



MATRICES

Introduction

Nyimas Dewi Sartika
Universitas Multimedia Nusantara
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DEFINITION

Matrix algebra has at least two advantages:

- Reduces complicated systems of equations to simple expressions
- Adaptable to systematic method of mathematical treatment and well suited to computers

Definition:

A matrix is a set or group of numbers arranged in a square or rectangular array enclosed by two brackets

$$\begin{bmatrix} 1 & -1 \end{bmatrix} \quad \begin{bmatrix} 4 & 2 \\ -3 & 0 \end{bmatrix} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

A matrix is denoted by a bold capital letter and the elements within the matrix are denoted by lower case letters

Matrix $[\mathbf{A}]$ with elements a_{ij}

$$\mathbf{A}_{m \times n} = \begin{bmatrix} a_{11} & a_{12} \cdots & a_{ij} & a_{in} \\ a_{21} & a_{22} \cdots & a_{ij} & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{ij} & a_{mn} \end{bmatrix}$$

i goes from 1 to m
j goes from 1 to n

PROPERTIES

Properties:

- A specified number of rows and a specified number of columns
- Two numbers (rows x columns) describe the dimensions or size of the matrix

Examples:

$$\begin{array}{l} 3 \times 3 \text{ matrix } \begin{bmatrix} 1 & 2 & 4 \\ 4 & -1 & 5 \\ 3 & 3 & 3 \end{bmatrix} \\ 2 \times 4 \text{ matrix } \begin{bmatrix} 1 & 1 & 3 & -3 \\ 0 & 0 & 3 & 2 \end{bmatrix} \\ 1 \times 2 \text{ matrix } \begin{bmatrix} 1 & -1 \end{bmatrix} \end{array}$$

TYPES OF MATRICES

1. Column matrix or vector

The number of rows may be any integer but the number of columns is always 1

$$\begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$$

2. Row matrix or vector

Any number of columns but only one row

$$\begin{bmatrix} 1 & 1 & 6 \end{bmatrix} \quad \begin{bmatrix} 0 & 3 & 5 & 2 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \cdots a_{1n} \end{bmatrix}$$

3. Rectangular matrix

Contains more than one element and number of rows is not equal to the number of columns

$$\begin{bmatrix} 1 & 1 \\ 3 & 7 \\ 7 & -7 \\ 7 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 2 & 0 & 3 & 3 & 0 \end{bmatrix}$$

$$m \neq n$$

4. Square matrix

The number of rows is equal to the number of columns (a square matrix $\mathbf{A}_{m \times m}$ has an order of m)

$$\begin{bmatrix} 1 & 1 \\ 3 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 1 \\ 9 & 9 & 0 \\ 6 & 6 & 1 \end{bmatrix}$$

The principal or main diagonal of a square matrix is composed of all elements a_{ij} for which $i = j$

5. Diagonal matrix

A square matrix where all the elements are zero except those on the main diagonal

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix}$$

$$a_{ij} = 0 \text{ for all } i \neq j$$

$$a_{ij} \neq 0 \text{ for some or all } i = j$$

6. Unit or Identity matrix - I

A diagonal matrix with ones on the main diagonal

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a_{ij} & 0 \\ 0 & a_{ij} \end{bmatrix}$$

$$a_{ij} = 0 \text{ for all } i \neq j$$

$$a_{ij} = 1 \text{ for all } i = j$$

7. Null (zero) matrix - 0

All elements in the matrix are zero

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$a_{ij} = 0 \quad \text{for all } i, j$$

8. Triangular matrix

A square matrix whose elements above or below the main diagonal are all zero

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 8 & 9 \\ 0 & 1 & 6 \\ 0 & 0 & 3 \end{bmatrix}$$

8a. Upper triangular matrix

A square matrix whose elements below the main diagonal are all zero

$$\begin{bmatrix} a_{ij} & a_{ij} & a_{ij} \\ 0 & a_{ij} & a_{ij} \\ 0 & 0 & a_{ij} \end{bmatrix} \quad \begin{bmatrix} 1 & 8 & 7 \\ 0 & 1 & 8 \\ 0 & 0 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 7 & 4 & 4 \\ 0 & 1 & 7 & 4 \\ 0 & 0 & 7 & 8 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$$a_{ij} = 0 \text{ for all } i > j$$

8b. Lower triangular matrix

A square matrix whose elements above the main diagonal are all zero

$$\begin{bmatrix} a_{ij} & 0 & 0 \\ a_{ij} & a_{ij} & 0 \\ a_{ij} & a_{ij} & a_{ij} \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix}$$

$$a_{ij} = 0 \text{ for all } i < j$$

9. Scalar matrix

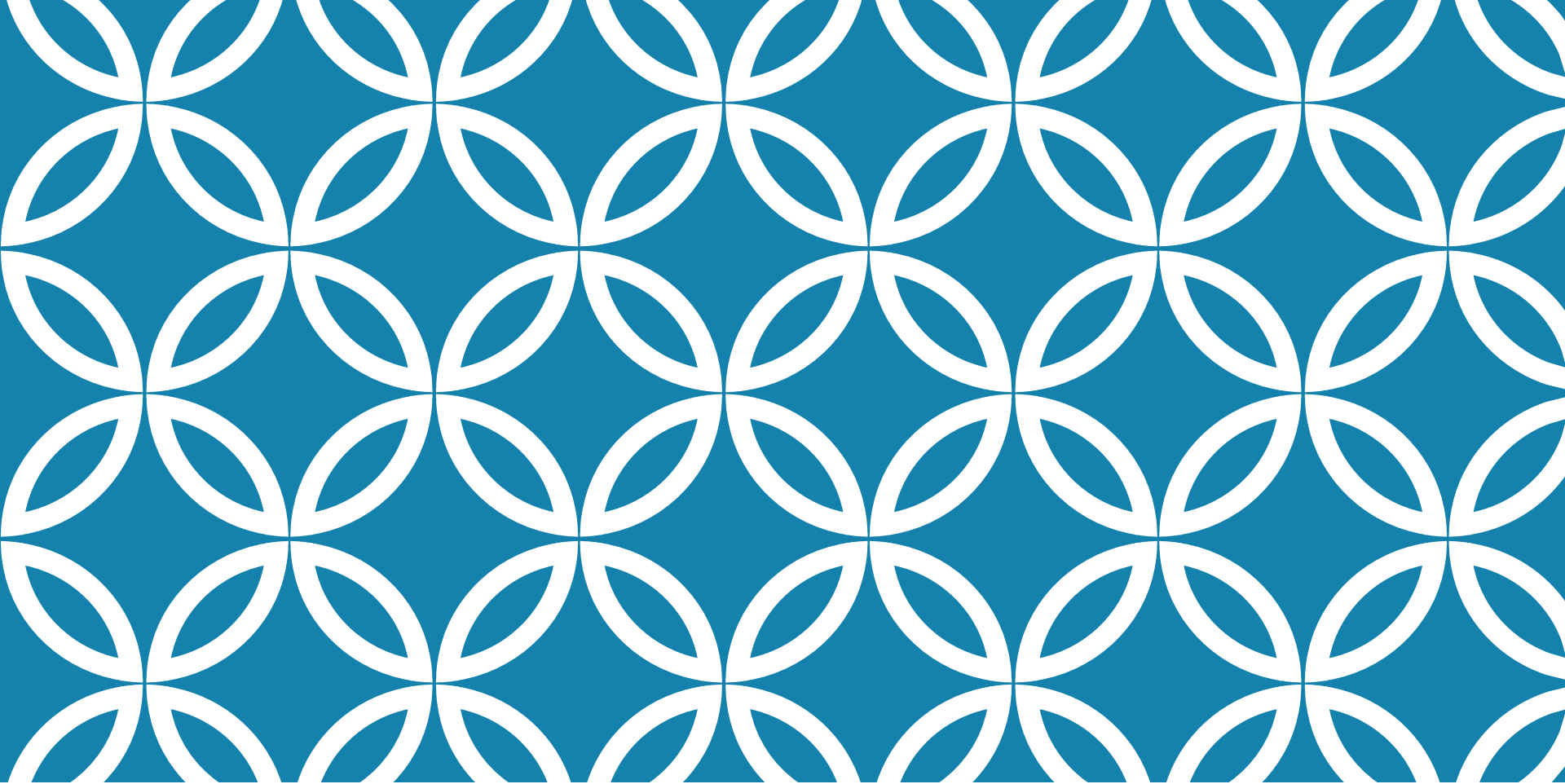
A diagonal matrix whose main diagonal elements are equal to the same scalar

A scalar is defined as a single number or constant

$$\begin{bmatrix} a_{ij} & 0 & 0 \\ 0 & a_{ij} & 0 \\ 0 & 0 & a_{ij} \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

$$a_{ij} = 0 \text{ for all } i \neq j$$

$$a_{ij} = a \text{ for all } i = j$$



MATRICES Operations

Nyimas DS - UMN

Nyimas Dewi Sartika
Universitas Multimedia Nusantara
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EQUALITY OF MATRICES

Two matrices are said to be equal only when all corresponding elements are equal

Therefore their size or dimensions are equal as well

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix} \quad \mathbf{A} = \mathbf{B}$$

Some properties of equality:

- If $\mathbf{A} = \mathbf{B}$, then $\mathbf{B} = \mathbf{A}$ for all \mathbf{A} and \mathbf{B}
- If $\mathbf{A} = \mathbf{B}$, and $\mathbf{B} = \mathbf{C}$, then $\mathbf{A} = \mathbf{C}$ for all \mathbf{A} , \mathbf{B} and \mathbf{C}

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 2 & 3 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$\text{If } \mathbf{A} = \mathbf{B} \text{ then } a_{ij} = b_{ij}$$

ADDITION AND SUBTRACTION OF MATRICES

The sum or difference of two matrices, **A** and **B** of the same size yields a matrix **C** of the same size

$$c_{ij} = a_{ij} + b_{ij}$$

Matrices of different sizes cannot be added or subtracted

Commutative Law:

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$$

Associative Law:

$$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + \mathbf{B} + \mathbf{C}$$

$$\begin{bmatrix} 7 & 3 & -1 \\ 2 & -5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 5 & 6 \\ -4 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 8 & 8 & 5 \\ -2 & -7 & 9 \end{bmatrix}$$

A
2x3

B
2x3

C
2x3

$$\mathbf{A} + \mathbf{0} = \mathbf{0} + \mathbf{A} = \mathbf{A}$$

$$\mathbf{A} + (-\mathbf{A}) = \mathbf{0}$$

(where $-\mathbf{A}$ is the matrix composed of $-a_{ij}$ as elements)

$$\begin{bmatrix} 6 & 4 & 2 \\ 3 & 2 & 7 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 5 & 2 & 2 \\ 2 & 2 & -1 \end{bmatrix}$$

SCALAR MULTIPLICATION OF MATRICES

Matrices can be multiplied by a scalar (constant or single element)

Let k be a scalar quantity; then $kA = Ak$

$$\text{If } k = 4 \text{ and } A = \begin{bmatrix} 3 & -1 \\ 2 & 1 \\ 2 & -3 \\ 4 & 1 \end{bmatrix}$$

$$4 \times \begin{bmatrix} 3 & -1 \\ 2 & 1 \\ 2 & -3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 2 & 1 \\ 2 & -3 \\ 4 & 1 \end{bmatrix} \times 4 = \begin{bmatrix} 12 & -4 \\ 8 & 4 \\ 8 & -12 \\ 16 & 4 \end{bmatrix}$$

Properties:

- $k(\mathbf{A} + \mathbf{B}) = k\mathbf{A} + k\mathbf{B}$
- $(k + g)\mathbf{A} = k\mathbf{A} + g\mathbf{A}$
- $k(\mathbf{AB}) = (k\mathbf{A})\mathbf{B} = \mathbf{A}(k)\mathbf{B}$
- $k(g\mathbf{A}) = (kg)\mathbf{A}$

MULTIPLICATION OF MATRICES


The product of two matrices is another matrix

Two matrices **A** and **B** must be **conformable** for multiplication to be possible

i.e. the number of columns of **A** must equal the number of rows of **B**

Example.

$$\begin{array}{ccccc} \mathbf{A} & \times & \mathbf{B} & = & \mathbf{C} \\ (1 \times 3) & & (3 \times 1) & & (1 \times 1) \end{array}$$


$$\begin{matrix} \mathbf{B} & \times & \mathbf{A} & = & \text{Not possible!} \\ (2 \times 1) & & (4 \times 2) \end{matrix}$$

$$\begin{matrix} \mathbf{A} & \times & \mathbf{B} & = & \text{Not possible!} \\ (6 \times 2) & & (6 \times 3) \end{matrix}$$

Example

$$\begin{matrix} \mathbf{A} & \times & \mathbf{B} & = & \mathbf{C} \\ (2 \times 3) & & (3 \times 2) & & (2 \times 2) \end{matrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

$$(a_{11} \times b_{11}) + (a_{12} \times b_{21}) + (a_{13} \times b_{31}) = c_{11}$$

$$(a_{11} \times b_{12}) + (a_{12} \times b_{22}) + (a_{13} \times b_{32}) = c_{12}$$

$$(a_{21} \times b_{11}) + (a_{22} \times b_{21}) + (a_{23} \times b_{31}) = c_{21}$$

$$(a_{21} \times b_{12}) + (a_{22} \times b_{22}) + (a_{23} \times b_{32}) = c_{22}$$

Successive multiplication of row i of **A** with column j of **B** – row by column multiplication

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & 7 \end{bmatrix} \begin{bmatrix} 4 & 8 \\ 6 & 2 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} (1 \times 4) + (2 \times 6) + (3 \times 5) & (1 \times 8) + (2 \times 2) + (3 \times 3) \\ (4 \times 4) + (2 \times 6) + (7 \times 5) & (4 \times 8) + (2 \times 2) + (7 \times 3) \end{bmatrix}$$

$$= \begin{bmatrix} 31 & 21 \\ 63 & 57 \end{bmatrix}$$

Remember also:

$$\mathbf{IA} = \mathbf{A}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 31 & 21 \\ 63 & 57 \end{bmatrix} = \begin{bmatrix} 31 & 21 \\ 63 & 57 \end{bmatrix}$$

Assuming that matrices **A**, **B** and **C** are conformable for the operations indicated, the following are true:

1. $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$
2. $\mathbf{A(BC)} = (\mathbf{AB})\mathbf{C} = \mathbf{ABC}$ - (associative law)
3. $\mathbf{A(B+C)} = \mathbf{AB} + \mathbf{AC}$ - (first distributive law)
4. $(\mathbf{A+B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$ - (second distributive law)

Caution!

1. \mathbf{AB} not generally equal to \mathbf{BA} , \mathbf{BA} may not be conformable
2. If $\mathbf{AB} = \mathbf{0}$, neither \mathbf{A} nor \mathbf{B} necessarily $= \mathbf{0}$
3. If $\mathbf{AB} = \mathbf{AC}$, \mathbf{B} not necessarily $= \mathbf{C}$

AB not generally equal to **BA**
BA may not be conformable

$$T = \begin{bmatrix} 1 & 2 \\ 5 & 0 \end{bmatrix}$$

$$S = \begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix}$$

$$TS = \begin{bmatrix} 1 & 2 \\ 5 & 0 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 15 & 20 \end{bmatrix}$$

$$ST = \begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 5 & 0 \end{bmatrix} = \begin{bmatrix} 23 & 6 \\ 10 & 0 \end{bmatrix}$$

If $\mathbf{AB} = \mathbf{0}$, neither \mathbf{A} nor \mathbf{B} necessarily $= \mathbf{0}$

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

TRANSPOSE OF A MATRIX

$$A = \begin{bmatrix} 2 & 4 & 7 \\ 5 & 3 & 1 \end{bmatrix}$$

Transpose of A, denoted A^T is:

$$A^T = \begin{bmatrix} 2 & 5 \\ 4 & 3 \\ 7 & 1 \end{bmatrix}$$

$$a_{ij} = a_{ji}^T \quad \text{for all } i \text{ and } j$$

To transpose:

Interchange rows and columns

The dimensions of \mathbf{A}^T are the reverse of the dimensions of \mathbf{A}

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & 7 \\ 5 & 3 & 1 \end{bmatrix} \quad 2 \times 3$$

$$\mathbf{A}^T = \begin{bmatrix} 2 & 5 \\ 4 & 3 \\ 7 & 1 \end{bmatrix} \quad 3 \times 2$$

Properties of transposed matrices:

1. $(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$

2. $(\mathbf{AB})^T = \mathbf{B}^T \mathbf{A}^T$

3. $(k\mathbf{A})^T = k\mathbf{A}^T$

4. $(\mathbf{A}^T)^T = \mathbf{A}$

$$(\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$$

$$\begin{bmatrix} 7 & 3 & -1 \\ 2 & -5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 5 & 6 \\ -4 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 8 & 8 & 5 \\ -2 & -7 & 9 \end{bmatrix} \rightarrow \begin{bmatrix} 8 & -2 \\ 8 & -7 \\ 5 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 7 & 2 \\ 3 & -5 \\ -1 & 6 \end{bmatrix} + \begin{bmatrix} 1 & -4 \\ 5 & -2 \\ 6 & 3 \end{bmatrix} = \begin{bmatrix} 8 & -2 \\ 8 & -7 \\ 5 & 9 \end{bmatrix}$$

$$(\mathbf{AB})^T = \mathbf{B}^T \mathbf{A}^T$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \end{bmatrix} \Rightarrow [2 \quad 8]$$

$$\begin{bmatrix} 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 0 & 3 \end{bmatrix} = [2 \quad 8]$$

A square matrix is symmetric if it is equal to its transpose:

$$\mathbf{A} = \mathbf{A}^T$$

$$A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$$
$$A^T = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$$

When the original matrix is square, transposition does not affect the elements of the main diagonal

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
$$A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

The identity matrix, **I**, a diagonal matrix **D**, and a scalar matrix, **K**, are equal to their transpose since the diagonal is unaffected.

INVERSE OF A MATRIX

Consider a scalar k . The inverse is the reciprocal or division of 1 by the scalar.

Example:

$K = 7$ the inverse of k or $k^{-1} = 1/k = 1/7$

Division of matrices is not defined since there may be $\mathbf{AB} = \mathbf{AC}$ while $\mathbf{B} \neq \mathbf{C}$

Instead matrix inversion is used.

The inverse of a square matrix, \mathbf{A} , if it exists, is the unique matrix \mathbf{A}^{-1} where:

$$\mathbf{AA}^{-1} = \mathbf{A}^{-1} \mathbf{A} = \mathbf{I}$$

Example:

$$A = A^2 = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$$

Because:

$$\begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Properties of the inverse:

$$(AB)^{-1} = B^{-1}A^{-1}$$

$$(A^{-1})^{-1} = A$$

$$(A^T)^{-1} = (A^{-1})^T$$

$$(kA)^{-1} = \frac{1}{k} A^{-1}$$

A square matrix that has an inverse is called a nonsingular matrix

A matrix that does not have an inverse is called a singular matrix

Square matrices have inverses except when the determinant is zero

When the determinant of a matrix is zero the matrix is singular

DETERMINANT OF A MATRIX

To compute the inverse of a matrix, the determinant is required

Each square matrix \mathbf{A} has a unit scalar value called the determinant of \mathbf{A} , denoted by $\det \mathbf{A}$ or $|\mathbf{A}|$

$$\text{If } A = \begin{bmatrix} 1 & 2 \\ 6 & 5 \end{bmatrix}$$

$$\text{then } |A| = \begin{vmatrix} 1 & 2 \\ 6 & 5 \end{vmatrix}$$

If $\mathbf{A} = [\mathbf{A}]$ is a single element (1x1), then the determinant is defined as the value of the element

$$\text{Then } |\mathbf{A}| = \det \mathbf{A} = a_{11}$$

If \mathbf{A} is (n x n), its determinant may be defined in terms of order (n-1) or less.

MINORS

If \mathbf{A} is an $n \times n$ matrix and one row and one column are deleted, the resulting matrix is an $(n-1) \times (n-1)$ submatrix of \mathbf{A} .

The determinant of such a submatrix is called a minor of \mathbf{A} and is designated by m_{ij} , where i and j correspond to the deleted row and column, respectively.

m_{ij} is the minor of the element a_{ij} in \mathbf{A} .

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Each element in A has a minor

Delete first row and column from A .

The determinant of the remaining 2 x 2 submatrix
is the minor of a_{11}

$$m_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

Therefore the minor of a_{12} is:

$$m_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

And the minor for a_{13} is:

$$m_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

COFACTORS

The cofactor C_{ij} of an element a_{ij} is defined as:

$$C_{ij} = (-1)^{i+j} m_{ij}$$

When the sum of a row number i and column j is even, $c_{ij} = m_{ij}$ and when $i+j$ is odd, $c_{ij} = -m_{ij}$

$$c_{11}(i = 1, j = 1) = (-1)^{1+1} m_{11} = +m_{11}$$

$$c_{12}(i = 1, j = 2) = (-1)^{1+2} m_{12} = -m_{12}$$

$$c_{13}(i = 1, j = 3) = (-1)^{1+3} m_{13} = +m_{13}$$

DETERMINANT

The determinant of an $n \times n$ matrix \mathbf{A} can now be defined as

$$|A| = \det A = a_{11}c_{11} + a_{12}c_{12} + \dots + a_{1n}c_{1n}$$

The determinant of \mathbf{A} is therefore the sum of the products of the elements of the first row of \mathbf{A} and their corresponding cofactors.

(It is possible to define $|\mathbf{A}|$ in terms of any other row or column but for simplicity, the first row only is used)

Therefore the 2 x 2 matrix :

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$c_{11} = m_{11} = |a_{22}| = a_{22}$$

$$c_{12} = -m_{12} = -|a_{21}| = -a_{21}$$

And the determinant of **A** is:

$$|A| = a_{11}c_{11} + a_{12}c_{12} = a_{11}a_{22} - a_{12}a_{21}$$

Example 1:

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$$

$$|A| = (3)(2) - (1)(1) = 5$$

For a 3 x 3 matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The cofactors of the first row are:

$$c_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{23}a_{32}$$

$$c_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = -(a_{21}a_{33} - a_{23}a_{31})$$

$$c_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}a_{32} - a_{22}a_{31}$$

The determinant of a matrix A is:

$$|A| = a_{11}c_{11} + a_{12}c_{12} = a_{11}a_{22} - a_{12}a_{21}$$

Which by substituting for the cofactors in this case is:

$$|A| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Example 2:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ -1 & 0 & 1 \end{bmatrix}$$

$$|A| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$|A| = (1)(2 - 0) - (0)(0 + 3) + (1)(0 + 2) = 4$$

ADJOINT MATRICES

A cofactor matrix **C** of a matrix **A** is the square matrix of the same order as **A** in which each element a_{ij} is replaced by its cofactor c_{ij} .

Example:

$$A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$$

The cofactor C of A is

$$C = \begin{bmatrix} 4 & 3 \\ -2 & 1 \end{bmatrix}$$

The adjoint matrix of \mathbf{A} , denoted by $\text{adj } \mathbf{A}$, is the transpose of its cofactor matrix

$$\text{adj}A = C^T$$

It can be shown that: $\mathbf{A}(\text{adj } \mathbf{A}) = (\text{adj } \mathbf{A}) \mathbf{A} = |\mathbf{A}| \mathbf{I}$

Example:

$$A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$$

$$|A| = (1)(4) - (2)(-3) = 10$$

$$\text{adj}A = C^T = \begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix}$$

$$A(adjA) = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} = 10I$$

$$(adjA)A = \begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} = 10I$$

USING THE ADJOINT MATRIX IN MATRIX INVERSION

Since

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

and

$$\mathbf{A}(\text{adj } \mathbf{A}) = (\text{adj } \mathbf{A})\mathbf{A} = |\mathbf{A}| \mathbf{I}$$

then

$$\mathbf{A}^{-1} = \frac{\text{adj } \mathbf{A}}{|\mathbf{A}|}$$

Example

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{10} \begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 0.4 & -0.2 \\ 0.3 & 0.1 \end{bmatrix}$$

To check $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$

$$\mathbf{A}\mathbf{A}^{-1} = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 0.4 & -0.2 \\ 0.3 & 0.1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$$

$$\mathbf{A}^{-1}\mathbf{A} = \begin{bmatrix} 0.4 & -0.2 \\ 0.3 & 0.1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$$

Example

$$A = \begin{bmatrix} 3 & -1 & 1 \\ 2 & 1 & 0 \\ 1 & 2 & -1 \end{bmatrix}$$

The determinant of \mathbf{A} is

$$|\mathbf{A}| = (3)(-1-0) - (-1)(-2-0) + (1)(4-1) = -2$$

The elements of the cofactor matrix are

$$\begin{array}{lll} c_{11} = +(-1), & c_{12} = -(-2), & c_{13} = +(3), \\ c_{21} = -(-1), & c_{22} = +(-4), & c_{23} = -(7), \\ c_{31} = +(-1), & c_{32} = -(-2), & c_{33} = +(5), \end{array}$$

The cofactor matrix is therefore

$$C = \begin{bmatrix} -1 & 2 & 3 \\ 1 & -4 & -7 \\ -1 & 2 & 5 \end{bmatrix}$$

$$\text{adj}A = C^T = \begin{bmatrix} -1 & 1 & -1 \\ 2 & -4 & 2 \\ 3 & -7 & 5 \end{bmatrix}$$

$$A^{-1} = \frac{\text{adj}A}{|A|} = \frac{1}{-2} \begin{bmatrix} -1 & 1 & -1 \\ 2 & -4 & 2 \\ 3 & -7 & 5 \end{bmatrix} = \begin{bmatrix} 0.5 & -0.5 & 0.5 \\ -1.0 & 2.0 & -1.0 \\ -1.5 & 3.5 & -2.5 \end{bmatrix}$$

The result can be checked using

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

The determinant of a matrix must not be zero for the inverse to exist as there will not be a solution

Nonsingular matrices have non-zero determinants

Singular matrices have zero determinants