Representation of Groups

HHH

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Introduction

§1

Definition 1.1. *Suppose now G is a finite group and k is a field.*

1. *A k-**linear representation** of G is a group homomorphism*

$$\rho : G \to \mathrm{GL}(V)$$

   where V is a k vector space of dimension n.

2. *Let ρ and ρ' be two representations of the same group G in vector spaces $\mathbf{V}$ and $\mathbf{V}'$. These representations are said to be similar (or isomorphic) if there exists a linear isomorphism $\tau : \mathbf{V} \to \mathbf{V}'$ which "transforms" ρ into ρ', that is, which satisfies the identity*

$$\tau \circ \rho(s) = \rho'(s) \circ \tau \quad \textit{for all } s \in \mathbf{G}.$$

   When ρ and ρ' are given in matrix form by R_s and R'_s respectively, this means that there exists an invertible matrix T such that

$$\mathrm{T} \cdot \mathrm{R}_s = \mathrm{R}'_s \cdot \mathrm{T}, \quad \textit{for all } s \in \mathrm{G},$$

   which is also written $\mathrm{R}'_s = \mathrm{T} \cdot \mathrm{R}_s \cdot \mathrm{T}^{-1}$. We can identify two such representations (by having each $x \in \mathrm{V}$ correspond to the element $\tau(x) \in \mathrm{V}'$); in particular, ρ and ρ' have the same degree.

Remark. *A k-linear representation of G is equivalent to a finite dimension $k[G]$-module*