# A Functorial and G-Set Theoretic Formulation of the Fundamental Theorem of Galois Theory

## Contents



# 1 Introduction

This note formulates the classical Fundamental Theorem of Galois Theory in a modern categorical language, and further describes the Galois correspondence via $G$-sets. We assume throughout that $L/K$ is a finite Galois extension with Galois group

$$G = \mathrm{Gal}(L/K).$$

The goal is to reinterpret:

$$\{K \subseteq F \subseteq L\} \quad \longleftrightarrow \quad \{H \leq G\}$$

as an equivalence between two categories and then as a correspondence between $G$-equivariant quotients of a canonical $G$-set.

# 2 Intermediate Fields and Subgroups as Categories

## 2.1 The category of intermediate fields

**Definition 2.1.** Let $\mathbf{Int}(L/K)$ be the category whose objects are intermediate fields

$$K \subseteq F \subseteq L,$$

and where there is a unique morphism $F_1 \to F_2$ iff $F_1 \subseteq F_2$. Thus $\mathbf{Int}(L/K)$ is a poset category.

## 2.2 The category of subgroups

**Definition 2.2.** Let $\mathbf{Sub}(G)$ be the category whose objects are subgroups $H \leq G$ and where there is a unique morphism $H_1 \to H_2$ iff $H_1 \subseteq H_2$. This is also a poset category.

# 3 Two Fundamental Contravariant Functors

## 3.1 The fixed field functor

**Definition 3.1.** Define the functor

$$\mathrm{Fix} : \mathbf{Sub}(G)^{op} \to \mathbf{Int}(L/K)$$

by sending a subgroup $H \leq G$ to its fixed field

$$L^H = \{x \in L : \sigma(x) = x \text{ for all } \sigma \in H\}.$$

If $H_1 \subseteq H_2$, then the morphism in $\mathbf{Sub}(G)$ is reversed in $\mathbf{Sub}(G)^{op}$, yielding the inclusion

$$L^{H_1} \subseteq L^{H_2}.$$

## 3.2 The Galois group functor

**Definition 3.2.** Define the functor

$$\mathrm{Gal} : \mathbf{Int}(L/K)^{op} \to \mathbf{Sub}(G)$$

by

$$\mathrm{Gal}(F) = \mathrm{Gal}(L/F).$$

If $F_1 \subseteq F_2$, then restriction gives

$$\mathrm{Gal}(L/F_1) \subseteq \mathrm{Gal}(L/F_2).$$

# 4 The Categorical Fundamental Theorem of Galois Theory

**Theorem 4.1** (Categorical Galois Correspondence)**.** There is a contravariant equivalence of categories

$$\mathrm{Fix} : \mathbf{Sub}(G)^{op} \rightleftarrows \mathbf{Int}(L/K) : \mathrm{Gal}\,.$$

Moreover,

$$\mathrm{Fix} \circ \mathrm{Gal} \cong \mathrm{id}_{\mathbf{Int}(L/K)}, \qquad \mathrm{Gal} \circ \mathrm{Fix} \cong \mathrm{id}_{\mathbf{Sub}(G)}.$$

# 5 Galois Correspondence via G-Sets

## 5.1 The canonical $G$-set

Let

$$X = \mathrm{Hom}_{K\text{-alg}}(L, \overline{K})$$

where $\overline{K}$ is a fixed algebraic closure.

**Proposition 5.1.** The group $G = \mathrm{Gal}(L/K)$ acts on $X$ by

$$(\sigma \cdot \varphi) = \sigma \circ \varphi.$$

This makes $X$ into a finite transitive $G$-set.

## 5.2 Intermediate fields correspond to quotients of this $G$-set

For every intermediate field $F$, restriction gives a natural $G$-equivariant map:

$$\mathrm{res}_{L/F} : X \to \mathrm{Hom}_K(F, \overline{K}).$$

**Proposition 5.2.** There is a $G$-equivariant bijection

$$\mathrm{Hom}_K(F, \overline{K}) \;\cong\; X/H_F, \qquad H_F = \mathrm{Gal}(L/F).$$

Thus:

**Theorem 5.3** (G-set formulation of Galois correspondence)**.** The category of intermediate fields satisfies

$$\mathbf{Int}(L/K) \;\simeq\; \mathbf{G\text{-}Set}_X,$$

the category of $G$-equivariant quotient maps $X \to Y$. Every such quotient is isomorphic to $X/H$ for a unique subgroup $H \leq G$.

## 5.3 Conclusion

Because $X$ is a transitive $G$-set, all its equivariant quotients are of the form $G/H$. Therefore,

$$\mathbf{G\text{-}Set}_X \;\simeq\; \mathbf{Sub}(G)^{op}.$$

Combining everything we obtain:

**Theorem 5.4** (Galois Correspondence via G-sets)**.**

$$\mathbf{Int}(L/K) \;\simeq\; \mathbf{Sub}(G)^{op} \;\simeq\; \mathbf{G\text{-}Set}_X.$$

Each intermediate field $F$ corresponds to the subgroup

$$H_F = \mathrm{Gal}(L/F)$$

and to the $G$-set quotient

$$\mathrm{Hom}_K(F, \overline{K}) \cong X/H_F.$$