Modelling and Simulation Assessment: Zombies! vs Plants

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1 Context and Scoping

Greetings, new neighbor! The name's Crazy Dave, but if you ask me, the real crazies are the realtors still selling houses in the middle of a zombie apocalypse – and you, for still buying them. Oh, the guy before you? He was testing a model he came up with... a few wrong assumptions later, his house was back on the market! Let me catch you up.

A few weeks ago, zombies started popping up in the neighborhood, but they weren't the only ones. For some reason, the plants in our yards started to mutate too. The first ones we noticed were the sunflowers, which started producing a harvestable resource called "Sun" which could be used to instantly grow plants, though different types required different amounts. The zombies weren't your run-of-the-mill brain-craving types either. In our observation, we listed down some quirks in their behavior:

1. The zombies seem to understand the concept of shelter, and beeline for occupied houses. As a result, typical approaches to the zombie apocalypse like doomsday prepping or becoming mercenaries aren't effective strategies.

2. The only thing that distracts them from this beeline are the presence of the mutated plants nearby. **So long as there is a surviving population of plants, the zombies will prioritise eating them before eating you.** Weirdly, their attention is only focused on mutated plants in your property line – the neighbors lots can't protect you.

3. Importantly, zombies seem to spawn using this "Sun" resource as well, which they harvest when they consume sunflowers. Thus, while the zombies are not converting the sunflowers into more zombies, zombie spawning is related to sunflowers eaten [1].

After we noticed these behaviors, we theorised that the situation was reminiscent of a predator-prey system as long as the prey population didn't hit zero, and we began working on strategies to survive.

2 Preliminary Modeling - Logistic Predator-Prey

From that second observation, we determined that a Lotka-Volterra predator prey model with logistic growth would be appropriate, as the yard is a finite space and can thus only host a finite number of sunflowers before they die off due to nutrient competition.

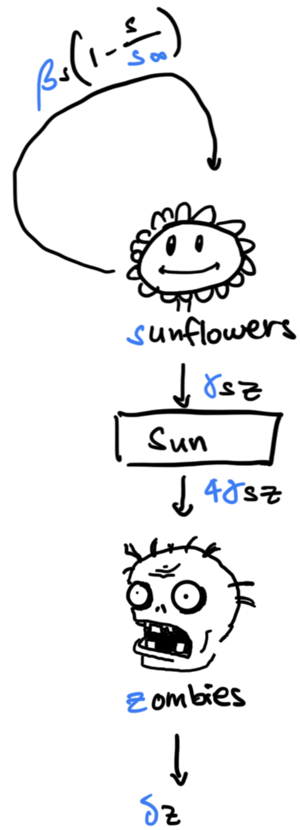


Figure 1. Preliminary logistic growth predator-prey based sunflower-zombie model.

2.1 Assumptions

The model rests on the following assumptions, which were adapted from previous population models [2]:

- If all produced Sun was directed to planting sunflowers without them being culled by zombies, the sunflower population would grow logistically until reaching the carrying capacity. Due to the magic properties of Sun, it's assumed that the planting/growth time is negligible.

- Zombies are extremely slow, so although all encounters between sunflower and zombie will always result in predation, the predation rate is significantly smaller than the sunflower spawn rate due to the zombies shambling from one flower to the next. While they might also be able to convert humans, the addition to zombie numbers is negligible compared to those spawned by Sun.

- The per-zombie predation rate is a linear function of the sunflower population, as more sunflowers means less walking.

- Zombies, modeled like natural predators, would expire after some time. The sunflowers do this too, but the rate at which that occurs is negligible relative to the loss of sunflowers due to predation.

2.2 Parameters

- β: the "birth" rate of sunflowers, which is the rate at which new sunflowers can be planted per arbitrary unit of time. In an infinite yard, the sunflowers would produce Sun which produces sunflowers, creating exponential growth βS. However, due to yard space constraints, this has a logistic term built in curbing new growths as it approaches carrying capacity.
- s_∞: The carrying capacity of the sunflower population in a yard of a given size.
- γ: The predation rate of sunflowers by zombies. Since the per-zombie rate is a linear function, this creates a non-linear term γSZ which represents the number of sunflowers eaten in the same arbitrary time step. For every sunflower eaten, enough Sun is produced to spawn 4 zombies, reflected in the zombie input term. [1]
- δ: The natural death rate of the zombies, forming the zombie expiration term δZ.

Based on these assumptions and parameters, the equations governing the system were formalised (where S is the sunflower population and Z is the zombie population):

$$\begin{aligned}\frac{dS}{dt} &= \beta S\left(1-\frac{S}{S_\infty}\right) - \gamma SZ \\ \frac{dZ}{dt} &= 4\gamma SZ - \delta Z\end{aligned} \tag{1}$$

2.3 System Characteristics

Nullclines are curves in phase space where the rate of change of the variables (S and Z) are zero. From the sunflower term, the nullclines are given by:

$$\begin{aligned}\frac{dS}{dt} = \beta S\left(1-\frac{S}{S_\infty}\right) - \gamma SZ &= 0 \\ S\left[\beta\left(1-\frac{S}{S_\infty}\right) - \gamma Z\right] &= 0 \\ S = 0, Z &= \frac{\beta}{\gamma}\left(1-\frac{S}{S_\infty}\right)\end{aligned} \tag{2}$$

Finding these for the zombie terms:

$$\begin{aligned}\frac{dZ}{dt} = 4\gamma SZ - \delta Z &= 0 \\ Z(4\gamma S - \delta) &= 0 \\ Z = 0, S &= \frac{\delta}{4\gamma}\end{aligned} \tag{3}$$

The equilibrium points that help characterise the behaviour of the system can be found at the intersections between these nullclines. These equilibrium points can be used to identify system characteristics by evaluating the stability of the equilibria as the parameters vary. This can be done by calculating the eigenvalues of the Jacobian at these points. Thus:

$$\begin{aligned}
J &= \begin{bmatrix} \frac{\delta f_1}{dS} & \frac{\delta f_1}{dZ} \\ \frac{\delta f_1}{dS} & \frac{\delta f_2}{dZ} \end{bmatrix} \\
&= \begin{bmatrix} \beta\left(1 - \frac{2S}{S_\infty}\right) - \gamma Z & -\gamma S \\ 4\gamma Z & 4\gamma S - \delta \end{bmatrix} \\
J - \lambda I &= \begin{bmatrix} \beta\left(1 - \frac{2S}{S_\infty}\right) - \gamma Z - \lambda & -\gamma S \\ 4\gamma Z & 4\gamma S - \delta - \lambda \end{bmatrix} \\
det(J - \lambda I) &= \left[\beta\left(1 - \frac{2S}{S_\infty}\right) - \gamma Z - \lambda\right] \cdot [4\gamma S - \delta - \lambda] - (-\gamma S) \cdot (4\gamma Z) = 0
\end{aligned} \tag{4}$$

From this, three characterised equilibria emerge:

- At the **trivial equilibrium** $(S, Z) = (0, 0)$,

$$\begin{aligned}
J &= \begin{bmatrix} \beta\left(1 - \frac{2\cdot 0}{S_\infty}\right) - \gamma \cdot 0 & -\gamma \cdot 0 \\ 4\gamma \cdot 0 & 4\gamma \cdot 0 - \delta \end{bmatrix} \\
&= \begin{bmatrix} \beta & 0 \\ 0 & -\delta \end{bmatrix} \\
\lambda_1 &= \beta, \lambda_2 = -\delta
\end{aligned} \tag{5}$$

  Since $\beta, \delta > 0$ as parameters modeling population, the trivial equilibrium is a saddle point (one positive real eigenvalue and one negative real eigenvalue) diverging in terms of sunflower growth and converging in terms of the zombie population along the nullclines.

- At the **boundary equilibrium** $(S, Z) = (S_\infty, 0)$,

$$\begin{aligned}
det(J - \lambda I) &= \left[\beta\left(1 - \frac{2S_\infty}{S_\infty}\right) - \gamma \cdot 0 - \lambda\right] \cdot [4\gamma S_\infty - \delta - \lambda] - (-\gamma S_\infty) \cdot (4\gamma \cdot 0) = 0 \\
&= (-\beta - \lambda) \cdot (4\gamma S_\infty - \delta - \lambda) = 0 \\
\lambda_1 &= -\beta, \lambda_2 = 4\gamma S_\infty - \delta
\end{aligned} \tag{6}$$

  Since all parameters > 0 as they are being used in a biological context, $\lambda_1 < 0$, meaning it is stable along the sunflower axis. This makes intuitive sense as, left alone, the sunflowers would grow until reaching the carrying capacity. λ_2 is of interest, however, as it has some bifurcation behaviour. If $4\gamma S_\infty > \delta, \lambda_2 > 0$, meaning the equilibrium is a saddle point, unstable in the direction of zombie growth. Conversely, if $4\gamma S_\infty < \delta, \lambda_2 < 0$, which makes the point a stable node, meaning nearby trajectories converge.

- Finally, at the **coexistence equilibrium** $(S^*, Z^*) = \left(\frac{\delta}{4\gamma}, \frac{\beta}{\gamma}\left(1 - \frac{\delta}{4\gamma S_\infty}\right)\right)$

$$
\begin{aligned}
det(J - \lambda I) &= \left[\beta\left(1 - \frac{2S^*}{S_\infty}\right) - \gamma Z^* - \lambda\right] \cdot [4\gamma S^* - \delta - \lambda] - (-\gamma S^*) \cdot (4\gamma Z^*) = 0 \\
&= \left[\beta\left(1 - \frac{\delta}{2\gamma S_\infty}\right) - \beta\left(1 - \frac{\delta}{4\gamma S_\infty}\right) - \lambda\right] \cdot [-\lambda] + \delta\beta\left(1 - \frac{\delta}{4\gamma S_\infty}\right) = 0 \\
&= \left[-\frac{\beta\delta}{4\gamma S_\infty} - \lambda\right] \cdot [-\lambda] + \delta\beta - \frac{\delta^2\beta}{4\gamma S_\infty} = 0 \\
&= \lambda^2 + \left[\frac{\beta\delta}{4\gamma S_\infty}\right]\lambda + \left[\delta\beta - \frac{\delta^2\beta}{4\gamma S_\infty}\right] = 0
\end{aligned} \tag{7}
$$

This can be characterised as a quadratic $\lambda^2 + a\lambda + b = 0$, where $a = 1, b = \frac{\beta\delta}{4\gamma S_\infty}, c = \frac{\delta^2\beta}{4\gamma S_\infty} - \delta\beta$. This allows us to use the quadratic formula to find the nature of the roots of the equation, which are the eigenvalues in this case.

$$
\begin{aligned}
\lambda_1, \lambda_2 &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-\frac{\beta\delta}{4\gamma S_\infty} \pm \sqrt{\left(\frac{\beta\delta}{4\gamma S_\infty}\right)^2 - 4\left(\delta\beta - \frac{\delta^2\beta}{4\gamma S_\infty}\right)}}{2} \\
&= -\frac{\beta\delta}{8\gamma S_\infty} \pm \frac{\sqrt{-\beta\delta\left(64S_\infty^2\gamma^2 - 16S_\infty\delta\gamma - \beta\delta\right)}}{8S_\infty\gamma}
\end{aligned} \tag{8}
$$

Since $\beta, \delta, \gamma, S_\infty > 0$, the first term of the expression in (8) is guaranteed to be negative. Upon finding this, we rejoiced! Our safety was guaranteed!

We came to that conclusion because the coexistence equilibrium point had two relevant bifurcation points. Firstly, the behaviour of the system can be broadly classified based on whether the second term of the eigenvalues were real or imaginary. This introduced the bifurcation point dependent on the determinant:

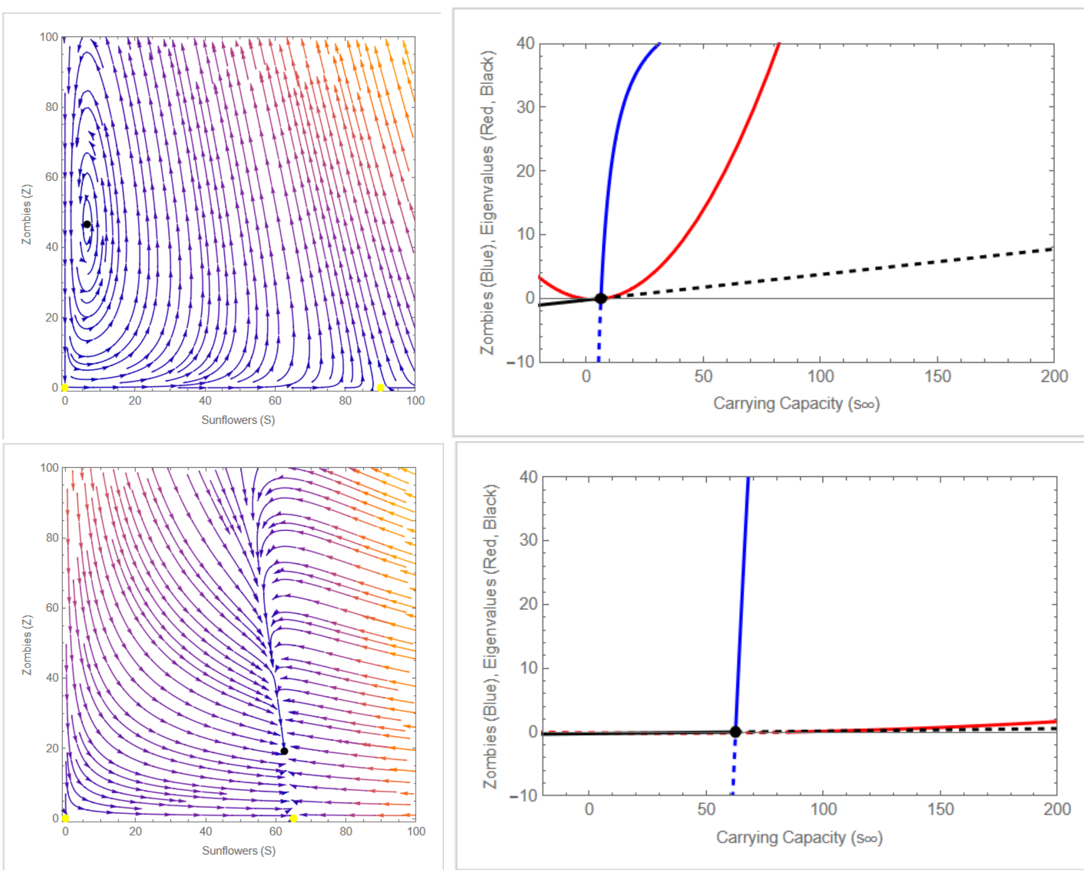
$$
\begin{aligned}
-\beta\delta\left(64S_\infty^2\gamma^2 - 16S_\infty\delta\gamma - \beta\delta\right) &< 0 \\
64S_\infty^2\gamma^2 - 16S_\infty\delta\gamma - \beta\delta &> 0
\end{aligned} \tag{9}
$$

When this condition was fulfilled, the second term was imaginary – meaning the coexistence point was a stable spiral. This meant both populations would enter damped oscillations before settling at $(S^*, Z^*) = \left(\frac{\delta}{4\gamma}, \frac{\beta}{\gamma}\left(1 - \frac{\delta}{4\gamma S_\infty}\right)\right)$.

Alternatively, when this condition wasn't fulfilled, the real second term meant that the point could either be a stable node or a saddle point (one eigenvalue guaranteed negative). This is where the second bifurcation point comes into play. The equilibrium point is a stable node when:

$$
\begin{aligned}
&-\beta\delta + \sqrt{-\beta\delta\left(64S_\infty^2\gamma^2 - 16S_\infty\delta\gamma - \beta\delta\right)} < 0 \\
&\beta\delta > -64S_\infty^2\gamma^2 + 16S_\infty\delta\gamma + \beta\delta \\
&S_\infty > \frac{\delta}{4\gamma}
\end{aligned}
\tag{10}
$$

When that condition isn't true, the equilibrium point would become a saddle point, with one positive eigenvalue and one negative. However, if you'd recall, the **boundary equilibrium** shares the same bifurcation point – and when that condition isn't true ($S < \frac{\delta}{4\gamma}$), it evolves into a stable node. Additionally, the zombie population of the coexistence equilibrium at this bifurcation point and below is ≤ 0, meaning it becomes irrelevant in a biological context. This means the boundary equilibrium dominates the system as a stable node, keeping us safe!



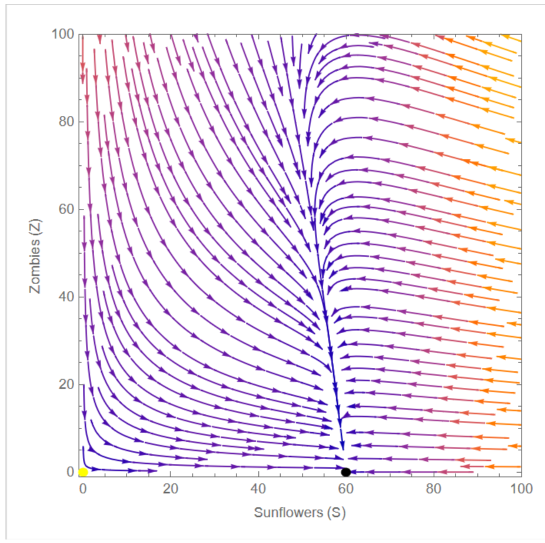


Figure 2A, B, C. The phase space diagram (left) and bifurcation plot (right, with red = coexistence, black = boundary) of (A) the system with $\beta = 0.5, \gamma = 0.01, \delta = 0.25, S_\infty = 90$*, showing the coexistence point as a stable spiral and (B) the system with* $\beta = 0.5, \gamma = 0.001, \delta = 0.25, S_\infty = 65$*, showing it as a stable node. (C) shows a point below bifurcation point, with the same parameters as (B) except* $S_\infty = 60$ *making the boundary a stable node. Interactive plots available in code.*

We verified this with numerical solutions, using a Runge-Kutta-Fehlberg method [3] to simulate the sunflower and zombie populations. This method has a local truncation error of $O(h^5)$ and a global error of $O(h^4)$, where h is the step size, meaning with a sufficiently small step this would be very accurate. We found that these behaved exactly as expected, maintaining a non-zero sunflower population.

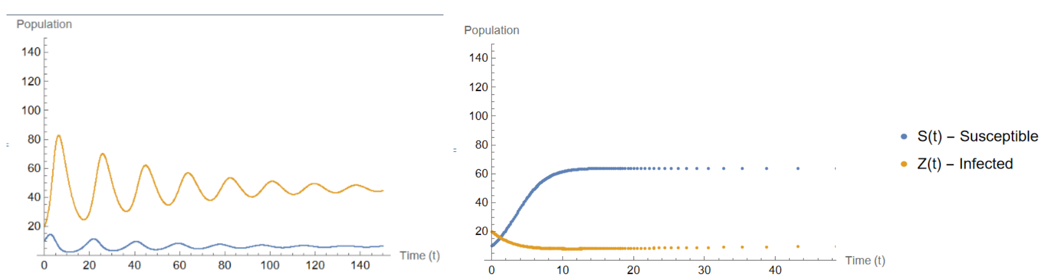


Figure 3A, B. (A) The numerical solution for the zombie (orange) and sunflower (blue) populations corresponding with Figure 2A, showing the stable spiral behaviour and (B) correseponding with 2B, showing the stable node behaviour. These were arbitrarily initialised with populations $(S_0, Z_0) = (10, 20)$ *but behaviour was not sensitive to initial conditions as long as they were greater than 0.*

To be thorough, we also looked conducted a numerical parametric sensitivity analysis using an ensemble simulation, in case our parameter estimates were off. However, the behaviour broadly persisted as expected, building our confidence that we were safe.

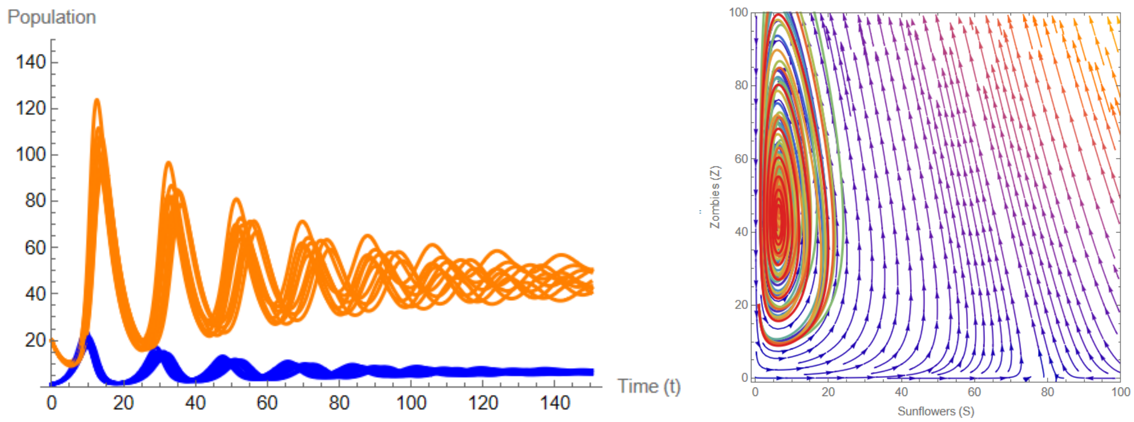


Figure 4A, B. (A) the time series plot of 10 iterations of varying all parameters by a random amount within ±10% on the base parameters defined in 2A and (B) the corresponding phase plot containing the trajectories of each solution. Perturbations and number of simulations can be changed in the code and run. Error tolerance was slackened for computation time in the ensemble simulations.

2.4 Tragedy!

Enboldened by this discovery, the old neighbour began to live his life confidently – that was, before he was eaten! It turns out that, while we had seen zombies die in our observation, it was not due to lack of prey or age. It was actually because of a different mutated plant: the pea(shooter)! Without this key knowledge, our assumption about the zombie death rate δ was entirely wrong, with the real value being $\delta = 0$ in our yards, which had no peashooters.

When $\delta = 0$, the coexistence equilibrium's eigenvalues given by:

$$\lambda_1, \lambda_2 = -\frac{\beta\delta}{8\gamma S_\infty} \pm \frac{\sqrt{-\beta\delta\left(64S_\infty^2\gamma^2 - 16S_\infty\delta\gamma - \beta\delta\right)}}{8S_\infty\gamma} \tag{11}$$

become 0, meaning it is a degenerate point and the behaviour is harder to characterise. Instead, we can look at the numerical solution given the same β, γ, S_∞ with this change.

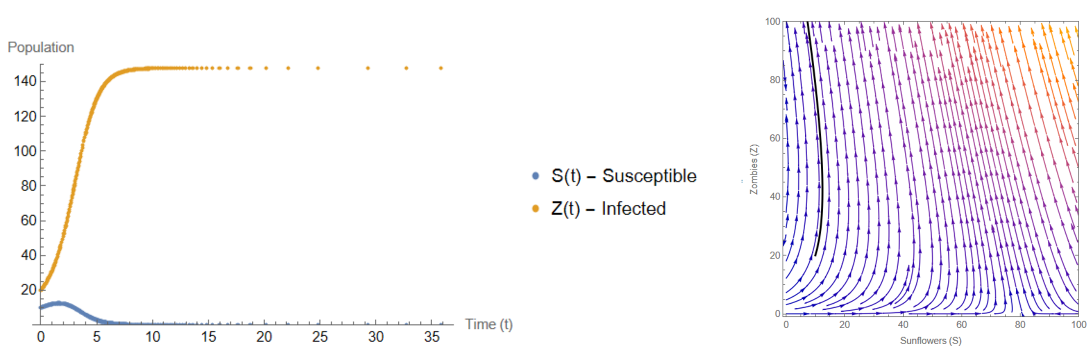


Figure 5A, B. The numerical solutions (A) time-series and (B) phase trajectory at the degenerate point when $\delta = 0$.

Clearly, the sunflower population reaches 0 and your would-be flatmate gets eaten.

3 Fighting Back!

Don't worry new neighbor! With that bit of trial and error complete, we've got this in the bag this time. Clearly, it's not enough to be on the defensive; we have to invest in some offense – and I know just the peas for the job. Let's update our model.

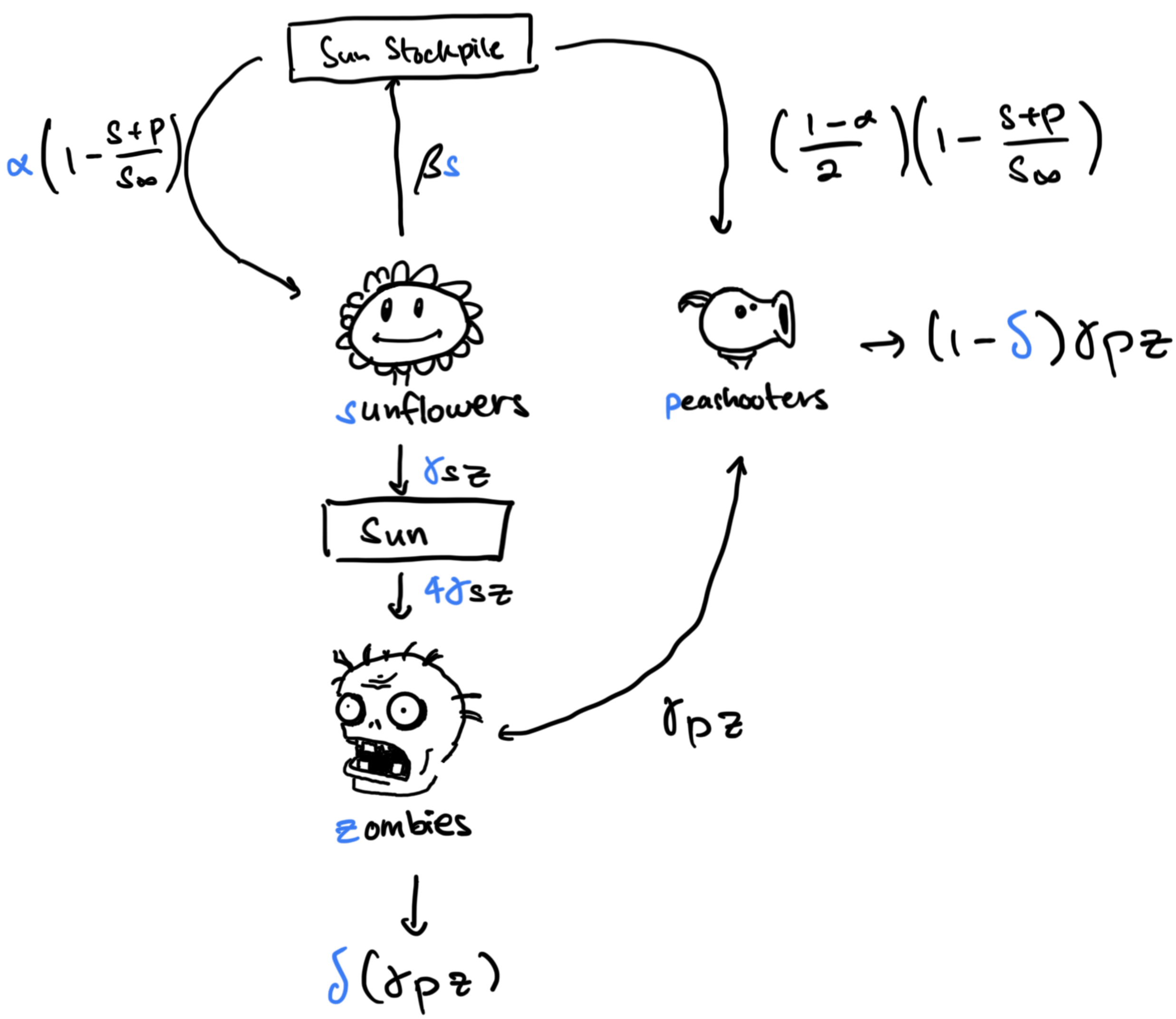


Figure 6. Iteration on logistic growth model with shared "food" resource dependent on one population. Sunflower-peashooter-zombie model.

3.1 Updated Assumptions

In this model, all the previous assumptions are carried forward with the introduction of a few more:

- Peashooters contribute to the carrying capacity of the yard exactly the same way sunflowers do, hence the shared logistic growth curve.
- Mutated peashooters are born of the same Sun resource as the sunflowers, meaning a new "food" stockpile feature should be added from which both sunflowers and peashooters can be spawned. Peashooters seem to require double the amount of Sun as sunflowers do, however [4].
- Zombies are equally likely to encounter peashooters as they are to encounter sunflowers. When a zombie encounters a peashooter, one of them is guaranteed to die. Either the zombie eats the peashooter or the peashooter kills the zombie.

3.2 Parameters

From these assumptions, some new parameters are defined:

- α is the allocation parameter, which determines how much of the Sun stockpile is committed to spawning new sunflowers while $\frac{1-\alpha}{2}$ shows how much is committed to spawning new peashooters.
- δ is functionally similar as a death rate, but is now an indicator of the efficacy of the peashooters. The more peashooters win an encounter with zombies, the greater the δ which means more zombie expiry.

This generates the new set of equations:

$$\begin{aligned}
\frac{dS}{dt} &= \alpha\beta S \cdot \left(1 - \frac{S+P}{s_\infty}\right) - \gamma SZ \\
\frac{dP}{dt} &= \beta S \cdot \left(1 - \frac{\alpha}{2}\right) \cdot \left(1 - \frac{S+P}{s_\infty}\right) - (1-\delta)\gamma PZ \\
\frac{dZ}{dt} &= 4\gamma SZ - \delta\gamma PZ
\end{aligned} \tag{12}$$

3.3 System Behaviour

Unlike the initial model, an analytical approach to this system shows that there is no three-way coexistence equilibrium. The only equilibrium points occur when $Z = 0, (S, P) = (0, 0)$ or $(S, P, Z) = (0, 0, 0)$ – either the plants or the zombies are going extinct.

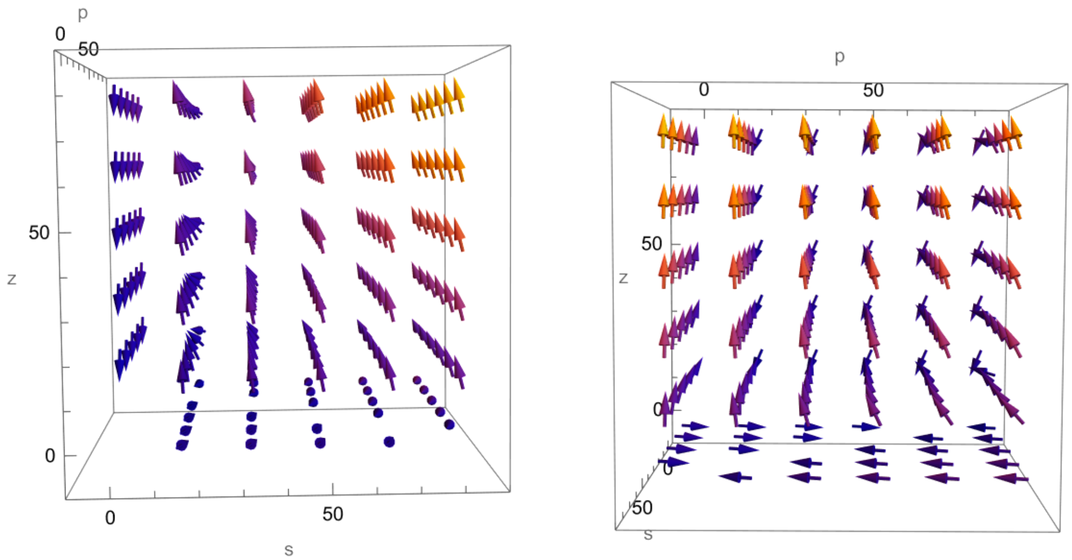


Figure 7A, B. 3D Vector graphs showing sunflower, zombie, and peashooter population interactions. Arrows are either pointing towards zombie eradication or towards plant eradication, more clearly shown in the interactive plots.

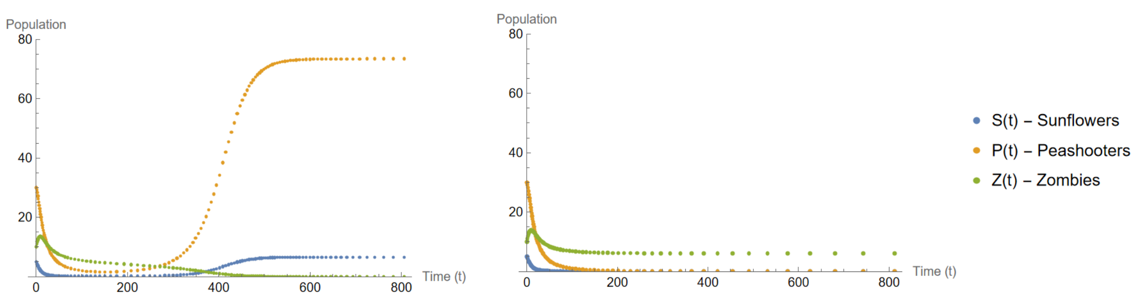


Figure 8A, B. RKF solved demonstrations of (A) Zombie extinction with parameters $(\alpha, \beta, \gamma, \delta, s_\infty, S_0, P_0, Z_0) = (0.1, 0.5, 0.01, 0.4, 80, 5, 30, 10)$ *and (B) plant extinction with the same parameters except* $\delta = 0.39$.

It's interesting to note that this system is very sensitive to changes in both initial parameters and initial populations. A $\pm 10\%$ perturbation ensemble study was conducted similar to the initial model, but this created largely different characteristic times and which equilibrium point the system approached. No reliable analytic expression for this bifurcation behavior was found, so these can instead be found through numerical methods.

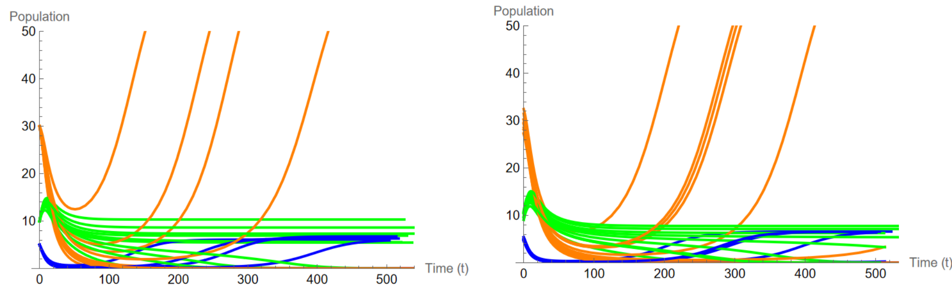


Figure 9A, B. Ensemble simulations with the base parameters and initial parameters $(\alpha, \beta, \gamma, \delta, S_\infty) = (0.1, 0.5, 0.01, 0.4, 80)$ *and initial conditions* $(S_0(blue), P_0(orange), Z_0(green)) = (5, 30, 10)$ *with (A) parameters being perturbed, and (B) populations being perturbed.*

3.4 Optimal Strategy

Finally, the section you've been waiting for: the one that'll save our lives! When the predation rate γ is relatively high ($\gamma \approx 0.01$), sunflowers are an active detriment. Better results are found when you have a decent initial peashooter population with no sunflowers! This is due to the large increase in the zombie population per sunflower encountered. However, as the encounter rate goes down ($\gamma \approx 0.001$), sunflowers become much more useful, even leading to zombie extinction without an initial peashooter population. Lower encounter rates also means the peashooter efficacy can afford to be lower, in case yours aren't straight shots. Ideally, you'd also want more peashooters than sunflowers; more protection never hurt anyone! Thus, lowering the Sun allocation for sunflowers (reducing α is beneficial).

It's worth noting that this observation occurred due to the model assuming that the plants are randomly scattered across your yard. If you were to protect your plants and place them deeper into your yard, then make zombies encounter peashooters before sunflowers, you minimise these encounter rates! It seems the best way to survive a zombie apocalypse is effective home planning, so let's get organizing!

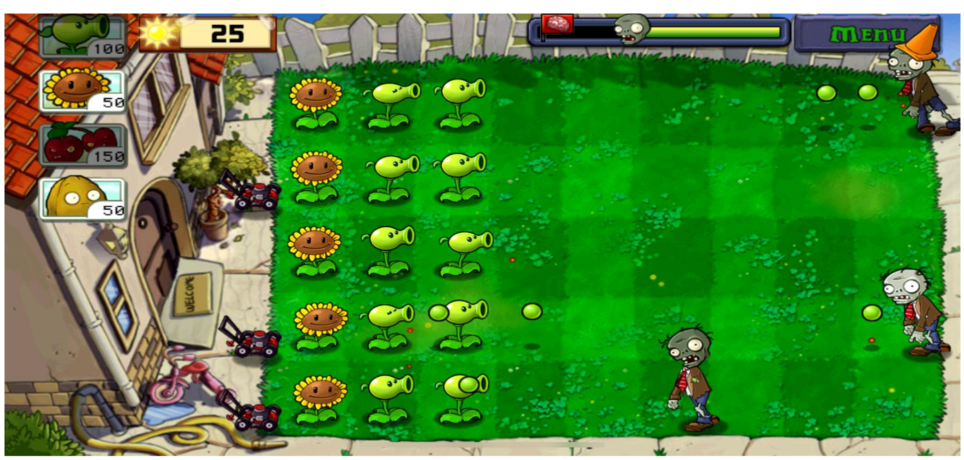


Figure 10. Implementing this strategy!

4 Sources

1. All i Zombie LEVELS! — PUZZLES — Plants vs Zombies (no date) YouTube. Available at: https://youtu.be/0jHwIe4p6as?si=lS-oPQ3l9w-b4H0 (Accessed: 24 November 2024).

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