# CHAPTER 1

## Hardware-Equivalent Dynamics Model

To simulate the dynamics of the hardware, a hardware-equivalent dynamics model was selected. The model was originally derived by Yamamoto [1] and has been successfully used in other control studies [2].

Figures 1.1 - 1.2, depict the physical model of the two-wheeled inverted pendulum as isometric and multiview projections. These figures use Yamamoto's original symbol notation; a legend is provided in Figure 1.2.

Yamamoto [1] makes the following assumptions in Figures 1.1 - 1.2:

- All mass geometries are uniform.
- All masses are uniformly distributed.
- The hardware consists of three principal masses:

  · A cuboid [*The body.*]
  · A cylinder [*The left wheel.*]
  · A cylinder [*The right wheel.*]

Tables 1.1 - 1.2, define the variables and the parameters, respectively, that the physical model of the two-wheeled inverted pendulum will use.

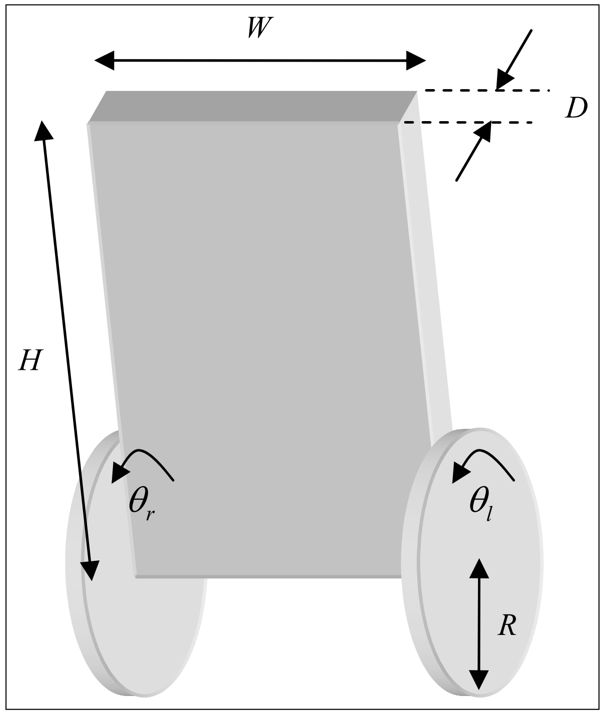


Figure 1.1: [Hardware-Equivalent Physical Dynamics Model]: Isometric[1]

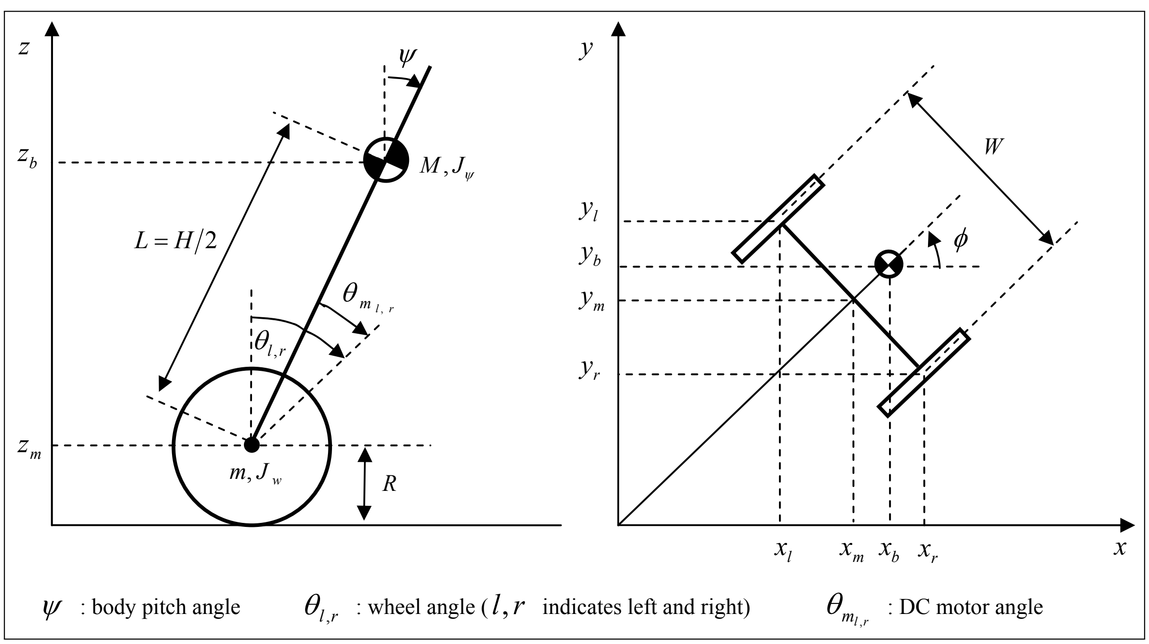


Figure 1.2: [Hardware-Equivalent Physical Dynamics Model]: Multiview[1]

Table 1.1: Hardware-Equivalent Physical Dynamics Model: Variables

| **Symbol** | **Definition** | **Unit** |
|---|---|---|
| $\theta$ | Angular position: Wheel $[\theta = \theta_{g,av}]$ [Measured from the wheel center of mass] | $rad$ |
| $\theta_g, \theta_b$ | Origin aligns with: $\begin{bmatrix} \text{global rejection vector} \\ (\textit{orthogonal from earth's surface}) \end{bmatrix}$ $\begin{bmatrix} \text{body pitch } \phi_x \end{bmatrix}$ | |
| $\theta_{av}, \theta_l, \theta_r$ | Component: [ average of left and right wheels ] [ left wheel ] [ right wheel ] | |
| $\phi$ | Angular position: Body [Measured from the wheels center of mass.] | $rad$ |
| $\phi_x, \phi_y, \phi_z$ | Dimension: [ X ] [ Y ] [ Z ] | |
| $p$ | Translational position $[p = p_w]$ [Measured from the corresponding center of mass] | $m$ |
| $p_x, p_y, p_z$ | Dimension: [ X ] [ Y ] [ Z ] | |
| $p_w, p_{wl}, p_{wr}, p_b$ | Component: [ both wheels ] [ left wheel ] [ right wheel ] [ body ] | |
| $v_{mtr}$ | Voltage: Motor Input | $V$ |
| $v_{mtr.l}, v_{mtr.r}$ | Component: [ left-wheel ] [ right-wheel ] | |

*Note*: When a subscript is unspecified, assume the first option is used by default.

Table 1.2: Hardware-Equivalent Physical Dynamics Model: Parameters

| **Symbol** | **Definition** | **Value** | **Unit** | **Source** |
|---|---|---|---|---|
| $a_g$ | Acceleration of gravity: Earth | 9.81 | $\frac{m}{s^2}$ | - |
| $m_w$ | Mass: Wheel [*Includes wheel axel.*] | 0.018 | $kg$ | [3] |
| $m_b$ | Mass: Body | 0.381 | $kg$ | [3] |
| $l_{b.h}$ | Length: Body: Height | ?? | $m$ | - |
| $l_{b.w}$ | Length: Body: Width | ?? | $m$ | - |
| $l_{b.d}$ | Length: Body: Depth | ?? | $m$ | - |
| $l_{b.c2a}$ | Length: Body: [Center of mass] to [Axis of Rotation] | ?? | $m$ | Sec. 1.4.3 |
| $r_w$ | Length: Wheel: Radius | 0.021 | $m$ | [3] |
| $J_{b.\phi_x}$ | Moment of Inertia: Wheel | $7.46 \cdot 10^{-6}$ | $kg \cdot m^2$ | [3] |
| $J_{b.\phi_x}$ | Moment of Inertia: Body: X-axis (pitch) | ?? | $kg \cdot m^2$ | Sec. 1.4.1 |
| $J_{b.\phi_y}$ | Moment of Inertia: Body: Y-axis (yaw) | ?? | $kg \cdot m^2$ | Sec. 1.4.2 |
| $R_{mtr}$ | Motor: Resistance | 4.4 | $\Omega$ | [3] |
| $k_{mtr.bEMF}$ | Motor: Coefficient of Back EMF | 0.495 | $\frac{V \cdot s}{rad}$ | [3] |
| $k_{mtr.T}$ | Motor: Coefficient of Torque | 0.470 | $\frac{N \cdot m}{A}$ | [3] |
| $k_{fr.m2w}$ | Motor: Coefficient of friction: [DC Motor] to [Wheel] | ?? | - | Sec. 1.4.4 |

## 1.1 Nonlinear model

The dynamic motion equations of the two-wheeled robot are derived using the Lagrangian method. The equations are based on the coordinate system provided in Figure 1.2.

### 1.1.1 Coordinate System

The coordinate system is explicitly defined in Equations (1.1) - (1.6).

$$\begin{bmatrix} \theta_{g.l} \\ \theta_{g.r} \\ \theta_{g.av} \\ \phi_y \end{bmatrix} = \begin{bmatrix} & & \theta_{b.l} & + & \phi_x \\ & & \theta_{b.r} & + & \phi_x \\ \frac{1}{2} & \cdot & (\theta_{g.l} & + & \theta_{g.r}) \\ \frac{r_w}{l_{b.w}} & \cdot & (\theta_{g.r} & - & \theta_{g.l}) \end{bmatrix} \tag{1.1}$$

$$\begin{bmatrix} \dot{p}_{w.x} \\ \dot{p}_{w.y} \\ \dot{p}_{w.z} \end{bmatrix} = \begin{bmatrix} r_w \cdot \dot{\theta}_{g.av} \cdot \cos(\phi_y) \\ r_w \cdot \dot{\theta}_{g.av} \cdot \sin(\phi_y) \\ 0 \end{bmatrix} \tag{1.2}$$

$$\begin{bmatrix} p_{w.x} \\ p_{w.y} \\ p_{w.z} \end{bmatrix} = \begin{bmatrix} \int \dot{p}_{w.x} \cdot \mathrm{d}t & + & p_{w.x}(0) \\ \int \dot{p}_{w.y} \cdot \mathrm{d}t & + & p_{w.y}(0) \\ \int \dot{p}_{w.z} \cdot \mathrm{d}t & + & p_{w.z}(0) \end{bmatrix} \tag{1.3}$$

$$\begin{bmatrix} p_{wl.x} \\ p_{wl.y} \\ p_{wl.z} \end{bmatrix} = \begin{bmatrix} p_{w.x} & - & \frac{l_{b.w}}{2} \cdot \sin(\phi_y) \\ p_{w.x} & + & \frac{l_{b.w}}{2} \cdot \cos(\phi_y) \\ & p_{w.z} & \end{bmatrix} \qquad \begin{bmatrix} p_{wr.x} \\ p_{wr.y} \\ p_{wr.z} \end{bmatrix} = \begin{bmatrix} p_{w.x} & + & \frac{l_{b.w}}{2} \cdot \sin(\phi_y) \\ p_{w.x} & - & \frac{l_{b.w}}{2} \cdot \cos(\phi_y) \\ & p_{w.z} & \end{bmatrix} \tag{1.4}$$

$$\begin{bmatrix} p_{b.x} \\ p_{b.y} \\ p_{b.z} \end{bmatrix} = \begin{bmatrix} p_{w.x} & + & l_{b.c2a} \cdot \sin(\phi_x) \cdot \cos(\phi_y) \\ p_{w.y} & + & l_{b.c2a} \cdot \sin(\phi_x) \cdot \sin(\phi_y) \\ p_{w.z} & + & l_{b.c2a} \cdot \cos(\phi_x) \end{bmatrix} \tag{1.5}$$

Typically, initial conditions are assumed to be as follows:

$$\begin{bmatrix} p_{w.x}(0) \\ p_{w.y}(0) \\ p_{w.z}(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ r_w \end{bmatrix} \tag{1.6}$$

## 1.2 Differential Equations

After creating a nonlinear model using the Lagrangian method, and then linearizing that model, Yamamoto [1] provides the differential equations (1.7) and (1.18), [*and their abbreviated term definitions*].

### 1.2.1 Wheel Angular Position $\theta$ and Body Pitch $\phi_x$

Equation (1.7) corresponds to wheel angular position $\theta$ and body pitch $\phi_x$.

$$\mathbf{K}_{1.\ddot{x}} \cdot \begin{bmatrix} \ddot{\theta} \\ \ddot{\phi}_x \end{bmatrix} + \mathbf{K}_{1.\dot{x}} \cdot \begin{bmatrix} \dot{\theta} \\ \dot{\phi}_x \end{bmatrix} + \mathbf{K}_{1.x} \cdot \begin{bmatrix} \theta \\ \phi_x \end{bmatrix} = \mathbf{K}_{1.v} \cdot \begin{bmatrix} v_{mtr.l} \\ v_{mtr.r} \end{bmatrix} \tag{1.7}$$

$$\mathbf{K}_{1.\ddot{x}} = \begin{bmatrix} +k_{1.1} & +k_{1.2} \\ +k_{1.2} & +k_{1.3} \end{bmatrix} \tag{1.8}$$

$$\mathbf{K}_{1.\dot{x}} = \begin{bmatrix} +k_{1.4} & -k_{1.4} \\ -k_{1.4} & +k_{1.4} \end{bmatrix} \tag{1.9}$$

$$\mathbf{K}_{1.x} = \begin{bmatrix} 0 & 0 \\ 0 & +k_{1.5} \end{bmatrix} \tag{1.10}$$

$$\mathbf{K}_{1.v} = \begin{bmatrix} +k_{1.6} & +k_{1.6} \\ -k_{1.6} & -k_{1.6} \end{bmatrix} \tag{1.11}$$

$$k_{1.1} = \left(2 \cdot m_w + m_b\right) \cdot r_w + J_w \tag{1.12}$$

$$k_{1.2} = m_b \cdot r_w \cdot l_{b.c2a} \tag{1.13}$$

$$k_{1.3} = m_b \cdot l_{b.c2a}^2 + J_{b.\phi_x} \tag{1.14}$$

$$k_{1.4} = 2 \cdot \left(\frac{k_{mtr.T} \cdot k_{mtr.bEMF}}{R_{mtr}} + k_{fr.m2w}\right) \tag{1.15}$$

$$k_{1.5} = -m_b \cdot a_g \cdot l_{b.c2a} \tag{1.16}$$

$$k_{1.6} = \frac{k_{mtr.T}}{R_{mtr}} \tag{1.17}$$

### 1.2.2 Body Yaw $\phi_y$

Equation (1.18) corresponds to body yaw $\phi_y$.

$$k_{2.\ddot{x}} \quad \cdot \begin{bmatrix} \ddot{\phi}_y \end{bmatrix} \quad + \quad k_{2.\dot{x}} \quad \cdot \begin{bmatrix} \dot{\phi}_y \end{bmatrix} \quad = \quad k_{2.v} \quad \cdot \begin{bmatrix} v_{mtr.r} - v_{mtr.l} \end{bmatrix} \tag{1.18}$$

$$k_{2.0} = \frac{l_{b.w}}{r_w} \tag{1.19}$$

$$k_{2.\ddot{x}} = \frac{1}{2} \cdot m_w \cdot l_{b.w}^2 + \frac{1}{2} \cdot k_{2.0}^2 \cdot J_{b.\phi_y} \tag{1.20}$$

$$k_{2.\dot{x}} = \frac{1}{2} \cdot k_{2.0}^2 \cdot k_{1.4} \tag{1.21}$$

$$k_{2.v} = \frac{1}{2} \cdot k_{2.0} \cdot k_{1.6} \tag{1.22}$$

## 1.3 State-Space Representation

The general form of state-space representation is exhibited in Equation (1.23).

$$
\begin{aligned}
\underset{nx1}{\dot{\mathbf{x}}} &= \underset{nxn}{\mathbf{A}} \cdot \underset{nx1}{\mathbf{x}} + \underset{nxp}{\mathbf{B}} \cdot \underset{px1}{\mathbf{u}} \\
\underset{mx1}{\mathbf{y}} &= \underset{mxn}{\mathbf{C}} \cdot \underset{nx1}{\mathbf{x}} + \underset{mxp}{\mathbf{D}} \cdot \underset{px1}{\mathbf{u}}
\end{aligned}
\tag{1.23}
$$

The designated $x$ states and $p$ inputs are exhibited in Equations (1.24) - (1.25).

$$
\underset{nx1}{\mathbf{x}} = \begin{bmatrix} \theta \\ \phi_x \\ \dot{\theta} \\ \dot{\phi}_x \\ \phi_y \\ \dot{\phi}_y \end{bmatrix} \tag{1.24}
$$

$$
\underset{px1}{\mathbf{u}} = \begin{bmatrix} v_{mtr.l} \\ v_{mtr.r} \end{bmatrix} \tag{1.25}
$$

The derivation of indices for the system matrices **A** and **B** which are nonintuitive are derived from Equations (1.7) - (1.18). in Equations (1.26) - (1.27).

$$\begin{aligned}
\mathbf{K}_{1.\ddot{x}} \cdot \begin{bmatrix} \ddot{\theta} \\ \ddot{\phi}_x \end{bmatrix} + \mathbf{K}_{1.\dot{x}} \cdot \begin{bmatrix} \dot{\theta} \\ \dot{\phi}_x \end{bmatrix} + \mathbf{K}_{1.x} \cdot \begin{bmatrix} \theta \\ \phi_x \end{bmatrix} &= \mathbf{K}_{1.v} \cdot \begin{bmatrix} v_{mtr.l} \\ v_{mtr.r} \end{bmatrix} \\
\begin{bmatrix} \ddot{\theta} \\ \ddot{\phi}_x \end{bmatrix} = \underbrace{-\mathbf{K}_{1.\ddot{x}}^{-1} \cdot \mathbf{K}_{1.\dot{x}}}_{\mathbf{A}_1} \cdot \begin{bmatrix} \dot{\theta} \\ \dot{\phi}_x \end{bmatrix} + \underbrace{-\mathbf{K}_{1.\ddot{x}}^{-1} \cdot \mathbf{K}_{1.x}}_{\mathbf{A}_0} \cdot \begin{bmatrix} \theta \\ \phi_x \end{bmatrix} &+ \underbrace{\mathbf{K}_{1.\ddot{x}}^{-1} \cdot \mathbf{K}_{1.v}}_{\mathbf{B}_1} \cdot \begin{bmatrix} v_{mtr.l} \\ v_{mtr.r} \end{bmatrix}
\end{aligned} \tag{1.26}$$

$$\begin{aligned}
k_{2.\ddot{x}} \cdot \begin{bmatrix} \ddot{\phi}_y \end{bmatrix} + k_{2.\dot{x}} \cdot \begin{bmatrix} \dot{\phi}_y \end{bmatrix} &= k_{2.v} \cdot \begin{bmatrix} v_{mtr.r} - v_{mtr.l} \end{bmatrix} \\
\begin{bmatrix} \ddot{\phi}_y \end{bmatrix} = \underbrace{-k_{2.\ddot{x}}^{-1} \cdot k_{2.\dot{x}}}_{A_2} \cdot \begin{bmatrix} \dot{\phi}_y \end{bmatrix} &+ \underbrace{k_{2.\ddot{x}}^{-1} \cdot k_{2.v}}_{B_2} \cdot \begin{bmatrix} v_{mtr.r} - v_{mtr.l} \end{bmatrix}
\end{aligned} \tag{1.27}$$

Note that $K_{1.\ddot{x}}$ must be invertible to perform the second step in in Equation (1.26). The derivation for matrix invertibility and the proof that $K_{1.\ddot{x}}$ is nonsingular [*and is therefore invertible*], are exhibited in Equations (1.28) - (1.31).

$$\underset{2x2}{\mathbf{X}} = \begin{bmatrix} +X_{(1,1)} & +X_{(1,2)} \\ +X_{(2,1)} & +X_{(2,2)} \end{bmatrix} \tag{1.28}$$

$$\mathbf{X}^{-1} = \frac{1}{\det(\mathbf{X})} \cdot \operatorname{adj}(\mathbf{X}) = \frac{1}{X_{(1,1)} \cdot X_{(2,2)} - X_{(1,2)} \cdot X_{(2,1)}} \cdot \begin{bmatrix} +X_{(2,2)} & -X_{(2,1)} \\ -X_{(1,2)} & +X_{(1,1)} \end{bmatrix} \tag{1.29}$$

$$\det(\mathbf{X}) \neq 0 \tag{1.30}$$

$$\det(\mathbf{K}_{1.\ddot{x}}) = k_{1.1} \cdot k_{1.3} - k_{1.2} \cdot k_{1.2} \neq 0 \tag{1.31}$$

The **A** matrix and the state vector **x** are exhibited in Equation (1.32).

$$\underset{nxn}{\mathbf{A}} \cdot \underset{nx1}{\mathbf{x}} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \mathbf{A}_{0\ (1,1)} & \mathbf{A}_{0\ (1,2)} & \mathbf{A}_{1\ (1,1)} & \mathbf{A}_{1\ (1,2)} & 0 & 0 \\ \mathbf{A}_{0\ (2,1)} & \mathbf{A}_{0\ (2,2)} & \mathbf{A}_{1\ (2,1)} & \mathbf{A}_{1\ (2,2)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & A_2 \end{bmatrix} \cdot \begin{bmatrix} \theta \\ \phi_x \\ \dot{\theta} \\ \dot{\phi}_x \\ \phi_y \\ \dot{\phi}_y \end{bmatrix} \tag{1.32}$$

The **B** matrix and the input vector **u** are exhibited in Equation (1.33).

$$\underset{nxp}{\mathbf{B}} \cdot \underset{px1}{\mathbf{u}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \mathbf{B}_{\mathbf{1}\ (\mathbf{1},\mathbf{1})} & \mathbf{B}_{\mathbf{1}\ (\mathbf{1},\mathbf{2})} \\ \mathbf{B}_{\mathbf{1}\ (\mathbf{2},\mathbf{1})} & \mathbf{B}_{\mathbf{1}\ (\mathbf{2},\mathbf{2})} \\ 0 & 0 \\ -B_2 & +B_2 \end{bmatrix} \cdot \begin{bmatrix} v_{mtr.l} \\ v_{mtr.r} \end{bmatrix} \tag{1.33}$$

The **C** matrix and the state vector **x** are exhibited in Equation (1.34).

$$\underset{mxn}{\mathbf{C}} \cdot \underset{nx1}{\mathbf{x}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \theta \\ \phi_x \\ \dot{\theta} \\ \dot{\phi}_x \\ \phi_y \\ \dot{\phi}_y \end{bmatrix} \tag{1.34}$$

The **D** matrix and the input vector **u** are exhibited in Equation (1.35).

$$\underset{mxp}{\mathbf{D}} \cdot \underset{px1}{\mathbf{u}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} v_{mtr.l} \\ v_{mtr.r} \end{bmatrix} \tag{1.35}$$

### 1.4 Calculation of Nonintuitive Parameters

Most of the hardware-equivalent dynamic model parameter values (*with respect to the MinSeg hardware*) were publicly available [3, 4], or were intuitive to obtain [*Example:* $l_h$, $l_w$, $l_d$].

Methods to determine those parameters which were not considered easily obtained are defined in the following sections.

#### 1.4.1 Moment of Inertia: Body: X-axis (Pitch) $J_{\phi_x}$

The moment of inertia of the body with respect to pitch, is assumed to be sufficiently equivalent to the moment of inertia of "an ideal thin rectangular plate with length $l_h$, width $l_w = 0$, an axis of rotation at one end of the plate".

This relation is exhibited in Equation (1.36).

$$J_{\phi_x} = \frac{m_b \cdot l_{b.c2a}^2}{3} \tag{1.36}$$

#### 1.4.2 Moment of Inertia: Body: Y-axis (Yaw) $J_{\phi_y}$

The moment of inertia of the body with respect to pitch, is assumed to be sufficiently equivalent to the moment of inertia of "an ideal thin rectangular plate with length $l_h$, width $l_w = 0$, an axis of rotation at one end of the plate".

This relation is exhibited in Equation (1.37).

$$J_{\phi_y} = \frac{m_b \cdot (l_{b.w}^2 + l_{b.d}^2)}{12} \tag{1.37}$$

### 1.4.3 Length From Body Center of Mass to Body Axis of Rotation $l_{b.c2a}$

The length from the body center of mass to the body axis of rotation $l_{b.c2a}$ may be determined using more than one method.

#### 1.4.3.1 Yamamoto Method

As seen in Figure 1.1 [*on page* 2], Yamamoto [1] assumes that the geometries of the wheels and the body are uniform. He also assumes that the masses of these geometries are uniform. He therefore defines length from the body center of mass to the body axis of rotation $l_{b.c2a}$, as exhibited in Equation (1.38)

$$l_{b.c2a} = \frac{l_{b.h}}{2} \tag{1.38}$$

#### 1.4.3.2 Vaccaro Method

Since the geometries of the actual hardware are assumed to significantly deviate from the assumption of uniform mass distribution, an alternative method is instead used to calculate length from the body center of mass to the body axis of rotation $l_{b.c2a}$, as exhibited in Equation (1.38)

If the hardware is mounted at both wheel axels *along the axis which is shared by both wheel axels*, and if the hardware is given a degree of freedom to rotate about the wheel axel axis, *without rotating the actual wheel axels*, then the hardware may be lifted slightly and then released to swing freely like a pendulum along that axis.

Allowing the hardware to freely swing like a pendulum along the wheel axel axis significantly simplifies the dynamic equations of motion of the hardware. Furthermore, if friction at the newly added mount coupling points is negligible, then there will not be a need to model and implement the friction into the dynamics equations.

If the hardware is freely swung like a pendulum along the wheel axel axis as described above, then the relations exhibited in Equations (1.39) - (1.40) become true.

$$\theta = \phi_x \tag{1.39}$$

$$\mathbf{u} = \mathbf{0} \tag{1.40}$$

The effects of these changes are exhibited in Equation (1.42) [*on page* 14]. This results in two relations, which are exhibited in Equation (1.41).

$$\begin{aligned} \ddot{\phi}_x \quad + \quad & 0 \quad = \quad 0 \\ \ddot{\phi}_x \quad + \quad & \underbrace{\frac{k_{1.5}}{k_{1.2} + k_{1.3}}}_{k_\omega} \cdot \phi_x \quad = \quad 0 \qquad \Leftarrow \end{aligned} \tag{1.41}$$

Of the two resulting relations in Equation (1.41), the former cannot be true while the hardware is in motion; thus, the latter is selected, as depicted on the right with a left-facing arrow.

$$
\begin{aligned}
&\mathbf{K}_{1.\ddot{x}} \cdot \begin{bmatrix} \ddot{\theta} \\ \ddot{\phi}_x \end{bmatrix} + \mathbf{K}_{1.\dot{x}} \cdot \begin{bmatrix} \dot{\theta} \\ \dot{\phi}_x \end{bmatrix} + \mathbf{K}_{1.x} \cdot \begin{bmatrix} \theta \\ \phi_x \end{bmatrix} = \mathbf{K}_{1.v} \cdot \begin{bmatrix} v_{mtr.l} \\ v_{mtr.r} \end{bmatrix} \\
&\mathbf{K}_{1.\ddot{x}} \cdot \begin{bmatrix} \ddot{\phi}_x \\ \ddot{\phi}_x \end{bmatrix} + \mathbf{K}_{1.\dot{x}} \cdot \begin{bmatrix} \dot{\phi}_x \\ \dot{\phi}_x \end{bmatrix} + \mathbf{K}_{1.x} \cdot \begin{bmatrix} \phi_x \\ \phi_x \end{bmatrix} = \underbrace{\mathbf{K}_{1.v} \cdot \begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\oslash} \\
&\begin{bmatrix} k_{1.1} & k_{1.2} \\ k_{1.2} & k_{1.3} \end{bmatrix} \cdot \begin{bmatrix} \ddot{\phi}_x \\ \ddot{\phi}_x \end{bmatrix} + \underbrace{2 \cdot k_{1.4} \cdot \begin{bmatrix} +1 & -1 \\ -1 & +1 \end{bmatrix} \cdot \begin{bmatrix} \dot{\phi}_x \\ \dot{\phi}_x \end{bmatrix}}_{\oslash} + \begin{bmatrix} 0 & 0 \\ 0 & k_{1.5} \end{bmatrix} \cdot \begin{bmatrix} \phi_x \\ \phi_x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
&\begin{bmatrix} \left(k_{1.1} + k_{1.2}\right) \cdot \ddot{\phi}_x \\ \left(k_{1.2} + k_{1.3}\right) \cdot \ddot{\phi}_x \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ k_{1.5} \cdot \phi_x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned} \tag{1.42}
$$

The coefficient term, abbreviated as $k_w$, is expanded in Equation (1.43). It may be expanded further with the use of Equation (1.36), as exhibited in Equation (1.44).

$$k_\omega = \frac{k_{1.5}}{k_{1.2} + k_{1.3}} = \frac{-m_b \cdot a_g \cdot l_{b.c2a}}{\left(m_b \cdot r_w \cdot l_{b.c2a}\right) + \left(m_b \cdot l_{b.c2a}^2 + J_{b.\phi_x}\right)} \tag{1.43}$$

$$k_\omega = \frac{-m_b \cdot l_{b.c2a} \cdot a_g}{m_b \cdot l_{b.c2a} \cdot r_w + m_b \cdot l_{b.c2a}^2 + \left(m_b \cdot l_{b.c2a}^2 \cdot \frac{1}{3}\right)} = \frac{-a_g}{r_w + l_{b.c2a} \cdot \left(1 + \frac{1}{3}\right)} \tag{1.44}$$

### Harmonic Oscillator

Notably, the selected relation in Equation (1.41) form-matches the equation for a harmonic oscillator [5, p. 119 - 120, 122 - 123], as is exhibited in Equation (1.45).

$$\begin{aligned} \ddot{y} + \omega^2 \cdot y &= \omega^2 \cdot u \\ \ddot{\phi}_x + k_\omega \cdot \phi_x &= k_\omega \cdot 0 \end{aligned} \tag{1.45}$$

This allows for the relation of the abbreviated term representing the system dynamics, $k_w$, to the natural angular frequency of the hardware [*a pendulum*] $\omega_p$, as is exhibited in Equation (1.46).

$$\omega_p^2 = k_\omega = \frac{-a_g}{r_w + l_{b.c2a} \cdot \frac{4}{3}} \tag{1.46}$$

This proves significant since $\omega_p$ represents the angular frequency of the pendulum, which is a measurable value, and since $k_\omega$ includes the desired unknown term $l_{b.c2a}$. [*All other terms are known*]. The relation may rewritten to solve for length from the body center of mass to the body axis of rotation $l_{b.c2a}$, as is exhibited as Equation (1.47).

$$l_{b.c2a} = -\frac{3}{4} \cdot \left( \frac{a_g}{\omega_p^2} + r_w \right) = -\frac{3}{4} \cdot \left( \frac{a_g}{\left(2 \cdot \pi \cdot f_p\right)^2} + r_w \right) \tag{1.47}$$

### 1.4.4 Motor: Coefficient of Friction: Wheel to DC Motor $k_{mtr}$

## 1.5 References


[1] Y. Yamamoto. (May 1, 2009). Nxtway-gs (self-balancing two-wheeled robot) controller design, [Online]. Available: `https://www.mathworks.com/matlabcentral/fileexchange/19147-nxtway-gs--self-balancing-two-wheeled-robot--controller-design` (visited on 07/03/2017).

[2] M. D. Peltier, "Trajectory control of a two-wheeled robot," Master's thesis, University of Rhode Island (URI): Department of Electrical Engineering, Jan. 2012.

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