PREDICATE LOGICS

Predicate Logic and Propositional Function

- Predicate logic uses the following new features:
  - Variables: x, y, z
  - Predicates: P(x), M(x)
- Propositional functions are a generalization of propositions.
  - They contain variables and a predicate: P(x)
  - Variables can be replaced by elements from their domain.
- Propositional functions become propositions (and have truth values) when their variables are each replaced by a value from the domain (or bound by a quantifier).

- The statement P(x) is said to be the value of the propositional function P at x.
- For example, let P(x) denote "$x > 0$" and the domain be the integers. Then:
  - P(-3) is false
  - P(0) is false
  - P(3) is true
- Let "$x + y = z$" be denoted by R(x, y, z) and U (for all three variables) be the integers. Find these truth values:
  - R(2,-1,5) is false
  - R(3,4,7) is true
  - R(x, 3, z) is not a proposition
- Often the domain is denoted by U. So, in this example U is the integers.

- Connectives from propositional logic carry over to predicate logic.
- If *P(x)* denotes "$x > 0$," find these truth values:

  $P(3) \lor P(-1)$ is T

  $P(3) \land P(-1)$ is F

  $P(3) \rightarrow P(-1)$ is F

  $P(3) \rightarrow P(1)$ is T
- Expressions with variables are not propositions and therefore do not have truth values. For example,

  $P(3) \land P(y)$

  $P(x) \rightarrow P(y)$
- When used with quantifiers (to be introduced next), these expressions (propositional functions) become propositions.

Quantifier

- Quantifiers are words that refer to quantities such as "some" or "all" and tell for how many elements a given predicate is true.
- We need quantifiers to express the meaning of English words including all and some:
  - "All men are Mortal."
  - "Some cats do not have fur."
- The two most important quantifiers are:
  - Universal Quantifier, "For all," symbol: $\forall$
  - Existential Quantifier, "There exists," symbol: $\exists$
- We write as in $\forall x\ P(x)$ and $\exists x\ P(x)$.
  - $\forall x\ P(x)$ asserts $P(x)$ is true for every x in the domain.
  - $\exists x\ P(x)$ asserts $P(x)$ is true for some x in the domain.
- The quantifiers are said to bind the variable x in these expressions.

Universal Quantifier

- The symbol $\forall$ denotes “for all” (or “for any”, “for every”, “for each”) and is called the universal quantifier.
- Let P(x) be a predicate and D the domain of x. A universal statement is a statement of the form “$\forall x \in D, P(x)$”.
  - It is defined to be true iff P(x) is true for every x in D.
  - It is defined to be false iff P(x) is false for at least one x in D.
- A value for x for which P(x) is false is called a counterexample.

Examples:

1. Let D = {1, 2, 3, 4, 5}, and consider the statement $\forall x \in D$, $x^2 \geq x$. Show that this statement is true.

   Check that "$x^2 \geq x$" is true for each x in D. $1^2 \geq 1$; $2^2 \geq 2$; $3^2 \geq 3$; $4^2 \geq 4$; $5^2 \geq 5$. Hence "$\forall x \in D, x^2 \geq x$" is true.

2. If P(x) denotes "$x > 0$" and U is the integers,

   then $\forall x\, P(x)$ is false.

3. If P(x) denotes "$x > 0$" and U is the positive integers,

   then $\forall x\, P(x)$ is true.

4. If P(x) denotes "x is even" and U is the integers,

   then $\forall x\, P(x)$ is false.

Existential Quantifier

- The symbol ∃ denotes "For some x, P(x)", or as "There is an x such that P(x)," or "For at least one x, P(x)." and is called the universal quantifier.
- Let P(x) be a predicate and D the domain of x. An **existential statement** is a statement of the form "$\exists x \in D$ such that P(x)".
  - It is defined to be true iff P(x) is true for at least one x in D.
  - It is defined to be false iff P(x) is false for all x in D.

Examples:

1. Show that $\exists m \in Z^+$ such that $m^2 = m$ is true.

   Observe that $1^2 = 1$. Thus "$m^2 = m$" is true for at least one integer m.

   Hence "$\exists m \in Z^+$ such that $m^2 = m$" is true.

2. Let $E = \{5, 6, 7, 8\}$. Show that $\exists m \in \mathbf{E}$ such that $m^2 = m$ is false.

   Note that $m^2 = m$ is not true for any integer m from 5 through 8:

   $5^2 = 25 \neq 5$; $6^2 = 36 \neq 6$; $7^2 = 49 \neq 7$; $8^2 = 64 \neq 8$.

   Hence "$\exists m \in E$ such that $m^2 = m$" is false.

3. If P(x) denotes "$x > 0$" and U is the integers, then $\exists x\, P(x)$ is true.

   It is also true if U is the positive integers.

4. If P(x) denotes "$x < 0$" and U is the positive integers, then $\exists x\, P(x)$ is false.

5. If P(x) denotes "x is even" and U is the integers, then $\exists x\, P(x)$ is true.

Uniqueness Quantifier

∃!x P(x) means that P(x) is true for <u>one and only one</u> x in the universe of discourse.

This is commonly expressed in English in the following equivalent ways:

- "There is a unique x such that P(x)"
- "There is one and only one x such that P(x)"

Examples:

1. If P(x) denotes "$x + 1 = 0$" and U is the integers, then ∃!x P(x) is true.
2. If P(x) denotes "$x > 0$," then ∃!x P(x) is false.

Universal Conditional Statement

A reasonable argument can be made that the most important form of statement in mathematics is the universal conditional statement: $\forall x$, if $P(x)$ then $Q(x)$.

Examples:

Rewrite the following statement informally, without quantifiers or variables: $\forall x \in \mathbf{R}$, if $x > 2$ then $x^2 > 4$.

- If a real number is greater than 2, then its square is greater than 4, or
- Whenever a real number is greater than 2, its square is greater than 4.

Properties of Quantifier

The truth value of $\exists x\, P(x)$ and $\forall x\, P(x)$ depend on both the propositional function $P(x)$ and on the domain U.

Examples:

1. If U is the positive integers and $P(x)$ is the statement "$x < 2$", then $\exists x\, P(x)$ is true, but $\forall x\, P(x)$ is false.
2. If U is the negative integers and $P(x)$ is the statement "$x < 2$", then both $\exists x\, P(x)$ and $\forall x\, P(x)$ are true.
3. If U consists of 3, 4, and 5, and $P(x)$ is the statement "$x > 2$", then both $\exists x\, P(x)$ and $\forall x\, P(x)$ are true.

   But if $P(x)$ is the statement "$x < 2$", then both $\exists x\, P(x)$ and $\forall x\, P(x)$ are false.

Precedence of Quantifier

- The quantifiers $\forall$ and $\exists$ have higher precedence than all the logical operators.
- $\forall x\ P(x) \lor Q(x)$ means $(\forall x\ P(x)) \lor Q(x)$.
- $\forall x\ (P(x) \lor Q(x))$ means something different.
- Unfortunately, often people write $\forall x\ P(x) \lor Q(x)$ when they mean $\forall x\ (P(x) \lor Q(x))$.

Equivalences in Predicate Logic

- Statements involving predicates and quantifiers are logically equivalent if and only if they have the same truth value
  - for every predicate substituted into these statements and
  - for every domain of discourse used for the variables in the expressions.
- The notation $S \equiv T$ indicates that S and T are logically equivalent.
- Example: $\forall x\ (S(x) \wedge T(x)) \equiv \forall x\ S(x) \wedge \forall x\ T(x)$

Relation among $\forall$, $\exists$, $\wedge$, and $\vee$

If $Q(x)$ is a predicate and the domain D of x is the set $\{x_1, x_2, \ldots, x_n\}$, then

$$\forall x \in D, Q(x) \equiv Q(x_1) \wedge Q(x_2) \wedge \ldots \wedge Q(x_n)$$

$$\exists x \in D, Q(x) \equiv Q(x_1) \vee Q(x_2) \vee \ldots \vee Q(x_n)$$

Thinking as A Programmer

- When the domain of discourse is finite, we can think of quantification as looping through the elements of the domain.
- To evaluate $\forall x\ P(x)$ loop through all x in the domain.
  - If at every step $P(x)$ is true, then $\forall x\ P(x)$ is true.
  - If at a step $P(x)$ is false, then $\forall x\ P(x)$ is false and the loop terminates.
- To evaluate $\exists x\ P(x)$ loop through all x in the domain.
  - If at some step, $P(x)$ is true, then $\exists x\ P(x)$ is true and the loop terminates.
  - If the loop ends without finding an x for which $P(x)$ is true, then $\exists x\ P(x)$ is false.
- Even if the domains are infinite, we can still think of the quantifiers this fashion, but the loops will not terminate in some cases.

Negating Quantified Expressions

The negation of a statement of the form $\forall x$ in D, P(x) is logically equivalent to a statement of the form $\exists x$ in D such that ~P(x).

Symbolically, $\sim(\forall x \text{ in } D, P(x)) \equiv \exists x \text{ in } D \text{ such that } \sim P(x)$

The negation of a universal statement ("all are") is logically equivalent to an existential statement ("some are not" or "there is at least one that is not").

The negation of a statement of the form $\exists x$ in D such that P(x) is logically equivalent to a statement of the form $\forall x$ in D, ~P(x)

Symbolically, $\sim(\exists x \text{ in } D \text{ such that } P(x)) \equiv \forall x \text{ in } D, \sim P(x)$

The negation of an existential statement ("some are") is logically equivalent to a universal statement ("none are" or "all are not").

De Morgan's Laws for Quantifiers:

$$\neg\forall x\, P(x) \equiv \exists x\, \neg P(x)$$

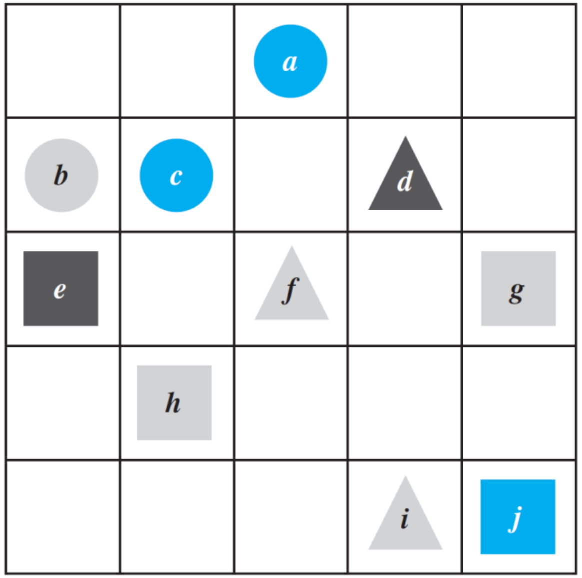
$$\neg\exists x\, P(x) \equiv \forall x\, \neg P(x)$$

Statement	Negation
$\forall x\, P(x)$ "Every student in your class has taken a course in Java." where $P(x)$ is "x has taken a course in Java" and the domain is students in your class.	$\exists x\, \neg P(x)$ It is not the case that every student in your class has taken Java." This implies that "There is a student in your class who has not taken Java."
$\exists x\, P(x)$ "There is a student in this class who has taken a course in Java." where $P(x)$ is "x has taken a course in Java."	$\forall x\, \neg P(x)$ "It is not the case that there is a student in this class who has taken Java." This implies that "Every student in this class has not taken Java"

Statements with Multiple Quantifiers

Statement	When True?	When False
$\forall x \forall y P(x,y)$ $\forall y \forall x P(x,y)$	P(x,y) is true for every pair x,y.	There is a pair x, y for which P(x,y) is false.
$\forall x \exists y P(x,y)$	For every x there is a y for which P(x,y) is true.	There is an x such that P(x,y) is false for every y.
$\exists x \forall y P(x,y)$	There is an x for which P(x,y) is true for every y.	For every x there is a y for which P(x,y) is false.
$\exists x \exists y P(x,y)$ $\exists y \exists x P(x,y)$	There is a pair x, y for which P(x,y) is true.	P(x,y) is false for every pair x,y

- Consider the Tarski's world.
- Show that the following statement is true:
  - For all triangles *x*, there is a square *y* such that *x* and *y* have the same color.
- The statement says that no matter which triangle someone gives you, you will be able to find a square of the same color.
- There are only 3 triangles *d*, *f*, and *i*.



Given x =	choose y =	and check that y is the same color as x.
d	e	yes •
f or i	h or g	yes •

- If you want to establish the truth of a statement of the form:

  $\forall x$ in D, $\exists y$ in E such that P(x, y)

  your challenge is to allow someone else to pick whatever element x in D they wish and then you must find an element y in E that "works" for that particular x.

- If you want to establish the truth of a statement of the form:

  $\exists x$ in D such that $\forall y$ in E, P(x, y)

  your job is to find one particular x in D that will "work" no matter what y in E anyone might choose to challenge you with.

Examples:

Statement	**P(x,y): x + y = 0** **U is R**	**P(x,y): x . y = 0** **U is R**	**P(x,y): x / y = 1** **U is R - {0}**
$\forall x\ \forall y\ P(x,y)$	False	False	False
$\forall x\ \exists y\ P(x,y)$	True	True	True
$\exists x\ \forall y\ P(x,y)$	False	True	False
$\exists x\ \exists y\ P(x,y)$	True	True	True