Assignment 1

ECS 122B — Spring 2016

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Problem 1

Recurrence Relations (Quickselect)

Given $B(n) \leq an + B(\frac{3}{4}n)$ We must show that $B(n) = O(n)$

$B(n) \leq an + B(\frac{3}{4}n) \leq \frac{3}{4}n + \frac{3}{16}n + B(\frac{3}{4^3}n) \leq 3(\frac{1}{4}n + \frac{1}{4^2}n + ... + \frac{1}{4^k})$ where $k = \lceil log_2 n \rceil$

We get $B(n) \leq \frac{3}{4}n(n + \frac{1}{4}n + ... + \frac{1}{4^{k-1}}) \leq \frac{3}{4}n(1 - \frac{1}{4^{k-1}} \leq \frac{3}{4}n) = O(n)$

Therefore, $B(n) = O(n)$ ■

Problem 2

Generating Functions

Part a.

Let $f(x) = \frac{1}{(1-x)}$ using proof by induction that $f^{(n)}(0) = n!$

Taking derivative, $f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}$

Doing proof by induction, proof base case.

let $f^0(0) = \frac{0!}{(1-0)!} = \frac{1}{1} = 1$ Satisfies.

Inductive step assuming $f^{(n)}(0) = n!$

Want to show that $f^{(n+1)}(x) = \frac{(n+1)!}{(1-x)^{n+2}}$

$f^{(n+1)}(x) = \frac{(n+1)!}{(1-x)^{n+2}}$ let $x = 0$ then $f^{(n+1)}(0) = \frac{(n+1)!}{(1-0)^{n+2}} = \frac{(n+1)!}{1} = (n+1)!$ ∎

Part b.

Compute $[x^n]\frac{3}{1-2x}$ using table 3.1.

$3(\frac{1}{1-2x}) = 3\sum_{n\geq 0} 2^n z^n = \sum_{n\geq 0} 3(2^n)z^n$

Thus, the coefficient of x^n is $3(2^n)x^n$

Part c.

Let $A^{(k)}(x) = \sum_{n\geq k} \frac{n!}{(n-k)!} a_n x^{n-k}$ where k^{th} derivative of $A(x)$

Using proof by induction:

Base case: $k = 0$ then $A^0(x) = \sum_{n\geq 0} \frac{n!}{(n-0)!} a_n x^{n-0} = \sum_{n\geq 0} a_n [x^n] = a_k$

Inductive step: Assume $A^{(k)}(x) = \sum_{n\geq k} \frac{n!}{(n-k)!} a_n x^{n-k}$ To be true.

Want to show that $A^{(k+1)}(x) = \sum_{n\geq k+1} \frac{n!}{(n-(k+1))!} a_n x^{n-(k+1)}$

To do this, we take the derivative with respect to x

$\frac{d}{dx}(A^k(x)) = \frac{d}{dx}\sum_{n\geq k} \frac{n!}{(n-k)!} a_n x^{n-k} = \sum_{n\geq k} \frac{d}{dx}(\frac{n!}{(n-k)!} a_n x^{n-k}) = \sum_{n\geq k+1} \frac{(n-k)n!}{(n-(k+1))!} a_n x^{n-(k+1)}$

$= \sum_{n\geq k+1} \frac{n!}{(n-(k+1))!} a_n x^{n-(k+1)}$ ∎

Part d.

I wish to solve $\frac{1}{(1-z)(1-bz)} = C_0\frac{1}{1-z} + C_1\frac{1}{1-bz}$ where $C_0 = -\frac{1}{b-1}$ and $C_1 = \frac{b}{b-1}$

Using the method of partial fractions: Cross-multiplying, we see that these constants must satisfy the simultaneous equations.

1. $C_0 + C_1 = 1$
2. $-bC_0 - C_1 = 0$

plugging in 1. we get $-\frac{1}{b-1} + \frac{b}{b-1} = \frac{b-1}{b-1} = 1$

Plugging in 2. we get $-\frac{b}{b-1} - \frac{b}{b-1} = \frac{b-b}{b-1} = 0$

Doing sanity checking, we evaluate $C_0\frac{1}{1-z} + C_1\frac{1}{1-bz} = \frac{1}{b-1}(-\frac{1}{1-z} + \frac{b}{1-bz})$

$\frac{1}{b-1}(\frac{-1+bz+b-bz}{(1-z)(1-bz)}) = \frac{b-1}{(1-z)(1-bz)}(\frac{1}{b-1}) = \frac{1}{(1-z)(1-bz)}$ ■

Part e.

We must show that $\sum_{n\geq 0} a_{n-1}z^n = z^2A(z)$

Given an OGF $A(z) = \sum_{n\geq 0} a_n z^n$, adding z on both sides.

We get, $zA(z) = \sum_{n\geq 1} a_{n-1}z^n$

Again we add z on both sides to get $z^2A(z) = \sum_{n\geq 2} a_{n-2}z^n$ ■

Part f.
i) Given $F(z) = zF(z) + z^2F(z) + z$

$z = F(z) - zF(z) - z^2F(z) = F(z)(1 - z - z^2)$

$F(z) = \frac{z}{1-z-z^2}$ $\blacksquare$

ii) To find the roots of $1 - z - z^2$, use quadratic formula

$z = \frac{1\pm\sqrt{1+4}}{2} = \frac{1\pm\sqrt{5}}{2}$

We get $-(z + (\frac{1-\sqrt{5}}{2}))(z + \frac{1+\sqrt{5}}{2}) = 0$

$-(z^2 + z - 1) = 1 - z - z^2$

Therefore, $F(z) = \frac{-z}{(z+\phi)(z+\phi')}$ $\blacksquare$

iii) Using partial fractions to show $F(z) = \frac{1}{\sqrt{5}}(\frac{\phi'}{z+\phi'} - \frac{\phi}{z+\phi})$
In other words, $F(z) = C_0\frac{1}{z+\phi} + C_1\frac{1}{z+\phi'}$

We get the simultaneous equations
$C_0 + C_1 = -1$
$(1 - \sqrt{5})C_0 + (1 + \sqrt{5}) = 0$

This implies $C_0 = \frac{-5-\sqrt{5}}{10}$ and $C_1 = \frac{-5+\sqrt{5}}{10}$

$F(z) = \frac{1}{10}(\frac{-5-\sqrt{5}}{z+\phi} + \frac{-5+\sqrt{5}}{z+\phi'}) = \frac{1}{\sqrt{5}}(\frac{-1-\sqrt{5}}{2(z+\phi)} + \frac{-1+\sqrt{5}}{2(z+\phi'}))$
$\frac{1}{\sqrt{5}}(\frac{\phi}{(z+\phi)} + \frac{\phi'}{(z+\phi'}))$

Therefore, $F(z) = \frac{1}{\sqrt{5}}(\frac{\phi}{(z+\phi)} + \frac{\phi'}{(z+\phi'}))$ $\blacksquare$

Problem 3

Algorithm 1 Subset & N-Tuples Enumeration

```
procedure AddOneMixedRadix(A, B, n + 1)
    a_0 = a_0 + 1
    for i = 0 in n do
        if a_0 ≥ b_0 then
            a_0 = 0
            a_1 = a_1 + 1
        end if
        if a_i ≥ b_i then
            a_i = 0
            a_i + 1 = a_i + 1 = 0
        end if
    end for
    for i = 1 in n do
        for j = 1 in i do
            p = p ∗ b_{j−1}
        end for
        sum = sum + (p ∗ a_i)
    end for
    Return (a_0 + sum)
end procedure
```