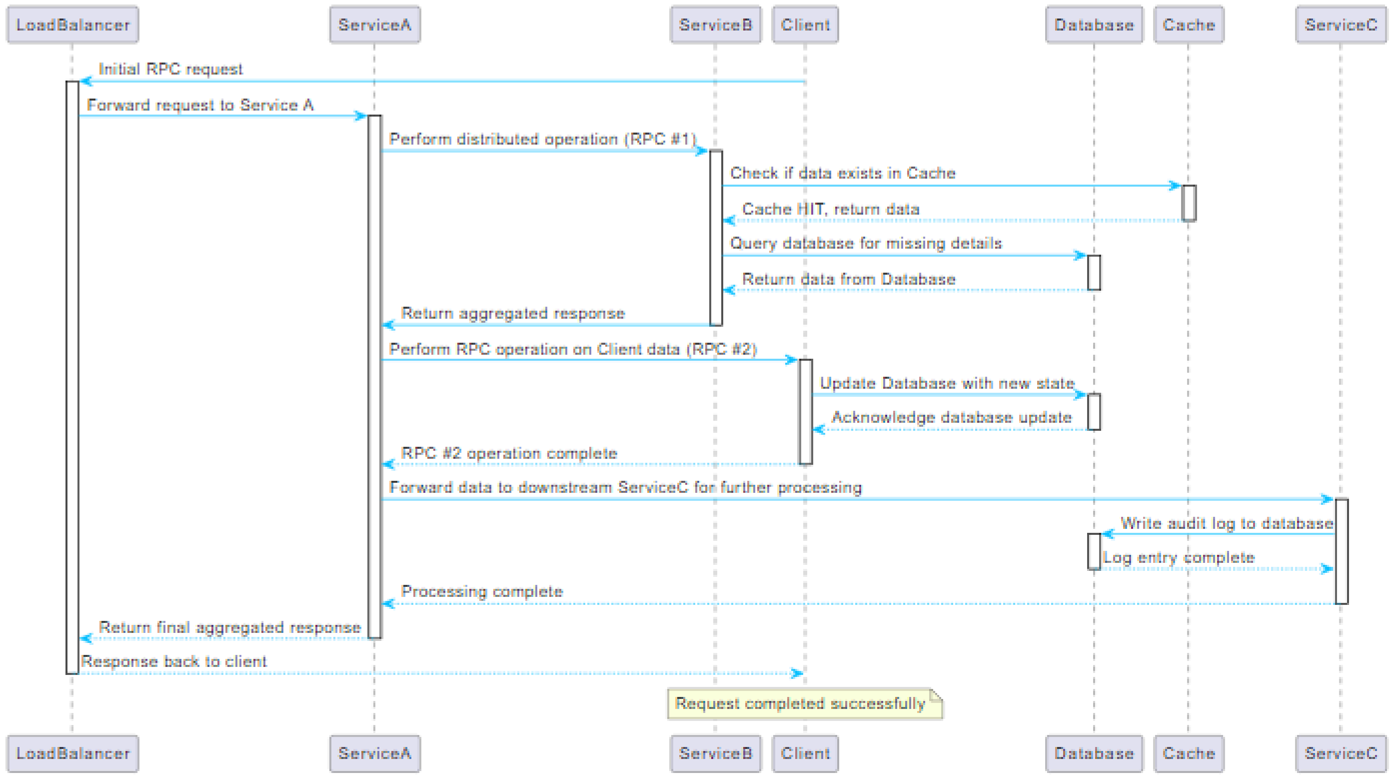
Introduction

A Comprehensive Presentation of Concepts in Machine Learning and Optimization

Visualization of PlantUML Sequence Diagram



Problem Setup

Suppose we have two classes, A and B, and we want to train a classifier to distinguish between them.

- **Core Idea**:
  - Extract a **core subset** S_A and S_B from A and B respectively.
  - Identify a decision boundary using compressed representations.

Encoder-Decoder Architecture

Step 1: Projection

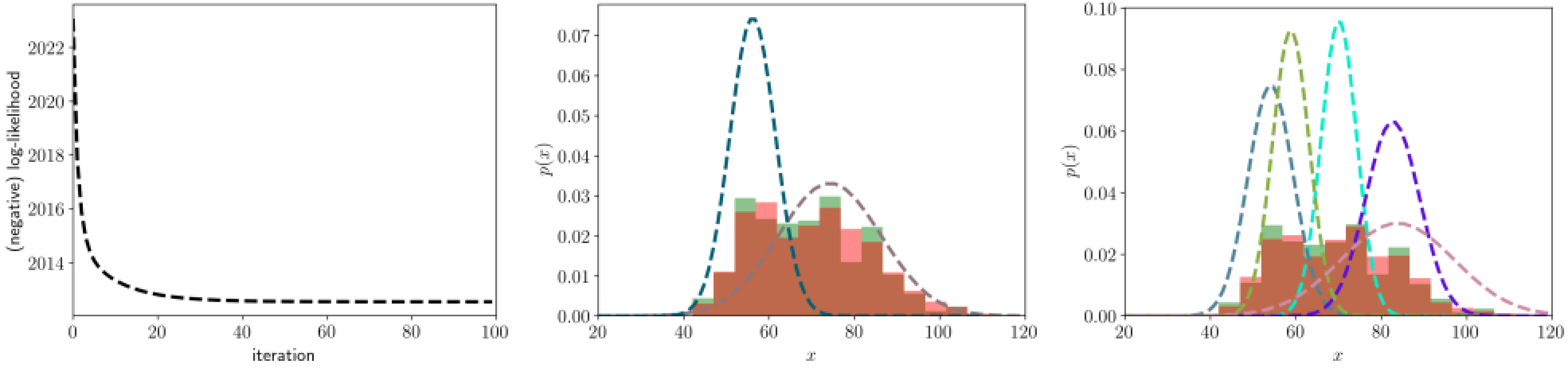
1. Use an **encoder** to project the dataset $R^n(A, B)$ into a lower-dimensional space R^k where $k \ll d$.
2. Reconstruct the dataset back to R^n.

Gaussian Mixture Modeling

- In the reduced space $R^k(A', B')$, the data looks smoother than in R^n.
- We can approximate the class distributions using **Gaussian Mixture Models (GMMs)**:
  - Represent the class distributions as linear combinations of Gaussian distributions.

$$N(\mu_A, \Sigma_A), \quad N(\mu_B, \Sigma_B)$$

Gaussian Mixture Model (GMM)



- The image shows the learning of Gaussian Mixture Models (GMMs) with different numbers of clusters ($k = 2$ and $k = 5$).
- **GMM Objective**:
  - Learn the parameters μ_k and Σ_k to best fit the data distributions.
- **Applications**:
  - Clustering
  - Dimensionality reduction
  - Density estimation

Compression Step

To compress the data:

- Learn μ_A, Σ_A, μ_B, Σ_B using **Gaussian Mixture Model Distribution Learning (GMM)**.
- Compressed dataset becomes:
  - $\mu_A, \Sigma_A, \mu_B, \Sigma_B$.
- The decoder maps $R^k \to R^n$.

Reconstruction Step

Given $\mu_A, \Sigma_A, \mu_B, \Sigma_B$, and the decoder:

Step 1: Sampling

- Sample from $N(\mu_A, \Sigma_A)$ and $N(\mu_B, \Sigma_B)$.

Step 2: Pass Samples Through Decoder

- The samples are passed through the decoder to reconstruct the dataset in R^n.

Reconstructed Dataset

- The reconstructed dataset approximates the original $R^n(A, B)$.
- This approach provides an efficient way to compress, store, and reconstruct datasets.

Kullback-Leibler (KL) Divergence

The formula for KL divergence between a reference probability distribution P and a second probability distribution Q is:

$$D_{KL}(P||Q) = \sum_{x \in \chi} P(x) \log\left(\frac{P(x)}{Q(x)}\right)$$

- **Interpretation**: The expected excess surprise from using Q as a model instead of P.

Reconstruction Loss

The code to compute the reconstruction loss:

```
from torch.nn import functional as F

def reconstruction_loss(x, x_hat):
    return F.binary_cross_entropy(x_hat, x, reduction="sum")
```

- Inputs: `x` and `x_hat` have dimensions $(N, 1, H, W)$.
- **Explanation**: This computes the negative log-likelihood of the Bernoulli distribution.

Adversarial Optimization

1. **Adversarial Loss Minimization**:

$$\min_{\delta \in C} L(x_t, y_{\text{adv}}, \theta(\delta))$$

2. **Training Loss**:

$$\theta(\delta) = \arg\min_{\theta} \sum_{i \in V} L(x_i + \delta_i, y_i, \theta)$$

3. **Perturbation Class**:

$$C = \{\delta \in \mathbb{R}^{n \times m} : ||\delta||_\infty \leq \epsilon, \delta_i = 0 \; \forall i \notin V_p\}$$

Second-Order Approximation

A second-order approximation of $F(w+\delta)$:

$$F(w+\delta) = \frac{1}{2}\delta^T H_S \delta + g_S^T \delta + F(w)$$

- $g_S = \nabla_w F(w, S)$: Gradient.
- H_S: Hessian matrix.

Federated Learning Objectives

1. **Global Objective**:

$$F_{\text{glob}}(w) = \sum_{k=1}^{m} p_k F_k(w)$$

2. **FedProx Regularization**:

$$\min_{w_{\text{loc}}} F_k(w_{\text{loc}}) + \frac{\mu}{2} ||w_{\text{loc}} - w_t||^2$$

3. **q-FFL Loss**:

$$\min_{w_M} F(w_M) = \frac{1}{q+1} \sum_{k=1}^{m} p_k F_k(w_M)^{q+1}$$

Submodularity Properties

1. **Diminishing Gains**:

$$F(A \cup \{e\}) - F(A) \geq F(B \cup \{e\}) - F(B)$$

2. **Union-Intersection**:

$$F(S) + F(T) \geq F(S \cup T) + F(S \cap T)$$

3. **Monotonicity**:

$$F(A) \leq F(B) \quad \forall A \subset B$$

Gradient-Based Clustering

The optimization problem is:

$$\arg\min_{S,\gamma_j \geq 0} |S| \quad \text{s.t.} \quad \max_{w \in W} ||\nabla_w F(w, V) - \nabla_w F(w, S)|| \leq \epsilon$$

Greedy Algorithms

1. **Subset Selection**:

$$F(S_k) \geq \left(1 - \frac{1}{e}\right) F(\mathrm{OPT})$$

2. **Complexity**:

- Time Complexity: $O(nk)$.

Hessian-Based Optimization

The bound is:

$$d_{ij} \leq \text{constant} \times ||x_i - x_j||$$

This guides the selection of core sets S^* for efficient learning.

Summary

- Explored concepts in optimization, adversarial learning, and federated learning.
- Applied submodularity and second-order approximations to enhance efficiency.
- Addressed practical scenarios with examples and mathematical rigor.

Autoencoders

- Autoencoders are neural networks designed for unsupervised learning.
- They consist of two main parts:
    1. **Encoder**: Maps input (x) to a lower-dimensional latent representation (z).
    2. **Decoder**: Reconstructs (x) from (z).

Architecture of Autoencoders

```
Input --> Encoder --> Latent Space --> Decoder --> Output
```

- **Objective**:

  Minimize reconstruction loss (L(x, \hat{x})).

Reconstruction Loss

The reconstruction loss measures how well the autoencoder reconstructs the input:

$$L(x, \hat{x}) = ||x - \hat{x}||^2$$

Alternatively, for binary data:

$$L(x, \hat{x}) = -\sum_i \left[x_i \log(\hat{x}_i) + (1 - x_i) \log(1 - \hat{x}_i)\right]$$

Compression with Autoencoders

- **Goal**: Reduce the dimensionality of the input while retaining essential information.
- Latent space acts as a compressed representation.
- Applications:
  - Image compression.
  - Dimensionality reduction.

Lemma: Gradient Descent Convergence

Lemma: For a differentiable convex function (f(w)) with a Lipschitz-continuous gradient (L), the gradient descent update:

$$w_{t+1} = w_t - \eta \nabla f(w_t)$$

with learning rate (\eta \leq \frac{1}{L}), satisfies the following convergence property:

$$f(w_{t+1}) - f(w^*) \leq \frac{1}{2\eta} ||w_t - w^*||^2 - \frac{1}{2\eta} ||w_{t+1} - w^*||^2$$

where (w^*) is the global minimum of (f(w)).

Limitations of Standard Autoencoders

- They struggle to learn meaningful latent spaces.
- Lack of control over latent representations.
- Enter **Variational Autoencoders (VAEs)**.

Variational Autoencoders (VAEs)

- **VAEs** extend autoencoders to probabilistic models.
- Instead of a fixed latent (z), VAEs learn a probability distribution (p(z|x)).
- Outputs are sampled from this distribution.

VAE Architecture

```
Input --> Encoder --> Latent Distribution --> Decoder --> Output
```

- **Encoder**: Produces (\mu) and (\sigma^2) for latent distribution.
- **Latent Distribution**: Sampled using reparameterization trick.
- **Decoder**: Reconstructs (x) from samples (z).

Latent Space in VAEs

- In VAEs, the latent space represents a distribution.
- Assumes $(z \sim \mathcal{N}(\mu, \sigma^2))$, a Gaussian distribution.

VAE Loss Function

The VAE loss combines two terms:

1. **Reconstruction Loss**:

$$L_{\text{recon}}(x, \hat{x}) = ||x - \hat{x}||^2$$

2. **KL Divergence** (regularizer):

$$L_{\text{KL}} = D_{\text{KL}}(q(z|x)||p(z))$$

KL Divergence in VAEs

The KL divergence regularizes the latent space:

$$D_{\mathrm{KL}}(q(z|x)||p(z)) = \int q(z|x) \log \left(\frac{q(z|x)}{p(z)} \right) dz$$

- Encourages (q(z|x)) to be close to the prior (p(z)).

Reparameterization Trick

To allow backpropagation through the sampling process:

- Replace $z \sim \mathcal{N}(\mu, \sigma^2)$ with:

$$z = \mu + \sigma \odot \epsilon, \quad \epsilon \sim \mathcal{N}(0, I)$$

Applications of VAEs

1. **Image Generation**:
   - Generate new samples by sampling (z) from (p(z)).
2. **Anomaly Detection**:
   - Use reconstruction loss to identify anomalies.
3. **Data Imputation**:
   - Fill in missing data.

Compression

- Compression aims to reduce data storage requirements while maintaining quality.
- Lossy Compression:
  - Examples: JPEG, MP3.
  - Allows small errors for higher compression.
- Lossless Compression:
  - Examples: PNG, FLAC.
  - Retains original data perfectly.

Information Bottleneck Principle

- A theoretical framework for compression in neural networks.
- Balances:
  - **Compression**: Reduce information from (x) to (z).
  - **Relevance**: Ensure (z) retains information about (y).

Objective of Information Bottleneck

Minimize the following loss:

$$\mathcal{L} = I(x; z) - \beta I(z; y)$$

Where:

- (I(x; z)): Mutual information between (x) and (z).
- (I(z; y)): Mutual information between (z) and (y).
- (\beta): Controls the trade-off.

Connection Between VAEs and Information Bottleneck

- VAEs implicitly optimize an information bottleneck objective.
- KL Divergence term in VAEs regularizes the latent space.

Challenges in VAEs

1. Balancing reconstruction loss and KL divergence.
2. Posterior collapse:
   - When $(q(z|x))$ becomes overly simple (e.g., close to prior $(p(z))$).

Key Takeaways

1. Autoencoders and VAEs are powerful tools for representation learning.
2. The latent space is critical for meaningful representations.
3. Information bottleneck provides a theoretical foundation for compression and relevance trade-offs.

Thank You!

Questions? 🤔