LIMIT

·L·I·M·I·T·

$x(t)$: position function

t_0 : given

The average velocity from t_0 to $t_0+\Delta t$ is $\frac{x(t_0+\Delta t)-x(t_0)}{\Delta t}$ if the $\underbrace{\text{average velocity}}_{\text{function}}$ close to a $\underbrace{\text{value}}_{\text{limit}}$ when $\underbrace{\Delta t}_{\text{variable}}$ close to 0, we call the value is the instantaneous velocity at time t_0.

Def. $f(x)$ is defined for x near a. If the function $f(x)$ close to L when x close to a, then we write $\lim_{x\to a} f(x)=L$ or $f(x)\to L$ as $x \xrightarrow{\text{close to}} a$.

$$\boxed{\lim_{x\to a} f(x)=L}$$

- guess L
- prove $\lim_{x\to a} f(x)=L$
- $L=f(a)$?

1. $\lim_{x\to a} f(x)=L \overset{?}{=} L=f(a)$ NO

   Proof :

   $f(x)=\frac{x-1}{x^2-1}$

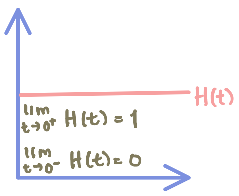
   $\lim_{x\to 1}\frac{x-1}{x^2-1}=\frac{1}{2}\neq f(1)$

   $f(1)=\frac{1-1}{1^2-1}$ is NOT defined

2. $F(a)=L \overset{?}{=} \lim_{x\to a} f(x)=L$ NO

   Proof :

   $H(t)=\begin{cases}0, & t<0\\ 1, & t\geq 0\end{cases}$



   $H(0)=1$

   When t approach 0 from right / left, $H(t)$ approach 1 / 0.

   $\therefore \lim_{t\to 0} H(t)$ does NOT exist.

Def. //one-sided limit if $f(x)$ close to L when x close to a from right / left we write $\lim_{x\to a^+} f(x)=L$ / $\lim_{x\to a^-} f(x)=L$.

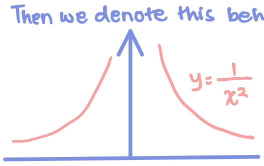
$\lim_{x\to a} f(x)=L$

$\Leftrightarrow \left.\begin{array}{l}\lim_{x\to a^+} f(x)=L\\ \lim_{x\to a^-} f(x)=L\end{array}\right\}$ if and only if

Ex. $\lim_{x\to 0}\frac{1}{x^2}$?

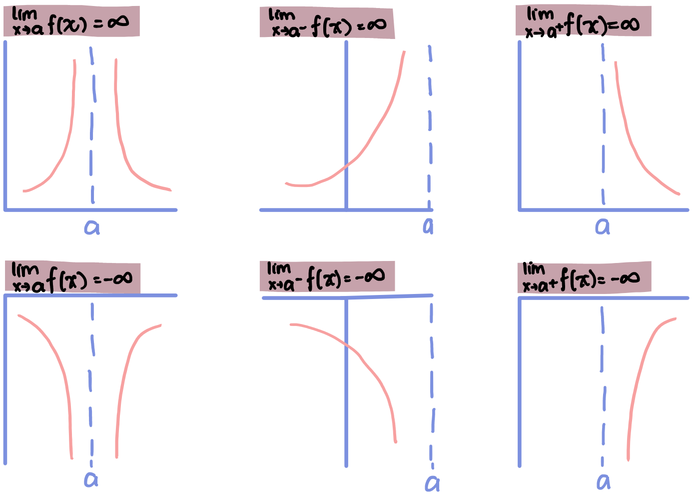
Sol. When x close to zero, we observe $\frac{1}{x^2}$ is increasing and we bounded.

Then we denote this behaviour $\lim_{x\to 0}\frac{1}{x^2}=\infty$.



$x=0$ is a vertical asymptote of $y=\frac{1}{x^2}$

Def. $\lim_{x\to 0} f(x)=\infty$ / $-\infty$ means $f(x)$ can be made arbitrarily large / negative large by taking x sufficiently close to a.



Def. The vertical line $x=a$ is called the asymptote of the curve $y=f(x)$ if one of the following holds: $\lim_{x\to a^+} f(x) = \pm\infty$ or $\lim_{x\to a^-} f(x) = \pm\infty$.

Ex. Find the vertical asymptotes of $f(x) = \tan x$

Sol. $\tan x = \frac{\sin x}{\cos x}$ we know $\cos x = 0$ for $x = \frac{\pi}{2} + n\pi$, $\cos x > 0$. Thus,

$$\lim_{x\to(\frac{\pi}{2})^-} \tan x = \lim_{x\to(\frac{\pi}{2})^-} \frac{\sin x}{\underbrace{\cos x}_{>0 \to 0}} = +\infty$$

$x = \frac{\pi}{2}$ is a vertical asymptote of $f(x) = \tan x$.

n is even: The proof is the same with $x = \frac{\pi}{2}$

n is odd: When $x < \frac{\pi}{2} + n\pi$ and close to $\frac{\pi}{2} + n\pi$, $\cos x < 0$, $\sin x < 0$.

$$\lim_{x\to(\frac{\pi}{2}+n\pi)^-} \tan x = \lim_{x\to(\frac{\pi}{2}+n\pi)^-} \frac{\sin x}{\cos x} = +\infty$$

$x = \frac{\pi}{2} + n\pi$ where n is an integer are all the vertical asymptote of $f(x) = \tan x$.

COMPUTING USING LIMIT LAWS

Suppose $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist

1. $\lim_{x\to a} [f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$

2. $\lim_{x\to a} [f(x) \cdot g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$

3. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, if $\lim_{x\to a} g(x) \neq 0$ // c is a constant

4. $\lim_{x\to a} c f(x) = c \lim_{x\to a} f(x)$

5. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$, if n is even, $\lim_{x\to a} f(x) \geq 0$.

Prop. If f is a polynomial or a rational function, and a is in the domain of f, then $\lim_{x\to a} f(x) = f(a)$ is defined.

1. If $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ where n is a positive integer, we call it the polynomial.

2. If $r(x) = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomial, we call it the rational function.

Ex. (a) $\lim_{x\to 5}(2x^2 - 3x + 4) = 2\cdot 25 - 3\cdot 5 + 4 = 39$

(b) $\lim_{x\to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x} = \frac{-8 + 8 - 1}{5 + 6} = \frac{-1}{11}$

$P(x), q(x)$: polynomials

$\lim_{x\to a} \frac{P(x)}{q(x)} = \frac{P(a)}{q(a)}$, if $q(a) \neq 0$

If $q(a) = 0$,

1. $P(a) \neq 0$, $\lim_{x\to a} \frac{P(x)}{q(x)} = +\infty$ or $-\infty$, does not exist.

   Ex: $p = 1$, $q = x^2$; $p = 1$, $q = -x^2$; $p = 1$, $q = x$

2. $P(a) = 0$

Prop. If $f(x) = g(x)$ when $x \neq a$, $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$ or $\lim_{x\to a} \frac{P(x)}{q(x)} = \lim_{x\to a} \frac{P_1(x)}{q_1(x)}$ where P_1 and q_1 satisfy $P(x) = (x-a)P_1(x)$ and $q(x) = (x-a)q_1(x)$.

Ex. $\lim_{x\to 1} \frac{x^2 - 1}{x - 1}$

Sol. When $x = 1$, $x^2 - 1 = 0$, and $x - 1 = 0$

$x^2 - 1 = (x-1)(x+1)$

$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{(x-1)(x+1)}{x-1} = \lim_{x\to 1}(x+1) = 2$

Ex. $\lim_{t\to 0} \frac{\sqrt{t^2+9} - 3}{t^2}$ // $a^2 - b^2 = (a-b)(a+b)$

Sol. When $t = 0$, $\sqrt{t^2+9} - 3 = 0$ and $t^2 = 0$.

$\lim_{t\to 0} \frac{\sqrt{t^2+9} - 3}{t^2} = \lim_{t\to 0} \frac{\sqrt{t^2+9} - 3}{t^2} \times \frac{\sqrt{t^2+9} + 3}{\sqrt{t^2+9} + 3} = \lim_{t\to 0} \frac{\cancel{t^2} \cancel{+9-9}}{\cancel{t^2}(\sqrt{t^2+9} + 3)} = \frac{1}{6}$

Ex. $\lim_{x\to 0} \frac{|x|}{x}$ does not exist

Proof. $\lim_{x\to 0} \frac{|x|}{x}$ for $x > 0$, $|x| = x$ so $\lim_{x\to 0^+} \frac{|x|}{x} = 1$

$\lim_{x\to 0} \frac{|x|}{x}$ for $x < 0$, $|x| = -x$ so $\lim_{x\to 0^-} \frac{|x|}{x} = -1$

Since $\lim_{x\to 0^+} \frac{|x|}{x} = 1 \neq -1 = \lim_{x\to 0^-} \frac{|x|}{x}$, $\lim_{x\to 0} \frac{|x|}{x}$ does not exist.

// $[\![x]\!]$ or $[x]$ or $\lfloor x \rfloor$ greatest integer function ; the largest integer that is less than x.

// $\lceil x \rceil$ least integer function ; the smallest integer that is more than x.

// Ex. (1) $\lceil -2.5 \rceil = -2$

(2) $\lfloor \pi \rfloor = 3$

1. If $f(x) \le g(x)$ when x is near a and both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then $\lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$.

   Ex. Assume $f(x) \le 0$ when x is near a and $\lim_{x\to a} f(x)$ exists. By <1>, $\lim_{x\to a} f(x) \le 0$.

2. Squeeze Theorem

   If $f(x) \le \boxed{g(x)} \le h(x)$ when x is near a and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$

   find

then $\lim_{x\to a} g(x)=L$.

Ex. Show $\lim_{x\to 0} x^2\sin\frac{1}{x}=0$

Proof. First we know that $-1\le \sin\frac{1}{x}\le 1$. So we have $\overset{f(x)}{\boxed{-x^2}}\le x^2\sin\frac{1}{x}\le \overset{h(x)}{\boxed{x^2}}$.

Also, we have $\lim_{x\to 0}(-x^2)=0=\lim_{x\to 0}x^2$. By Squeeze Theorem,

$\lim_{x\to 0} x^2\sin\frac{1}{x}=0$.

PRECISE DEFINITION

If $f(x)$ is arbitrarily close to L by restricting x to be sufficiently close to a,

$\lim_{x\to a} f(x)=L$. For every $\varepsilon>0$, $\exists\,\delta>0$ such that if $0<|x-a|<\delta$ then $|f(x)-L|<\varepsilon$.

To show $\lim_{x\to a} f(x)=L$, given any $\varepsilon>0$, our goal is to find $\delta>0$ such that the statement if $0<|x-a|<\delta$ then $|f(x)-L|<\varepsilon$ holds.

Known: a, $f(x)$, L, ε

Unknown: δ

1. Observe the relation between $|f(x)-L|$ and $|x-a|$.

2. Use the relation found in <1> to guess δ.

3. Show the δ works.

Ex. Show $\lim_{x\to 3} x^2=9$

Proof. $|x^2-9|=|x-3||x+3|$

For $|x-3|<1$, we have $2<x<4$ and $5<|x+3|<7$ so $|x^2-9|<7|x-3|$. Since we want $|x^2-9|<\varepsilon$, we ask for $|x-3|<\frac{\varepsilon}{7}$. Now we take $\delta=\min\{1,\frac{\varepsilon}{7}\}$.

If $0<|x-3|<\delta$, then $|x^2-9|=|x-3||x+3|<7|x-3|$

since $|x-3|<\delta<1\to|x+3|<7$

$<7\delta\le 7\cdot\frac{\varepsilon}{7}=\varepsilon$

$\delta=\min\{1,\frac{\varepsilon}{7}\}$

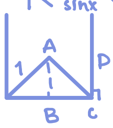
Therefore, $\lim_{x\to 3}x^2=9$.

$\lim_{x\to a} f(x)=\infty\,/\,-\infty$ means that for every $N>0\,/\,N<0$ there is a $\delta>0$ such that if $0<|x-a|<\delta$, then $f(x)>N\,/\,f(x)<N$.

Ex. Show $\lim_{x\to 0}\frac{\sin x}{x}=1$

Proof. For $x>0$, since $\overline{AB}<\overline{AC}<\overline{DC}$ we have $\sin x<x<\tan x$. It shows us

$1<\frac{x}{\sin x}<\frac{1}{\cos x}\longrightarrow 1>\frac{\sin x}{x}>\cos x$



Since $\lim_{x\to 0^+}1=1$, $\lim_{x\to 0^+}\cos x=1$ by Squeeze Theorem, we obtain $\lim_{x\to 0^+}\frac{\sin x}{x}=1$. It remains to show $\lim_{x\to 0^-}\frac{\sin x}{x}=1$. Set $y=-x$. As $x\to 0^-$, we have $y\to 0^+$.

$\lim_{x\to 0^-}\frac{\sin x}{x}=\lim_{y\to 0^+}\frac{\overset{\to -\sin y}{\sin(-y)}}{-y}=\lim_{y\to 0^+}\frac{\sin y}{y}=1$

From the results shown in the above, we have $\lim_{x\to 0}\frac{\sin x}{x}=1$.

Corollary: $\lim_{x\to 0}\frac{\sin ax}{bx}=\frac{a}{b}$

Proof. $\lim_{x\to 0}\frac{\sin ax}{bx}=\lim_{x\to 0}\frac{a}{b}\cdot\frac{\sin ax}{ax}=\frac{a}{b}\cdot\lim_{x\to 0}\frac{\sin ax}{ax}=\frac{a}{b}$