A Primer of Algebraic D-modules

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Contents

Throughout this book, K denotes a field of characteristic zero and $K[X]$ the ring of polynomials $K\left[x_1, \ldots, x_n\right]$ in n commuting indeterminates over K.

Chapter I

Weyl Algebra

§1 Definition

Definition 1.1. *The ring $K[X]$ is a vector space of infinite dimension over K. Its algebra of K-linear operators is denoted by $\operatorname{End}_K(K[X])$. Let $\hat{x}_1, \ldots, \hat{x}_n$ be the operators of $K[X]$ which are defined on a polynomial $f \in K[X]$ by the formulae $\hat{x}_i(f) = x_i \cdot f$. Similarly, $\partial_1, \ldots, \partial_n$ are the operators defined by $\partial_i(f) = \partial f / \partial x_i$. These are linear operators of $K[X]$.*

*The n**-th Weyl algebra** A_n is the K-subalgebra of $\operatorname{End}_K(K[X])$ generated by the operators $\hat{x}_1, \ldots, \hat{x}_n$ and $\partial_1, \ldots, \partial_n$. For the sake of consistency, we write $A_0 = K$.*

Proposition 1.2.

$$[\partial_i, \hat{x}_j] = \delta_{ij} \cdot 1$$
$$[\partial_i, \partial_j] = [\hat{x}_i, \hat{x}_j] = 0$$
$$[\partial_i, f] = \frac{\partial f}{\partial x_i}$$

Proposition 1.3. *We have*

1. $$\partial^\beta x^\alpha = \sum_{\substack{\beta_1 + \beta_2 = \beta \\ \beta_i \in \mathbb{N}^n}} \binom{\beta}{\beta_1} \frac{\partial x^\alpha}{\partial x^{\beta_1}} \cdot \partial^{\beta_2}$$

   thus $\partial^\beta x^\alpha = x^\alpha \partial^\beta + D'$ where $\deg D' \leq |\alpha| + |\beta| - 2$

2. *The set $\mathbf{B} = \left\{ x^\alpha \partial^\beta : \alpha, \beta \in \mathbb{N}^n \right\}$ is a basis of A_n as a vector space over K.*

Theorem 1.4. *Let the free K-algebra $K\{z_1, \ldots, z_{2n}\}$ and J be the two-sided ideal generated by $[z_{i+n}, z_i] - 1$ for $i = 1, 2, \ldots, n$ and $[z_i, z_j]$ for $j \neq i + n$ and $1 \leq i, j \leq 2n$. We may define a surjective homomorphism $\phi : K\{z_1, \ldots, z_{2n}\} \to A_n$ by $\phi(z_i) = x_i$ and $\phi(z_{i+n}) = \partial_i$, for $i = 1, 2, \ldots, n$.*

$$\begin{tikzcd}
\{z_1, \ldots, z_n\} \arrow[r] \arrow[d, "\iota"] & A_n \\
K\{z_1, \ldots, z_n\} \arrow[ur, "\phi"] &
\end{tikzcd}$$

It follows that $J \subseteq \ker \phi$. *Thus* ϕ *induces a homomorphism* $\bar{\phi} : K\{z_1, \ldots, z_{2n}\}/J \longrightarrow A_n$.

Proof. We may use the relations to show that every element of $K\{z_1, \ldots, z_{2n}\}/J$ may be written as a linear combination of monomials of the form

$$z_1^{m_1} \ldots z_{2n}^{m_{2n}} + J$$

by 1.3 the images of these monomials under $\bar{\phi}$ form a basis of A_n as a vector space over K. In particular, the monomials must be linearly independent in $K\{z_1, \ldots, z_{2n}\}/J$. Hence $\bar{\phi}$ is an isomorphism of vector spaces and, a fortiori, an isomorphism of rings. □

Corollary 1.5. *Let* $m < n$ *be positive integers. Choose polynomials* $f_i \in K[X]$*, for* $1 \leq i \leq n$*, as follows: if* $i \leq m$*, then* f_i *is a polynomial in the variables* $x_{m+1}, \ldots, x_n$*; otherwise* $f_i = 0$*. The map* $\sigma : A_n \longrightarrow A_n$ *defined by the formulae*

$$\sigma(x_i) = x_i + f_i$$
$$\sigma(\partial_i) = \partial_i - \sum_1^n \frac{\partial f_k}{\partial x_i}\partial_k$$

induce an automorphism of A_n.

Proof. Define a homomorphism ϕ of $K\{z_1, \ldots, z_{2n}\}$ to A_n by $\phi(z_i) = x_i + f_i$ and

$$\phi(z_{i+n}) = \partial_i - \sum_1^n \frac{\partial f_k}{\partial x_i}\partial_k$$

Choose i, j such that $1 \leq i, j \leq n$. It is clear that $\phi([z_i, z_j]) = 0$.

Let us calculate $\phi([z_{i+n}, z_j])$. Since ϕ is a ring homomorphism, this is the same as $[\phi(z_{i+n}), \phi(z_j)]$, which equals

$$\begin{aligned}
\phi([z_{i+n}, z_j]) &= [\phi(z_{i+n}), \phi(z_j)] \\
&= \left[\partial_i - \sum_1^n \frac{\partial f_k}{\partial x_i}\partial_k, x_j + f_j\right] \\
&= \delta_{ij} + \frac{\partial f_j}{\partial x_i} - \frac{\partial f_j}{\partial x_i} - \sum_1^n \frac{\partial f_k}{\partial x_i} \cdot \frac{\partial f_j}{\partial x_k} \\
&= \delta_{ij}
\end{aligned}$$

A similar calculation shows that $\phi([z_{i+n}, z_{j+n}]) = 0$. Thus ϕ induces an endomorphism $\sigma = \bar{\phi}$ of A_n.

A similar argument shows that the map τ defined by the formulae

$$\tau(x_i) = x_i - f_i$$
$$\tau(\partial_i) = \partial_i + \sum_1^n \frac{\partial f_k}{\partial x_i}\partial_k$$

is an endomorphism of A_n. It is now easy to check that τ is the inverse of σ. Thus σ is an automorphism of A_n. $\square$

§2 Degree

Definition 2.1. *Let $D \in A_n$. The* ***degree*** *of D is the largest length of the multi-indices $(\alpha, \beta) \in \mathbb{N}^n \times \mathbb{N}^n$ for which $x^\alpha \partial^\beta$ appears with non-zero coefficient in the canonical form of D. It is denoted by* $\deg(D)$*. As with the degree of a polynomial, we use the convention that the zero polynomial has degree $-\infty$.*

Lemma 2.2. *We have*

$$\partial^\beta x^\alpha = \sum_{\substack{\beta_1+\beta_2=\beta \\ \beta_i \in \mathbb{N}^n}} \partial^{\beta_1}(x^\alpha)\partial^{\beta_2}$$

thus $\partial^\beta x^\alpha = x^\alpha \partial^\beta + D'$ where $\deg D' \leq |\alpha| + |\beta| - 2$

Theorem 2.3. *The degree satisfies the following properties; for $D, D' \in A_n$:*

1. $\deg(D + D') \leq \max\{\deg(D), \deg(D')\}$. *if* $\deg(D) \neq \deg(D')$ *then we have equality in the above formula.*
2. $\deg(DD') = \deg(D) + \deg(D')$.
3. $\deg[D, D'] \leq \deg(D) + \deg(D') - 2$.

Corollary 2.4. *The algebra A_n is a domain.*

§3 Ideal Structure

Theorem 3.1. *The algebra A_n is simple.*

Proof. Let I be a non-zero two-sided ideal of A_n. Choose an element $D \neq 0$ of smallest degree in I. If D has degree 0 , it is a constant, and $I = A_n$. Assume that D has degree $k > 0$ and let us aim at a contradiction.

Suppose that (α, β) is a multi-index of length k. If $x^\alpha \partial^\beta$ is a summand of D with non-zero coefficient and $\beta_i \neq 0$, then $[x_i, D]$ is non-zero and has degree $k - 1$. Since I is a two-sided ideal of A_n, we have that $[x_i, D] \in I$. But this contradicts the minimality of D. Thus $\beta = (0, \ldots, 0)$. Since $k > 0$, we must have that $\alpha_i \neq 0$, for some $i = 1, 2, \ldots, n$. Hence $[\partial_i, D]$ is a non-zero element of I of degree $k - 1$, and once again we have a contradiction. $\square$

Corollary 3.2. *Every (ring, algebra) endomorphism of A_n is injective.*

Theorem 3.3. *Every left ideal of A_n is generated by two elements.*

Chapter II

Rings of Differential Operators

Let R be a commutative K-algebra.

§1 Definition

Definition 1.1. *Let R be a commutative K-algebra.*

(i) We will identify an element $a \in R$ with the operator of $\mathrm{End}_K(R)$ defined by the rule $r \mapsto ar$, for every $r \in R$. An operator $P \in \mathrm{End}_K(R)$ has order zero if $[a, P] = 0$, for every $a \in R$.

(ii) Suppose we have defined operators of order $< n$. An operator $P \in \mathrm{End}_K(R)$ has order n if it does not have order less than n and $[a, P]$ has order less than n for every $a \in R$.

Let $D_n(R)$ denote the K-vector space of all operators of $\mathrm{End}_K(R)$ of order $< n$. The ***ring of differential operators*** *of R is defined as the subring $D(R) = \bigcup_{n \geq 0} D_n(R)$.*

Remark. *It follows that $D_n(R) = \{P \in \mathrm{End}_K R : [\cdots[[P, a_1], a_2], \cdots, a_{n+1}] = 0$ for any $a_i \in R\}$ and the set of order n is $D_n - D_{n-1}$.*

Definition 1.2. *A* ***derivation*** *of the K-algebra R is a K-linear operator D of R which satisfies Leibniz's rule:*

$$D(ab) = aD(b) + bD(a) \quad \text{for every } a, b \in R$$

Let $\mathrm{Der}_K(R)$ denote the K-vector space of all derivations of R. If $D \in \mathrm{Der}_K(R)$ and $a \in R$, we define a new derivation aD by $(aD)(b) = aD(b)$ for every $b \in R$. The K-vector space $\mathrm{Der}_K(R)$ is a left R-module for this action.

Proposition 1.3. *Let R be a commutative K-algebra.*

1. *The elements of order zero are the elements of R i.e. $D^0(R) = R$*

2. *The operators of order ≤ 1 correspond to the elements of $\mathrm{Der}_K(R) + R$ .i.e.*

$$D^1(R) \subset \mathrm{Der}_K(R) + R$$

Proof. 2. Let $Q \in D^1(R)$ and put $P = Q - Q(1)$. Note that $P(1) = 0$ and that P has order ≤ 1. Hence $[P, a]$ has order zero for every $a \in R$. Thus for every $b \in R$, we have that $[[P, a], b] = 0$. Writing the commutators explicitly, one obtains the equality

$$(Pa)b - (aP)b - b(Pa) + b(aP) = 0$$

Applying this operator to $1 \in R$, we end up with $P(ab) = aP(b) + bP(a)$, it follows that P is a derivation of R. But $Q = P + Q(1) \in \mathrm{Der}_K(R) + R$, as required. □

Proposition 1.4. *Let $P \in D^n(R)$ and $Q \in D^m(R)$, then $P \cdot Q \in D^{n+m}(R)$.*

Proof. The proof is by induction on $m + n$. If $m + n = 0$ the result is obvious. Suppose the result true whenever $m + n < k$. If $m + n = k$ and $a \in R$, we have that

$$[PQ, a] = P[Q, a] + [P, a]Q$$

The definition of order implies that $[Q, a] \in D^{m-1}(R)$ and $[P, a] \in D^{n-1}(R)$. Thus, by the induction hypothesis $P[Q, a], [P, a]Q \in D^{n+m-1}$. Hence $[PQ, a]$ belongs to D^{n+m-1}, as required. □

§2 The Weyl Algebra

Proposition 2.1. *Every derivation of $K[X]$ is of the form $\sum_1^n f_i \partial_i$, for some $f_1, \ldots, f_n \in K[X]$.*

Lemma 2.2. *Let $P \in D(K[X])$. If $[P, x_i] = 0$ for every $i = 1, \ldots, n$, then $P \in K[X]$.*

Definition 2.3. *Define C_r to be the set of operators in A_n which can be written in the form $\sum_\alpha f_\alpha \partial^\alpha$ with $|\alpha| \leq r$. A simple calculation shows that*

$$C_r = C_{r+1} \cap D^r(K[X])$$

By Proposition 1.3, we have that $C_1 = \mathrm{Der}_K(K[X]) + K[X]$ and that $C_0 = K[X]$. We will use the convention that if $k < n$ then $\mathbb{N}^k$ is embedded in $\mathbb{N}^n$ as the set of n-tuples whose last $n - k$ components are zero.

Lemma 2.4. *It follows from the identity $\left[\partial^\beta, x_n\right] = \beta_n \partial^{\beta - e_n}$ that $\left[\partial^\beta, x_n\right] = 0 \Leftrightarrow \beta_n = 0$. Thus $[G, x_n] = 0$ implies that G can be written as a linear combination of monomials of the form $x^\alpha \partial^\beta$ with $\beta \in \mathbb{N}^{n-1}$.*

Lemma 2.5. *Let $P_1, \ldots, P_n \in C_{r-1}$ and assume that $[P_i, x_j] = [P_j, x_i]$ whenever $1 \leq i, j \leq n$. Then there exists $Q \in C_r$ such that $P_i = [Q, x_i]$, for $i = 1, \ldots, n$.*

Proof. Suppose, by induction, that we have determined $Q' \in C_r$ such that $[Q', x_i] = P_i$ for $k+1 \leq i \leq n$. Write $G = [Q', x_k] - P_k$, then

$$G = \sum_{\alpha \in \mathbb{N}^k} f_\alpha \partial^\alpha.$$

since $[G, x_i] = [[Q', x_k], x_i] - [P_k, x_i] = [[Q', x_i], x_k] - [P_k, x_i] = 0$ for $k+1 \leq i \leq n$.

Now write

$$Q'' = \sum_{\alpha \in \mathbb{N}^k} (\alpha_k + 1)^{-1} f_\alpha \partial^{\alpha + e_k} \in C^r$$

We have $[Q'', x_k] = G$ and $[Q'', x_i] = 0$. Thus $[Q' - Q'', x_i] = P_i$, for $k \leq i \leq n$; and the induction is complete. □

Theorem 2.6. *The ring of differential operators of $K[X]$ is $A_n(K)$ i.e. . Besides this, .*

1. $D^k(K[X]) = C_k$
2. $D(K[X]) = A_n(K)$

Proof. It is enough to prove that $D^k(K[X]) \subseteq C_k$. Let $P \in D(K[X])$. If $P \in D^1(K[X])$ then by Lemma 1.1, $P \in \mathrm{Der}_K(K[X]) + K[X]$. Thus $P \in C_1$ by Proposition 1.3.

Suppose, by induction, that $D^k(K[X]) = C_k$ for $k \leq m-1$. Let $P \in D^m(K[X])$. Write $P_i = [P, x_i]$. Since P_i has order $k \leq m-1$, it follows that $P_i \in C_{n-1}$. But, for all $1 \leq i, j \leq n$,

$$[P_i, x_j] = [[P, x_i], x_j] = [[P, x_j], x_i] = [P_j, x_i].$$

Thus by Lemma 2.2 there exists $Q \in C_m$ such that $[Q, x_i] = P_i, 1 \leq i \leq n$. Hence $[Q - P, x_i] = 0$ in $D(K[X])$. Since this holds whenever $1 \leq i \leq n$, we conclude by Lemma 2.1 that $Q - P \in K[X] = C_0$. Therefore $P \in C_m$. Hence $D^m(K[X]) \subseteq C^m$, as we wanted to prove. □

Theorem 2.7. ord $(PQ) = \text{ord}(P) + \text{ord}(Q)$

Chapter III

Jacobian Conjecture

§1 Polynomial Maps

Definition 1.1. *Let $F : K^n \to K^m$. We say that F is* ***polynomial*** *if there exist $F_1, \ldots, F_m \in K[x_1, \ldots, x_n]$ such that $F(p) = (F_1(p), \ldots, F_m(p))$ for all $p \in K^n$.*

A polynomial map is called an isomorphism or a ***polynomial isomorphism*** *if it has an inverse which is also a polynomial map.*

For the rest of the section we shall write X, Y for the spaces K^n and K^m; and $K[X], K[Y]$ for the polynomial rings $K[x_1, \ldots, x_n]$ and $K[y_1, \ldots, y_m]$.

Definition 1.2. *Suppose that $F : X \to Y$ is a polynomial map. We may define a map,*

$$F^\sharp : K[Y] \to K[X], \quad \text{given that } g \mapsto g \circ F$$

The map $F^\sharp$ is called the ***comorphism*** *of F.*

Suppose that a ring homomorphism $\phi : K[Y] \to K[X]$ is given. Then we may use it to construct a polynomial map from X to Y. Now let

$$\phi_\sharp : X \to Y, \quad \mathbf{x} \mapsto (\phi(y_1)(\mathbf{x}), \ldots, \phi(y_m)(\mathbf{x}))$$

Theorem 1.3. *Let $F : X \to Y$ and $G : Y \to Z$ be polynomial maps, then*

1. $\left(F^\sharp\right)_\sharp = F$
2. *$G \cdot F : X \to Z$ is a polynomial map and $(G \cdot F)^\sharp = F^\sharp \cdot G^\sharp$*

Theorem 1.4. *If $\phi : K[Y] \to K[X]$ and $\psi : K[Z] \to K[Y]$ are homomorphism of polynomial rings, then*

1. $(\phi_\sharp)^\sharp = \phi$.
2. $(\phi \cdot \psi)_\sharp = \psi_\sharp \cdot \phi_\sharp$.

Corollary 1.5. *A polynomial map $F : X \to Y$ is an isomorphism if and only if $F^\sharp$ is an isomorphism.*

§2 Jacobian Conjecture

Jacobian conJecture. Let $F : K^n \to K^n$ be a polynomial map. If $\Delta F = 1$ on K^n then F has a polynomial inverse on the whole of K^n.

Lemma 2.1. *Let $F : X \to X$ be a polynomial map and suppose that $\Delta F \neq 0$ everywhere in X. Then $F^\sharp$ is injective.*

Proof. Suppose that $F^\sharp$ is not injective, and choose the non-constant polynomial $g \in K[X]$ of smallest degree such that $F^\sharp(g) = 0$. Then $g \circ F = 0$. Let $g_i = \partial g / \partial x_i$ and

$$\mathbf{v} = \left(g_1\left(F_1, \ldots, F_n\right), \ldots, g_n\left(F_1, \ldots, F_n\right)\right).$$

Hence, by the chain rule,

$$D\left(g \circ F\right) = \mathbf{v}(p) \cdot JF(p) = 0$$

for every $p \in X$. Since

$$\Delta F(p) = \det JF(p) \neq 0,$$

we conclude that $\mathbf{v}(p) = 0$ for every $p \in X$. Thus $g_i\left(F_1, \ldots, F_n\right) = 0$ for $1 \leq i \leq n$. Since g is not constant, at least one of the g_i must be non-zero. But g_1 has degree smaller than g, a contradiction. $\square$

Proposition 2.2. *Denote by $K\left[F_1, \ldots, F_n\right]$ the subalgebra of $K[X]$ generated by the coordinate functions of F. This is the image of the homomorphism $F^\sharp$. Thus the Jacobian conjecture may be rephrased as follows.*

Let $F : K^n \to K^n$ be a polynomial map and assume that $\Delta F = 1$ in K^n. Then $K\left[F_1, \ldots, F_n\right] = K\left[x_1, \ldots, x_n\right]$.

§3 Derivation

Definition 3.1. *Let D be a derivation of a K-algebra S.*

1. *It follows from Leibniz's rule that the kernel of D is a subring (subalgebra) of S, it is called the **ring of constants of** D.*

2. *The derivation D is **locally nilpotent** if given $a \in S$, there exists $k \in \mathbb{N}$ such that $D^k(a) = 0$.*

3. *Let S be a ring and D a locally nilpotent derivation. Define a map $\phi : S \longrightarrow S[x]$ by the rule*

$$\phi(a) = \sum_0^\infty \frac{D^n(a)}{n!} x^n$$

for every $a \in S$. It is easy to check that ϕ is a ring homomorphism which satisfies

$$\phi \cdot D = \partial \cdot \phi$$

Proposition 3.2. *Let S be a K-algebra and $D_1, \ldots, D_n$ be commuting locally nilpotent derivations of S. Suppose that there exist $t_1, \ldots, t_n \in S$ such that $D_i\left(t_j\right) = \delta_{ij}$. Then*

1. $S = R\left[t_1, \ldots, t_n\right]$, *where R is the ring of constants with respect to $D_1, \ldots, D_n$,*

2. $t_1, \ldots, t_n$ *are algebraically independent over R,*

3. $D_i = \partial / \partial t_i$ *for $i = 1, \ldots, n$.*

Proof. Proof: We firstly prove it whene $n = 1$. Put $\bar{S} = S/St$. Let $\rho : S \longrightarrow \bar{S}[x]$ be the composition of ϕ defined above and the projection $S[x] \longrightarrow \bar{S}[x]$. We want to show that ρ is an isomorphism. Note that $\rho(t) = x$.

To prove that ρ is surjective it is enough to prove that its image contains $\bar{S}$. Let $a \in S$. Denote by $\bar{a}$ its image in $\bar{S}$. Since D is locally nilpotent, there exists $n \in \mathbb{N}$ such that $D^k(a) = 0$ for $k > n$. Thus,

$$\rho(a) = \sum_0^n \frac{\overline{D^i(a)}}{i!} x^i.$$

If $n = 0$, then $\rho(a) = \bar{a}$. If $n > 0$ put $a_0 = a$ and define $a_{j+1} = a_j - D^{n-j}\left(a_j\right) t^{n-j}/(n-j)!$, for $j = 1, \ldots, n$. It is easy to show, by induction on j, that $D^k\left(a_j\right) = 0$ for $k > n - j$ and that

$$\rho\left(a_j\right) = \sum_0^{n-j} \frac{\overline{D^i\left(a_j\right)}}{i!} x^i.$$

Thus $\rho\left(a_n\right) = \bar{a}_n$. However, since $\bar{t} = 0$, we have that $\rho\left(a_n\right) = \bar{a}$. Thus ρ is surjective.

Let us prove that ρ is injective. If not, then there exists a non-zero $a \in S$ such that $\rho(a) = 0$. Thus $D^k(a) \in tS$, for every $k \in \mathbb{N}$. Hence $a = a_1 \cdot t$, for some $a_1 \in S$. Since $\rho(t) = x$, we have that $\rho\left(a_1\right) = 0$. Thus $a_1 \in tS$ and $a = a_2 \cdot t^2$, for some $a_2 \in S$. Continuing this way we conclude that t^n divides a for all $n \geq 0$. But this is impossible, unless $a = 0$. Indeed, ϕ maps t to $t + x$. Thus if t^n divides a, we also have that $\phi\left(t^n\right) = (t + x)^n$ divides $\phi(a)$ in the polynomial ring $S[x]$. Hence, if $a \neq 0$ we have that $\deg(\phi(a)) \geq n$ for every $n > 0$, which is clearly impossible. Thus $a = 0$, as required.

We conclude that the homomorphism $\rho : S \longrightarrow \bar{S}[x]$ is an isomorphism. Since $\rho \cdot D = d/dx \cdot \rho$, we have that $R = \rho^{-1}(\bar{S})$. The result now follows if we recall that $\rho(t) = x$.

We proceed by induction on the number n of derivations. By Lemma 3.2, $S = R_1\left[t_1\right]$, where R_1 is the ring of constants of D_1. But t_1 is algebraically independent over R_1 and $D_1 = d/dt_1$. Since D_1 commutes with D_i for $i > 1$, we have that $D_i\left(R_1\right) \subseteq R_1$. Thus, by the induction hypothesis, $R_1 = R\left[t_2, \ldots t_n\right]$, and the proposition follows. □

§4 Automorphisms

Definition 4.1. *Let $X = K^n$. The rational function field of $K[X]$ will be denoted by $K(X)$. Let $F : X \to X$ be a polynomial map with coordinate functions $F_1, \ldots, F_n$. Assume that*

$$\Delta = \Delta F \neq 0$$

everywhere on X.

1. *Define a map $D_i : K(X) \to K(X)$ by*

   $$D_i(g) = \Delta^{-1} \det J\left(F_1, \ldots, F_{i-1}, g, F_{i+1}, \ldots, F_n\right).$$

   It is easy to check that D_i is a derivation of $K(X)$.

2. *Now let $K\left[X, \Delta^{-1}\right]$ be the K-subalgebra of $K(X)$ of all rational functions whose denominator is a power of Δ. Then D_i restricts to a derivation of $K\left[X, \Delta^{-1}\right]$, since $D_i\left(\Delta^{-1}\right) = -\Delta^2 D_i(\Delta) \in K\left[X, \Delta^{-1}\right]$*

Proposition 4.2. *Let R be a commutative ring and D, D' be derivations of R, then $[D, D']$ is a derivation of R.*

Proposition 4.3. *Let D be a K-derivation of $K\left[x_1, \ldots, x_n\right]$.*

1. *D can be extended to the power series ring $K\left[\left[x_1, \ldots, x_n\right]\right]$.*

2. *If Δ is a power series such that $\Delta(0) \neq 0$ then $\Delta^{-1} \cdot D$ is a derivation of the power series ring $K\left[\left[x_1, \ldots, x_n\right]\right]$.*

Lemma 4.4. *As derivations of $K\left[X, \Delta^{-1}\right]$ the D_i satisfy:*

1. *$D_i\left(F_j\right) = \delta_{ij}$.*

2. *The D_i commute pairwise.*

Proof. Note first that $\Delta(0) \neq 0$. Thus Δ is invertible as a power series and $K\left[X, \Delta^{-1}\right] \subseteq K[[X]]$. On the other hand, $\Delta \cdot D_i$ is a derivation of $K\left[x_1, \ldots, x_n\right]$ which can be extended to a derivation on the power series ring $K[[X]] = K\left[\left[x_1, \ldots, x_n\right]\right]$. Since Δ is invertible as a power series, then D_i can also be extended to a derivation of $K[[X]]$.

Put derivation $B = \left[D_i, D_j\right]$. We want to show that $B = 0$ on $K\left[X, \Delta^{-1}\right]$. It is enough to show that $B = 0$ on the power series ring $K[[X]]$.

Moreover $B\left(F_k\right) = 0$, for $1 \leq k \leq n$; and so B is zero in the subalgebra $K\left[F_1, \ldots, F_n\right]$. But $F_1, \ldots, F_n$ are algebraically independent, by Lemma 2.2. Hence we may consider B as a derivation on the power series ring $K\left[\left[F_1, \ldots, F_n\right]\right]$. By (1), B is zero on $K\left[\left[F_1, \ldots, F_n\right]\right]$.

For $1 \leq i \leq n$ let $a_i = F_i(0)$. The jacobian matrices of $(F_1 - a_1, \ldots, F_n - a_n)$ and F coincide. Since the latter is invertible in $K[[x_1, \ldots, x_n]]$, we conclude from the local inversion theorem (see Appendix 2) that

$$K[[x_1, \ldots, x_n]] = K[[F_1 - a_1, \ldots, F_n - a_n]] = K[[F_1, \ldots, F_n]].$$

Thus B is zero on $K[[x_1, \ldots, x_n]]$, as required. □

Definition 4.5. *Let $a \in A_n$. The map $\mathrm{ad}_a : A_n \to A_n$ is defined by*

$$\mathrm{ad}_a(b) = [a, b].$$

This is a K-linear map, but it is not a K-algebra homomorphism.

Theorem 4.6. *Let $F : K^n \to K^n$ be a polynomial map and assume that $\Delta F = 1$ everywhere on K^n. If every endomorphism of A_n is an automorphism, then the Jacobian conjecture holds.*

Proof. Since $\Delta F = 1$, it follows from Lemma 4.1 that $D_1, \ldots, D_n$ are derivations of $K[X]$ which satisfy

$$[D_i, F_j] = D_i(F_j) = \delta_{ij} \text{ and } [D_i, D_j] = 0$$

for $1 \leq i, j \leq n$. By, there exists an endomorphism $\phi : A_n \to A_n$ such that $\phi(x_i) = F_i$ and $\phi(\partial_i) = D_i$, for $1 \leq i \leq n$. Note that for $b \in A_n$,

$$\deg(\mathrm{ad}_{\partial_i}(b)) = \deg[\partial_i, b] \leq \deg b - 1$$

Thus given $b \in A_n$, there exists $k \in \mathbb{N}$ such that $(\mathrm{ad}_{\partial_1})^k(b) = 0$. Since

$$\phi(\mathrm{ad}_{\partial_i}(b)) = \mathrm{ad}_{D_i} \phi(b)$$

we have that $(\mathrm{ad}_{D_i})^k(\phi(b)) = 0$. Assuming that ϕ is an automorphism, we conclude that D_i is locally nilpotent. It then follows by Proposition 3.1 that $K[F_1, \ldots, F_n] = K[x_1, \ldots, x_n]$, which is the Jacobian conjecture as stated in 2.3. □

Chapter IV

Modules Over The Weyl Algebra

§1 The Polynomial Ring

Proposition 1.1. *Let R be a ring and M an irreducible left R-module.*

1. *$M = Rm$ for every $0 \neq m \in M$.*
2. *If $0 \neq u \in M$, then $M \cong R/ann_R(u)$.*
3. *If R is not a division ring, then M is a torsion module.*

Proposition 1.2. *The A_n-module $K[X]$ is an irreducible, torsion A_n-module. Besides this, we have isomorphism of A_n-module*

$$K[X] \cong A_n / \sum_{1}^{n} A_n \partial_i.$$

Proof. First of all 1 is clearly a generator of A_n-module $K[X]$ and the annihilator of 1 is the left ideal generated by $\partial_1, \ldots, \partial_n$.

$$\begin{tikzcd} A_n \arrow[r] \arrow[d] & K[X] \\ A_n / \sum_1^n A_n \partial_i \arrow[ur] & \end{tikzcd}$$

□

Remark. *Choose $g_1, \ldots, g_n \in K[X]$ and consider the left ideal J of A_n generated by $\partial_1 - g_1, \ldots, \partial_n - g_n$. It is easy to check that the map*

$$\psi : A_n/J \longrightarrow K[X] \text{ defined by } \psi(f + J) = f$$

is an isomorphism of A_n-modules.

Proposition 1.3. *Another A_n-module is $A_n / \sum_1^n A_n \cdot x_i$.*

$$\overline{x^\alpha \partial^\beta} = (-1)^{|\alpha|}\, \overline{\partial^{\beta - \alpha}}$$

As a K-vector space it is isomorphic to $K[\partial] = K[\partial_1, \ldots, \partial_n]$, the set of polynomials in $\partial_1, \ldots, \partial_n$. Using this isomorphism, we may identify the action of A_n directly on $K[\partial]$: the ∂ 's act by multiplication, whilst x_i acting on ∂_j gives $-\delta_{ij} \cdot 1$.

§2 Twisting

Definition 2.1. *Let R be a ring and M a left R module. Suppose that σ is an automorphism of R. We shall define a new left module M_σ, as follows. As an abelian group, $M_\sigma = M$. Let $a \in R$ and $u \in M$, define*

$$r \cdot m = \sigma(r)m$$

*It is called the **twisted module of** M **by** σ.*

Proposition 2.2. *Let R be a ring, M a left R-module and σ an automorphism of R. Then:*

1. *M_σ is irreducible if and only if M is irreducible.*
2. *M_σ is a torsion module if and only if M is a torsion module.*
3. *$M_\sigma \oplus M'_\sigma \cong (M \oplus M')_\sigma$.*
4. *If N is a submodule of M then $(M/N)_\sigma \cong M_\sigma / N_\sigma$.*
5. *Let J be a left ideal of R. Set $\sigma(J) = \{\sigma(r) : r \in J\}$. Then $\sigma(J)$ is a left ideal of J and $(R/J)_\sigma \cong R/\sigma^{-1}(J)$.*

Proposition 2.3. *The Fourier transform of $K[X]$ is $K[\partial]$.*

Proof. Let $\mathcal{F}$ be the Fourier transform automorphism of A_n. By Proposition 5.2, the twisted module $(K[X])_{\mathcal{F}}$ is isomorphic to

$$A_n / \sum_1^n A_n \cdot \mathcal{F}^{-1}(\partial_i) = A_n / \sum_1^n A_n \cdot x_i \cong K[\partial]$$

□

Theorem 2.4. *For every positive integer r let σ_r be the automorphism of A_n which satisfies $\sigma_r(x_i) = x_i$ and $\sigma_r(\partial_i) = \partial_i - x_i^r$. The modules $K[X]_{\sigma_r}$ form an infinite family of pairwise nonisomorphic irreducible modules over A_n.*

Proof. Let $r < t$, and suppose that there exists an isomorphism, $\phi : K[X]_{\sigma_r} \longrightarrow K[X]_{\sigma_t}$. Since $K[X]_{\sigma_r}$ is irreducible, it is generated by 1 . Thus ϕ is completely determined by the image of 1 ; say $\phi(1) = f \neq 0$. Now the equation $\phi(\partial_i \bullet 1) = \partial_i \bullet \phi(1)$ translates as the differential equation

$$\phi(\partial_i \bullet 1) = \phi((\partial_i - x_i^r)(1)) = \phi(-x_i^r) = -\phi(x_i^r) \text{ and } \partial_i \bullet \phi(1) = (\partial_i - x^r)(f)$$

thus

$$\frac{\partial f}{\partial x_i} = \left(x_i^t - x_i^r\right) f.$$

The left hand side of the equation has degree $\leq \deg f - 1$. Since $f \neq 0$ and $r < t$, the right hand side has degree $\deg f + t$. This is a contradiction, so the theorem is proved. □

§3 Holomorphic Functions

Lemma 3.1. *Let $h(z)$ be the holomorphic function* $\exp(\exp(z))$. *For every positive integer m there exists a polynomial $F_m(x) \in \mathbb{C}[x]$ of degree m such that*

$$d^m h/dz^m = F_m\left(e^z\right) h(z).$$

Proof. The proof is by induction on m. If $m = 1$ then $dh/dz = e^z h(z)$, so we may take $F_1(x) = x$. Suppose that the result is true for $m = k$. Then

$$d^{k+1} h/dz^{k+1} = d/dz\left(F_k\left(e^z\right) h(z)\right) = \left(e^z F_k'\left(e^z\right) + e^z F_k\left(e^z\right)\right) h(z).$$

Thus we may take

$$F_{k+1}(x) = x F_k'(x) + x F_k(x)$$

which is a polynomial of degree $k + 1$. □

Proposition 3.2. *The function $h(z) = \exp(\exp(z))$ is not a torsion element of the $A_1(\mathbb{C})$-module $\mathcal{H}(U)$.*

Proof. Suppose that there exists a non-zero operator $P \in A_1(\mathbb{C})$ such that $P \cdot h = 0$. Write

$$P = \sum_{i=0}^{m} f_i(z) \partial^i$$

with $f_m(z) \neq 0$. By Lemma 6.1, we have

$$0 = P \cdot h = \sum_{i=0}^{m} f_i(z) F_i\left(e^z\right) h(z).$$

Since $h(z) \neq 0$, we conclude that

$$\sum_{i=0}^{m} f_i(z) F_i\left(e^z\right) = 0.$$

But this is impossible since F_m has degree m and $f_m(z) \neq 0$. □

Chapter V

Differential Equations

§1 The D-Modules of Equations

Definition 1.1. *Let P be an operator in A_n. This differential operator gives rise to the equation*

$$P(f) = \sum_{\alpha} g_\alpha \partial_\alpha (f) = 0 \text{ in } K[X],$$

where $f \in K[X]$. More generally, if $P_1, \ldots, P_m$ are differential operators in A_n, then we have a system of differential equations

$$P_i(f) = 0, \quad i = 1, \ldots, m. \tag{1}$$

*The A_n-**module associated to the system (1)** is $A_n / \sum_1^m A_n P_i$. A polynomial solution of (1) is a polynomial $f \in K[X]$ which satisfies $P_i(f) = 0$, for $i = 1, \ldots, m$. The set of all polynomial solutions of (1) forms a K-vector space.*

Theorem 1.2. *Let M be the A_n-module associated with the system (1). The K-vector space of polynomial solutions of the system (1) is isomorphic to* $\mathrm{Hom}_{A_n}(M, K[X])$.

Proof. Step1. Let f be a polynomial solution of (1). Define a map $\phi_f : A_n \to K[X]$ by the rule

$$\phi_f(D) = D(f)$$

It is easy to check that ϕ_f is an A_n-module homomorphism. Moreover, if $D \in \sum_1^m A_n P_i$, then

$$\phi_f(D) = D(f) = \sum_1^m D_i P_i(f) = 0.$$

that is $\sum_1^m A_n P_i \subseteq \ker \phi_f$. Thus ϕ_f induces an A_n-module homomorphism $\bar{\phi}_f : M \to K[X]$.

$$\begin{tikzcd} A_n \arrow[r, "\phi_f"] \arrow[d, "\pi"] & K[X] \\ M \arrow[ur, "\bar{\phi}_f"] & \end{tikzcd}$$

Step 2. Conversely, let $\psi : M \to K[X]$ be an A_n-module homomorphism. Define $f = \psi(\overline{1})$, where $\overline{1}$ is the image of 1 in M. Then for $i = 1, \ldots, m$,

$$P_i(f) = P_i(\psi(\overline{1})) = \psi(P_i(\overline{1})) = 0.$$

Thus f is a polynomial solution of (1).

It is easy to check that the maps $f \mapsto \bar{\phi}_f$ and $\psi \mapsto \psi(\overline{1})$ are inverse to each other. Hence we have established the required isomorphism. □

Definition 1.3. *Let $\mathcal{S}$ be a left A_n-module; and let $M = (P_i)$ be a finitely generated left A_n-module. We will call* $\mathrm{Hom}_{A_n}(M, \mathcal{S})$ *the solution space of M in $\mathcal{S}$ which isomorphic to K-vector space of solutions of*

$$P_i\,(f) = 0 \textit{ in } \mathcal{S}, \quad i = 1, \ldots, m.$$

§2 Direct Limit of Modules

Definition 2.1. *Let I be a set with a relation $\leq$. We say that I is* ***partially ordered*** *if $\leq$ is reflexive, antisymmetric, and transitive.*

A partially ordered set I is ***directed*** *if, given $i, j \in I$, there exists $k \in I$ such that $i \leq k$ and $j \leq k$.*

Definition 2.2. *Let R be a ring and I be a directed set. Suppose that to every $i \in I$ we associate a left R-module M_i. A* ***directed family*** *of left R-modules consist of the data of*

(i) For every $i \in I$, there exist a left R-module M_i

(ii) given any $i, j \in I$, satisfying $i \leq j$, there exists a homomorphism of R-modules

$$f_{ij} : M_i \longrightarrow M_j,$$

such that

(iii) For every $i \in I$, $f_{ii} = \mathrm{id}_{M_i}$

(iv) if $i \leq j \leq k$, then

$$f_{ij} \circ f_{jk} = f_{ik}.$$

Definition 2.3. *Let (M_i, f_{ij}) be a directed family of left R-modules. Denote by M the disjoint union of the modules M_i*

$$M = \bigsqcup_{i \in I} M_i.$$

We define an equivalence relation in M as follows: $(u,i),(v,j) \in U$ are equivalent if and only if there exists $k \in I$ such that $i \leq k, j \leq k$ and $\pi_{ik}(u) = \pi_{jk}(v)$. The ***direct limit*** *of the family $\{M_i : i \in I\}$, denoted by $\varinjlim M_i$, is the quotient set of M by this equivalence relation.*

Remark. *The operation on M is defined*

$$\overline{(a_i,i)} + \overline{(a_j,j)} = \overline{(f_{ik}(a_i) + f_{jk}(a_i),k)}$$

for some $k \in I$ such that $i \leq k$ and $j \leq k$,

$$r\overline{(a_i,i)} = \overline{(ra_i,i)}$$

We sometimes simply write $[a_i]$ to replace $\overline{(a_i,i)}$ in M

§3 Microfunctions

Let $D(\epsilon)$ be the open disk of $\mathbb{C}$ of centre 0 and radius ϵ, $D'(\epsilon) = D(\epsilon)\backslash 0$ and $\mathcal{H}(\Omega)$ be the set of holomorphic functions in the open set $\Omega \subseteq \mathbb{C}$ viewed as an $A_1(\mathbb{C})$-module.

Proposition 3.1. *Let directed set $I = \{D(\epsilon) : \epsilon \in \mathbb{R}\}$ such that $D(\epsilon) \leq D(\epsilon')$ iif $D(\epsilon) \supseteq D(\epsilon')$, and a directed family $\{\mathcal{H}(D(\epsilon)) : D(\epsilon) \in I\}$. The homomorphisms $\tau_{\epsilon\epsilon'} : \mathcal{H}(D(\epsilon)) \longrightarrow \mathcal{H}(D(\epsilon'))$ are defined by restriction of holomorphic functions.*

The elements of $\mathcal{H}_0 = \varinjlim \mathcal{H}(D(\epsilon))$ are called ***germs of holomorphic functions*** *at 0.*

Proposition 3.2. *The* ***universal cover*** *of $D'(\epsilon)$ is the set $\tilde{D}(\epsilon) = \{z \in \mathbb{C} : \mathrm{Re}(z) < \log(\epsilon)\}$. The projection π of $\tilde{D}(\epsilon)$ on $D'(\epsilon)$ is defined by $\pi(z) = e^z$. We have the commutative diagram*

$$\begin{array}{ccc} \tilde{D}(\epsilon) & & \\ \downarrow{\scriptstyle \pi} & & \\ D'(\epsilon) & \hookrightarrow & D(\epsilon) \end{array}$$

Proposition 3.3. *Let $h \in \mathcal{H}(\tilde{D}(\epsilon))$. The action of a polynomial $f \in \mathbb{C}[x]$ on h is given by $f \bullet h = f(e^z)\,h(z)$. The operator $\partial = d/dx$ acts on h by the formula $\partial \bullet h = h'(z)e^{-z}$. Then the map*

$$\pi^* : \mathcal{H}(D'(\epsilon)) \longrightarrow \mathcal{H}(\tilde{D}(\epsilon))$$

defined by $\pi^*(h) = h \circ \pi$ is an injective homomorphism of $A_1(\mathbb{C})$ modules.

Definition 3.4. *Let $\mathcal{M}_\epsilon$ denote the quotient module $\mathcal{H}(\tilde{D}(\epsilon))/\pi^*(\mathcal{H}(D(\epsilon)))$. If $D(\epsilon) \leq D(\epsilon')$, then $\tilde{D}\,(\epsilon') \subseteq \tilde{D}(\epsilon)$ and $\mathcal{H}(\tilde{D}(\epsilon)) \subseteq \mathcal{H}\left(\tilde{D}\,(\epsilon')\right)$. This induces a homomorphism of $A_1(\mathbb{C})$-modules*

$$\tau_{\epsilon\epsilon'} : \mathcal{M}_\epsilon \longrightarrow \mathcal{M}_{\epsilon'}.$$

*Hence $\{\mathcal{M}_\epsilon : \epsilon \in \mathbb{R}\}$ is a directed family of $A_1(\mathbb{C})$-modules, and we may take its direct limit called the **module of microfunctions**, denoted by $\mathcal{M}$.*

Chapter VI

§1 Graded and filtered

§1.1 Graded

Definition 1.1. *Let R be a ring.*

1. *If we have abelian group direct sum*

$$R = \bigoplus_{i \geq 0} R_i$$

*with $R_i R_j \subseteq R_{i+j}$ for all i, j, then R is called a **graded ring**. The R_i are called the **homogeneous components** of R.*

2. *A ideal (left, right, two-sided) I of R is called **graded ideal** if $I = \bigoplus_{i \geq 0}(R_i \cap I)$.*

3. *If I is a two-sided graded ideal of R, then quotient graded ring $R/I = \bigoplus_{i \geq 0} R_i / (R_i \cap I)$ is well-defined.*

Remark. *It clear that a ideal I is graded if and only if it is generated by homogeneous elements.*

Definition 1.2. *Let $R = \bigoplus_{i \geq 0} R_i$ and $S = \bigoplus_{i \geq 0} S_i$ be graded rings, the **graded ring homomorphism** is a ring homomorphism $f : R \to S$ such that $f(R_i) \subseteq S_i$ for all i.*

Proposition 1.3. *Let $R = \bigoplus_{i \geq 0} R_i$ and $S = \bigoplus_{i \geq 0} S_i$ be graded rings and $f : R \to S$ a graded ring homomorphism*

1. $\ker f$ *(the kernel of ring homomorphism) is a graded ideal.*

Definition 1.4. *Let $R = \bigoplus_{i \geq 0} R_i$ be a graded ring. A left R-module M is called a **graded left R-module** if*

(i) it is a direct sum $M = \bigoplus_{i \geq 0} M_i$ of abelian groups

(ii) such that $R_i M_j \subseteq M_{i+j}$ for all i, j.

*The M_i are called the **homogeneous components** of* degree *i of M.*

§1.2 Filtered

Definition 1.5. *Let R be a ring. A **increasing filtration** of R is a sequence of subgroups $\{F_iR\}$ of R such that*

(i) $F_0R \subset F_1R \subset F_2R \subset \cdots \subset F_nR \subset \cdots \subset R$

(ii) $\bigcup_{i\geq 0} F_iR = R$.

(iii) $F_iR \cdot F_jR \subseteq F_{i+j}R$ *for all* i, j.

*R is called a **filtered ring** if it has an increasing filtration.*

Proposition 1.6. *Let R be a ring.*

1. *If $R = \bigoplus_{i\geq 0} R_i$ is a graded ring, then the sequence of subgroups $\left\{F_kR := \bigoplus_{i=0}^{k} R_i\right\}$ is an increasing filtration of R.*
2. *If $\mathcal{F} = \{F_iR\}$ is a filtration of R, then*

$$\operatorname{gr}_{\mathcal{F}} R := \bigoplus_{i\geq 0} F_{i+1}R/F_iR$$

*is a graded ring (multiplication follows from R), called the **associated graded ring of** R **associated with the filtration** $\mathcal{F}$.*

Definition 1.7. *Let R be a filtered ring with filtration $\mathcal{F} = \{F_iR\}$. A left R-module M is called a **filtered left R-module** if it has a sequence of subgroups $\Gamma = \{\Gamma_iM\}$ such that*

(i) $\Gamma_0M \subset \Gamma_1M \subset \Gamma_2M \subset \cdots \subset \Gamma_nM \subset \cdots \subset M$

(ii) $\bigcup_{i\geq 0} \Gamma_iM = M$

(iii) $F_iR \cdot \Gamma_jM \subseteq \Gamma_{i+j}M$ *for all* i, j.

Proposition 1.8. *The **graded module of** M **associated with the filtration** Γ is defined by*

$$\operatorname{gr}_{\Gamma} M = \bigoplus_{i\geq 0} \Gamma_{i+1}M/\Gamma_iM$$

§1.3 Induced filtrations

Definition 1.9. *Let R be a filtered ring with filtration $\mathcal{F} = \{F_iR\}$ and M be a left filtered R-module.*

1. *If N is a submodule of M, then the **induced filtration** on N is a filtration $\Gamma' = \{\Gamma_iN\}$ defined by*

$$\Gamma_iN := N \cap \Gamma_iM.$$

Remark. *associated graded module* $\operatorname{gr}_{\Gamma'} N$ *of* N *and graded module homomorphism* i_k : $N \cap \Gamma_k M / N \cap \Gamma_{k-1} M \to \Gamma_k M / \Gamma_{k-1} M$

2. *the* ***quotient filtration*** $\Gamma'' = \{\Gamma_i(M/N)\}$ *on* M/N *is defined by*

$$\Gamma_i(M/N) := \Gamma_i M / (N \cap \Gamma_i M).$$

Remark. *associated graded module* $\operatorname{gr}_{\Gamma''}(M/N)$ *of* M/N *and graded module homomorphism* $\pi_k : \Gamma_k M / \Gamma_{k-1} M \to (\Gamma_k M / (N \cap \Gamma_k M)) \,/\, (\Gamma_{k-1} M / (N \cap \Gamma_{k-1} M))$ *(note that the right hand side is isomorphic to* $\Gamma_k M / \left(\Gamma_{k-1} M + \Gamma_k M \cap N\right)$*,* $(A_1/B_1) \,/\, (A_2/B_2) \cong A_1 / \left(B_1 + A_2\right)$*)*

Proposition 1.10. *Let* R *be a filtered ring with filtration* $\mathcal{F} = \{F_i R\}$ *and* M *be a left filtered* R*-module with filtration* $\Gamma = \{\Gamma_i M\}$*. If* N *is a submodule of* M*, then there exists a short exact sequence of graded* $\operatorname{gr}_{\mathcal{F}} R$*-modules*

$$0 \longrightarrow \operatorname{gr}_{\Gamma'} N \xrightarrow{i} \operatorname{gr}_{\Gamma} M \xrightarrow{\pi} \operatorname{gr}_{\Gamma''}(M/N) \longrightarrow 0.$$

is exact.

Proof. It is clear that the each

$$0 \longrightarrow \Gamma'_k \cap \Gamma'_{k-1} \xrightarrow{\ i_k\ } \Gamma_k/\Gamma_{k-1} \xrightarrow{\ \pi_k\ } \Gamma''_k/\Gamma''_{k-1} \longrightarrow 0$$

is exact in **Ab**. Thus the proposition follows. □