Topology

HHH

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Contents

Chapter 1

Topological Space

1.1 Topological Space

1.1.1 Basic Definition

Definition 1.1.1. *A **topology** on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties:*

(a) $\varnothing$ and X are in $\mathcal{T}$.

(b) The union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

(c) The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

*A set X for which a topology $\mathcal{T}$ has been specified is called a **topological space**.*

Proposition 1.1.2. *Let X be a topological space. Then the following conditions hold:*

    (1) $\varnothing$ and X are closed.
    (2) Arbitrary intersections of closed sets are closed.
    (3) Finite unions of closed sets are closed.

Definition 1.1.3. *Suppose that $\mathcal{T}$ and $\mathcal{T}'$ are two topologies on a given set X. If $\mathcal{T}' \supset \mathcal{T}$, we say that $\mathcal{T}'$ is **finer** than $\mathcal{T}$; if $\mathcal{T}'$ properly contains $\mathcal{T}$, we say that $\mathcal{T}'$ is **strictly finer** than $\mathcal{T}$.*

We also say that $\mathcal{T}$ is **coarser** than $\mathcal{T}'$, or strictly coarser, in these two respective situations.

We say $\mathcal{T}$ is **comparable** with $\mathcal{T}'$ if either $\mathcal{T}' \supset \mathcal{T}$ or $\mathcal{T} \supset \mathcal{T}'$.

1.1.2 Limit Points, Closure and Interior

Definition 1.1.4. *Given a subset A of a topological space $(X, \mathscr{T})$, the **interior** of A is defined as the union of all open sets contained in A, denoted by $\operatorname{Int} A$. And the **closure** of A is defined as the intersection of all closed sets containing A, denoted by $\bar{A}$.*

Obviously $\operatorname{Int} A$ is an open set and $\bar{A}$ is a closed set; furthermore,

$$\operatorname{Int} A \subset A \subset \bar{A}$$

Definition 1.1.5. *If A is a subset of the topological space X and if x is a point of X, we say that x is a **limit point** of A if every neighborhood of x intersects A in some point other than x itself.*

*The sets A' consisting of all limit points of A is called **derived set** of A.*

Theorem 1.1.6. *Let A be a subset of the topological space X.*
(1) Then $x \in \bar{A}$ if and only if every open set U containing x intersects A.
(2) x is a limit point of A if and only if it belongs to the closure of $A - \{x\}$.
(3) $\bar{A} = A \cup A'$

Corollary 1.1.7. *A subset of a topological space is closed if and only if it contains all its limit points.*

1.1.3 Basis For a Topological Space

Definition 1.1.8. *If X is a set,*
*(1) A **basis** for a topology on X is a collection $\mathscr{B}$ of subsets of X (called basis elements) such that*

(a) For each $x \in X$, there is at least one basis element B containing x.

(b) If x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 containing x such that $B_3 \subset B_1 \cap B_2$.

(2) If $\mathscr{B}$ satisfies these two conditions, then we define the topology $\mathcal{T}$ **generated by basis** *$\mathcal{B}$ as follows: A subset U of X is said to be open in X if for each $x \in U$, there is a basis element $B \in \mathscr{B}$ such that $x \in B$ and $B \subset U$.*

Proposition 1.1.9. *Let X be a set; let $\mathscr{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ equals the collection of all unions of elements of $\mathscr{B}$.*

Definition 1.1.10. *A **subbasis** $\mathcal{S}$ for a topology on X is a collection of subsets of X whose union equals X. The topology generated by the subbasis S is defined to be the collection $\mathcal{T}$ of all unions of finite intersections of elements of $\mathcal{S}$.*

1.1.4 Subspace Topology

Definition 1.1.11. *Let X be a topological space with topology $\mathcal{T}$. If Y is a subset of X, the collection*

$$\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}\}$$

*is a topology on Y, called the **subspace topology**. With this topology, Y is called a subspace of X; its open sets consist of all intersections of open sets of X with Y.*

Theorem 1.1.12. *Let X be a topological space and Y a subset of X*
   (1) If $\mathscr{B}$ is a basis for the topology of X then the collection

$$\mathscr{B}_Y = \{B \cap Y \mid B \in \mathscr{B}\}$$

is a basis for the subspace topology on Y.
   (2) Then a set A is closed in Y if and only if it equals the intersection of a closed set of X with Y.
   (3) If A is closed in Y and Y is closed in X, then A is closed in X.

Proof. (1)
   (2) Assume that $A = C \cap Y$, where C is closed in X. Then $X - C$ is open in X, so that the complement of A in Y, $Y - A = (X - C) \cap Y$, is open in Y, then so A is closed in Y.
   Conversely, assume that A is closed in Y. Then $Y - A$ is open in Y, so that it equals the intersection of an open set U of X with Y. The set $X - U$ is closed in

X, and $A = Y \cap (X - U)$, so that A equals the intersection of a closed set of X with Y, as desired. $\square$

Chapter 2

Countability

2.1 The Countability Axioms

Definition 2.1.1. *A space X is said to have a countable basis at x if there is a countable collection $\mathscr{B}$ of neighborhoods of x such that each neighborhood of x contains at least one of the elements of $\mathscr{B}$.*

*A space that has a countable basis at each of its points is said to satisfy the first countability axiom, or to be **first-countable**.*

Theorem 2.1.2. *Let X be a topological space.*

(1) Let A be a subset of X. If there is a sequence of points of A converging to x, then $x \in \bar{A}$; the converse holds if X is first-countable.

(2) Let $f : X \to Y$. If f is continuous, then for every convergent sequence $x_n \to x$ in X, the sequence $f(x_n)$ converges to $f(x)$. The converse holds if X is first countable.

Definition 2.1.3. *If a space X has a countable basis for its topology, then X is said to satisfy the second countability axiom, or to be **second-countable**.*

Definition 2.1.4. *A subset A of a space X is said to be **dense** in X if $\bar{A} = X$. A space having a countable dense subset is often said to be **separable**.*

Theorem 2.1.5 (Hereditarity and Countable Multiplicativity). *A subspace of a first-countable space is first-countable, and a countable product of first-countable spaces is first-countable.*

A subspace of a secondcountable space is second-countable, and a countable product of second-countable spaces is second-countable.

Definition 2.1.6. *A space for which every open coverng contains a countable subcovering is called a **Lindelöf** space.*

Theorem 2.1.7. *Suppose that X has a countable basis. Then:*
(1) *X is a Lindelöf space.*
(2) *X is separable.*

2.2 The Separation Axioms

Definition 2.2.1. *A space X is said to be **Frechét** (or T_1) if for each pair of distinct point x and y, there exist neighborhood of x which not contains y.*

*A space X is said to be **Hausdorff** (or T_2) if for each pair consisting of distinct point x and y, there exist disjoint open sets containing x and y, respectively.*

*A space X is said to be **regular** (or T_3) if for each pair consisting of a point x and a closed set B disjoint from x, there exist disjoint open sets containing x and B, respectively.*

*The space X is said to be **normal** (or T_4) if for each pair A, B of disjoint closed sets of X, there exist disjoint open sets containing A and B, respectively.*

Theorem 2.2.2. *Let (X, τ) be topology space*
(1) *X is T_1 if and only if single point set $\{x\}$ is closed.*
(2) *X is T_2 if and only if $\Delta = \{(x, x) : x \in X\}$ is closed in $X \times X$*
(3) *X is T_3 if and only if for every $x \in X$ and open set U containing x, there exists open V that $x \in V \subset \overline{V} \subset U$.*
(4) *X is T_4 if and only if for every closed $A \subset X$ and open set U containing A, there exists open V that $A \subset V \subset \overline{V} \subset U$.*

Proposition 2.2.3. *Let X be a space. Then*
(1) $T_2 \Rightarrow T_1$
(2) *Compact* $+T_2 \Rightarrow T_3$
(3) *Compact* $+T_3 \Rightarrow T_4$; *thus Compact* $+T_2 \Rightarrow T_4$
(4) *Lindelof* $+T_3 \Rightarrow T_4$; *Thus* $A_2 + T_3 \Rightarrow T_4$ *and σ-compact* $+T_3 \Rightarrow T_4$

2.3 Urysohn Lemma

Theorem 2.3.1 (Urysohn lemma)**.** *Let X be a normal (T_4) space, let A and B be disjoint closed subsets of X. Let $[a, b]$ be a closed interval in the real line. Then there exists a continuous map*

$$f : X \longrightarrow [a, b]$$

such that $f(A) = a$ and $f(B) = b$.

Proof. Put $r_1 = 0, r_2 = 1$, and let $r_3, r_4, r_3, \ldots$ be an enumeration of the rationals in $(0, 1)$. By Theorem 2.7, we can find open sets V_0 and then V_1 such that $\overline{V}_0$ is compact and

$$A \subset V_1 \subset \overline{V}_1 \subset V_0 \subset \overline{V}_0 \subset B^c$$

Suppose $n \geq 2$ and $V_{r_1}, \ldots, V_{r_n}$ have been chosen in such a manner that $r_i < r_j$ implies $\overline{V}_{r_j} \subset V_{r_1}$. Then one of the numbers $r_1, \ldots, r_n$, say r_i, will be the largest one which is smaller than r_{n+1}, and another, say r_j, will be the smallest one larger than r_{n+1}. Using Theorem 2.7 again, we can find $V_{r_{n+1}}$ so that

$$\overline{V}_{r_j} \subset V_{r_{n+1}} \subset \overline{V}_{r_{n+1}} \subset V_{r_j}$$

Continuing, we obtain a collection $\{V_r\}$ of open sets, one for every rational $r \in [0, 1]$, with the following properties: $A \subset V_1, \overline{V}_0 \subset B^c$, each $\overline{V}_r$, and

$$s > r \text{ implies } \overline{V}_s \subset V_r$$

Define

$$f_r(x) = r\chi_{V_r} \quad g_s(x) = s + (1 - s)\chi_{\overline{V}_s}$$

and

$$f = \sup_r f_r, \quad g = \inf_s g_s$$

then f is lower semicontinuous and that g is upper semicontinuous. It is clear that $0 \leq f \leq 1$, that $f(A) = 1$ and that $f(B) = 0$. The proof will be completed by showing that $f = g$.

The inequality $f_r(x) > g_s(x)$ is possible only if $r > s, x \in V_r$, and $x \notin \bar{V}_s$. But $r > s$ implies $V_r \subset V_s$. Hence $f_r \leq g$, for all r and s, so $f \leq g$. Suppose $f(x) < g(x)$ for some x. Then there are rationals r and s such that $f(x) < r < s < g(x)$. Since $f(x) < r$, we have $x \notin V_r$; since $g(x) > s$, we have $x \in \bar{V}_s$. By (3), this is a contradiction. Hence $f = g$. $\qquad\square$

Corollary 2.3.2. *Suppoes that X is a T_4 space, then there exists $f \in C(X)$ that $f^{-1}(0) = A$ if and only if A is a closed G_δ's.*

Proof. 1.
$$A = f^{-1}(0) = \bigcup_n f^{-1}\left(\left(-\frac{1}{n}, \frac{1}{n}\right)\right)$$

2. Assume that A is a closed G_δ's, there exists open sers U_i that $A = \bigcap U_i$. By Urysohn lemma, there is $g_n : X \to [0, 1]$ that

$$A \subset g_n^{-1}(0), \quad U_n^c \subset g_n^{-1}(1)$$

Define

$$f = \sum_{n=1}^{\infty} \frac{g_n(x)}{2^n}$$

thus $f^{-1}(0) = A$

Theorem 2.3.3 (Strong Urysohn lemma)**.** *Let X be a normal (T_4) space, let A and B be disjoint closed G_δ's subsets of X. Let $[a, b]$ be a closed interval in the real line. Then there exists a continuous map $f : X \to [a, b]$ such that*

$$f^{-1}(a) = A, \quad f^{-1}(b) = B$$

2.4 Tietze Extension Theorem

Theorem 2.4.1 (Tietze extension theorem)**.** *Let X be a normal T_4 space; let A be a closed subspace of X. Then any continuous map $f : A \to \mathbb{R}$ may be extended to a continuous map $F : X \to \mathbb{R}$ that $F|_A = f$*

2.5 Urysohn Metrization Theorem

Proposition 2.5.1. *The topology space (X, τ) is metrizable if and only if X can be topologically embedded into a metric space.*

Theorem 2.5.2. *If X is T_1, T_4 and A_2 (thus T_i, A_j), then X can be imbedding into $[0,1]^{\omega}$ or $\mathbb{R}^{\omega}$; thus metrizable.*

Chapter 3

Connectness

3.1 Connected Space

Definition 3.1.1. *Let X be a topological space. A **separation** of X is a pair U, V of disjoint nonempty open subsets of X whose union is X. The space X is said to be **connected** if there does not exist a separation of X.*

Theorem 3.1.2. *If Y is a subspace of X, a separation of Y is a pair of disjoint nonempty sets A and B whose union is Y, neither of which contains a limit point of the other. The space Y is connected if there exists no separation of Y.*

Proof. Suppose first that A and B form a separation of Y. Then A is both open and closed in Y. The closure of A in Y is the set $\bar{A} \cap Y$ (where $\bar{A}$ as usual denotes the closure of A in X). Since A is closed in $Y, A = \bar{A} \cap Y$; or to say the same thing, $\bar{A} \cap B = \varnothing$. Since $\bar{A}$ is the union of A and its limt points, B contains no limit points of A. A similar argument shows that A contains no limt points of B.

Conversely, suppose that A and B are disjoint nonempty sets whose union is Y, neither of which contains a limt point of the other. Then $\bar{A} \cap B = \varnothing$ and $A \cap \bar{B} = \varnothing$; therefore, we conclude that $\bar{A} \cap Y = A$ and $\bar{B} \cap Y = B$. Thus both A and B are closed in Y, and since $A = Y - B$ and $B = Y - A$, they are open in Y as well. □

Theorem 3.1.3. *Let (X, τ) be a topology space.*

(1) If the sets C and D form a separation of X, and if Y is a connected subspace of X, then Y lies entirely within either C or D.

(2) The union of a collection of connected subspaces of X that have a point in common is connected.

(3) Let A be a connected subspace of X. If $A \subset B \subset \bar{A}$, then B is also connected.

Proof. We only prove (3). Let A be connected and let $A \subset B \subset \bar{A}$. Suppose that $B = C \cup D$ is a separation of B, then A must lie entirely in C or in D; suppose that $A \subset C$. Then $\bar{A} \subset \bar{C}$; since $\bar{C}$ and D are disjoint, B cannot intersect D. This contradicts the fact that D is a nonempty subset of B. $\square$

Theorem 3.1.4. *A finite cartesian product of connected spaces is connected.*

Proof. We prove the theorem first for the product of two connected spaces X and Y. Choose a "base point" $a \times b$ in the product $X \times Y$. Note that the "horizontal slice" $X \times b$ is connected, and each "vertical slice" $x \times Y$ is connected. As a result, each "T-shaped" space

$$T_x = (X \times b) \cup (x \times Y)$$

is connected, being the union of two connected spaces that have the point $x \times b$ in common. Now the union

$$\bigcup_{x \in X} T_x = X \times Y$$

is connected because it is the union of a collection of connected spaces that have the point $a \times b$ in common.

The proof for any finite product of connected spaces follows by induction, using the fact that $X_1 \times \cdots \times X_n$ is homeomorphic with $(X_1 \times \cdots \times X_{n-1}) \times X_n$. $\square$

Theorem 3.1.5. *The image of a connected space under a continuous map is connected.*

3.2

Lemma 3.2.1. *Given X, define an equivalence relation on X by setting $x \sim y$ if there is a connected subspace of X containing both x and y. The equivalence*

*classes are called the **connected components** of X.*

Theorem 3.2.2. *The components of X are connected disjoint subspaces of X whose union is X, such that each nonempty connected subspace of X intersects only one of them.*

Definition 3.2.3. *We define another equivalence relation on the space X by defining $x \sim y$ if there is a path in X from x to y. The equivalence classes are called the **path components** of X.*

Theorem 3.2.4. *The path components of X are path-connected disjoint subspaces of X whose union is X, such that each nonempty path-connected subspace of X intersects only one of them.*

Definition 3.2.5. *A space X is said to be **locally connected at** x if for every neighborhood U of x, there is a connected neighborhood V of x contained in U. If X is locally connected at each of its points, it is said simply to be **locally connected**.*

*Similarly, a space X is said to be locally path connected at x if for every neighborhood U of x, there is a path-connected neighborhood V of x contained in U. If X is locally path connected at each of its points, then it is said to be **locally path connected**.*

Theorem 3.2.6. *A space X is locally connected if and only if for every open set U of X, each connected component of U is open in X.*

Proof. Suppose that X is locally connected; let U be an open set in X; let C be a component of U If x is a point of C, we can choose a connected neighborhood V of x such that $V \subset U$. Since V is connected, it must lie entirely in the component C of U. Therefore, C is open in X.

Conversely, suppose that components of open sets in X are open. Given a point x of X and a neighborhood U of x, let C be the component of U containing x. Now C is connected; since it is open in X by hypothesis, X is locally connected at x. $\qquad\square$

Theorem 3.2.7. *A space X is locally path connected if and only if for every open set U of X, each path connected component of U is open in X.*

Theorem 3.2.8. *Let (X, τ) be a topology space.*

   (1) If X is path connected, then X is connected.

   (2) Each path component of X lies in a component of X

   (3) If X is locally path connected, then the components and the path components of X are the same.

Chapter 4

Compactness

4.1 Basic Definition

Definition 4.1.1. *A collection $\mathcal{A}$ of subsets of a space X is said to cover X, or to be a covering of X, if the union of the elements of $\mathcal{A}$ is equal to X. It is called an open covering of X if its elements are open subsets of X.*

Definition 4.1.2. *A space X is said to be **compact** if every open covering $\mathcal{A}$ of X contains a finite subcollection that also covers X.*

*A space X is said to be **sequentially compact** if every sequence of X contains a convergent subsequence.*

Theorem 4.1.3 (Criterion for compact sets)**.** *Let Y be a subspace of X. Then Y is compact if and only if every covering of Y by sets open in X contains a finite subcollection covering Y.*

Proof. Suppose that Y is compact and $\mathcal{A} = \{A_\alpha\}_{\alpha \in J}$ is a covering of Y by sets open in X. Then the collection

$$\{A_\alpha \cap Y \mid \alpha \in J\}$$

is a covering of Y by sets open in Y; hence a finite subcollection

$$\{\boldsymbol{A}_{\boldsymbol{\alpha}_1} \cap Y, \ldots, \boldsymbol{A}_{\boldsymbol{\alpha}_n} \cap Y\}$$

covers Y. Then $\{A_{\alpha_1}, \ldots, A_{\alpha_n}\}$ is a subcollection of $\mathscr{A}$ that covers Y.

Conversely, suppose the given condition holds; we wish to prove Y compact. Let $\mathcal{A}' = \{A'_{\alpha}\}$ be a covering of Y by sets open in Y. For each α, choose a set A_{α} open in X such that

$$A'_{\alpha} = A_{\alpha} \cap Y.$$

The collection $\mathcal{A} = \{A_{\alpha}\}$ is a covering of Y by sets open in X By hypothesis, some finite subcollection $\{A_{\alpha_1}, \ldots, A_{\alpha_n}\}$ covers Y. Then $\{A'_{\alpha_1}, \ldots, A'_{\alpha_n}\}$ is a subcollection of $\mathcal{A}'$ that covers Y $\qquad\square$

Theorem 4.1.4. *Let X be a topology space.*

(1) If X is compact, then closed set of X is compact.

(2) Every compact subspace of a Hausdorff space is closed.

(3) If Y is a compact subspace of the T_2 space X and $x_0 \notin Y$, then there exist disjoint open sets U and V of X containing x_0 and Y, respectively.

(4) The image of a compact space under a continuous map is compact.

(5) The product of finitely many compact spaces is compact.

Lemma 4.1.5 (The tube lemma)**.** *Consider the product space $X \times Y$, where Y is compact. If N is an open set of $X \times Y$ containing the slice $x_0 \times Y$, then N contains some tube $W \times Y$, where W is a neighborhood of x_0 in X.*

Definition 4.1.6. *A collection $\mathcal{C}$ of subsets of X is sad to have the **finite intersection property** if for every finite subcollection*

$$\{C_1, \ldots, C_n\}$$

of $\mathcal{C}$, the intersection $C_1 \cap \cdots \cap C_n$ is nonempty.

Theorem 4.1.7. *Let X be a topological space. Then X is compact if and only if for every collection $\mathcal{C}$ of closed sets in X having the finite intersection property, the intersection $\bigcap_{C \in \mathcal{C}} C$ is nonempty.*

4.2 Tychonoff Theorem

Theorem 4.2.1 (Tychonoff theorem)**.** *An arbitrary product of compact spaces is compact in the product topology*

Proof. Let

$$X = \prod_{\alpha \in J} X_\alpha$$

where each space X_α is compact. Let $\mathcal{A}$ be a collection of subsets of X having the finite intersection property. We prove that the intersection is nonempty.

Compactness of X follows. Applying Lemma 37.1, choose a collection $\mathcal{D}$ of subsets of X such that $\mathcal{D} \supset \mathcal{A}$ and D is maximal with respect to the finite intersection property It will suffice to show that the intersection $\bigcap_{D \in D} \bar{D}$ is nonempty.

Given $\alpha \in J$, let $\pi_\alpha : X \to X_\alpha$ be the projection map, as usual Consider the collection

$$\{\pi_\alpha(D) \mid D \in D\}$$

§37 The Tychonoff Theorem 235 of subsets of X_α. This collection has the finite intersection property because $\mathcal{D}$ does. By compactness of X_α, we can for each α choose a point x_α of X_α such that

$$x_\alpha \in \bigcap_{D \in D} \overline{\pi_\alpha(D)}$$

Let x be the point $(x_\alpha)_{\alpha \in J}$ of X. We shall show that $\mathrm{x} \in \bar{D}$ for every $D \in D$; then our proof will be finished.

First we show that if $\pi_\beta^{-1}(U_\beta)$ is any subbasis element (for the product topology on X) containing x , then $\pi_\beta^{-1}(U_\beta)$ intersects every element of $\mathcal{D}$. The set U_β is a neighborhood of x_β in X_β. Since $x_\beta \in \overline{\pi_\beta(D)}$ by definition, U_β intersects $\pi_\beta(D)$ in some point $\pi_\beta(\mathbf{y})$, where $\mathbf{y} \in D$ Then it follows that $\mathbf{y} \in \pi_\beta^{-1}(U_\beta) \cap D$.

It follows from (b) of Lemma 37.2 that every subbasis element containing $\mathbf{x}$ belongs to $\mathfrak{D}$. And then it follows from (a) of the same lemma that every basis element containing $\mathbf{x}$ belongs to $\mathcal{D}$. Since $\mathcal{D}$ has the finite intersection property, this means that every basis element containing $\mathbf{x}$ intersects every element of $\mathcal{D}$; hence $\mathbf{x} \in \bar{D}$ for every $D \in D$ as desired. $\qquad \square$

4.3 Compactness in Metric Space

Theorem 4.3.1. *If X is a metric space, then the following*

(1) X is compact

(2) X is sequentially compact

(3) X is complete and totally bounded.

(4) Bolzano-Weierstrass property. Every infinite subset $A \subset X$ has a limit point at least.

Chapter 5

The Fundamental Group

5.1 Homotopy of Paths

Definition 5.1.1. *If f and f' are continuous maps of the space X into the space Y, we say that f is homotopic to f' if there is a continuous map $F : X \times I \to Y$ such that*

$$F(x, 0) = f(x) \quad and \quad F(x, 1) = f'(x)$$

*for each x. (Here $I = [0, 1]$.) The map F is called a **homotopy** between f and f'. If f is homotopic to f', we write $f \simeq f'$. If $f \simeq f'$ and f' is a constant map, we say that f is **nulhomotopic**.*

*Two paths f and f', mapping the interval $I = [0, 1]$ into X, are said to be **path homotopic** if they have the same initial point x_0 and the same final point x_1, and if there is a continuous map $F : I \times I \to X$ such that*

$$F(s, 0) = f(s) \quad and \quad F(s, 1) = f'(s)$$
$$F(0, t) = x_0 \quad and \quad F(1, t) = x_1$$

*for each $s \in I$ and each $t \in I$. We call F a **path homotopy** between f and f'. If f is path homotopic to f', we write $f \simeq_p f'$.*

Lemma 5.1.2. *The relations $\simeq$ and $\simeq_p$ are equivalence relations.*

Definition 5.1.3. *If f is a path in X from x_0 to x_1, and if g is a path in X from x_1 to x_2, we define the product $f * g$ of f and g to be the path h given by the*

equations

$$h(s) = \begin{cases} f(2s) & for \ s \in \left[0, \frac{1}{2}\right] \\ g(2s - 1) & for \ s \in \left[\frac{1}{2}, 1\right] \end{cases}$$

The function h is well-defined and continuous, by the pasting lemma; it is a path in X from x_0 to x_2.

Theorem 5.1.4. *The operation $*$ has the following properties:*

(1) The product operation on paths induces a well-defined operation on path-homotopy classes, defined by the equation

$$[f] * [g] = [f * g]$$

(2) Associativity. If $[f] * ([g] * [h])$ is defined, so is $([f] * [g]) * [h]$, and they are equal.

(3) Right and left identities. Given $x \in X$, let e_x denote the constant path $e_x : I \to X$ carrying all of I to the point x. If f is a path in X from x_0 to x_1, then

$$[f] * [e_{x_1}] = [f] \quad and \quad [e_{x_0}] * [f] = [f]$$

(4) Inverse. Given the path f in X from x_0 to x_1, let $\bar{f}$ be the path defined by $\bar{f}(s) = f(1 - s)$. It is called the reverse of f Then

$$[f] * [\bar{f}] = [e_{x_0}] \quad and \quad [\bar{f}] * [f] = [e_{x_1}]$$

Definition 5.1.5. *Let X be a space; let x_0 be a point of X. A path in X that begins and ends at x_0 is called a **loop based at** x_0. The set of path homotopy classes of loops based at x_0, with the operation $*$, is called the **fundamental group of** X **relative to the base point** x_0. It is denoted by $\pi_1(X, x_0)$.*

Theorem 5.1.6. *Let α be a path in X from x_0 to x_1. Then α induces a group isomorphism by*

$$\hat{\alpha} : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1)$$

by the equation

$$\hat{\alpha}([f]) = [\bar{\alpha}] * [f] * [\alpha] = [\bar{\alpha}]^{-1} * [f] * [\alpha]$$

Proposition 5.1.7. *Let α be a path in X from x_0 to x_1; let β be a path in X from x_1 to x_2. Show that if $v = \alpha * \beta$, then $\hat{v} = \hat{\beta} \circ \hat{\alpha}$.*

Corollary 5.1.8. *If X is path connected and x_0 and x_1 are two points of X, then $\pi_1(X, x_0)$ is isomorphic to $\pi_1(X, x_1)$.*

Definition 5.1.9. *A space X is said to be **simply connected** if it is a path-connected space and if $\pi_1(X, x_0)$ is the trivial (one-element) group for some $x_0 \in X$, and hence for every $x_0 \in X$. We often express the fact that $\pi_1(X, x_0)$ is the trivial group by writing $\pi_1(X, x_0) = 0$.*

Definition 5.1.10. *Let $h : (X, x_0) \to (Y, y_0)$ be a continuous map. Define*

$$h_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$$

by the equation

$$h_*([f]) = [h \circ f]$$

*The map h_* is called the **homomorphism induced by** h, relative to the base point x_0.*

Theorem 5.1.11. *If $h : (X, x_0) \to (Y, y_0)$ and $k : (Y, y_0) \to (Z, z_0)$ are continuous, then $(k \circ h)_* = k_* \circ h_*$ If $i : (X, x_0) \to (X, x_0)$ is the identity map, then i_* is the identity homomorphism.*

Corollary 5.1.12. *If $h : (X, x_0) \to (Y, y_0)$ is a homeomorphism of X with Y, then h_* is an isomorphism of $\pi_1(X, x_0)$ with $\pi_1(Y, y_0)$.*

5.2 Covering Space

Definition 5.2.1. *Let $p : E \to B$ be a continuous surjective map. The open set U of B is said to be **evenly covered by** p if the inverse image $p^{-1}(U)$ can be written as the union of disjoint open sets V_α in E such that for each α, the restriction of p to V_α is a homeomorphism of V_α onto U. The collection $\{V_\alpha\}$ will be called a partition of $p^{-1}(U)$ into slices*

*If every point b of B has a neighborhood U that is evenly covered by p, then p is called a **covering map**, and E is said to be a **covering space** of B*

Theorem 5.2.2 (Hereditarity). *Let $p : E \rightarrow B$ be a coverng map. If B_0 is a subspace of B, and if $E_0 = p^{-1}(B_0)$, then the map $p_0 : E_0 \rightarrow B_0$ obtained by restructing p is a covering map.*

Proof. Given $b_0 \in B_0$, let U be an open set in B containing b_0 that is evenly covered by p; let $\{V_\alpha\}$ be a partition of $p^{-1}(U)$ into slices. Then $U \cap B_0$ is a neighborhood of b_0 in B_0, and the sets $V_\alpha \cap E_0$ are disjoint open sets in E_0 whose union is $p^{-1}(U \cap B_0)$, and each is mapped homeomorphically onto $U \cap B_0$ by p. $\quad\square$

Theorem 5.2.3 (Finite Multiplicativity). *It $p : E \rightarrow B$ and $p' : E' \rightarrow B'$ are coverng maps, then*

$$p \times p' : E \times E' \rightarrow B \times B'$$

is a covering map.

Proof. Given $b \in B$ and $b' \in B'$, let U and U' be neighborhoods of b and b', respectively, that are evenly covered by p and p', respectively. Let $\{V_\alpha\}$ and $\{V'_\beta\}$ be partitions of $p^{-1}(U)$ and $(p')^{-1}(U')$, respectively, into slices. Then the inverse image under $p \times p'$ of the open set $U \times U'$ is the union of all the sets $V_\alpha \times V'_\beta$. These are disjoint open sets of $E \times E'$, and each is mapped homeomorphically onto $U \times U'$ by $p \times p'$. $\quad\square$

5.3 Lifting Lemma

Definition 5.3.1. *Let $p : E \rightarrow B$ be a map. If f is a continuous mapping of some space X into B, a **lifting** of f is a map $\bar{f} : X \rightarrow E$ such that $p \circ \bar{f} = f$.*

Theorem 5.3.2 (Path-lifting lemma). *Let $p : (E, e_0) \rightarrow (B, b_0)$ be a covering map. Any path $f : [0, 1] \rightarrow B$ beginning at b_0 has a unique lifting to a path $\bar{f}$ in E beginning at e_0.*

Theorem 5.3.3. *Let* $p : (E, e_0) \to (B, b_0)$ *be a covering map. Let the map* $F : I \times I \to B$ *be continuous, with* $F(0,0) = b_0$*. There is a unique lifting of* F *to a continuous map*

$$\tilde{F} : I \times I \to E$$

such that $\bar{F}(0,0) = e_0$*. If* F *is a path homotopy, then* $\tilde{F}$ *is a path homotopy.*

Definition 5.3.4. *Let* $p : (E, e_0) \rightarrow (B, b_0)$ *be a covering map. Given an element* $[f]$ *of* $\pi_1 (B, b_0)$, *let* $\bar{f}$ *be the lifting of* f *to a path in* E *that begins at* e_0. *Let*

$$\phi([f]) = \tilde{f}(1)$$

Then ϕ *is a well-defined set map*

$$\phi : \pi_1 (B, b_0) \rightarrow p^{-1} (b_0)$$

We call ϕ *the* **lifting correspondence derived from the covering map** p.

5.4

Definition 5.4.1. *If* $A \subset X$, *a* **retraction** *of* X *onto* A *is a continuous map* $r : X \rightarrow A$ *such that* $r|_A$ *is the identity map of* A. *If such a map* r *exists, we say that* A *is a* **retract** *of* X.

Proposition 5.4.2. *If* A *is a retract of* X, *then the homomorphism of fundamental groups induced by inclusion* $j : A \rightarrow X$ *is injective.*

    Proof: If $r : X \rightarrow A$ *is a retraction, then the composite map* $r \circ j$ *equals the identity map of* A. *It follows that* $r_* \circ j_*$ *is the identity map of* $\pi_1(A, a)$, *so that* j_* *must be injective.*

Theorem 5.4.3 (No-retraction theorem)**.** *There is no retraction of* B^2 *onto* S^1.

    Proof: If S^1 *were a retract of* B^2, *then the homomorphism induced by inclusion* $j : S^1 \rightarrow B^2$ *would be injective. But the fundamental group of* S^1 *is nontrivial and the fundamental group of* B^2 *is trivial.*

Lemma 5.4.4. *Let* $h : S^1 \rightarrow X$ *be a continuous map Then the following conditions are equivalent:*

    (1) h is nulhomotopic.
    (2) h extends to a continuous map $k : B^2 \rightarrow X$.
    (3) h_* *is the trivial homomorphism of fundamental groups.*

Proof: (1) $\Rightarrow$ (2). *Let* $H : S^1 \times I \to X$ *be a homotopy between* h *and a constant map. Let* $\pi : S^1 \times I \to B^2$ *be the map*

$$\pi(x,t) = (1-t)x$$

Then π *is continuous, closed and surjective, so it is a quotient map; it collapses* $S^1 \times 1$ *to the point 0 and is otherwise injective. Because* H *is constant on* $S^1 \times 1$, *it induces, via the quotient map* π, *a continuous map* $k : B^2 \to X$ *that is an extension of* h.

(2) $\Rightarrow$ (3). *If* $j : S^1 \to B^2$ *is the inclusion map, then* h *equals the composite* $k \circ j$. *Hence* $h_* = k_* \circ j_*$. *But*

$$j_* : \pi_1\left(S^1, b_0\right) \to \pi_1\left(B^2, b_0\right)$$

is trivial because the fundamental group of B^2 *is trivial. Therefore* h_* *is trivial.*

(3) $\Rightarrow$ (1). *Let* $p : \mathbb{R} \to S^1$ *be the standard covering map, and let* $p_0 : I \to S^1$ *be its restriction to the unit interval. Then* $[p_0]$ *generates* $\pi_1\left(S^1, b_0\right)$ *because* p_0 *is a loop in* S^1 *whose lift to* $\mathbb{R}$ *begins at 0 and ends at 1 .*

Let $x_0 = h\left(b_0\right)$. *Because* h_* *is trivial, the loop* $f = h \circ p_0$ *represents the identity element of* $\pi_1\left(X, x_0\right)$. *Therefore, there is a path homotopy* F *in* X *between* f *and the constant path at* x_0. *The map* $p_0 \times$ *id* $\cdot I \times I \to S^1 \times I$ *is a quotient map, being continuous, closed, and surjective; it maps* $0 \times t$ *and* $1 \times t$ *to* $b_0 \times t$ *for each* t, *but is otherwise injective. The path homotopy* F *maps* $0 \times I$ *and* $1 \times I$ *and* $I \times 1$ *to the point* x_0 *of* X, *so it induces a continuous map* $H : S^1 \times I \to X$ *that is a homotopy between* h *and a constant map. See Figure 55.2.*