Introduction

To realistically describe rotating drum flows requires mechanistically accurate models of the bed geometry (free surface and basal interface) and velocity profile. Unfortunately, non-equilibrium state solutions are complicated and remain elusive. It is also anticipated that even if such models were available, solving them will prove difficult. For this reason we necessarily employ depth-integrated formulations and invoke the *shallow flow approximation* to tame the resulting (ordinary differential) equations. To this end, we seek solutions to the free surface $\tilde{z}(x)$, basal interface $\underset{\sim}{z}(x)$, and depth-averaged velocity $\overline{u}(x)$ for arbitrary transient and non-uniform flow; or equivalently, we seek solutions to the free surface $\tilde{z}(x)$, flowing layer depth $h(x)$, and depth averaged, streamwise velocity $\overline{u}(x)$ for arbitrary transient and non-uniform flow. To still recover the full velocity profile of the flowing layer, we will assume that the velocity profile remains similar to the equilibrium profile, even for non-equilibrium flows.

Setting the scene

Consider a batch flow of granular material (diameter D and density ρ_s), in a half-filled cylindrical drum (radius R and width L), rotating at an angular speed of ω radians per second about the horizontal drum axis. For a limited range of ω, the constant rotation produces two distinct granular regions as shown in figure (1):

(a) A deep bed undergoing rigid body rotation and flowing over it,

(b) a shallow avalanching layer undergoing shear flow.

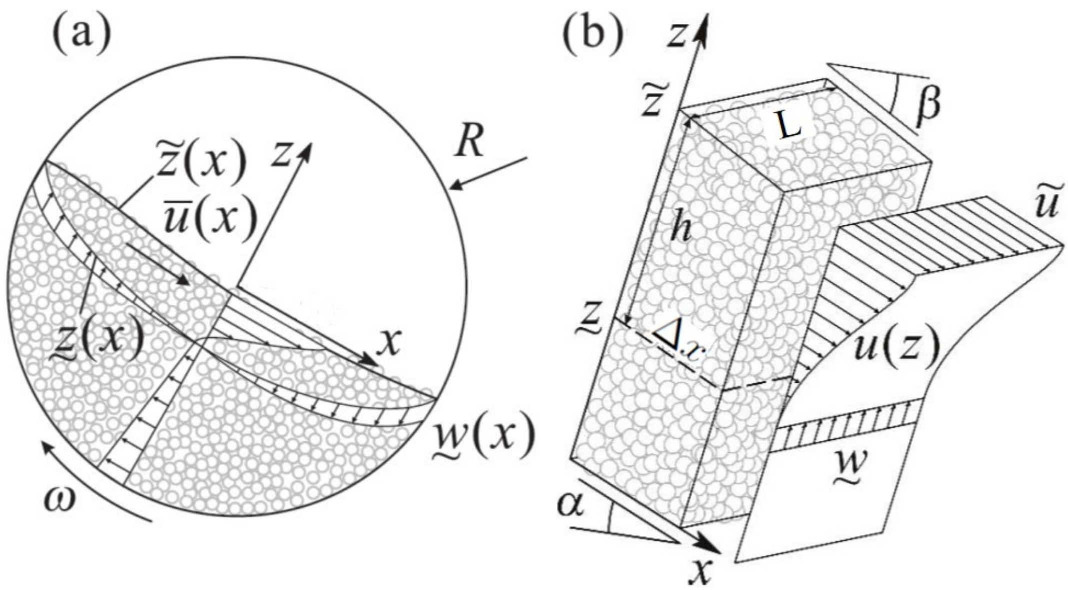

Figure 1: Definition sketch for rotating drum flow: (a) Front overview; (b) local cutaway spanning a short length Δx along the free surface.

The shallow avalanching or *flowing layer* is bounded on top by a free surface $\tilde{z}(x)$ and basal interface[1] $\underset{\sim}{z}(x)$ at the bottom, where tildes above and below variables denote values sampled along the free surface and basal interface respectively. The flow is such that no flux occurs across the free surface[2], however, the erodible basal interface is a yield locus across which grains are continuously entrained and detrained at rate $\underset{\sim}{w}(x) = -\omega x$. The basal layer is reposed at an average angle α to the horizontal while the free surface is oriented at angle $\beta(x)$. Granular motions are assumed to be nearly parallel to the free surface[3] which automatically constrains our study to the case $\alpha \approx \beta$. The coordinate x is parallel to the free surface with origin $x = 0$ at the drum centre and z denotes the perpendicular direction along which the local flowing layer depth at coordinate x is given by $h(x) = \tilde{z}(x) - \underset{\sim}{z}(x)$. The corresponding orthogonal velocity components in the x and z directions are u and w respectively with v denoting the component of velocity in the y direction. Finally, $g_\perp = g\cos(\beta)$ and $g_\parallel = g\sin(\beta)$ are the perpendicular (in the z direction) and parallel (in the x direction) components of gravitational acceleration.

Model assumptions & boundary conditions

The following additional assumptions and boundary conditions apply:

(i) The flowing layer is governed by the *shallow flow approximation*, implying that:

  (a) $h(x) \ll \ell$, where $\ell = R$ for a half filled drum.

  (b) Spatial and temporal acceleration involving w can be neglected, i.e.,

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial z} = \frac{\partial w}{\partial t} = 0.$$

(ii) Under the shallow flow approximation, the total granular pressure σ in the flowing layer is assumed *isotropic* and *lithostatic*[4].

(iii) Flow is incompressible.

(iv) Flow is uniform along the drum length L.

(v) Using the notation τ for the internal shear stress and σ for confining pressure, we define τ within the granular bed by a linearised viscoplastic dense granular flow rheology

$$\tau_{xz} = \mu_0\sigma + \chi D\dot{\gamma}_{xz}\sqrt{\rho\sigma} \tag{1}$$

expressed as the sum of two components: A plastic yield stress $\tau_0 = \mu_0\sigma$ along the basal interface with $\mu_0 = \tan(\alpha)$ (a bulk material property) and a viscous stress that varies linearly with shear rate $\dot{\gamma}$ while exhibiting an effective viscosity $\chi D\sqrt{\rho\sigma}$ that depends on an experimentally determined dimensionless rheological coefficient χ.

[1] Also called the equilibrium surface in minerals engineering literature.
[2] No flux across free surface implies that it is part of a streamline.
[3] Notwithstanding, we will test the validity of the model beyond this limitation.
[4] Isotropy is implied for confining pressure while the lithostatic assumption only applies in the shallow flow approximation

NB: The index notation for, say, τ_{xz} reads: The force per unit area in the x-direction on a surface area whose normal points in the z-direction. This sometimes gets reversed depending on the field of study. We will drop the index notation for convenience and only use it when necessary.

(vi) The shear stress between the side walls and the granular bed τ_W are governed by a Coulomb-like frictional force that depends on the normal stress[5] and a constant friction coefficient μ_W which reflects the bulk frictional properties between the sidewalls and granular material.

(vii) Along the free surface

$$\widetilde{w} = \frac{\partial \widetilde{z}}{\partial t} + \widetilde{u}\frac{\partial \widetilde{z}}{\partial x}, \tag{2}$$

$$\widetilde{\sigma} = 0, \tag{3}$$

$$\widetilde{\tau} = 0. \tag{4}$$

(viii) Along the basal interface

$$\underset{\sim}{u} = 0, \tag{5}$$

$$\underset{\sim}{w} = -\omega x, \tag{6}$$

$$\underset{\sim}{\tau} \equiv \tau_0 = \mu_0 \underset{\sim}{\sigma}. \tag{7}$$

Question 1: Kinematic boundary condition (10 marks)

Let $\widetilde{z}(x,t)$ denote positions along the free surface. By considering small time-dependent motions along the free surface that result in small changes $\delta\widetilde{z}$, prove the kinematic boundary condition given by equation (2).

NB: This proof should help you better understand the coordinate dependencies of $\widetilde{z}$.

[10 marks]

Question 2: Mass balance (50 marks)

(2.1) Use the *shell balance method* and the incompressibility condition to show that

[10 marks]

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0. \tag{8}$$

(2.2) Applying the basal boundary conditions to equation (8), show that

$$\frac{\partial}{\partial x}(h\overline{u}) + \frac{\partial \widetilde{z}}{\partial t} = -\omega x, \tag{9}$$

[5] The isotropic pressure assumption is useful here!

where $\overline{u}(x) = \frac{1}{h(x)} \int\limits_{\underset{\sim}{z}}^{\widetilde{z}} u(z)dz$ is the depth-averaged velocity along depth $h(x)$.

Hint: *Use the Leibniz integration rule.*

[10 marks]

(2.3) By *suitable choice* of a finite control volume, perform an integral mass balance to show that the maximum mass efflux at steady-state in the flowing layer yields

[10 marks]

$$h\overline{u} = \frac{\omega R^2}{2}; \tag{10}$$

and hence show that $[h(x) \ll R] \Rightarrow \left[\underset{\sim}{w}(x) \ll \overline{u}(x)\right]$.

NB: Equation (10) can also be derived directly from equation (9) with permeability boundary condition:

$$\overline{u}(x)\Big|_{x=\pm R} = 0. \tag{11}$$

(2.4) The incompressible flow assumption implies that the granular bed neither compresses nor dilates. Accordingly, the drum filling fraction $\left(\frac{\text{volume of grains + interstitial volume}}{\text{internal volume of drum}}\right)$ remains constant[6]. Show that this leads to the integral boundary condition

$$\int\limits_{-R}^{R} \widetilde{z}(x)dx = 0 \tag{12}$$

when the drum is half-filled. [10 marks]

(2.5) Assuming an initial (and incorrect) value of $\widetilde{z}(x{=}{-R}) = 0$ to numerically solve the free surface profile $\widetilde{z}_n(x)$, use the integral boundary condition to show that the correct initial value $\widetilde{z}(x{=}{-R}) = \widetilde{z}_c$ is given by

$$\widetilde{z}_c = \frac{1}{2R} \int\limits_{-R}^{R} \widetilde{z}_n(x)dx, \tag{13}$$

where $\int\limits_{-R}^{R} \widetilde{z}_n(x)dx$ is typically evaluated using the trapezoidal integration method, and the final free surface profile is $\widetilde{z}_n(x) - \widetilde{z}_c$ [10 marks]

NB: Physically, the integral boundary condition of equation (12) expresses mass conservation in the sense that the drum rotation leads to material being added to one part of the drum (like the upper half of the drum) while simultaneously being removed from elsewhere (like the lower half of the drum). The incompressibility assumption can also be exploited to study drum fill fractions different from half. In that case the integration limits to equation (12) are simply the intersection of the chord subtending the known drum filling fraction ε and the circle of radius R.

[6] This is true even if the free surface profile is asymmetric.

Useful observations from question 2

(a) Noting that $Q(x) \triangleq \rho h(x)\overline{u}(x)$ is the mass efflux per unit length of drum, we define the granular discharge[7] $q(x) \triangleq \frac{Q(x)}{\rho}$ as

$$q(x) = h(x)\overline{u}. \tag{14}$$

(b) The simplest approach to determining the maximum granular discharge $q(x = R)$ is to simply integrate equation (9) from 0 to R at steady state flow conditions; however, the current approach makes explicit two important points:

(i) That we are actually evaluating the maximum mass efflux.

(ii) That the free surface profile $\widetilde{z}(x)$ is a streamline and hence no flux passes through it.

(c) Noting that $\ell = R$ for a half full drum, the condition $[h(x) \ll \ell] \Rightarrow \left[\underset{\sim}{w}(x) \ll \overline{u}(x)\right]$ is an additional confirmation that we can apply the *shallow flow approximation*.

Question 3: Differential equations (40 marks)

The final steady-state, depth-integrated equations for the momentum

$$\frac{d}{dx}\left(\gamma h\overline{u}^2\right) = -g_\perp h\frac{d\widetilde{z}}{dx} - \frac{\mu_W}{L}g_\perp h^2 \tag{15}$$

and kinetic energy

$$\frac{d}{dx}\left(\kappa h\overline{u}^3\right) = -g_\perp h\overline{u}\frac{d\widetilde{z}}{dx} - \frac{5}{9}\frac{\mu_W}{L}g_\perp h^2\overline{u} - \frac{35}{9}\chi D g_\perp^{\frac{1}{2}}\frac{\overline{u}^2}{h^{\frac{1}{2}}}, \tag{16}$$

can be derived by careful application of the Leibniz integration rule[8], where $\gamma = \frac{77}{48}$ and $\kappa\frac{342853}{233376}$.
This system of equations, including the mass balance from equation (9), can be non-dimensionlised by introducing $\hat{x} = x/R$, $\hat{h} = h/h_c$, $\hat{\overline{u}} = \overline{u}/\overline{u}_c$ as well as expressions for the slope of the free surface $\frac{d\widetilde{z}}{dx} = -S$ and $\hat{S} = S/S_c$. In non-dimensionalised form, the system of equations reduces to:

$$\frac{d}{d\hat{x}}\left(\hat{h}\hat{\overline{u}}\right) = -\hat{\omega}\hat{x}, \tag{17}$$

$$\frac{d}{d\hat{x}}\left(\gamma\hat{h}\hat{\overline{u}}^2\right) = \hat{h}\hat{S} - \hat{h}^2, \tag{18}$$

$$\frac{d}{d\hat{x}}\left(\kappa\hat{h}\hat{\overline{u}}^3\right) = \hat{h}\hat{\overline{u}}\hat{S} - \frac{5}{9}\hat{h}^2\hat{\overline{u}} - \frac{35}{9}\frac{\hat{\overline{u}}^2}{\hat{h}^{\frac{1}{2}}}. \tag{19}$$

On normalising in this way, the characteristic (normalised) response variables

$$h_c = (\chi D)^{\frac{1}{2}} R^{\frac{1}{4}}\left(\frac{L}{\mu_W}\right)^{\frac{1}{4}}, \tag{20}$$

[7] $q(x)$ is also dimensionally equivalent to the volumetric flow rate (or mass efflux) per unit length of drum
[8] try it for yourself!

$$t_c = \frac{h_c^{3/2}}{gcos(\alpha)\chi D}, \tag{21}$$

$$\bar{u}_c = g_\perp^{\frac{1}{2}} (\chi D)^{\frac{1}{4}} R^{\frac{5}{8}} \left(\frac{L}{\mu_W}\right)^{-\frac{3}{8}}, \tag{22}$$

$$S_c = (\chi D)^{\frac{1}{2}} R^{\frac{1}{4}} \left(\frac{L}{\mu_W}\right)^{-\frac{3}{4}}, \tag{23}$$

depend on the single dimensionless number $\hat{\omega}$, termed the Entrainment number.

$$\hat{\omega} = \frac{\omega R^{\frac{9}{8}}}{g_\perp^{\frac{1}{2}} (\chi D)^{\frac{3}{4}}} \left(\frac{L}{\mu_W}\right)^{\frac{1}{8}}. \tag{24}$$

Introducing the mass efflux per unit length of drum $q(x) = h\bar{u}$ and its non-dimensional version $\hat{q}(\hat{x}) = \hat{h}\hat{\bar{u}}$, the mass balance equation becomes

$$\frac{d\hat{q}}{d\hat{x}} = -\hat{\omega}\hat{x}, \tag{25}$$

with solution $\hat{q}(\hat{x}) = \frac{1}{2}\hat{\omega}\left(1 - \hat{x}^2\right)$.

Finally, the non-dimensionalised ordinary differential equations (ODEs) governing the flowing layer depth $\left(\frac{d\hat{h}}{d\hat{x}}\right)$ and the free surface $\left(\frac{d\hat{\tilde{z}}}{d\hat{x}}\right)$ can be derived.

$$\frac{d\hat{h}}{d\hat{x}} = \frac{6\left(72\kappa - 77\right)\hat{h}\hat{q}\left(\frac{d\hat{q}}{d\hat{x}}\right) + 560\sqrt{\hat{h}}\hat{q} - 64\hat{h}^4}{3\hat{q}^2\left(96\kappa - 77\right)}, \tag{26}$$

$$\frac{d\hat{\tilde{z}}}{d\hat{x}} = \frac{2695\sqrt{\hat{h}}\hat{q} - \hat{h}^4\left(864\kappa - 385\right) - 693\kappa\hat{q}\hat{h}\left(\frac{d\hat{q}}{d\hat{x}}\right)}{9\hat{h}^3\left(96\kappa - 77\right)}. \tag{27}$$

Numerically solve and plot the ordinary differential equations describing the free and basal interface subject to the integral boundary condition (expressing mass conservation)

$$\int_{-R}^{R} \tilde{z}(x)dx = 0, \tag{28}$$

and the initial (upstream) boundary conditions:

$$\begin{aligned}
\hat{h}\left(\hat{x} = -1\right) &= 1 \times 10^{-6} && (29)\\
\hat{q}\left(\hat{x} = -1\right) &= 1 \times 10^{-6} && (30)\\
\hat{\tilde{z}}\left(\hat{x} = -1\right) &= 0. && (31)
\end{aligned}$$

Your plot should include $\tilde{z}(x)$ and $\underset{\sim}{z}(x)$.
To make your plot quantitative, use the values given in table (1).

Question 4: Differential equation (20 marks)

Repeat your numerical solution in question (3) for a fill fraction of $\zeta = 0.3$.

Parameter	Description	SI Units
$g = 9.81$	Local gravitational acceleration	$\frac{\mathrm{m}}{\mathrm{s}^2}$
$R = 0.2$	Drum radius	m
$L = 0.04$	Drum length	m
$\omega = 10$	Angular drum speed	RPM
$d = 2 \times 10^{-3}$	Particle diameter	m
$\alpha = 18$	Repose angle at basal interface	deg
$\zeta = 0.5$	Fill fraction	$[-]$
$\mu_w = 0.35$	Mohr-Coulomb wall friction coefficient	$[-]$
$\mu_0 = 0.33$	Basal friction coefficient	$[-]$
$\chi = 0.52$	Dimensionless rheological coefficient	$[-]$

Table 1: Input parameters for the numerical solution of the ODEs