MATH-UA 120 Discrete Mathematics: Problem Set 9

Queen Elizabeth

Due Monday, December 9th, 2024

Assignment Instructions

- These are to be written up in LaTeX and turned in on Gradescope.
- **Click here to duplicate this `.tex` file in Overleaf**.
- Write your solutions inside the `solution` environment.
- You are always encouraged to talk problems through with your peers and your instructor, but your write up should be done independently.
- Problems are graded on correctness and fluency.
- Unless stated otherwise, all calculations require justification.
- Some tutorials on how to use LaTeX can be found here. If you have any questions about LaTeX commands you can always ask your instructor for advice.

Statement on generative AI

In this and other mathematics courses, you are expected to construct clear and concise mathematical arguments based on statements proven in our text and class notes. Large language models such as ChatGPT are unable to produce this kind of solution. They also frequently generate circular logic and outright false results.

You may use AI to summarise content, generate study plans, create problems, or do other study-related activities. You may not ask a chatbot to solve your quiz or homework problems, or do any assessment-related activities.

You may use AI tools to edit your grammar and punctuation, but remember that mathematical English is not the same as academic English in other disciplines.

Problem 1

Let G and H be graphs. We say that G is *isomorphic* to H if there is a bijection from $f : V(G) \longrightarrow V(H)$ such that for all $a, b \in V(G)$ we have $a \sim b$, in G, if and only if $f(a) \sim f(b)$, in H. The function f is called an *isomorphism* from G to H.

a) Prove that isomorphic graphs have the same number of vertices.

b) Prove that if $f : G \longrightarrow H$ is an isomorphism and $v \in V(G)$, then the degree of v in G is equal to the degree of $f(v)$ in H.

c) Prove that isomorphic graphs have the same number of edges.

Problem 2

Suppose G is a subgraph of H. Prove or disprove:

a) $\alpha(G) \leq \alpha(H)$

b) $\omega(G) \leq \omega(H)$

Problem 3

Let $G = (V, E)$ be a graph with $V = \{v_1, v_2, \ldots, v_n\}$. Its **degree sequence** is the list of degrees of its vertices, arranged in non-increasing order. That is, the degree sequence of G is $(d(v_1), d(v_2), \ldots, d(v_n))$ with the vertices arranged such that $d(v_1) \geq d(v_2) \geq \cdots \geq d(v_n)$. Below are different lists of possible degree sequences. Determine whether each case can be a graph with n vertices. If not, explain why not. If so, describe a graph with these degrees: is the graph a complete graph, a cycle, a path, contains specific subgraphs, connected, etc?

a) $n = 7$ and $(6, 5, 4, 3, 2, 1, 0)$

b) $n = 6$ and $(2, 2, 2, 2, 2, 2)$

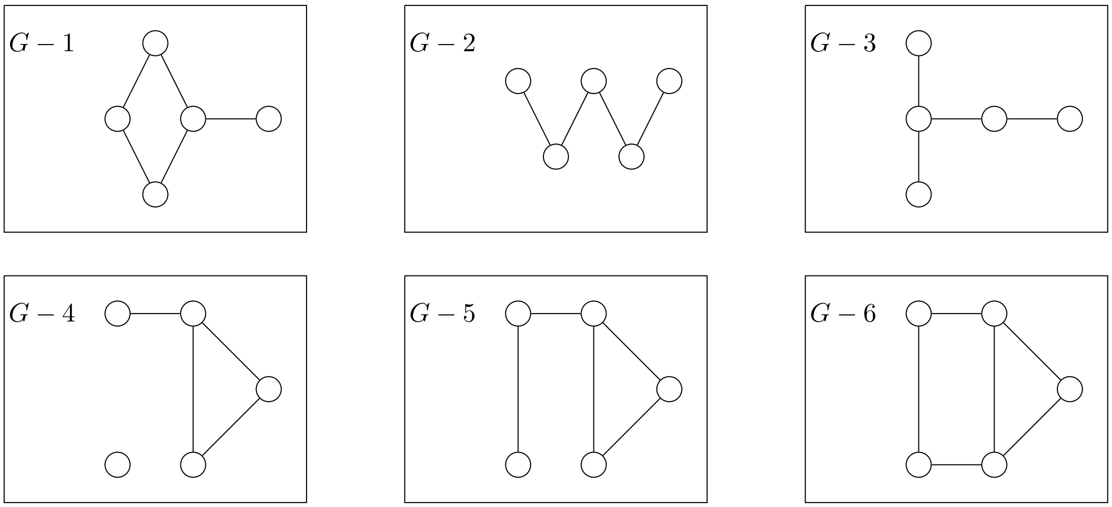
c) $n = 6$ and $(3, 2, 2, 2, 2, 2)$

d) $n = 6$ and $(1, 1, 1, 1, 1, 1)$

e) $n = 6$ and $(5, 3, 3, 3, 3, 3)$

Problem 4

Let $G = (V, E)$ be a graph with $V = \{1, 2, 3, 4, 5, 6\}$. In the figures below we show the graphs (up to isomorphism) of $G - 1$, $G - 2$ and so on, but we do not have the names on the vertices. The goal of this problem is to reconstruct the graph G.



a) Determine the number of edges in G.

b) Using your answer in a) and the figures above, determine the degrees of each of the six vertices of G.

c) Determine G and sketch it.

There are instructions in the .tex file to help you draw the graph in LaTeX.

Problem 5

a) Given a graph with n vertices. First, what is the maximum number of edges can the graph have and be disconnected? Then, what is the minimum number of edges we need to add to the previous graph to be connected?

b) A *complete bipartite graph* $K_{m,n}$ is a graph whose vertices can be partitioned $V = X \cup Y$ such that $|X| = m$ and $|Y| = n$ for positive integers m, n, and $\{x, y\}$ is an edge in $K_{m,n}$ if and only if $x \in X$ and $y \in Y$. What is the number of edges in $K_{m,n}$?

c) Given a cycle graph C_4, how many possible subgraphs of C_4 can there be?

Problem 6

Prove that given a graph with exactly two vertices of odd degree, there must be a path joining these two vertices.

Problem 7

Prove by induction on n: Given integer $n \geq 1$. If T is a tree with n vertices, then T has $n-1$ edges.

Problem 8

a) Prove that if a tree has n vertices where $n \geq 4$ and is not a path graph P_n, then it has at least three vertices of degree 1.

b) A **complete bipartite graph** $K_{m,n}$ is a graph whose vertices can be partitioned $V = X \cup Y$ such that $|X| = m$ and $|Y| = n$ for positive integers m, n, and $\{x, y\}$ is an edge in $K_{m,n}$ if and only if $x \in X$ and $y \in Y$. Prove that every cycle in $K_{m,n}$ has an even number of edges.

Problem 9

Given a tree G with two vertices of degree 2, four vertices of degree 3, three vertices of degree 4, and the remaining vertices of degree 1. How many vertices does G have?