Real Analysis

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Chapter I

Measure

§1 Measurable space

§1.1 Definition

Definition 1.1. *Let X be a nonempty set. An **algebra** of sets on X is a nonempty collection $\mathcal{A}$ of subsets of X that is closed under finite unions and complements.*

Definition 1.2. *A family of sets $\mathcal{R} \subset \mathcal{P}(X)$ is called a **ring** if it is closed under finite unions and differences.*

*A ring that is closed under countable unions is called a σ-**ring**.*

Proposition 1.3. .

1. *Rings (resp. σ-rings) are closed under finite (resp. countable) intersections.*
2. *If $\mathcal{R}$ is a ring (resp. σ-ring), then $\mathcal{R}$ is an algebra (resp. σ-algebra) iff $X \in \mathcal{R}$.*
3. *If $\mathcal{R}$ is a σ-ring, then $\{E \subset X : E \in \mathcal{R} \text{ or } E^c \in \mathcal{R}\}$ is a σ-algebra.*
4. *If $\mathcal{R}$ is a σ-ring, then $\{E \subset X : E \cap F \in \mathcal{R} \text{ for all } F \in \mathcal{R}\}$ is a σ-algebra.*

Definition 1.4. *A collection $\mathcal{M}$ of subsets of a set X is said to be a σ-**algebra** or σ-**field** in X if $\mathcal{M}$ has the following properties:*

(i) If $E_n \in \mathcal{M}$, then $E = \bigcup\limits_{n=1}^{\infty} E_n \in \mathcal{M}$

(ii) If $E \in \mathcal{M}$, then $E^c \in \mathcal{M}$

*If $\mathcal{M}$ is a σ-algebra in X, then $(X, \mathcal{M})$ is called a **measurable space**, and the members of $\mathcal{M}$ are called the **measurable sets** in X.*

Proposition 1.5. *Let $(X, \mathcal{M})$ be a Measurable space.*

1. *$\varnothing \in \mathcal{M}$, $X \in \mathcal{M}$*

2. $A_1 \cup A_2 \cup \ldots \cup A_n \in \mathcal{M}$ *if* $A_i \in \mathcal{M}$ *for* $i = 1, \ldots, n$.

3. $\bigcap_{n=1}^{\infty} A_n = \left(\bigcup_{n=1}^{\infty} A_n^c\right)^c$ *is measurable if* $A_i \in \mathcal{M}$ *for* $i = 1, \ldots, n$

4. $A - B \in \mathcal{M}$ *if* $A \in \mathcal{M}$ *and* $B \in \mathcal{M}$.

Definition 1.6. *If $\mathcal{E}$ is any collection of subsets of X, there exists a smallest σ-algebra $\mathcal{M}(\mathcal{E})$ in X such that $\mathcal{E} \subset \mathcal{M}(\mathcal{E})$ called **the σ-algebra generated by** $\mathcal{E}$.*

*Let X be a topological space. There exists a smallest σ-algebra $\mathcal{B}$ in X such that every open set in X belongs $\mathcal{B}$. The members of $\mathcal{B}$ are called the **Borel set of** X. All countable unions of closed set are called F_σ set, all countable intersection of open set are called G_δ set.*

Proposition 1.7. *Then $\mathcal{M}(\mathcal{E})$ is the union of the σ-algebras generated by $\mathcal{F}$ as $\mathcal{F}$ ranges over all countable subsets of $\mathcal{E}$ i.e.*

$$\mathcal{M}(\mathcal{E}) = \bigcup_{\substack{\mathcal{F} \subset \mathcal{E} \\ \mathcal{F} \text{ countable}}} \mathcal{M}(\mathcal{F})$$

§1.2 Product of measurable space

Definition 1.8. *Let $\{(X_\alpha, \mathcal{M}_\alpha)\}_{\alpha \in A}$ be an indexed collection of measurable space, $X = \prod_{\alpha \in A} X_\alpha$, and $\pi_\alpha : X \to X_\alpha$ the coordinate maps. The **product σ-algebra** on X is the σ-algebra generated by*

$$\left\{\pi_\alpha^{-1}(E_\alpha) : E_\alpha \in \mathcal{M}_\alpha, \alpha \in A\right\}$$

We denote this σ-algebra by $\bigotimes_{\alpha \in A} \mathcal{M}_\alpha$.

Proposition 1.9. *Let $\{(X_\alpha, \mathcal{M})\}_{\alpha \in A}$ be an indexed collection of measurable space, and product $\bigotimes_{\alpha \in A} \mathcal{M}_\alpha$*

1. *If A is countable, then $\bigotimes_{\alpha \in A} \mathcal{M}_\alpha$ is generated by $\left\{\prod_{\alpha \in A} E_\alpha : E_\alpha \in \mathcal{M}_\alpha\right\}$.*

2. *Suppose that $\mathcal{M}_\alpha$ is generated by $\mathcal{E}_\alpha$. Then $\bigotimes_{\alpha \in A} \mathcal{M}_\alpha$ is generated by $\mathcal{F}_1 = \{\pi_\alpha^{-1}(E_\alpha) : E_\alpha \in \mathcal{E}_\alpha, \alpha \in A$*

3. *If A is countable and $X_\alpha \in \mathcal{E}_\alpha$ for all α, $\bigotimes_{\alpha \in A} \mathcal{M}_\alpha$ is generated by $\mathcal{F}_2 = \left\{\prod_{\alpha \in A} E_\alpha : E_\alpha \in \mathcal{E}_\alpha\right\}$.*

Proposition 1.10. *Let $X_1, \ldots, X_n$ be topological spaces and let $X = \prod_1^n X_j$, equipped with the product topology. Then $\bigotimes_1^n \mathcal{B}_{X_j} \subset \mathcal{B}_X$. If the X_j are C_2, then $\bigotimes_1^n \mathcal{B}_{X_j} = \mathcal{B}_X$.*

Proof. By 1.9, $\bigotimes_1^n \mathcal{B}_{X_j}$ is generated by $\left\{\pi_j^{-1}(U_j) : 1 \leq j \leq n, U_j \text{ is open in } X_j\right\}$, thus $\bigotimes_1^n \mathcal{B}_{X_j} \subset \mathcal{B}_X$.

Let $\mathcal{E}_j$ be a countable basis of X_j, then $\mathcal{E}_j = \{\prod U_i : U_i \in \mathcal{E}_i\}$ is a countable basis of X, It follows that $\mathcal{B}_{X_j}$ is generated by $\mathcal{E}_j$ and $\mathcal{B}_X$ is generated by $\{\prod_1^n E_j : E_j \in \mathcal{E}_j\}$. Therefore $\mathcal{B}_X = \bigotimes_1^n \mathcal{B}_{X_j}$ by 1.9. $\square$

Corollary 1.11. $\mathcal{B}_{\mathbb{R}^n} = \mathcal{B}_{\mathbb{R}}^{\otimes n}$.

§1.3 Elementary family

Definition 1.12. *We define an **elementary family** to be a collection $\mathcal{E}$ of subsets of X such that*

(i) $\varnothing \in \mathcal{E}$,

(ii) if $E, F \in \mathcal{E}$ *then* $E \cap F \in \mathcal{E}$,

(iii) if $E \in \mathcal{E}$ *then* E^c *is a finite disjoint union of members of* $\mathcal{E}$.

Proposition 1.13. *If $\mathcal{E}$ is an elementary family, the collection $\mathcal{A}$ of finite disjoint unions of members of $\mathcal{E}$ is an algebra.*

§2 Measure

Definition 2.1. *Let $(X, \mathcal{M})$ be a measurable space.*

1. *A **(positive) measure** on $(X, \mathcal{M})$ is a function μ, s.t.*

   (i) $\mu : \mathcal{M} \to [0, +\infty]$ *and* $\mu(\varnothing) = 0$

   (ii) If E_i is a sequence of disjoint sets in $\mathcal{M}$, then $\mu\left(\bigcup\limits_{i=1}^{\infty} E_i\right) = \sum\limits_{i=1}^{\infty} \mu(E_i)$

   *A **measure space** is a measurable space which has a positive measure defined on the σ-algebra, denoted by $(X, \mathcal{M}, \mu)$*

2. *If $\mu(X) < \infty$, μ is called **finite**.*

3. *If $E = \bigcup_1^\infty E_j$ where $E_j \in \mathcal{M}$ and $\mu(E_j) < \infty$ for all j, the set E is said to be σ**-finite** for μ. If X is σ-finite, μ is called σ**-finite**.*

4. *If for each E with $\mu(E) = \infty$ there exists a measurable subset F of E s.t. $0 < \mu(F) < \infty$, then μ is called **semifinite**.*

   Remark. *In this case, $\mu(F)$ can indeed be made arbitrarily large (though still finite).*

5. *A set $E \in \mathcal{M}$ such that $\mu(E) = 0$ is called a **null set**.*

6. *If a statement about points $x \in X$ is true except for x in some null set, we say that it is true μ-**almost everywhere** (abbreviated a.e.), or for almost every x.*

Theorem 2.2. *Let μ be a measure on $(X, \mathcal{M})$, define μ_0 on $\mathcal{M}$ by $\mu_0(E) = \sup\{\mu(F) : F \subset E$ and $\mu(F) < \infty\}$.*

1. *μ_0 is a semifinite measure. It is called the **semifinite part** of μ.*

2. *If μ is semifinite, then $\mu = \mu_0$.*

3. *There is a measure ν on $\mathcal{M}$ (in general, not unique) which assumes only the values 0 and ∞ such that $\mu = \mu_0 + \nu$.*

Proposition 2.3. *Let μ be a measure on a σ-algebra $\mathcal{M}$.*

1. *(finitely additivity) $\mu(A_1 \cup \ldots \cup A_n) = \mu(A_1) + \ldots + \mu(A_n)$ if $A_1, \ldots, A_n$ are pairwise disjoin members of $\mathcal{M}$*
2. *(Monotonicity) $A \subset B$ implies $\mu(A) \leq \mu(B)$ if $A \in \mathcal{M}, B \in \mathcal{M}$*
3. *(Subadditivity) If $\{E_j\}_1^\infty \subset \mathcal{M}$, then $\mu\left(\bigcup_1^\infty E_j\right) \leq \sum_1^\infty \mu\left(E_j\right)$.*
4. *(Continuity from below) If $\{E_j\}_1^\infty \subset \mathcal{M}$ and $E_1 \subset E_2 \subset \cdots$, then $\mu\left(\bigcup_1^\infty E_j\right) = \lim_{j\to\infty} \mu\left(E_j\right)$.*
5. *(Continuity from above) If $\{E_j\}_1^\infty \subset \mathcal{M}, E_1 \supset E_2 \supset \cdots$, and $\mu\left(E_1\right) < \infty$, then $\mu\left(\bigcap_1^\infty E_j\right) = \lim_{j\to\infty} \mu\left(E_j\right)$.*

Definition 2.4 (Completion)**.** *Let $(X, \mathcal{M}, \mu)$ be a measure space, let $\overline{\mathcal{M}}$ be the collection of all $E \subset X$ for which there exist sets A and $B \in \mathcal{M}$ such that $A \subset E \subset B$ and $\mu(B - A) = 0$, and define $\overline{\mu}(E) = \mu(A)$ in this situation.*

Remark. *It follows that $\overline{\mathcal{M}} = \{E \cup N : E \in \mathcal{M}, N \text{ a subset of a null set}\}$*

Proposition 2.5. *Let $(X, \mathcal{M}, \mu)$ be a measure space and $E \in \mathcal{M}$, define $\mu_E(A) = \mu(A \cap E)$ for $A \in \mathcal{M}$ and $\mathcal{M}_E = \{A \cap E : A \in \mathcal{M}\}$. Then*

1. *$(X, \mathcal{M}, \mu_E)$ is a measure space.*
2. *$(E, \mathcal{M}_E, \mu_E)$ is a measure space.*

§3 Outer Measures

Definition 3.1. *An **outer measure** on a nonempty set X is a function that satisfies*

(i) $\mu^* : \mathcal{P}(X) \to [0, \infty]$ *and* $\mu^*(\varnothing) = 0$,

(ii) $\mu^*(A) \leq \mu^*(B)$ *if* $A \subset B$,

(iii) $\mu^*\left(\bigcup_1^\infty A_j\right) \leq \sum_1^\infty \mu^*\left(A_j\right)$.

Proposition 3.2. *Let $\mathcal{E} \subset \mathcal{P}(X)$ and $\rho : \mathcal{E} \to [0, \infty]$ be such that $\varnothing \in \mathcal{E}, X \in \mathcal{E}$, and $\rho(\varnothing) = 0$. For any $A \subset X$, define*

$$\mu^*(A) = \inf\left\{\sum_1^\infty \mu\left(E_j\right) : E_j \in \mathcal{E} \text{ and } A \subset \bigcup_1^\infty E_j\right\}.$$

Then μ^* is an outer measure.

Proof. For any $A \subset X$ there exists $\{E_j\}_1^\infty \subset \mathcal{E}$ such that $A \subset \bigcup_1^\infty E_j$ (take $E_j = X$ for all j) so the definition of μ^* makes sense. Obviously $\mu^*(\varnothing) = 0$ (take $E_j = \varnothing$ for all j), and $\mu^*(A) \leq \mu^*(B)$ for $A \subset B$ because the set over which the infimum is taken in the definition of $\mu^*(A)$ includes the corresponding set in the definition of $\mu^*(B)$. To prove the countable subadditivity, suppose $\{A_j\}_1^\infty \subset \mathcal{P}(X)$ and $\epsilon > 0$. For each j there exists $\left\{E_j^k\right\}_{k=1}^\infty \subset \mathcal{E}$ such that $A_j \subset \bigcup_{k=1}^\infty E_j^k$ and $\sum_{k=1}^\infty \rho\left(E_j^k\right) \leq \mu^*\left(A_j\right) + \epsilon 2^{-j}$. But then if $A = \bigcup_1^\infty A_j$, we have $A \subset \bigcup_{j,k=1}^\infty E_j^k$ and $\sum_{j,k} \rho\left(E_j^k\right) \leq \sum_j \mu^*\left(A_j\right) + \epsilon$, whence $\mu^*(A) \leq \sum_j \mu^*\left(A_j\right) + \epsilon$. Since ϵ is arbitrary, we are done. □

Definition 3.3. *Let μ^* be an outer measure on X, a set $A \subset X$ is called **μ^*-measurable** if*

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*\left(E \cap A^c\right) \text{ for all } E \subset X.$$

So we see that A is μ^*-measurable iff

$$\mu^*(E) \geq \mu^*(E \cap A) + \mu^*\left(E \cap A^c\right) \text{ for all } E \subset X \text{ such that } \mu^*(E) < \infty.$$

Some motivation for the notion of μ^*-measurability can be obtained by referring to the discussion at the beginning of this section. If E is a "well-behaved" set such that $E \supset A$, the equation $\mu^*(E) = \mu^*(E \cap A) + \mu^*\left(E \cap A^c\right)$ says that the outer measure of A, $\mu^*(A)$, is equal to the "inner measure" of A, $\mu^*(E) - \mu^*\left(E \cap A^c\right)$. The leap from "well-behaved" sets containing A to arbitrary subsets of X a large one, but it is justified by the following theorem.

Theorem 3.4 (Carathéodory's Theorem)**.** *Let μ^* be an outer measure on X, then the collection $\mathcal{M}$ of μ^*-measurable sets is a σ-algebra, and $(X, \mathcal{M}, \mu = \mu^*|_{\mathcal{M}})$ is a complete measure space.*

Premeasure

Definition 3.5. *Let $\mathcal{A} \subset \mathcal{P}(X)$ be an algebra on X, a function μ will be called a **premeasure** if*

1. *$\mu_0 : \mathcal{A} \to [0, \infty]$ and $\mu_0(\varnothing) = 0$,*
2. *if $\{A_j\}_1^\infty$ is a sequence of disjoint sets in $\mathcal{A}$ such that $\bigcup_1^\infty A_j \in \mathcal{A}$, then $\mu_0\left(\bigcup_1^\infty A_j\right) = \sum_1^\infty \mu_0\left(A_j\right)$.*

If μ_0 is a premeasure on $\mathcal{A} \subset \mathcal{P}(X)$, it induces an outer measure on X in accordance with 3.2, namely,

$$\mu^*(E) = \inf\left\{\sum_1^\infty \mu_0\left(A_j\right) : A_j \in \mathcal{A}, E \subset \bigcup_1^\infty A_j\right\}.$$

§4 Product Measure

Chapter II

Integration

Contents

§1 Measurable Function

§1.1 Basic Definition

Definition 1.1. *Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be measurable spaces, a mapping $f : X \to Y$ is called $(\mathcal{M}, \mathcal{N})$-**measurable** if $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{N}$.*

*If X and Y are topological spaces, $f : X \to Y$ is called **Borel measurable** if it is $(\mathcal{B}_X, \mathcal{B}_Y)$-measurable.*

Lemma 1.2. *Let $f : X \to Y$ be a map inducing a mapping $f^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X)$, defined by $f^{-1}(E) = \{x \in X : f(x) \in E\}$, which preserves countable unions, intersections, and complements. Thus, if $\mathcal{N}$ is a σ-algebra on Y, $\{f^{-1}(E) : E \in \mathcal{N}\}$ is a σ-algebra on X.*

Proposition 1.3. *Let $(X, \mathcal{M})$, $(Y, \mathcal{N})$ and $(Z, \mathcal{O})$ be measurable spaces.*

1. *If $f : X \to Y$ is $(\mathcal{M}, \mathcal{N})$-measurable and $g : Y \to Z$ is $(\mathcal{N}, \mathcal{O})$-measurable, then $g \circ f$ is $(\mathcal{M}, \mathcal{O})$-measurable.*

2. *If $\mathcal{N}$ is generated by $\mathcal{E}$, then $f : X \to Y$ is $(\mathcal{M}, \mathcal{N})$-measurable iff $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{E}$.*

3. *If X and Y are topological spaces, every continuous $f : X \to Y$ is $(\mathcal{B}_X, \mathcal{B}_Y)$-measurable.*

Definition 1.4. *Let $(X, \mathcal{M})$ be a measurable space, f is a function on X, and $E \in \mathcal{M}$, we say that f is measurable on E if $f^{-1}(B) \cap E \in \mathcal{M}$ for all Borel sets B. (Equivalently, $f|_E$ is $\mathcal{M}_E$-measurable, where $\mathcal{M}_E = \{F \cap E : F \in \mathcal{M}\}$.)*

Proposition 1.5. *Let $(X, \mathcal{M})$ and $(Y_\alpha, \mathcal{N}_\alpha)$ $(\alpha \in A)$ be measurable spaces, $Y = \prod_{\alpha \in A} Y_\alpha, \mathcal{N} = \otimes_{\alpha \in A} \mathcal{N}_\alpha$, and $\pi_\alpha : Y \to Y_\alpha$ the coordinate maps. Then $f : X \to Y$ is $(\mathcal{M}, \mathcal{N})$-measurable iff $f_\alpha = \pi_\alpha \circ f$ is $(\mathcal{M}, \mathcal{N}_\alpha)$-measurable for all α.*

§1.2 Complex-valued measurable function

Definition 1.6. *Let $(X, \mathcal{M})$ be a measurable space, a complex-, real- or extended real-valued function f on X will be called $\mathcal{M}$-measurable, if it is $(\mathcal{M}, \mathcal{B}_{\mathbb{C}})$, $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$ or $(\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}})$ measurable.*

*In particular, $f : \mathbb{R}^n \to \mathbb{C}, \mathbb{R}$ or $\overline{\mathbb{R}}$ is **Lebesgue measurable** if it is $(\mathcal{L}, \mathcal{B}_{\mathbb{C}})$, $(\mathcal{L}, \mathcal{B}_{\mathbb{R}})$ or $(\mathcal{L}, \mathcal{B}_{\overline{\mathbb{R}}})$ measurable respectively.*

Proposition 1.7. *Let $(X, \mathcal{M})$ be a measurable space*

1. *A function $f : X \to \overline{\mathbb{R}}$ is measurable iff $f^{-1}(\{-\infty\}) \in \mathcal{M}$, $f^{-1}(\{\infty\}) \in \mathcal{M}$, and $f\chi_{\{|f|<\infty\}} : X \to \mathbb{R}$ is measurable.*

2. *A function $f : X \to \mathbb{C}$ is measurable iff $\operatorname{Re} f$ and $\operatorname{Im} f : X \to \mathbb{R}$ are measurable.*

Proposition 1.8. *2. Suppose $f, g : X \to \overline{\mathbb{R}}$ are measurable. a. fg is measurable (where $0 \cdot (\pm\infty) = 0$). b. Fix $a \in \overline{\mathbb{R}}$ and define $h(x) = a$ if $f(x) = -g(x) = \pm\infty$ and $h(x) = f(x) + g(x)$ otherwise. Then h is measurable.*

Proposition 1.9. *Let $f_n : (X, \mathcal{M}) \to \overline{\mathbb{R}}$ be measurable. Then*

1. *$g = \sup f_n, h = \inf f_n, \liminf f_n$ and $\limsup f_n$ are measurable.*

2. *The limit of every pointwise convergent sequence of complex measurable functions is measurable.*

Definition 1.10. *Let $(X, \mathcal{M})$ be a measurable space.*

1. *if $f : X \to \overline{\mathbb{R}}$, we define the* ***positive and negative parts*** *of f to be*

$$f^+(x) = \max(f(x), 0), \quad f^-(x) = \max(-f(x), 0)$$

Then $f = f^+ - f^-$. If f is measurable, so are $f^+$and f^-, by Corollary 2.8.

2. *if $f : X \to \mathbb{C}$, we have its* ***polar decomposition****:*

$$f = (\operatorname{sgn} f)|f|, \quad \textit{where} \quad \operatorname{sgn} z = \begin{cases} z/|z| & \textit{if } z \neq 0 \\ 0 & \textit{if } z = 0 \end{cases}$$

if f is measurable, so are $|f|$ and sgn *f.*

§1.3 Simple Functions

Definition 1.11. *Let $s : X \longrightarrow \mathbb{C}$ a complex function on a measurable space X. If the range of s consists of only finitely many points, s is called a* ***simple function***. *If $\alpha_1, \ldots, \alpha_n$ are the distinct values of a simple function s, then clearly*

$$s = \sum_{i=1}^{n} \alpha_i \chi_{f^{-1}(\alpha_i)}$$

It is also clear that s is measurable if and only if each of the sets A_ι is measurable.

Among these are the nonnegative simple functions, whose range is a finite subset of $[0, \infty)$.

Theorem 1.12. *Let $f : (X, \mathcal{M}) \to [0, \infty]$ be measurable. There exist simple measurable functions s_n on X such thet*

(i) $0 \le s_1 \le s_2 \le \ldots \le f$.

(ii) $s_n(x) \to f(x)$ *as $n \to \infty$ for every $x \in X$. pointwise, and $\phi_n \to f$ uniformly on any set on which f is bounded.*

Proof. To each positive integer n and $t \in R$, define

$$\varphi_n(t) = \begin{cases} 2^{-n}[2^n t] & 0 \le t < n \\ n & n \le t \le \infty \end{cases}$$

Each φ_n is then a Borel function on $[0, \infty]$, $0 \le \varphi_1 \le \varphi_2 \le \ldots \varphi_n \le t$ and $\varphi_n \to t$ as $n \to \infty$ for every $t \in [0, \infty]$. It follows that the function

$$s_n = \varphi_n \circ f$$

are measurable simple function satisfied (i) and (ii). □

Corollary 1.13. *Let $(X, \mathcal{M})$ be a measurable space. If $f : X \to \mathbb{C}$ or $\overline{\mathbb{R}}$ is measurable, there is a sequence $\{\phi_n\}$ of simple functions s.t. $0 \le |\phi_1| \le |\phi_2| \le \cdots \le |f|, \phi_n \to f$ pointwise, and $\phi_n \to f$ uniformly on any set on which f is bounded.*

§1.4 Completion

Proposition 1.14. *Let $(X, \mathcal{M}, \mu)$ be a measure space. Then the following proposition are equivalent*

1. *$(X, \mathcal{M}, \mu)$ is complete.*
2. *If f is measurable and $f = g$ a.e., then g is measurable.*
3. *If f_n is measurable and $f_n \to f$ a.e., then f is measurable.*

Proposition 1.15. *Let $(X, \mathcal{M}, \mu)$ be a measure space and $(X, \overline{\mathcal{M}}, \bar{\mu})$ be its completion. If f is an $\overline{\mathcal{M}}$-measurable function on X, there is an $\mathcal{M}$-measurable function g such that $f(x) = g(x)$ a.e. $\bar{\mu}$.*

§2 Lebesgue Integration

Definition 2.1. *Let $(X, \mathcal{M}, \mu)$ be a measure space.*

1. *If $s : X \longrightarrow [0, \infty)$ is a measurable simple function of the form*

$$s = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$$

where $\alpha_1, \ldots, \alpha_n$ are the distinct values of s, and if $E \in \mathcal{M}$, we define

$$\int_E s \, \mathrm{d}\mu = \sum_{i=1}^{n} \alpha_i \, \mu(A_i \cap E)$$

2. *If $f : X \to [0, \infty]$ is a nonnegative measurable function, and $E \in \mathcal{M}$, we defne*

$$\int_E f \, \mathrm{d}\mu = \sup \int_E s \, \mathrm{d}\mu$$

the supremum being taken over all simple measurable functions such that $0 \leq s \leq f$.

3. *If $f : X \to [-\infty, \infty]$ is measurable, and $E \in \mathcal{M}$, we define*

$$\int_E f \, \mathrm{d}\mu = \int_E f^+ \, \mathrm{d}\mu - \int_E f^- \, \mathrm{d}\mu$$

*provided that at least one of the integrals on the right is finite. The left side is then a number in $[-\infty, \infty]$ called the **Lebesgue integral** of f over E, with respect to the measure μ. We say f is integrablr if both $\int_E f^+ \, \mathrm{d}\mu$ and $\int_E f^- \, \mathrm{d}\mu$ are finite.*

Proposition 2.2. *Let $(X, \mathcal{M}, \mu)$ be a measure space and let $(X, \overline{\mathcal{M}}, \bar{\mu})$ be its completion. If f is an $\overline{\mathcal{M}}$-measurable function on X, there is an $\mathcal{M}$-measurable function g such that $f = g$ $\bar{\mu}$-almost everywhere.*

§2.1 Convergence Theorem

Theorem 2.3 (Lebesgue's Monotone Convergence Theorem)**.** *Let $\{f_n\}$ be a sequence of measurable functions on X, and suppose that*

(i) $0 \leq f_1(x) \leq f_2(x) \leq \cdots \leq \infty$ *for every $x \in X$*

(ii) $\lim f_n(x) \to f(x)$ *for every $x \in X$*

Then f is measurable, and

$$\lim_{n\to\infty} \int_X f_n \, \mathrm{d}\mu = \int_X f \, \mathrm{d}\mu$$

Proof. Since $\int f_n \leq \int f_{n+1} \leq \int f$, there exists an $\alpha \in [0, \infty]$ such that

$$\int_X f_n \, \mathrm{d}\mu \to \alpha \leq \int_X f d\mu$$

Let s be any simple measurable function such that $0 \leq s \leq f$, let $c \in (0, 1)$ be a constant, and define

$$E_n = \{x : f_n(x) \geq cs(x)\} \quad (n = 1, 2, \ldots)$$

Each E_n is measurable, $E_1 \subset E_2 \subset E_3 \subset \ldots$, and $X = \bigcup E_n$. Also,

$$\alpha \geq \int_X f_n \, \mathrm{d}\mu \geq \int_{E_n} f_n \, \mathrm{d}\mu \geq c \int_{E_n} s \, \mathrm{d}\mu \quad (n = 1, 2, \ldots)$$

Let $n \to \infty$ and then $c \to 1^-$, the result is

$$\alpha \geq \int_X s \, \mathrm{d}\mu$$

for every simple measurable s satisfying $0 \leq s \leq f$, so that

$$\alpha \geq \int_X f \, \mathrm{d}\mu$$

□

Corollary 2.4. *Let $\{f_n\}$ be a sequence of measurable functions on X, and suppose that*

1. $\infty \geq f_1(x) \geq f_2(x) \geq \cdots \geq 0$ *for every $x \in X$*
2. $\lim f_n(x) = f(x)$ *for every $x \in X$*
3. $f_1 \in L^1(\mu)$

Then f is measurable, and

$$\int_X f_n \, \mathrm{d}\mu \to \int_X f \, \mathrm{d}\mu \qquad as \; n \to \infty$$

Corollary 2.5. *If $f_n : X \longrightarrow [0,\infty]$ is measurable, for $n = 1, 2, \dots$, and*

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \quad (x \in X),$$

is measurable, then

$$\int_X f \, \mathrm{d}\mu = \sum_{n=1}^{\infty} \int_X f_n \, \mathrm{d}\mu$$

Theorem 2.6 (Fatou's Lemma). *If $f_n : X \to [0,\infty]$ is measurable for each positive integer n, then*

$$\int_X \left(\liminf_{n\to\infty} f_n \right) \mathrm{d}\mu \le \liminf_{n\to\infty} \int_X f_n \, \mathrm{d}\mu$$

Proof. Put

$$g_k(x) = \inf_{i \ge k} f_i(x) \quad (k = 1, 2, \dots; x \in X)$$

Then $g_k \le f_k$. Also, $0 \le g_1 \le g_2 \le \cdots$, each g_k is measurable, so that

$$\int_X \liminf f_n \xleftarrow[\text{Monotone}]{\text{lim inf}} \int_X g_k \le \int_X f_k \xrightarrow{\text{lim inf}} \liminf \int_X f_n$$

$\square$

Theorem 2.7 (Lebesgue's Dominated Convergence Theorem). *Suppose $\{f_n\}$ is a sequence of complex measurable functions on X such that*

(i) $\lim_{n\to\infty} f_n(x) = f(x)$ *exists for every $x \in X$.*

(ii) there is a function $g \in L^1(\mu)$ such that

$$|f_n(x)| \le |g(x)| \quad (n = 1, 2, \dots; x \in X)$$

then $f \in L^1(\mu)$ and

$$\lim_{n\to\infty} \int_X |f_n - f| \, d\mu = 0$$

Proof. Since $|f| \le g$ and f is measurable, $f \in L^1(\mu)$. Since $|f_n - f| \le 2g$, Fatou's lemma applies to the functions $2g - |f_n - f|$ and yields

$$\begin{aligned}
\int_X 2g \, \mathrm{d}\mu &\le \liminf \int_X (2g - |f_n - f|) \, \mathrm{d}\mu \\
&= \int_X 2g \, \mathrm{d}\mu + \liminf \left(- \int_X |f_n - f| \, \mathrm{d}\mu \right) \\
&= \int_X 2g \, \mathrm{d}\mu - \limsup \int_X |f_n - f| \, \mathrm{d}\mu
\end{aligned}$$

Since $\int 2g\,\mathrm{d}\mu$ is finite, we may subtract it and obtain

$$\limsup \int_X |f_n - f|\,\mathrm{d}\mu \le 0$$

Thus $\lim \int_X |f_n - f|\,\mathrm{d}\mu \le 0$. $\square$

Theorem 2.8 (A generalized Dominated Convergence Theorem)**.** *Suppose $g_n, g \in L^1$. If*

(i) $f_n \to f$ for all $x \in X$

(ii) $g_n \to g$ for all $x \in X$, $|f_n| \le g_n$ and $\int g_n \to \int g$

then $f \in L^1$ and $f \xrightarrow{L^1} f$

Proof. Apply Fatou's lemma to

$$g_n + g - |f_n - f|$$

$\square$

Theorem 2.9. *Suppose f_n is a sequence of complex measurable functions on $(X, \mathcal{M}, \mu)$ s.t,*

$$\sum_{n=1}^{\infty} \int_X |f_n|\,\mathrm{d}\mu < \infty$$

Then the series

$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$

converges absolutely for a.e. $x \in X$ (thus f is well-defined a.e.), and

$$\int_X f\,\mathrm{d}\mu = \sum_{n=1}^{\infty} \int_X f_n\,\mathrm{d}\mu$$

Proof. Let S_n be the set on which f_n is defined, so that $\mu(S_n^c) = 0$. Put

$$\varphi(x) = \sum_{n=1}^{\infty} |f_n(x)|$$

for $x \in S = \bigcap S_n$. Then $\mu(S^c) = 0$ and φ is defined a.e. $[\mu]$. By monotone convergence theorem,

$$\int_S \varphi\,\mathrm{d}\mu = \sum_{n=1}^{\infty} \int_S |f_n|\,\mathrm{d}\mu < \infty$$

If $E = \{x \in S : \varphi(x) < \infty\} \subset S$, it follows that $\mu(S - E) = 0$. The series $f(x) = \sum f_n(x)$ converges absolutely for every $x \in E$, and if $f(x)$ is defined for $x \in E$ then $|f(x)| \le \varphi(x)$ on E, so that $f \in L^1(\mu)$ on E. If $g_n = \sum_{k=1}^{n} f_k$ then $|g_n| \leq \varphi,\ g_n(x) \to f(x)$ for all $x \in E$, then

$$\sum_{n=1}^{\infty} \int_X f_n \,\mathrm{d}\mu = \sum_{n=1}^{\infty} \int_E f_n \,\mathrm{d}\mu = \int_E f \,\mathrm{d}\mu = \int_X f \,\mathrm{d}\mu$$

□

Corollary 2.10. *Let $\{E_k\}$ be a sequence of measurable sets n X, such that*

$$\sum_{k=1}^{\infty} \mu(E_k) < \infty$$

Then almost all $x \in X$ lie in at mostf finitely many of the sets E_k.

§2.2 $L^1(\mu)$ Space

Definition 2.11. *Let (X, μ) be a measure. We define $L^1(\mu)$ to be the collection of all complex measurable functions f on X for which*

$$\int_X |f| \,\mathrm{d}\mu < \infty$$

*The members of $L^1(\mu)$ are called **Lebesgue integrable functions (with respect to μ)** or summable functions.*

If $f = u + \mathrm{i}v$, where u and v are real measurable functions on X, and if $f \in L^1(\mu)$, we define

$$\int_E f \,\mathrm{d}\mu = \int_E u^+ \,\mathrm{d}\mu - \int_E u^- \,\mathrm{d}\mu + \mathrm{i} \int_E v^+ \,\mathrm{d}\mu - \mathrm{i} \int_E v^- \,\mathrm{d}\mu$$

for every measurable set E.

Theorem 2.12. *The operator $\int \cdot \,\mathrm{d}\mu$ is a bounded $\mathbb{C}$-linear functional in L^1 with norm 1.*

Proof. Put $z = \int_X f \,\mathrm{d}\mu$. Since z is a complex number, there is a complex number α, with $|\alpha| = 1$,such that $\alpha z = |z|$. Let u be the real part of αf. Then $u \leq |\alpha f| = |f|$. Hence

$$\left| \int_X f \,\mathrm{d}\mu \right| = \alpha \int_X f \,\mathrm{d}\mu = \int_X \alpha f \,\mathrm{d}\mu = \int_X u \,\mathrm{d}\mu \leq \int_X |f| \,\mathrm{d}\mu$$

□

Theorem 2.13.

(1) Suppose $f : X \longrightarrow [0, \infty]$ is measurable, $E \in \mathcal{M}$, and $\int_E f \,\mathrm{d}\mu = 0$. Then $f = 0$ a.e. on E.

(2) Suppose $f \in L^1(\mu)$ and $\int_E f \,\mathrm{d}\mu = 0$ for every $E \in \mathcal{M}$. Then $f = 0$ a.e. on X.

(3) Suppose $f \in L^1(\mu)$ and

$$\int_X f \,\mathrm{d}\mu = \int_X |f| \,\mathrm{d}\mu$$

Then there is a constant α such that $f = \alpha\,|f|$ a.e. on X.

Theorem 2.14 (Absolutely continuous)**.** *Suppoes $f \in L^1(\mu)$. Then to each $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$\int_E |f|\, d\mu < \varepsilon$$

whenever $\mu(E) < \delta$.

§3 Modes of Convergence

§3.1 Convergence in measure

Definition 3.1. *Let $(X, \mathcal{M}, \mu)$ be a measure space. We say that a sequence $\{f_n\}$ of measurable complex-valued functions on $(X, \mathcal{M}, \mu)$ is* ***Cauchy in measure*** *if for every $\varepsilon > 0$,*

$$\mu\left(\{x : |f_n(x) - f_m(x)| \geq \varepsilon\}\right) \to 0 \textit{ as } m, n \to \infty,$$

and that $\{f_n\}$ converges in measure to f if for every $\varepsilon > 0$,

$$\mu\left(\{x : |f_n(x) - f(x)| \geq \varepsilon\}\right) \to 0 \textit{ as } n \to \infty.$$

Remark. *If we relax the condition so that the function f_n and f can take ∞, then this definition is not well-defined. Therefore, we only consider $\mathbb{R}$-valued functions here, and furthermore, $\mathbb{C}$-valued functions.*

Theorem 3.2. *Let $(X, \mathcal{M}, \mu)$ be a measure space. Then $f_n \to f$ in measure iff for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\mu\left(\{x : |f_n(x) - f(x)| \geq \epsilon\}\right) < \epsilon$ for $n \geq N$.*

Theorem 3.3. *Suppose that $\{f_n\}$ is Cauchy in measure. Then there is a unique (in the sense of a.e.) measurable function f such that $f_n \to f$ in measure, and there is a subsequence $\left\{f_{n_j}\right\}$ that converges to f a.e.*

Proof. We can choose a subsequence $\{g_j\}$ of $\{f_n\}$ such that $E_j = \{x : |g_j(x) - g_{j+1}(x)| \geq 2^{-j}\}$ of measure $\mu(E_j) \leq 2^{-j}$. If $F_k = \bigcup_{j=k}^{\infty} E_j$, then $\mu(F_k) \leq \sum_k^{\infty} 2^{-j} = 2^{1-k}$, and if $x \notin F_k$, for $i \geq j \geq k$ we have

$$|g_j(x) - g_i(x)| \leq \sum_{l=j}^{i-1} |g_{l+1}(x) - g_l(x)| \leq \sum_{l=j}^{i-1} 2^{-l} \leq 2^{1-j}$$

Thus $\{g_j\}$ is pointwise Cauchy on F_k^c. Let $F = \bigcap_1^{\infty} F_k = \limsup E_j$. Then $\mu(F) = 0$, and if we set $f(x) = \lim g_j(x)$ for $x \notin F$ and $f(x) = 0$ for $x \in F$, then f is measurable and $g_j \to f$ a.e.

Also, $|g_j(x) - f(x)| \leq 2^{1-j}$ for $x \notin F_k$ and $j \geq k$. Since $\mu(F_k) \to 0$ as $k \to \infty$, it follows that $g_j \to f$ in measure. But then $f_n \to f$ in measure, because

$$\{x : |f_n(x) - f(x)| \geq \varepsilon\} \subset \left\{x : |f_n(x) - g_j(x)| \geq \frac{1}{2}\varepsilon\right\} \cup \left\{x : |g_j(x) - f(x)| \geq \frac{1}{2}\varepsilon\right\},$$

and the sets on the right both have small measure when n and j are large. Likewise, if $f_n \to g$ in measure,

$$\{x : |f(x) - g(x)| \geq \varepsilon\} \subset \left\{x : |f(x) - f_n(x)| \geq \frac{1}{2}\varepsilon\right\} \cup \left\{x : |f_n(x) - g(x)| \geq \frac{1}{2}\varepsilon\right\}$$

for all n, hence $\mu(\{x : |f(x) - g(x)| \geq \varepsilon\}) = 0$ for all ε. Letting ε tend to zero through some sequence of values, we conclude that $f = g$ a.e. □

Proposition 3.4. *$f_n \to f$ in L^1 $\Rightarrow$ $f_n \to f$ in measure $\Rightarrow$ $f_{n_j} \to f$ a.e.*

Proposition 3.5. *Let $(X, \mathcal{M}, \mu)$ be a finite measure space. If f and g are complex-valued measurable functions on X, define*

$$\rho(f, g) = \int \frac{|f - g|}{1 + |f - g|} \, \mathrm{d}\mu.$$

Then ρ is a metric on $L^0(\mu)$, and $f_n \to f$ with respect to this metric iff $f_n \to f$ in measure.

Proposition 3.6. *Suppose $f_n \to f$ in measure and $g_n \to g$ in measure. Then*

1. *$f_n + g_n \to f + g$ in measure.*
2. *$f_n g_n \to fg$ in measure if $\mu(X) < \infty$.*

Theorem 3.7. *Let $(X, \mathcal{M}, \mu)$ be a finite measure space, then $L^0(\mu)$ is a linear topological space with complete metric.*

§3.2 Almost uniform convergence

Theorem 3.8 (Egoroff's Theorem)**.** *Let $(X, \mathcal{M}, \mu)$ be a finite measure space, $f_1, f_2, \ldots$ and f be measurable complex-valued functions on X such that $f_n \to f$ a.e. Then for every $\varepsilon > 0$ there exists $E \subset X$ such that $\mu(E) < \varepsilon$ and $f_n \to f$ uniformly on E^c.*

Remark. *The type of convergence involved in the conclusion of Egoroff's theorem is sometimes called **almost uniform convergence**.*

The hypothesis " $\mu(X) < \infty$ " can be replaced by " $|f_n| \leq g$ for all n, where $g \in L^1(\mu)$."

Proof. Without loss of generality we may assume that $f_n \to f$ everywhere on X. For $k, n \in \mathbb{N}$ let

$$E_n(k) = \bigcup_{m=n}^{\infty} \{x : |f_m(x) - f(x)| \geq k^{-1}\}$$

Then, for fixed $k, E_n(k)$ decreases as n increases, and $\bigcap_{n=1}^{\infty} E_n(k) = \varnothing$, so since $\mu(X) < \infty$ we conclude that $\mu\left(E_n(k)\right) \to 0$ as $n \to \infty$. Given $\varepsilon > 0$ and $k \in \mathbb{N}$, choose n_k so large that $\mu\left(E_{n_k}(k)\right) < \varepsilon 2^{-k}$ and let $E = \bigcup_{k=1}^{\infty} E_{n_k}(k)$. Then $\mu(E) < \varepsilon$, and we have $|f_n(x) - f(x)| < k^{-1}$ for $n > n_k$ and $x \notin E$. Thus $f_n \to f$ uniformly on E^c. $\square$

§3.3 Relation between different convergence modes

Proposition 3.9 (The Vitali Convergence Theorem)**.** *Suppose* $1 \leq p < \infty$ *and* $\{f_n\}_1^{\infty} \subset L^p$*. Then* $\{f_n\}$ *is Cauchy in the* L^p *iff*

(i) $\{f_n\}$ *is Cauchy in measure;*

(ii) the sequence $\{|f_n|^p\}$ *is uniformly integrable.*

(iii) for every $\epsilon > 0$ *there exists* $E \subset X$ *such that* $\mu(E) < \infty$ *and* $\int_{E^c} |f_n|^p < \epsilon$ *for all* n*.*

Remark. *It follows that*

$$f_n \xrightarrow{L^p} f \Rightarrow f_n \xrightarrow{\mu} f$$

On the other hand,

$$f_n \xrightarrow{\mu} f \text{ and } |f_n| \leq g \in L^p \Rightarrow f_n \xrightarrow{L^p} f$$

Proposition 3.10. *Suppose* $1 \leq p < \infty$*. If* $f_n, f \in L^p$ *and* $f_n \to f$ *a.e., then* $\|f_n - f\|_p \to 0$ *iff* $\|f_n\|_p \to \|f\|_p$*.*

Proof. Apply Fatou's lemma to

$$h_n = 2^{n-1}\left(|f_n|^p + |f|^p\right) - |f_n - f|^p \geq 0$$

$\square$

Chapter III

Radon measure

§1 Some topology Preliminaries

Definition 1.1. *Let X be a topological space.*

*X is **locally compact** if every point of X has a neighborbood whose closure is compact.*

If X is itself compact, then X is called a compact space.

Theorem 1.2. *Suppose K is compact and F is closed, in a topological space X. If $F \subset K$, then F is compact.*

Corollary 1.3. *If $A \subset B$ and if B has compact closure, so does A.*

Theorem 1.4. *Suppose X is a Hausdorff space, a compact $K \subset X$ and $p \notin K$. Then there are open sets $p \in U$ and $K \subset V$ such that $U \cap V = \varnothing$.*

Corollary 1.5. *Let X be a Hausdorff space.*

(1) Compact subsets are closed.

(2) If F is closed and K is compact, then $F \cap K$ is compact.

Theorem 1.6. *If $\{K_\alpha\}$ is a collection of compact subsets of a Hausdorff space and if $\bigcap_a K_\alpha = \varnothing$, then some finite subcollection of $\{K_{p_k}\}_{k=1}^n$ also has empty intersection.*

Proof: Fix a member K_1 of $\{K_a\}$. Since no point of K_1 belongs to every K_a, then $\{K_\alpha^c\}$ is an open cover of K_1. Hence $K_1 \subset K_{\alpha_1}^c \cup \cdots \cup K_{\alpha_n}$ for some finite collection $\{K_{\alpha_i}\}$. This implies that

$$K_1 \cap K_{\alpha_1} \cap \cdots \cap K_{\alpha_n} = \varnothing$$

Theorem 1.7. *Suppose X is a locally compact Hausdorff space X, $K \subset U$, U is open, and K is compact. Then there is an open set V with compact closure such that*

$$K \subset V \subset \overline{V} \subset U$$

Proof: Since every point of K has a neighborhood with compact closure, and since K is covered by the union of finitely many of these neighborhoods, K lies in an open set G with compact closure. If $U = X$, take $V = G$.

Otherwise, theorem 2.1.4 shows that to each $p \in U^c \subset K^c$ there corresponds an open set W_p such that $K \subset W_p$ and $p \notin \overline{W}_p$ (). Hence $\{U^c \cap \overline{G} \cap \overline{W}_p\}_{p \in U^c}$ is a collection of compact sets with empty intersection. By Theorem 2.6 there are points $p_1, \ldots, p_n \in C$ such that

$$U^c \cap \overline{G} \cap \overline{W}_{p_1} \cap \ldots \cap \overline{W}_{p_n} = \varnothing$$

then

$$\overline{G} \cap \overline{W}_{p_1} \cap \ldots \cap \overline{W}_{p_n} \subset U$$

The set $V = G \cap W_{p_1} \cap \ldots \cap W_{p_n}$

§2 Convex Function and Inequalities

Definition 2.1. *A real function φ defined on a segment (a, b), where $-\infty \leq a < b \leq \infty$, is called convex if the inequality*

$$\varphi((1-\lambda)x + \lambda y) \leq (1-\lambda)\varphi(x) + \lambda\varphi(y)$$

holds whenever $a < x < b, a < y < b$, and $0 \leq \lambda \leq 1$.

Also, it is equivalent to the requirement that

$$\frac{\varphi(t) - \varphi(s)}{t - s} \leq \frac{\varphi(u) - \varphi(t)}{u - t}$$

whenever $a < s < t < u < b$.

Theorem 2.2. *If φ is convex on (a, b) then φ is continuous on (a, b)*

Theorem 2.3. *Let $\{\varphi_\alpha\}$ be a collection of convex function on (a, b), then*

(1) $f(x) = \sup\limits_{\alpha} \varphi_\alpha(x)$, assume that it is finite, is convex (and lower semicontinuous).

(2) $g(x) = \overline{\lim}\psi_n(x)$ is convex (and lower semicontinuous)

Proof: (i)Suppose that $x, y \in (a, b)$ and $0 \leq \lambda \leq 1$. Since f_α is convex

$$\varphi_\alpha((1-\lambda)x + \lambda y) \leq (1-\lambda)\varphi_\alpha(x) + \lambda\varphi_\alpha(y) \leq (1-\lambda)f(x) + \lambda f(y)$$

for all α. We have

$$f((1-\lambda)x + \lambda y) \leq (1-\lambda)f(x) + \lambda f(y)$$

As for (ii)

$$\psi_n((1-\lambda)x + \lambda y) \leq (1-\lambda)\psi_n(x) + \lambda\psi_n(y)$$

and let b

Corollary 2.4.

§2.1 Inequality

Theorem 2.5 (Jensen's Inequality)**.** *Let $(\Omega, \mu, \mathcal{M})$ be a positive measure space with $\mu(\Omega) = 1$. If $f(x)$ is a real function in $L^1(\mu)$, if $a < f(x) < b$ for all $x \in \Omega$, and if φ is convex on (a, b), then*

$$\varphi\left(\int_\Omega f \,\mathrm{d}\mu\right) \le \int_\Omega \varphi \circ f \,\mathrm{d}\mu$$

Note: The cases $a = -\infty$ and $b = \infty$ are not excluded. It may happen that $\varphi \circ f$ is not in $L^1(\mu)$; in that case, the integral of $\varphi \circ f$ exists in the extended sense, and its value is $+\infty$.

Proof: Put $t = \int_\Omega f \,\mathrm{d}\mu$, then $a < t < b$. Since φ is convex on (a, b), there is a $\beta \in \mathbb{R}$ that

$$\varphi(s) \ge \varphi(t) + \beta(s - t) \quad (a < s < b) \tag{2}$$

Hence

$$\varphi(f(x)) - \varphi(t) - \beta[f(x) - t] \ge 0 \tag{3}$$

for every $x \in \Omega$. Since φ is continuous, $\varphi \circ f$ is measurable. If we integrate both sides with respect to μ, then we get the inequality.

Definition 2.6. *If p and q are positive real number such that*

$$\frac{1}{p} + \frac{1}{q} = 1$$

then we call p and q a pair of conjugate exponents. 1 and ∞ are also regarded as a pair of conjugate exponents.

Theorem 2.7 (Hölder's inequality)**.** *Let p and q be conjugate exponents, $1 < p < \infty$. Let X be a measure space, with measure μ. Let f and g be measurable functions on X, with range in $[0, \infty]$. Then*

$$\int_X fg \,\mathrm{d}\mu \le \left\{\int_X f^p \,\mathrm{d}\mu\right\}^{1/p} \left\{\int_X g^q \,\mathrm{d}\mu\right\}^{1/q} \tag{1}$$

Proof: Let A and B be the two factors on the right of (1). If $A = 0$, then $f = 0$ a.e. ; hence $fg = 0$ a.e. , so (1) holds. If $A > 0$ and $B = \infty$, (1) is again trivial. So we need consider only the case $0 < A < \infty, 0 < B < \infty$. Put

$$F = \frac{f}{A}, \quad G = \frac{g}{B} \tag{3}$$

This gives

$$\int_X F^p \,\mathrm{d}\mu = \int_X G^q \,\mathrm{d}\mu = 1 \tag{4}$$

Since $1/p + 1/q = 1$, the convexity of the exponential function implies that

$$F(x)G(x) \leq \frac{F(x)^p}{p} + \frac{G(x)^q}{q} \tag{6}$$

for every $x \in X$. Integration of (6) yields

$$\int_X FG \,\mathrm{d}\mu \leq p^{-1} + q^{-1} = 1 \tag{7}$$

by (4); inserting (3) into (7), we obtain (1).

Theorem 2.8.

$$\left\{\int_X (f+g)^p \,\mathrm{d}\mu\right\}^{1/p} \leq \left\{\int_X f^p \,\mathrm{d}\mu\right\}^{1/p} + \left\{\int_X g^p \,\mathrm{d}\mu\right\}^{1/p} \tag{2}$$

Proof: We write

$$(f+g)^p = f \cdot (f+g)^{p-1} + g \cdot (f+g)^{p-1} \tag{8}$$

Hölder's inequality gives

$$\int f \cdot (f+g)^{p-1} \leq \left\{\int f^p\right\}^{1/p} \left\{\int (f+g)^{(p-1)q}\right\}^{1/q} \tag{9}$$

and

$$\int g \cdot (f+g)^{p-1} \leq \left\{\int g^p\right\}^{1/p} \left\{\int (f+g)^{(p-1)q}\right\}^{1/q} \tag{9'}$$

Since $(p-1)q = p$, addition of (9) and (9') gives

$$\int (f+g)^p \leq \left\{\int (f+g)^p\right\}^{1/q} \left[\left\{\int f^p\right\}^{1/p} + \left\{\int g^p\right\}^{1/p}\right] \tag{10}$$

Clearly, it is enough to prove (2) in the case that the left side is greater than 0 and the right side is less than ∞. The convexity of the function t^p for $0 < t < \infty$ shows that

$$\left(\frac{f+g}{2}\right)^p \leq \frac{1}{2}\left(f^p + g^p\right)$$

Hence the left side of (2) is less than ∞, and (2) follows from (10) if we divide by the first factor on the right of (10), bearing in mind that $1 - 1/q = 1/p$. This completes the proof.

§3 Semicontinwous

Definition 3.1. *Let f be a real (or extended-real) function on a topological space. If*

$$\{x : f(x) > \alpha\}$$

is open for every real α, f is said to be **lower semicontinuous**. *If*

$$\{x : f(x) < \alpha\}$$

is open for every real α, f is said to be upper semicontinuous

Theorem 3.2. *Let X be topology space.*

(1) $\chi_V(x)$ is lower semicontinuous iff V is open.

(2) $\chi_F(x)$ is uper semicontinuous iff F is closed.

(3) The supremum of any collection of lower semicontinuous functions is lower semicontinuous.

(4) The infimum of any collection of upper semicontinuous functions is upper semicontinwous.

Theorem 3.3. *Let X be a topological space and $f : X \to \mathbb{R}$*

(1) f is lower semicontinuous if and only if given $x \in X$, for every $\{x_n\} \subseteq X \setminus \{x\}$ converging to x

$$\liminf_{n\to\infty} f(x_n) \geq f(x)$$

(2) f is upper semicontinuous if and only if given $x \in X$, for every $\{x_n\} \subseteq X \setminus \{x\}$ converging to x

$$\limsup_{n\to\infty} f(x_n) \leq f(x)$$

Theorem 3.4. *Let $\{f_n\}$ be a sequence of real functions on $\mathbb{R}$, and consider the following four statements:*

(1) If f_1 and f_2 are upper semicontinuous, then $f_1 + f_2$ is upper semicontinuous.

(2) If f_1 and f_2 are lower semicontinuous, then $f_1 + f_2$ is lower semicontinuous.

(3) If each f_n is lower semicontinuous and nonnegative, then $\sum_1^\infty f_n$ is lower semicontinuous.

Proof: Noticed that

$$\{f_1 + f_2 > \alpha\} = \bigcup_{\beta_1+\beta_2>\alpha} (\{f_1 > \beta_1\} \cap \{f_2 > \beta_2\})$$

is open if f_1 and f_2 are lower semicontinuous. Then $g_n = \sum_1^n f_k$ is lower semicontinuous and

$$f = \sum_1^\infty f_n = \sup_n g_n$$

is lower semicontinuous.

Theorem 3.5 (Extreme Value Theorem for semicontinuous function)**.** *If X is compact and $f : X \longrightarrow R$ is upper (lower) semicontinuous, then f attains its maximum (minimum) at some point of X.*

Theorem 3.6. *Suppose that X is a metric space with metric d, and let $f : X \longrightarrow [0, \infty]$ that $f(p) < \infty$ for at least one $p \in X$ (or $f : X \longrightarrow \mathbb{R}$). Then $f(x)$ is lower semicontinuous if and* *only if there is real nonnegative continuous function sequence $\{f_n\}$ that*

$$\lim_{n\to\infty} f_n(x) \to f(x) \quad \textit{for all } x \in X$$

Proof: For $n = 1, 2, 3, \ldots, x \in X$, define

$$f_n(x) = \inf\{f(p) + nd(x,p) : p \in X\}$$

and prove that

$$|f_n(x) - f_n(y)| \le nd(x,y)$$

and

$$0 \le f_1 \le f_2 \le \cdots \le f$$

(iii) $f_n(x) \to f(x)$ as $n \to \infty$, for all $x \in X$.

§4 The completion of $C_0(X)$

Definition 4.1. *A complex function f on a locally compact Hausdorff space X is said to* ***vanish at infinity*** *if to every $\varepsilon > 0$ there exists a compact set $K \subset X$ such that $|f(x)| < \varepsilon$ for all $x \in X\backslash K$. The class of all continuous f on X which vanish at infinity is called $C_0(X)$.*

It is clear that $C_c(X) \subset C_0(X)$, and that the two classes coincide if X is compact. In that case we write $C(X)$ for either of them.

Theorem 4.2. *Let X be locally compact Hausdorff space, then $C_0(X)$ is Banach space with norm $\cdot_\infty$*

Theorem 4.3. *If X is a locally compact Hausdorff space, then $C_0(X)$ is the completion of $C_c(X)$, relative to the metric defined by the supremum norm*

$$\|f\| = \sup_{x\in X} |f(x)|$$

Proof: An elementary verification shows that $C_0(X)$ satisfies the axioms of a metric space if the distance between f and g is taken to be $\|f - g\|$. We have to show that (a) $C_c(X)$ is dense in $C_0(X)$ and (b) $C_0(X)$ is a complete metric space.

Given $f \in C_0(X)$ and $\varepsilon > 0$, there is a compact set K so that $|f(x)| < \varepsilon$ outside K. Urysohn's lemma gives us a function $g \in C_c(X)$ such that $0 \le g \le 1$ and $g(x) = 1$ on K. Put $h = fg$. Then $h \in C_c(X)$ and $\|f - h\| < \varepsilon$. This proves (a).

To prove (b), let $\{f_n\}$ be a Cauchy sequence in $C_0(X)$, i.e, assume that $\{f_n\}$ converges uniformly. Then its pointwise limit function f is continuous. Given $\varepsilon > 0$, there exists an n so that $\|f_n - f\| < \varepsilon/2$ and there is a compact set K so that $|f_n(x)| < \varepsilon/2$ outside K. Hence $|f(x)| < \varepsilon$ outside K, and we have proved that f vanishes at infinity. Thus $C_0(X)$ is complete.

§5 Urysohn's Lemma (LCH's version)

Definition 5.1. *Let X be a topological space.*

(1) The notation

$$K \prec f$$

will mean that K is a compact subset of X, that $f \in C_c(X)$, that $0 \leq f(x) \leq 1$ for all $x \in X$, and that $f(x) = 1$ for all $x \in K$.

(2) The notation

$$f \prec V$$

will mean that V is open, that $f \in C_c(X), 0 \leq f \leq 1$, and that the support of f lies in V.

(3) The notation

$$K \prec f \prec V$$

will be used to indicate that the both hold.

Theorem 5.2 (Urysohn's Lemma)**.** *Suppose X is a locally compact Hausdorff space, F is closed, K is compact in X, and $K \cap V \neq \varnothing$. Then there exists an $f \in C_c(X)$ maps X into $[0,1]$, such that*

$$f(K) = 1, \quad f(F) = 0$$

Corollary 5.3. *Suppose X is a locally compact Hausdorff space, V is open, K is compact in X, and $K \subset V$. Then there exists an $f \in C_c(X)$, such that*

$$K \prec f \prec V$$

Theorem 5.4. *Suppose $V_1, \ldots, V_n$ are open subsets of a locally compact Hausdorff space X, K is compact, and*

$$K \subset V_1 \cup \cdots \cup V_n$$

Then there exist functions $h_i \prec V_i \quad (i = 1, \ldots, n)$ such that

$$h_1(x) + \cdots + h_n(x) = 1 \quad (x \in K)$$

*The collection $\{h_1, \ldots, h_n\}$ is called a **partition of unity on** K, subordinate to the cover $\{V_1, \ldots, V_n\}$.*

Proof: Each $x \in K$ has a neighborhood W_x with compact closure $\overline{W}_x \subset V_i$ for some i_x. There are points $x_1, \ldots, x_m$ such that $W_{x_1} \cup \cdots \cup W_{x_m} \supset K$. If $1 \leq i \leq n$, let H_1 be the union of those

$\overline{W}_x$ which lie in V_i. By Urysohn's lemma, there are functions g_i such that $H_i \prec g_i \prec V_i$. Define

$$\begin{aligned} h_1 &= g_1 \\ h_2 &= (1-g_1)\,g_2 \\ &\cdots\cdots\cdots \\ h_n &= (1-g_1)\,(1-g_2)\cdots(1-g_{n-1})\,g_n. \end{aligned}$$

Then $h_i \prec V_i$. It is easily verified, by induction, that

$$h_1 + h_2 + \cdots + h_n = 1 - (1-g_1)\,(1-g_2)\cdots(1-g_n)\,.$$

Since $K \subset H_1 \cup \cdots \cup H_n$, at least one $g_(x) = 1$ at each point $x \in K$; hence (3) shows that (1) holds.

Throughout this chapter, X will denote an LCH space and μ a positive Borel measure on X.

§6 Basic Definition

Definition 6.1. *Let μ be a Borel measure on locally compact Hausdorff space $(X, \tau, \mathcal{B})$ and E a Borel set of X.*

1. *The measure μ is called **outter regular** on E if*

$$\mu(E) = \inf\{\mu(V) : E \subset V,\ V \text{ open}\}$$

2. *is called **inner regular** on E if*

$$\mu(E) = \sup\{\mu(K) : K \subset E,\ K \text{ compact}\}$$

3. *If μ is outer and inner regular on all Borel sets, μ is called **regular**.*

Definition 6.2. *A **Radon measure** μ on X is a Borel measure that*

1. *is finite on all compact sets,*
2. *is outer regular on all Borel sets,*
3. *is inner regular on all open sets.*

Remark. *It follows the definition that μ is inner regular on σ-finite sets.*

Corollary 6.3. *Suppose that μ is a σ-finite Radon measure on X and E is a Borel set in X.*

1. *For every $\varepsilon > 0$ there exist an open U and a closed F with $F \subset E \subset U$ and $\mu(U \backslash F) < \varepsilon$.*
2. *There exist an F_σ set A and a G_δ set B such that $A \subset E \subset B$ and $\mu(B \backslash A) = 0$.*

Corollary 6.4. *σ-compact $\Rightarrow$ σ-finite $\Rightarrow$ regular*

§7 Riesz Representation Theorem

Theorem 7.1 (Riesz Representation Theorem)**.** *Let (X, τ) be a locally compact Hausdorff space, and let Λ be a positive linear functional on $C_c(X)$. Then there is a unique Radon measure μ on X such that $\Lambda(f) = \int_X f \, \mathrm{d}\mu$ for all $f \in C_c(X)$.*

Proof. Step 1. Construction of μ and $\mathcal{M}$ For every open set V in X, define

$$\mu(V) = \sup \{\Lambda f : f \prec V\} \tag{1}$$

If $V_1 \subset V_2$, it is clear that (1) implies $\mu(V_1) \leq \mu(V_2)$. Hence

$$\mu(E) = \inf \{\mu(V) : E \subset V, V \text{ open }\} \tag{2}$$

if E is an open set, and it is consistent with (1) to define $\mu(E)$ by (2), for every $E \subset X$.

Let $\mathcal{M}_F$ be the class of all $E \subset X$ which satisfy two conditions: $\mu(E) < \infty$, and

$$\mu(E) = \sup \{\mu(K) : K \subset E, K \text{ compact }\} \tag{3}$$

Finally, let $\mathcal{M}$ be the class of all $E \subset X$ such that $E \cap K \in \mathcal{M}_F$ for every compact K.

Step 2. Subadditivity. If $E_1, E_2, E_3, \ldots$ are arbitrary subsets of X, then

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mu(E_j) \tag{4}$$

We first show that

$$\mu(V_1 \cup V_2) \leq \mu(V_1) + \mu(V_2)$$

if V_1 and V_2 are open. For any $g \prec V_1 \cup V_2$, there are functions h_1 and h_2 such that $h_i \prec V_i$ and $h_1(x) + h_2(x) = 1$ for all x in the support of g. Hence $h_i g \prec V_i$, $g = h_1 g + h_2 g$, and so

$$\Lambda g = \Lambda(h_1 g) + \Lambda(h_2 g) \leq \mu(V_1) + \mu(V_2)$$

It follows that $\mu(V_1 \cup V_2) \leq \mu(V_1) + \mu(V_2)$.

If $\mu(E_i) = \infty$ for some i, then (4) is trivially true. Suppose therefore that $\mu(E_i) < \infty$ for every i. Choose $\varepsilon > 0$. By (2) there are open sets $V_i \supset E_i$ such that

$$\mu(V_i) < \mu(E_i) + 2^{-i}\varepsilon \quad (i = 1, 2, 3, \ldots)$$

Put $V = \bigcup_i^{\infty} V_i$, and choose $f \prec V$. Since f has compact support, we see that $f \prec V_1 \cup \cdots \cup V_n$ for some n. we therefore obtain

$$\Lambda f \leq \mu(V_1 \cup \cdots \cup V_n) \leq \mu(V_1) + \cdots + \mu(V_n) \leq \sum_{i=1}^{\infty} \mu(E_i) + \varepsilon$$

Since this holds for every $f \prec V$, and since $\bigcup E_i \subset V$, it follows that

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \mu(V) \leq \sum_{i=1}^{\infty} \mu\left(E_i\right)+\varepsilon,$$

which proves (4), since ε was arbitrary.

Step 3. If K is compact, then $K \in \mathcal{M}_F$ and

$$\mu(K)=\inf \{\Lambda f: K \prec f\} \tag{5}$$

If $K \prec f$ and $0<\alpha<1$, let $V_\alpha=\{x: f(x)>\alpha\}$. Then $K \subset V_a$, and $\alpha g \leq f$ whenever $g \prec V_a$. Hence

$$\mu(K) \leq \mu\left(V_a\right)=\sup \left\{\Lambda g: g \prec V_a\right\} \leq \alpha^{-1} \Lambda f$$

Let $\alpha \rightarrow 1^{-}$, to conclude that

$$\mu(K) \leq \Lambda f$$

Thus $\mu(K)<\infty$. Since K evidently satisfies (3), $K \in \mathcal{M}_F$.

If $\varepsilon>0$, there exists $V \supset K$ with $\mu(V)<\mu(K)+\varepsilon$. By Urysohn's lemma, $K \prec f \prec V$ for some f. Thus

$$\Lambda f \leq \mu(V)<\mu(K)+\varepsilon$$

which gives (5).

Step 4. Every open set satisfies (3) is inner regular. Hence $\mathcal{M}_F$ contains every open set V with $\mu(V)<\infty$.

Let α be a real number such that $\alpha<\mu(V)$. There exists an $f \prec V$ with $\alpha<\Lambda f$. If W is any open set which contains the support K of f, then $f \prec W$, hence $\Lambda f \leq \mu(W)$. Thus

$$\Lambda f \leq \inf \{\mu(W): K \subset W, W \text { is open}\}=\mu(K)$$

This exhibits a compact $K \subset V$ with $\alpha<\mu(K)$, so that (3) holds for V.

Step 5. Suppose $E=\bigcup_{i=1}^{\infty} E_i$, where $E_1, E_2, E_3, \ldots$ are pairwise disjoint members of $\mathcal{M}_F$. Then

$$\mu(E)=\sum_{i=1}^{\infty} \mu\left(E_i\right)$$

If, in addition, $\mu(E)<\infty$, then also $E \in \mathcal{M}_F$.

We first show that

$$\mu\left(K_1 \cup K_2\right)=\mu\left(K_1\right)+\mu\left(K_2\right)$$

if K_1 and K_2 are disjoint compact sets. Choose $\varepsilon>0$. By Urysohn's lemma, there exists $f \in C_c(X)$ such that $f(x)=1$ on K_1, $f(x)=0$ on K_2, and $0 \leq f \leq 1$. By 3. there exists g such that

$$K_1 \cup K_2<g \text { and } \Lambda g<\mu\left(K_1 \cup K_2\right)+\varepsilon$$

Note that $K_1 \prec fg$ and $K_2 \prec (1-f)g$. Since Λ is linear, it follows from that

$$\mu\left(K_1\right)+\mu\left(K_2\right) \leq \Lambda(fg)+\Lambda(g-fg)=\Lambda g<\mu\left(K_1 \cup K_2\right)+\varepsilon$$

Since ε was arbitrary, (10) follows now from Step 1.

If $\mu(E)=\infty$, (9) follows from Step I. Assume therefore that $\mu(E)<\infty$, and choose $\varepsilon>0$. Since $E_i \in \mathcal{M}_F$, there are compact sets $H_i \subset E_i$ with

$$\mu\left(H_i\right)>\mu\left(E_i\right)-2^{-i} E \quad (i=1,2,3, \ldots)$$

Putting $K_n=H_1 \cup \cdots \cup H_n$ and using induction on (10), we obtain

$$\mu(E) \geq \mu\left(K_n\right)=\sum_{i=1}^n \mu\left(H_i\right)>\sum_{i=1}^n \mu\left(E_i\right)-\varepsilon$$

Since (12) holds for every n and every $\varepsilon>0$, the left side of (9) is not smaller than the right side, and so (9) follows from Step I. But if $\mu(E)<\infty$ and $\varepsilon>0$, (9) shows that

$$\mu(E) \leq \sum_{i=1}^N \mu\left(E_i\right)+\varepsilon$$

for some N. By (12), it follows that $\mu(E) \leq \mu\left(K_N\right)+2 \varepsilon_,$ and this shows that E satisfies (3); hence $E \in \mathcal{M}_F$.

Step 6. If $E \in \mathcal{M}_{\mathrm{F}}$ and $\in>0$, there is a compact K and an open V such that $K \subset E \subset V$ and $\mu(V-K)<\varepsilon$.Our definitions show that there exist $K \subset E$ and $V \supset E$ so that

$$\mu(V)-\frac{\varepsilon}{2}<\mu(E)<\mu(K)+\frac{\varepsilon}{2}$$

Since $V-K$ is open, $V-K \in \mathcal{M}_F$, by Step III. Hence Step IV implies that

$$\mu(K)+\mu(V-K)=\mu(V)<\mu(K)+\varepsilon$$

Step 7. If $A \in \mathcal{M}_F$ and $B \in \mathcal{M}_F$, then $A-B, A \cup B$, and $A \cap B$ belong to $\mathcal{M}_F$.

If $\varepsilon>0$, Step 6 shows that there are sets K_i and V_i such that $K_1 \subset A \subset V_1, K_2 \subset B \subset V_2$, and $\mu\left(V_i-K_i\right)<\varepsilon$, for $i=1,2$. Since

$$A-B \subset V_1-K_2 \subset\left(V_1-K_1\right) \cup\left(K_1-V_2\right) \cup\left(V_2-K_2\right),$$

Step I shows that

$$\mu(A-B) \leq \varepsilon+\mu\left(K_1-V_2\right)+\varepsilon .$$

Since K_1-V_2 is a compact subset of $A-B$, (14) shows that $A-B$ satisfies (3), so that $A-B \in \mathbb{M}_F$. Since $A \cup B=(A-B) \cup B$, an application of Step IV shows that $A \cup B \in \mathcal{M}_f$. Since

$A \cap B = A - (A - B)$, we also have $A \cap B \in \mathcal{M}_F$. $\square$

Theorem 7.2. *Let X be a locally compact Hausdorff space in which every open set is σ-compact (implies X is σ-compact). Let λ be any positive Borel measure on X such that $\lambda(K) < \infty$ for every compact set K. Then λ is regular.*

Proof: Put $\Lambda f = \int_X f \,\mathrm{d}\lambda$, for $f \in C_c(X)$. Since $\lambda(K) < \infty$ for every compact K, Λ is a positive linear functional on $C_c(X)$, and there is a regular measure μ, satisfying the conclusions of previous Theorem , such that

$$\int_X f \,\mathrm{d}\lambda = \int_X f \,\mathrm{d}\mu \quad f \in C_c(X).$$

We will show that $\lambda = \mu$. Let V be open in X. Then $V = \bigcup K_i$, where K_i is compact, $i = 1, 2, 3, \ldots$. By Urysohn's lemma we can choose f_i so that $K_i \prec f_i \prec V$. Let $g_n = \max(f_1, \ldots, f_n)$. Then $g_n \in C_c(X)$ and $g_n(x)$ increases to $\chi_V(x)$ at every point $x \in X$. Hence (1) and the monotone convergence theorem imply

$$\lambda(V) = \lim_{n\to\infty} \int_X g_n \,\mathrm{d}\lambda = \lim_{n\to\infty} \int_X g_n \,\mathrm{d}\mu = \mu(V)$$

Now let E be a Borel set in X, there is a closed set F and an open set V such that $F \subset E \subset V$ and $\mu(V - F) < \varepsilon$. Hence $\mu(V) \leq \mu(F) + \varepsilon \leq \mu(E) + \varepsilon$.

Since $V - F$ is open, shows that $\lambda(V - F) < \varepsilon$, hence $\lambda(V) \leq \lambda(E) + \varepsilon$. Consequently and

$$\lambda(E) \leq \lambda(V) = \mu(V) \leq \mu(E) + \varepsilon$$
$$\mu(E) \leq \mu(V) = \lambda(V) \leq \lambda(E) + \varepsilon$$

so that $|\lambda(E) - \mu(E)| < \varepsilon$ for every $\varepsilon > 0$. Hence $\lambda(E) = \mu(E)$.

§8 Approximation

Proposition 8.1. *If μ is a Radon measure on X, $C_c(X)$ is dense in $L^p(\mu)$ for $1 \leq p < \infty$.*

Proof. Since the L^p simple functions are dense in L^p, it suffices to show that for any Borel set E with $\mu(E) < \infty$, χ_E can be approximated in the L^p norm by elements of $C_c(X)$. Given $\varepsilon > 0$, by Proposition 7.5 we can choose a compact $K \subset E$ and an open $U \supset E$ such that $\mu(U \backslash K) < \varepsilon$, and by Urysohn's lemma we can choose $f \in C_c(X)$ such that $\chi_K \leq f \leq \chi_U$. Then $\|\chi_E - f\|_p \leq \mu(U \backslash K)^{1/p} < \varepsilon^{1/p}$, so we are done. $\square$

Theorem 8.2 (Lusin's Theorem)**.** *Suppose that μ is a Radon measure on X and $f : X \to \mathbb{C}$ is a measurable function that vanishes outside a set of finite measure. Then for any $\varepsilon > 0$ there exists $\phi \in C_c(X)$ such that $\phi = f$ except on a set of measure $< \varepsilon$. If f is bounded, ϕ can be taken to satisfy $\|\phi\|_u \leq \|f\|_u$.*

Proof. Let $E = \{x : f(x) \neq 0\}$ of finite measure, and suppose to begin with that f is bounded. Then $f \in L^1(\mu)$, so by 8.1 there is a sequence $\{g_n\}$ in $C_c(X)$ that converges to f in L^1, and hence by Corollary 2.32 a subsequence (still denoted by $\{g_n\}$) that converges to f a.e. By Egoroff's theorem there is a set $A \subset E$ such that $\mu(E \backslash A) < \varepsilon/3$ and $g_n \to f$ uniformly on A, and there exist a compact $K \subset A$ and an open $U \supset E$ such that $\mu(A \backslash K) < \varepsilon/3$ and $\mu(U \backslash E) < \varepsilon/3$. Since $g_n \to f$ uniformly on B, $f|_B$ is continuous, so by 5.2 there exists $h \in C_c(X)$ such that $K \prec h \prec U$. But then $\{x : f(x) \neq h(x)\}$ is contained in $U \backslash B$, which has measure $< \varepsilon$.

To complete the proof for f bounded, define $\beta : \mathbb{C} \to \mathbb{C}$ by $\beta(z) = z$ if $|z| \leq \|f\|_u$ and $\beta(z) = \|f\|_u \operatorname{sgn} z$ if $|z| > \|f\|_u$, and set $\phi = \beta \circ h$. Then $\phi \in C_c(X)$ since β is continuous and $\beta(0) = 0$. Moreover, $\|\phi\|_u \leq \|f\|_u$, and $\phi = f$ on the set where $h = f$, so we are done.

If f is unbounded, let $A_n = \{x : 0 < |f(x)| \leq n\}$. Then A_n increases to E as $n \to \infty$, so $\mu(E \backslash A_n) < \varepsilon/2$ for sufficiently large n. By the preceding argument there exists $\phi \in C_c(X)$ such that $\phi = f\chi_{A_n}$ except on a set of measure $< \varepsilon/2$, and hence $\phi = f$ except on a set of measure $< \varepsilon$. □

Theorem 8.3 (Borel measurable version)**.** *Let f be a real-valued Lebesgue measurable function on $\mathbb{R}^k$. Prove that there exist Borel functions g and h such that $g(x) = h(x)$ a.e. $[m]$, and $g(x) \leq f(x) \leq h(x)$ for every $x \in \mathbb{R}^k$.*

Theorem 8.4 (The Vitali-Caratheodory Theorem)**.** *Suppose $f \in L^1(\mu)$, f is real-valued, and $\varepsilon > 0$. Then there exist functions u and v on X such that $u \leq f \leq v$, u is upper semicontinuous and bounded above, v is lower semicontinuous and bounded below, and*

$$\int_X (v - u)d\mu < \varepsilon \tag{1}$$

Proof: Assume first that $f \geq 0$ and that f is not identically 0 . Since f is the pointwise limit of an increasing sequence of simple functions s_n, f is the sum of the simple functions $t_n = s_n - s_{n-1}$ (taking $s_0 = 0$), and since t_n is a linear combination of characteristic functions, we see that there are measurable sets E_i (not necessarily disjoint) and constants $c_i > 0$ such that

$$f(x) = \sum_{i=1}^{\infty} c_i \chi_{E_i}(x) \quad (x \in X) \tag{2}$$

Since

$$\int_X f du = \sum_{i=1}^{\infty} c_i \mu(E_i) \tag{3}$$

the series in (3) converges. There are compact sets K_i and open sets V_i such that $K_i \subset E_i \subset V_i$ and

$$c_i \mu(V_i - K_i) < 2^{-i-1}\varepsilon \quad (i = 1, 2, 3, \ldots) \tag{4}$$

Put

$$v=\sum_{i=1}^{\infty} c_i \chi_{V_i} \quad u=\sum_{i=1}^{N} c_i \chi_{K_i}, \tag{5}$$

where N is chosen so that

$$\sum_{N+1}^{\infty} c_i \mu\left(E_j\right)<\frac{\varepsilon}{2}. \tag{6}$$

Then v is lower semicontinuous, u is upper semicontinuous, $u \leq f \leq 0$, and

$$\begin{aligned} v-u & =\sum_{i=1}^{N} c_i\left(\chi_{V_i}-\chi_{K_i}\right)+\sum_{N+1}^{\infty} c_i \chi_{V_i} \\ & \leq \sum_{i=1}^{\infty} c_i\left(\chi_{V_i}-\chi_{K_i}\right)+\sum_{N+1}^{\infty} c_i \chi_{E_i}\end{aligned}$$

so that (4) and (6) imply (1).

In the general case, write $f=f^{+}-f^{-}$, attach u_1 and v_1 to f^{+}, attach u_2 and v_2 to f^{-}, as above, and put $u=u_1-v_2, v=v_1-u_2$. Since $-v_2$ is upper semicontinuous and since the sum of two upper semicontinuous functions is upper semicontinuous (similarly for lower semicontinuous; we leave the proof of this as an exercise), u and v have the desired properties.

§9 Lebesgue Measure

Theorem 9.1 (Lebesgue Measure on $\mathbb{R}^n$)**.** *There exists a positive complete measure m defined on a σ-algebra $\mathcal{M}$ in $\mathbb{R}^k$, with the following properties:*

(a) $m(W)=\operatorname{vol}(W)$ for every k-cell W.

(b) $\mathcal{M}$ contains all Borel sets in $\mathbb{R}^k$; more precisely, $E \in \mathfrak{P}$ if and only if there are sets A and $B \subset R^k$ such that $A \subset E \subset B, A$ is an F_σ, B is $a G_d$, and $m(B-A)=0$. Also, m is regular.

(c) m is translation-invariant, i.e.,

$$m(E+x)=m(E)$$

for every $E \in \mathcal{M}$ and every $x \in \mathbb{R}^k$.

(e) To every linear transformation T of R^k into R^k corresponds a real number $\Delta(T)$ such that

$$m(T(E))=\Delta(T) m(E)$$

for every $E \in \mathbb{M}$. In particular, $m(T(E))=m(E)$ when T is a rotation.

The members of $\mathcal{M}$ are the Lebesgue measurable sets in $\mathbb{R}^k$; m is the Lebesgue measure on $\mathbb{R}^k$.

Proof: *If f is any complex function on $\mathbb{R}^k$, with compact support, define*

$$\Lambda_n f = 2^{-nk} \sum_{2^n x \in \mathbb{Z}} f(x) \quad (n = 1, 2, 3, \ldots) \tag{1}$$

Now suppose $f \in C_c\left(R^k\right)$, f is real, W is an open k-cell which contains the support of f, and $\varepsilon > 0$. The uniform continuity of f shows that there is an integer N and that there are functions g and h with support in W, such that (i) g and h are constant on each box belonging to Ω_N, (ii) $g \leq f \leq h$, and (iii) $h - g < \varepsilon$. If $n > N$, then

$$\Lambda_N g = \Lambda_{\mathrm{n}} g \leq \Lambda_{\mathrm{n}} f \leq \Lambda_n h = \Lambda_N h \tag{2}$$

Thus the upper and lower limits of $\{\Lambda_n f\}$ differ by at most $\operatorname{vol}(W)\varepsilon$, *and since ε was arbitrary, we have proved the existence of*

$$\Lambda f = \lim_{n \to \infty} \Lambda_n f \quad f \in C_c\left(R^k\right) \tag{3}$$

It is immediate that Λ is a positive linear functional on $C_c\left(R^k\right)$. (In fact, Λf is precisely the Riemann integral of f over R^k.)

2. We define m and $\mathcal{M}$ to be the measure and σ-algebra associated with this Λ as in Riesz Representation Theorem. Since Theorem 2.14 gives us a complete measure and since R^k is σ compact, Theorem 2.17 011implies assertion (b).

3. To prove (a), let W be the open cell , let E_r be the union of those boxes belonging to Ω_r whose closures lie in W, choose f_r so that $\overline{E}_r \prec f_r \prec W$, and put $g_r = \max\{f_1, \ldots, f_r\}$. Our construction of Λ shows that

$$\operatorname{vol}(E_r) \leq \Lambda f_r \leq \Lambda g_r \leq \operatorname{vol}(W) \tag{4}$$

As $r \to \infty, \operatorname{vol}(E_r) \to \operatorname{vol}(W)$, and

$$\Lambda g_r = \int g_r \, \mathrm{d}m \to m(W) \tag{5}$$

by the monotone convergence theorem, since $g_r(x) \to \chi_W(x)$ for all $x \in R^k$. Thus $m(W) =$ $\operatorname{vol}(W)$ *for every open cell W, and since every k-cell is the intersection of a decreasing sequence of open k-cells, we obtain (a).*

The proofs of (c), (d), and (e) will use the following observation: If λ is a positive Borel measure on R^k and $\lambda(E) = m(E)$ for all boxes E, then the same equality holds for all open sets E, by property $2.19(d)$, *and therefore for all Borel sets E, since λ and m are regular (Theorem 2.18).*

To prove (c), fix $x \in R^k$ and define $\lambda(E) = m(E + x)$. It is clear that λ is then a measure; by (a), $\lambda(E) = m(E)$ for all boxes, hence $m(E + x) = m(E)$ for all Borel sets E. The same equality holds for every $E \in \mathbb{P}$, because of (b).

Suppose next that μ satisfies the hypotheses of (d). Let Q_0 be a 1 -box, put $c = \mu(Q_0)$. Since

Q_0 is the union of 2^{nk} disjoint 2^{-n} boxes that are translates of each other, we have

$$2^{nk}\mu(Q) = \mu\left(Q_0\right) = cm\left(Q_0\right) = c \cdot 2^{nk} m(Q)$$

for every 2^{-n}-box Q. Property 2.19(d) implies now that $\mu(E) = cm(E)$ for all open sets $E \subset R^k$. This proves (d).

To prove (e), let $T : R^k \to R^k$ be linear. If the range of T is a subspace Y of lower dimension, then $m(Y) = 0$ and the desired conclusion holds with $\Delta(T) = 0$. In the other case, elementary linear algebra tells us that T is a one-to-one map of R^k onto R^k whose inverse is also linear. Thus T is a homeomorphism of R^k onto R^k, so that $T(E)$ is a Borel set for every Borel set E, and we can therefore define a positive Borel measure μ on R^k by

$$\mu(E) = m(T(E))$$

The linearity of T, combined with the translation-invariance of m, gives

$$\mu(E + x) = m(T(E + x)) = m(T(E) + Tx) = m(T(E)) = \mu(E).$$

Thus μ is translation-invariant, and the first assertion of (e) follows from (d), first for Borel sets E, then for all $E \in \mathcal{M}$ by (b).

To find $\Delta(T)$, we merely need to know $m(T(E)/m(E)$ for one set E with $0 < m(E) < \infty$. If T is a rotation, let E be the unit ball of R^k; then $T(E) = E$, and $\Delta(T) = 1$.

Theorem 9.2. *If μ is any positive translation-invariant Borel measure on R^k such that $\mu(K) < \infty$ for every compact set K, then there is a constant c such that $\mu(E) = cm(E)$ for all Borel sets $E \subset R^k$.*

Chapter IV

L^p Space

§1 Definition

Definition 1.1. *Let* $(X, \mathcal{M}, \mu)$ *be a measure space. If* $0 < p < \infty$ *and if* f *is a complex measurable function on* X*, define*

$$\|f\|_p = \left\{\int_X |f|^p \,\mathrm{d}\mu\right\}^{1/p}$$

and let $L^p(\mu)$ *consist of all* f *for which*

$$\|f\|_p < \infty$$

We call $\|f\|_p$ *the* L^p*-norm of* f*. If* μ *is Lebesgue measure on* $\mathbb{R}^k$*, we write* $L^p\left(\mathbb{R}^k\right)$ *instead of* $L^p(\mu)$*.*

Definition 1.2. *Suppose* $g : X \longrightarrow [0, \infty]$ *is measurable. Let* S *be the set of all real* α *such that*

$$\mu\left(g^{-1}((\alpha, \infty])\right) = 0$$

If $S = \varnothing$*, put* $\beta = \infty$*. If* $S \neq \varnothing$*, put* $\beta = \inf S$*. Since*

$$g^{-1}((\beta, \infty]) = \bigcup_{n=1}^{\infty} g^{-1}\left((\beta + \frac{1}{n}, \infty]\right)$$

and since the union of a countable collection of sets of measure 0 has measure 0 , we see that $\beta \in S$*. We call* β *the* ***essential supremum*** *of* g*.*

If f *is a complex measurable function on* X*, we define* $\|f\|_\infty$ *to be the essential supremum of* $|f|$*, and we let* $L^\infty(\mu)$ *consist of all* f *for which* $\|f\|_\infty < \infty$*. The members of* $L^\infty(\mu)$ *are sometimes called essentially bounded measurable functions on* X*.*

Remark. *It follows from this definition that the inequality* $|f(x)| \leq \lambda$ *holds for almost all* x *if and only if* $\lambda \geq \|f\|_\infty$*.*

Proposition 1.3. *If* f *is a measurable function on* X*, define the* ***essential range*** R_f *of* f *to be the set of all* $z \in \mathbb{C}$ *such that* $\{x : |f(x) - z| < \epsilon\}$ *has positive measure for all* $\epsilon > 0$*. Then*

1. *R_f is closed.*

2. *If $f \in L^\infty$, then R_f is compact and $\|f\|_\infty = \max\{|z| : z \in R_f\}$.*

Theorem 1.4 (Holder's Inequality). *Suppoes p and q are conjugate exponents, $1 \le p \le \infty$, and if $f \in L^p(\mu)$ and $g \in L^g(\mu)$, then $fg \in L^1(\mu)$, and*

$$\|fg\|_1 \le \|f\|_p \|g\|_q$$

and in this case equality holds iff

$$|f|^p \text{ and } |g|^q \text{ are linear dependent in } L^0.\ 1 < p < \infty$$
$$|g(x)| = \|g\|_\infty \text{ a.e. on the set } \{x : f(x) \neq 0\}.$$

Corollary 1.5 (Generalized). *For $\sum \frac{1}{p_i} = \frac{1}{p}$ with $1 \le p_i$ we have*

$$\left\|\prod f_i\right\|_p \le \prod \|f_i\|_{p_i}$$

Corollary 1.6 (Interpolation). *Suppose $\frac{\theta}{p} = \sum \frac{\theta_i}{p_i}$ with $\sum \theta_i = \theta$*

$$\|f\|_p^\theta \le \prod \|f\|_{p_i}^{\theta_i}$$

Theorem 1.7 (Minkowski's Inequality). *Suppose $1 \le p \le \infty$, and $f \in L^p(\mu), g \in L^p(\mu)$. Then $f + g \in L^p(\mu)$, and*

$$\|f+g\|_p \le \|f\|_p + \|g\|_p$$

§2 L^p Space

Theorem 2.1. *Let $(X, \mathcal{M}, \mu)$ be a measure space. Then $L^p(\mu)$ is a banach space for $1 \le p \le \infty$.*

Proof. Assume first that $1 \le p < \infty$. Let $\{f_n\}$ be a Cauchy sequence in $L^p(\mu)$. There is a subsequence $\{f_n\}, n_1 < n_2 < \cdots$, such that

$$\left\|f_{n_{i+1}} - f_{n_i}\right\|_p < 2^{-i}$$

Put

$$g_k = \sum_{i=1}^k \left|f_{n_{i-1}} - f_{n_i}\right|, \quad g = \sum_{i=1}^\infty |f_{n_i+1} - f_{m_i}|$$

the Minkowski inequality shows that $\|g_k\|_p < 1$ for $k = 1, 2, 3, \ldots$. Hence an application of Fatou's lemma to $\{g^p\}$ gives $\|g\|_p \le 1$. In particular, $g(x) < \infty$ a.e., so that the series

$$f_{n_1}(x) + \sum_{i=1}^\infty \left(f_{n_{i+1}}(x) - f_{n_i}(x)\right)$$

converges absolutely for a.e. $x \in X$. Denote the sum of (3) by $f(x)$, for those x at which (3) converges; put $f(x) = 0$ on the remaining set of measure zero. Since

$$f_{n+1} + \sum_{i=1}^{k-1} (f_{n+1} - f_{ni}) = f_{nt}$$

we see that

$$f(x) = \lim_{i\to\infty} f_m(x) \quad \text{a.e.}$$

Having found a function f which is the pointwise limit a.e. of $\{f_n\}$, we now have to prove that this f is the L^p-limit of $\{f_m\}$. Choose $\varepsilon > 0$. There exists an N such that $\|f_n - f_m\|_p < \varepsilon$ if $n > N$ and $m > N$. For every $m > N$, Fatou's lemma shows therefore that

$$\int_X |f - f_m|^p \, d\mu \leq \liminf_{i\to\infty} \int_X |f_{m_i} - f_m|^p \, d\mu \leq \varepsilon^p$$

We conclude from (6) that $f - f_m \in L^p(\mu)$, hence that $f \in L(\mu)$ [since $f = (f - f_m) + f_m$, and finally that $\|f - f_m\|_p \to 0$ as $m \to \infty$. This completes the proof for the case $1 \leq p < \infty$.

In $L^\infty(\mu)$ the proof is much easier. Suppose $\{f_n\}$ is a Cauchy sequence in $L^\infty(\mu)$, let A_k and $B_{m,n}$ be the sets where $|f_k(x)| > \|f_k\|_\infty$ and where $|f_m(x) - f_m(x)| > \|f_n - f_m\|_\infty$, and let E be the union of these sets, for $k, m, n = 1, 2, 3, \ldots$ Then $\mu(E) = 0$, and on the complement of E the sequence $\{f_n\}$ converges uniformly to a bounded function f. Define $f(x) = 0$ for $x \in E$. Then $f \in L^\infty(\mu)$, and $\|f_n - f\|_\infty \to 0$ as $n \to \infty$. $\square$

Proposition 2.2. *If* $1 \leq p < q < r \leq \infty$,

1. $L^p \cap L^r$ *is a Banach space with norm* $\|f\| = \|f\|_p + \|f\|_r$
2. *The inclusion map* $L^p \cap L^r \to L^q$ *is continuous.*

Remark. *It follows* $\|f\|_q \leq \|f\|_p^{\theta_1} \|f\|_r^{\theta_2} \leq \theta_1 \|f\|_p + \theta_2 \|f\|_r \leq \|f\|$ *that* $\|\cdot\|$ *and* $\|\cdot\|_q$ *are equivalent in* $L^p \cap L^r$.

Proposition 2.3. *If* $1 \leq p < q < r \leq \infty$,

1. $L^p + L^r$ *is a Banach space with norm* $\|f\| = \inf\left\{\|g\|_p + \|h\|_r : f = g + h\right\}$
2. *The inclusion map* $L^q \to L^p + L^r$ *is continuous.*

Proof. First, given $f \in L^p$, we have

$$f = \chi_E f + \chi_{E^c} f$$

where $E = \{x : |f(x)| > 1\}$ of finite measure. Thus $g = \chi_E f \in L^p$, $h = \chi_{E^c} f \in L^r$ and $L^q \subset L^p + L^r$. Furthermore,

$$\|f\| \leq \|\chi_E f\|_p + \|\chi_{E^c} f\|_r$$

$\square$

§3 Approximation in L^p

Theorem 3.1 (Simple function)**.** *For $1 \le p < \infty$, the set $S = \left\{f = \sum_1^n a_j \chi_{E_j} : \mu(E_j) < \infty\right\}$ is dense in $L^p(\mu)$.*

Theorem 3.2 ($C_c(X)$ on local compact Hausdorff space X)**.** *Let X be a LCHS and μ be a measure on a σ-algebra $\mathcal{M}$ on X with the properties stated in Riesz Representation Theorem. For $1 \le p < \infty$, $C_c(X)$ is dense in $L^p(\mu)$.*

Proof: Suppose $f \in L^p(\mu)$. Let s be a complex, measurable, simple functions with

$$\|s - f\|_p < \frac{\varepsilon}{2}$$

Also, there exists a $g \in C_c(X)$ such that $g(x) = s(x)$ except on a set of measure $< \varepsilon$, and $|g| \le \|s\|_\infty$ (Lusin's theorem). Hence

$$\|g - s\|_p \le 2\varepsilon^{1/p}\|s\|_\infty.$$

This completes the proof.

Thus $L^p(\mu)$ is the completion of the metric space which is obtained by endowing $C_c(\mu)$ with the L^p-metric.

Theorem 3.3 (Step function on $\mathbb{R}$)**.** *A step function is, by definition, a finite linear combination of characteristic functions of bounded intervals in $\mathbb{R}$.*

§4 Distribution Functions and Weak L^p

Definition 4.1. *If f is a measurable function on $(X, \mathcal{M}, \mu)$, we define its* ***distribution function*** *$\lambda_f : (0,\infty) \to [0,\infty]$ by*

$$\lambda_f(\alpha) = \mu(\{x : |f(x)| > \alpha\}).$$

Proposition 4.2. .

1. *λ_f is decreasing and right continuous.*
2. *If $|f| \le |g|$, then $\lambda_f \le \lambda_g$.*
3. *If $|f_n|$ increases to $|f|$, then λ_{f_n} increases to λ_f.*
4. *If $f = g + h$, then $\lambda_f(\alpha) \le \lambda_g\left(\frac{1}{2}\alpha\right) + \lambda_h\left(\frac{1}{2}\alpha\right)$.*

Theorem 4.3. *If $\lambda_f(\alpha) < \infty$ for all $\alpha > 0$ and ϕ is a nonnegative Borel measurable function on $(0,\infty)$, then*

$$\int_X \phi \circ |f| d\mu = -\int_0^\infty \phi(\alpha) d\lambda_f(\alpha)$$

Definition 4.4. *Let $(X, \mathcal{M}, \mu)$ be measure space. If f is a measurable function on X and $0 < p < \infty$, we define*

$$[f]_p = \left(\sup_{\alpha>0} \alpha^p \lambda_f(\alpha)\right)^{1/p}$$

and we define weak L^p to be the set of all f such that $[f]_p < \infty$□ $[\cdot]_p$ is a quasinorm ()

However, weak L^p is a topological vector space; see Exercise 35.

The relationship between L^p and weak L^p is as follows. On the one hand,

$$L^p \subset \text{ weak } L^p, \quad \text{and} \quad [f]_p \leq \|f\|_p$$

(This is just a restatement of Chebyshev's inequality.) On the other hand, if we replace $\lambda_f(\alpha)$ by $([f]_p/\alpha)^p$ in the integral $p\int_0^\infty \alpha^{p-1}\lambda_f(\alpha)d\alpha$, which equals $\|f\|_p^p$, we obtain a constant times $\int_0^\infty \alpha^{-1}d\alpha$, which is divergent at both 0 and ∞ - but

§5 Mapping from L^p to L^1

Lemma 5.1. *For $x \geq 0, y \geq 0$, that*

$$|x^p - y^p| \leq \begin{cases} |x-y|^p & \text{if } 0 < p < 1 \\ p|x-y|\left(x^{p-1} + y^{p-1}\right) & \text{if } 1 \leq p < \infty \end{cases}$$

Note that (a) and (b) establish the continuity of the mapping $f \to |f|^\rho$ that carries $E(\mu)$ into $L^1(\mu)$.

Theorem 5.2. *Suppose μ is a positive measure, $f, g \in L^p(\mu)$. Let the natural map*

$$f \to |f|^p$$

carries $L^p(\mu)$ into $L^1(\mu)$

(1) Lipschitz continuous. If $0 < p < 1$, prove that

$$\int ||f|^p - |g|^p|\, d\mu \leq \int |f-g|^p d\mu$$

that $\Delta(f,g) = \int |f-g|^p d\mu$ defines a metric on $L^p(\mu)$, and that the resulting metric space is complete.

(2) Local lipschitz continuous. If $1 \leq p < \infty$ and $\|f\|_p \leq R$, $\|g\|_p \leq R$, prove that

$$\int ||f|^p - |g|^p|\, d\mu \leq 2pR^{p-1}\|f-g\|_p$$

§6

Theorem 6.1 (Young's). *Let $p, q \in [1, \infty]$ and $f \in L^p$ and $g \in L^q$. Then $f * g \in L^r$ and*

$$\|f * g\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}$$

where

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$$

Proof: Show that the desired inequality is equivalent to

$$\left| \int f(y) g(x-y) h(x) \, \mathrm{d}x \, \mathrm{d}y \right| \leq \|f\|_{L^p} \|g\|_{L^q} \|h\|_{L^s}$$

for all $h \in L^s$ with $\frac{1}{r} + \frac{1}{s} = 1$ (prove that $f^* g \in (L^r)^*$)

$$\begin{aligned}
&\left| \int f(y) g(x-y) h(x) \, \mathrm{d}x \, \mathrm{d}y \right| \\
&\leq \int |f(y) g(x-y) h(x)| \, \mathrm{d}x \, \mathrm{d}y \\
&\leq \left\| \left(|g(x-y)|^q |h(x)|^s\right)^{\frac{p-1}{p}} \right\|_{\frac{p}{p-1}(\mathbb{R}^2)} \left\| \left(|f(y)|^p |h(x)|^s\right)^{\frac{q-1}{q}} \right\|_{L^{\frac{q}{q-1}}(\mathbb{R}^2)} \\
&\quad \cdot \left\| \left(|f(y)|^p |g(x-y)|^q\right)^{\frac{s-1}{s}} \right\|_{L^{\frac{s}{s-1}}(\mathbb{R}^2)}^{\frac{p-1}{p}} \left(\int_{(x,y)\in\mathbb{R}^2} |f(y)|^p |h(x)|^s dxdy \right)^{\frac{q-1}{q}} \\
&= \left(\int_{(x,y)\in\mathbb{R}^2} |g(x-y)|^q |h(x)|^s dxdy \right)^{\frac{s-1}{s}} \\
&\quad \cdot \left(\int_{(x,y)\in\mathbb{R}^2} |f(y)|^p |g(x-y)|^q dxdy \right)^{\frac{q-1}{q} + \frac{s-1}{s}} \left(\int_{y\in\mathbb{R}} |g(y)|^q dy \right)^{\frac{p-1}{p} + \frac{s-1}{s}} \left(\int_{x\in\mathbb{R}} |h(x)|^s dy \right)^{\frac{p-1}{p} + \frac{q-1}{q}} \\
&= \left(\int_{y\in\mathbb{R}} |f(y)|^p dy \right)
\end{aligned}$$

(from Fubini's theorem of integrating with respect to x or y first)

$$= \left(\int_{y\in\mathbb{R}} |f(y)|^p dy \right)^{\frac{1}{p}} \left(\int_{y\in\mathbb{R}} |g(y)|^q dy \right)^{\frac{1}{q}} \left(\int_{x\in\mathbb{R}} |h(x)|^s dy \right)^{\frac{1}{s}}$$

which is the inequality we would like to prove, where the last step comes from

$$\frac{q-1}{q} + \frac{s-1}{s} = \frac{1}{p}, \quad \frac{s-1}{s} + \frac{p-1}{p} = \frac{1}{q}, \quad \textit{and} \quad \frac{p-1}{p} + \frac{q-1}{q} = \frac{1}{s}.$$

Another proof: Indeed, $\frac{1}{r} + \frac{1}{q'} = \frac{1}{p}$, $\frac{1}{r} + \frac{1}{p'} = \frac{1}{q}$ *where* $\frac{1}{p} + \frac{1}{p'} = 1$ *and* $\frac{1}{q} + \frac{1}{q'} = 1$

$$\begin{aligned}
|f * g(x)| &\leq \int_{\mathbb{R}^n} |f(y)g(x-y)| \ \mathrm{d}y \\
&= \int_{\mathbb{R}^n} |f(y)^p|^{\frac{1}{q'}} \ |f(y)^p g\,(y-x)^q|^{\frac{1}{r}} \ |g\,(y-x)^q|^{\frac{1}{p'}} \ \mathrm{d}y \\
&\leq \left\| f^{\frac{p}{q'}} \right\|_{L^{q'}} \left(\int_{\mathbb{R}^n} f(y)^p g(x-y)^q \, \mathrm{d}y \right)^{\frac{1}{r}} \left\| g^{\frac{q}{p'}} \right\|_{L^{p'}}
\end{aligned}$$

Thus

$$\begin{aligned}
\|f * g\|_{L^r} &\leq \|f\|_{L^p}^{\frac{p}{q'}} \|g\|_{L^q}^{\frac{q}{p'}} \left(\int \int_{\mathbb{R}^n} f(y)^p g(x-y)^q \, \mathrm{d}y \, \mathrm{d}x \right)^{\frac{1}{r}} \\
&= \|f\|_{L^p} \|g\|_{L^q}
\end{aligned}$$

Chapter V

Signed Measures

§1

Definition 1.1. *Let $(X, \mathcal{M})$ be a measurable space. A* ***signed measure*** *ν on $(X, \mathcal{M})$ is a function such that*

(i) $\nu : \mathcal{M} \to [-\infty, \infty]$ *and ν assumes at most one of the values $\pm\infty$;*

(ii) $\nu(\varnothing) = 0$*;*

(iii) if $\{E_j\}$ is a sequence of disjoint sets in $\mathcal{M}$, then $\nu\left(\bigcup_1^\infty E_j\right) = \sum_1^\infty \nu\left(E_j\right)$, where the latter sum converges absolutely if $\nu\left(\bigcup_1^\infty E_j\right)$ is finite.

Thus every measure is a signed measure; for emphasis we shall sometimes refer to measures as ***positive measures****.*

Proposition 1.2. *Let $(X, \mathcal{M})$ be a measurable space.*

1. *First, if μ_1, μ_2 are measures on $\mathcal{M}$ and at least one of them is finite, then $\nu = \mu_1 - \mu_2$ is a signed measure.*

2. *Second, if μ is a measure on $\mathcal{M}$ and $f : X \to [-\infty, \infty]$ is a measurable function such that at least one of $\int f^+ d\mu$ and $\int f^- d\mu$ is finite (in which case we shall call f an extended μ-integrable function), then the set function ν defined by $\nu(E) = \int_E f d\mu$ is a signed measure.*

In fact, we shall see shortly that these are really the only examples: Every signed measure can be represented in either of these two forms.

Theorem 1.3 (The Hahn Decomposition Theorem)**.** *Let ν be a signed measure on $(X, \mathcal{M})$, there exist a positive set P and a negative set N for ν such that $P \cup N = X$ and $P \cap N = \varnothing$. If P', N' is another such pair, then $P \Delta P'$ $(= N \Delta N')$ is null for ν.*

Definition 1.4 (The Jordan Decomposition Theorem)**.** *Let ν be a signed measure on $(X, \mathcal{M})$, there exist unique positive measures $\nu^+$and ν^-such that $\nu = \nu^+ - \nu^-$and $\nu^+ \perp \nu^-$.*

Proof. *Let* $X = P \cup N$ *be a Hahn decomposition for* ν, *and define* $\nu^+(E) = \nu(E \cap P)$ *and* $\nu^-(E) = -\nu(E \cap N)$. *Then clearly* $\nu = \nu^+ - \nu^-$ *and* $\nu^+ \perp \nu^-$. *If also* $\nu = \mu^+ - \mu^-$ *and* $\mu^+ \perp \mu^-$, *let* $E, F \in \mathcal{M}$ *be such that* $E \cap F = \varnothing, E \cup F = X$, *and* $\mu^+(F) = \mu^-(E) = 0$. *Then* $X = E \cup F$ *is another Hahn decomposition for* ν, *so* $P \Delta E$ *is* ν*-null. Therefore, for any* $A \in \mathcal{M}, \mu^+(A) = \mu^+(A \cap E) = \nu(A \cap E) = \nu(A \cap P) = \nu^+(A)$, *and likewise* $\nu^- = \mu^-$.

The measures ν^+ *and* ν^- *are called the* ***positive and negative variations*** *of* ν, *and* $\nu = \nu^+ - \nu^-$ *is called the* ***Jordan decomposition*** *of* ν. *Furthermore, we define the* ***total variation*** *of* ν *to be the measure* $|\nu|$ *defined by*

$$|\nu| = \nu^+ + \nu^-.$$

Proposition 1.5. *Let* ν *be a signed measure on* $(X, \mathcal{M})$

It is easily verified that $E \in \mathcal{M}$ *is* ν*-null iff* $|\nu|(E) = 0$, *and* $\nu \perp \mu$ *iff* $|\nu| \perp \mu$ *iff* $\nu^+ \perp \mu$ *and* $\nu^- \perp \mu$ *(Exercise 2.)*

We observe that if ν *omits the value* ∞ *then* $\nu^+(X) = \nu(P) < \infty$, *so that* ν^+ *is a finite measure and* ν *is bounded above by* $\nu^+(X)$; *similarly if* ν *omits the value* $-\infty$. *In particular, if the range of* ν *is contained in* $\mathbb{R}$, *then* ν *is bounded.*

Integration with respect to a signed measure ν *is defined in the obvious way: We set*

$$L^1(\nu) = L^1\left(\nu^+\right) \cap L^1\left(\nu^-\right)$$
$$\int f d\nu = \int f d\nu^+ - \int f d\nu^- \quad \left(f \in L^1(\nu)\right)$$

One more piece of terminology: a signed measure ν *is called finite (resp.* σ*-finite) if* $|\nu|$ *is finite (resp.* σ*-finite).*

Chapter VI

Complex Measure

§1 Total Variation

Definition 1.1. *Let $\mathcal{M}$ a σ-algebra in a set X. Call a countable collection $\{E_i\}$ of members of $\mathcal{M}$ a partition of E if $E_i \cap E_j = \varnothing$ whenever $i \neq j$, and if $E = \cup E_i$.*

A complex measure μ on $\mathcal{M}$ is then a complex function on $\mathcal{M}$ such that

$$\mu(E) = \sum_{i=1}^{\infty} \mu\left(E_i\right) \quad (E \in \mathcal{M})$$

for every partition $\{E_i\}$ of E.

Remark *Since the union of the sets E_i is not changed if the subscripts are permuted, every rearrangement of the series must also converge. Hence the series actually converges absolutely.*

Definition 1.2. *The function $|\mu|$ on $\mathcal{M}$ called the total variation of complex measure μ, defined by*

$$|\mu|(E) = \sup \sum_{i=1}^{\infty} \left|\mu\left(E_i\right)\right| \quad (E \in \mathcal{M})$$

is a positive measure on $\mathcal{M}$.

Proof: Let $\{E_i\}$ be a partition of $E \in \mathcal{M}$. Let any $\varepsilon > 0$, then each E_i has a partition $\{A_{ij}\}_{j=1}^{\infty}$ such that

$$|\mu|(E_i) - \frac{\varepsilon}{2^i} < \sum_j \left|\mu\left(A_{ij}\right)\right|$$

for $i = 1, 2, 3, \ldots$. Since $\{A_{ij}\}$ $(i, j = 1, 2, 3, \ldots)$ is a partition of E, it follows that

$$|\mu|(E) \geq \sum_{i,j} \left|\mu\left(A_{ij}\right)\right| \geq \sum_i |\mu|(E_i) - \varepsilon$$

We see that

$$\sum_i |\mu|\left(E_i\right) \leq |\mu|(E)$$

To prove the opposite inequality, let $\{A_j\}$ be any partition of E. Then for any fixed j, $\{A_j \cap E_i\}$ is a partition of A_j, and for any fixed i, $\{A_j \cap E_j\}$ is a partition of E_i. Hence

$$\begin{aligned}
\sum_j |\mu(A_j)| &= \sum_j \left| \sum_i \mu(A_j \cap E_i) \right| \\
&\leq \sum_j \sum_i |\mu(A_j \cap E_j)| \\
&= \sum_i \sum_j |\mu(A_j \cap E_i)| \leq \sum_i |\mu|(E_i)
\end{aligned}$$

holds for every partition $\{A_j\}$ of E, we have

$$|\mu|(E) \leq \sum_i |\mu|(E_i). \tag{5}$$

Lemma 1.3. *If $z_1, \ldots, z_N$ are complex numbers then there is a subset S of $\{1, \ldots, N\}$ for which*

$$\left| \sum_{k \in S} z_k \right| \geq \frac{1}{\pi} \sum_{k=1}^{N} |z_k|$$

Proof: Write $z_k = |z_k| e^{i\alpha_k}$. For $-\pi \leq \theta \leq \pi$, let $S(\theta)$ be the set of all k for which $\cos(\alpha_k - \theta) > 0$. *Then*

$$\left| \sum_{S(\theta)} z_k \right| = \left| \sum_{S(\theta)} e^{-i\theta} z_k \right| \geq \operatorname{Re} \sum_{S(\theta)} e^{-i\theta} z_k = \sum_{k=1}^{N} |z_k| \cos^+(\alpha_k - \theta).$$

Choose θ_0 so as to maximize the last sum, and put $S = S(\theta_0)$. This maximum is at least as large as the average of the sum over $[-\pi, \pi]$, and this average is $\pi^{-1} \sum |z_k|$, because

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \cos^+(\alpha - \theta) d\theta = \frac{1}{\pi}$$

for every α.

Theorem 1.4. *If μ is a complex measure on X, then*

$$|\mu|(X) < \infty$$

Definition 1.5. *Let us now consider a real measure (signed measures) μ on a σ-algebra $\mathcal{M}$. Define $|\mu|$ as before, and define*

$$\mu^+ = \frac{1}{2}(|\mu| + \mu), \quad \mu^- = \frac{1}{2}(|\mu| - \mu)$$

Then both μ^+ and μ^- are positive measures on $\mathcal{M}$, and they are bounded, by Theorem 6.4. Also,

$$\mu = \mu^+ - \mu^-, \quad |\mu| = \mu^+ + \mu^-$$

The measures $\mu^+$and $\mu^-$are called the positive and ***negative variations*** *of μ, respectively. This representation of μ as the difference of the positive measures μ^+ and $\mu^-$is known as the Jordan decomposition of μ.*

§1.1 Absolutely Continuous

Definition 1.6. *Let μ be a positive measure on a σ-algebra $\mathcal{M}$, and let λ be an arbitrary measure on $\mathcal{M}$; λ may be positive or complex.*

(1) We say that λ ***is absolutely continuous with respect to*** *μ, and write*

$$\lambda \ll \mu$$

if $\lambda(E) = 0$ for every $E \in \mathcal{M}$ for which $\mu(E) = 0$.

(2) If there is a set $A \in \mathcal{M}$ such that $\lambda(E) = \lambda(A \cap E)$ for every $E \in \mathcal{M}$, we say that λ ***is concentrated on*** *A.*

(3) Suppose λ_1 and λ_2 are measures on $\mathcal{M}$, and suppose there exists a pair of disjoint sets A and B such that λ_1 is concentrated on A and λ_2 is concentrated on B. Then we say that λ_1 ***and*** *λ_2* ***are mutually singular****, and write*

$$\lambda_1 \perp \lambda_2$$

Theorem 1.7. *Suppose, μ, λ, λ_1 and λ_2 are measures on a σ-algebra $\mathcal{M}$, and μ is positive.*

(1) If λ is concentrated on A, so is $|\lambda|$.

(2) If $\lambda_1 \perp \lambda_2$, then $|\lambda_1| \perp |\lambda_2|$.

(3) If $\lambda_1 \perp \mu$ and $\lambda_2 \perp \mu$, then $\lambda_1 + \lambda_2 \perp \mu$.

(4) If $\lambda_1 \ll \mu$ and $\lambda_2 \ll \mu$, then $\lambda_1 + \lambda_2 < \mu$.

(5) If $\lambda \ll \mu$, then $|\lambda| \ll \mu$.

(6) If $\lambda_1 \ll \mu$ and $\lambda_2 \perp \mu$, then $\lambda_1 \perp \lambda_2$.

(7) If $\lambda \ll \mu$ and $\lambda \perp \mu$, then $\lambda = 0$.

Lemma 1.8 (Borel-Cantelli Lemma)**.** *Suppose E_k is a countable family of measurable sets on $(X, \mathcal{M}, \mu)$. Let*

$$\begin{aligned} E &= \{x \in X : x \in E_k, \text{ for infinitely many } k\} \\ &= \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} E_i \end{aligned}$$

and

$$\mu(E) = \lim_{n \to \infty} \mu\Big(\bigcup_{i=n}^{\infty} E_i\Big)$$

Theorem 1.9. *Suppose μ and λ are measures on a σ-algebra $\mathcal{M}$, μ is positive, and λ is complex. Then the following two conditions are equivalent:*

1. $\lambda \ll \mu$.

2. *To every* $\varepsilon > 0$ *corresponds a* $\delta > 0$ *such that* $|\lambda(E)| < \varepsilon$ *for all* $E \in \mathcal{M}$ *with* $\mu(E) < \delta$.

Proof: Suppose (ii) holds. If $\mu(E) = 0$, *then* $\mu(E) < \delta$ *for every* $\delta > 0$, *hence* $|\lambda(E)| < \varepsilon$ *for every* $\varepsilon > 0$, *so* $\lambda(E) = 0$. *Thus (ii) implies (i).*

Suppose (ii) is false. Then there exists an $\varepsilon > 0$ *and there exist sets* $E_n \in \mathcal{M}(n = 1, 2, 3, \ldots)$ *such that* $\mu(E_n) < 2^{-n}$ *but* $|\lambda(E_n)| \geq \varepsilon$. *Hence* $|\lambda|(E_n) \geq \varepsilon$. *Put*

$$A = \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} E_i$$

Then $\mu(A) = 0$ *and*

$$|\lambda|(A) = \lim_{n \to \infty} |\lambda| \left(\bigcup_{i=n}^{\infty} E_i \right) \geq \varepsilon > 0,$$

It follows that we do not have $|\lambda| \ll \mu$, *hence (ii) is false.*

§2 Lebesgue-Radon-Nikodym

Theorem 2.1. *If* μ *is a positive* σ*-finite measure on a* σ*-algebra* $\mathcal{M}$ *in a set* X, *then there is a function* $w \in L^1(\mu)$ *such that* $0 < w(x) < 1$ *for every* $x \in X$.

Proof: To say that μ *is* σ*-finite means that* X *is the union of countably many sets* $E_n \in \mathcal{M}(n = 1, 2, 3, \ldots)$ *for which* $\mu(E_n)$ *is finite. Put*

$$w_n(x) = \frac{1}{2^n[1 + \mu(E_n)]} \chi_{E_n}(x)$$

Then $w = \sum_1^{\infty} w_n$ *has the required properties.*

Remark *The point of the theorem is that* μ *can be replaced by a finite measure* $\tilde{\mu}$ *(namely,* $d\tilde{\mu} = w d\mu$ *) which, because of the strict positivity of* w, *has precisely the same sets of measure 0 as* μ.

Theorem 2.2 (Lebesgue-Radon-Nikodym)**.** *Let* μ *be a positive* σ*-finite measure on a* σ*-algebra* $\mathcal{M}$ *in a set* X, *and let* λ *be a complex measure on* $\mathcal{M}$.

(1) There is then a unique pair of complex measures λ_a *and* λ_s *on* $\mathcal{M}$ *such that*

$$\lambda = \lambda_a + \lambda_s, \quad \lambda_a \ll \mu, \quad \lambda_s \perp \mu$$

If λ *is positive and finite, then so are* λ_a *and* λ_b.

(2) There is a unique $h \in L^1(\mu)$ *such that*

$$\lambda_a(E) = \int_E h \, \mathrm{d}\mu$$

for every set $E \in \mathcal{M}$. We may express this in the form $d\lambda_{\mathrm{a}} = hd\mu$, or even in the form $h = d\lambda_{\mathrm{a}}/d\mu$.

*The pair (λ_a, λ_b) is called the **Lebesgue decomposition of** λ **relative to** μ. The function h which occurs in (2) is called the **Radon-Nikodym derivative of** λ_a **with respect to** μ.*

Proof: The uniqueness of (a) and (b) is easily seen.

Assume first that λ is a positive bounded measure on M. Associate w to μ as in previous theorem. Then

$$d\varphi = d\lambda + w\,\mathrm{d}\mu$$

defines a positive bounded measure φ on $\mathcal{M}$.

The definition of the sum of two measures shows that

$$\int_X f d\varphi = \int_X f d\lambda + \int_X f w\,\mathrm{d}\mu$$

for $f = \chi_E$, hence for simple f, hence for any nonnegative measurable f. If $f \in L^2(\varphi)$, the Schwarz inequality gives

$$\left|\int_X f\,\mathrm{d}\lambda\right| \leq \int_X |f|\,\mathrm{d}\lambda \leq \int_X |f|\,\mathrm{d}\varphi \leq \left\{\int_X |f|^2\,\mathrm{d}\varphi\right\}^{1/2} \{\varphi(X)\}^{1/2}$$

Since $\varphi(X) < \infty$, we see that

$$f \to \int_X f\,\mathrm{d}\lambda$$

is a bounded linear functional on $L^2(\varphi)$. Hence there exists a $g \in L^2(\varphi)$ such that

$$\int_X f\,\mathrm{d}\lambda = \int_X fg\,\mathrm{d}\varphi \tag{1}$$

for every $f \in L^2(\varphi)$. Observe also that although g is defined uniquely as an element of $L^2(\varphi)$, g is determined only a.e. $[\varphi]$ as a point function on X.

Put $f = \chi_E$ in (1), for any $E \in \mathcal{M}$ with $\varphi(E) > 0$, and since $0 \leq \lambda \leq \varphi$, we have

$$\varphi(E) \geq \lambda(E) = \int_E g\,\mathrm{d}\varphi$$

$$0 \leq \frac{1}{\varphi(E)} \int_E g\,\mathrm{d}\varphi = \frac{\lambda(E)}{\varphi(E)} \leq 1$$

Hence $g(x) \in [0,1]$ for almost all x with respect to φ. We may therefore assume that $0 \leq g(x) \leq 1$ for every $x \in X$, without affecting (1), and we rewrite (1) in the form

$$\int_X (1-g) f\,\mathrm{d}\lambda = \int_X fgw\,\mathrm{d}\mu \tag{2}$$

Put

$$A = \{x : 0 \leq g(x) < 1\}, \quad B = \{x : g(x) = 1\}$$

and define measures λ_a and λ_s by

$$\lambda_a(E) = \lambda(A \cap E), \quad \lambda_s(E) = \lambda(B \cap E),$$

for all $E \in \mathcal{M}$. If $f = \chi_B$ in (2), the left side is 0 , the right side is $\int_B w \,\mathrm{d}\mu$. Since $w(x) > 0$ for all x, we conclude that $\mu(B) = 0$. Thus $\lambda_s \perp \mu$.

Since g is bounded, (2) holds if f is replaced by

$$(1 + g + \cdots + g^n)\,\chi_E$$

for $n = 1, 2, 3, \ldots, E \in \mathcal{M}$. For such f, (2) becomes

$$\int_E \left(1 - g^{n+1}\right) \mathrm{d}\lambda = \int_E g\,(1 + g + \cdots + g^n)\, w \,\mathrm{d}\mu \tag{3}$$

Note that $1 - g^{n+1}(x) < \chi_A(x)$ and $\lambda(A) < \lambda(X) < \infty$; $\chi_A(x) \in L^1(\lambda)$. The Dominated Convergence Theorem shows that the left side of (3) converges therefore to $\lambda(A \cap E) = \lambda_a(E)$ as $n \to \infty$. The monotone convergence theorem shows that the right side of (2) tends to $\int_E h \,\mathrm{d}\mu$ as $n \to \infty$,where $h = \frac{g}{1-g}$.

$$\lambda_a(E) = \int_E h \,\mathrm{d}\mu \tag{4}$$

Taking $E = X$, we see that $h \in L^1(\mu)$, since $\lambda_a(X) < \infty$. Finally, (4) shows that $\lambda_a \ll \mu$, and the proof is complete for positive λ.

If λ is a complex measure on $\mathcal{M}$, then $\lambda = \lambda_1 + i\lambda_2$, with λ_1 and λ_2 real, and we can apply the preceding case to the positive and negative variations of λ_1 and λ_2.

Remark *If λ (and μ) are positive and σ-finite, we can now write*

$$X = \bigcup_{n=1}^{\infty} X_n$$

where $X_1 \subset X_2 \subset \cdots$ and $\lambda\,(X_n) < \infty$, for $n = 1, 2, 3, \ldots$. Considering the Lebesgue decompositions of the measures

$$\lambda_n(E) = \lambda\,(E \cap X_n)$$

on $\mathcal{M}$ that is positive and finite, we still get a function $h_n \in L^1(\mu)$ which satisfies

$$\lambda_{n,a}(E) = \int_E h_n \,\mathrm{d}\mu$$

Let $n \to \infty$, we have

$$\lambda_a(E) = \int_E h \,\mathrm{d}\mu$$

by the monotone convergence theorem, where $h = \lim h_n$

We observe that $\lambda_n(E)$ however, it is no longer true that $h \in L^1(\mu)$, although h is "locally in

L," i.e., $\int_{x_n} h d\mu < \infty$ *for each* n.

If we go beyond σ*-finiteness of* λ*, we meet situations where the two theorems under consideration actually fail. For example, let* μ *be Lebesgue measure on* $(0, 1)$*, and let* λ *be the counting measure on the* σ*-algebra of all Lebesgue*

§3 Consequence Radon-Nikodym Theorem

Theorem 3.1 (Polar Representation of Complex Measure μ)**.** *Let* μ *be a complex measure on a* σ*-algebra* $\mathcal{M}$ *in* X*. Then there is a measurable function* $h \in L^1(|\mu|)$ *such that* $|h(x)| = 1$ *for all* $x \in X$ *and such that*

$$d\mu = h\,\mathrm{d}\,|\mu|$$

Proof: It is trivial that $\mu \ll |\mu|$ *and* $|\mu|$ *is finite, therefore the Radon-Nikodym theorem guarantees the existence of some* $h \in L^1(|\mu|)$ *which satisfies* $d\mu = h\,\mathrm{d}|\mu|$ *.*

Let $A_r = \{x : |h(x)| < r\}$*, where r is some positive number, and let* $\{E_j\}$ *be a partition of* A_r*. Then*

$$\sum_j |\mu(E_j)| = \sum_j \left| \int_{E_j} h\,\mathrm{d}|\mu| \right| \leq \sum_j r|\mu|(E_j) = r|\mu|(A_r)$$

for all partition of A_r*, so that* $|\mu|(A_r) \leq r|\mu|(A_r)$*. If* $r < 1$*, this forces* $|\mu|(A_r) = 0$*. Thus* $|h| \geq 1$ *a.e.* $|\mu|$

On the other hand, if $|\mu|(E) > 0$

$$\left| \frac{1}{|\mu|(E)} \int_E h\,\mathrm{d}\,|\mu| \right| = \frac{|\mu(E)|}{|\mu|(E)} \leq 1$$

We now apply Theorem 1.40 (with the closed unit disc in place of S) and conclude that $|h| \leq 1$ *a.e.* $|\mu|$

Let $B = \{x \in X : |h(x)| \neq 1\}$*. We have shown that* $|\mu|(B) = 0$*, and if we redefine* h *on* B *so that* $h(x) = 1$ *on* B*, we obtain a function with the desired properties.*

Theorem 3.2 (The Hahn Decomposition Theorem)**.** *Let* μ *be a real measure on a* σ *algebra* $\mathcal{M}$ *in a set* X*. Then there exist sets* A *and* $B \in \mathcal{M}$ *such that* $A \cup B = X, A \cap B = \varnothing$*, and such that the positive and negative variations* μ^+*and* μ^-*of* μ *satisfy*

$$\mu^+(E) = \mu(A \cap E), \quad \mu^-(E) = -\mu(B \cap E) \quad (E \in \mathcal{M})$$

The pair (A, B) *is called a* ***Hahn decomposition*** *of* X*, induced by* μ*.*

Proof: By previous theorem, $d\mu = h\,\mathrm{d}|\mu|$*, where* $|h| = 1$*. Since* μ *is real, it follows that* h *is real (a.e.* $|\mu|$*), hence* $h = \pm 1$*. Put*

$$A = \{x : h(x) = 1\}, \quad B = \{x : h(x) = -1\}.$$

Since $\mu^+ = \frac{1}{2}(|\mu| + \mu)$, we have, for any $E \in \mathcal{M}$,

$$\mu^+(E) = \frac{1}{2}\int_E (1+h)\,\mathrm{d}|\mu| = \int_{E\cap A} h\,\mathrm{d}|\mu| = \mu(E\cap A)$$

Since $\mu(E) = \mu(E\cap A) + \mu(E\cap B)$ and since $\mu = \mu^+ - \mu^-$, the second half follows from the first.

Corollary 3.3. *Let μ be a real measure on $\mathcal{M}$. If $\mu = \lambda_1 - \lambda_2$, where λ_1 and λ_2 are positive measures, then $\lambda_1 \geq \mu^+$ and $\lambda_2 \geq \mu^-$.*

Proof: Since $\mu \leq \lambda_1$, we have

$$\mu^+(E) = \mu(E\cap A) \leq \lambda_1(E\cap A) \leq \lambda_1(E).$$

Remark *This is the minimum property of the Jordan decomposition.*

Corollary 3.4. *Suppose μ is a positive measure on $\mathcal{M}$, $g \in L^1(\mu)$, and*

$$\lambda(E) = \int_E g\,\mathrm{d}\mu \quad (E \in \mathcal{M})$$

Then

$$|\lambda|(E) = \int_E |g|\,d\mu \quad (E \in \mathcal{M})$$

Proof: By previous Theorem , there is a function h, of absolute value 1 , such that $d\lambda = h\,\mathrm{d}|\lambda|$. By hypothesis, $d\lambda = g\,\mathrm{d}\mu$. Hence

$$h\,\mathrm{d}|\lambda| = g\,\mathrm{d}\mu$$

This gives $d|\lambda| = \overline{h}g\,\mathrm{d}\mu$. Since $|\lambda| \geq 0$ and $\mu \geq 0$, it follows that $\overline{h}g \geq 0$ a.e. $[\mu]$, so that $\overline{h}g = |g|$ a.e. $[\mu]$.

§4 Riesz Theorem

Theorem 4.1. *Let X be a locally compact Hausdorff space, then every bounded linear functional Φ on $C_0(X)$ is represented by a unique regular complex Borel measure μ_Φ, in the sense that*

$$\Phi f = \int_X f\,\mathrm{d}\mu$$

for every $f \in C_0(X)$. Moreover, $\|\Phi\| = \|\mu\| = |\mu|(X)$.

Proof: We first settle the uniqueness question. Suppose μ is a regular complex Borel measure on X and $\int f\,\mathrm{d}\mu = 0$ for all $f \in C_0(X)$. By Theorem 6.12 there is a Borel function h, with $|h| = 1$, such that $d\mu = h\,\mathrm{d}|\mu|$. For any sequence $\{f_n\}$ in $C_0(X)$ we then have

$$|\mu|(X) = \int_X \left(\bar{h} - f_k\right) h\,\mathrm{d}|\mu| \leq \int_X \left|\bar{h} - f_n\right| d|\mu|$$

and since $C_c(X)$ is dense in $L^1(|\mu|)$, thus $|\mu|(X)=0$, and $\mu=0$.

Now consider a given bounded linear functional Φ on $C_0(X)$. Assume $\|\Phi\|=1$, without loss of generality. We shall construct a positive linear functional Λ on $C_c(X)$, such that

$$|\Phi(f)| \leq \Lambda(|f|) \leq \|f\| \quad (f \in C_c(X))$$

where $\|f\|$ denotes the supremum norm. Once we have this Λ, we associate with it a positive Borel measure λ, as in Theorem 2.14. The conclusion of Theorem 2.14 shows that λ is regular if $\lambda(X)<\infty$. Since

$$\lambda(X)=\sup \{\Lambda f: 0 \leq f \leq 1, f \in C_c(X)\}$$

and since $|\Lambda f| \leq 1$ if $\|f\| \leq 1$, we see that actually $\lambda(X) \leq 1$. We also deduce from (4) that

$$|\Phi(f)| \leq \Lambda(|f|)=\int_x|f| d \lambda=\|f\|_1 \quad (f \in C_c(X))$$

The last norm refers to the space $L^1(\lambda)$. Thus Φ is a linear functional on $C_c(X)$ of norm at most 1 , with respect to the $L^1(\lambda)$-norm on $C_c(X)$. There is a normpreserving extension of Φ to a linear functional on $L^1(\lambda)$, and therefore Theorem 6.16 (the case $p=1$) gives a Borel function g, with $|g| \leq 1$, such that

$$\Phi(f)=\int_X f g d \lambda \quad (f \in C_c(X))$$

Each side of (6) is a continuous functional on $C_0(X)$, and $C_c(X)$ is dense in $C_0(X)$. Hence (6) holds for all $f \in C_0(X)$, and we obtain the representation (1) with $d\mu=g d \lambda$.

Since $\|\mathbf{\Phi}\|=1,(6)$ shows that

$$\int_X|g| d \lambda \geq \sup \{|\Phi(f)|: f \in C_0(X),\|f\| \leq 1\}=1$$

We also know that $\lambda(X) \leq 1$ and $|g| \leq 1$. These facts are compatible only if $\lambda(X)=1$ and $|g|=1$ a.e. [λ]. Thus $d|\mu|=|g| d \lambda=d \lambda$, by Theorem 6.13, and

$$|\mu|(X)=\lambda(X)=1=\|\Phi\|$$

which proves (2). So all depends on finding a positive linear functional Λ that satisfies (4). If $f \in C_c^{+}(X)$ [the class of all nonnegative real members of $C_c(X)$], define

$$\Lambda f=\sup \{|\Phi(h)|: h \in C_c(X),|h| \leq f\}$$

Then $\Lambda f \geq 0, \Lambda$ satisfies (4), $0 \leq f_1 \leq f_2$ implies $\Delta f_1 \leq \Delta f_2$, and $\Lambda(c f)=c \Delta f$ if c is a positive constant. We have to show that

$$\Lambda(f+g)=\Lambda f+\Lambda g \quad \left(f \text{ and } g \in C_c^{+}(X)\right)$$

and we then have to extend Λ to a linear functional on $C_c(X)$. Fix f and $g \in C_c^+(X)$. If $\varepsilon > 0$, there exist h_1 and $h_2 \in C_c(X)$ such that $|h_1| \leq f, |h_2| \leq g$, and

$$\Lambda f \leq |\Phi(h_1)| + \varepsilon, \quad \Lambda g \leq |\Phi(h_2)| + \varepsilon$$

There are complex numbers $\alpha_i, |\alpha_i| = 1$, so that $\alpha_i \Phi(h_i) = |\Phi(h_i)|, i = 1, 2$. Then

$$\begin{aligned}
\Lambda f + \Lambda g &\leq |\Phi(h_1)| + |\Phi(h_2)| + 2\varepsilon \\
&= \Phi(\alpha_1 h_1 + \alpha_2 h_2) + 2\varepsilon \\
&\leq \Lambda(|h_1| + |h_2|) + 2\varepsilon \\
&\leq \Lambda(f+g) + 2\varepsilon
\end{aligned}$$

so that the inequality $\geq$ holds in (10). Next, choose $h \in C_c(X)$, subject only to the condition $|h| \leq f + g$, let $V = \{x : f(x) + g(x) > 0\}$, and define

$$\begin{aligned}
&h_1(x) = \frac{f(x)h(x)}{f(x)+g(x)}, \quad h_2(x) = \frac{g(x)h(x)}{f(x)+g(x)} \quad (x \in V), \\
&h_1(x) = h_2(x) = 0 \quad (x \notin V).
\end{aligned}$$

It is clear that h_1 is continuous at every point of V. If $x_0 \notin V$, then $h(x_0) = 0$; since h is continuous and since $|h_1(x)| \leq |h(x)|$ for all $x \in X$, it follows that x_0 is a point of continuity of h_1. Thus $h_1 \in C_c(X)$, and the same holds for h_2.

Since $h_1 + h_2 = h$ and $|h_1| \leq f, |h_2| \leq g$, we have

$$|\Phi(h)| = |\Phi(h_1) + \Phi(h_2)| \leq |\Phi(h_1)| + |\Phi(h_2)| \leq \Lambda f + \Lambda g.$$

Hence $\Lambda(f+g) \leq \Lambda f + \Lambda g$, and we have proved (10). If f is now a real function, $f \in C_c(X)$, then $2f^+ = |f| + f$, so that $f^+ \in C_c^+(X)$; likewise, $f^- \in C_c^+(X)$; and since $f = f^+ - f^-$, it is natural to define

$$\Lambda f = \Lambda f^+ - \Lambda f^- \quad (f \in C_{\mathrm{r}}(X), f \text{ real })$$

and

$$\Lambda(u + iv) = \Lambda u + i\Lambda v.$$

Simple algebraic manipulations, just like those which occur in the proof of Theorem 1.32, show now that our extended functional Λ is linear on $C_c(X)$.

This completes the proof.

Chapter VII

Integration On Product Spaces

§1 Application

Theorem 1.1 (Minkowski). *Suppose that $(X, \mathscr{L}, \mu)$ and $(Y, \mathscr{T}, \nu)$ are σ-finite measure spaces. Let f be an $(\mathscr{L} \times \mathscr{T})$ measurable function on $X \times Y$. If $0 \leq f \leq \infty$ and $1 \leq p < \infty$, then we have the analogy of Minkowski's inequality*

$$\left\{\int \left[\int f(x,y)\,\mathrm{d}\lambda(y)\right]^p \mathrm{d}\mu(x)\right\}^{\frac{1}{p}} \leq \int \left[\int f(x,y)^p\,\mathrm{d}\lambda(y)\right]^{\frac{1}{p}} \mathrm{d}\mu(x)$$

Proof: If $p = 1$, then it is exactly The Fubini Theorem. Suppose that $1 < p < \infty$ and q is the conjugate exponent of p. Since the inequality holds trivially when

$$\int \left[\int f^p(x,y)\,\mathrm{d}\mu(x)\right]^{\frac{1}{p}} \mathrm{d}\lambda(y) = \infty$$

so without loss of generality, we assume further that this integral is finite. Let $g \in L^q(\mu)$, we define Φ by

$$\Phi(g) = \int \left[\int f(x,y)\,\mathrm{d}\lambda(y)\right] \cdot g(x)\,\mathrm{d}\mu(x)$$

we have

$$\begin{aligned}
\Phi(g) &= \iint f(x,y) \cdot g(x)\,\mathrm{d}\mu(x)\,\mathrm{d}\lambda(y) \\
&\leq \int \left[\int f^p(x,y)\,\mathrm{d}\mu(x)\right]^{\frac{1}{p}} \left[\int |g|^q\,\mathrm{d}\mu(x)\right]^{\frac{1}{q}} \mathrm{d}\lambda(y) \\
&= \|g\|_q \int \left[\int f^p(x,y)\,\mathrm{d}\mu(x)\right]^{\frac{1}{p}} \mathrm{d}\lambda(y)
\end{aligned}$$

then Φ is a bounded linear functional on $L^q(\mu)$ and there exists a unique $h \in L^p(\mu)$ such that

$$\Phi(g) = \int hg\,\mathrm{d}\mu(x)$$

the uniqueness of h and the comparison of the representations and show that

$$h(x) = \int f(x,y)\,\mathrm{d}\lambda(y)$$

Furthermore, we deduce that

$$\begin{aligned}\left\{\int \left[\int f(x,y)\,\mathrm{d}\lambda(y)\right]^p \mathrm{d}\mu(x)\right\}^{\frac{1}{p}} &= \|h\|_p \\ &= \|\Phi\| \\ &\le \int \left[\int f^p(x,y)\,\mathrm{d}\mu(x)\right]^{\frac{1}{p}} \mathrm{d}\lambda(y)\end{aligned}$$

which completes the proof of the problem.

Theorem 1.2 (Hardy's Inequality)**.** *Suppose $f \ge 0$ on $(0,\infty)$, $f \in L^p$, $1 \le p \le \infty$, and*

$$F(x) = \frac{1}{x}\int_0^x f(t)\,\mathrm{d}t$$

Proof. *Since $f(t)t^{\alpha}$ and $t^{-\alpha}$ are nonnegative, we have estimate by Holder's*

$$\begin{aligned}xF(x) &= \int_0^x f(t)t^{\alpha}t^{-\alpha}\,\mathrm{d}t \\ &\le \left(\int_0^x f^p t^{\alpha p}\,\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^x t^{\alpha q}\,\mathrm{d}t\right)^{\frac{1}{q}} \\ &= \left(\int_0^x f^p t^{\alpha p}\,\mathrm{d}t\right)^{\frac{1}{p}} \times \left(\frac{x^{-\alpha q+1}}{-\alpha q+1}\right)^{\frac{1}{q}}\end{aligned}$$

which gives

$$F^p(x) \le (1-\alpha q)^{1-p} x^{-1-\alpha p}\int_0^x f^p t^{\alpha p}\,\mathrm{d}t$$

and then

$$\begin{aligned}\int_0^{\infty} F^p(x)\,\mathrm{d}x &\le (1-\alpha q)^{1-p}\int_0^{\infty} x^{-1-\alpha p}\int_0^x f^p t^{\alpha p}\,\mathrm{d}t\,\mathrm{d}x \\ &= (1-\alpha q)^{1-p}\int_0^{\infty} f^p t^{\alpha p}\int_0^{\infty}\chi_{(0,x)}(t)x^{-1-\alpha p}\,\mathrm{d}x\,\mathrm{d}t \\ &= (1-\alpha q)^{1-p}\int_0^{\infty} f^p t^{\alpha p} \times \frac{t^{-\alpha p}}{\alpha p}\,\mathrm{d}t \\ &= (1-\alpha q)^{1-p}(\alpha p)^{-1}\int_0^{\infty} f^p\,\mathrm{d}t \\ &= \left(\frac{p}{p-1}\right)^p \int_0^{\infty} f^p\,\mathrm{d}t \qquad (\textit{if we choose } \alpha = \frac{1}{pq})\end{aligned}$$

as the space $(0,\infty)$ is σ-finite, $\chi_{(0,x)}x^{-1-\alpha p}f^p t^{\alpha p} \ge 0$ and measurable on $(0,\infty)\times(0,\infty)$.

Lemma 1.3. *Suppose that f and K are measurable functions on $\mathbb{R}$, then $F(x,y) = K(x-y)f(y)$ is measurable on $\mathbb{R}^2$*

Proof: There exist Borel functions f_0 and K_0 such that $f_0 = f$ a.e. and $K_0 = K$ a.e. We claim that $F_0(x, y) = K_0(x - y) f_0(y) = F(x, y)$ a.e. $(x, y) \in \mathbb{R}^2$.

As we can see, the set $\{(x, y) : F_0(x, y) \neq F_(x, y)\}$ is contained in

$$\{(x, y) : f_0(y) \neq f(y)\} \cup \{(x, y) : K_0(x - y) \neq K(x - y)\}$$

where $\{(x, y) : f_0 \neq f\} = \mathbb{R} \times \{y : f_0 \neq f\}$ is 0 measure and

$$\begin{aligned} m\left(\{(x, y) : K_0(x - y) \neq K(x - y)\}\right) &= \int_{\mathbb{R}^2} \chi_E(x - y)\, \mathrm{d}x\, \mathrm{d}y \\ &= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} \chi_{y+E}(x)\, \mathrm{d}x\right] \mathrm{d}y \\ &= 0 \end{aligned}$$

Consequently, we conclude that $m\left(\{(x, y) : F_0(x, y) \neq F(x, y)\}\right) = 0$ which implies that $F_0(x, y) = F(x, y)$ a.e. $(x, y) \in \mathbb{R}^2$.

Next, we define $\varphi : \mathbb{R}^2 \to \mathbb{R}^1$ and $\psi : \mathbb{R}^2 \to \mathbb{R}^1$ by

$$\varphi(x, y) = x - y, \quad \psi(x, y) = y.$$

Then $f_0(x - y) = (f_0 \circ \varphi)(x, y)$ and $g_0(y) = (g_0 \circ \psi)(x, y)$ are Borel functions on $\mathbb{R}^2$ since φ and ψ are Borel(continuous) functions. Hence so is their product F_0 is a Borel functions on $\mathbb{R}^2$. Finally we conclude that F is measurable on $\mathbb{R}^2$

Theorem 1.4 (Young's convolution inequality)**.** *Suppose $1 \leq p \leq \infty$, $K \in L^1$, and $f \in L^p$. Then the integral defining*

$$(K * f)(x) = \int_{\mathbb{R}} K(x - y) f(y)\, \mathrm{d}y$$

exists for almost every $x \in \mathbb{R}$, $f * K \in L^1$, and

$$\|K * f\|_p \leq \|K\|_1 \|f\|_p$$

Proof: Suppoes that $p = 1$ and write $F(x, y) = K(x - y) f(y)$. We observe that

$$\int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} |F(x, y)| dx = \int_{-\infty}^{\infty} |g(y)| dy \int_{-\infty}^{\infty} |f(x - y)| dx = \|f\|_1 \|g\|_1$$

Thus $F \in L^1\left(\mathbb{R}^2\right)$, and Fubini's theorem implies that the integral

$$\left(K * f\right)(x) = \int_{\mathbb{R}} F(x, y)\, \mathrm{d}y$$

exists for almost all $x \in \mathbb{R}^1$ *and that* $K * f \in L^1(\mathbb{R}^1)$*. Finally,*

$$
\begin{aligned}
\|K * f\|_1 &= \int_{-\infty}^{\infty} \left| \int_{\mathbb{R}} F(x, y) \, \mathrm{d}y \right| \mathrm{d}x \\
&\leq \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} |F(x, y)| \, \mathrm{d}y \\
&= \|f\|_1 \, \|g\|_1
\end{aligned}
$$

Now suppose that $1 < p < \infty$*. By Hölder's Inequality, we derive that*

$$
\begin{aligned}
\int_{\mathbb{R}} |K(x-y) f(y)| \, \mathrm{d}y &= \int_{-\infty}^{\infty} |K(x-y)|^{\frac{1}{q}} \cdot |K(x-y)|^{\frac{1}{p}} |f(y)| \, \mathrm{d}y \\
&\leq \left(\int_{\mathbb{R}} |K(x-y)| \, \mathrm{d}y \right)^{\frac{1}{q}} \left(\int_{-\infty}^{\infty} |K(x-y)| |f(y)|^p \, \mathrm{d}y \right)^{\frac{1}{p}} \\
&= \|K\|_1^{\frac{1}{q}} \cdot \left(\int_{-\infty}^{\infty} |K(x-y)| |f(y)|^p \, \mathrm{d}y \right)^{\frac{1}{p}}
\end{aligned}
$$

which gives

$$
\left(\int_{-\infty}^{\infty} |K(x-y) f(y)| \, \mathrm{d}y \right)^p \leq \|K\|_1^{\frac{p}{q}} \int_{-\infty}^{\infty} K|(x-y)| \cdot |f(y)|^p \, \mathrm{d}y
$$

According to the case $p = 1$*, we have* $\int_{-\infty}^{\infty} |f(x-y)| \cdot |g(y)|^p \, \mathrm{d}y$ *exist a.e.* $x \in \mathbb{R}$ *since* f^p *and* K *are belong to* L^1*. Then* $K * f(x)$ *exist a.e. by this estimate. Also,*

$$
\begin{aligned}
\|K * f\|_p^p &= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} K(x-y) f(y) \, \mathrm{d}y \right|^p \mathrm{d}x \\
&\leq \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |K(x-y) f(y)| \, \mathrm{d}y \right)^p \mathrm{d}x \\
&\leq \|K\|_1^{\frac{p}{q}} \int \left(\int_{-\infty}^{\infty} |K(x-y)| |f(y)|^p \, \mathrm{d}y \right) \mathrm{d}x \\
&= \|K\|_1^p \, \|f\|_p^p
\end{aligned}
$$

which implies that $\|K * f\|_p \leq \|K\|_1 \cdot \|f\|_p < \infty$*.*

Remark

Chapter VIII

Fourier Transforms

§1 Formal Properties

§1.1 Definition

Definition 1.1. *If $f, g \in L^1(\mathbb{R}^n)$, we define Fourier transform*

$$\hat{f}(\xi) = \mathcal{F}f(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi} \, \mathrm{d}x \quad (\xi \in \mathbb{R}^n)$$

and

$$\check{g}(\xi) = \mathcal{F}^{-1}g(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} g(x) e^{ix\cdot\xi} \, \mathrm{d}x \quad (\xi \in \mathbb{R}^n)$$

Definition 1.2. *If $f \in L^2(\mathbb{R}^n)$, we choose a sequence $\{f_k\}_{k=1}^{\infty} \subset L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ with*

$$f_k \xrightarrow{L^2(\mathbb{R}^n)} f$$

Then

$$\left\| \hat{f}_k - \hat{f}_j \right\|_{L^2(\mathbb{R}^n)} = \left\| \widehat{f_k - f_j} \right\|_{L^2(\mathbb{R}^n)} = \|f_k - f_j\|_{L^2(\mathbb{R}^n)}$$

and thus $\left\{ \hat{f}_k \right\}_{k=1}^{\infty}$ is a Cauchy sequence in $L^2(\mathbb{R}^n)$. This sequence consequently converges to a limit u in $L^2(\mathbb{R}^n)$, which we define to be $\mathcal{F}f = u$.

The definition of u does not depend upon the choice of approximating sequence $\{f_k\}$. We can choose $f_k(x) = \chi_{[-k,k]^n}(x) \cdot f(x)$ then

$$\mathcal{F}f(\xi) = \lim_{k\to+\infty} \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{[-k,k]^n} f(x) e^{-ix\cdot\xi} \, \mathrm{d}x \quad \text{in } L^2(\mathbb{R}^n)$$

We similarly define $\mathcal{F}^{-1}$

Theorem 1.3 (Plancherel's Theorem in $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$)**.** *Assume $u \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Then $\hat{u}, \check{u} \in L^2(\mathbb{R}^n)$ and*

$$\|\hat{u}\|_{L^2(\mathbb{R}^n)} = \|\check{u}\|_{L^2(\mathbb{R}^n)} = \|u\|_{L^2(\mathbb{R}^n)}$$

Proof: 1. First we note that if $v, w \in L^1(\mathbb{R}^n)$*, then* $\hat{v}, \hat{w} \in L^\infty(\mathbb{R}^n)$*. Also*

$$\int_{\mathbb{R}^n} v(x)\hat{w}(x)dx = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-ix\cdot y} v(x)w(y)dxdy = \int_{\mathbb{R}^n} \hat{v}(y)w(y)dy$$

Consequently if $\varepsilon > 0$ *and* $v_\varepsilon(x) := e^{-\varepsilon|x|^2}$*, we have* $\hat{v}_\varepsilon(y) = \frac{e^{-\frac{|y|^2}{\varepsilon}}}{(2\varepsilon)^{n/2}}$*. Thus implies for each* $\varepsilon > 0$ *that*

$$\int_{\mathbb{R}^n} \hat{w}(y)e^{-\varepsilon|y|^2}dy = \frac{1}{(2\varepsilon)^{n/2}} \int_{\mathbb{R}^n} w(x)e^{-\frac{|x|^2}{4\varepsilon}}dx$$

2. Now take $u \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ *and set* $v(x) := \bar{u}(-x)$*. Let*

$$w := u * v \in L^1(\mathbb{R}^n) \cap C(\mathbb{R}^n)$$

and check that

$$\hat{w} = (2\pi)^{n/2}\hat{u}\hat{v} = (2\pi)^{n/2}|\hat{u}|^2 \in L^1(\mathbb{R}^n)$$

Since $w \in C(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$*, we have*

$$\lim_{\varepsilon\to 0} \frac{1}{(2\varepsilon)^{n/2}} \int_{\mathbb{R}^n} w(x)e^{-\frac{|x|^2}{4\varepsilon}}dx = (2\pi)^{n/2}w(0) = (2\pi)^{n/2}\|u\|_{L^2}$$

Also,

$$\lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^n} \hat{w}(y)e^{-\varepsilon|y|^2}dy = \int_{\mathbb{R}^n} \hat{w}(y)dy = (2\pi)^{n/2}\|\hat{u}\|_{L^2}$$

The proof for $\check{u}$ *is similar.*

§1.2 Properties

Theorem 1.4. *Assume* $\mathcal{F}f$ *exist, and* h *and* λ *belong to* $\mathbb{R}^n$*. Then*

(i)

$$\mathcal{F}(f(x)e^{ih\cdot x})(\xi) = \hat{f}(\xi - h)$$

(ii)

$$\mathcal{F}(f(x+h))(\xi) = \hat{f}(\xi)e^{ih\cdot\xi}$$

(iii)

$$\mathcal{F}(f(-x))(\xi) = \mathcal{F}^{-1}f(\xi)$$

(vi)

$$\mathcal{F}\bar{f} = \overline{\mathcal{F}^{-1}f}$$

(v)

$$\mathcal{F}\left(\frac{1}{|\lambda|^n} f\left(\frac{x}{\lambda}\right)\right)(\xi) = \mathcal{F}f(\lambda\xi)$$

Theorem 1.5. *(i) If $f, g \in L^1$, then $f * g \in L^1$ and*

$$\mathcal{F}(f * g) = \frac{1}{(2\pi)^{\frac{n}{2}}}\mathcal{F}(f) \cdot \mathcal{F}(g)$$

(ii) If $g(x) = (-ix)^\alpha f(x)$ and $g \in L^1$, then is differentiable and

$$\mathcal{F}((-ix)^\alpha f(x)) = D^\alpha(\mathcal{F}f)$$

(iii) If $D^\alpha u \in L^2(\mathbb{R}^n)$ for multiindex α, then

$$\mathcal{F}D^\alpha u = (iy)^\alpha \mathcal{F}u$$

§2 p=1

Theorem 2.1. *Suppose $1 \leq p < \infty$ and any function f on $L^p(\mathbb{R}^n)$, the mapping*

$$y \to f_y(x) = f(x - y)$$

is a uniformly continuous mapping of $\mathbb{R}^n$ into $L^p(\mathbb{R}^n)$.

Proof: Fix $\varepsilon > 0$. Since $f \in L^p$, there exists a continuous function g whose support lies in a bounded interval $[-A, A]^n$, such that

$$\|f - g\|_p < \varepsilon$$

The uniform continuity of g shows that there exists a $\delta > 0$ such that $|s - t| < \delta$ implies

$$|g(s) - g(t)| < \frac{\varepsilon}{(3A)^{\frac{1}{p}}}$$

If $|s - t| < \delta$, it follows that

$$\int_{\mathbb{R}^n} |g(x - s) - g(x - t)|^p \,\mathrm{d}x < \frac{\varepsilon^p}{3A}(2A + \delta)^n < \varepsilon^p$$

Thus

$$\begin{aligned}
\|f_s - f_t\|_p &\leq \|f_s - g_s\|_p + \|g_s - g_t\|_p + \|g_t - f_t\|_p \\
&= \|(f - g)_s\|_p + \|g_s - g_t\|_p + \|(g - f)_t\|_p \\
&< 3\varepsilon
\end{aligned}$$

whenever $|s - t| < \delta$. This completes the proof.

Theorem 2.2. *If $f \in L^1$, then $\hat{f} \in C_0$ and*

$$\|\hat{f}\|_\infty \leq \|f\|_1$$

Proof: 1. If $t_n \to t$, then

$$|\hat{f}(t_n) - \hat{f}(t)| \le \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} |f(x)||e^{-it_n x} - e^{-itx}| \,\mathrm{d}x$$

The integrand is bounded by $2|f(x)|$ and tends to 0 for every x, as $n \to \infty$. Hence $\hat{f}(t_n) \to \hat{f}(t)$, by the dominated convergence theorem. Thus $\hat{f}$ is continuous.

2. Since $e^{i\pi} = -1$

$$f(t) = -\frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} f(x) e^{-it(x+\frac{\pi}{t})} \,\mathrm{d}x = -\frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} f(x - \frac{\pi}{t}) e^{-itx} \,\mathrm{d}x$$

Hence

$$2f(t) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \left\{ f(x) - f\left(x - \frac{\pi}{t}\right) \right\} e^{-itx} \,\mathrm{d}x$$

so that

$$2|\hat{f}(t)| \le \left\| f - f_{\frac{\pi}{t}} \right\|_1$$

which tends to 0 as $t \to \pm\infty$, by previous theorem.

Theorem 2.3 (The Inversion Theorem for $L^1(\mathbb{R}^n)$)**.** *If $f \in L^1$ and $\hat{f} \in L^1$, then $\mathcal{F}^{-1}(f)(x)$ exist for all $x \in \mathbb{R}^n$ and*

$$\mathcal{F}^{-1}\hat{f} = f$$

for almost every $x \in \mathbb{R}^n$.

Proof: Consider

$$\begin{aligned} f * \frac{1}{(2\varepsilon)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4\varepsilon}} &= f * \mathcal{F}^{-1}(e^{\varepsilon|y|^2}) \\ &= \int_{\mathbb{R}^n} f(x-y) \left(\frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\varepsilon|t|^2} e^{it\cdot y} dt \right) dy \\ &= \int_{\mathbb{R}^n} e^{-\varepsilon|t|^2} \left(\frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} f(x-y) e^{it\cdot y} dy \right) dt \\ &= \int_{\mathbb{R}^n} e^{-\varepsilon|t|^2} \hat{f}(t) e^{it\cdot x} dt \end{aligned}$$

Let $\varepsilon \to 0^+$, we have

$$f(x) = \mathcal{F}^{-1}(\hat{f})(x)$$

a.e. $x \in \mathbb{R}^n$ since $f \in L^1(\mathbb{R}^n)$ and $\hat{f} \in C^0(\mathbb{R}^n)$.

Remark *We can see f has a version $\mathcal{F}^{-1}f \in C^0(\mathbb{R}^n)$ in this case.*

Theorem 2.4 (The Uniqueness Theorem)**.** *If $f \in L^1$ and $\hat{f}(t) = 0$ for all $t \in R$, then $f(x) = 0$ a.e. $[m]$*

§3 p=2

Theorem 3.1 (The Plancherel Theorem on $L^2(\mathbb{R}^n)$)**.** *$\mathcal{F}$ is a unitary operator on $L^2(\mathbb{R}^n)$.*

Theorem 3.2. *$\mathcal{F}$ is a self-adjoint operator on $L^2(\mathbb{R}^n)$.*

Proof: We observe that if $u, v \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, then

$$\int_{\mathbb{R}^n} \hat{u} v dx = \int_{\mathbb{R}^n} u \hat{v} dx$$

since both sides equal

$$\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} u(y) v(x) e^{-ix \cdot y} dx dy$$

Theorem 3.3 (The Inversion Theorem for $L^2(\mathbb{R}^n)$)**.** *In $L^2(\mathbb{R}^n)$*

$$\mathcal{F}^{-1} \mathcal{F} = \mathcal{F} \mathcal{F}^{-1} = I$$

Proof: Ues the Plancherel Theorem

$$\int_{\mathbb{R}^n} \mathcal{F}^{-1}(\hat{u}) v dx = \int_{\mathbb{R}^n} \hat{u} \check{v} dx = \int_{\mathbb{R}^n} \hat{u} \overline{\mathcal{F}(\overline{v})} dx = \int_{\mathbb{R}^n} u v dx$$

This holds for all $v \in L^2(\mathbb{R}^n)$, and so statement follows.

§4 The Schwarz Space

Definition 4.1. *The Schwarz space $\mathcal{S}(\mathbb{R}^n)$ consists of all infinitely differentiable functions f on $\mathbb{R}^n$ such that*

$$\sup_{x \in \mathbb{R}^n} \left| x^\alpha \left(\frac{\partial}{\partial x} \right)^\beta f(x) \right| < \infty$$

for every multi-index α and β. In other words, f and all its derivatives are required to be rapidly decreasing.

Noted that $\mathcal{S} \subset L^1 \cap L^\infty \cap C^\infty$

Theorem 4.2 (Poisson Summation Formula)**.** *Let $f \in \mathcal{S}(\mathbb{R})$, then*

$$\sum_{k=-\infty}^{\infty} f(2k\pi) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \mathcal{F} f(n)$$

Proof: Let

$$F(x) = \sum_{k=-\infty}^{\infty} f(x + 2k\pi)$$

then F is periodic, with period 2π, the n-th Fourier coefficient of F is

$$\begin{aligned}\hat{F}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} F(x)e^{-inx}\,\mathrm{d}x \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} f(x)e^{-inx}\,\mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}}\mathcal{F}f(n)\end{aligned}$$

hence $F(x) = \sum \frac{1}{\sqrt{2\pi}}\mathcal{F}f(n)e^{inx}$. In particular, let $x = 0$, we have

More generally, if $\alpha > 0, \beta > 0, \alpha\beta = 2\pi$, we have

$$\sum_{k=-\infty}^{\infty} f(k\beta) = \frac{\alpha}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \varphi(n\alpha)$$

Chapter IX

Differentiation

§1 Derivates

§1.1 Derivates of Measures

Definition 1.1. *Let us fix a dimension k, denote the open ball with center $x \in R^k$ and radius $r > 0$ by*

$$B(x,r) = \left\{ y \in R^k : |y - x| < r \right\}$$

associate to any complex Borel measure μ on $\mathbb{R}^k$ the quotients

$$\left(Q_r \mu\right)(x) = \frac{\mu(B(x,r))}{m(B(x,r))}$$

where m is Lebesgue measure on R^k, and define the ***symmetric derivative*** *of μ at x to be*

$$(D\mu)(x) = \lim_{r \to 0^+} \left(Q_r \mu\right)(x)$$

at those points $x \in R^k$ at which this limit exists.

We shall study $D\mu$ by means of the ***maximal function*** *$M\mu$ that is defined by*

$$(M\mu)(x) = \sup_{0<r<\infty} \left(Q_r |\mu|\right)(x)$$

Theorem 1.2. *The functions $M\mu : R^k \to [0, \infty]$ are lower semicontinuous, hence measurable.*

Proof: To see this, assume $\mu \geq 0$, pick $\lambda > 0$, let $E = \{M\mu > \lambda\}$, and fix $x \in E$. Then there is an $r > 0$ such that

$$\mu(B(x,r)) = tm(B(x,r))$$

for some $t > \lambda$. Then there are small δ and r that

$$\begin{aligned}\mu(B(y, r+\delta)) &\geq \mu(B(x, r)) \\ &= tm(B(x, r)) \\ &= (t-\varepsilon)m(B(y, r+\delta)) \\ &> \lambda m(B(y, r+\delta))\end{aligned}$$

Thus $B(x, \delta) \subset E$.

Lemma 1.3. *If W is the union of a finite collection of balls $B\left(x_i, r_i\right) \quad i=1,2\ldots, N$, then there is a set $S \subset \{1, \ldots, N\}$ so that*

(i) the balls $B\left(x_i, r_i\right)$ with $i \in S$ are disjoint,

(ii) $W \subset \bigcup\limits_{i \in S} B\left(x_i, 3r_i\right)$

Proof: Order the balls $B_i = B\left(x_i, r_i\right)$ so that $r_1 \geq r_2 \geq \ldots \geq r_N$. Put $i_1 = 1$. Discard all B_j that intersect B_{i_1}. Let B_{i_2} be the first of the remaining B_j, if there are any. Discard all B_j with $j > i_2$ that intersect B_{i_2}, let B_{i_3} be the first of the remaining ones, and so on, as long as possible. This process stops after a finite number of steps and gives $S = \{i_1, i_2, \ldots\}$.

It is clear that (a) holds. Every discarded B_j is a subset of $B\left(x_i, 3r_i\right)$ for some $i \in S$, for if $r' \leq r$ and $B\left(x', r'\right)$ intersects $B(x, r)$, then $B\left(x', r'\right) \subset B(x, 3r)$. This proves (b)

Theorem 1.4. *If μ is a complex Borel measure on R^k and λ is a positive number, then*

$$m\{M\mu > \lambda\} \leq 3^k \lambda^{-1} \|\mu\|$$

Proof: Fix μ and λ. Let K be a compact subset of the open set $\{M\mu > \lambda\}$. Each $x \in K$ is the center of an open ball B for which

$$|\mu|(B) > \lambda m(B)$$

Some finite collection of these Balls covers K, and previous Lemma gives us a disjoint subcollection, say $\{B_1, \ldots, B_n\}$, that satisfies

$$\begin{aligned} m(K) &\leq \sum_1^n m(B_i(x_i, 3r_i)) \\ &= 3^k \sum_1^n m(B_i) \\ &\leq 3^k \lambda^{-1} \sum_1^n |\mu|(B_i) \leq 3^k \lambda^{-1} \|\mu\| \end{aligned}$$

Now taking the supremum over all compact $K \subset \{M\mu > \lambda\}$.

Definition 1.5. *Any measurable function f for which*

$$\lambda \cdot m\{|f| > \lambda\}$$

*is a bounded function of λ on $(0, \infty)$ is said to belong to **weak** L^1. Thus weak L^1 contains L^1.*

§1.2 Derivates of functions

Definition 1.6. *We associate to each $f \in L^1(R^k)$ its **maximal function** $Mf : R^k \to [0, \infty]$, by setting*

$$(Mf)(x) = \sup_{0<r<\infty} \frac{1}{m(B_r)} \int_{B(x,r)} |f| \, \mathrm{d}m$$

If we identify f with the measure μ given by $d\mu = f dm$, we see that Mf agrees with the previously defined $M\mu$. Theorem 7.4 states therefore that the "maximal operator " M sends L^1 to weak L^1, with a bound that depends only on the space R^k : For every $f \in L^1(R^k)$ and every $\lambda > 0$,

$$m\{Mf > \lambda\} \leq 3^k \lambda^{-1} \|f\|_1$$

Theorem 1.7 (Lebesgue points)**.** *If $f \in L^1(R^k)$, any $x \in R^k$ for which it is true that*

$$\lim_{r \to 0} \frac{1}{m(B_r)} \int_{B(x,r)} |f(y) - f(x)| \, \mathrm{d}m(y) = 0$$

*is called a **Lebesgue point** of f. If $f \in L^1(R^k)$, then almost every $x \in R^k$ is a Lebesgue point of*

f.

Proof: Define

$$(T_r f)(x) = \frac{1}{m(B_r)} \int_{B(x,r)} |f - f(x)| \, \mathrm{d}m$$

for $x \in R^k, r > 0$, *and put*

$$(Tf)(x) = \limsup_{r \to 0} \left(T_r f\right)(x)$$

We have to prove that $Tf = 0$ *a.e.* [m].

Pick $y > 0$. *Let* n *be a positive integer. By Theorem 3.14, there exists* $g \in C\left(R^k\right)$ *so that* $\|f - g\|_1 < 1/n$. *Put* $h = f - g$, *then*

$$\left(T_r h\right)(x) \leq \frac{1}{m\left(B_r\right)} \int_{B(x,r)} |h| dm + |h(x)|$$

Let $r \to 0$, *we have*

$$Th \leq Mh + |h|$$

Since g *is continuous,* $Tg = 0$. *Since* $T_r f \leq T_r g + T_r h$, *it follows that*

$$Tf \leq Th \leq Mh + |h|$$

Therefore

$$\{Tf > 2y\} \subset \{Mh > y\} \cup \{|h| > y\}$$

Denote the union on the right $E(y, n)$. *Since* $\|h\|_1 < 1/n$, *Theorem 7.4 and the inequality* 7.5(1) *show that*

$$\begin{aligned} m(E(y,n)) &\leq m(\{Mh > y\}) + m(\{|h| > y\}) \\ &\leq 3^k y^{-1} \|h\|_1 + y^{-1} \|h\|_1 \\ &\leq \frac{3^k + 1}{yn} \end{aligned}$$

The left side is independent of n. *Hence*

$$\{Tf > 2y\} \subset \bigcap_{n=1}^{\infty} E(y, n)$$

This intersection has measure 0, so that $\{Tf > 2y\}$ *has measure 0 . This holds for every positive* y. *Hence* $Tf = 0$ *a.e.* $[m]$.

Definition 1.8. *A sequence* $\{E_i\}$ *of Borel sets in* R^k *is said to* ***shrink to*** x ***nicely*** *if there is a number* $\alpha > 0$ *with the following property: There is a sequence of balls* $B\left(x, r_i\right)$, *with* $\lim r_i = 0$, *such that* $E_i \subset B\left(x, r_i\right)$ *and*

$$m\left(E_i\right) \geq \alpha \cdot m\left(B\left(x, r_j\right)\right)$$

for $i = 1, 2, 3, \ldots$.

Theorem 1.9. *Associate to each $x \in R^k$ a sequence $\{E_i(x)\}$ that shrinks to x nicely, and let $f \in L^1\left(R^k\right)$. Then*

$$f(x) = \lim_{i\to\infty} \frac{1}{m\left(E_i(x)\right)} \int_{E_i(x)} f \, \mathrm{d}m$$

at every Lebesgue point of f, hence a.e. $[m]$.

Proof: Let x be a Lebesgue point of f and let $\alpha(x)$ and $B\left(x, r_j\right)$ be the positive number and the balls that are associated to the sequence $\{E_i(x)\}$. Then, because $E_i(x) \subset B\left(x, r_i\right)$,

$$\frac{\alpha(x)}{m\left(E_i(x)\right)} \int_{E_i(x)} |f - f(x)| \, \mathrm{d}m \leq \frac{1}{m\left(B\left(x, r_i\right)\right)} \int_{B(x, r_i)} |f - f(x)| \, \mathrm{d}m$$

The right side converges to 0 as $i \to \infty$, because $r_i \to 0$ and x is a Lebesgue point of f. Hence the left side converges to 0.

§1.3 Consequence

Theorem 1.10. *If $f \in L^1\left(R^1\right)$ and*

$$F(x) = \int_{-\infty}^{x} f \, \mathrm{d}m$$

then $F'(x) = f(x)$ at every Lebesgue point of f, hence a.e. $[m]$.

Proof: Let $\{\delta_i\}$ be a sequence of positive numbers that converges to 0 . Theorem 7.10 , with $E(x) = [x, x + \delta_i]$, shows then that

$$\frac{F(x + \delta_i) - F(x)}{\delta_i} = \frac{1}{\delta} \int_{E_i(x)} f \, \mathrm{d}m \to f(x)$$

at all Lebesgue points of x of f.

If we let $E_i(x)$ be $[x - \delta_i, x]$ instead, we obtain the same result for the let-hand derivative of F at x.

Theorem 1.11 (Metric density)**.** *Let E be a Lebesgue measurable subset of R^k. The **metric density** of E at a point $x \in R^k$ is defined to be*

$$\lim_{r\to 0} \frac{m(E \cap B(x, r))}{m(B(x, r))}$$

provided, of course, that this limit exists.

If we let f be the characteristic function of E and apply Theorem 7.8 or Theorem 7.10, we see that the metric density of E is 1 at almost every point of E, and that it is 0 at almost every point of the complement of E.

Proposition 1.12. *If $A \subset \mathbb{R}^1$ and $B \subset \mathbb{R}^1$, define $A + B = \{a + b : a \in A, b \in B\}$. Suppose $m(A) > 0, m(B) > 0$. Then $A + B$ contains a segment.*

Proof: There are points a_0 and b_0 where A and B have metric density 1. *Put $c_0 = a_0 + b_0$. We can assume that $a_0 = b_0 = c_0$, and prove that $A + B$ contains segment $(-\delta, \delta)$ for some small $\delta > 0$.*

Question 1.13. *Suppose G is a proper subgroup of $(\mathbb{R}^1, +)$ and G is Lebesgue measurable. Prove that then $m(G) = 0$.*

§2 The Fundamental Theorem for Calcus

§2.1 Bounded Variation

Definition 2.1. *If f is any (complex) function on $I = [a, b]$, define*

$$V_a^b(f) = \sup \sum_{i=1}^{N} |f(t_i) - f(t_{i-1})|$$

where the supremum is taken over all N and over all choices of $\{t_i\}_{i=1}^N$ such that

$$a = t_0 < t_1 < \cdots < t_N = x$$

*If $V_a^b(f) < \infty$ called **total variation function of** f, then f is said to have **bounded variation** on I (briefly, f is BV on I; $f \in BV[a, b]$).*

Proposition 2.2. *Fix $I = [a, b]$,*

(1) Bounded variation function is bounded on I

(2) Monotonic function on I has total variation $|f(b) - f(a)|$

(3) $BV[a, b]$ is a linear space on $\mathbb{R}$ (or $\mathbb{C}$)

(4) $f \in BV[a, b]$ implies $|f| \in BV[a, b]$

(5) for any $c \in (a, b)$

$$V_a^c(f) + V_c^b(f) = V_a^b(f)$$

Theorem 2.3. *Suppose $f : I \to R$ is BV on $I = [a, b]$. Let*

$$F(x) = V_a^x(f) \quad (a \leq x \leq b)$$

The functions $F, F + f, F - f$ are then nondecreasing and BV on I.

Proof: Step 1. If $x < y \leq b$, then

$$F(y) \geq |f(y) - f(x)| + \sum_{i=1}^{N} |f(t_i) - f(t_{i-1})|.$$

Hence $F(y) \geq |f(y) - f(x)| + F(x)$. In particular

$$F(y) \geq f(y) - f(x) + F(x) \text{ and } F(y) \geq f(x) - f(y) + F(x)$$

This proves that $F, F+f, F-f$ are nondecreasing.

Step 2. Since sums of two BV functions are obviously BV, it only remains to be proved that F is BV on I.

Theorem 2.4 (Jordan Decomposition)**.** *$f \in BV[a,b]$ if and only if $f = g - h$ where g and h are nondecreasing function on $[a,b]$*

Differentiable of nondecreasing function

Definition 2.5. *A collection $\mathcal{B}$ of balls $\{B\}$ is said to be a **Vitali covering** of a set E if for every $x \in R$ and any $\varepsilon > 0$ there is a ball $B \in \mathcal{B}$, such that $x \in B$ and $m(B) < \varepsilon$.*

Theorem 2.6 (Vitali covering theorem)**.**

Theorem 2.7 (Lebesgue Theorem)**.** *Nondecreasing function f on $I = [a,b]$ is differentiable a.e. on I. And*

$$\int_{[a,b]} f' \, \mathrm{d}x \leq f(b) - f(a)$$

Proof: Let $E = \{x \in [a,b] : D^+ f(x) > D_- f(x)\}$, we prove that $\mu(E) = 0$. For every pair (p,q), define

$$E_{p,q} = \left\{x \in E : D_- f(x) < p < q < D^+ f(x)\right\}$$

we have

$$E = \bigcup_{p,q \in \mathbb{Q}} E_{p,q}$$

For any $E_{p,q}$ and $x \in E_{p,q}$, there exists arbitrarily small $h > 0$ such that

$$\frac{f(x) - f(x-h)}{h} < p$$

Thus $\{[x-h,x]\}$ is a Vitali covering of E, then there exists finite disjoint sets $\{[x_i - h_i, x_i]\}$ such that

$$m\left(E_{p,q} - \bigcup_i [x_i - h_i, x_i]\right) < \varepsilon$$

We have the estimate for left intervals

$$\sum_i h_i \leq m\left(\bigcup_i [x_i - h_i, x_i]\right) \leq m(E_{p,q}) + \varepsilon$$

it follows that the estimate

$$\sum_i f(x_i) - f(x_i - h_i) < p \sum_i h_i \leq p(m(E_{p,q}) + \varepsilon)$$

Similarly,

$$q(m(E_{p,q} - 2\varepsilon)) \geq q \sum_j k_j < \sum_j f(x_i + k_i) - f(x_i)$$

Corollary 2.8. *Bounded variation functions are differentiable a.e. on I*

§2.2 Absolutely Continuous

Definition 2.9. *A complex function f, defined on an interval $I=[a, b]$, is said to be **absolutely continuous** on I (briefly, f is AC on I) if there corresponds to every $\varepsilon>0$ a $\delta>0$ so that*

$$\sum_{i=1}^{n}\left|f\left(\beta_{i}\right)-f\left(\alpha_{i}\right)\right|<\varepsilon$$

for any n and any disjoint collection of segments $\left(\alpha_{1}, \beta_{1}\right), \ldots,\left(\alpha_{n}, \beta_{n}\right)$ in I whose lengths satisfy

$$\sum_{i=1}^{n}\left(\beta_{i}-\alpha_{i}\right)<\delta .$$

Proposition 2.10. *Fix $I=[a, b]$,*

(1) $AC[a, b] \subset BV[a, b]$

(2) $AC[a, b] \subset C(I)$

(3) $AC[a, b]$ is a linear space over $\mathbb{R}$

(4) If $f \in AC[a, b]$, then $F(x)=V_{a}^{x}(f)$ is nondecreasing and AC .

Proof: If $(a, \beta) \subset I$ then

$$F(\beta)-F(\alpha)=\sup \sum_{i}^{n}\left|f\left(t_{i}\right)-f\left(t_{i-1}\right)\right|,$$

the supremum being taken over all $\left\{t_{i}\right\}$ that satisfy $\alpha=t_{0}<\cdots<t_{n}=\beta$. Note that $\sum\left(t_{i}-t_{i-1}\right)=\beta-\alpha$. Now pick $\varepsilon>0$, associate $\delta>0$ to f and ε , choose disjoint segments $\left(\alpha_{j}, \beta_{j}\right) \subset I$ with $\sum\left(\beta_{j}-\alpha_{j}\right)<\delta$, and apply (5) to each $\left(\alpha_{j}, \beta_{j}\right)$. It follows that

$$\sum_{j}\left(F\left(\beta_{j}\right)-F\left(\alpha_{j}\right)\right) \leq \varepsilon$$

by our choice of δ. Thus F is AC on I.

Theorem 2.11. *Let $I=[a, b]$, let $f: I \rightarrow R^{1}$ be continuous and nondecreasing. Each of the following three statements about fimplies the other two:*

(1) f is AC on I.

(2) Lusin's property. f maps sets of measure 0 to sets of measure 0 .

(3) f is differentiable a.e. on I, $f^{\prime} \in L^{1}$, and

$$f(x)-f(a)=\int_{a}^{x} f^{\prime}(t) \mathrm{d} t \quad(\alpha \leq x \leq b).$$

Proof: $(1) \rightarrow(2)$. *Assume f is AC on I, pick $E \subset I$ so that $E \in \mathcal{M}$ and $m(E)=0$. We have to show that $f(E) \in \mathcal{M}$ and $m(f(E))=0$. Without loss of generality, assume that neither a nor*

b lie in E.

Choose $\varepsilon > 0$. Associate $\delta > 0$ to f and ε. There is then an open set V with $m(V) < \delta$, so that $E \subset V \subset I$. Let (α_i, β_i) be the disjoint segments whose union is V. Then $\sum(\beta_1 - \alpha_i) < \delta$, and our choice of δ shows that therefore

$$\sum_i (f(\beta_i) - f(\alpha_i)) \leq \varepsilon$$

Since $E \subset V, f(E) \subset \bigcup [f(\alpha_i), f(\beta_i)]$, then $m(f(E)) = 0$.

$(2) \to (3)$. *Assume next that (2) holds. Define*

$$g(x) = x + f(x) \quad (a \leq x \leq b).$$

If the f-image of some segment of length η has length η', then the g-image of that same segment has length $\eta + \eta'$. From this it follows easily that g satisfies (2), since f does.

Now suppose $E \subset I, E \in \mathcal{M}$. Then $E = E_1 \cup E_0$ where $m(E_0) = 0$ and E_1 is an F_g. Thus E_1 is a countable union of compact sets, and so is $g(E_1)$, because g is continuous. Since g satisfies $(b), m(g(E_0)) = 0$. Since $g(E) = g(E_1) \cup g(E_0)$, we conclude: $g(E) \in \mathcal{M}$.

Therefore we can define

$$\mu(E) = m(g(E)) \quad (E \subset I, E \in \mathcal{M})$$

Since g is one-to-one, disjoint sets in I have disjoint g-images. The countable additivity of m shows therefore that μ is a positive, bounded measure on $\mathcal{M}$. Also, $\mu \ll m$, because g satisfies (2). Thus

$$d\mu = h \, \mathrm{d}m$$

for some $h \in L^1(m)$, by the Radon-Nikodym theorem. If $E = [a, x]$, then $g(E) = [g(a), g(x)]$, and

$$g(x) - g(a) = m(g(E)) = \mu(E) = \int_E h \, \mathrm{d}m = \int_a^x h(t) \, \mathrm{d}t$$

If we now use $g(x) = f(x) + x$, we conclude that

$$f(x) - f(a) = \int_a^x [h(t) - 1] dt \quad (\alpha \leq x \leq b).$$

Thus $f'(x) = h(x) - 1$ a.e. $[m]$, by Theorem 7.11.

The discussion that preceded Definition 7.17 showed that (3) *implies* (1).

§2.3 Main Objects

Theorem 2.12. *If f is a (complex) function that is AC on $I = [a, b]$, then*

(1) f is differentiable at almost all points of I

(2) $f' \in L^1[a,b]$, and

(3)

$$f(x) - f(a) = \int_a^x f'(t)\,\mathrm{d}t \quad (a \le x \le b).$$

(4) Lusin's property

Proof: It is of course enough to prove this for real f. Let F be its total variation function, define

$$f_1 = \frac{1}{2}(F + f), \quad f_2 = \frac{1}{2}(F - f),$$

and then f_1 and f_2 are nondecreasing and AC. Since

$$f = f_1 - f_2$$

this yields (1).

Theorem 2.13. *If $f : [a,b] \to \mathbb{R}^1$ is differentiable at every point of $[a,b]$ and $f' \in L^1$ on $[a,b]$, then*

$$f(x) - f(a) = \int_a^x f'(t)\,\mathrm{d}t \quad (a \le x \le b)$$

Proof: It is clear that it is enough to prove this for $x = b$. Fix $\varepsilon > 0$. Theorem 2.25 ensures the existence of a lower semicontinuous function g on $[a,b]$ such that $f' < g$ and

$$\int_a^b g(t)dt < \int_a^b f'(t)dt + \varepsilon.$$

For any $\eta > 0$, define

$$F_\eta(x) = \int_a^x g(t)\,\mathrm{d}t - [f(x) - f(a)] + \eta(x - a) \quad (a \le x \le b).$$

To each $x \in [a,b)$ there corresponds a $\delta_x > 0$ such that

$$g(t) > f'(x) \text{ and } \frac{f(t) - f(x)}{t - x} < f'(x) + \eta$$

for all $t \in (x, x + \delta_x)$, since g is lower semicontinuous and $g(x) > f'(x)$. For any such t we therefore have

$$\begin{aligned} F_\eta(t) - F_\eta(x) &= \int_x^t g(s)\,\mathrm{d}s - [f(t) - f(x)] + \eta(t - x) \\ &> (t - x)f'(x) - (t - x)\,[f'(x) + \eta] + \eta(t - x) \\ &= 0 \end{aligned}$$

Thus F_η is increasing on $[a, b)$. Since $F_\eta(a) = 0$ and F_η is continuous

$$F_\eta(b) = \int_a^b g(t)\,\mathrm{d}t - [f(b) - f(a)] + \eta(b-a) \geq 0$$

Since this holds for every $\eta > 0$, we have

$$f(b) - f(a) \leq \int_a^b g(t)dt < \int_a^b f'(t)dt + \varepsilon,$$

and since ε was arbitrary, we conclude that

$$f(b) - f(a) \leq \int_a^b f'(t)\,\mathrm{d}t$$

Similarly, we conclude that

$$-f(b) - [-f(a)] \leq \int_a^b -f'(t)\,\mathrm{d}t$$

Finally,

$$f(b) - f(a) = \int_a^b f'(t)\,\mathrm{d}t$$

§3 Differentiable Transforms

Theorem 3.1. *Suppose that*

(i) $V \subset R^k$, *V is open, and $T : V \to R^k$ is continuous;*

(ii) *$X \subset V$ is Lebesgue measurable, T is one-to-one on X, and T is differentiable at every point of X;*

(iii) $m(T(V - X)) = 0$.

Then, setting $Y = T(X)$,

$$\int_Y f\,\mathrm{d}m = \int_X (f \circ T)\,|J_T|\;\mathrm{d}m$$

for every measurable $f : R^k \to [0, \infty]$.

Corollary 3.2. *Suppose $\varphi : [a, b] \to [\alpha, \beta]$ is AC, monotonic, $\varphi(a) = \alpha, \varphi(b) = \beta$, and $f \geq 0$ is Lebesgue measurable. Then*

$$\int_a^\beta f(t)\,\mathrm{d}t = \int_a^b f(\varphi(x))\varphi'(x)\,\mathrm{d}x$$

§4 Problem

Question 4.1. *Suppose that E is a measurable subset of $\mathbb{R}$ with arbitrarily small periods. Prove that then either E or its complement has measure 0.*

Proof: Pick $\alpha \in \mathbb{R}^1$ and period p_i, put $F(x) = m\left(E \cap [\alpha, x]\right)$ for $x > \alpha$, show that

$$F\left(x + p_i\right) - F\left(x - p_i\right) = F\left(y + p_i\right) - F\left(y - p_j\right)$$

if $\alpha + p_i < x < y$. It follows that $F'(x) = C$ a.e. on $\mathbb{R}$.

Question 4.2. *Suppose f is a real Lebesgue measurable function with periods s and t whose quotient s/t is irrationial. Prove that there is a constant c such that $f(x) = c$ a.e. on $\mathbb{R}$.*

Hint: Apply Exercise 3 to the sets $\{f > \lambda\}$.