IF120
Discrete Mathematics

02 Logics

Nyimas Dewi Sartika, Januar Wahjudi, Seng Hansun

REVIEW

- Sets
- Basic Operation on Sets

OUTLINE

- Propositions
- Logics Operators and Truth Table
- Conditional Propositions and Logical Equivalence
- Arguments and Rules of Inference
- Quantifiers

PROPOSITIONS

- A sentence that is either true or false, but not both, is called a **proposition**.
  1. The only positive integers that divide 7 are 1 and 7 itself.
  2. For every positive integer n, there is a prime number larger than n.
  3. Earth is the only planet in the universe that contains life.
  4. Buy two tickets to the "Unhinged Universe" rock concert for Friday. (x)
  5. x + 4 = 6. (x)
- A proposition is typically expressed as a **declarative** sentence (as opposed to a question, command, etc.).
- Propositions are the basic building blocks of any theory of logic.
- We will use variables, such as p, q, and r, to represent propositions, much as we use letters in algebra to represent numbers.
- We will also use the notation

$$p: 1 + 1 = 3$$

  to define p to be the proposition 1 + 1 = 3.

CONJUNCTION & DISJUNCTION

- Let p and q be propositions.
- The conjunction of p and q, denoted **p ∧ q**, is the proposition p **and** q.
- The disjunction of p and q, denoted **p ∨ q**, is the proposition p **or** q.
- The truth values of propositions such as conjunctions and disjunctions can be described by **truth tables**.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

- The **inclusive-or** of propositions p and q is true if p or q, or both, is true, and false otherwise.
- There is also an **exclusive-or** that defines p exor q to be true if p or q, but not both, is true, and false otherwise.

NEGATION

- The **negation** of p, denoted ¬p, is the proposition not p.
- The truth value of the proposition ¬p is defined by the truth table

p	$\neg p$
T	F
F	T

- In expressions involving some or all of the operators ¬, ∧, and ∨, in the absence of parentheses, we first evaluate ¬, then ∧, and then ∨.
- We call such a convention **operator precedence**.

❑ Given that proposition p is false, proposition q is true, and proposition r is false, determine whether the proposition ¬p ∨ q ∧ r is true or false.

CONDITIONAL PROPOSITIONS

- If p and q are propositions, the proposition

$$if\ p\ then\ q$$

  is called a **conditional proposition** and is denoted

$$p \rightarrow q$$

- The proposition p is called the **hypothesis** (or **antecedent**) and the proposition q is called the **conclusion** (or **consequent**).
- The proposition $p \rightarrow q$ can be true while the proposition $q \rightarrow p$ is false. We call the proposition $q \rightarrow p$ the **converse** of the proposition $p \rightarrow q$.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Assuming that p is true, q is false, and r is true, find the truth value of each proposition.

- ❑ $p \wedge q \rightarrow r$
- ❑ $p \vee q \rightarrow \neg r$
- ❑ $p \wedge (q \rightarrow r)$
- ❑ $p \rightarrow (q \rightarrow r)$

BICONDITIONAL PROPOSITIONS

- If p and q are propositions, the proposition

$$p \ if \ and \ only \ if \ q$$

is called a **biconditional proposition** and is denoted

$$p \leftrightarrow q$$

- The proposition 1 < 5 if and only if 2 < 8

can be written symbolically as

$$p \leftrightarrow q$$

if we define

$$p : 1 < 5, q : 2 < 8$$

Since both p and q are true, the proposition $p \leftrightarrow q$ is true.

LOGICALLY EQUIVALENT

- Suppose that the propositions P and Q are made up of the propositions $p_1, \ldots, p_n$. We say that P and Q are **logically equivalent** and write

$$P \equiv Q,$$

provided that, given any truth values of $p_1, \ldots, p_n$, either P and Q are both true, or P and Q are both false.

- We will verify the first of **De Morgan's** laws $\neg(p \lor q) \equiv \neg p \land \neg q$, by using the truth table.

p	q	$\neg(p \lor q)$	$\neg p \land \neg q$
T	T	F	F
T	F	F	F
F	T	F	F
F	F	T	T

- The **contrapositive** (or **transposition**) of the conditional proposition $p \rightarrow q$ is the proposition $\neg q \rightarrow \neg p$.
- Both of them are logically equivalent.

ARGUMENTS

- An **argument** is a sequence of propositions written

$$\begin{array}{l} p_1 \\ p_2 \\ \vdots \\ \underline{p_n} \\ \therefore q \end{array}$$

or

$$p_1, p_2, \ldots, \frac{p_n}{\therefore q}$$

- The symbol $\therefore$ is read "therefore."
- The propositions $p_1, \ldots, p_n$ are called the **hypotheses** (or **premises**), and the proposition q is called the **conclusion**.
- The argument is **valid** provided that if p_1 and p_2 and $\cdots$ and p_n are all true, then q must also be true; otherwise, the argument is **invalid** (or a **fallacy**).

RULES OF INFERENCE

- Each step of an extended argument involves drawing intermediate conclusions.
- For the argument as a whole to be valid, each step of the argument must result in a valid, intermediate conclusion.
- **Rules of inference**, brief, valid arguments, are used within a larger argument.

❑ Determine whether the argument

$$\begin{array}{c} p \to q \\ \underline{\quad p \quad} \\ \therefore q \end{array}$$

is valid.

We construct a truth table for all the propositions involved:

p	q	$p \to q$	p	q
T	T	T	T	T
T	F	F	T	F
F	T	T	F	T
F	F	T	F	F

We observe that whenever the hypotheses $p \to q$ and p are true, the conclusion q is also true; therefore, the argument is valid.

RULE OF INFERENCES

Rule of Inference	Name	Rule of Inference	Name
$p \rightarrow q$ p $\therefore q$	Modus ponens	p q $\therefore p \wedge q$	Conjunction
$p \rightarrow q$ $\neg q$ $\therefore \neg p$	Modus tollens	$p \rightarrow q$ $q \rightarrow r$ $\therefore p \rightarrow r$	Hypothetical syllogism
p $\therefore p \vee q$	Addition	$p \vee q$ $\neg p$ $\therefore q$	Disjunctive syllogism
$p \wedge q$ $\therefore p$	Simplification		

RULE OF INFERENCES

- Represent the argument

  The bug is either in module 17 or in module 81.

  The bug is a numerical error.

  Module 81 has no numerical error.

  ∴ The bug is in module 17.

  given at the beginning of this section symbolically and show that it is valid.

If we let p : The bug is in module 17, q : The bug is in module 81, r : The bug is a numerical error, the argument may be written

$$\begin{array}{c} p \lor q \\ r \\ \underline{r \to \neg q} \\ \therefore\ p \end{array}$$

From $r \to \neg q$ and r, we may use modus ponens to conclude $\neg q$. From $p \lor q$ and $\neg q$, we may use the disjunctive syllogism to conclude p. Thus the conclusion p follows from the hypotheses and the argument is valid.

UNIVERSAL QUANTIFIERS

- Let P(x) be a statement involving the variable x and let D be a set.
- We call P a **propositional function** or **predicate** (with respect to D) if for each $x \in D$, P(x) is a proposition.
- We call D the **domain of discourse** of P.
- Let P be a propositional function with domain of discourse D.
- The statement

$$for\ every\ x, P(x)$$

  is said to be a universally quantified statement.
- The symbol ∀ means "**for every,**" "**for all**," "**for any.**" Thus the statement for every x, P(x) may be written

$$\forall x\ P(x).$$

- The symbol ∀ is called a **universal quantifier**.

COUNTEREXAMPLE

- The universally quantified statement $\forall x\, P(x)$ is false if for at least one x in the domain of discourse, the proposition P(x) is false.
- A value x in the domain of discourse that makes P(x) false is called a **counterexample** to the statement $\forall x\, P(x)$.

❑Consider the universally quantified statement $\forall x(x^2 - 1 > 0)$.

The domain of discourse is R.

The statement is false since, if x = 1, the proposition $1^2 - 1 > 0$ is false.

The value 1 is a counterexample to the statement $\forall x(x^2 - 1 > 0)$.

Although there are values of x that make the propositional function true, the counterexample provided shows that the universally quantified statement is false.

EXISTENTIAL QUANTIFIERS

- Let P be a propositional function with domain of discourse D.
- The statement

$$there\ exists\ x, P(x)$$

  is said to be an existentially quantified statement.
- The symbol ∃ means "**there exists**." "**for some**," "**for at least one.**" Thus the statement there exists x, P(x) may be written

$$\exists x\ P(x).$$

- The symbol ∃ is called an **existential quantifier.**
- The existentially quantified statement $\exists x\ P(x)$ is false if for every x in the domain of discourse, the proposition P(x) is false.

GENERALIZED DE MORGAN'S LAWS

- If P is a propositional function, each pair of propositions in (a) and (b) has the same truth values (i.e., either both are true or both are false).

a) $\neg(\forall x\, P(x))$; $\exists x\, \neg P(x)$

b) $\neg(\exists x\, P(x))$; $\forall x\, \neg P(x)$

Rules of Inference for Quantified Statements

Rule of Inference	Name
$\dfrac{\forall x\, P(x)}{\therefore P(d) \text{ if } d \in D}$	Universal instantiation
$\dfrac{P(d) \text{ for every } d \in D}{\therefore \forall x\, P(x)}$	Universal generalization
$\dfrac{\exists x\, P(x)}{\therefore P(d) \text{ for some } d \in D}$	Existential instantiation
$\dfrac{P(d) \text{ for some } d \in D}{\therefore \exists x\, P(x)}$	Existential generalization

NESTED QUANTIFIERS

- Consider writing the statement

  $$The\ sum\ of\ any\ two\ positive\ real\ numbers\ is\ positive,$$

  symbolically.
- We first note that since two numbers are involved, we will need two variables, say x and y. The assertion can be restated as: $If\ x > 0\ and\ y > 0, then\ x + y > 0$. The given statement says that the sum of any two positive real numbers is positive, so we need two universal quantifiers. If we let P(x, y) denote the expression $(x > 0) \land (y > 0) \rightarrow (x + y > 0)$, the given statement can be written symbolically as

  $$\forall x \forall y\ P(x, y).$$
- In words, for every x and for every y, $if\ x > 0\ and\ y > 0, then\ x + y > 0$. The domain of discourse of the two-variable propositional function P is R × R, which means that each variable x and y must belong to the set of real numbers.
- Multiple quantifiers such as $\forall x \forall y$ are said to be **nested quantifiers**.

NESTED QUANTIFIERS

- By definition, the statement $\forall x \forall y\, P(x, y)$, with domain of discourse X ×Y , is true if, for every x ∈ X and for every y ∈ Y , P(x, y) is true.
- The statement $\forall x \forall y\, P(x, y)$ is false if there is at least one x ∈ X and at least one y ∈ Y such that P(x, y) is false.

- By definition, the statement $\forall x \exists y\, P(x, y)$, with domain of discourse X × Y , is true if, for every x ∈ X, there is at least one y ∈ Y for which P(x, y) is true.
- The statement $\forall x \exists y\, P(x, y)$ is false if there is at least one x ∈ X such that P(x, y) is false for every y ∈ Y.

NESTED QUANTIFIERS

- By definition, the statement $\exists x \forall y\ P(x, y)$, with domain of discourse X × Y , is true if there is at least one x ∈ X such that P(x, y) is true for every y ∈ Y .
- The statement $\exists x \forall y\ P(x, y)$ is false if, for every x ∈ X, there is at least one y ∈ Y such that P(x, y) is false.

- By definition, the statement $\exists x \exists y\ P(x, y)$, with domain of discourse X × Y , is true if there is at least one x ∈ X and at least one y ∈ Y such that P(x, y) is true.
- The statement $\exists x \exists y\ P(x, y)$ is false if, for every x ∈ X and for every y ∈ Y , P(x, y) is false.

NESTED QUANTIFIERS

- Write the negation of $\exists x \forall y(x\,y < 1)$, where the domain of discourse is R×R. Determine the truth value of the given statement and its negation.

Using the generalized De Morgan's laws for logic, we find that the negation is

$$\neg(\exists x \forall y(x\,y < 1)) \equiv \forall x \neg(\forall y(x\,y < 1)) \equiv \forall x \exists y \neg(x\,y < 1) \equiv \forall x \exists y(x\,y \geq 1).$$

The given statement $\exists x \forall y(x\,y < 1)$ is true because there is at least one x (namely x = 0) such that x y < 1 for every y.

Since the given statement is true, its negation is false.

PRACTICE

NEXT WEEK'S OUTLINE

- Direct Proofs and Indirect Proofs
- Other Proofs Methods
- Proofs Strategy
- Mathematical Induction

REFERENCES

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- Lipschutz, Seymour, Lipson, Marc Lars, *Schaum's Outline of Theory and Problems of Discrete Mathematics*, McGraw-Hill.
- Liu, C.L., 1995, *Dasar-Dasar Matematika Diskret*, Jakarta: Gramedia Pustaka Utama.
- Other offline and online resources.

Visi

Menjadi Program Studi Strata Satu Informatika **unggulan** yang menghasilkan lulusan **berwawasan internasional** yang **kompeten** di bidang Ilmu Komputer (*Computer Science*), **berjiwa wirausaha** dan **berbudi pekerti luhur**.

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