0.1 State-Space Representation

The general form of state-space representation is exhibited in Equation (0.1).

$$\begin{aligned} \underset{nx1}{\dot{\mathbf{x}}} &= \underset{nxn}{\mathbf{A}} \cdot \underset{nx1}{\mathbf{x}} + \underset{nxp}{\mathbf{B}} \cdot \underset{px1}{\mathbf{u}} \\ \underset{mx1}{\mathbf{y}} &= \underset{mxn}{\mathbf{C}} \cdot \underset{nx1}{\mathbf{x}} + \underset{mxp}{\mathbf{D}} \cdot \underset{px1}{\mathbf{u}} \end{aligned} \tag{0.1}$$

The designated x states and p inputs are exhibited in Equations (0.2) - (0.3).

$$\underset{nx1}{\mathbf{x}} = \begin{bmatrix} \theta \\ \phi_x \\ \dot{\theta} \\ \dot{\phi}_x \\ \phi_y \\ \dot{\phi}_y \end{bmatrix} \tag{0.2}$$

$$\underset{px1}{\mathbf{u}} = \begin{bmatrix} v_{mtr.l} \\ v_{mtr.r} \end{bmatrix} \tag{0.3}$$

The derivation of indices for the system matrices **A** and **B** which are nonintuitive are derived from Equations (**??**) - (**??**). in Equations (0.4) - (0.5).

$$\begin{aligned}
\mathbf{K}_{1.\ddot{x}} \cdot \begin{bmatrix} \ddot{\theta} \\ \ddot{\phi}_x \end{bmatrix} + \mathbf{K}_{1.\dot{x}} \cdot \begin{bmatrix} \dot{\theta} \\ \dot{\phi}_x \end{bmatrix} + \mathbf{K}_{1.x} \cdot \begin{bmatrix} \theta \\ \phi_x \end{bmatrix} &= \mathbf{K}_{1.v} \cdot \begin{bmatrix} v_{mtr.l} \\ v_{mtr.r} \end{bmatrix} \\
\begin{bmatrix} \ddot{\theta} \\ \ddot{\phi}_x \end{bmatrix} &= \underbrace{-\mathbf{K}_{1.\ddot{x}}^{-1} \cdot \mathbf{K}_{1.\dot{x}}}_{\mathbf{A}_1} \cdot \begin{bmatrix} \dot{\theta} \\ \dot{\phi}_x \end{bmatrix} + \underbrace{-\mathbf{K}_{1.\ddot{x}}^{-1} \cdot \mathbf{K}_{1.x}}_{\mathbf{A}_0} \cdot \begin{bmatrix} \theta \\ \phi_x \end{bmatrix} + \underbrace{\mathbf{K}_{1.\ddot{x}}^{-1} \cdot \mathbf{K}_{1.v}}_{\mathbf{B}_1} \cdot \begin{bmatrix} v_{mtr.l} \\ v_{mtr.r} \end{bmatrix}
\end{aligned} \tag{0.4}$$

$$\begin{aligned}
k_{2.\ddot{x}} \cdot \begin{bmatrix} \ddot{\phi}_y \end{bmatrix} + k_{2.\dot{x}} \cdot \begin{bmatrix} \dot{\phi}_y \end{bmatrix} &= k_{2.v} \cdot \begin{bmatrix} v_{mtr.r} - v_{mtr.l} \end{bmatrix} \\
\begin{bmatrix} \ddot{\phi}_y \end{bmatrix} &= \underbrace{-k_{2.\ddot{x}}^{-1} \cdot k_{2.\dot{x}}}_{A_2} \cdot \begin{bmatrix} \dot{\phi}_y \end{bmatrix} + \underbrace{k_{2.\ddot{x}}^{-1} \cdot k_{2.v}}_{B_2} \cdot \begin{bmatrix} v_{mtr.r} - v_{mtr.l} \end{bmatrix}
\end{aligned} \tag{0.5}$$

Note that $K_{1.\ddot{x}}$ must be invertible to perform the second step in in Equation (0.4). The derivation for matrix invertibility and the proof that $K_{1.\ddot{x}}$ is nonsingular [*and is therefore invertible*], are exhibited in Equations (0.6) - (0.9).

$$\underset{2x2}{\mathbf{X}} = \begin{bmatrix} +X_{(1,1)} & +X_{(1,2)} \\ +X_{(2,1)} & +X_{(2,2)} \end{bmatrix} \tag{0.6}$$

$$\mathbf{X}^{-1} = \frac{1}{\det(\mathbf{X})} \cdot \operatorname{adj}(\mathbf{X}) = \frac{1}{X_{(1,1)} \cdot X_{(2,2)} - X_{(1,2)} \cdot X_{(2,1)}} \cdot \begin{bmatrix} +X_{(2,2)} & -X_{(2,1)} \\ -X_{(1,2)} & +X_{(1,1)} \end{bmatrix} \tag{0.7}$$

$$\det(\mathbf{X}) \neq 0 \tag{0.8}$$

$$\det(\mathbf{K}_{1.\ddot{x}}) = k_{1.1} \cdot k_{1.3} - k_{1.2} \cdot k_{1.2} \neq 0 \tag{0.9}$$

The $\mathbf{A}$ matrix and the state vector $\mathbf{x}$ are exhibited in Equation (0.10).

$$\underset{nxn}{\mathbf{A}} \cdot \underset{nx1}{\mathbf{x}} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \mathbf{A}_{0\ (1,1)} & \mathbf{A}_{0\ (1,2)} & \mathbf{A}_{1\ (1,1)} & \mathbf{A}_{1\ (1,2)} & 0 & 0 \\ \mathbf{A}_{0\ (2,1)} & \mathbf{A}_{0\ (2,2)} & \mathbf{A}_{1\ (2,1)} & \mathbf{A}_{1\ (2,2)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & A_2 \end{bmatrix} \cdot \begin{bmatrix} \theta \\ \phi_x \\ \dot{\theta} \\ \dot{\phi_x} \\ \phi_y \\ \dot{\phi_y} \end{bmatrix} \tag{0.10}$$

The $\mathbf{B}$ matrix and the input vector $\mathbf{u}$ are exhibited in Equation (0.11).

$$\underset{nxp}{\mathbf{B}} \cdot \underset{px1}{\mathbf{u}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \mathbf{B_{1\ (1,1)}} & \mathbf{B_{1\ (1,2)}} \\ \mathbf{B_{1\ (2,1)}} & \mathbf{B_{1\ (2,2)}} \\ 0 & 0 \\ -B_2 & +B_2 \end{bmatrix} \cdot \begin{bmatrix} v_{mtr.l} \\ v_{mtr.r} \end{bmatrix} \tag{0.11}$$

The $\mathbf{C}$ matrix and the state vector $\mathbf{x}$ are exhibited in Equation (0.12).

$$\underset{mxn}{\mathbf{C}} \cdot \underset{nx1}{\mathbf{x}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \theta \\ \phi_x \\ \dot{\theta} \\ \dot{\phi_x} \\ \phi_y \\ \dot{\phi_y} \end{bmatrix} \tag{0.12}$$

The $\mathbf{D}$ matrix and the input vector $\mathbf{u}$ are exhibited in Equation (0.13).

$$\underset{mxp}{\mathbf{D}} \cdot \underset{px1}{\mathbf{u}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} v_{mtr.l} \\ v_{mtr.r} \end{bmatrix} \tag{0.13}$$