Convolution

Let me show a general context.

The convolution b/w two signals $x[n]$ and $h[n]$ is

$$y[n] = x[n] * h[n]$$

$$= \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

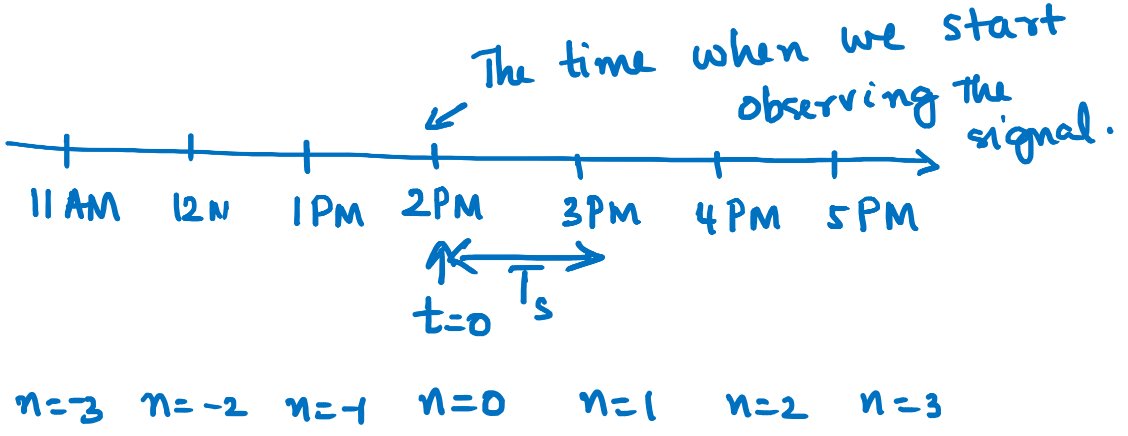
$$= \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

$$= h[n] * x[n].$$

We can think of the discrete-time signal $x[n]$ as one obtained by sampling a continuous time signal $x(t)$ at $t = nT_s$.

ie $$x[n] = x(t)\Big|_{t=nT_s}$$

where T_s is the sampling interval.



Now, Let us take a specific example.

$$x[n] = \{\underset{\substack{\uparrow \\ n=0}}{1}, 2, 3, 4\}$$

$$h[n] = \{1, \underset{\substack{\uparrow \\ n=0}}{2}, 1\}$$

There can as well be another signal

$$h_1[n] = \{\underset{\substack{\uparrow \\ n=0}}{1}, 2, 1\}.$$

But in our example, we will take $h[n]$.

$$\therefore \quad y[n] = x[n] * h[n]$$

$$= \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

Since $x[k] = 0$ when $k < 0$ and $k > 3$

we have

$$y[n] = \sum_{k=0}^{3} x[k]\, h[n-k]$$

Now, $y[0] = \sum_{k=0}^{3} x[k]\, h[0-k]$

$$= x[0]\, h[0] + x[1]\, h[-1] + x[2]\, \underbrace{h[-2]}_{0}$$

$$+ x[3]\, \underbrace{h[-3]}_{0}$$

$$= x[0]\, h[0] + x[1]\, h[-1]$$

Graphically,

$$y[0] = x[0]h[0] + x[1]h[-1] + x[2]\underbrace{h[-2]}_{0} + x[3]\underbrace{h[-3]}_{0}$$

$$\begin{array}{cccc} & x[0] & x[1] & x[2] & x[3] \\ & \uparrow & & & \\ h[1] & h[0] & h[-1] & & \end{array}$$

Flip the $h[n]$ and then multilpy.

Now $y[-1]$

$$y[-1] = \sum_{k=0}^{3} x[k]\, h[-1-k]$$

i.e. $y[-1] = x[0]h[-1] + x[1]\underbrace{h[-2]}_{0} + x[2]\underbrace{h[-3]}_{0} + x[3]\underbrace{h[-4]}_{0}$

$$\begin{array}{cccccc} & & x[0] & x[1] & x[2] & x[3] \\ h[1] & h[0] & h[-1] & & & \end{array}$$

Flip the signal and move it to left by 1 unit to obtain $y[-1]$.

$y[1]$

$$y[1] = \sum_{k=0}^{3} x[k]\, h[1-k]$$

$$= x[0]h[1] + x[1]h[0] + x[2]h[-1] + x[3]\underbrace{h[-2]}_{0}$$

$$\begin{array}{cccc} x[0] & x[1] & x[2] & x[3] \\ h[1] & h[0] & h[-1] & \end{array}$$

Flip and move it to right by 1 unit

Summarizing,

① $\underline{y[-1]}$

$$\begin{array}{cccccc} & & x[0] & x[1] & x[2] & x[3] \\ h[1] & h[0] & h[-1] & & & \end{array}$$

② $\underline{y[0]}$

$$\begin{array}{ccccc} & x[0] & x[1] & x[2] & x[3] \\ h[1] & h[0] & h[-1] & & \end{array}$$

③ $\underline{y[1]}$

$$\begin{array}{cccc} x[0] & x[1] & x[2] & x[3] \\ h[1] & h[0] & h[-1] & \end{array}$$

④ $\underline{y[2]}$

$$\begin{array}{cccc} x[0] & x[1] & x[2] & x[3] \\ & h[1] & h[0] & h[-1] \end{array}$$

⑤ $\underline{y[3]}$

$$\begin{array}{ccccc} x[0] & x[1] & x[2] & x[3] & \\ & & h[1] & h[0] & h[-1] \end{array}$$

⑥ $\underline{y[4]}$

$$\begin{array}{cccccc} x[0] & x[1] & x[2] & x[3] & & \\ & & & h[1] & h[0] & h[-1] \end{array}$$

→ observe that $h[0]$ aligns with the 'n' where we want to find the o/p.

For eg: For $y[2] \rightarrow h[0]$ is with $x[2]$

$y[-1] \rightarrow h[0]$ is with $x[-1]$

Observations

1) Total length of $y[n]$ = Length of $x[n]$ + Length of $h[n]$ − 1

In our example,

$$\text{Length of } y[n] = 4 + 3 - 1 = \underline{\underline{6}}$$

2) Starting point (value of n) of $y[n]$ = Starting 'n' of $x[n]$ + Starting 'n' of $h[n]$

In our example,

$$y[n] \text{ starts at } n = 0 + -1 = \underline{\underline{-1}}$$

3) Ending point of $y[n]$ = Ending point of $x[n]$ + Ending point of $h[n]$

Here,

$$y[n] \text{ ends at } n = 3 + 1 = \underline{\underline{4}}$$

$$\therefore\ y[n] = 0 \text{ when } \underline{\underline{n < -1 \text{ and } n > 4}}$$