# LOGICS

# Logics

- Logic = the study of correct reasoning.
- Use of logic
  - In mathematics: to prove theorems.
  - In computer science: to prove that programs do what they are supposed to do.

# Proposition Logic

A declarative sentence or statement that is either true or false, but not both.

Examples:

- Jakarta is the capital of Indonesia (proposition)
- $2 + 2 = 5$ (proposition)
- $x + 1 = 2$ (not proposition)
- What time is it? (not proposition)
- Read this carefully (not proposition)

# Proposition Logic

- Propositional variables: p, q, r, s
- Truth value: true (T) or false (F)
- Truth table: displays the relationship between the truth values of propositions.
  - Truth table is useful as a visual display for the workings of a logical operator
  - Truth table can also be used to determine the truth value of a compound proposition based on the truth values of its component propositions.

# Compound Proposition

- Compound propositions: news propositions formed from existing propositions using logical operators (connectives)
- Most common connectives:

| Connectives | Symbol |
|---|---|
| Conjunction | $\wedge$ |
| Inclusive disjunction | $\vee$ |
| Exclusive disjunction | $\oplus$ / $\underline{\vee}$ |
| Negation | $\neg$ / $\sim$ |
| Implication | $\rightarrow$ |
| Double implication | $\leftrightarrow$ |

# Compound Proposition

By convention:

| Logical Operator | Precedence |
|---|---|
| $\neg$ | 1 |
| $\wedge$ | 2 |
| $\vee$ | 3 |
| $\rightarrow$ | 4 |
| $\leftrightarrow$ | 5 |

Examples:

- $\neg p \wedge q \rightarrow r$ is equivalent to $((\neg p) \wedge q) \rightarrow r$
- $p \leftrightarrow q \rightarrow r \wedge s$ is equivalent to $p \leftrightarrow (q \rightarrow (r \wedge s))$

# Negation

| $p$ | $\neg p$ |
|---|---|
| T | F |
| F | T |

Negation:

$\neg$ p is false when p is true,

$\neg$ p is true when p is false.

Examples:

p: Today is Friday

Negation ($\neg$ p) means:

- ❖ It is not the case that today is Friday
- ❖ Today is not Friday

# Conjunction

| $p$ | $q$ | $p \wedge q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | F |

Conjunction: $p \wedge q$ is true when both p and q are true. False otherwise.

Examples:

p: Today is Friday, q: $1 + 1 = 2$

- ❖ Today is Friday and $1 + 1 = 2$ (T)
- ❖ Today is Friday and $1 + 1 \neq 2$ (F)
- ❖ Today is not Friday and $1 + 1 = 2$ (F)
- ❖ Today is not Friday and $1 + 1 \neq 2$ (F)

# Disjunction

| $p$ | $q$ | $p \vee q$ |
|---|---|---|
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | F |

Disjunction: $p \vee q$ is false when both p and q are false.

True otherwise.

Examples:

p: Today is Friday, q: $1 + 1 = 2$

- Today is Friday or $1 + 1 = 2$ (T)
- Today is Friday or $1 + 1 \neq 2$ (T)
- Today is not Friday or $1 + 1 = 2$ (T)
- Today is not Friday or $1 + 1 \neq 2$ (F)

# Exclusive Disjunction

| $p$ | $q$ | $p \oplus q$ |
|---|---|---|
| T | T | F |
| T | F | T |
| F | T | T |
| F | F | F |

Exclusive disjunction: $p \oplus q$ is true when exactly one of p, q is true. False otherwise.

Examples:

p: Today is Friday, q: 1 + 1 = 2

- Today is Friday xor 1 + 1 = 2 (F)
- Today is Friday xor 1 + 1 ≠ 2 (T)
- Today is not Friday xor 1 + 1 = 2 (T)
- Today is not Friday xor 1 + 1 ≠ 2 (F)

# Conditional Statement

| $p$ | $q$ | $p \to q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

Conditional statement: if p, then q

- p: premise (or antecedent)
- q: conclusion (or consequent)
- $p \to q$ is false when p is true and q is false.
- True otherwise

Examples:

p: you get a 100 on the final, q: you will get an A in the class

- This statement is only violated (False) if you do get a 100 on the final but you do not get an A in the class. This corresponds to p being true while q is false.
- Consider the 3 other scenarios where this statement is satisfied:
  - p:T, q:T: you get a 100 on the final and you get an A in the class.
  - p:F, q:T: you do not get a 100 on the final but you do get an A.
  - p:F, q:F: you do not get a 100 on the final and you do not get an A.
- All the original statement does (if I were to say it to you) is to rule out the possibility of getting a 100 on the final but not getting an A. The other 3 scenarios can all occur with my original statement still being satisfied.

# Different ways to Expressing $p \to q$

| | | |
|---|---|---|
| If p, then q | p implies q | p is sufficient for q |
| if p, q | q unless $\neg$ p | q is necessary for p |
| q if p | q follows from p | a necessary condition for p is q |
| p only if q | q when p | a sufficient condition for q is p |

Examples:

p: You get a 100 on the final, q: you will get an A in the class

- If you get a 100 on the final, then you will get an A in the class (if p, then q)
- If you get a 100 on the final, you will get an A in the class (if p, q)
- You will get an A in the class, if you get a 100 on the final (q if p)
- You get a 100 on the final only if you will get an A in the class (p only if q)
- You get a 100 on the final implies you will get an A in the class (p implies q)

# Converse, Inverse and Contrapositive

- Conditional statements:
  - Implication : $p \rightarrow q$
  - Converse : $q \rightarrow p$
  - Inverse : $\neg p \rightarrow \neg q$
  - Contrapositive : $\neg q \rightarrow \neg p$

- Implication and contrapositive are equivalent
- Converse and inverse are equivalent

| $p$ | $q$ | $\neg p$ | $\neg q$ | $p \rightarrow q$ | $q \rightarrow p$ | $\neg p \rightarrow \neg q$ | $\neg q \rightarrow \neg p$ |
|---|---|---|---|---|---|---|---|
| T | T | F | F | T | T | T | T |
| T | F | F | T | F | T | T | F |
| F | T | T | F | T | F | F | T |
| F | F | T | T | T | T | T | T |

# Converse, Inverse and Contrapositive

- The converse of $p \rightarrow q$ is $q \rightarrow p$.
- "If you get an A in this class, then you get a 100 on the final" is the converse of our previous example.
- The only way this converse is violated is if you get an A in the class, but don't get a 100 on the final.
- So now, the only way to get an A in the class is to get a 100 on the final (and even getting a 100 on the final doesn't guarantee you an A in the class).

# Converse, Inverse and Contrapositive

- The inverse of $p \rightarrow q$ is $\neg p \rightarrow \neg q$.
- "If you don't get a 100 on the final, then you don't get an A in the class"
- This statement means the same thing as the converse.
- Basically to be eligible to get an A in the class, you must get a 100 on the final (again, a 100 on the final doesn't guarantee an A in the class).

# Converse, Inverse and Contrapositive

- The contrapositive of $p \rightarrow q$ is $\neg q \rightarrow \neg p$.
- "If you don't get an A in this class, then you don't get a 100 on the final" is the contrapositive of our original example.
- This statement means the same thing as the original implication, $p \rightarrow q$.
- That is, if you don't end up with an A in the class, then you didn't get a 100 on the final (original: getting a 100 on the final implies you get an A in the class).

# Biconditional Statement (Bi-implication)

| $p$ | $q$ | $p \leftrightarrow q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | T |

Biconditional statement: p if and only if q

- $p \leftrightarrow q$ means: $p \rightarrow q$ and $q \rightarrow p$
- $p \leftrightarrow q$ is true when p , q have the same truth value.
- False otherwise

| | |
|---|---|
| p if and only if q | p iff q |
| If p then q, and conversely | p is necessary and sufficient for q |

Example:

p: you can take the flight, q: you buy a ticket

- ❖ $p \leftrightarrow q$: you can take the flight if and only if you buy a ticket
- ❖ This statement is true:
  - If you buy a ticket and take the flight
  - If you do not buy a ticket and you cannot take the flight

# Translating English Sentences

- Often we need to translate a statement in English into a compound proposition involving symbols and logical connectives.
- English statements are often ambiguous so we translate them in order to specify precisely what we mean and to be able to reason.
- Due to the ambiguity in an English statement, there is often more than one way to translate it into a logical proposition.
- Steps to convert an English sentence to a statement in propositional logic
  - Identify atomic propositions and represent using propositional variables.
  - Determine appropriate logical connectives

# Translating English Sentences

Example:

“You can access the Internet from campus only if you are a computer science major or you are not a freshman.”

a : you can access the internet from campus

c : you are a computer science major

f : you are a freshman

# Logic Fuzzle

Example:

- An island has two kinds of inhabitants: knights, who always tell the truth, and knaves, who always lie.
- You go to the island and meet A and B.
  - A says "B is a knight."
  - B says "The two of us are of opposite types."
- What are the types of A and B?
- Let p and q be the statements that A is a knight and B is a knight, respectively.
- So, then $\neg p$ represents the proposition that A is a knave and $\neg q$ that B is a knave.
  - If A is a knight, then p is true. Since knights tell the truth, q must also be true. Then $(p \wedge \neg q) \vee (\neg p \wedge q)$ would have to be true, but it is not. So, A is not a knight and therefore $\neg p$ must be true.
  - If A is a knave, then B must not be a knight since knaves always lie. So, then both $\neg p$ and $\neg q$ hold since both are knaves.

| $p$ | $q$ | $\neg p$ | $\neg q$ | $\neg p \wedge q$ | $p \wedge \neg q$ | $(\neg p \wedge q) \vee (p \wedge \neg q)$ |
|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 0 | 0 | 0 |
| 0 | 1 | 1 | 0 | 1 | 0 | 1 |
| 1 | 0 | 0 | 1 | 0 | 1 | 1 |
| 1 | 1 | 0 | 0 | 0 | 0 | 0 |

# Logic and Bit Operation

- Computers represent information using bits.
- A bit is a symbol with two possible values: 0 and 1.
- A variable is called a Boolean variable if its value is either true or false.
- Bit operation: replace true by 1 and false by 0 in logical operations (by convention).
- Computer bit operations correspond to logical connectives.
- We can use bit operations to efficiently compute the logical not, and, or, xor connectives.

| $p$ | $q$ | $\neg p$ | $p \land q$ | $p \lor q$ | $p \oplus q$ |
|---|---|---|---|---|---|
| 0 | 0 | 1 | 0 | 0 | 0 |
| 0 | 1 | 1 | 0 | 1 | 1 |
| 1 | 0 | 0 | 0 | 1 | 1 |
| 1 | 1 | 0 | 1 | 1 | 0 |

# Logic and Bit Operation

- We can use bit operations to efficiently compute the logical not, and, or, xor connectives.

| not | and | or | xor |
|---|---|---|---|
| $\neg 1 = 0$ | $1 \wedge 1 = 1$ | $1 \vee 1 = 1$ | $1 \oplus 1 = 0$ |
| | $1 \wedge 0 = 0$ | $1 \vee 0 = 1$ | $1 \oplus 0 = 1$ |
| $\neg 0 = 1$ | $0 \wedge 1 = 0$ | $0 \vee 1 = 1$ | $0 \oplus 1 = 1$ |
| | $0 \wedge 0 = 0$ | $0 \vee 0 = 0$ | $0 \oplus 0 = 0$ |

- A bit string is a sequence of zero or more bits.
- The length of this string is the number of bits in the string.

Example:

Find the bitwise OR, bitwise AND, and bitwise XOR of the bit string

01 1011 0110 and 11 0001 1101.

```
01 1011 0110
11 0001 1101
------------------
11 1011 1111      bitwise OR
01 0001 0100      bitwise AND
10 1010 1011      bitwise XOR
```

# Tautology, Contradiction, Contingency

- A *tautology* is a proposition that is always true.

| $p$ | $\neg p$ | $p \vee \neg p$ |
|---|---|---|
| T | F | T |
| F | T | T |

- A *contradiction* is a proposition that is always false.

| $p$ | $\neg p$ | $p \wedge \neg p$ |
|---|---|---|
| T | F | F |
| F | T | F |

- A *contingency* is a proposition that is neither a tautology nor a contradiction.

| $p$ | $\neg p$ | $p \rightarrow \neg p$ |
|---|---|---|
| T | F | F |
| F | T | T |

# Logical Equivalences

- p and q are logically equivalent if, and only if, they have identical truth values for each possible substitution of statements for their statement variables.
- p and q are logically equivalent if $p \leftrightarrow q$ is a tautology.
- Denoted $p \equiv q$ or $p \Leftrightarrow q$

# Logical Equivalences

| | | |
|---|---|---|
| Identity | $p \wedge T \equiv p$ | $p \vee F \equiv p$ |
| Domination | $p \vee T \equiv T$ | $p \wedge F \equiv F$ |
| Idempotence | $p \vee p \equiv p$ | $p \wedge p \equiv p$ |
| Commutativity | $p \vee q \equiv q \vee p$ | $p \wedge q \equiv q \wedge p$ |
| Associativity | $(p \vee q) \vee r \equiv p \vee (q \vee r)$ | $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ |
| Distributivity | $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ | $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ |
| De Morgan's | $\neg(p \wedge q) \equiv \neg p \vee \neg q$ | $\neg(p \vee q) \equiv \neg p \wedge \neg q$ |
| Double negation | $\neg\neg p \equiv p$ | |
| Excluded Middle | $p \vee \neg p \equiv T$ | |
| Uniqueness | $p \wedge \neg p \equiv F$ | |
| Useful LE | $p \rightarrow q \equiv \neg p \vee q$ | |

# Logical Equivalences

Use known logical equivalences to prove that two propositions are logically equivalent.

Example:

$\neg(\neg p \wedge \neg q) \equiv p \vee q$

$\neg(\neg p \wedge \neg q) \equiv \neg(\neg (p \vee q))$ (De Morgan's)

$\equiv p \vee q$ (Double negation)

# Truth Table of Compound Propositions

Example:

1. Construct the truth table for this statement form: $(p \lor \neg q) \rightarrow (p \land q)$

2. Show that: $p \lor (q \land r)$ and $(p \lor q) \land (p \lor q)$ are logically equivalent

# Logical Equivalences

1. Show that $\neg(p \lor (\neg p \land q))$ is logically equivalent to $\neg p \land \neg q$

2. Show that $(p \land q) \to (p \lor q)$ is a tautology.

# Logical Equivalences

$$p \to q \equiv \neg p \vee q$$

$$p \to q \equiv \neg q \to \neg p$$

$$p \vee q \equiv \neg p \to q$$

$$p \wedge q \equiv \neg (p \to \neg q)$$

$$\neg(p \to q) \equiv p \wedge \neg q$$

$$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$$

$$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$$

$$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$$

$$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$$

$$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$$

$$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$$

$$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$$

$$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$$

# Argument

- An argument is a sequence of statements (statement forms). All statements in an argument (argument form), except for the final one, are called premises (or assumptions or hypothesis). The final statement (statement form) is called the conclusion (or consequent).
  - Premises : $p_1, \ldots, p_n$
  - Conclusion : $q$
- An argument form is called valid if, and only if, whenever statements are substituted that make all the premises true, the conclusion is also true.
- An argument is valid,

  if $p_1 \wedge p_2 \wedge_{\ldots} \wedge p_n \rightarrow q$ is true when $p_1, \ldots, p_n$ are true.

# Argument

Testing an Argument Form for Validity

1. Identify the premises and conclusion of the argument form.
2. Construct a truth table showing the truth values of all the premises and the conclusion.
3. A row of the truth table in which all the premises are true is called a critical row.
   - If there is a critical row in which the conclusion is false $\Rightarrow$ the argument form is invalid.
   - If the conclusion in every critical row is true $\Rightarrow$ the argument form is valid.

# Argument

$$p \rightarrow q \vee \sim r$$
$$q \rightarrow p \wedge r$$
$$\therefore p \rightarrow r$$

*Invalid argument*

*premises* *conclusion*

| p | q | r | ~r | q ∨ ~r | p ∧ r | p → q ∨ ~r | q → p ∧ r | p → r |
|---|---|---|---|---|---|---|---|---|
| T | T | T | F | T | T | T | T | T |
| T | T | F | T | T | F | T | F | |
| T | F | T | F | F | T | F | T | |
| T | F | F | T | T | F | T | T | F |
| F | T | T | F | T | F | T | F | |
| F | T | F | T | T | F | T | F | |
| F | F | T | F | F | F | T | T | T |
| F | F | F | T | T | F | T | T | T |

*Critical rows*

# Modus Ponens

p = I have a total score over 96

q = I get an A for the class

- I have a total score over 96
- If I have a total score over 96, then I get an A for the class
- Therefore, I get an A for this class

$$\begin{array}{l} p \\ p \rightarrow q \\ \hline \therefore q \end{array}$$

Tautology:

$(p \wedge (p \rightarrow q)) \rightarrow q$

# Modus Tollens

p = The power supply fails

q = The lights go out

- If the power supply fails then the lights go out
- The lights are on
- Therefore, the power supply has not failed

$$\begin{array}{l} p \rightarrow q \\ \neg q \\ \hline \therefore \neg p \end{array}$$

Tautology:

$$((p \rightarrow q) \wedge \neg q) \rightarrow \neg p$$

# Addition

p = I am a student

q = I am a soccer player

- I am a student
- Therefore, I am a student or I am a soccer player

$$\frac{p}{\therefore p \vee q}$$

Tautology:

$p \rightarrow (p \vee q)$

# Simplification

p = I am a student

q = I am a soccer player

- I am a student and I am a soccer player
- Therefore, I am a student

$$\frac{p \land q}{\therefore p}$$

Tautology:

$(p \land q) \rightarrow p$

# Conjunction

- p = I am a student.
- q = I am a soccer player.
  - I am a student
  - I am a soccer player
  - Therefore, I am a student and I am a soccer player

$$\begin{array}{c} p \\ q \\ \hline \therefore p \wedge q \end{array}$$

Tautology:

$((p) \wedge (q)) \rightarrow p \wedge q$

# Disjunctive Syllogism

p = I am a student

q = I am a soccer player

- I am a student or I am a soccer player
- I am a not soccer player
- Therefore,  I am a student

$$\begin{array}{l} p \vee q \\ \neg\, q \\ \hline \therefore\, p \end{array}$$

Tautology:

$((p \vee q) \wedge \neg q) \rightarrow p$

# Hypothetical Syllogism

p = I get a total score over 96

q = I get an A in the course

r = I have a 4.0 semester average

- If I get a total score over 96, I will get an A in the course
- If I get an A in the course, I will have a 4.0 semester average
- If I get a total score over 96 then I will have a 4.0 semester average.

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array}$$

Tautology:

$((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$

# Resolution

p = I am taking discrete mathematics

q = I am taking calculus

r = I am taking linear algebra

- I am taking discrete mathematics or I am taking calculus
- I am not taking discrete mathematics or I am taking linear algebra
- Therefore, I am taking calculus or I am taking linear algebra

$$\begin{array}{c} p \vee q \\ \neg p \vee r \\ \hline \therefore q \vee r \end{array}$$

Tautology:

$((p \vee q) \wedge (\neg p \vee r)) \rightarrow (q \vee r)$

# Proof by Cases

p = I am taking discrete mathematics

q = I am taking calculus

r = I am taking linear algebra

- I am taking discrete mathematics or I am taking calculus
- If I am taking discrete mathematics then I am taking linear algebra
- If I am taking calculus then I am taking linear algebra
- Therefore, I am taking linear algebra

$$\begin{array}{l} p \vee q \\ p \rightarrow r \\ q \rightarrow r \\ \hline \therefore \; r \end{array}$$

Tautology:

$((p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r)) \rightarrow r$

# Exercises

**Exercise 1:**

- “It is not sunny this afternoon and it is colder than yesterday”
- “We will go swimming only if it is sunny”
- “If we do not go swimming, then we will take a canoe trip”
- “If we take a canoe trip, then we will be home by sunset”
- Conclusion: “We will be home by sunset”

## Exercise 2:

You are about to leave for school in the morning and discover that you don't have your glasses. You know the following statements are true:

a. If I was reading the newspaper in the kitchen, then my glasses are on the kitchen table.
b. If my glasses are on the kitchen table, then I saw them at breakfast.
c. I did not see my glasses at breakfast.
d. I was reading the newspaper in the living room or I was reading the newspaper in the kitchen.
e. If I was reading the newspaper in the living room then my glasses are on the coffee table.

So, where are your glasses?