Application of Derivative

MAX & MIN

Def. Let c be a number in the domain D of a function f. Then $f(c)$ is called absolute maximum value of f on D if $f(c) \geq f(x)$ for all x in D and absolute minimum value if $f(c) \leq f(x)$.

Def. The number $f(c)$ is a local maximum value of f if $f(c) \geq f(x)$ when x is near c and local minimum if $f(c) \leq f(x)$.

See Example 1

EXTREME VALUE THEOREM

If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ for some c, d in $[a, b]$.

Q. Where may the maximum or minimum occur?

FERMET THEOREM

If f has a local maxima or local minima at c and $f'(c)$ exists, then $f'(c) = 0$. 0 is a local minima of f, $f'(0)$ doesn't exist.

Def. A critical number of a function f is a number c such that

1. $f(c)$ is defined
2. $f'(c) = 0$ or $f'(c)$ doesn't exist

See Example 2

If f has a local maximum or local minimum at c, then c is a critical number.

Proof. If $f'(c)$ does not exist, c is a critical number. If $f'(c)$ exists, by Fermet's Theorem, $f'(c) = 0$. c is a critical number of f.

Q. If f is a critical number, is $f(c)$ a local maxima, or a local minima?

To find the absolute maximum and absolute minimum, of a continuous function f on $[a, b]$:

1. Find the critical numbers and evaluate them
2. Evaluate $f(a)$ and $f(b)$
3. Absolute maximum : largest value found in (1) and (2)

   Absolute minimum : smallest value found in (1) and (2)

See Example 3

EXAMPLES

1. $f(x) = 3x^4 - 16x^3 + 18x^2 \qquad [-1, 4]$. What are the absolute maximum, absolute minimum, local maximum, and local minimum?

   Sol. Critical points happen when $f'(x) = 0$.

   $f'(x) = 12x^3 - 48x^2 + 36x = 12x(x^2 - 4x + 3) = 0 \Rightarrow x = 0, x = 1, x = 3$

   We find all the values of all the x that we got from above (because they are in the interval) and the x from the interval:

$$\underbrace{f(-1)=37}_{\text{abs. max}}, \quad \underbrace{f(4)=32}_{\text{local max}}, \quad \underbrace{f(1)=5}_{\text{local min}}, \quad \underbrace{f(0)=0}_{\text{local min}}, \quad \underbrace{f(3)=-27}_{\text{abs. min}}$$

2. Find the critical number of

   a. $f(x) = x^{\frac{3}{5}}(4-x)$

   Sol. $f'(x) = \dfrac{3}{5}x^{-\frac{2}{5}}(4-x) - x^{\frac{3}{5}} = \dfrac{4}{5}x^{-\frac{2}{5}}(3-2x)$

   It is easy to see that $f'(0)$ doesn't exist and $f(0)=0$. Hence, 0 is a critical number of f. To solve $f'(x)=0$, we obtain $x=\dfrac{3}{2}$, $f\left(\dfrac{3}{2}\right)$ is defined. $x=\dfrac{3}{2}$ is a critical number of f.

   b. $g(x) = \dfrac{3x^{\frac{2}{3}}}{x-1}$

   Sol. $g'(x) = \dfrac{-x-2}{x^{\frac{1}{3}}(x-1)^2}$

   So we have $g'(0)$ and $g'(1)$ are not defined. Since $g(0)=0$ and $g(1)$ is not defined, $x=0$ is a critical number and $x=1$ is not a critical number. Next to solve $g'(x)=0$, we have $x=-2$. Since $g(-2)$ is defined, $x=-2$ is a critical number.

3. Find the absolute maximum & absolute minimum of $f(x) = x^3 - 3x^2 + 1 \qquad [-\frac{1}{2}, 4]$

   Sol. $f'(x) = 3x^2 - 6x$ is defines for $[-\dfrac{1}{2}, 4]$. To solve $f'(x)=0$, we have $x=0$ or 2. $f(0)=1, f(2)=-3, f(-\dfrac{1}{2})=\dfrac{1}{8}, f(4)=17$. The absolute maximum of f is 17 and absolute minimum of f is -3.