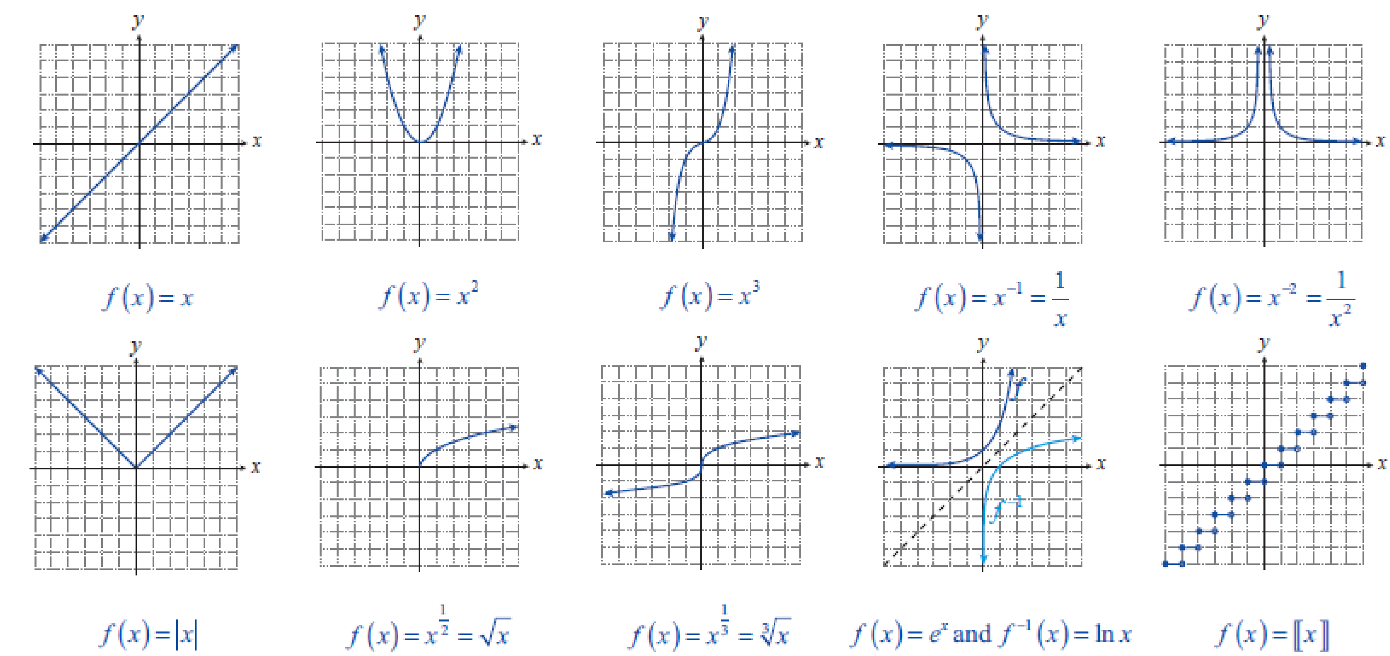
Chapter 4 Working with Functions

GRAPHS OF BASIC FUNCTIONS 3.4, 6.1, 6.3



Sec. 4.1 Transformations of Functions

Three Basic Transformations of Graphs

1. Shifting graphs vertically and horizontally
2. Reflecting graphs
3. Stretching graphs vertically and horizontally

Theorem: Horizontal Shifting (Translations)

Let $f\ (x)$ be a function, and let h be a fixed real number.

If we replace x with $x-\mathrm{h}$, we obtain a new function $g(x)=f(x-\mathrm{h})$.

The graph of g has the same shape as the graph of f, but shifted h units to the right if $\mathrm{h}>0$ and shifted $|h|$ units to the left if $\mathrm{h}<0$.

Note: Utilize Desmos.com/calculaor to understand the transformation.

Caution:

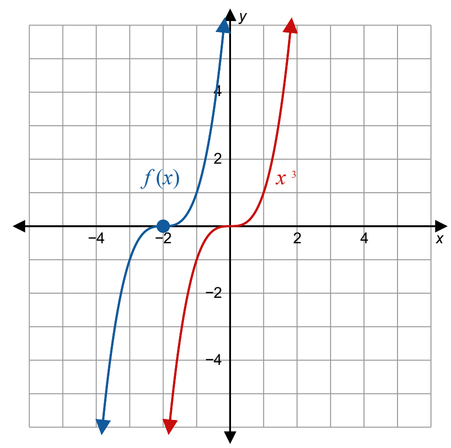
The minus sign in the expression $x - \mathrm{h}$ is critical. When you see an expression in the form $x + \mathrm{h}$ you must think of it as $x - (-\mathrm{h})$.

For example, replacing x with $x - 5$ shifts the graph 5 units to the *right*, since 5 is positive. Replacing x with $x + 5$ shifts the graph 5 units to the *left*, since we have actually replaced x with $x - (-5)$.

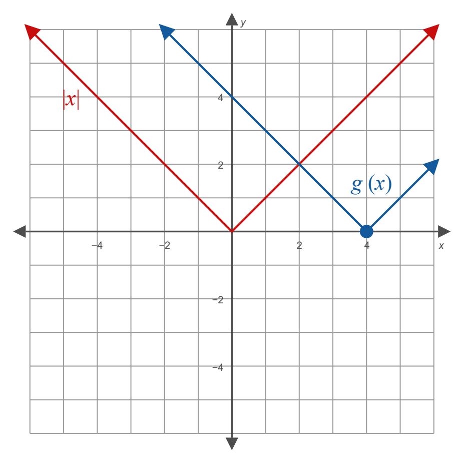
Example 1

Sketch the graphs of the following functions.

a. $f(x) = (x + 2)^3$



b. $g(x) = |x - 4|$



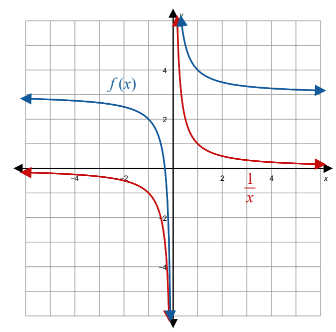
Theorem: Vertical Shifting (Translations)

Let $f\ (x)$ be a function whose graph is known, and let k be a fixed real number. The graph of the function $g(x) = f(x) + k$ is the same shape as the graph of f, but shifted k units up if $k > 0$ and $|k|$ units down if $\mathrm{k} < 0$.

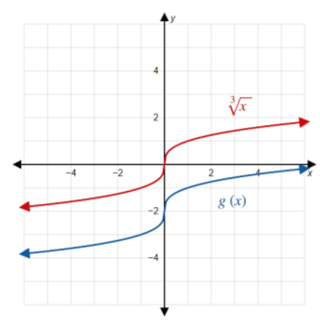
Example 2

Sketch the graphs of the following functions.

(a) $f(x) = \frac{1}{x} + 3$

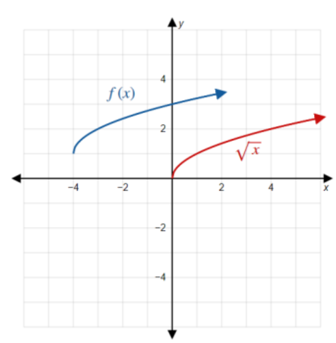


(b) $g(x) = \sqrt[3]{x} - 2$



Example 3

Sketch the graph of the function $f(x) = \sqrt{x+4} + 1$.



Theorem: Reflecting with Respect to the Axes

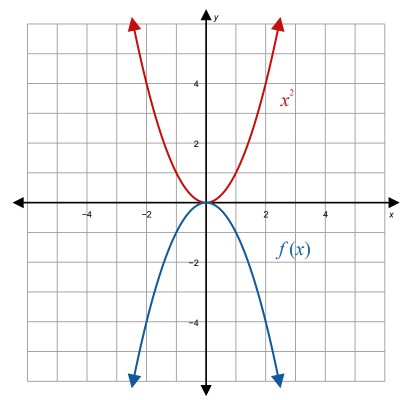
Given a function $f(x)$,

1. the graph of the function $g(x) = -f(x)$ is the reflection of the graph of f with respect to the x-axis;
2. the graph of the function $g(x) = f(-x)$ is the reflection of the graph of f with respect to the y-axis.

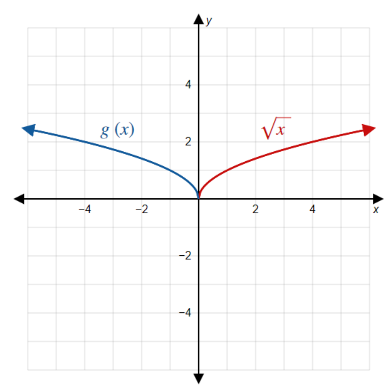
Example 4

Sketch the graphs of the following functions.

a. $f(x) = -x^2$



b. $\mathrm{g(x)} = \sqrt{-\mathrm{x}}$



Theorem: Vertical Stretching and Compressing

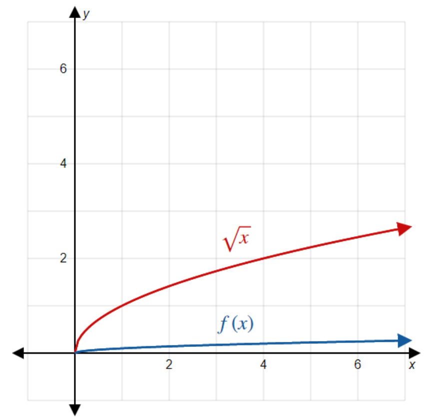
Let $f\ (x)$ be a function and let a be a positive real number.

1. The graph of the function $g(x) = a\,f(x)$ is stretched vertically compared to the graph of f by a factor of a if $a > 1$.
2. The graph of the function $g(x) = a\,f(x)$ is compressed vertically compared to the graph of f by a factor of a if $0 < a < 1$.

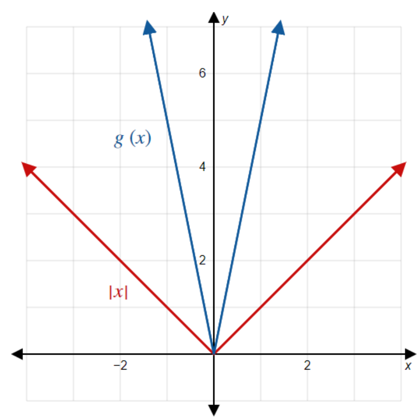
Example 5

Sketch the graphs of the following functions.

a. $f(x) = \frac{\sqrt{x}}{10}$



b. $g(x) = 5|x|$



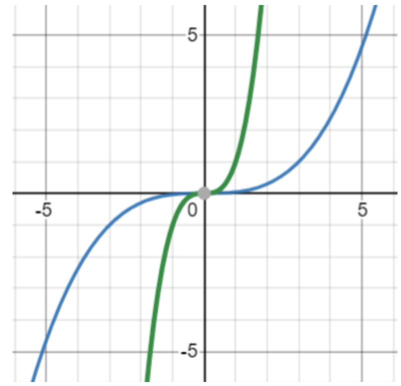
Horizontal Stretching and Compressing

Let $f(x)$ be a function and let a be a positive real number.

1. The graph of the function $g(x) = f(ax)$ is stretched horizontally compared to the graph of f by a factor of $\frac{1}{a}$ if $0 < a < 1$.
2. The graph of the function $g(x) = f(ax)$ is compressed horizontally compared to the graph of f by a factor of $\frac{1}{a}$ if $a > 1$.

Example 6 (Watch the lecture video)

Sketch the graph of the function $g(x) = \left(\frac{x}{3}\right)^3$.



Procedure: Order of Transformations

If a function g has been constructed from a simpler function f through a number of transformations, g can be understood by looking for transformations in the following order:

1. Horizontal shifts
2. Horizontal and vertical stretching and compressing
3. Reflections
4. Vertical shifts

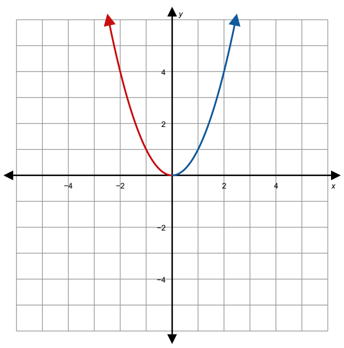
Read Example 7 and Example 8

Sec. 4.2 Properties of Functions

The graph of a function f has **y-axis symmetry**, or is **symmetric with respect to the y-axis**, if

$$f(-x) = f(x)$$

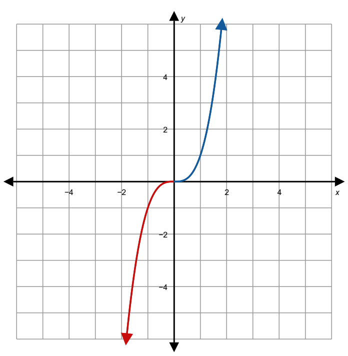
for all x in the domain of f. Such functions are called **even functions**.



The graph of a function f has **origin symmetry**, or is **symmetric with respect to the origin**, if

$$f(-x) = -f\,(x)$$

for all x in the domain of f. Such functions are called **odd functions**.



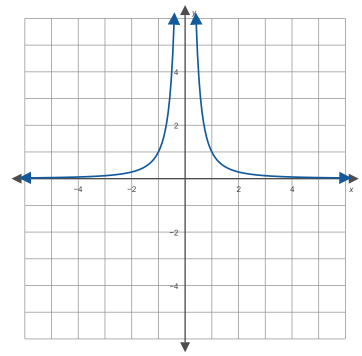
We say that an equation in x and y is **symmetric with respect to**

1. the **y-axis** if replacing x with $-x$ results in an equivalent equation; $\boldsymbol{f(-x) = f(x)}$
2. the **x-axis** if replacing y with $-y$ results in an equivalent equation;
3. the **origin** if replacing x with −x and y with −y results in an equivalent equation; $\boldsymbol{f(-x) = -f(x)}$

Example 1 Sketch the graphs of the following relations, making use of symmetry.

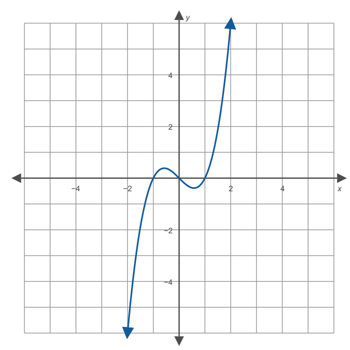
(a) $f(x) = \frac{1}{x^2}$

$f(-x) =$



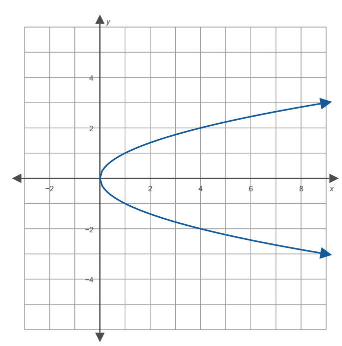
(b) $g(x) = x^3 - x$

$g(-x) =$



(c) $\mathrm{x} = \mathrm{y}^2$

$(-y)^2 =$

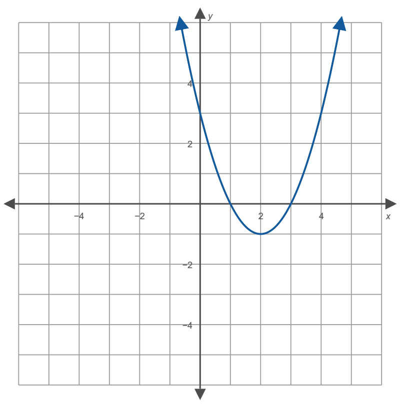


We say that a function f is

1. **increasing on an interval** if for any x_1 and x_2 in the interval with $x_1 < x_2$, it is the case that $f(x_1) < f(x_2)$;
2. **decreasing on an interval** if for any x_1 and x_2 in the interval with $x_1 < x_2$, it is the case that $f(x_1) > f(x_2)$;
3. **constant on an interval** if for any x_1 and x_2 in the interval, it is the case that $\mathrm{f}(\mathrm{x}_1) = \mathrm{f}(\mathrm{x}_2)$.

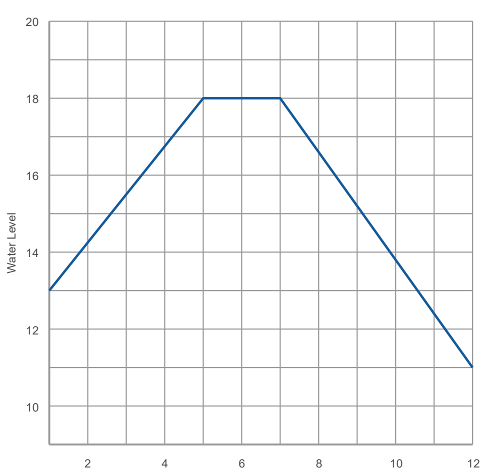
Example 2

Determine the open intervals of monotonicity of the function $f(x) = (x-2)^2 - 1$.



Example 3

The water level of a certain river varied over the course of a year as follows. In January, the level was 13 feet. From that level, the water increased linearly to a level of 18 feet in May. The water remained constant at that level until July, at which point it began to decrease linearly to a final level of 11 feet in December. Graph the water level as a function of time and determine the open intervals of monotonicity.



Given a function f defined on an interval $[a, b]$, $a \neq b$, the **average rate of change** of f over $[a, b]$ is

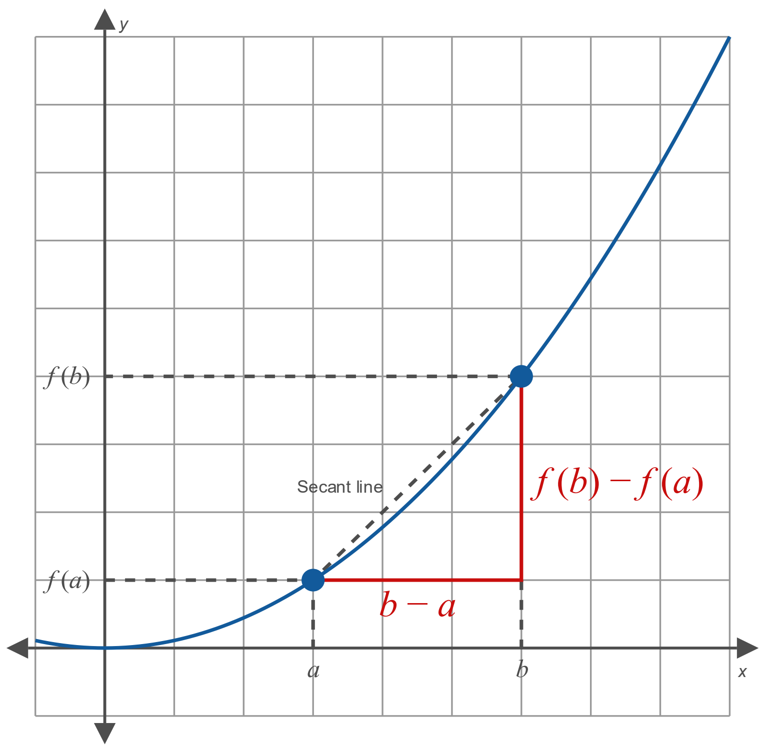
$$\frac{\mathbf{f(b)-f(a)}}{\mathbf{b-a}}.$$

If $\mathrm{y = f(x)}$,then any of the following expressions may be used to represent the average rate of change of f over $\mathrm{[a, b]}$:

$$\frac{\text{change in f}}{\text{change in x}} = \frac{\text{change in y}}{\text{change in x}} = \frac{\Delta \mathrm{f}}{\Delta \mathrm{x}} = \frac{\Delta \mathrm{y}}{\Delta \mathrm{x}} = \frac{\mathrm{f(b) - f(a)}}{\mathrm{b - a}},$$

where the Greek letter Δ is used in this context to denote a change in the variable that follows it.

The average rate of change of f over $\mathrm{[a, b]}$ represents the slope of the **secant line** drawn between the points $\mathrm{(a, f(a))}$ and $\mathrm{(b, f(b))}$ on the graph of f, as shown in Figure 15.



Example 5 Given the function $f(x) = 3x^2 - 5x + 2$, determine the average rate of change over each of the following intervals.

(a) $[1,3]$

(b) $[-2,2]$

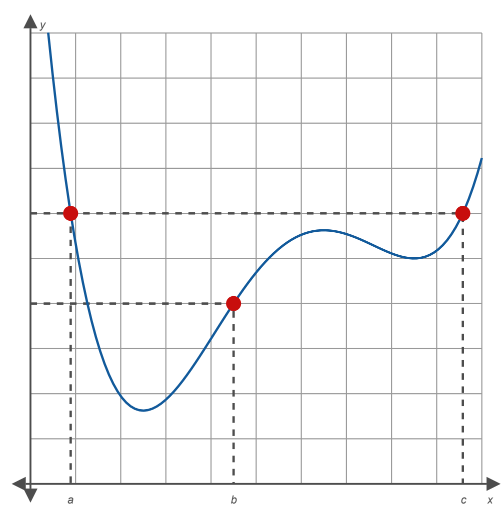
(c) $[\mathrm{c}, \mathrm{c}+\mathrm{h}]$, where $\mathrm{h} \neq 0$

Example 6 Use the given graph of the function f and the marked locations on the x-axis as endpoints to determine intervals over which the average rate of change of f is

(a) Positive

(b) Negative

(c) Zero



Sec. 4.3 Combining Functions

Let f and g be two functions. The **sum** $f + g$, **difference** $f - g$, **product** fg and **quotient** $\frac{f}{g}$ are four new functions defined as follows.

1. Sum: $(\mathrm{f} + \mathrm{g})(\mathrm{x}) = \mathrm{f}(\mathrm{x}) + \mathrm{g}(\mathrm{x})$
2. Difference: $(\mathrm{f} - \mathrm{g})(\mathrm{x}) = \mathrm{f}(\mathrm{x}) - \mathrm{g}(\mathrm{x})$
3. Product: $(\mathrm{fg})(\mathrm{x}) = \mathrm{f}(\mathrm{x})\,\mathrm{g}(\mathrm{x})$
4. Quotient: $\left(\frac{\mathrm{f}}{\mathrm{g}}\right)(\mathrm{x}) = \frac{\mathrm{f}(\mathrm{x})}{\mathrm{g}(\mathrm{x})}$, provided that $\mathrm{g}(\mathrm{x}) \neq 0$

The domain of each of these new functions consists of the common elements (or the intersection of elements) of the domains of f and g individually, with the added condition that in the quotient function we have to omit those elements for which $\mathrm{g}(\mathrm{x}) = 0$.

Example 1

Given that $f(-2) = 5$ and $g(-2) = -3$, find

$(f - g)(-2) =$

$\left(\frac{f}{g}\right)(-2) =$

Example 2

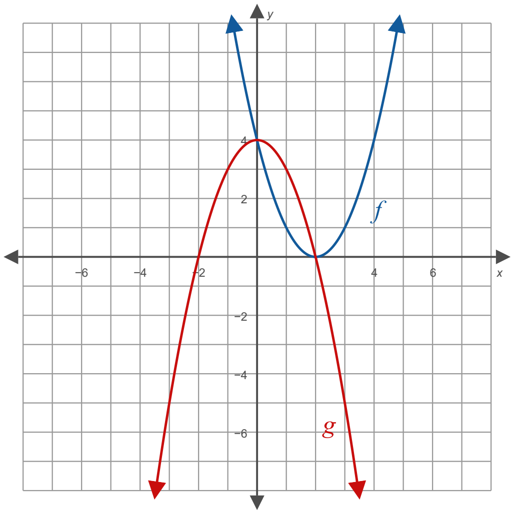
Given the two functions $f(x) = 4x^2 - 1$ and $g(x) = \sqrt{x}$, find

$(f + g)(x) =$

$(fg)(x) =$

Example 3

Given the graphs of f and g in Figure 1, determine the domain of $f + g$ and $\frac{f}{g}$ and evaluate $(f + g)\,(1)$ and $\left(\frac{f}{g}\right)\,(1)$.



$(f + g)(1) =$

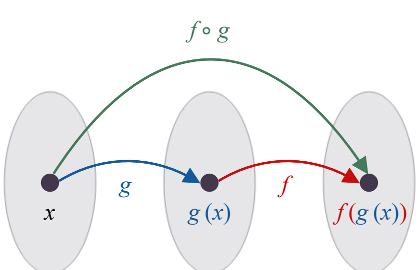
$\left(\frac{f}{g}\right)(1) =$

Let f and g be two functions.

The **composition** of f and g, denoted $f \circ g$, is the function defined by $\boldsymbol{(f \circ g)(x) = f(g(x))}$.

The domain of $f \circ g$ consists of all x in the domain of g for which $g\,(x)$ is in turn in the domain of f.

The function $f \circ g$ is read "f composed with g", or "f of g."



Note: The order of f and g is important. In general, we can expect the function $f \circ g$ to be *different* from the function $g \circ f$. In formal terms, the composition of two functions, unlike the sum and product of two functions, is not commutative.

Caution:

When evaluating the composition $(f \circ g)\,(x)$ at a point x, there are two reasons the value might be undefined:

1. If x is not in the domain of g, then $g\,(x)$ is undefined and we can't evaluate $f\,(g(x))$.
2. If $g\,(x)$ is not in the domain of f, then $f\,(g(x))$ is undefined and we can't evaluate it.

In either case, $(\mathrm{f} \circ \mathrm{g})(\mathrm{x}) = \mathrm{f}(\mathrm{g}(\mathrm{x}))$ is undefined, and x is not in the domain of $(\mathrm{f} \circ \mathrm{g})\,(\mathrm{x})$.

Example 4 Given $f(x) = x^2$ and $g(x) = x - 3$, find the following.

(a) $(f \circ g)\,(6)$

(b) $(g \circ f)\,(6)$

(c) $(f \circ g)\,(x)$

(d) $(\mathrm{g} \circ \mathrm{f})\,(\mathrm{x})$

Example 5 Let $f(x) = \sqrt{x-5}$ and $g(x) = \frac{2}{x+1}$. Evaluate the following.

(a) $(f \circ g)(-1)$

(b) $(\mathrm{f} \circ \mathrm{g})(1)$

Example 6 Let $f(x) = x^2 - 4$ and $g(x) = \sqrt{x}$. Find formulas and state the domains for the following.

(a) $f \circ g$

(b) $\mathrm{g} \circ \mathrm{f}$

Example 7 Decompose the function $f(x) = |x^2 - 3| + 2$ into the following. There may be several ways.

(a) a composition of two functions

(b) a composition of three functions

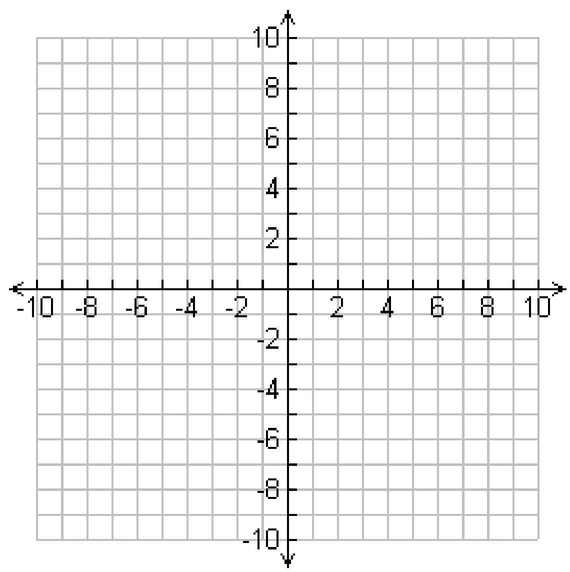
Sec. 4.4 Inverse Functions

Let R be a relation. The **inverse of** R, denoted R^{-1}, is the relation defined by switching the first and second coordinates of each ordered pair that is an element of R.

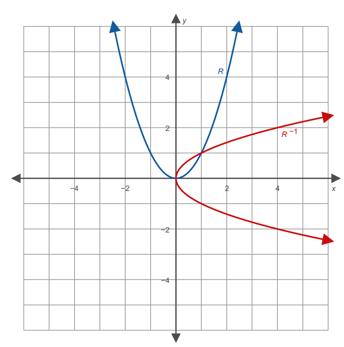
$$\mathrm{R}^{-1} = \{(\mathrm{b}, \mathrm{a}) | (\mathrm{a}, \mathrm{b}) \in \mathrm{R}\}$$

Example 1 Determine the inverse of each of the following relations. Then graph each relation and its inverse, and determine the domain and range of both.

(a) $R = \{(4, -1), (-3,2), (0,5)\}$



(b) $\mathrm{y} = \mathrm{x}^2$



Caution:

We are faced with another example of the reuse of notation. Note that f^{-1} does *not* stand for $\frac{1}{f}$ when f is a function! We use an exponent of -1 to indicate the reciprocal of a number or an algebraic expression, but when applied to a function or a relation it stands for the inverse relation.

A function f is **one-to-one** if, for every pair of distinct elements x_1 and x_2 in the domain of f, we have $\mathrm{f(x_1) \neq f(x_2)}$. This means that every element of the range of f is paired with exactly one element of the domain of f.

Theorem: The Horizontal Line Test

Let f be a function. We say that the graph of f passes the **horizontal line test** if every horizontal line in the plane intersects the graph no more than once.

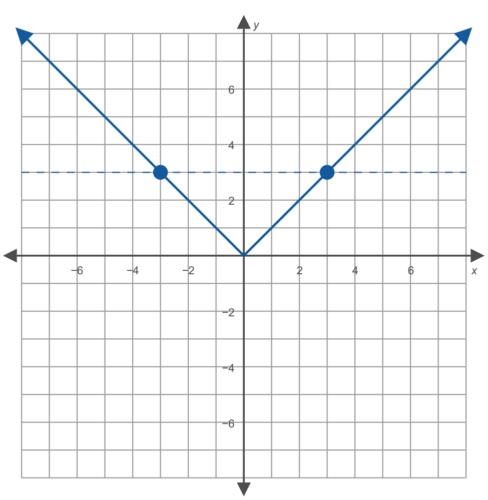
If the graph of a function passes the horizontal line test, it is a one-to-one function.

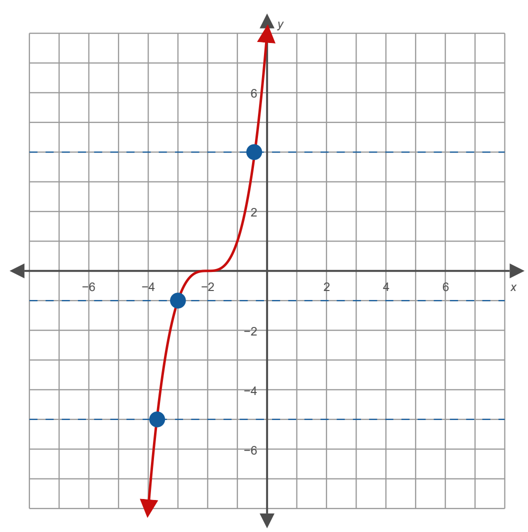
If the function is one-to-one, it has an inverse function.

Example 2 Determine if the following functions have inverse functions.

(a) $f(x) = |x|$

(b) $\mathrm{g(x) = (x + 2)^3}$





Procedure: Finding Formulas of Inverse Functions

Let f be a one-to-one function, and assume that f is defined by a formula. To find a formula for f^{-1}, perform the following steps.

Step 1: Replace $f\,(x)$ in the definition of f with the variable y. The result is an equation in x and y that is solved for y at this point.

Step 2: Solve the equation for x.

Step 3: Replace the x in the resulting equation with $f^{-1}\,(x)$ and replace each occurrence of y with x.

Example 3 Find the inverse of each of the following functions.

(a) $f(x) = (x-1)^3 + 2$

(b) $\mathrm{g(x)} = \frac{\mathrm{x-3}}{\mathrm{2x+1}}$

Theorem: Composition of Functions and Inverses

Given a function f and its inverse f^{-1}, the following statements are true:

$\mathrm{f}(\mathrm{f}^{-1}(\mathrm{x})) = \mathrm{x}$ for all $\mathrm{x} \in \mathrm{Dom}\ (\mathrm{f}^{-1})$, and

$\mathrm{f}^{-1}(\mathrm{f}(\mathrm{x})) = \mathrm{x}$ for all $\mathrm{x} \in \mathrm{Dom}\ (\mathrm{f})$.

Example 5 Use the functions f and g of Example 3 to demonstrate that the composition of a function and its inverse leaves any input unchanged.

(a) $f(x) = (x-1)^3 + 2$ and $f^{-1}(x) = (x-2)^{\frac{1}{3}} + 1$

(b) $\mathrm{g}(\mathrm{x}) = \frac{\mathrm{x}-3}{2\mathrm{x}+1}$ and $\mathrm{g}^{-1}(\mathrm{x}) = \frac{-\mathrm{x}-3}{2\mathrm{x}-1}$