Rational Root Theorem

Claim 1:

$$f(z) = 0 \longleftrightarrow z \mid a_0$$

$$a_n z^n + a_{n-1} z^{n-1} \ldots a_1 z + a_0 = 0$$

$$\Rightarrow -a_0 = a_n z^n + a_{n-1} z^{n-1} \ldots a_1 z$$

$$= z\,(\underbrace{a_n z^{n-1} + a_{n-1} z^{n-2} \ldots a_1}_{})$$

Since all coefficients are $\mathbb{Z}$ then this is also $\mathbb{Z}$

$\therefore$ z divides a_0

Claim 2:

$$a_n\left(\frac{p}{q}\right)^n + a_{n-1}\left(\frac{p}{q}\right)^{n-1} + \dots a_0 = 0$$

$\cancel{\Rightarrow -a_0 = p(a_n}$ multiply by $q^n \rightarrow$

$$a_n p^n + a_{n-1} p^{n-1} q \dots a_0 q^n = 0$$

$$\Rightarrow -a_0 q^n = p\left(a_n p^{n-1} \underbrace{+ a_{n-1} p^{n-2} q}_{\text{integer}} \dots\right)$$

$\therefore$ p divides a_0 since p & q are relatively prime on 1 $\longrightarrow$ subtract $a_n p^n$ $\longrightarrow$

$$a_{n-1}p^{n-1}q + a_{n-2}p^{n-2}q^2 \dots \ a_0 q^n = -a_n p^n$$

$$\Rightarrow \ -a_n p^n = q\left(a_{n-1}p^{n-1} + a_{n-2}p^{n-2}q \dots \ a_0 q^{n-1}\right)$$

Since p and q are co-primes,
a_n is divisible by q