Optimization – Programming Assignment – Part II - Solution

Leg distance:

$$d_i = \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}$$

Leg time:

$$t_i = \frac{d_i}{v_i}$$

Objective function – total time:

$$T = \sum_{i=1}^{n} t_i = \sum_{i=1}^{n} \frac{d_i}{v_i} = \sum_{i=1}^{n} \frac{1}{v_i}\sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}$$

Gradient:

Recall that x_0 and x_n are constants given by x_s and x_f, thus only derivatives of T w.r.t x_k $\forall k \in \{1, \dots, n-1\}$ should be taken.

$$\frac{\partial T}{\partial x_k} = \sum_{i=1}^{n} \frac{1}{v_i}\frac{1}{2}\left((x_i - x_{i-1})^2 + (y_i - y_{i-1})^2\right)^{-\frac{1}{2}} \cdot 2(x_i - x_{i-1})\left(\delta_{i,k} - \delta_{i-1,k}\right) =$$

$$= \frac{x_k - x_{k-1}}{v_k\sqrt{(x_k - x_{k-1})^2 + (y_k - y_{k-1})^2}} - \frac{x_{k+1} - x_k}{v_{k+1}\sqrt{(x_{k+1} - x_k)^2 + (y_{k+1} - y_k)^2}} =$$

$$= \frac{x_k - x_{k-1}}{v_k d_k} - \frac{x_{k+1} - x_k}{v_{k+1} d_{k+1}}$$

$$\boxed{\frac{\partial T}{\partial x_k} = \frac{x_k - x_{k-1}}{v_k d_k} - \frac{x_{k+1} - x_k}{v_{k+1} d_{k+1}} \;, \forall k \in \{1, \dots, n-1\}}$$

Hessian:

$$\frac{\partial^2 T}{\partial x_m \partial x_k} =$$

$$= \frac{1}{v_k}\frac{\left(\delta_{k,m} - \delta_{k-1,m}\right)d_k - (x_k - x_{k-1})\frac{1}{2}\left((x_k - x_{k-1})^2 + (y_k - y_{k-1})^2\right)^{-\frac{1}{2}} \cdot 2(x_k - x_{k-1})\left(\delta_{k,m} - \delta_{k-1,m}\right)}{(x_k - x_{k-1})^2 + (y_k - y_{k-1})^2}$$

$$- \frac{1}{v_{k+1}}\frac{\left(\delta_{k+1,m} - \delta_{k,m}\right)d_{k+1} - (x_{k+1} - x_k)\frac{1}{2}\left((x_{k+1} - x_k)^2 + (y_{k+1} - y_k)^2\right)^{-\frac{1}{2}} \cdot 2(x_{k+1} - x_k)\left(\delta_{k+1,m} - \delta_{k,m}\right)}{(x_{k+1} - x_k)^2 + (y_{k+1} - y_k)^2}$$

$$= \frac{1}{v_k}\left(\delta_{k,m} - \delta_{k-1,m}\right)\frac{d_k - \frac{(x_k - x_{k-1})^2}{d_k}}{d_k^2} - \frac{1}{v_{k+1}}\left(\delta_{k+1,m} - \delta_{k,m}\right)\frac{d_{k+1} - \frac{(x_{k+1} - x_k)^2}{d_{k+1}}}{d_{k+1}^2} =$$

$$= \frac{1}{v_k}\left(\delta_{k,m} - \delta_{k-1,m}\right)\frac{d_k^2 - (x_k - x_{k-1})^2}{d_k^3} - \frac{1}{v_{k+1}}\left(\delta_{k+1,m} - \delta_{k,m}\right)\frac{d_{k+1}^2 - (x_{k+1} - x_k)^2}{d_{k+1}^3} =$$

$$= \frac{1}{v_k}\left(\delta_{k,m} - \delta_{k-1,m}\right)\frac{(y_k - y_{k-1})^2}{d_k^3} - \frac{1}{v_{k+1}}\left(\delta_{k+1,m} - \delta_{k,m}\right)\frac{(y_{k+1} - y_k)^2}{d_{k+1}^3}$$

$$h_{m,k}(\underline{x}) = \begin{cases} m = k: & \frac{1}{v_k}\frac{(y_k - y_{k-1})^2}{d_k^3} + \frac{1}{v_{k+1}}\frac{(y_{k+1} - y_k)^2}{d_{k+1}^3} \\ m = k-1: & -\frac{1}{v_k}\frac{(y_k - y_{k-1})^2}{d_k^3} \\ m = k+1: & -\frac{1}{v_{k+1}}\frac{(y_{k+1} - y_k)^2}{d_{k+1}^3} \end{cases}, \forall m, k \in \{1, \ldots, n-1\}$$