}{2}\right)$, and functions $v_i \in C^\infty\left(\bar{V}_i\right)$ $(i = 1, \ldots, N)$ such that $\partial\Omega \subset \bigcup_{i=1}^N B^0\left(x_i^0, \frac{r_i}{2}\right)$ and

$$\|v_i - u\|_{W^{k,p}(V_i)} \leq \delta$$

Take an open set

$$V_0 = \Omega_\delta \subset\subset V_0' = \Omega_{\delta'} \subset\subset \Omega$$

such that $\Omega \subset \bigcup_{i=0}^{N} V_i$ ($0 < \delta' < \delta < \min\{\frac{r_i}{2}\}$) and select, using Theorem 1, a function $v_0 \in C^\infty(V_0') \subset C^\infty(\bar{V}_0)$ satisfying

$$\|v_0 - u\|_{W^{k,p}(V_0)} \leq \delta$$

5. Now let $\{\zeta_i\}_{i=0}^N$ be a smooth partition of unity on $\bar{\Omega}$, subordinate to the open sets $\left\{V_0, B^0\left(x_1^0, \frac{r_1}{2}\right), \cdots, B^0\right.$ Define $v := \sum_{i=0}^N \zeta_i v_i$. Then clearly $v \in C^\infty(\bar{\Omega})$. In addition, since $u = \sum_{i=0}^N \zeta_i u$, we see using Leibniz formula that for each $|\alpha| \leq k$

$$\begin{aligned}\|D^\alpha v - D^\alpha u\|_{L^p(\Omega)} &\leq \sum_{i=0}^{N} \|D^\alpha(\zeta_i v_i) - D^\alpha(\zeta_i u)\|_{L^p(V_i)} \\ &\leq C_{\zeta,\alpha} \sum_{i=0}^{N} \|v_i - u\|_{W^{k,p}(V_i)} = C(N+1)\delta\end{aligned}$$

□

§6 Trace

Theorem 6.1 (Trace Theorem)**.** *Let Ω be a bounded Lipschitz open set; $1 \leq p \leq \infty$.; $1 \leq p < \infty$. Then there exists a bounded linear operator*

$$T : W^{1,p}(\Omega) \to L^p(\partial\Omega)$$

such that

(i) $Tu = u|_{\partial\Omega}$ if $u \in C(\bar{\Omega}) \cap W^{1,p}(\Omega)$

(ii) T is bounded : $\|Tu\|_{L^p(\partial\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)}$ with the constant C depending only on p and Ω.

*We call Tu the **trace** of u on $\partial\Omega$.*

Remark. *By the theorem 5.3 , the trace operator is uniquely determined by property (i).*

Proof. **Step 1: Local flattening of the boundary.** Since Ω is Lipschitz, for each point $x_i^0 \in \partial\Omega$, there exists a neighborhood U_i and a Lipschitz function $\gamma_i : \mathbb{R}^{n-1} \to \mathbb{R}$ such that, after a suitable rotation of coordinates,

$$\Omega \cap U_i = \{x = (x', x_n) \in U_i : x_n > \gamma_i(x')\}.$$

By compactness of $\partial\Omega$, finitely many such neighborhoods U_i cover $\partial\Omega$.

Step 2: Reduction to the upper half-space. Consider the local change of coordinates

$$\Phi_i : U_i \to \tilde{U}_i \subset \mathbb{R}^n, \quad \Phi_i(x', x_n) = (x', x_n - \gamma_i(x')).$$

Then locally $\Phi_i(\Omega \cap U_i) \subset \mathbb{R}^n_+ := \{(x', x_n) : x_n > 0\}$.

For $u \in W^{1,p}(\Omega)$, define $v_i = u \circ \Phi_i^{-1}$ in $\mathbb{R}^n_+$. Classical one-dimensional argument (Hardy inequality) shows

$$\|v_i(\cdot,0)\|_{L^p(\mathbb{R}^{n-1})} \leq C\|v_i\|_{W^{1,p}(\mathbb{R}^n_+)}.$$

Step 3: Partition of unity. Take a smooth partition of unity $\{\phi_i\}_{i=1}^N$ subordinate to $\{U_i\}$, such that $\sum_i \phi_i = 1$ near $\partial\Omega$.

Define the local traces:

$$T_i u := v_i(\cdot,0) = (u \circ \Phi_i^{-1})(\cdot,0) \in L^p(\mathbb{R}^{n-1}).$$

Then the global trace is

$$Tu := \sum_{i=1}^N (\phi_i T_i u) \circ \Phi_i \in L^p(\partial\Omega).$$

Step 4: Verification.

(a) Linearity and boundedness:

$$\|Tu\|_{L^p(\partial\Omega)} \leq C\sum_i \|T_i u\|_{L^p(\mathbb{R}^{n-1})} \leq C\sum_i \|v_i\|_{W^{1,p}(\mathbb{R}^n_+)} \leq C\,\|u\|_{W^{1,p}(\Omega)}\,.$$

(b) Consistency with smooth functions: If $u \in C(\overline{\Omega}) \cap W^{1,p}(\Omega)$, then locally $T_i u = u|_{\partial\Omega \cap U_i}$, so globally

$$Tu = u|_{\partial\Omega}.$$

Hence T defines a bounded linear trace operator on $W^{1,p}(\Omega)$. □

Theorem 6.2 (Trace-zero functions in $W^{1,p}$)**.** *Let Ω be a bounded Lipschitz open set; $1 \leq p < \infty$. Suppose furthermore that $u \in W^{1,p}(\Omega)$. Then*

$$u \in W_0^{1,p}(\Omega) \quad \textit{iff} \quad Tu = 0 \textit{ on } \partial\Omega.$$

§7 The space $W^{k,2}$

§7.1 $W^{k,2}(\mathbb{R}^n)$

Lemma 7.1. *Suppoes $u \in L_{\text{loc}}(\mathbb{R}^n)$ has a weak derivative $D^\alpha u \in L^1(\mathbb{R}^n)$ or $L^2(\mathbb{R}^n)$, then*

$$\mathcal{F}D^\alpha u = (iy)^\alpha \hat{u}$$

Theorem 7.2 (Characterization of $W^{k,2}(\mathbb{R}^n)$ by Fourier transform)**.** *Let k be a nonnegative integer.*

A function $u \in L^2(\mathbb{R}^n)$ belongs to $W^{k,2}(\mathbb{R}^n)$ if and only if

$$\left(1+|y|^k\right)\hat{u} \in L^2(\mathbb{R}^n)$$

Remark. *In addition, we have the inequalities*

$$\left\|\left(1+|y|^k\right)\hat{u}\right\|_{L^2(\mathbb{R}^n)} \leq \sqrt{2}\,\|u\|_{W^{k,2}(\mathbb{R}^n)}$$

for each $u \in W^{k,2}(\mathbb{R}^n)$. Thus the norm $\|\cdot\| := \left\|\left(1+|y|^k\right)\mathcal{F}\cdot\right\|_{L^2(\mathbb{R}^n)}$ is equivalent to $\|\cdot\|_{W^{k,2}(\mathbb{R}^n)}$ by the norm equivalent theorem.

Proof. Assume first $u \in W^{k,2}(\mathbb{R}^n)$. Then for each multiindex $|\alpha| \leq k$, we have $D^\alpha u \in L^2(\mathbb{R}^n)$ and

$$\mathcal{F}(D^\alpha u) = (iy)^\alpha \hat{u}$$

belongs to $L^2(\mathbb{R}^n)$. Also,

$$\|D^\alpha u\|_{L^2(\mathbb{R}^n)} = \|(iy)^\alpha \hat{u}\|_{L^2(\mathbb{R}^n)}$$

Thus

$$\begin{aligned}
\|\left(1+|y|^k\right)\hat{u}\|_{L^2(\mathbb{R}^n)} &= \left(\int_{\mathbb{R}^n}\left(1+2|y|^k+|y|^{2k}\right)\hat{u}^2\,\mathrm{d}y\right)^{\frac{1}{2}} \\
&\leq \left(\int_{\mathbb{R}^n} 2\left(1+|y|^{2k}\right)\hat{u}^2\,\mathrm{d}y\right)^{\frac{1}{2}} \\
&\leq \sqrt{2}\left(\|\hat{u}\|_{L^2(\mathbb{R})} + \sum_{|\alpha|=k}\|y^\alpha \hat{u}\|_{L^2(\mathbb{R})}\right) \\
&= \sqrt{2}\left(\|u\|_{L^2(\mathbb{R})} + \sum_{|\alpha|=k}\|D^\alpha \hat{u}\|_{L^2(\mathbb{R})}\right) \\
&\leq \sqrt{2}\,\|u\|_{H^k(\mathbb{R}^n)}
\end{aligned}$$

belongs to $L^2(\mathbb{R}^n)$ and

2. Suppose conversely $\left(1+|y|^k\right)\hat{u} \in L^2(\mathbb{R}^n)$ and $|\alpha| \leq k$. Then

$$\|(iy)^\alpha \hat{u}\|^2_{L^2(\mathbb{R}^n)} \leq \int_{\mathbb{R}^n}|y|^{2|\alpha|}|\hat{u}|^2 dy \leq C_\alpha \left\|\left(1+|y|^k\right)\hat{u}\right\|^2_{L^2(\mathbb{R}^n)} < \infty$$

thus $(iy)^\alpha \hat{u} \in L^2(\mathbb{R}^n)$. Set

$$u_\alpha := \mathcal{F}^{-1}\left((iy)^\alpha \hat{u}\right)$$

Then for each $\phi \in C_c^\infty(\mathbb{R}^n)$

$$(D^\alpha \phi, u) = (\mathcal{F}D^\alpha \phi, \hat{u}) = \left((iy)^\alpha \hat{\phi}, \hat{u}\right) = (-1)^{|\alpha|}\left(\hat{\phi}, (iy)^\alpha \hat{u}\right) = (-1)^{|\alpha|}(\phi, u_\alpha)$$

Thus $u_\alpha = D^\alpha u$ in the weak sense and $D^\alpha u \in L^2(\mathbb{R}^n)$. Hence $u \in H^k(\Omega)$, as required. $\square$

Definition 7.3. *Let $s > 0$ be a noninteger real number. We define the fractional Sobolev space $H^s(\mathbb{R}^n)$ as follows:*

$$H^s(\mathbb{R}^n) := \left\{u \in L^2(\mathbb{R}^n) : (1 + |y|^s)\, \hat{u} \in L^2(\mathbb{R}^n)\right\}$$

with the norm

$$\|u\|_{H^s(\mathbb{R}^n)} := \|(1 + |y|^s)\, \hat{u}\|_{L^2(\mathbb{R}^n)}$$

§7.2 Gelfand triple

Theorem 7.4. *Let $\Omega \subset \mathbb{R}^n$ be a open set. Then the following **Gelfand triple** holds:*

$$H_0^k(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-k}(\Omega)$$

where $H^{-k}(\Omega)$ denotes the dual space of $H_0^k(\Omega)$. Moreover, both embeddings are continuous and dense.

Chapter VII

Sobolev Inequality; Imbedding Theorem

Contents

§1 Sobolev inequalities

Theorem 1.1. *Let Ω be a a bounded Lipschitz domain in $\mathbb{R}^n$. If $u \in W_0^{k,p}(\Omega)$ and $kp < n$, then there exists a constant $C = C(k,p,n,\Omega)$ such that*

$$\|u\|_{L^{p^*}(\Omega)} \leq C \|u\|_{W^{k,p}(\Omega)}$$

where $p^* = \frac{np}{n-kp}$ ($\frac{1}{p^*} = \frac{1}{p} - \frac{k}{n}$) is the Sobolev conjugate of p.

§1.1 Proof of Sobolev inequalities

Definition 1.2. *Let n-multiindex $\mathbf{p}_i, \boldsymbol{\theta}_i$ and $\mathbf{p}$ be given such that*

$$\frac{\boldsymbol{\theta}}{\mathbf{p}} = \sum_{i=1}^{k} \frac{\boldsymbol{\theta}_i}{\mathbf{p}_i}$$

meaning that $\frac{\boldsymbol{\theta}^j}{\mathbf{p}^j} = \sum_{i=1}^{k} \frac{\boldsymbol{\theta}_i^j}{\mathbf{p}_i^j}$ *for each component* $j = 1, \ldots, n.$ *, and let* f_i

$$\left\| \prod f_i \right\|_{L^{\mathbf{p}}} \leq \prod \|f_i\|_{L^{\mathbf{p}_i}}^{\boldsymbol{\theta}_i}$$

where

$$\|u\|_{L^{\mathbf{p}}}^{\boldsymbol{\theta}} := \left\| \cdots \left\| \|f\|_{L^{\mathbf{p}^1}(dx_1)}^{\boldsymbol{\theta}^1} \right\|_{L^{\mathbf{p}^2}(dx_2)}^{\boldsymbol{\theta}^2} \cdots \right\|_{L^{\mathbf{p}^n}(dx_n)}^{\boldsymbol{\theta}^n}$$

Definition 1.3 (Multi-index Hölder inequality)**.** *Let* n*-multiindices* $\mathbf{p}_i, \boldsymbol{\theta}_i$ *and* $\mathbf{p}$ *be given such that*

$$\frac{\boldsymbol{\theta}}{\mathbf{p}} = \sum_{i=1}^{k} \frac{\boldsymbol{\theta}_i}{\mathbf{p}_i},$$

meaning that

$$\frac{\boldsymbol{\theta}^j}{\mathbf{p}^j} = \sum_{i=1}^{k} \frac{\boldsymbol{\theta}_i^j}{\mathbf{p}_i^j}, \quad j = 1, \ldots, n.$$

For functions f_i*, define the nested* $L^{\mathbf{p}}$*-norm with weights* $\boldsymbol{\theta}$ *as*

$$\|f\|_{L^{\mathbf{p}}}^{\boldsymbol{\theta}} := \left\| \cdots \left\| \|f\|_{L^{\mathbf{p}^1}(dx_1)}^{\boldsymbol{\theta}^1} \right\|_{L^{\mathbf{p}^2}(dx_2)}^{\boldsymbol{\theta}^2} \cdots \right\|_{L^{\mathbf{p}^n}(dx_n)}^{\boldsymbol{\theta}^n}.$$

Then the multi-index Hölder inequality states that

$$\left\| \prod_{i=1}^{k} f_i \right\|_{L^{\mathbf{p}}} \leq \prod_{i=1}^{k} \|f_i\|_{L^{\mathbf{p}_i}}^{\boldsymbol{\theta}_i}.$$

Theorem 1.4 (Estimates for $C_c^1(\mathbb{R}^n)$)**.** *Assume* $1 \leq p < n$*. There exists a constant* $C = \frac{p(n-1)}{n-p}$*, such that*

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C \|Du\|_{L^p(\mathbb{R}^n)}$$

for all $u \in C_c^1(\mathbb{R}^n)$*.*

Proof. 1. First Assume $p = 1$. Since u has compact support, for each $i = 1, \ldots, n$ and $x \in \mathbb{R}^n$ we have

$$u(x) = \int_{-\infty}^{x_i} u_{x_i}(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n) \, \mathrm{d}y_i$$

and so

$$|u(x)| \leq \int_{-\infty}^{\infty} |Du(x_1, \ldots, y_i, \ldots, x_n)| \, \mathrm{d}y_i$$

Consequently

$$|u(x)|^{\frac{n}{n-1}} \leq \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} |Du(x_1, \ldots, y_i, \ldots, x_n)| \, \mathrm{d}y_i \right)^{\frac{1}{n-1}} \tag{1}$$

Integrate with respect to x_1 :

$$\begin{aligned}\int_{-\infty}^{\infty}|u|^{\frac{n}{n-1}}\,\mathrm{d}x_1 &\leq \int_{-\infty}^{\infty}\prod_{i=1}^{n}\left(\int_{-\infty}^{\infty}|Du|\,\mathrm{d}y_i\right)^{\frac{1}{n-1}}\,\mathrm{d}x_1\\ &= \left(\int_{-\infty}^{\infty}|Du|\,\mathrm{d}y_1\right)^{\frac{1}{n-1}}\int_{-\infty}^{\infty}\prod_{i=2}^{n}\left(\int_{-\infty}^{\infty}|Du|\,\mathrm{d}y_i\right)^{\frac{1}{n-1}}\,\mathrm{d}x_1\\ &\leq \left(\int_{-\infty}^{\infty}|Du|\,\mathrm{d}y_1\right)^{\frac{1}{n-1}}\prod_{i=2}^{n}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|Du|\,\mathrm{d}x_1\,\mathrm{d}y_i\right)^{\frac{1}{n-1}}\end{aligned}$$

Then integrate this inequality with respect to x_2:

We continue by integrating with respect to $x_3,\dots,x_n$, eventually to find

$$\begin{aligned}\int_{\mathbb{R}^n}|u|^{\frac{n}{n-1}}\,\mathrm{d}x &\leq \prod_{i=1}^{n}\left(\int_{\mathbb{R}^n}|Du|dx_1\dots dy_i\dots dx_n\right)^{\frac{1}{n-1}}\\ &= \left(\int_{\mathbb{R}^n}|Du|dx\right)^{\frac{n}{n-1}}\end{aligned}$$

This is estimate for $p=1$.

2. Consider now the case that $1<p<n$. We apply previous estimate to $v:=|u|^{\gamma}$, where $\gamma=\frac{p(n-1)}{n-p}>1$. Then

$$\begin{aligned}\left(\int_{\mathbb{R}^n}|u|^{p^*}\,\mathrm{d}x\right)^{\frac{n-1}{n}} &= \left(\int_{\mathbb{R}^n}|u|^{\frac{\gamma n}{n-1}}dx\right)^{\frac{n-1}{n}}\\ &\leq \int_{\mathbb{R}^n}\left|D\,|u|^{\gamma}\right|\,\mathrm{d}x\\ &= \gamma\int_{\mathbb{R}^n}|u|^{\gamma-1}|Du|dx\\ &\leq \gamma\left(\int_{\mathbb{R}^n}|u|^{(\gamma-1)\frac{p}{p-1}}dx\right)^{\frac{p-1}{p}}\|Du\|_p\\ &= \gamma\left(\int_{\mathbb{R}^n}|u|^{p^*}\,\mathrm{d}x\right)^{\frac{p-1}{p}}\|Du\|_p\end{aligned}$$

in which case $\frac{\gamma n}{n-1}=(\gamma-1)\frac{p}{p-1}=\frac{np}{n-p}=p^*$. So we have

$$\left(\int_{\mathbb{R}^n}|u|^{p^*}dx\right)^{\frac{1}{p^*}}\leq\gamma\left(\int_{\mathbb{R}^n}|Du|^{p}dx\right)^{\frac{1}{p}}$$

□

Theorem 1.5 (Estimates for $W_0^{1,p}$)**.** *Assume* Ω *is a bounded, open subset of* $\mathbb{R}^n$. *Suppose* $u\in W_0^{1,p}(\Omega)$ *for some* $1\leq p<n$. *Then we have the estimate*

$$\|u\|_{L^{p^*}(\Omega)}\leq C\|Du\|_{L^p(\Omega)}$$

Furthermore, $\|u\|_{L^q(\Omega)} \leq C'\|Du\|_{L^p(\Omega)}$ *for each* $q \in [1, p^*]$*, the constant* C' *depending only on* p, q, n *and* Ω*.*

Proof: Since $u \in W_0^{1,p}(\Omega)$*, there exist functions* $u_m \in C_c^\infty(\Omega)$ *converging to* u *in* $W^{1,p}(\Omega)$*. We extend each function* u_m *to be 0 on* $\mathbb{R}^n - \bar{\Omega}$ *(that* $C_c^\infty(\Omega) \subset C_c^\infty(\mathbb{R}^n)$*) and apply the estimate for* $C_c^\infty(\mathbb{R}^n)$ *to discover*

$$\|u_m\|_{L^{p^*}(\Omega)} = \|u_m\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|Du_m\|_{L^p(\mathbb{R}^n)} = C\|Du_m\|_{L^p(\Omega)}$$

Thus

$$\|u_m - u_n\|_{L^{p^*}(\Omega)} \leq C\|Du_m - Du_n\|_{L^p(\Omega)} \leq C\|u_m - u_n\|_{W^{m,p}(\Omega)} \to 0$$

Since $L^{p^*}(\Omega)$ *is complete, we conclude that* $u_m \xrightarrow{L^{p^*}(\Omega)} u$ *and*

$$\|u\|_{L^{p^*}(\Omega)} \leq C\|Du\|_{L^p(\Omega)}$$

Remark *In view of this estimate, on* $W_0^{1,p}(\Omega)$ *the norm* $\|Du\|_{L^p(\Omega)}$ *is equivalent to* $\|u\|_{W^{1,p}(\Omega)}$*, if* Ω *is a bounded, open subset of* $\mathbb{R}^n$*.*

$$\|Du\|_{L^p(\Omega)} \leq \|u\|_{W^{1,p}(\Omega)}$$

$$\|u\|^p_{W^{1,p}(\Omega)} = \|u\|^p_{L^p(\Omega)} + \|Du\|^p_{L^p(\Omega)} \leq C\|Du\|^p_{L^p(\Omega)}$$

since $p \in [1, p^*]$*.*

Theorem 1.6 (Estimates for $W^{1,p}, 1 \leq p < n$)**.** *Let* Ω *be a bounded, open subset of* $\mathbb{R}^n$*, and suppose* $\partial\Omega$ *is* C^1*. If* $1 \leq p < n$*. and* $u \in W^{1,p}(\Omega)$*. Then* $u \in L^{p^*}(\Omega)$*, with the estimate*

$$\|u\|_{L^{p^*}(\Omega)} \leq C\,\|u\|_{W^{1,p}(\Omega)}$$

the constant C *depending only on* p, n*, and* Ω*.*

Proof: Since $\partial\Omega$ *is* C^1*, there exists an extension* $Eu = \bar{u} \in W^{1,p}(\mathbb{R}^n)$*, such that*

$$\begin{cases} \bar{u} = u \text{ in } \Omega, \bar{u} \text{ has compact support within } V \\ \|\bar{u}\|_{W^{1,p}(\mathbb{R}^n)} \leq C\,\|u\|_{W^{1,p}(\Omega)} \end{cases}$$

Because $\bar{u}$ *has compact support within* V *(*$V \subset \Omega_\varepsilon$*), we know that there exist functions* $u_m \in C^\infty(V) \subset C_c^\infty(\mathbb{R}^n)$ *such that*

$$u_m \to \bar{u} \quad \text{in } W^{1,p}(\mathbb{R}^n)$$

Now according to inequality for $C_c^\infty(\mathbb{R}^n)$*,*

$$\|u_m - u_l\|_{L^{p^*}(\mathbb{R}^n)} \leq C\,\|Du_m - Du_l\|_{L^p(\mathbb{R}^n)} \leq C\,\|u_m - u_l\|_{W^{1,p}(\mathbb{R}^n)}$$

and

$$\|u_m\|_{L^{p^*}(\mathbb{R}^n)} \le C\,\|Du_m\|_{L^p(\mathbb{R}^n)}$$

for all $l, m \ge 1$. *Thus*

$$u_m \xrightarrow{L^{p^*}(\mathbb{R}^n)} \bar{u}$$

and

$$\|\bar{u}\|_{L^{p^*}(\mathbb{R}^n)} \le C\|D\bar{u}\|_{L^p(\mathbb{R}^n)}$$

Then we conclude that

$$\|u\|_{L^{p^*}(\Omega)} \le \|\bar{u}\|_{L^{p^*}(\mathbb{R}^n)} \le C\|D\bar{u}\|_{L^p(\mathbb{R}^n)} \le C\|\bar{u}\|_{W^{1,p}(\mathbb{R}^n)} \le C\|u\|_{W^{1,p}(\Omega)}$$

§2 Morrey's inequality

Theorem 2.1. *Let* Ω *be a a bounded Lipschitz domain in* $\mathbb{R}^n$. *If* $u \in W^{k,p}(\Omega)$ *and* $kp > n$ *then there exists a constant* $C = C(k, p, n, \Omega)$ *such that*

$$\|u\|_{C^{m,\gamma}(\bar{\Omega})} \le C\,\|u\|_{W^{k,p}(\Omega)}$$

where $m = k - 1 - \left[\frac{n}{p}\right]$ *and* $\gamma = \frac{n}{p} - \left[\frac{n}{p}\right]$.

§2.1 Proof of Morrey's inequality

Lemma 2.2. *For all* $u \in C^1$, *we claim there exists a constant* $C = \frac{1}{n\alpha(n)}$, *depending only on* n, *such that*

$$⨍_{B(x,r)} |u(y) - u(x)|\, \mathrm{d}y \le C \int_{B(x,r)} \frac{|Du(y)|}{|y-x|^{n-1}}\, \mathrm{d}y$$

$$\begin{aligned}
⨍_{B(x,r)} |u(y) - u(x)|\, \mathrm{d}y &= \frac{1}{\alpha(n) r^n} \int_0^r \int_{\partial B(0,1)} |u(x+s\omega) - u(x)| s^{n-1}\, \mathrm{d}S_\omega\, \mathrm{d}s \\
&= \frac{1}{\alpha(n) r^n} \int_0^r s^{n-1} \int_{\partial B(0,1)} \left| \int_0^s \frac{d}{dt} u(x+tw)\, \mathrm{d}t \right| \mathrm{d}S_\omega\, \mathrm{d}s \\
&\le \frac{1}{\alpha(n) r^n} \int_0^r s^{n-1} \int_{\partial B(0,1)} \int_0^s |Du(x+tw)|\, \mathrm{d}t\, \mathrm{d}S_\omega\, \mathrm{d}s \\
&= \frac{1}{\alpha(n) r^n} \int_0^r s^{n-1} \int_{B(x,s)} \frac{|Du(y)|}{|y-x|^{n-1}}\, \mathrm{d}y ds \\
&\le \frac{1}{\alpha(n) r^n} \int_0^r s^{n-1} \int_{B(x,r)} \frac{|Du(y)|}{|y-x|^{n-1}}\, \mathrm{d}y ds \\
&= \frac{1}{n\alpha(n)} \int_{B(x,r)} \frac{|Du(y)|}{|y-x|^{n-1}}\, \mathrm{d}y
\end{aligned}$$

Theorem 2.3 **(**$C^1(\mathbb{R}^n) \cap W^{1,p}(\mathbb{R}^n)$**).** *Assume* $n < p \le \infty$. *Then there exists a constant* C, *depending only on p and n, such that*

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \le C\,\|u\|_{W^{1,p}(\mathbb{R}^n)}$$

for all $u \in C^1(\mathbb{R}^n)$, where $\gamma = 1 - n/p$

Proof: 1. Now fix $x \in \mathbb{R}^n$. We apply lemma as follows:

$$\begin{aligned}
|u(x)| &\le ⨍_{B(x,r)} |u(x) - u(y)|\,\mathrm{d}y + ⨍_{B(x,r)} |u(y)|\,\mathrm{d}y \\
&\le C \int_{B(x,r)} \frac{|Du(y)|}{|x-y|^{n-1}}\,\mathrm{d}y + C\,\|u\|_{L^p(B(x,r))} \\
&\le C\|Du\|_{L^p(\mathbb{R}^n)} \left(\int_{B(x,r)} \frac{1}{|x-y|^{(n-1)\frac{p}{p-1}}}\,\mathrm{d}y \right)^{\frac{p-1}{p}} + C\,\|u\|_{L^p(\mathbb{R}^n)} \\
&\le C\,\|u\|_{W^{1,p}(\mathbb{R}^n)}
\end{aligned}$$

The last estimate holds since $p > n$ implies $(n-1)\frac{p}{p-1} < n$. As $x \in \mathbb{R}^n$ is arbitrary, it follows that

$$\|u\|_{C(\bar{\Omega})} = \sup_{x \in \mathbb{R}^n} |u| \le C\,\|u\|_{W^{1,p}(\mathbb{R}^n)}$$

2. Next, choose any two points $x, y \in \mathbb{R}^n$ and write $r := |x-y|$. Let $W := B(x,r) \cap B(y,r)$. Then

$$|u(x) - u(y)| \le ⨍_W |u(x) - u(z)|\,\mathrm{d}z + ⨍_W |u(y) - u(z)|\,\mathrm{d}z$$

But lemma allows us to estimate

$$\begin{aligned}
⨍_W |u(x) - u(z)|\,\mathrm{d}z &\le ⨍_{B(x,r)} |u(x) - u(z)|\,\mathrm{d}z \\
&\le C \int_{B(x,r)} \frac{|Du(z)|}{|z-x|^{n-1}}\,\mathrm{d}z \\
&\le C\|Du\|_{L^p(\mathbb{R}^n)} \left(\int_{B(x,r)} \frac{dz}{|x-z|^{(n-1)\frac{p}{p-1}}} \right)^{\frac{p-1}{p}} \\
&= C\frac{n\alpha(n)(p-1)}{pn} r^{1-\frac{n}{p}} \|Du\|_{L^p(\mathbb{R}^n)}
\end{aligned}$$

Likewise,

$$⨍_W |u(y) - u(z)|\,\mathrm{d}z \le C r^{1-\frac{n}{p}} \|Du\|_{L^p(\mathbb{R}^n)}$$

These estimates yield

$$|u(x) - u(y)| \le C r^{1-\frac{n}{p}} \|Du\|_{L^p(\mathbb{R}^n)} = C|x-y|^{\gamma} \|Du\|_{L^p(\mathbb{R}^n)}$$

Thus

$$[u]_{C^{0,1-n/p}(\mathbb{R}^n)} = \sup_{x\neq y}\left\{\frac{|u(x)-u(y)|}{|x-y|^{1-n/p}}\right\} \leq C\|Du\|_{L^p(\mathbb{R}^n)}$$

This complete the proof .

Theorem 2.4 (Estimates for $W^{1,p}$)**.** *Let Ω be a bounded, open subset of $\mathbb{R}^n$, and suppose $\partial\Omega$ is C^1. Assume $n < p \leq \infty$ and $u \in W^{1,p}(\Omega)$. Then u has a version $u^* \in C^{0,\gamma}(\bar{\Omega})$, for $\gamma = 1 - \frac{n}{p}$, with the estimate*

$$\|u^*\|_{C^{0,\gamma}(\bar{\Omega})} \leq C\,\|u\|_{W^{1,p}(\Omega)}$$

The constant C depends only on p, n and Ω. In view of this Theorem, we will henceforth always identify a function $u \in W^{1,p}(\Omega)$ $(p > n)$ with its continuous version.

Proof: Since $\partial\Omega$ is C^1, there exists an extension $Eu = \bar{u} \in W^{1,p}(\mathbb{R}^n)$ such that

$$\begin{cases} \bar{u} = u \text{ in } \Omega \\ \bar{u} \text{ has compact support, and} \\ \|\bar{u}\|_{W^{1,p}(\mathbb{R}^n)} \leq C\,\|u\|_{W^{1,p}(\Omega)} \end{cases}$$

Assume first $n < p < \infty$. Since $\bar{u}$ has compact support, we obtain functions $u_m \in C_c^\infty(\mathbb{R}^n)$ such that

$$u_m \xrightarrow{W^{1,p}(\mathbb{R}^n)} \bar{u}$$

Now according to previous theorem ,

$$\|u_m - u_l\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C\,\|u_m - u_l\|_{W^{1,p}(\mathbb{R}^n)}$$

and

$$\|u_m\|_{C^{0,\gamma}} \leq C\,\|u_m\|_{W^{1,p}(\mathbb{R}^n)}$$

for all $l, m \geq 1$, whence there exists a function $u^* \in C^{0,\gamma}(\mathbb{R}^n)$ such that

$$u_m \xrightarrow{C^{0,\gamma}(\mathbb{R}^n)} u^*$$

We see that $u^* = u$ a.e. on Ω, so that u^* is a version of u and

$$\|u^*\|_{C^{0,1-n/p}(\mathbb{R}^n)} \leq C\|\bar{u}\|_{W^{1,p}(\mathbb{R}^n)} \leq C\,\|u\|_{W^{1,p}(\Omega)}$$

2. If $p = \infty$, $\gamma = 1$ we can assume that $\bar{u} \leq \|\bar{u}\|_{L^\infty(\mathbb{R}^n)}$ for all $x \in \mathbb{R}^n$. Then

$$\sup_{x\in\mathbb{R}^n} \bar{u} \leq \|\bar{u}\|_{L^\infty(\mathbb{R}^n)}$$

and

$$|\bar{u}(y)-\bar{u}(x)| = \left|\int_0^{|y-x|} \frac{d}{dt}\bar{u}(x+t\frac{y-x}{|y-x|})\,\mathrm{d}t\right|$$
$$\leq \left|\frac{y-x}{|y-x|}\cdot D\bar{u}(x+t\frac{y-x}{|y-x|})\right| |y-x|$$
$$\leq \|D\bar{u}\|_{L^\infty(\mathbb{R}^n)}\,|y-x|$$

for erery $x, y \in \mathbb{R}^n$. Thus $\bar{u} \in C^{0,1}(\mathbb{R}^n)$ and $\bar{u}|_\Omega$ is a version of u. Also,

$$\|u\|_{C^{0,1}(\bar{\Omega})} \leq \|\bar{u}\|_{C^{0,1}(\mathbb{R}^n)} \leq \|\bar{u}\|_{W^{1,\infty}(\mathbb{R}^n)} \leq C\,\|u\|_{W^{1,\infty}(\Omega)}$$

§3 Poincaré's inequality

Theorem 3.1 (Poincaré's inequality)**.** *Let Ω be a bounded, connected, open subset of $\mathbb{R}^n$, with a C^1 boundary $\partial\Omega$. Assume $1 \leq p \leq \infty$. Then there exists a constant C, depending only on n, p and Ω, such that*

$$\|u-(u)_\Omega\|_{L^p(\Omega)} \leq C\|Du\|_{L^p(\Omega)}$$

for each function $u \in W^{1,p}(\Omega)$.

Proof: We argue by contradiction. Were the stated estimate false, there would exist for each integer $k = 1, \ldots$ a function $u_k \in W^{1,p}(\Omega)$ satisfying

$$\|u_k-(u_k)_\Omega\|_{L^p(\Omega)} > k\,\|Du_k\|_{L^p(\Omega)}$$

We renormalize by defining

$$v_k := \frac{u_k-(u_k)_\Omega}{\|u_k-(u_k)_\Omega\|_{L^p(\Omega)}} \quad (k=1,\ldots)$$

Then

$$(v_k)_\Omega = 0, \|v_k\|_{L^p(\Omega)} = 1$$

and (2) implies

$$\|Dv_k\|_{L^p(\Omega)} < \frac{1}{k} \quad (k=1,2,\ldots)$$

In particular the functions $\{v_k\}_{k=1}^\infty$ are bounded in $W^{1,p}(\Omega)$. Then there exist a subsequence $\{v_{k_j}\}_{j=1}^\infty \subset \{v_k\}_{k=1}^\infty$ and a function $v \in L^p(\Omega)$ such that

$$v_{k_j} \to v \quad \text{in } L^p(\Omega)$$

It follows that

$$(v)_\Omega = 0, \|v\|_{L^p(\Omega)} = 1$$

On the other hand, since $\|Dv_k\|_{L^p(\Omega)} \to 0$

$$\int_\Omega v\phi_{x_i}dx = \lim_{k_j\to\infty}\int_\Omega v_{k_j}\phi_{x_i}dx = -\lim_{k_j\to\infty}\int_\Omega v_{k_j,x_i}\phi dx = 0$$

for each $\phi \in C_c^\infty(\Omega)$*. Consequently* $v \in W^{1,p}(\Omega)$*, with* $Dv = 0$ *a.e.*

Thus v *is constant, since* Ω *is connected. Since* v *is constant and* $(v)_\Omega = 0$*, we must have* $v \equiv 0$*, in which case* $\|v\|_{L^p(\Omega)} = 0$*. This contradiction establishes estimate.*

§4 Compactness

Definition 4.1. *Let* X *and* Y *be Banach spaces,* $X \subset Y$*. We say that* X *is* ***compactly embedded*** *in* Y*, written*

$$X \subset\subset Y$$

provided the imbedding operator is a linear compact operator i.e.

(i) $\|u\|_Y \le C\|u\|_X$ *for all* $u \in X$*, where* C *is a constant,*

(ii) each bounded sequence in X *is precompact in* Y*.*

Theorem 4.2 (Rellich-Kondrachov Compactness Theorem)**.** *Assume* Ω *is a bounded open subset of* $\mathbb{R}^n$ *and* $\partial\Omega$ *is* C^1*. Suppose* $1 \le p < n$*. Then*

$$W^{1,p}(\Omega) \subset\subset L^q(\Omega)$$

for each $1 \le q < p^*$*.*

Proof: Step 1. Fix $1 \le q < p^*$ *and note that since* Ω *is bounded and* $\partial\Omega$ *is* C^1*, Gagliardo-Nirenberg-Sobolev inequality for* $W^{1,p}(\Omega)$ *implies*

$$W^{1,p}(\Omega) \subset L^q(\Omega), \quad \|u\|_{L^q(\Omega)} \le C\|u\|_{W^{1,p}(\Omega)}.$$

It remains therefore to show that if $\{u_m\}_{m=1}^\infty$ *is a bounded sequence in* $W^{1,p}(\Omega)$*, there exists a subsequence* $\{u_{m_j}\}_{j=1}^\infty$ *which converges in* $L^q(\Omega)$*.*

Step 2. In view of the Extension Theorem , we may with no loss of generality assume that $\Omega = \mathbb{R}^n$ *and the functions* $\{u_m\}_{m=1}^\infty$ *all have compact support in some bounded open set* $V \subset \mathbb{R}^n$*. We also may assume*

$$\sup_m \|u_m\|_{W^{1,p}(V)} < \infty.$$

Step 3. Let us first study the smoothed functions

$$u_m^\varepsilon := \eta_\varepsilon * u_m \quad (\varepsilon > 0,\ m = 1, 2, \ldots),$$

where η_ε *denotes the usual mollifier. We may suppose the functions* $\{u_m^\varepsilon\}_{m=1}^\infty$ *all have support in* V *as well.*

Step 4. We first claim

$$u_m^\varepsilon \xrightarrow{L^q(V)} u_m \text{ as } \varepsilon \to 0, \text{ uniformly in } m.$$

To prove this, we first note that if u_m is smooth, then

$$\begin{aligned}
u_m^\varepsilon(x) - u_m(x) &= \frac{1}{\varepsilon^n}\int_{B(x,\varepsilon)} \eta\left(\frac{x-z}{\varepsilon}\right)(u_m(z) - u_m(x))\,dz \\
&= \int_{B(0,1)} \eta(y)(u_m(x-\varepsilon y) - u_m(x))\,dy \\
&= \int_{B(0,1)} \eta(y)\int_0^1 \frac{d}{dt}(u_m(x-\varepsilon ty))\,dt\,dy \\
&= -\varepsilon\int_{B(0,1)} \eta(y)\int_0^1 Du_m(x-\varepsilon ty)\cdot y\,dt\,dy
\end{aligned}$$

Thus

$$\begin{aligned}
\int_V |u_m^\varepsilon(x) - u_m(x)|\,dx &\le \varepsilon\int_{B(0,1)} \eta(y)\int_0^1\int_V |Du_m(x-\varepsilon ty)|\,dx\,dt\,dy \\
&\le \varepsilon\int_V |Du_m(z)|\,dz.
\end{aligned}$$

By approximation, this estimate holds if $u_m \in W^{1,p}(V)$. Hence

$$\|u_m^\varepsilon - u_m\|_{L^1(V)} \le \varepsilon\|Du_m\|_{L^1(V)} \le \varepsilon C\|Du_m\|_{L^p(V)},$$

We thereby discover

$$u_m^\varepsilon \to u_m \quad \text{in } L^1(V), \text{ uniformly in } m.$$

But then since $1 \le q < p^*$, we see using the interpolation inequality for L^p-norms that

$$\|u_m^\varepsilon - u_m\|_{L^q(V)} \le \|u_m^\varepsilon - u_m\|_{L^1(V)}^\theta \|u_m^\varepsilon - u_m\|_{L^{p^*}(V)}^{1-\theta},$$

where $\frac{1}{q} = \frac{\theta}{1} + \frac{1-\theta}{p^*}, \quad 0 < \theta < 1.$ Consequently,the Gagliardo-Nirenberg-Sobolev inequality imply

$$\|u_m^\varepsilon - u_m\|_{L^q(V)} \le C\|u_m^\varepsilon - u_m\|_{L^1(V)}^\theta,$$

whence assertion (2) follows from (3).

Step 5. Next we claim

$$\begin{cases} \text{for each fixed } \varepsilon > 0, \text{ the sequence } \{u_m^\varepsilon\}_{m=1}^\infty \\ \text{is uniformly bounded and equicontinuous.} \end{cases}$$

Indeed, if $x \in \mathbb{R}^n$, then

$$|u_m^\varepsilon(x)| \le \int_{B(x,\varepsilon)} \eta_\varepsilon(x-y)|u_m(y)|\,dy$$

$$\leq \|\eta_\varepsilon\|_{L^\infty(\mathbb{R}^n)}\|u_m\|_{L^1(V)} \leq \frac{C}{\varepsilon^n} < \infty$$

for $m = 1, 2, \ldots$. *Similarly,*

$$|Du_m^\varepsilon(x)| \leq \int_{B(x,\varepsilon)} |D\eta_\varepsilon(x-y)||u_m(y)|\,dy$$

$$\leq \|D\eta_\varepsilon\|_{L^\infty(\mathbb{R}^n)}\|u_m\|_{L^1(V)} \leq \frac{C}{\varepsilon^{n+1}} < \infty,$$

for $m = 1, \ldots$. *Assertion (4) follows from these two estimates.*

Step 6. We now observe that since the functions $\{u_m\}_{m=1}^\infty$, *and thus the functions* $\{u_m^\varepsilon\}_{m=1}^\infty$, *have support in some fixed bounded set* $V \subset \mathbb{R}^n$, *we may utilize (4) and the Arzela-Ascoli compactness criterion, to obtain a subsequence* $\left\{u_{m_j}^\varepsilon\right\}_{j=1}^\infty \subset \{u_m^\varepsilon\}_{m=1}^\infty$ *which converges uniformly on* V. *In particular therefore*

$$\limsup_{j,k\to\infty} \left\|u_{m_j}^\varepsilon - u_{m_k}^\varepsilon\right\|_{L^q(V)} = 0$$

Now fix $\delta > 0$. *We will show there exists a subsequence* $\{u_{m_j}\}_{j=1}^\infty \subset \{u_m\}_{m=1}^\infty$ *such that*

$$\limsup_{j,k\to\infty} \|u_{m_j} - u_{m_k}\|_{L^q(V)} \leq \delta.$$

To see this, let us first employ assertion (2) to select $\varepsilon > 0$ *so small that*

$$\|u_m^\varepsilon - u_m\|_{L^q(V)} \leq \delta/2$$

for $m = 1, 2, \ldots$.

But then (6) and (7) imply

$$\limsup_{j,k\to\infty} \left\|u_{m_j} - u_{m_k}\right\|_{L^q(V)} \leq \delta$$

and so (5) is proved.

Step 7. We next employ assertion (5) with $\delta = 1, \frac{1}{2}, \frac{1}{3}, \ldots$ *and use a standard diagonal argument to extract a subsequence* $\{u_{m_l}\}_{l=1}^\infty \subset \{u_m\}_{m=1}^\infty$ *satisfying*

$$\limsup_{l,k\to\infty} \|u_{m_l} - u_{m_k}\|_{L^q(V)} = 0.$$

Corollary 4.3. *Assume* Ω *is a bounded open subset of* $\mathbb{R}^n$ *and* $\partial\Omega$ *is* C^1. *We have in particular*

$$W^{1,p}(\Omega) \subset\subset L^p(\Omega)$$

for all $1 \leq p \leq \infty$.

Proof: 1. If $1 \leq p < n$, *it is obvious from Rellich-Kondrachov Compactness Theorem.*

2. If $n < p \leq \infty$, *there is a version* $u_n^* \in C^{0,\gamma}(\bar{\Omega})$ *of* u_n *for bounded sequence* $\{u_n\}$ *in* $W^{1,p}(\Omega)$

and also

$$\|u_n^*\|_{C^{0,\gamma}(\bar{\Omega})} \leq C\|u_n\|_{W^{1,p}(\Omega)} \leq M$$

which implies u_n^* is uniformly bounded and equicontinuous. By Arzela-Ascoli theorem, $\{u_{n_k}\}$ convergent uniformly to $u \in C(\Omega)$ and also

$$u_{n_k} \to u \quad \text{in } L^p(\Omega)$$

since Ω is bounded.

Theorem 4.4. *Assume Ω is a bounded open subset of $\mathbb{R}^n$. Suppose $1 \leq p < n$. Then*

$$W^{1,p}(\Omega) \subset\subset L^q(\Omega)$$

for each $1 \leq q < p^*$.

§5 Difference quotients and $\mathrm{W}^{1,p}$

Definition 5.1. *Assume $u : \Omega \to \mathbb{R}$ is a locally summable function and $V \subset\subset \Omega$.*

(1) The i^{th} -difference quotient of size h is

$$D_i^h u(x) = \frac{u(x + he_i) - u(x)}{h} \quad (i = 1, \ldots, n)$$

for $x \in V$ and $h \in \mathbb{R}, 0 < |h| < \operatorname{dist}(V, \partial\Omega)$.

(2) $D^h u := \left(D_1^h u, \ldots, D_n^h u\right)$.

Proposition 5.2. *Choose $i = 1, \ldots, n, \phi \in C_c^\infty(V)$, and note for small enough h that*

$$\int_V u(x) \left[\frac{\phi(x + he_i) - \phi(x)}{h}\right] dx = -\int_V \left[\frac{u(x) - u(x - he_i)}{h}\right] \phi(x) dx$$

that is,

$$\int_V u\left(D_i^h \phi\right) dx = -\int_V \left(D_i^{-h} u\right) \phi dx$$

Theorem 5.3 (Difference quotients and weak derivatives)**.** *Let Ω be a open sets of $\mathbb{R}^n$.*

(1) Suppose $1 \leq p < \infty$ and $u \in W^{1,p}(\Omega)$. Then for each $V \subset\subset \Omega$

$$\left\|D^h u\right\|_{L^p(V)} \leq C\|Du\|_{L^p(\Omega)}$$

for some constant C depending on Ω, V and all $0 < |h| < \frac{1}{2} \operatorname{dist}(V, \partial\Omega)$.

(2) Assume $1 < p < \infty, u \in L^p(V)$, and there exists a constant C such that

$$\left\|D^h u\right\|_{L^p(V)} \leq C$$

for all $0<|h|<\frac{1}{2}\operatorname{dist}(V, \partial \Omega)$. *Then*

$$u \in W^{1,p}(V), \quad \textit{with} \quad \|Du\|_{L^p(V)} \leq C$$

Proof: Step1. Assume $1 \leq p<\infty$, *and temporarily suppose* u *is smooth. Then for each* $x \in V, i=1, \ldots, n$, *and* $0<|h|<\frac{1}{2}\operatorname{dist}(V, \partial \Omega)$, *we have*

$$u\left(x+h e_i\right)-u(x)=h \int_0^1 u_{x_i}\left(x+t h e_i\right) d t$$

so that

$$\left|u\left(x+h e_i\right)-u(x)\right| \leq|h| \int_0^1\left|D u\left(x+t h e_i\right)\right| d t$$

Consequently

$$\begin{aligned}
\int_V\left|D^h u\right|^p d x & \leq C \sum_{i=1}^n \int_V \int_0^1\left|D u\left(x+t h e_i\right)\right|^p d t d x \\
& =C \sum_{i=1}^n \int_0^1 \int_V\left|D u\left(x+t h e_i\right)\right|^p d x d t
\end{aligned}$$

Thus

$$\int_V\left|D^h u\right|^p d x \leq C \int_{\Omega}|D u|^p d x$$

This estimate holds should u *be smooth, and thus is valid by approximation for arbitrary* $u \in W^{1,p}(\Omega)$.

2. Now suppose estimate holds for all $0<|h|<\frac{1}{2}\operatorname{dist}(V, \partial \Omega)$ *and some constant* C. *Estimate (9) implies*

$$\sup_h\left\|D_i^{-h} u\right\|_{L^p(V)}<\infty$$

and therefore, since $1<p<\infty$, *there exists a function* $v_i \in L^p(V)$ *and a subsequence* $h_k \rightarrow 0$ *such that*

$$D_i^{-h_k} u \rightharpoonup v_i \quad \textit{weakly in } L^p(V)$$

But then

$$\begin{aligned}
\int_V u \phi_{x_i} d x & =\int_{\Omega} u \phi_{x_i} d x \\
& =\lim _{h_k \rightarrow 0} \int_{\Omega} u D_i^{h_k} \phi d x \\
& =-\lim _{h_k \rightarrow 0} \int_V D_i^{-h_k} u \phi d x \\
& =-\int_V v_i \phi d x \\
& =-\int_{\Omega} v_i \phi d x
\end{aligned}$$

Thus $v_i=u_{x_i}$ *in the weak sense (* $i=1, \ldots, n$ *), and so* $D u \in L^p(V)$. *As* $u \in L^p(V)$, *we deduce therefore that* $u \in W^{1,p}(V)$.

§5.1 Lipschitz functions and $\mathrm{W}^{1,\infty}$

Theorem 5.4 (Characterization of $W^{1,\infty}$)**.** *Let Ω be open and bounded, with $\partial\Omega$ of class C^1. Then $u : \Omega \to \mathbb{R}$ is Lipschitz continuous if and only if $u \in W^{1,\infty}(\Omega)$.*

Proof. Step 1. First assume $\Omega = \mathbb{R}^n$ and u has compact support. Suppose $u \in W^{1,\infty}(\mathbb{R}^n) \subset C^{0,\gamma}$. Then $u^\varepsilon := \eta_\varepsilon * u$ is smooth and satisfies

$$\begin{cases} u^\varepsilon \to u \text{ uniformly as } \varepsilon \to 0 \\ \|Du^\varepsilon\|_{L^\infty(\mathbb{R}^n)} \leq \|Du\|_{L^\infty(\mathbb{R}^n)} \end{cases}$$

Choose any two points $x, y \in \mathbb{R}^n, x \neq y$. We have

$$\begin{aligned} u^\varepsilon(x) - u^\varepsilon(y) &= \int_0^1 \frac{d}{dt} u^\varepsilon(tx + (1-t)y)dt \\ &= \int_0^1 Du^\varepsilon(tx + (1-t)y)dt \cdot (x - y) \end{aligned}$$

and so

$$|u^\varepsilon(x) - u^\varepsilon(y)| \leq \|Du^\varepsilon\|_{L^\infty(\mathbb{R}^n)} |x - y| \leq \|Du\|_{L^\infty(\mathbb{R}^n)}|x - y|.$$

We let $\varepsilon \to 0$ to discover

$$|u(x) - u(y)| \leq \|Du\|_{L^\infty(\mathbb{R}^n)}|x - y|.$$

Hence u is Lipschitz continuous.

Step 2. On the other hand assume now u is Lipschitz continuous; we must prove that u has essentially bounded weak first derivatives. Since u is Lipschitz, we see

$$\left\|D_i^{-h} u\right\|_{L^\infty(\mathbb{R}^n)} \leq \mathrm{Lip}(u)$$

and thus there exists a function $v_i \in L^\infty(\mathbb{R}^n)$ and a subsequence $h_k \to 0$ such that

$$D_i^{-h_k} u \rightharpoonup v_i \quad \text{weakly in } L^2_{loc}(\mathbb{R}^n)$$

Consequently

$$\begin{aligned} \int_{\mathbb{R}^n} u\phi_{x_i} dx &= \lim_{h_k \to 0} \int_{\mathbb{R}^n} u D_i^{h_k} \phi dx \\ &= -\lim_{h_k \to 0} \int_{\mathbb{R}^n} D_i^{-h_k} u \phi dx \\ &= -\int_{\mathbb{R}^n} v_i \phi dx \end{aligned}$$

by (12). The above equality holds for all $\phi \in C_c^\infty(\mathbb{R}^n)$, and so $v_i = u_{x_i}$ in the weak sense $(i = 1, \dots, n)$. Consequently $u \in W^{1,\infty}(\mathbb{R}^n)$.

3. In the general case that Ω is bounded, with $\partial\Omega$ of class C^1, we as usual extend u to $Eu = \bar{u}$ and apply the above argument.

Corollary 5.5. *Let Ω be a open set $u \in W^{1,\infty}_{loc}(\Omega)$ if and only if u is locally Lipschitz continuous in Ω.*

Theorem 5.6 (Differentiability almost everywhere). *Assume $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ for some $n < p \leq \infty$. Then u is differentiable a.e. in Ω, and its gradient equals its weak gradient a.e.*

Proof: Recall that we always identify u with its continuous version.

1. Assume first $n < p < \infty$. From the remark after the proof of Theorem 4 in §5.6.2, we recall Morrey's estimate

$$|v(y) - v(x)| \leq Cr^{1-\frac{n}{p}} \left(\int_{B(x,2r)} |Dv(z)|^p dz \right)^{1/p} \quad (y \in B(x,r))$$

valid for any C^1 function v and thus, by approximation, for any $v \in W^{1,p}$.

2. Choose $u \in W^{1,p}_{loc}(\Omega)$. Now for a.e. $x \in \Omega$, a version of Lebesgue's Differentiation Theorem (§E.4) implies

$$f_{B(x,r)} |Du(x) - Du(z)|^p dz \to 0$$

as $r \to 0$, Du denoting as usual the weak derivative of u. Fix any such point x and set

$$v(y) := u(y) - u(x) - Du(x) \cdot (y - x)$$

in estimate (14), where

$$r = |x - y|$$

We find

$$\begin{aligned} &| \, u(y) - u(x) - Du(x) \cdot (y - x) \, | \\ &\leq Cr^{1-n/p} \left(\int_{B(x,2r)} |Du(x) - Du(z)|^p dz \right)^{1/p} \\ &\leq Cr \left(f_{B(x,2r)} |Du(x) - Du(z)|^p dz \right)^{1/p} \\ &= o(r) \quad \text{by (15)} \\ &= o(|x - y|) \quad \text{by (16).} \end{aligned}$$

Thus u is differentiable at x, and its gradient equals its weak gradient at x. 3. In case $p = \infty$, we note $W^{1,\infty}_{loc}(\Omega) \subset W^{1,p}_{loc}(\Omega)$ for all $1 \leq p < \infty$ and apply the reasoning above.

Theorem 5.7 (Rademacher's Theorem). *Let u be locally Lipschitz continuous in Ω. Then u is differentiable almost everywhere in Ω.*

Part IV

Second Order Elliptic Equatons