Securely Training Decision Trees Efficiently

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ABSTRACT

Decision trees are an important class of supervised learning algorithms. When multiple entities contribute data to train a decision tree (e.g. when training models for fraud detection in the financial sector), data privacy concerns necessitate the use of a privacy-enhancing technology such as secure multi-party computation (MPC) in order to perform the training securely. Prior state-of-the-art (Hamada *et al.* [16]) showed how to construct an MPC protocol for this problem with a communication overhead of $O(hmN \log N)$, when building a decision tree of height h, with a training set having N data points with m attributes each.

In this work, we significantly improve upon this result by constructing an MPC protocol for secure decision tree training with communication complexity $O(mN \log N + hmN + hN \log N)$ hence achieving an improvement of $\approx \min(h, m, \log N)$. At the core of our technique is an improved protocol to regroup sorted private elements further into additional groups (according to a flag vector) while maintaining the relative ordering of the elements. We implement our protocol in the MP-SPDZ framework and show it to be 10× more communication efficient and 9× faster than [16].

KEYWORDS

decision tree training, secure multi-party computation

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1 INTRODUCTION

One of the most important and popular machine learning algorithms is decision tree learning. In this supervised learning approach, a classification or regression tree is built based on a set of features or attributes present in the training dataset. As with many learning algorithms, the accuracy of decision trees can be greatly improved with larger volumes of data. However, this can be a challenge since data may be present with many different parties who are unwilling to share their data in the clear with each other. Consider, for example, a decision tree model being built to analyze the risk profile associated with banking transactions. While the model built could benefit significantly from data from multiple banks, this is realistically difficult given the privacy concerns.

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The natural question is: *can multiple parties, holding subsets of a training dataset, jointly train a decision tree model?* Of course, the cryptographic technique of secure multi-party computation (MPC) [8, 15, 34] provides a generic way to compute any function securely when multiple mutually untrusted parties contribute private data and hence provides a solution to the above problem. Typically, the most significant drawback of MPC protocols is the amount of communication between the participating parties (which scales with the function being computed). While generic MPC protocols can be inefficient, much work over the last few decades has brought down their overheads for specific functions.

Perhaps the first work to consider secure decision tree training was that of Lindell and Pinkas [22]. The work of Hoogh *et al.* [13] provided the first concretely efficient MPC protocol for decision tree training for the specific case when all attributes are categorical (or discrete) values. For the more general (and useful) case of continuous attributes, the work of Abspoel [3] showed how to build a concretely efficient MPC protocol. When there are a total of N datapoints, with m attributes each, in order to build a decision tree of height h, this protocol required a total communication of $O(2^h mN \log N)$. The work of Hamada *at al.* [16] introduced a new secure data structure for decision trees and showed how to use that to obtain a protocol with communication cost of $O(hmN \log N)$.

1.1 Our Results

In this work, we make the following contributions:

- First, we significantly improve upon the prior state-of-the-art [16] and provide a secure decision tree training algorithm with cost $O(mN \log N + hmN + hN \log N)$, improving upon the communication complexity[1] by a factor of $\min(h, m, \log N)$.
- At the heart of our new protocol is a protocol that securely enables us to divide multiple groups of sorted secret elements into further sub-groups, according to a flag vector, while retaining their relative ordering (at the same time not revealing the group boundaries) with a complexity of $O(N)$. Such a protocol might be of independent interest.
- Finally, we implement our protocol for the setting of 3 parties tolerating one semi-honest corruption (the same setting of [16]) and show that our protocol is concretely 10× more communication frugal and 9× faster. Our code will be made publicly available.

1.2 Our Techniques

To describe our techniques, let us revisit the protocol of Hamada *et al.* [16]. At a very high level, their protocol worked by constructing the tree layer by layer for a tree with h layers (i.e., tree of height h). At every layer, the data elements are grouped (privately) according to what was computed in previous layers. Now at this layer, within

[1] All asymptotic communication overheads presented in the paper include an implicit factor of ℓ, the bitlength of elements. We omit this everywhere for the sake of readability.

each group, one needs to compute (what is known as) the Gini index for all possible attributes and thresholds. Then, again within each group, one must compute the attribute and threshold which has the minimum Gini index. Finally, the nodes must be grouped further according to this. Computing the Gini index requires sorting the data elements within each group according to the attribute value (after which the Gini index can be computed with $O(N)$ communication). The cost of secure sorting is $O(N \log N)$ and hence [16] paid communication cost of $O(hmN \log N)$ as this sorting had to be performed for all attributes at every level. Additionally, computing the attribute and threshold which has the minimum Gini index was done in [16] in 2 steps - first, by computing the minimum Gini index for each attribute in each group using a GroupMax protocol (with cost $O(hmN \log N)$) and then computing the minimum over all groups with a Max protocol (with cost $O(hmN)$).

Our protocol makes 2 fundamental changes to the above protocol. First, we sort the data elements according to each attribute value *once* (and not at each level). Naturally, this leads to a major problem – at every level, we need to somehow get the data elements *within each group* once again sorted without having to sort them again and without revealing which elements went to which group. To achieve this, we construct a new protocol that reduces this functionality to computing a new permutation of the data elements given the original permutation and the grouping. We show how to do this with communication overhead of $O(N)$. Thus, we only must perform the secure sorting once (irrespective of the number of layers in the tree). This results in a communication overhead of $O(mN \log N)$ as opposed to $O(hmN \log N)$. Second, in order to compute the minimum Gini index, we reverse the two operations performed in [16]. That is, we first compute the minimum value over every index (over all attributes) and then apply the GroupMax protocol *once* over the resulting vector. We show that doing so preserves functionality since the group boundaries remain the same. Crucially, this results in a reduced communication overhead of $O(hmN + hN \log N)$ (instead of $O(hmN \log N)$).

Our protocols make use of several lower level protocols for multiplication, comparisons, oblivious shuffling, and other groupwise operations and we rely on prior works for the constructions of these lower level protocols. Finally, we implement our protocol for the case of 3-party secure computation tolerating 1 semi-honest corruption. This is the same setting as considered in [3, 16] and is based on replicated secret sharing techniques.

1.3 Other Related Work

Protocols for secure training over horizontally (or vertically) partitioned dataset are proposed in [25, 27, 33] ([23, 30, 31]). These works assume that the computing servers know the partition of the training dataset in the clear. Moreover, the main efficiency improvements of these protocols comes from leaking intermediate outputs in the training which can allow an adversary to learn non trivial information about the data (e.g. an adversary can learn the training data distribution from the leakage in [30]).

In contrast, training algorithms that guarantee the complete privacy of the dataset have been designed either using Fully Homomorphic Encryption [14] or MPC based techniques (such as our work and the works discussed in the previous subsection). In the FHE based decision tree protocols [4, 5], a data owner sends encrypted dataset to a server which runs the training algorithm. While these algorithms have low communication and round complexity, they have two major limitations: a) the training algorithm is modified to approximate various functions such as comparisons; and b) the computational overhead is exorbitantly high making it impractical (e.g. the runtime in [4] for training even a small decision tree of height 4 with only $10,000$ samples is ≈ 1.5 days).

A long line of work has focused on secure decision tree evaluation only [10, 12, 18, 24, 28, 29, 32]. Our trained decision tree is compatible with these works and can be augmented to facilitate both secure training and evaluation of decision trees.

1.4 Organization

Section 2 introduces notation, the security model and provides a description of the cleartext algorithm for decision tree training. It also introduces secret sharing schemes and specifically the replicated secret sharing scheme used in this work. In Section 3, we describe the various crypto building blocks needed to construct our secure decision tree protocols. This includes element wise operations (such as multiplication, comparison etc.) and also vector and groupwise operations (e.g. shuffling, GroupMax etc.) for which protocols were presented in prior works. Section 4 provides the complete description of our secure decision tree training protocol, while Section 5 describes the security proof of our new protocols. In Section 6, we present details of our implementation and all experimental results.

2 PRELIMINARIES

2.1 Notation

Let $\mathbb{N}$ be the set of natural numbers. $[N]$ denotes the set $\{1, \ldots, N\}$ for $N \in \mathbb{N}$. $\mathbb{Z}_{2^\ell}$ is a ring of integers modulo 2^ℓ for $\ell \in \mathbb{N}$. Subset B of set A is denoted by $B \subseteq A$. We use n to denote $\log_2 N$.

2.2 Security Model

While all our protocols can be instantiated in any model that provides protocols for secure multiplications, comparisons, and equality, for concreteness, we consider an honest majority setting with semi-honest corruption. Specifically, there are three parties S_0, S_1, S_2. All parties follow the protocol specification faithfully (semi-honest behavior) and we show that the view of no single party reveals any information. All recent works on privacy preserving decision tree training [3, 16] also consider this setting.

Our protocols satisfy the simulation based security definition [21]. Suppose Π is the 3-party protocol that computes the functionality $\mathcal{F}$. We have a semi honest adversary $\mathcal{A}$ that can corrupt at most one party out of S_0, S_1 and S_2. For security, we require that the view of adversary $\mathcal{A}$ and the outputs of honest parties in the real world are computationally indistinguishable from the view and output of a simulator in an ideal world which has access to the ideal functionality $\mathcal{F}$. $\mathcal{F}$ simply receives inputs from all three parties and provides the function output (as defined by the functionality) to all parties i.e. for any semi honest non-uniform PPT adversary $\mathcal{A}$ that corrupts at most one party, there exists an ideal world PPT simulation Sim such that for any input inp to the protocol Π that computes $\mathcal{F}$, we have $\mathsf{Real}(\Pi, \mathcal{A}, \mathsf{inp}) \approx_c \mathsf{Ideal}(\mathcal{F}, \mathsf{Sim}, \mathsf{inp})$ where Real is the joint distribution of the view of adversary $\mathcal{A}$ and output

of honest parties in the real world, and Ideal is the joint distribution of the view of the simulator Sim and output of honest parties in the ideal world.

2.3 Cleartext Decision Tree Algorithms

We begin with a description of the cleartext decision tree algorithms we use in this work (which are the same as the ones in [3, 16]). We have a labelled dataset $\mathbf{D}$ with m attributes $\mathbf{x} = (x_1, \ldots, x_m)$ and label y. $\mathbf{D}[i] = (x_1[i], \ldots, x_m[i], y[i])$ are the attribute and label values for the i^{th} sample in the dataset. A decision tree is a machine learning model that predicts the value of output variable y given the input attributes $\mathbf{x}$.

A binary decision tree has two kinds of nodes (1) internal nodes associated with a test of the form $x_j < t$ (2) leaf nodes associated with a label b. Each edge out of an internal node denotes a possible outcome of the test i.e. true or false, and a corresponding child node (true child or false child). Typically, the left child is the false child and right child is the true child.

Decision Tree Evaluation. Given input attributes $\mathbf{x}$ of a sample, we can predict its label as follows. Starting from the root node, the test at each internal node is evaluated for the sample and based on the outcome of test, we trace an outgoing edge to reach the child node i.e. if the test $x_j < t$ holds for the input attributes, we traverse to the right child, else, we traverse to the left child. Traversing in this fashion, we reach a leaf node and output the leaf label as the predicted value of the label of the sample. The attributes can be continuous where attribute values are real numbers, or discrete where the attribute values are from a finite set.

Decision tree training. The algorithm to securely train decision trees with discrete attributes is straightforward and given in [13]. We focus on the more complex case of continuous attributes as our algorithm can be easily extended to handle discrete attributes as well. A more detailed discussion on how to handle discrete attributes is given in Appendix A.

Algorithm 14 (Appendix B) describes how to train a decision tree of height h with continuous attributes. The decision tree is trained from root node to leaf nodes in a recursive manner. To distinguish leaf nodes from internal nodes, the algorithm checks the height of the current node. If the height is less than h, the current node is an internal node. In that case, the algorithm selects a test $x_j < t$ to be performed at that node that splits the dataset in a way that minimizes the Gini index of the resulting split. The Gini index G for splitting a dataset $\mathbf{D}$ using test $x_j < t$ is defined as:

$$G_{x_j<t}(\mathbf{D}) = \frac{|\mathbf{D}_{x_j<t}|}{|\mathbf{D}|}\,\mathrm{Gini}(\mathbf{D}_{x_j<t}) + \frac{|\mathbf{D}_{x_j\geq t}|}{|\mathbf{D}|}\,\mathrm{Gini}(\mathbf{D}_{x_j\geq t})$$

where $\mathrm{Gini}(\mathbf{D}) = 1 - \sum_{b\in\{0,1\}} (|\mathbf{D}_{y=b}|^2/|\mathbf{D}|^2)$. The best split has the minimum Gini index and Abspoel et al. [3] showed that minimizing Gini index is same as maximizing the following expression:

$$\mathrm{G}'_{x_j<t}(\mathbf{D}) = \left(|\mathbf{D}_{x_j\geq t}|\left(|\mathbf{D}_{x_j<t\wedge y=0}|^2 + |\mathbf{D}_{x_j<t\wedge y=1}|^2\right)\right. \quad (1)$$
$$\left. + |\mathbf{D}_{x_j<t}|\left(|\mathbf{D}_{x_j\geq t\wedge y=0}|^2 + |\mathbf{D}_{x_j\geq t\wedge y=1}|^2\right)\right) / \left(|\mathbf{D}_{x_j<t}|\,|\mathbf{D}_{x_j\geq t}|\right),$$

where $\mathbf{D}_{x_j<t\wedge y=b} = \{(\mathbf{x}, y) \in \mathbf{D} \mid x_j < t \wedge y = b\}$. The algorithm selects the test $x_j < t$ that maximizes this expression.

Once the test is selected, the algorithm partitions the training dataset into $\mathbf{D}_{x_j<t}$ and $\mathbf{D}_{x_j\geq t}$. The left subtree $\mathcal{T}_l$ is trained on the partition $\mathbf{D}_{x_j<t}$ and right subtree $\mathcal{T}_r$ is trained on the partition $\mathbf{D}_{x_j\geq t}$. The algorithm outputs a decision tree with the current node as root node, $\mathcal{T}_l$ as left child and $\mathcal{T}_r$ as right child.

If the height the of current node is h, it is a leaf node. Let $\mathbf{D}_L$ be the partition of dataset $\mathbf{D}$ corresponding to the L^{th} leaf node. Then the label of leaf node is set to the most occuring label in the partition $\mathbf{D}_L$. In case of binary classification, we can compute the label of L^{th} leaf node as follows:

$$\text{Label} = \sum_{i\in\mathbf{D}_L} y[i] \overset{?}{>} \sum_{i\in\mathbf{D}_L} (1 - y[i])$$

Number encodings. Real numbers are represented using fixed or floating point numbers. However, we do not perform any computation on attribute values except for comparisons in decision tree training. This means that we can map these attribute values to elements of a sufficiently large ring as long as the ordering is preserved. We use $\mathbb{Z}_{2^\ell}$ ring (setting $\ell = 64$ in the experiments) to represent the continuous attribute values and class labels. The training algorithm can be extended to handle class labels from a larger domain $\mathbb{Z}_c, c > 2$ by using $\mathbf{y}_1, \ldots, \mathbf{y}_c \in \{0,1\}^N$ where $(\mathbf{y}_1[i], \ldots, \mathbf{y}_c[i])$ is a one-hot encoding of the label of the i^{th} sample and by appropriately using a measure such as mean squared error instead of Gini index. We focus on the case of binary labels in this work.

2.4 Secret Sharing Schemes

A secret sharing scheme allows a dealer to distribute a secret value amongst a group of parties such that only allowed subsets of parties can reconstruct the secret value and any unauthorized subset of parties do not learn anything about the secret.

Consider ring $\mathbb{Z}_{2^\ell}$ for $\ell \in \mathbb{N}$. An (η, t)-threshold secret sharing scheme over $\mathbb{Z}_{2^\ell}$, where $\eta, t \in \mathbb{N}, t \leq \eta$ is a pair of algorithms (Share, Reconstruct) such that:

- Share is a randomized algorithm that on input $v \in \mathbb{Z}_{2^\ell}$ outputs η-tuple of shares $(v_1, \ldots, v_\eta)$.
- Reconstruct is a deterministic algorithm that on input of t-tuple of shares outputs a value $v' \in \mathbb{Z}_{2^\ell}$.

The sharing scheme satisfies the following correctness property:

$\forall v \in \mathbb{Z}_{2^\ell}$ and $\forall A \subseteq [\eta]$ such that $|A| = t$

$$\Pr\left[v' = v \,|\, \text{Share}(v) = \{v_i\}_{i\in[\eta]}, \text{Reconstruct}\left(\{v_i\}_{i\in A}\right) = v'\right] = 1$$

Additionally, the secret sharing scheme satisfies the following security property: $\forall v \in \mathbb{Z}_{2^\ell}, \forall A \subseteq [\eta]$ s.t. $|A| \leq t-1$, the following distributions are perfectly indistinguishable

$$\left\{\{v_i\}_{i\in A} \,|\, \text{Share}(v) = (v_1, \ldots, v_\eta)\right\}$$
$$\left\{\{r_i\}_{i\in A} \,|\, \text{Share}(0) = (r_1, \ldots, r_\eta)\right\}$$

(3, 2) Replicated Secret Sharing (RSS) is an instance of a threshold linear secret sharing scheme that can be used to distribute a secret among 3 parties with corruption threshold $t - 1 = 1$. (3, 2)-RSS over ring $\mathbb{Z}_{2^\ell}$ has the following algorithms:

- Share: On input $v \in \mathbb{Z}_{2^\ell}$, sample $v_0, v_1, v_2 \in \mathbb{Z}_{2^\ell}$ such that $v_0 + v_1 + v_2 = v \bmod 2^\ell$ and output (x_0, x_1, x_2) where
$$x_0 = (v_0, v_1), x_1 = (v_1, v_2), x_2 = (v_2, v_0)$$
- Reconstruct: On input $x_i = (v_0, v_1), x_j = (v_1, v_2) \in \mathbb{Z}_{2^\ell}^2$ where $i \neq j; i, j \in \{1, 2, 3\}$, output $v' = v_0 + v_1 + v_2 \bmod 2^\ell$.

The above described pair of algorithms satisfy the correctness and security guarantees of secret sharing schemes [17]. We denote the replicated shares of $v \in \mathbb{Z}_{2^\ell}$ by $\langle v \rangle$ and the boolean replicated shares $v \in \mathbb{Z}_2$ by $\langle v \rangle^{\mathbf{B}}$. The share of party S_i is (v_i, v_{i+1}).

3 CRYPTO BUILDING BLOCKS

In this section, we describe the functionalities that serve as basic building blocks to construct a secure computation protocol for the decision tree training algorithm presented in Section 2.3. All functionalities are 3-party functionalities where parties begin with shares of inputs according to the (3,2)-RSS scheme described in Section 2.4 and receive shares of outputs according to the same secret sharing scheme. We also describe the (existing) protocols realizing these functionalities and their efficiency.

3.1 Element wise operations

We require the following primitives to build our protocols:

- **Secure Multiplication:** denoted by $\langle z \rangle = \Pi_{\mathsf{MULT}}(\langle u \rangle, \langle v \rangle)$.
- **Secure Bit Decomposition:** denoted by $\langle u \rangle^{\mathbf{B}} = \Pi_{\mathsf{A2B}}(\langle u \rangle)$.
- **Secure Comparison:** denoted by $\langle b \rangle = \Pi_{\mathsf{LT}}(\langle u \rangle, \langle v \rangle)$.
- **Secure Equality:** denoted by $\langle b \rangle = \Pi_{\mathsf{EQ}}(\langle u \rangle, \langle v \rangle)$.

These functionalities can be instantiated in Arithmetic Black Box model using $(3, 2)$ replicated sharing. We discuss the protocols for these and their communication cost in Appendix C.

3.2 Permuting Secret Vectors

We represent a permutation over N elements as a length N vector $\mathbf{P}$ where $\mathbf{P}[i]$ denotes the position of the i^{th} element in the permuted order. Let $\mathbf{A} \in \mathbb{Z}_{2^\ell}^N$ be a vector and $\mathbf{P}$ be a permutation. Let $\mathbf{A}^{\mathbf{P}}$ be the rearrangement of $\mathbf{A}$ according to $\mathbf{P}$ i.e. $\mathbf{A}^{\mathbf{P}} = \text{Permute}(\mathbf{A}, \mathbf{P})$. Since $\mathbf{P}[i]$ denotes the new position of i^{th} element, we have
$$\mathbf{A}^{\mathbf{P}}[\mathbf{P}[i]] = \mathbf{A}[i] \quad \text{for all } i \in [N]$$
The secret share of a permutation vector is defined as
$$\langle \mathbf{P} \rangle = (\langle \mathbf{P}[1] \rangle, \ldots, \langle \mathbf{P}[N] \rangle)$$
We make use of four functionalities that permute secret vectors – a) randomly shuffle a secret vector; b) apply a secret permutation to a secret vector; c) compose secret permutations; and d) stably sort a vector – and we describe them below. The protocols, with linear communication cost realizing these functionalities were given in [6, 7] (see Appendix D). Table 1 gives the concrete cost.

Oblivious Shuffling. (Functionality 1) Party $S_i (i \in \{0, 1, 2\})$ inputs replicated share of vector $\langle \mathbf{X} \rangle_i \in \mathbb{Z}_{2^\ell}^N$ and pair of permutations $(\pi_i, \pi_{(i+1) \bmod 3})$. The functionality permutes $\mathbf{X}$ according to $\pi = \pi_2 \circ \pi_1 \circ \pi_0$ and outputs replicated shares of the permuted vector $\mathbf{Y}$ to parties. Protocol Π_{Shuffle} in [6] (Figure 5) computes the functionality with $4N\ell$ bits of communication in 2 rounds.

The Shuffle functionality can be used to shuffle a vector by permutation π^{-1} by applying the permutations in reverse order i.e. apply π_2^{-1}, then π_1^{-1} and finally π_0^{-1} as $\pi^{-1} = \pi_0^{-1} \circ \pi_1^{-1} \circ \pi_2^{-1}$. A typical use of shuffling is to hide data dependent memory accesses. To do this, parties shuffle a vector using π, perform the memory accesses and then shuffle by π^{-1} to get back the same ordering.

Functionality 1: Shuffle

Input: Party S_i, $i \in \{0, 1, 2\}$ inputs $\langle \mathbf{X} \rangle_i$ and (π_i, π_{i+1})
Output: Parties receive $\langle \mathbf{Y} \rangle$ where $\mathbf{Y}$ is a rearrangement of $\mathbf{X}$ according to $\pi = \pi_2 \circ \pi_1 \circ \pi_0$.

1 Reconstruct $\mathbf{X}$ and $\pi = \pi_2 \circ \pi_1 \circ \pi_0$.
2 Compute $\mathbf{Y} = \text{Permute}(\mathbf{X}, \pi)$.
3 Output $\langle \mathbf{Y} \rangle$.

Apply permutation. Functionality 2 takes shares of a vector $\mathbf{X}$ and permutation $\mathbf{P}$ and outputs shares of $\mathbf{X}^{\mathbf{P}}$ where $\mathbf{X}^{\mathbf{P}} = \text{Permute}(\mathbf{X}, \mathbf{P})$. Protocol $\Pi_{\mathsf{ApplyPerm}}$ (Algorithm 15) in [7] realizes ApplyPerm with $8N\ell$ bits of communication in 2 rounds.

Functionality 2: ApplyPerm

Input: Parties input $\langle \mathbf{X} \rangle$ and $\langle \mathbf{P} \rangle$
Output: Parties receive $\langle \mathbf{X}^{\mathbf{P}} \rangle$ where $\mathbf{X}^{\mathbf{P}} = \text{Permute}(\mathbf{X}, \mathbf{P})$.

1 Reconstruct $\mathbf{X}$ and $\mathbf{P}$.
2 Compute $\mathbf{X}^{\mathbf{P}}$ where $\mathbf{X}^{\mathbf{P}} = \text{Permute}(\mathbf{X}, \mathbf{P})$.
3 Output $\langle \mathbf{X}^{\mathbf{P}} \rangle$.

Composing permutations. Functionality 3 takes as input shares of permutations $\mathbf{P}, \mathbf{Q}$ and outputs shares of permutation $\mathbf{R} = \mathbf{P} \circ \mathbf{Q}$ where $\mathbf{P} \circ \mathbf{Q}$ denotes applying $\mathbf{Q}$ followed by applying $\mathbf{P}$ so $\mathbf{R}[i] = \mathbf{P}[\mathbf{Q}[i]]$ i.e. i^{th} element goes to $\mathbf{Q}[i]$ position which then goes to $\mathbf{P}[\mathbf{Q}[i]]$ position. Protocol Π_{PermComp} (Algorithm 16) in [7] computes PermComp with $8N\ell$ bits communication in 4 rounds.

Composing permutation is particularly helpful when we have multiple permutations (say k permutations) that we have to apply to multiple vectors (say t vectors of length N) sequentially. We would have to apply k permutations to each vector which will cost $kt \times 8N\ell$ bits using ApplyPerm. Instead we use PermComp to compute the final permutation to be applied to vectors and apply this permutation to all l vectors which costs $(k + t) \times 8N\ell$ bits.

Functionality 3: PermComp

Input: Parties input $\langle \mathbf{P} \rangle$ and $\langle \mathbf{Q} \rangle$
Output: Parties receive $\langle \mathbf{R} \rangle$ where $\mathbf{R} = \mathbf{P} \circ \mathbf{Q}$.

1 Reconstruct $\mathbf{P}, \mathbf{Q}$.
2 Compute $\mathbf{R}$ where $\mathbf{R}[i] = \mathbf{P}[\mathbf{Q}[i]] \ \forall i \in [N]$.
3 Output $\langle \mathbf{R} \rangle$.

Sorting permutation for binary key. Functionality 4 takes a binary vector $\mathbf{b} \in \{0, 1\}^N$ as input and outputs shares of permutation $\mathbf{P}$ that stably sorts $\mathbf{b}$.

Definition 3.1. Permutation $\mathbf{P}$ stably sorts a list $\mathbf{b}$ if $\forall i, j \in [N]$

$$\mathbf{P}[i] < \mathbf{P}[j] \implies (\mathbf{b}[i] < \mathbf{b}[j]) \text{ OR } (\mathbf{b}[i] = \mathbf{b}[j] \text{ AND } i < j)$$

i.e. $\mathbf{P}$ sorts the vector while preserving the order of entries with same value. Protocol $\Pi_{\mathsf{SortPermBit}}$ (Algorithm 17) in [7] computes SortPermBit with $3N\ell$ bits communication in 1 round.

Functionality 4: SortPermBit

Input: Parties input $\langle \mathbf{b} \rangle$ where $\mathbf{b} \in \{0,1\}^N$.
Output: Parties receive $\langle \mathbf{P} \rangle$ where $\mathbf{P}$ stably sorts $\mathbf{b}$.

1 Reconstruct $\mathbf{b}$.
2 Compute permutation $\mathbf{P}$ that stably sorts $\mathbf{b}$.
3 Output $\langle \mathbf{P} \rangle$.

Sorting permutation for arbitrary key. Functionality 5 takes a vector $\mathbf{x} \in \mathbb{Z}_{2^\ell}^N$ as input and outputs shares of the permutation that stably sorts $\mathbf{x}$. Protocol Π_{SortPerm} (Algorithm 18) in [7] securely computes SortPerm with N calls to Π_{A2B}, l calls to $\Pi_{\mathsf{ApplyPerm}}$, l calls to $\Pi_{\mathsf{SortPermBit}}$ and l calls to Π_{PermComp}.

Functionality 5: SortPerm

Input: Parties input $\langle \mathbf{x} \rangle$ where $\mathbf{x} \in \mathbb{Z}_{2^\ell}^N$.
Output: Parties receive $\langle \mathbf{P} \rangle$ where $\mathbf{P}$ stably sorts $\mathbf{x}$.

1 Reconstruct $\mathbf{x}$. Compute permutation $\mathbf{P}$ that stably sorts $\mathbf{x}$.
2 Output $\langle \mathbf{P} \rangle$.

3.3 Groupwise Operations

Groupwise operations, first described in [16], are important constructions used in decision tree training protocols.

Privately grouped vector. Let $\mathbf{x}$ be a vector of length N secret shared among parties and grouped internally. This means that the vector is divided into groups of different sizes. To represent these internal groups, parties also have a secret shared *group flag vector* $\mathbf{g}$ of length N such that $\mathbf{g}[i] = 1$ if $\mathbf{x}[i]$ is the first element of any group. For example, let $\mathbf{x} = [2, 5, 6, 8, 10, 13]$ and $\mathbf{g} = [1, 0, 0, 0, 1, 0]$. Then entries with index 1 to 4 belong to first group and entries with index 5 to 6 belong to second group as $\mathbf{g}[5] = 1$ (Assume 1-indexing). Note that $\mathbf{g}[1]$ is always 1 as it is the first entry of the first group. This data structure allows parties to secret share internally grouped vectors and compute groupwise operations efficiently.

The authors of [16] give efficient secure protocols to compute various groupwise operations like Group Sum, Group Prefix Sum and Group Max. We will require these group wise operations as building blocks. Let $i \in [N]$ be an index and l_i, r_i be the leftmost and rightmost index of the group i belongs to. In the above example, $l_i = 1, r_i = 4$ for $1 \le i \le 4$ and $l_i = 5, r_i = 6$ for $5 \le i \le 6$. Then we can define the following group wise operations:

(1) $\langle \mathbf{a} \rangle = \Pi_{\mathsf{GroupSum}}(\langle \mathbf{x} \rangle, \langle \mathbf{g} \rangle)$
(2) $\langle \mathbf{b} \rangle = \Pi_{\mathsf{GroupPrefixSum}}(\langle \mathbf{x} \rangle, \langle \mathbf{g} \rangle)$
(3) $\langle \mathbf{c} \rangle = \Pi_{\mathsf{GroupMax}}(\langle \mathbf{x} \rangle, \langle \mathbf{g} \rangle)$
(4) $\langle \mathbf{d} \rangle = \Pi_{\mathsf{GroupFirstOne}}(\langle \mathbf{x} \rangle, \langle \mathbf{g} \rangle)$ defined when $\mathbf{x} \in \{0,1\}^N$

Protocol	Input	Cost	Rounds
ApplyPerm	$\langle \mathbf{x} \rangle, \langle \mathbf{P} \rangle \in \mathbb{Z}_{2^\ell}^N$	$8N\ell$	2
PermComp	$\langle \mathbf{P} \rangle, \langle \mathbf{Q} \rangle \in \mathbb{Z}_{2^\ell}^N$	$8N\ell$	4
SortPermBit	$\langle \mathbf{b} \rangle \in \mathbb{Z}_{2^\ell}^N$	$3N\ell$	1
SortPerm	$\langle \mathbf{x} \rangle \in \mathbb{Z}_{2^\ell}^N$	$42nN\ell$	n
GroupSum	$\langle \mathbf{x} \rangle, \langle \mathbf{b} \rangle \in \mathbb{Z}_{2^\ell}^N$	$25N\ell$	6
GroupPrefixSum	$\langle \mathbf{x} \rangle, \langle \mathbf{g} \rangle \in \mathbb{Z}_{2^\ell}^N$	$28N\ell$	7
GroupMax	$\langle \mathbf{x} \rangle, \langle \mathbf{g} \rangle \in \mathbb{Z}_{2^\ell}^N$	$24nN\ell$	$n \log \ell$
GroupFirstOne	$\langle \mathbf{b} \rangle, \langle \mathbf{g} \rangle \in \mathbb{Z}_{2^\ell}^N$	$51N\ell$	$2 \log \ell$

Table 1: Communication (in bits) and Rounds of Protocols

where $\forall i \in [N]$,

$$\mathbf{a}[i] = \sum_{j=l_i}^{r_i} \mathbf{x}[j]$$
$$\mathbf{b}[i] = \sum_{j=l_i}^{i} \mathbf{x}[j]$$
$$\mathbf{c}[i] = \mathrm{Max}_{j \in [l_i, r_i]}\, \mathbf{x}[j]$$
$$\mathbf{d}[i] = 1 \text{ if } \mathbf{x}[i] = 1 \text{ and } \mathbf{x}[j] = 0 \; \forall \; j \in [l_i, i)$$

By instantiating SortPermBit and ApplyPerm with protocols from Appendix D, we can compute groupwise operations with $O(N)$ communication in $O(1)$ rounds. In contrast, the communication complexity of these protocols in [16] was $O(N \log N)$ and $O(\log N)$ rounds. Hence, we save a factor of $\log N$ in both communication and round complexity in groupwise operations. The concrete complexities are summarised in Table 1.

4 SECURE DECISION TREE TRAINING

We begin with a description of the input and output formats of our secure training algorithm.

Input. To begin with, the 3 parties have (3, 2) replicated secret shares of a labelled dataset $\mathbf{D} \in \left(\mathbb{Z}_{2^\ell}^m \times \{0,1\}\right)^N$ where m is the number of attributes and N is the number of samples in dataset $\mathbf{D}$. The height h of the decision tree to be built is public information. $\mathbf{D}[j] = (\mathbf{x}_1[j], \ldots, \mathbf{x}_m[j], \mathbf{y}[j])$ are the attribute and label values for the j^{th} sample. $\mathbf{x}_i \in \mathbb{Z}_{2^\ell}^N$ is a vector of all the values of the i^{th} attribute and $\mathbf{y} \in \{0,1\}^N$ is the vector of labels.

Output. We setup some notation. $\mathsf{Layer}^{(k)}$, parameterized by $k \in \{1, \ldots, h+1\}$, is a collection of nodes at a distance of $k-1$ from root node. There are $n_k = 2^{k-1}$ nodes in $\mathsf{Layer}^{(k)}$. $\mathsf{Layer}^{(1)}$ consists of only the root node. $\mathsf{Layer}^{(2)}$ consists of child nodes of the root node and so on. If a node belongs to $\mathsf{Layer}^{(k)}$, its child nodes belong to $\mathsf{Layer}^{(k+1)}$. The leaf nodes belong to $\mathsf{Layer}^{(h+1)}$.

Each internal node has three variables associated with it: node id (NID), splitting attribute ($A \in [m]$) and splitting threshold ($T \in \mathrm{domain}(A)$). Each leaf node has two variables: node id (NID) and label ($L \in \{0,1\}$). The NID is unique for each node within a layer. For a node with node id NID in $\mathsf{Layer}^{(k)}$, the left child has node id NID and right child has node id $NID + 2^{k-1}$ in $\mathsf{Layer}^{(k+1)}$.

Final output. The final output of the secure training algorithm Π_{Train} (Algorithm 19) is the trained binary decision tree of height h in the format described below. For all internal node layers $k \in \{1, \ldots, h\}$, parties output $\left\langle \mathsf{Layer}^{(k)} \right\rangle = (\langle \mathbf{NID}_k \rangle, \langle \mathbf{A}_k \rangle, \langle \mathbf{T}_k \rangle)$ where

(1) $\mathbf{NID}_k$ is a vector of length n_k that stores the node id of nodes in the k^{th} layer. If the tree constructed is an incomplete binary tree i.e. the actual number of nodes are less than n_k, then the remaining elements of $\mathbf{NID}_k$ are initialized to 0.
(2) $\mathbf{A}_k$ is a vector of length n_k that stores the attribute for splitting at nodes in k^{th} layer. If $\mathbf{NID}_k[i] = 0$, then $\mathbf{AID}_k[i] = 0$.
(3) $\mathbf{T}_k$ is a vector of length n_k that stores the threshold for splitting at nodes in k^{th} layer. If $\mathbf{NID}_k[i] = 0$, then $\mathbf{T}_k[i] = 0$.

For a given layer, $(\mathbf{NID}_k[i], \mathbf{A}_k[i], \mathbf{T}_k[i])$ are the node id, attribute and threshold of $i^{th}(1 \le i \le n_k)$ node in the layer.

For the leaf node layer i.e. $k = h+1$, parties output $\left\langle \mathsf{Layer}^{(h+1)} \right\rangle = (\langle \mathbf{NID}_{h+1} \rangle, \langle \mathbf{L}_{h+1} \rangle)$ where $\mathbf{L}_{h+1}$ is a vector of length n_{h+1} that stores labels corresponding to leaf nodes. An example output of training algorithm with the corresponding decision tree is given in Figure 1 to illustrate how the output of training algorithm encodes a binary decision tree.

Intermediary outputs. The layers of the decision tree are trained sequentially. To train the k^{th} layer ($k \in [h+1]$), parties input a tuple of vectors $\mathsf{State}^{(k)}$ to $\Pi_{\mathsf{TrainInternalLayer}}$ (Algorithm 12) and receive $\mathsf{Layer}^{(k)}$ and $\mathsf{State}^{(k+1)}$ as output where $\mathsf{State}^{(k+1)}$ will serve as input to train the next layer. We call $\mathsf{State}^{(k)}$, $k \in [h+1]$, an intermediary output of the training protocol. $\mathsf{State}^{(k)}$, $k \in [h+1]$, is a tuple of $m+3$ vectors $\{\mathbf{D}_k, \mathbf{g}_k, \{\mathbf{P}_k^i\}_{i\in[m]}, \mathbf{NID}\}$, each of length N, where

(1) $\mathbf{D}_k$ is the training dataset grouped by nodes in layer k,
(2) $\mathbf{g}_k$ is the group flag vector encoding groups in $\mathbf{D}_k$,
(3) $\mathbf{P}_k^i$ (for all $i \in [m]$) is the permutation that sorts $\mathbf{D}_k$ based on the i^{th} attribute value within each group; and
(4) vector $\mathbf{NID}$ is the node id of each sample in $\mathbf{D}_k$. For example if $\mathbf{D}_k[i]$ belongs to the subset of the dataset of the j^{th} node, $\mathbf{NID}[i] = j$.

As described in the cleartext decision tree algorithm, each node has a subset of the dataset on which it is trained. Nodes in the k^{th} layer partition the dataset which is encoded by a group flag vector $\mathbf{g}_k$ of length N as described in Section 3.3. The samples in the dataset belonging to the same group appear consecutively. That means for $i < j$ s.t. $\mathbf{g}_k[i] = \mathbf{g}_k[j] = 1$ and $\mathbf{g}_k[l] = 0$ for all $i < l < j$, all samples from $\mathbf{D}_k[i]$ to $\mathbf{D}_k[j-1]$ belong to the same group. As a layer is trained and new groups are formed, the partition of the dataset changes and the dataset $\mathbf{D}_k$ has to be rearranged to ensure that samples in the same group appear consecutively. Vectors $\mathbf{g}_k, \mathbf{P}_k^i$ and $\mathbf{NID}$ are also updated accordingly. The output $\mathsf{State}^{(k+1)}$ stores these updated vectors which can be used to train the next layer.

Table 2 summarizes the different variables that are used in the training algorithm.

$\mathsf{Layer}^{(1)}$	
NID	1
A	2
T	10

$\mathsf{Layer}^{(2)}$		
NID	1	2
A	1	3
T	50	20

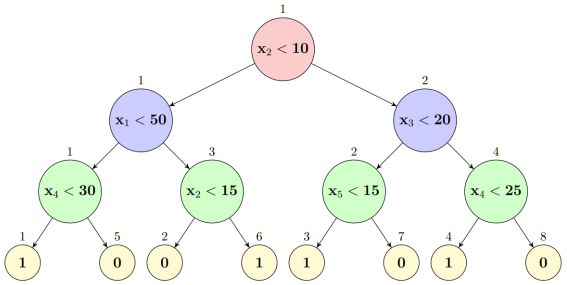
$\mathsf{Layer}^{(3)}$				
NID	1	2	3	4
A	4	5	2	4
T	30	15	15	25

Internal Node Layers

$\mathsf{Layer}^{(4)}$								
NID	1	2	3	4	5	6	7	8
L	1	0	1	1	0	1	0	0

Leaf Node Layer

(a) Example output of training algorithm for height 3.



(b) Decision Tree constructed from output.

Figure 1: (a) is a sample output of our training algorithm for height 3 which takes as input 5 attribute values and outputs a binary label. There are 3 internal node layers and 1 leaf node layer. (b) is the decision tree encoded in the output. The layers are color coded. Each node has a node id (written on top of the node), splitting attribute ($\mathsf{A} \in [1, 5]$) and splitting threshold (T). In the output, $\mathsf{Layer}^{(1)}$ has one node with value $(1, 2, 10)$ which means there is one node in 1st layer with node id 1 and test $\mathsf{x}_2 < 10$. From the numbering of node ids, the left child (right child) of root node will have node id 1 (2) respectively in $\mathsf{Layer}^{(2)}$. From the description of $\mathsf{Layer}^{(2)}$ in output, the left child will have test $\mathsf{x}_1 < 50$ and right child will have test $\mathsf{x}_3 < 20$ as shown in (b). Continuing this way, we can map the output description (a) to the decision tree (b).

4.1 Efficiency Bottleneck in the State-of-art

The communication complexity of the state-of-art secure decision tree training protocol [16] is $O(hmN \log N)$. We identify and discuss the efficiency bottlenecks in this algorithm and the challenges in removing those bottlenecks. Then, we redesign the training algorithm to solve these challenges and remove the bottlenecks, achieving both asymptotic and concrete improvements in communication complexity over the state-of-art.

Recall that all the nodes in a layer are trained together and the layers of the tree are trained sequentially using the train layer subroutine. Hence, we focus on training of one internal node layer. The training of an internal node layer consists of the following steps:

(1) Compute Gini index for all possible attributes and thresholds within each group.
(2) Compute splitting attribute and threshold associated with minimum Gini index in each group. This gives the output Layer.

Variable	Description
h	height of tree
m	number of attributes
N	number of training samples
ℓ	bitlength of attributes
D	Labeled training dataset
$\mathbf{x}_i$	i^{th} attribute vector
y	Label vector
g	Group flag vector
NID	node id
A	splitting attribute
T	splitting threshold
L	Leaf label
$\text{Layer}^{(k)}$	k^{th} layer information
$\text{State}^{(k)}$	Invariant state at k^{th} layer
n_k	Number of nodes in $\text{Layer}^{(k)}$

Table 2: Variables and their description.

(3) Compute new partition of the dataset based on the splitting attributes and thresholds. This gives the output State which serves as input to the next layer.

Step (1): To compute the Gini index for all possible thresholds of one attribute, the parties need to sort the attribute and label vector $(\mathbf{x}, \mathbf{y})$ according to the attribute values within each group. Then the parties can compute expression 1 securely for all groups with $O(N)$ communication. The cost of secure sorting is $O(N \log N)$ so the total cost in training due to step 1 is $O(hmN \log N)$. Note that the cost of computing Gini index for all attributes and thresholds over an *unsorted* dataset requires $O(hmN^2)$ [3] commmunication. So even though the secure sorting is a bottleneck in this step, it is an essential operation.

Step (2): Minimum Gini index (maximum expression 1) for each group is computed in two steps. First, parties compute the minimum Gini index for each attribute in each group using Π_{GroupMax} and then compute the minimum Gini index among all attributes using Π_{Max}([3], Fig. 13). Π_{GroupMax} contributes $O(hmN \log N)$ to the total cost while Π_{Max} contributes $O(hmN)$.

Step (3): To compute the new partition, parties securely compare the attribute values of each sample with the corresponding threshold. This splits each group into two new groups based on whether the test result is true or false. The dataset is sorted based on the outputs of the test i.e. all samples with false (0) result followed by all samples with true (1) result. This sorting arranges the dataset according to the new groups. The sorted dataset along with the updated group flag vector marking new groups is the input to train the next layer. Using the optimized $\Pi_{\text{SortPermBit}}$ (Algorithm 4), this step costs $O(hmN)$.

In [16], the secure sorting performed for every layer in **Step (1)** and GroupMax in **Step (2)** are the two bottlenecks in efficiency contributing $O(hmN \log N)$ each to communication.

4.2 Technical Overview

In this section, we discuss the challenges in removing these bottlenecks and how we can overcome these challenges.

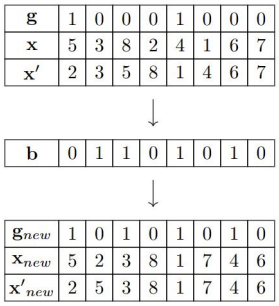


Figure 2: $\mathbf{x}$ is an attribute vector with two groups and $\mathbf{x}'$ is sorted $\mathbf{x}$ according to attribute value within groups. Let $\mathbf{b}$ be the test results. Then $\mathbf{x}_{\text{new}}$ is the new attribute vector with new groups which is obtained by stably sorting $\mathbf{x}$ based on $\mathbf{b}$. $\mathbf{x}'_{\text{new}}$ is the new sorted attribute vector. As ordering of $\mathbf{x}$ changes due to new groups, the permutation that sorts $\mathbf{x}_{\text{new}}$ to $\mathbf{x}'_{\text{new}}$ also changes.

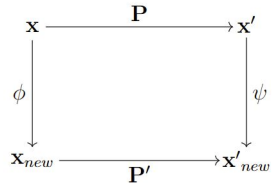


Figure 3: Borrowing notations from Figure 2, updated sorting permutation for $\mathbf{x}_{\text{new}}$ is $\mathbf{P}' = \psi \circ \mathbf{P} \circ \phi^{-1}$.

Idea 1. First, we note that in [16], for each layer and attribute, parties have to sort, within the groups, the attribute values and label vector $(\mathbf{x}, \mathbf{y})$ according to the attribute values in **Step (1)**. Since the groups are different and contain different sets of elements in each layer, [16] had to run a sorting protocol to sort the attribute values and label vector in every layer (thus leading to a communication complexity of $O(hmN \log N)$ for this step). See Figure 2 for an example on how the attribute vector is updated from $\mathbf{x}$ to $\mathbf{x}_{\text{new}}$ as new groups are created due to the split.

The key observation to address this challenge is that even though $\mathbf{x}$ and $\mathbf{x}_{\text{new}}$ are different, $\mathbf{x}_{\text{new}}$ is simply a permutation of $\mathbf{x}$. Similarly, $\mathbf{x'}_{\text{new}}$ is a permutation of $\mathbf{x}'$. This means that there are permutations ϕ, ψ such that $\mathbf{x}_{\text{new}} = \text{Permute}(\mathbf{x}, \phi)$ and $\mathbf{x'}_{\text{new}} = \text{Permute}(\mathbf{x}', \psi)$. Given the old sorting permutation $\langle \mathbf{P} \rangle$ (that sorts $\mathbf{x}$), $\langle \phi^{-1} \rangle$ and $\langle \psi \rangle$, the new sorting permutation (that sorts $\mathbf{x}_{\text{new}}$) is $\mathbf{P}' = \psi \circ \mathbf{P} \circ \phi^{-1}$ (see Figure 3) which parties can compute with $O(N)$ bits using Π_{PermComp}. Moreover, we show that $\langle \phi^{-1} \rangle$ and $\langle \psi \rangle$ can be computed with $O(N)$ bits using $\Pi_{\text{SortPermBit}}$. Thus, given the old sorting permutation $\langle \mathbf{P} \rangle$, parties can compute the new sorting permutation $\langle \mathbf{P}' \rangle$ with linear communication cost. This allows parties to update the sorting permutation whenever a layer is trained and new groups are formed. These sorting permutations can be applied to $(\mathbf{x}, \mathbf{y})$ directly with a cost of $O(N)$. Thus, the cost of **Step (1)** becomes $O(hmN)$ improving by a factor of $\log N$ over [16]. Parties will also

update the sorting permutations in **Step (3)** whose cost remains $O(hmN)$. This now only requires a *one time* secure sorting to compute the sorting permutations initially (which costs $O(mN \log N)$).

Idea 2. The bottleneck in **Step (2)** is the GroupMax operation which requires $O(N \log N)$ communication. In [16], the parties invoke m instances of Π_{GroupMax} (over length N vectors) to compute the threshold with minimum Gini index (maximum of expr. 1) for each attribute. Then, for each $i \in [N]$, parties compute attribute (and threshold) with minimum Gini index by invoking N instances of Π_{Max} (over length m vectors). We make two important observations here; (1) The cost of computing maximum of N values ($O(N)$) is less than the cost of computing group wise maximum (($O(N \log N)$)); and (2) reversing the order of operations does not change the output i.e. we can replace the m invocations of Π_{GroupMax} (on N length vector) followed by N invocations of Π_{Max} (on m length vector) with N invocations of Π_{Max} (on m length vector) and 1 invocation of Π_{GroupMax} (on N length vector). Reversing the order of these operations preserves functionality crucially because all attributes are grouped according to the same group boundaries. We formally show this in the proof of security of the protocol (Lemma 5.2). The proof relies on the fact that the sorting according to different attributes in Step (1) is within groups and does not change the starting and ending index of a group. Thus, now the cost of step (2) becomes $O(hmN + hN \log N)$ from $O(hmN \log N)$ saving a factor of $\min(m, \log N)$.

Protocol overview. Putting these together, at a high-level, our protocol can be broken down into the following steps:

(1) Compute and store the sorting permutations for all attributes. This is a one-time (not per layer) set-up with communication cost $O(mN \log N)$.
(2) For all internal node layers:
   (a) Apply sorting permutation and compute Gini index. The communication cost is $O(mN)$.
   (b) Compute the splitting attribute and threshold in each group which has minimum Gini index. Communication cost is $O(mN + N \log N)$.
   (c) Compute the new partition of dataset based on splitting attribute and threshold, and the updated sorting permutation for all attributes. Communication is $O(mN)$.
(3) For the leaf node layer, compute the most occuring labels in each group. Communication cost is $O(N)$.

Hence, the total communication cost of our new training protocol is $O(mN \log N + hmN + hN \log N)$, which asymptotically improves upon the state-of-the-art [16] by a factor of $\min(h, m, \log N)$.

We describe the subprotocols to compute each step described above along with the improvements in efficiency in detail in the following sections. In Section 4.6, we combine all these subprotocols and present our end-to-end secure decision tree training protocol.

4.3 Set Up Phase

Parties compute the shares of permutation $\mathbf{P}_1^i$ for all $i \in [m]$ that sorts $\mathbf{D}$ according to the i^{th} attribute. These are computed once in the beginning and updated for each layer. Updating the permutation is a part of the internal layer training subprotocols and described later. We use Π_{SortPerm} (Algorithm 18) that computes the sorting permutation with $O(N \log N)$ communication in $O(\log N)$ rounds for a vector of length N. $\Pi_{\mathsf{SetupPerm}}$ (Algorithm 6) requires m calls to Π_{SortPerm}.

Algorithm 6: $\Pi_{\mathsf{SetupPerm}}$

Input: Dataset $\langle \mathbf{D} \rangle = (\langle \mathbf{x}_1 \rangle, \ldots, \langle \mathbf{x}_m \rangle, \langle \mathbf{y} \rangle)$
Output: Sorting permutation $\left\langle \mathbf{P}_1^i \right\rangle$ for all $i \in [m]$
Cost: $O(mnN)$

for $i \in [m]$ **do**
  $\left\langle \mathbf{P}_1^i \right\rangle = \Pi_{\mathsf{SortPerm}}(\langle \mathbf{x}_i \rangle)$
end
Output $\left\{ \left\langle \mathbf{P}_1^i \right\rangle \right\}_{i \in [m]}$.

4.4 Training Internal Layers

Consider that we have trained nodes till layer $k-1$ and that we are now training nodes at layer k. The parties will have secret share of $\mathsf{State}^{(k)} = \{\mathbf{D}_k, \mathbf{g}_k, \{\mathbf{P}_k^i\}_{i \in [m]}, \mathbf{NID}\}$ which satisfies the following invariant:

(1) $\mathbf{D}_k$ is the training dataset grouped by nodes in layer k,
(2) $\mathbf{g}_k$ is the group flag vector that marks the starting of groups in $\mathbf{D}_k$,
(3) $\mathbf{P}_k^i$ (for all $i \in [m]$) is the permutation that sorts $\mathbf{D}_k$ based on i^{th} attribute within each group and
(4) vector $\mathbf{NID}$ is the node id (or group id) of each sample in $\mathbf{D}_k$. For example if $\mathbf{D}_k[i]$ belongs to group of j^{th} node, $\mathbf{NID}[i] = j$. There are $n_k = 2^{k-1}$ nodes in k^{th} layer.

The output of training the k^{th} internal node layer will be $\left\langle \mathsf{Layer}^{(k)} \right\rangle$ defined in Section 4 and $\left\langle \mathsf{State}^{(k+1)} \right\rangle$ i.e. updated state required to train the next layer.

For the base case, $\langle \mathbf{D}_1 \rangle = \langle \mathbf{D} \rangle, \langle \mathbf{g}_1 \rangle = \langle [1, 0, \ldots, 0] \rangle, \langle \mathbf{NID} \rangle = \langle [1, 1, \ldots, 1] \rangle$ as Layer 1 only has the root node. Hence, there is no partitioning of nodes. Moreover, from the setup phase, we have $\left\langle \mathbf{P}_1^i \right\rangle = \Pi_{\mathsf{SetupPerm}}(\langle \mathbf{x}_i \rangle)$. Thus,

$$\left\langle \mathsf{State}^{(1)} \right\rangle = \left\{ \langle \mathbf{D}_1 \rangle, \langle \mathbf{g}_1 \rangle, \{ \left\langle \mathbf{P}_k^i \right\rangle \}_{i \in [m]}, \langle \mathbf{NID} \rangle \right\}$$

satisfies the invariant for the base case.

The $\mathsf{TrainInternalLayer}$ functionality takes as input k and shares of $\mathsf{State}^{(k)}$, and outputs shares of $\mathsf{Layer}^{(k)}, \mathsf{State}^{(k+1)}$ where $\mathsf{Layer}^{(k)}$ is the k^{th} decision tree layer and $\mathsf{State}^{(k+1)}$ is the intermediary output used to train the next layer. Algorithm 12 securely computes this functionality.

In Lemma 5.4, we show that if the input $\mathsf{State}^{(k)}$ to algorithm 12 satisfies the invariant described above, the output $\mathsf{State}^{(k+1)}$ also satisfies the invariant. Therefore by induction, $\left\langle \mathsf{State}^{(k)} \right\rangle$ satisfies the invariant for all $k \in [h+1]$.

The $\Pi_{\mathsf{TrainInternalLayer}}$ protocol has total $O(mN + N \log N)$ communication cost and consists of the following 4 steps:

(1) Compute all possible thresholds and Gini index from the resulting split for each attribute. (Section 4.4.1)

(2) Select attribute and threshold for best split for all layer nodes. (Section 4.4.2)
(3) Apply tests to compute new groups after split. (Section 4.4.3)
(4) Update the invariant state. (Section 4.4.4)
(5) Output the final output of decision tree training $\mathsf{Layer}^{(k)}$. (Section 4.4.5)

We construct subprotocols for each of these steps in the following sections that outline various optimizations over [16].

4.4.1 Computing Gini index for an attribute. ComputeGini takes an attribute ($\mathbf{x}$) and label ($\mathbf{y}$) vector sorted by attribute values and computes all possible thresholds and the corresponding Gini index of the split. $\Pi_{\mathsf{ComputeGini}}$ (Algorithm 7) securely computes the functionality. The sorted attribute vector allows us to compute the split for each threshold efficiently. If threshold t is such that $\mathbf{x}[i] < t < \mathbf{x}[i+1]$, then the two partitions of the dataset are $\{\mathbf{x}[1], \ldots, \mathbf{x}[i]\}$ and $\{\mathbf{x}[i+1], \ldots, \mathbf{x}[N]\}$ respectively since the attribute vector is sorted.

Protocol $\Pi_{\mathsf{ComputeGini}}$ is the same as the protocol for computing Gini index in [16] except for one optimization. We take advantage of the observation that while the Group Prefix Sum of $\mathbf{y}$ labels changes based on ordering of samples, the Group Sum remains same. In [16], the Group Sum of $\mathbf{y}$ labels is computed for each attribute (m GroupSum invocations). We instead compute Group Sum of $\mathbf{y}$ labels only once resulting in saving $25(m-1)Nhl$ bits of communication. This is only a minor optimization and does not change the asymptotic complexity of the protocol.

Let $\langle \mathbf{sy}_0 \rangle, \langle \mathbf{sy}_1 \rangle$ be Group Sum of $\mathbf{y}$ and $\neg\mathbf{y}$ respectively which are computed once and passed as arguments to $\Pi_{\mathsf{ComputeGini}}$. $\langle \mathbf{x} \rangle, \langle \mathbf{y} \rangle$ are attribute values and labels (sorted according to attribute) respectively. The protocol outputs shares of $\mathbf{t}, \mathbf{s}$ where $\mathbf{t}[i]$ is a threshold that splits the group at the i^{th} sample and $\mathbf{s}[i]$ is the Gini index resulting from this split.

The correctness follows from expression 1 which is derived in [3]. $\Pi_{\mathsf{ComputeGini}}$ requires 2 calls to $\Pi_{\mathsf{GroupPrefixSum}}$, $11N$ calls to Π_{MULT} and N calls to Π_{EQ}.

Note. The Gini index (s) is stored as a tuple of integers (p, q) where $s = p/q$. This is done to avoid the expensive secure division operation. Since we only need to perform comparisons on Gini indices, $s_1 < s_2$ is equivalent to $p_1 q_2 < p_2 q_1$ where $s_i = (p_i, q_i)$.

4.4.2 Test Selection. Functionality TestSelection takes as input the database $\langle \mathbf{D}_k \rangle$, group flag vector $\langle \mathbf{g}_k \rangle$ and sorting permutations $\left\{\left\langle \mathbf{P}_k^i \right\rangle\right\}_i$, and outputs the splitting attributes $\langle \mathbf{A} \rangle$ and thresholds $\langle \mathbf{T} \rangle$ where $\mathbf{A}[j], \mathbf{T}[j]$ is the attribute and threshold with minimum Gini index for the node i belongs to. Protocol $\Pi_{\mathsf{TestSelection}}$ (Algorithm 8) computes the functionality.

$\Pi_{\mathsf{TestSelection}}$ invokes $\Pi_{\mathsf{ComputeGini}}$ as a subprotocol to compute the Gini Index for all possible attributes and thresholds, $\{\langle \mathbf{s}_i \rangle, \langle \mathbf{t}_i \rangle\}_{i \in [m]}$ where $\mathbf{t}_i[j]$ for $j \in [N]$ is the threshold for splitting the dataset on the i^{th} attribute and $\mathbf{s}_i[j]$ is the resulting Gini Index from this split.

In [16], Test Selection is done by invoking m instances of Group Max over length N vectors $\langle \mathbf{S'}_i \rangle, \langle \mathbf{T'}_i \rangle = \Pi_{\mathsf{GroupMax}}(\mathbf{g}, \mathbf{s}_i; \mathbf{t}_i)$ to compute the best threshold for the i^{th} attribute for all $i \in [m]$ followed by N invocations of Π_{Max} ($\forall j \in [N]$) over length m vectors to compute the best attribute and corresponding threshold

$$\mathbf{A}[j], \mathbf{T}[j] = \Pi_{\mathsf{Max}}\left(\left\{\mathbf{S'}_i[j]; i, \mathbf{T'}_i[j]\right\}_{i \in [m]}\right)$$

We note that the order of operations can be reversed, reducing the number of comparisons from $mN \log N + mN$ to $N \log N + mN$. Note that this optimization alone does not reduce the asymptotic cost of [16] and is only useful when combined with our new training algorithm that uses secret shared permutations instead of repeated secure sorting.

We know that $\{\mathbf{s}_i, \mathbf{t}_i\}_{i \in [m]}$ are correctly computed from the correctness of $\Pi_{\mathsf{ComputeGini}}$. To prove correctness of $\Pi_{\mathsf{TestSelection}}$, we have to show that $\mathbf{A}, \mathbf{T}$ computed in step $6-9$ of the algorithm are correct. Step 10 merely checks that if the group of a node is already homogenous (all labels are equal), then no more splitting is required and therefore, it sets the threshold to the minimum value. We show correctness of $\Pi_{\mathsf{TestSelection}}$ in Lemma 5.2 where we prove that reversing the order of Π_{GroupMax} and Π_{Max} does not affect the output.

$\Pi_{\mathsf{TestSelection}}$ requires $2m$ calls to $\Pi_{\mathsf{ApplyPerm}}$, m calls to $\Pi_{\mathsf{ComputeGini}}$, N calls to Π_{VectMax}, 1 call to Π_{GroupMax}, $3N$ calls to Π_{MULT} and $2N$ calls to Π_{EQ}.

4.4.3 Applying Tests. Once parties have $\langle \mathbf{A} \rangle, \langle \mathbf{T} \rangle$; parties compute the partition of nodes based on comparison with selected attributes and thresholds.

The cost of $\Pi_{\mathsf{ApplyTest}}$ is $O(hmN)$, which is insignificant in the total communication and we directly use the subprotocol from [16]. If $\mathbf{A}[j] = i^*$, the i^* attribute value has to be selected for the j^{th} sample (i.e. $\mathbf{x}_{i^*}[j]$) to compare with the threshold $\mathbf{T}[j]$. To fetch $\langle \mathbf{x}_{i^*}[j] \rangle$ in an oblivious manner, parties have to compute $\langle \mathbf{e}_{i^*} \rangle \in \{0,1\}^m$ from $\langle i^* \rangle$ which is its one hot encoding. We compute the one hot vector using m comparisons as:

$$\langle \mathbf{e}_{i^*} \rangle = \left(\langle i^* \rangle \stackrel{?}{=} 1, \langle i^* \rangle \stackrel{?}{=} 2, \ldots, \langle i^* \rangle \stackrel{?}{=} m\right)$$

Then the parties can compute $\mathbf{x}_{i^*}$ with dot product $\mathbf{x} \cdot \mathbf{e}_{i^*}$ and compare it with the threshold. $\Pi_{\mathsf{ApplyTest}}$ (Algorithm 9) requires N calls to Π_{MULT}, N calls to Π_{LT} and mN calls to Π_{EQ}.

Algorithm 7: $\Pi_{\mathsf{ComputeGini}}$

Input: $\langle \mathbf{g} \rangle, \langle \mathbf{x} \rangle, \langle \mathbf{y} \rangle, \langle \mathbf{sy}_0 \rangle, \langle \mathbf{sy}_1 \rangle$
Output: $\langle \mathbf{s} \rangle, \langle \mathbf{t} \rangle$ of length N.
Cost: $O(N)$

1 $\langle \mathbf{u_0} \rangle = \text{GroupPrefixSum}(\langle \mathbf{y} \rangle, \langle \mathbf{g} \rangle)$
2 $\langle \mathbf{u_1} \rangle = \text{GroupPrefixSum}(\langle \neg\mathbf{y} \rangle, \langle \mathbf{g} \rangle)$
3 $\langle \mathbf{w_b} \rangle = \langle \mathbf{sy}_b \rangle - \langle \mathbf{u_b} \rangle$ for $b \in \{0,1\}$
4 $\langle \mathbf{w} \rangle = \langle \mathbf{w_0} \rangle + \langle \mathbf{w_1} \rangle$ and $\langle \mathbf{u} \rangle = \langle \mathbf{u_0} \rangle + \langle \mathbf{u_1} \rangle$
5 $\langle \mathbf{p} \rangle = \langle \mathbf{w} \rangle \times \left(\langle \mathbf{u_0} \rangle^2 + \langle \mathbf{u_1} \rangle^2\right) + \langle \mathbf{u} \rangle \times \left(\langle \mathbf{w_0} \rangle^2 + \langle \mathbf{w_1} \rangle^2\right)$
6 $\langle \mathbf{q} \rangle = \langle \mathbf{u} \rangle \times \langle \mathbf{w} \rangle$
7 $\langle \mathbf{s} \rangle = (\langle \mathbf{p} \rangle, \langle \mathbf{q} \rangle)$
8 $\langle \mathbf{t}[i] \rangle = \langle \mathbf{x}[i] \rangle + \langle \mathbf{x}[i+1] \rangle$ for all $i \in [N]$,
$\langle \mathbf{t}[N] \rangle = \text{MinValue}$
9 $\langle \mathbf{r}[i] \rangle = \langle \mathbf{g}[i+1] \rangle$ OR $\left\langle \mathbf{x}[i] \stackrel{?}{=} \mathbf{x}[i+1] \right\rangle$ for $i \in [N]$,
$\langle \mathbf{r}[N] \rangle = 1$
10 $\langle \mathbf{s} \rangle, \langle \mathbf{t} \rangle = \text{IfElse}(\langle \mathbf{r} \rangle; \text{MinValue}, \text{MinValue}; \langle \mathbf{s} \rangle, \langle \mathbf{t} \rangle)$

Algorithm 8: $\Pi_{\text{TestSelection}}$

Input: $\langle \mathbf{D}_k \rangle, \langle \mathbf{g}_k \rangle, \left\{ \left\langle \mathbf{P}_k^i \right\rangle \right\}_i, \langle \mathbf{sy}_0 \rangle, \langle \mathbf{sy}_1 \rangle$
Output: $\langle \mathbf{A} \rangle, \langle \mathbf{T} \rangle$ of size N.
Cost: $O(mn + nN)$

for $i \in [m]$ **do**
  $\langle \mathbf{u}_i \rangle = \Pi_{\text{ApplyPerm}} \left(\langle \mathbf{x}_i \rangle, \left\langle \mathbf{P}_k^i \right\rangle \right)$
  $\langle \mathbf{v}_i \rangle = \Pi_{\text{ApplyPerm}} \left(\langle \mathbf{y} \rangle, \left\langle \mathbf{P}_k^i \right\rangle \right)$
  $\langle \mathbf{s}_i \rangle, \langle \mathbf{t}_i \rangle = \Pi_{\text{ComputeGini}} \left(\langle \mathbf{g}_k \rangle, \langle \mathbf{u}_i \rangle, \langle \mathbf{v}_i \rangle \right)$
end
for $j \in [N]$ **do**
  $\langle \mathbf{s}[j] \rangle, \langle \mathbf{t}[j] \rangle, \langle \mathbf{a}[j] \rangle =$
  $\Pi_{\text{Max}} \left(\{ \langle \mathbf{s}_i[j] \rangle ; \langle \mathbf{t}_i[j] \rangle, i \}_{i \in [m]} \right)$
end
$\langle \mathbf{A} \rangle, \langle \mathbf{T} \rangle = \Pi_{\text{GroupMax}} \left(\langle \mathbf{g}_k \rangle, \langle \mathbf{s} \rangle ; \langle \mathbf{a} \rangle, \langle \mathbf{t} \rangle \right)$
$\langle \mathbf{f} \rangle = \left(\langle \mathbf{sy}_0 \rangle \stackrel{?}{=} 0 \right) \text{OR} \left(\langle \mathbf{sy}_1 \rangle \stackrel{?}{=} 0 \right)$
$\langle \mathbf{A} \rangle, \langle \mathbf{T} \rangle = \text{IfElse} \left(\langle \mathbf{f} \rangle ; 1, \text{MinValue}; \langle \mathbf{A} \rangle, \langle \mathbf{T} \rangle \right)$

Algorithm 9: $\Pi_{\text{ApplyTest}}$

Input: $\langle \mathbf{D}_k \rangle, \langle \mathbf{A} \rangle, \langle \mathbf{T} \rangle$
Output: $\langle \mathbf{b} \rangle \in \{0,1\}^N$
Cost: $O(mN)$

for $i \in [N]$ **do**
  $\langle \mathbf{e}_{i^*} \rangle = \text{OneHotEnc} \left(\langle \mathbf{A}[i] \rangle \right)$
  $\langle \mathbf{b}[i] \rangle = \left(\langle \mathbf{D}_k[i] \cdot \mathbf{e}_{i^*} \rangle \stackrel{?}{<} \langle \mathbf{T}[i] \rangle \right)$
end
Output $\langle \mathbf{b} \rangle$.

4.4.4 Updating flag vector and state. After applying tests, each group is divided into two new groups based on whether $\mathbf{b}$ is equal to 0 or 1. To avoid sorting the dataset at each layer, parties have to update $\text{State}^{(k+1)}$ such that the new groups satisfy the invariant. Functionality UpdateState takes shares of $\text{State}^{(k)}$ and $\mathbf{b}$ as input and outputs $\text{State}^{(k+1)}$ such that $\text{State}^{(k+1)}$ satisfies the invariant described in Section 4.4. In [16], the parties only update the group flag vector and rearrange the dataset so that entries of same group appear together. UpdateState has to additionally update the sorting permutations so that we can avoid sorting and use the output $\text{State}^{(k+1)}$ to train the next layer of decision tree. $\Pi_{\text{UpdateState}}$ (Algorithm 10) securely computes this functionality.

Updating sorting permutations results in additional $O(mN)$ communication for a layer. The cost of update in [16] is $O(hmN)$ so the asymptotic cost does not increase with this change. Moreover, we are able to eliminate repeated calls to secure sorting, eliminating the efficiency bottleneck.

We create a new flag vector $\mathbf{g}$ that marks the first i such that $\mathbf{b}[i] = 0$ and $\mathbf{b}[i] = 1$ in each group using $\Pi_{\text{GroupFirstOne}}$ protocol from [16] described in section 3.3. We do so as the first 0 and 1 in each group from the comparison results is the starting of new groups now. We sort $\mathbf{D}_k$ and $\mathbf{g}$ based on values of $\mathbf{b}$ to obtain $\langle \mathbf{D}_{k+1} \rangle, \langle \mathbf{g}_{k+1} \rangle$ We compute the updated permutations $\left\{ \mathbf{P}_{k+1}^i \right\}_{i \in [m]}$.

$\Pi_{\text{UpdateState}}$ requires $3m + 3$ calls to $\Pi_{\text{ApplyPerm}}$ and m calls to Π_{PermComp}, 2 calls to $\Pi_{\text{GroupFirstOne}}$ and $m + 1$ calls to $\Pi_{\text{SortPermBit}}$. We provide a security proof for the UpdateState protocol in Lemma 5.4.

Algorithm 10: $\Pi_{\text{UpdateState}}$

Input: $\left\langle \text{State}^{(k)} \right\rangle = \left(\langle \mathbf{D}_k \rangle, \langle \mathbf{g}_k \rangle, \left\{ \left\langle \mathbf{P}_k^i \right\rangle \right\}_i, \langle \text{NID} \rangle \right), \langle \mathbf{b} \rangle, k$
Output: $\left\langle \text{State}^{(k+1)} \right\rangle$
Cost: $O(mN)$

$\langle \mathbf{Q} \rangle = \Pi_{\text{SortPermBit}} \left(\langle \mathbf{b} \rangle \right)$
$\langle \mathbf{D}_{k+1} \rangle = \Pi_{\text{ApplyPerm}} \left(\langle \mathbf{D}_k \rangle, \langle \mathbf{Q} \rangle \right)$
$\langle \mathbf{NID} \rangle = 2^{k-1} \langle \mathbf{b} \rangle + \langle \mathbf{NID} \rangle$
$\left\langle \mathbf{NID}' \right\rangle = \Pi_{\text{ApplyPerm}} \left(\langle \mathbf{NID} \rangle, \langle \mathbf{Q} \rangle \right)$
$\langle \mathbf{g} \rangle = \Pi_{\text{GroupFirstOne}} \left(\langle \mathbf{g}_k \rangle, \langle \mathbf{b} \rangle \right)$
  $+ \Pi_{\text{GroupFirstOne}} \left(\langle \mathbf{g}_k \rangle, \langle \neg \mathbf{b} \rangle \right)$
$\langle \mathbf{g}_{k+1} \rangle = \Pi_{\text{ApplyPerm}} \left(\langle \mathbf{g} \rangle, \langle \mathbf{Q} \rangle \right)$
for $i \in [m]$ **do**
  $\left\langle \mathbf{b}^i \right\rangle = \Pi_{\text{SortPermBit}} \left(\langle \mathbf{b} \rangle, \left\langle \mathbf{P}_k^i \right\rangle \right)$
  $\left\langle \mathbf{Q}^i \right\rangle = \Pi_{\text{ApplyPerm}} \left(\left\langle \mathbf{P}_k^i \right\rangle, \langle \mathbf{Q} \rangle \right)$
  $\left\langle \mathbf{R}^i \right\rangle = \Pi_{\text{ApplyPerm}} \left(\left\langle \mathbf{b}^i \right\rangle \right)$
  $\left\langle \mathbf{P}_{k+1}^i \right\rangle = \Pi_{\text{PermComp}} \left(\left\langle \mathbf{R}^i \right\rangle, \left\langle \mathbf{Q}^i \right\rangle \right)$
end
Output $\left\langle \text{State}^{(k+1)} \right\rangle =$
$\left(\langle \mathbf{D}_{k+1} \rangle, \langle \mathbf{g}_{k+1} \rangle, \left\{ \left\langle \mathbf{P}_{k+1}^i \right\rangle \right\}_{i \in [m]}, \left\langle \mathbf{NID}' \right\rangle \right)$.

4.4.5 Storing values of splitting nodes. Training each layer also outputs the splitting attribute and threshold for all nodes in the layer (stored in tuple $\text{Layer}^{(k)}$). Subprotocol $\Pi_{\text{StoreLayer}}$ (Algorithm 11) which outputs $\text{Layer}^{(k)}$ tuple is not a bottleneck in communication efficiency ($O(hmN)$) and we use it as is from [16].

Parties have $\langle \mathbf{NID} \rangle, \langle \mathbf{A} \rangle, \langle \mathbf{T} \rangle$ containing node id, attribute and threshold but the values repeat (same value for all elements in one group). We want to store the output of training of internal nodes in $\langle \mathbf{NID}_k \rangle, \langle \mathbf{A}_k \rangle, \langle \mathbf{T}_k \rangle$ where $\mathbf{NID}_k[i], \mathbf{A}_k[i], \mathbf{T}_k[i]$ stores the node id, attribute and threshold of i^{th} node. For the leaf node layer, parties output $\langle \mathbf{NID}_{h+1} \rangle, \langle \mathbf{L}_{h+1} \rangle$.

$\Pi_{\text{StoreLayer}}$ requires 1 call to $\Pi_{\text{SortPermBit}}$ and 3 calls to $\Pi_{\text{ApplyPerm}}$.

4.4.6 End to end internal layer training. With the subprotocols constructed in Sections 4.4.1 to 4.4.5, we can describe the complete internal training protocol in Algorithm 12. Training one internal layer requires 2 calls to Π_{GroupSum}, 1 call to $\Pi_{\text{TestSelection}}$, 1 call to $\Pi_{\text{ApplyTests}}$, 1 call to $\Pi_{\text{StoreLayer}}$ and 1 call to $\Pi_{\text{UpdateState}}$.

4.5 Training Leaf Layer

After training all internal layers, we have to train the final leaf node layer. The decision tree halts on the leaf node and outputs the label corresponding to the leaf label as the sample's classification result.

Algorithm 11: $\Pi_{\mathsf{StoreLayer}}$

Input: $\langle \mathbf{NID} \rangle, \langle \mathbf{A} \rangle, \langle \mathbf{T} \rangle, \langle \mathbf{g}_k \rangle$ of length N
Output: $\langle \mathbf{NID}_k \rangle, \langle \mathbf{A}_k \rangle, \langle \mathbf{T}_k \rangle$ of length $n_k = \min(2^{k-1}, N)$
Cost: $O(N)$

1 $\langle \mathbf{P} \rangle = \Pi_{\mathsf{SortPermBit}}(\neg \langle \mathbf{g}_k \rangle)$
2 $\langle \mathbf{NID}_k \rangle = \Pi_{\mathsf{ApplyPerm}}(\langle \mathbf{NID} \rangle, \langle \mathbf{P} \rangle)$ //First n_k terms
3 $\langle \mathbf{A}_k \rangle = \Pi_{\mathsf{ApplyPerm}}(\langle \mathbf{A} \rangle, \langle \mathbf{P} \rangle)$ //First n_k terms
4 $\langle \mathbf{T}_k \rangle = \Pi_{\mathsf{ApplyPerm}}(\langle \mathbf{T} \rangle, \langle \mathbf{P} \rangle)$ //First n_k terms
5 Output $\left\langle \mathsf{Layer}^{(k)} \right\rangle = (\langle \mathbf{NID}_k \rangle, \langle \mathbf{A}_k \rangle, \langle \mathbf{T}_k \rangle)$.

Algorithm 12: $\Pi_{\mathsf{TrainInternalLayer}}$

Input: $\left\langle \mathsf{State}^{(k)} \right\rangle = \left(\langle \mathbf{D}_k \rangle, \langle \mathbf{g}_k \rangle, \left\{ \left\langle \mathbf{P}_k^i \right\rangle \right\}_i, \langle \mathbf{NID} \rangle \right), k$
Output: $\left\langle \mathsf{Layer}^{(k)} \right\rangle, \left\langle \mathsf{State}^{(k+1)} \right\rangle$
Cost: $O(mN + nN)$

1 $\langle \mathbf{sy}_0 \rangle = \mathrm{GroupSum}(\langle \mathbf{y}_k \rangle, \langle \mathbf{g}_k \rangle)$
2 $\langle \mathbf{sy}_1 \rangle = \mathrm{GroupSum}(\langle \neg \mathbf{y}_k \rangle, \langle \mathbf{g}_k \rangle)$
3 $\langle \mathbf{A} \rangle, \langle \mathbf{T} \rangle = \Pi_{\mathsf{TestSelection}}\left(\langle \mathbf{D}_k \rangle, \langle \mathbf{g}_k \rangle, \left\{ \left\langle \mathbf{P}_k^i \right\rangle \right\}_i \right)$
4 $\left\langle \mathsf{Layer}^{(k)} \right\rangle = \Pi_{\mathsf{StoreLayer}}(\langle \mathbf{NID} \rangle, \langle \mathbf{A} \rangle, \langle \mathbf{T} \rangle, \langle \mathbf{g}_k \rangle, k)$
5 $\langle \mathbf{b} \rangle = \Pi_{\mathsf{ApplyTest}}(\langle \mathbf{D}_k \rangle, \langle \mathbf{A} \rangle, \langle \mathbf{T} \rangle)$
6 $\left\langle \mathsf{State}^{(k+1)} \right\rangle = \Pi_{\mathsf{UpdateState}}\left(\left\langle \mathsf{State}^{(k)} \right\rangle, \langle \mathbf{b} \rangle, k \right)$
7 Output $\left\langle \mathsf{Layer}^{(k)} \right\rangle, \left\langle \mathsf{State}^{(k+1)} \right\rangle$

The label of a leaf node is the majority of $\mathbf{y}$ labels for samples in the training dataset belonging to the node. This can be computed using Π_{GroupSum}. The final layer labels are stored in two vectors $\mathbf{NID}, \mathbf{L}$ of length n_{h+1} where $\mathbf{NID}$ stores the node id and $\mathbf{L}$ stores the corresponding labels. We can modify the $\Pi_{\mathsf{StoreLayer}}$ accordingly to store the leaf layer output. $\Pi_{\mathsf{TrainLeafLayer}}$ (Algorithm 13) requires 2 calls to Π_{GroupSum}, 1 call to $\Pi_{\mathsf{StoreLayer}}$ and N calls to Π_{LT}.

Algorithm 13: $\Pi_{\mathsf{TrainLeafLayer}}$

Input: $\left\langle \mathsf{State}^{(k)} \right\rangle$
Output: $\left\langle \mathsf{Layer}^{(k)} \right\rangle = (\langle \mathbf{NID}_k \rangle, \langle \mathbf{L}_k \rangle)$
Cost: $O(N)$

1 $\langle \mathbf{L} \rangle = \mathrm{GroupSum}(\langle \mathbf{y}_k \rangle, \langle \mathbf{g}_k \rangle) > \mathrm{GroupSum}(\langle \neg \mathbf{y}_k \rangle, \langle \mathbf{g}_k \rangle)$
2 $\langle \mathbf{NID}_k \rangle, \langle \mathbf{L}_k \rangle = \Pi_{\mathsf{StoreLayer}}(\langle \mathbf{NID} \rangle, \langle \mathbf{L} \rangle, \langle \mathbf{g}_k \rangle, k)$
3 Output $\left\langle \mathsf{Layer}^{(k)} \right\rangle = (\langle \mathbf{NID}_k \rangle, \langle \mathbf{L}_k \rangle)$.

4.6 End to end training

Finally, we combine sub-protocols $\Pi_{\mathsf{SetupPerm}}$, $\Pi_{\mathsf{TrainInternalLayer}}$ and $\Pi_{\mathsf{TrainLeafLayer}}$ to obtain an end to end privacy preserving decision tree training algorithm Π_{Train} (Algorithm 19 in Appendix E).

The communication cost of $\Pi_{\mathsf{SetupPerm}}$, $\Pi_{\mathsf{TrainInternalLayer}}$ and $\Pi_{\mathsf{TrainLeafLayer}}$ are $O(mN \log N)$, $O(mN + N \log N)$ and $O(N)$ respectively. The total communication cost of $\Pi_{\mathsf{TrainDecisionTree}}$ is $O(mN \log N + h(mN + N \log N))$ which improves by a factor of $\min(h, m, \log N)$ over the state-of-the-art [16].

5 SECURITY PROOF

Let Train be the functionality that takes as input the shares of labelled dataset $\langle \mathbf{D} \rangle$ and height h and outputs shares of trained decision tree of height h i.e. $\left\{ \mathsf{Layer}^{(i)} \right\}_i$. To prove the security of our training protocol, we show that the view of a semi-honest adversary corrupting one of the parties in the real execution of protocol Π_{Train} can be simulated by a simulator with access to functionality Train in the ideal world. All subprotocols in our training algorithm will be proven simulation secure under the universal composability [11] framework and the security of the end-to-end protocol will be through a simple invocation of the composability theorem [11].

Π_{Shuffle}, $\Pi_{\mathsf{ApplyPerm}}$ and Π_{PermComp} were already proven secure in [9, 20]. The security of subprotocols $\Pi_{\mathsf{ComputeGini}}$, $\Pi_{\mathsf{ApplyTest}}$ and $\Pi_{\mathsf{StoreLayer}}$ was shown in [16]. Thus, for each of these protocols, there exists a simulator that can simulate the view of the adversary. The only remaining subprotocols are $\Pi_{\mathsf{SetupPerm}}$, $\Pi_{\mathsf{TestSelection}}$ and $\Pi_{\mathsf{UpdateState}}$. We discuss the security of these subprotocols.

5.1 Security of SetupPerm

Lemma 5.1. *Protocol* $\Pi_{\mathsf{SetupPerm}}$ *defined in Algorithm 6 securely realizes the* SetupPerm *functionality from Section 4.3.*

Proof. Correctness of the protocol trivially holds. We prove security. Without loss of generality, we assume that the adversary $\mathcal{A}$ corrupts party S_0. Let $\mathcal{S}$ be the simulator for protocol $\Pi_{\mathsf{SetupPerm}}$. Let $\mathcal{S}'$ be the simulator for protocol Π_{SortPerm} (this simulator was shown in [20]). $\mathcal{S}$ (with S_0's input shares $\langle \mathbf{D} \rangle = (\langle \mathbf{x}_1 \rangle, \ldots, \langle \mathbf{x}_m \rangle)$) simply invokes m copies of $\mathcal{S}'$ with inputs $\langle \mathbf{x}_1 \rangle, \ldots, \langle \mathbf{x}_m \rangle$ respectively and outputs the output returned by $\mathcal{S}'$. Since the real view of the adversary in $\Pi_{\mathsf{SortPerm}}(\langle \mathbf{x}_i \rangle)$ is indistinguishable from the simulated view provided by $\mathcal{S}'(\langle \mathbf{x}_i \rangle)$, it follows from a simple hybrid argument that the real view of $\mathcal{A}$ in $\Pi_{\mathsf{SetupPerm}}$ is indistinguishable from the simulated view output by $\mathcal{S}$. □

5.2 Security of TestSelection

Lemma 5.2. *Protocol* $\Pi_{\mathsf{TestSelection}}$ *defined in Algorithm 8 securely realizes the* SetupPerm *functionality from Section 4.4.2.*

Proof. First, to demonstrate correctness of $\Pi_{\mathsf{TestSelection}}$, we prove that reversing the order of invocations of Π_{GroupMax} and Π_{Max} does not change the output in steps 6 – 9 in $\Pi_{\mathsf{TestSelection}}$ (Algorithm 8). $\{\mathbf{s}_i\}_{i \in [m]}$ are the length N vectors for Gini index for all attributes which are correctly computed from the correctness of $\Pi_{\mathsf{ComputeGini}}$ protocol. Private group vector $\mathbf{g}$ is of length N.

Suppose $\{\mathbf{s}_i\}_{i \in [m]}$ are m vectors each of length N, $\mathbf{g}$ is group flag vector denoting groups in $\mathbf{s}$ and $j \in [N]$ is an index. l_j, r_j are the leftmost and rightmost indices of the group that j belongs to (see Section 3.3). GroupMax and Max operations are defined as follows:

$$\mathbf{a} = \Pi_{\mathsf{Max}}(\mathbf{s}_1, \ldots, \mathbf{s}_m)$$
$$\mathbf{b}_i = \Pi_{\mathsf{GroupMax}}(\mathbf{g}, \mathbf{s}_i)$$

where

$$\mathbf{a}[j] = \text{Max}_{i\in[m]}\,(\mathbf{s}_i[j])$$
$$\mathbf{b}_i[j] = \text{Max}_{l_j\le k\le r_j}\,(\mathbf{s}_i[k])$$

Lemma 5.3. *Borrowing notations from above, we have*

$$\Pi_{\text{GroupMax}}(\mathbf{g},\mathbf{a}) = \Pi_{\text{Max}}(\mathbf{b}_1,\ldots,\mathbf{b}_m)$$

Proof. Let LHS = $\mathbf{c}$ and RHS = $\mathbf{d}$ where $\mathbf{c},\mathbf{d}$ are vectors of length N. We show that for all $j\in[N]$, $\mathbf{c}[j]=\mathbf{d}[j]$. For any $j\in[N]$:

$$\mathbf{c}[j] = \text{Max}_{l_j\le k\le r_j}\,(\mathbf{a}[k]) = \text{Max}_{l_j\le k\le r_j}\left(\text{Max}_{i\in[m]}\,(\mathbf{s}_i[k])\right)$$
$$= \text{Max}_{\substack{i\in[m]\\ l_j\le k\le r_j}}\,(\mathbf{s}_i[k])$$

Similarly we have

$$\mathbf{d}[j] = \text{Max}_{i\in[m]}\,(\mathbf{b}_i[j]) = \text{Max}_{i\in[m]}\left(\text{Max}_{l_j\le k\le r_j}\,(\mathbf{s}_i[k])\right)$$
$$= \text{Max}_{\substack{i\in[m]\\ l_j\le k\le r_j}}\,(\mathbf{s}_i[k])$$

Therefore, LHS = RHS. □

To prove security, we construct a simulator $\mathcal{S}$ that simulates the view of adversary $\mathcal{A}$ corrupting party S_0. Protocols for ComputeGini, Π_{GroupMax} and Π_{Max} have been proven simulation secure in [3, 16]. Steps $10-11$ in $\Pi_{\text{TestSelection}}$ use secure comparison (Π_{LT}) and multiplication (Π_{MULT}) protocols which are also simulation secure.

Since each step in $\Pi_{\text{TestSelection}}$ is simulatable, simulator $\mathcal{S}$ simply invokes the simulators for all subprotocols and outputs the view generated by all simulators. By universal composability theorem [11], the view generated by $\mathcal{S}$ is indistinguishable from view of adversary $\mathcal{A}$ in real execution of protocol $\Pi_{\text{TestSelection}}$. □

5.3 Security of UpdateState

Lemma 5.4. *Protocol $\Pi_{\text{UpdateState}}$ defined in Algorithm 10 securely realizes the* UpdateState *functionality from Section 4.4.4.*

We prove this Lemma in Appendix E.

Applying the composability theorem [11], we can then show:

Theorem 5.5. *Protocol Π_{Train} defined in Algorithm 19 securely realizes the* Train *functionality.*

6 EXPERIMENTS

We used the MP-SPDZ framework [1, 19] to implement our protocols. Concretely, we used the Replicated secret-sharing based three-party computation protocols over $\mathbb{Z}_{2^{64}}$. We ran our experiments using three Standard F16s v2 Azure instances, each of 32 GB RAM in the LAN and WAN settings. Our network bandwidth and ping latency were respectively 9.5 Gbps and 1.2 ms in the LAN setting, and 287 Mbps and 61 ms in the WAN setting.

For various values of N, m and h, Table 3 compares the communication cost incurred by our protocol vs [16] (based of the implementation at [2]). As can be observed from the table, our protocol is between $4-10\times$ more communication efficient than [16], with larger improvements on larger datasets. Tables 4 and 5 compare the runtimes in the LAN and WAN settings. Our protocol is $3.1-9\times$ faster than [16] in the LAN setting. In the WAN setting, we only ran the smaller benchmarks (as [16] had an incredibly high run time); even then our improvements range between $2.9-7.3\times$.

Input (n, m, h)	Cost ([16])	Cost (Ours)	IF
(13, 11, 4)	7.3	1.8	4.0
(13, 20, 4)	13.2	3.1	4.2
(13, 11, 10)	18.3	3.4	5.4
(13, 11, 20)	36.5	5.9	6.2
(16, 11, 10)	158.8	27.5	5.8
(16, 20, 20)	574.3	78.6	7.3
(16, 40, 50)	2,861.6	328.0	8.7
(18, 20, 20)	2,419.7	317.8	7.6
(18, 50, 50)	15,065.1	1,619.6	9.3
(18, 100, 100)	60,184.2	5,988.9	10.0
(19, 11, 10)	1,371.9	225.1	6.1
(19, 20, 50)	12,404.4	1,453.6	8.5
(19, 40, 40)	19,786.5	2,158.1	9.2

Table 3: Communication (GB) of Decision Tree Training for a dataset containing $N=2^n$ training samples, m attributes to construct a decision tree of height h. IF represents the Improvement Factor i.e. Cost ([16]) / Cost (Ours)

Input (n, m, h)	Runtime ([16])	Runtime (Ours)	IF
(13, 11, 4)	11.3	3.7	3.1
(13, 20, 4)	19.0	5.6	3.4
(13, 11, 10)	29.1	7.3	4.0
(13, 11, 20)	63.4	12.9	4.9
(16, 11, 10)	208.1	45.8	4.5
(16, 20, 20)	681.6	118.3	5.8
(16, 40, 50)	3,332.3	459.6	7.3
(18, 20, 20)	2,965.3	494.0	6.0
(18, 50, 50)	18,730.3	2,361.2	7.9
(18, 100, 100)	71,999.8	8,115.2	8.9
(19, 11, 10)	1,858.5	415.8	4.5
(19, 20, 50)	15,324.4	2,436.0	6.3
(19, 40, 40)	24,280.4	3,287.2	7.4

Table 4: Runtime (s) of Decision Tree Training for a dataset containing $N=2^n$ training samples, m attributes to construct a decision tree of height h in LAN setup. IF represents the Improvement Factor i.e. Runtime([16]) / Runtime (Ours)

Input (n, m, h)	Runtime ([16])	Runtime (Ours)	IF
(13, 11, 4)	417.3	144.2	2.9
(13, 20, 4)	676.2	205.7	3.3
(13, 11, 10)	1,039.3	291.5	3.6
(13, 11, 20)	2,073.3	536.3	3.9
(16, 11, 10)	2,906.9	691.2	4.2
(16, 20, 20)	9,764.7	1,751.6	5.6
(16, 40, 50)	48,171.5	6,622.2	7.3
(18, 20, 20)	42,502.8	6,680.0	6.4
(19, 11, 20)	23,850.7	4,923.4	4.8

Table 5: Runtime (s) of Decision Tree Training for a dataset containing $N=2^n$ training samples, m attributes to construct a decision tree of height h in WAN setup. IF represents the Improvement Factor i.e. Runtime([16]) / Runtime (Ours)

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A TRAINING PROTOCOL FOR DISCRETE ATTRIBUTES

Our protocol can be easily extended to handle discrete attributes as well. Using ideas of [13], we compute the Gini index for discrete attributes. Suppose dataset $\mathbf{D}$ has attribute values for a binary attribute $d \in \{0, 1\}$. Then the Gini index of splitting from test $d \stackrel{?}{=} 0$ (or $d \stackrel{?}{=} 1$) is defined as

$$G_d = \frac{|\mathbf{D}_{d=0}|}{|\mathbf{D}|}\,\mathrm{Gini}(\mathbf{D}_{d=0}) + \frac{|\mathbf{D}_{d=1}|}{|\mathbf{D}|}\,\mathrm{Gini}(\mathbf{D}_{d=1})$$

where $\mathrm{Gini}(\mathbf{D}) = 1 - \sum_{b\in\{0,1\}} (|\mathbf{D}_{y=b}|^2/|\mathbf{D}|^2)$. Minimizing G_d is same as maximizing G'_d:

$$\begin{aligned} G'_d(\mathbf{D}) = &\left(|\mathbf{D}_{d=1}|\left(\left|\mathbf{D}_{d=0\wedge y=0}\right|^2 + \left|\mathbf{D}_{d=0\wedge y=1}\right|^2\right)\right. \\ &\left. + |\mathbf{D}_{d=0}|\left(\left|\mathbf{D}_{d=1\wedge y=0}\right|^2 + \left|\mathbf{D}_{d=1\wedge y=1}\right|^2\right)\right) / (|\mathbf{D}_{d=0}|\,|\mathbf{D}_{d=1}|) \end{aligned}$$

where $\mathbf{D}_{d=b'\wedge y=b} = \{(\mathbf{d}, y) \in \mathbf{D} \mid d = b' \wedge y = b\}$.

This expression is similar to that of the Gini index of continuous variables and can be computed securely by simply modifying

$\Pi_{\mathsf{ComputeGini}}$. Once we have Gini index for both continuous and discrete attributes, simply choose the splitting test which has the maximum Gini index and the rest of the training protocol proceeds as described earlier.

If the attribute is discrete but non-binary i.e. $\text{domain}(d) = \mathbb{Z}_k$, then each value $i \in \text{domain}(d)$ induces a partition of dataset as $\{\mathbf{D}_{d=i}, \mathbf{D}_{d\neq i}\}$. We can then proceed to compute the Gini index of all such possible splits using the expression above and select the split associated with minimum Gini index.

B PLAINTEXT DECISION TREE TRAINING

The algorithm for cleartext decision tree training is presented in Algorithm 14.

Algorithm 14: Plaintext decision tree training

Input: Labelled training dataset $\mathbf{D}$, maximum height h.
Output: Decision tree $\mathcal{T}$

if $h = 0$ **then**
  Let v be a leaf node.
  Label(v) is the most occuring label in $\mathbf{D}$. Output tree with v as root node.
else
  Find j, t such that $x_j < t$ minimizes Gini index of the split.
  Let $\mathbf{D}_l = \mathbf{D}_{x_j<t}$ and $\mathbf{D}_r = \mathbf{D}_{x_j\geq t}$ be partition of $\mathbf{D}$ based on test.
  Compute $\mathcal{T}_l = \mathsf{Train}(\mathbf{D}_l, h-1)$ and $\mathcal{T}_r = \mathsf{Train}(\mathbf{D}_r, h-1)$.
  Let v be an internal node with test $x_j < t$ and with left child $\mathcal{T}_l$, right child $\mathcal{T}_r$.
  Output tree with v as root node.
end

C PROTOCOLS FOR ELEMENT WISE OPERATIONS

The protocols realizing element wise operations are as follows:

- **Secure Multiplication:** denoted by $\langle z\rangle = \Pi_{\mathsf{MULT}}(\langle u\rangle, \langle v\rangle)$. Parties have shares $\langle u\rangle, \langle v\rangle \in \mathbb{Z}_{2^\ell}$ and want to compute shares $\langle z\rangle \in \mathbb{Z}_{2^\ell}$ where $w = u \cdot v$. Then

$$\begin{aligned} z &= (u_0 + u_1 + u_2)\cdot(v_0 + v_1 + v_2) \\ &= (u_0v_0 + u_0v_1 + u_1v_0) + (u_1v_1 + u_1v_2 + u_2v_1) \\ &\quad + (u_2v_2 + u_2v_0 + u_0v_2) \\ &= z_0 + z_1 + z_2 \end{aligned}$$

  where party S_i can compute z_i locally. Each party can obtain replicated shares of z with communication of 3ℓ bits (each party sends its share to the next party).
- **Secure Bit Decomposition:** denoted by $\langle u\rangle^{\mathbf{B}} = \Pi_{\mathsf{A2B}}(\langle u\rangle)$. Parties have RSS $\langle u\rangle \in \mathbb{Z}_{2^\ell}$ and want to compute shares of bit decomposition $\langle u\rangle^{\mathbf{B}} = \left(\langle u_{\ell-1}\rangle^{\mathbf{B}}, \ldots, \langle u_0\rangle^{\mathbf{B}}\right)$ where $u = \sum_{i=0}^{\ell-1} 2^i u_i$. We use the method described in [26] which uses $\ell \log \ell$ AND gates requiring $1 + \log \ell$ rounds. The concrete cost (including preprocessing phase) of bit decomposition is $\approx 212\ell$ bits.
- **Secure Comparison:** denoted by $\langle b\rangle = \Pi_{\mathsf{LT}}(\langle u\rangle, \langle v\rangle)$. Parties have RSS $\langle u\rangle, \langle v\rangle \in \mathbb{Z}_{2^\ell}$ and want to compute $\langle b\rangle \in \mathbb{Z}_{2^\ell}$ where $b = (u < v)$. Parties compute $\langle z\rangle = \langle u - v\rangle$ locally and compute $b = (z < 0)$ by extracting most significant bit of z using 2ℓ AND gates in $\log \ell$ rounds. Note that $\mathsf{MSB}(z) = 1$ if $z < 0$ and $\mathsf{MSB}(z) = 0$ otherwise. The concrete cost (including preprocessing phase) of secure comparison using $(3, 2)$ RSS is $\approx 15\ell$ bits.
- **Secure Equality:** denoted by $\langle b\rangle = \Pi_{\mathsf{EQ}}(\langle u\rangle, \langle v\rangle)$. Parties have RSS $\langle u\rangle, \langle v\rangle \in \mathbb{Z}_{2^\ell}$ and want to compute $\langle b\rangle \in \mathbb{Z}_{2^\ell}$ where $b = \left(u \stackrel{?}{=} v\right) = \left((u - v) \stackrel{?}{=} 0\right)$. Parties compute $\langle z\rangle = \langle u - v\rangle$ locally, compute its bit decomposition and then check if z is zero using a binary circuit. The concrete cost (including preprocessing phase) of secure equality using $(3, 2)$ RSS is $\approx 20\ell$ bits.

D PROTOCOLS FOR OBLIVIOUS PERMUTATIONS

Algorithm 15: $\Pi_{\mathsf{ApplyPerm}}$

Input: Vector $\langle \mathbf{X}\rangle$, permutation $\langle \mathbf{P}\rangle$ of length N.
Output: Vector $\langle \mathbf{X}^{\mathbf{P}}\rangle$.

Parties compute $\langle \mathbf{X}^{\pi}\rangle, \mathbf{P}^{\pi} = \text{Shuffle}(\langle \mathbf{X}\rangle, \langle \mathbf{P}\rangle)$.
for $i \in [N]$ **do**
  $\langle \mathbf{X}^{\mathbf{P}}[\mathbf{P}^{\pi}[i]]\rangle = \langle \mathbf{X}^{\pi}[i]\rangle$
end
Output $\langle \mathbf{X}^{\mathbf{P}}\rangle$

Algorithm 16: Π_{PermComp}

Input: Permutations $\langle \mathbf{P}\rangle$ and $\langle \mathbf{Q}\rangle$ of length N.
Output: Permutation $\langle \mathbf{G}\rangle$ where $\mathbf{G} = \mathbf{P} \circ \mathbf{Q}$.

Parties compute $\langle \mathbf{Q}^{\pi}\rangle = \text{Shuffle}(\langle \mathbf{Q}\rangle)$ and reconstruct $\mathbf{Q}^{\pi}$.
Let $\langle \mathbf{G}^{\pi}[i]\rangle = \langle \mathbf{P}[\mathbf{Q}^{\pi}[i]]\rangle$ for all $i \in [N]$.
Parties compute $\langle \mathbf{G}\rangle = \text{Unshuffle}(\langle \mathbf{G}^{\pi}\rangle)$.
Parties output $\langle \mathbf{G}\rangle$.

Algorithm 17: $\Pi_{\mathsf{SortPermBit}}$

Input: Vector $\langle \mathbf{b}\rangle \in \{0, 1\}^N$.
Output: Permutation $\langle \mathbf{P}\rangle$ that stably sorts $\mathbf{b}$

$\langle nz\rangle = \text{Sum}(\neg \langle \mathbf{b}\rangle)$
$\langle \mathbf{r}\rangle = \text{PrefixSum}(\langle \mathbf{b}\rangle) - \langle \mathbf{b}\rangle$ //Local compute
for $i \in \{1, \ldots, N\}$ **do**
  $\langle \mathbf{P}[i]\rangle = i - (i-1)\cdot\langle \mathbf{b}[i]\rangle - \langle \mathbf{r}[i]\rangle + \langle \mathbf{b}[i]\rangle \cdot \langle nz + 2\mathbf{r}[i]\rangle$
end
Output $\langle \mathbf{P}\rangle$.

Algorithm 18: Π_{SortPerm}

Input: Vector $\langle \mathbf{x} \rangle \in \mathbb{Z}_{2^\ell}^N$.
Output: Permutation $\langle \mathbf{P} \rangle$ that stably sorts $\mathbf{x}$

for $i = 1$ *to* N **do**
  Let $\langle x_i \rangle = \langle \mathbf{x}[i] \rangle$
  $(\langle x_i[\ell-1] \rangle^{\mathbf{B}}, \ldots, \langle x_i[0] \rangle^{\mathbf{B}}) = \Pi_{\mathsf{A2B}}(\langle x_i \rangle)$
end
Let $\langle \mathbf{P} \rangle = \langle [1, \ldots, N] \rangle$.
for $j = 0$ *to* $\ell - 1$ **do**
  Let $\langle \mathbf{y}_j \rangle = \left[\langle x_1[j] \rangle^{\mathbf{B}}, \ldots, \langle x_N[j] \rangle^{\mathbf{B}} \right]$
  $\left\langle \mathbf{y}_j^{\mathbf{P}} \right\rangle = \Pi_{\mathsf{ApplyPerm}} \left(\langle \mathbf{y}_j \rangle, \langle \mathbf{P} \rangle \right)$
  $\langle \sigma \rangle = \Pi_{\mathsf{SortPermBit}} \left(\left\langle \mathbf{y}_j^{\mathbf{P}} \right\rangle \right)$
  $\langle \mathbf{P} \rangle = \Pi_{\mathsf{PermComp}}(\langle \sigma \rangle, \langle \mathbf{P} \rangle)$
end
Output $\langle \mathbf{P} \rangle$.

E END-TO-END SECURE TRAINING ALGORITHM

We combine $\Pi_{\mathsf{SetupPerm}}$, $\Pi_{\mathsf{TrainInternalLayer}}$ and $\Pi_{\mathsf{TrainLeafLayer}}$ to design our end-to-end secure decision training algorithm in 19.

Algorithm 19: Π_{Train}

Input: dataset $\langle \mathbf{D} \rangle$, height h
Output: $\left\{ \left\langle \mathsf{Layer}^{(i)} \right\rangle \right\}_{i \in [h+1]}$
Cost: $O(mN \log N + h(mN + N \log N))$

$\langle \mathbf{D}_1 \rangle = \langle \mathbf{D} \rangle$
$\langle \mathbf{g}_1[1] \rangle = 1$ and $\langle \mathbf{g}_1[i] \rangle = 0$ for $i \in [2, N]$
$\langle \mathbf{NID}[i] \rangle = 1$ for all $i \in [N]$
$\left\{ \left\langle \mathbf{P}_1^i \right\rangle \right\} = \Pi_{\mathsf{SetupPerm}}(\langle \mathbf{D}_1 \rangle)$
$\left\langle \mathsf{State}^{(1)} \right\rangle = \left(\langle \mathbf{D}_1 \rangle, \langle \mathbf{g}_1 \rangle, \left\{ \left\langle \mathbf{P}_1^i \right\rangle \right\}_{i \in [m]}, \langle \mathbf{NID} \rangle \right)$
for $k = 1$ *to* h **do**
  $\left\langle \mathsf{Layer}^{(k)} \right\rangle, \left\langle \mathsf{State}^{(k+1)} \right\rangle = \Pi_{\mathsf{TrainInternalLayer}} \left(\left\langle \mathsf{State}^{(k)} \right\rangle, k \right)$
end
$\left\langle \mathsf{Layer}^{(h+1)} \right\rangle = \Pi_{\mathsf{TrainLeafLayer}} \left(\left\langle \mathsf{State}^{(h+1)} \right\rangle \right)$
Output $\left\{ \left\langle \mathsf{Layer}^{(i)} \right\rangle \right\}_{i \in [h+1]}$.

F CORRECTNESS LEMMA

In $\Pi_{\mathsf{UpdateState}}$, parties compute the shares of updated database $\mathbf{D}_{k+1}$ by stably sorting $\mathbf{D}_k$ based on $\mathbf{b}$. We prove that $\mathbf{D}_{k+1}$ has correctly grouped elements.

Lemma F.1. *Entries of the same group appear consecutively in* $\mathbf{D}_{k+1}$ *where* $\mathbf{D}_{k+1}$ *is obtained by sorting* $\mathbf{D}_k$ *based on values of* $\mathbf{b}$.

Proof. We prove by contradiction. Suppose the theorem statement is not true. This means that there exist $d, e, f \in [N]$ such that $d < e < f$ and $\mathbf{D}_{k+1}[d]$, $\mathbf{D}_{k+1}[f]$ belong to the same group (call it (x, y)) and $\mathbf{D}_{k+1}[e]$ belongs to a different group (call it (x', y')). This implies that either

(1) $y \neq y'$: This is a contradiction as $\mathbf{D}_{k+1}$ is obtained by sorting based on y so for all $e \in [d, f]$, $y' = y$.
(2) $x' \neq x$: We already know that $y' = y$. Suppose we have d', e', f' such that

$$\mathbf{D}_k[d'] = \mathbf{D}_{k+1}[d]$$
$$\mathbf{D}_k[e'] = \mathbf{D}_{k+1}[e]$$
$$\mathbf{D}_k[f'] = \mathbf{D}_{k+1}[f]$$

Then $d' < e' < f'$ (stable sorting) and for all $e' \in [d', f']$, $x' = x$ (all entries of a group appear consecutively in $\mathbf{D}_k$). This is also a contradiction.

Hence, all entries in the same group appear consecutively in $\mathbf{D}_{k+1}$. □

Lemma 5.4. *Protocol* $\Pi_{\mathsf{UpdateState}}$ *defined in Algorithm 10 securely realizes the* UpdateState *functionality from Section 4.4.4.*

Proof. It is easy to see that by applying universal composability theorem [11], $\Pi_{\mathsf{UpdateState}}$ is simulatable as the composing subprotocols are all simulation secure. We formally prove correctness.

In the UpdateState protocol, parties update the shares of dataset, group flag and node id vectors. Parties also update the permutations that sort the updated dataset according to each attribute within groups. To prove correctness of $\Pi_{\mathsf{UpdateState}}$, we show that these updates are computed correctly and updated vectors satisfy the invariant described in Section 4.4.

Correctness of $\mathbf{D}_{k+1}$. Suppose $\mathbf{D}_k$ has n_k groups and satisfies the invariant. Each group will be further subdivided into 2 new groups based on output of comparison i.e. $\mathbf{b} = 0$ or 1. We can denote the new groups of samples as a tuple (x, y), $x \in [1, n_k]$ and $y \in \{0, 1\}$ where x is the sample's group in $\mathbf{D}_k$ and y is the output of comparison.

To obtain $\mathbf{D}_{k+1}$, $\Pi_{\mathsf{UpdateState}}$ stably sorts $\mathbf{D}_k$ based on values of $\mathbf{b}$. The order in which groups appear in $\mathbf{D}_k$ is $1, 2, \ldots, n_k$ i.e. in $\mathbf{D}_k$ we first have elements of group 1 followed by group 2 and so on. The order in which groups appear in $\mathbf{D}_{k+1}$ is $(1, 0), \ldots, (n_k, 0)$, $(1, 1), \ldots, (n_k, 1)$. The ordering of groups is irrelevant as long as entries of the same group appear consecutively in $\mathbf{D}_{k+1}$.

In lemma F.1, we show that sorting $\mathbf{D}_k$ according to $\mathbf{b}$ indeed gives us $\mathbf{D}_{k+1}$ with correctly grouped entries.

Correctness of $\mathbf{g}_{k+1}$. We compute $\mathbf{g}$ from $\mathbf{g}_k$ in step 5 in $\Pi_{\mathsf{UpdateState}}$ using $\Pi_{\mathsf{GroupFirstOne}}$ protocol. From the description of $\Pi_{\mathsf{GroupFirstOne}}$ (Section 3.3), we can see that $\mathbf{g}[i] = 1$ if $\mathbf{b}[i]$ is the first instance of a 0 or 1 in any group. This means $\mathbf{g}[i] = 1$ if i is the first element of group $(j, 0)$ or $(j, 1)$ for any $j \in [1, n_k]$.

Finally $\Pi_{\mathsf{UpdateState}}$ sorts $\mathbf{g}$ according to $\mathbf{b}$ to obtain $\mathbf{g}_{k+1}$ in step 6. As we are stably sorting, the first element of any new group in $\mathbf{D}_k$ will be the first element of the group in $\mathbf{D}_{k+1}$. Hence, $\mathbf{g}_{k+1}[i] = 1$ if i is the first element of a group in $\mathbf{D}_{k+1}$.

Correctness of NID. Suppose we have a node with $NID = d$. Then the left child will be numbered d and the right child will be numbered $d + 2^{k-1}$. This is computed in step 3 in $\Pi_{\mathsf{UpdateState}}$ as $\mathbf{NID} = \mathbf{NID} + 2^{k-1} \cdot \mathbf{b}$. This numbering is consistent with the

ordering of groups in $\mathbf{D}_{k+1}$ as the left children of all the nodes are numbered followed by all the right children.

Finally we apply permutation $\mathbf{Q}$ in step 4 to keep the ordering of **NID** and $\mathbf{D}_{k+1}$ same. Same ordering is necessary so that $\mathbf{NID}[i]$ corresponds to the node id of sample $\mathbf{D}_{k+1}[i]$. Hence, the updated vector **NID** satisfies the invariant.

Correctness of sorting permutations $\mathbf{P}^i_{k+1}$. We introduce a few more notations as follows:

(1) $\mathbf{D}_k$ is the dataset grouped by nodes in k^{th} layer. $\mathbf{P}^i_k$ is the permutation that sorts $\mathbf{D}_k$ according to i^{th} attribute. Let $\mathbf{D}^i_k = \text{Permute}\left(\mathbf{D}_k, \mathbf{P}^i_k\right)$.

(2) $\mathbf{D}_{k+1}$ is the new dataset grouped by nodes in $(k+1)^{th}$ layer. $\mathbf{P}^i_{k+1}$ is the new permutation that sorts $\mathbf{D}_{k+1}$ according to i^{th} attribute. Let $\mathbf{D}^i_{k+1} = \text{Permute}\left(\mathbf{D}_{k+1}, \mathbf{P}^i_{k+1}\right)$.

Suppose we are given $\mathbf{P}^i_k$, $\mathbf{D}_{k+1}$ and have to compute $\mathbf{P}^i_{k+1}$. From the definition of $\mathbf{D}^i_k, \mathbf{D}^i_{k+1}$, we have

$$\mathbf{D}^i_k\left[\mathbf{P}^i_k[j]\right] = \mathbf{D}_k[j] \quad \forall i \in [m], j \in [N] \tag{2}$$

$$\mathbf{D}^i_{k+1}\left[\mathbf{P}^i_{k+1}[j]\right] = \mathbf{D}_{k+1}[j] \quad \forall i \in [m], j \in [N] \tag{3}$$

Just like how we obtain $\mathbf{D}_{k+1}$ from $\mathbf{D}_k$ by sorting it based on $\mathbf{b}$, we can obtain $\mathbf{D}^i_{k+1}$ by sorting $\mathbf{D}^i_k$ based on $\mathbf{b}^i$ (can be proved similar to lemma F.1) where $\mathbf{b}^i = \text{Permute}(\mathbf{b}, \mathbf{P}^i_k)$. Let $\mathbf{R}^i$ be the permutation that sorts $\mathbf{b}^i$. Then, we have

$$\mathbf{b}^i\left[\mathbf{P}^i_k[j]\right] = \mathbf{b}[j] \quad \forall i \in [m], j \in [N] \tag{4}$$

$$\mathbf{D}^i_{k+1}\left[\mathbf{R}^i[j]\right] = \mathbf{D}^i_k[j] \quad \forall i \in [m], j \in [N] \tag{5}$$

Let $\mathbf{Q}$ be the permutation that sorts $\mathbf{b}$. Then we have

$$\mathbf{D}_{k+1}[\mathbf{Q}[j]] = \mathbf{D}_k[j] \quad \forall j \in [N] \tag{6}$$

Let $\mathbf{Q}^i$ be the permutation obtained by applying $\mathbf{Q}$ to $\mathbf{P}^i_k$ i.e.

$$\mathbf{Q}^i[\mathbf{Q}[j]] = \mathbf{P}^i_k[j] \quad \forall i \in [m], j \in [N] \tag{7}$$

Using these equations, we have

$$\begin{aligned}
&\mathbf{D}^i_{k+1}\left[\mathbf{R}^i[j]\right] = \mathbf{D}^i_k[j] && \text{from (5)}\\
\implies &\mathbf{D}^i_{k+1}\left[\mathbf{R}^i\left[\mathbf{P}^i_k[j]\right]\right] = \mathbf{D}^i_k\left[\mathbf{P}^i_k[j]\right]\\
\implies &\mathbf{D}^i_{k+1}\left[\mathbf{R}^i\left[\mathbf{P}^i_k[j]\right]\right] = \mathbf{D}_k[j] && \text{from (2)}\\
\implies &\mathbf{D}^i_{k+1}\left[\mathbf{R}^i\left[\mathbf{Q}^i[\mathbf{Q}[j]]\right]\right] = \mathbf{D}_{k+1}[\mathbf{Q}[j]] && \text{from (6, 7)}\\
\implies &\mathbf{D}^i_{k+1}\left[\mathbf{R}^i\left[\mathbf{Q}^i[j]\right]\right] = \mathbf{D}_{k+1}[j]
\end{aligned}$$

Therefore, $\mathbf{P}^i_{k+1} = \mathbf{R}^i \circ \mathbf{Q}^i$ where $\mathbf{R}^i$ is the permutation that sorts $\mathbf{b}^i$ and $\mathbf{Q}^i$ is the permutation obtained by applying $\mathbf{Q}$ to $\mathbf{P}^i_k$. $\Pi_{\mathsf{UpdateState}}$ computes $\mathbf{R}^i$ and $\mathbf{Q}^i$ using $\Pi_{\mathsf{ApplyPerm}}$ in steps 9, 10 and $\mathbf{P}^i_{k+1}$ using Π_{PermComp} in step 11.

This concludes our correctness proof as all components of updated $\text{State}^{(k+1)}$ satisfy the invariant.

□

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