Logic and Complexity

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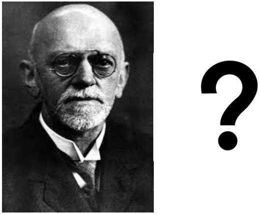
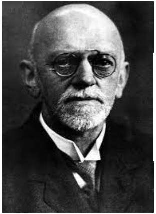
A System of Proofs

- A Formal System of Mathematical Proofs

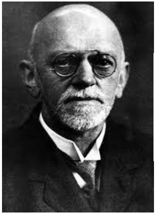


A System of Proofs



- A Formal System of Mathematical Proofs
  - Axioms $\mathcal{A} = \{A_1, A_2, \ldots, A_k\}$

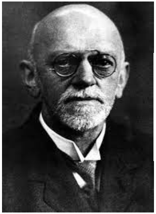
A System of Proofs



- A Formal System of Mathematical Proofs
  - Axioms $\mathcal{A} = \{A_1, A_2, \ldots, A_k\}$
  - Truth-preserving rules $\mathcal{I} = \{I_1, \ldots, I_m\}$

$$I_i = \frac{S_1, S_2, \ldots, S_n}{C}$$

A System of Proofs



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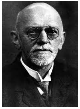
$$I_i = \frac{S_1, S_2, \ldots, S_n}{C}$$

- Objective: Anything true has a proof

$$\begin{tikzcd}
 & & & T & & & \\
 & T_1 \arrow[urr, "I_{k'}"] & & & & T_2 \arrow[ull] & \\
A_i \arrow[ur, "I_{i'}"] & & A_j \arrow[ul] & & A_k \arrow[ur] & & A_\ell \arrow[ul, "I_{j'}"']
\end{tikzcd}$$

Hilbert's Questions



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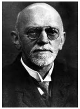
1.

2.

3.

Hilbert's Questions



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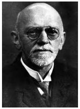
1. Complete
2. Consistent
3. Decidable

Hilbert's Questions



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1. Complete X

Some Truths cannot be proven

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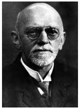
2. Consistent

?

3. Decidable



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1. Complete X

Some Truths cannot be proven

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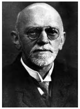
2. Consistent ?

Proving its own consistency beyond system

?

3. Decidable

Hilbert's Questions



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1. Complete X

Some Truths cannot be proven

2. Consistent ?

Proving its own consistency beyond system

3. Decidable 🤔 This talk

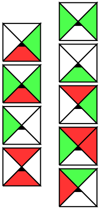
Given a statement and axioms, is there a proof?

Computability

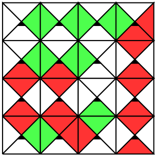
- **NEED:** Given S and $\mathcal{A}$, "procedure" to decide whether there is a proof
  - For any axiom system $(\mathcal{A}, \mathcal{I})$ and any statement S
    $$f(S, \mathcal{A}, \mathcal{I}) = \begin{cases} 1 & \mathcal{A} \vdash_{\mathcal{I}} S \\ 0 & \mathcal{A} \nvdash_{\mathcal{I}} S \end{cases}$$
  - Is f *computable*?
- What is *computability*?
- **Example 1**: $\forall x \in \mathbb{N}, f(x) = x^2 + 2x + 1$
  - $f(13) = ?$
- **Example 2**: $\forall x \in \mathbb{N}, \forall y \in \mathbb{N} f(x, y) = x^y + xy + 1$
  - $f(13, 12) = ?$
  - **HARDER**

Computability

- Proper Tiling:
  - Given a set of tiles $\mathcal{T}$



  - A Proper tiling on a finite grid:



- $f(\mathcal{T}) = \begin{cases} 1 & \text{Proper tiling covering infinite grid} \\ 0 & \text{No such proper tiling} \end{cases}$

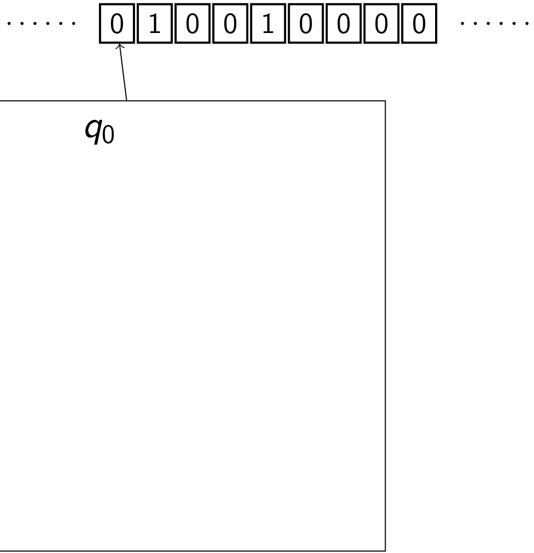
Model of Computability

- $f(x) = x^2 + 2x + 1$.
- $f(13) = 13^2 + (2x13) + 1$
- Local rules:
  - Rule of multiplication
  - Rule of addition
- Local states:
  1. Initial $13^2 + (2x13) + 1$
  2. All multiplication done $169 + 26 + 1$
  3. All addition done 196
  4. Final value 196
- **NEED: MODEL** Anything that model can compute is computable
- Turing Machine, Lambda Calculus

The Turing Machine

$$\cdots\cdots \quad \boxed{0}\boxed{1}\boxed{0}\boxed{0}\boxed{1}\boxed{0}\boxed{0}\boxed{0}\boxed{0} \quad \cdots\cdots$$

The Turing Machine



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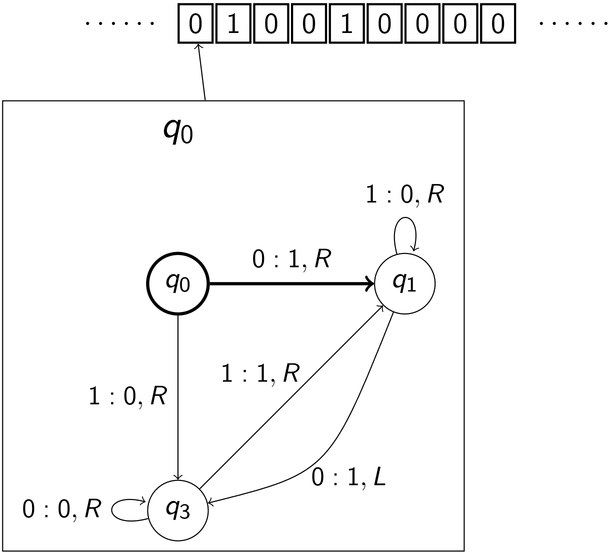

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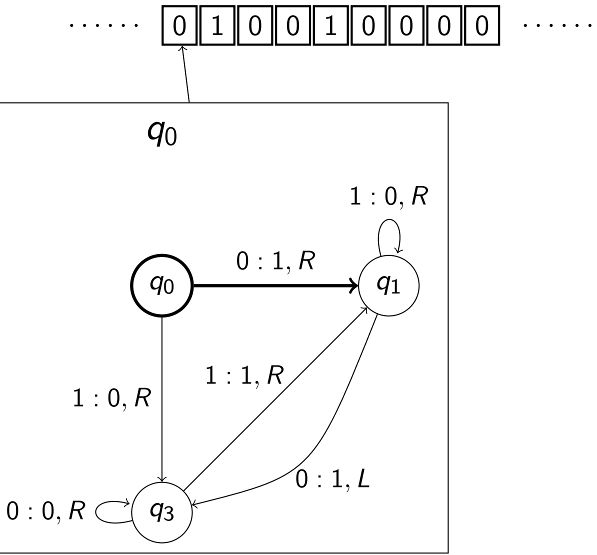
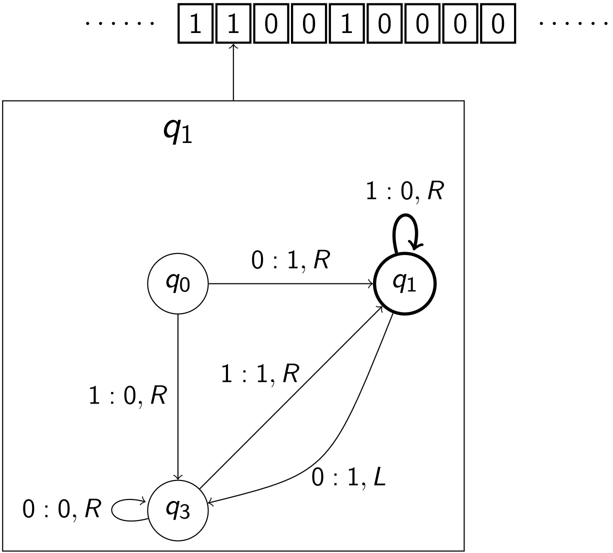
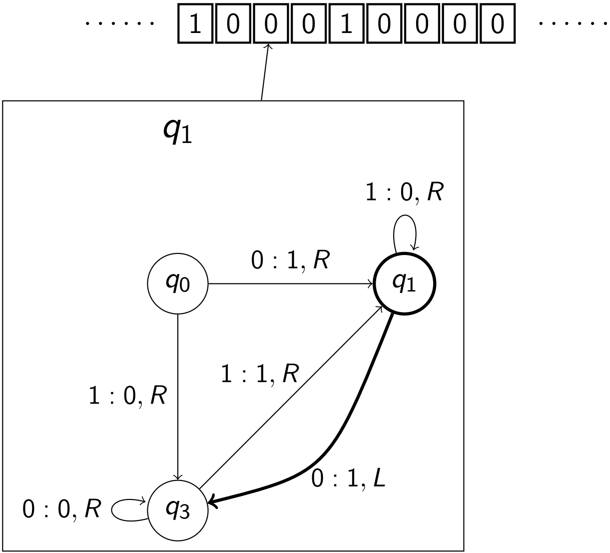


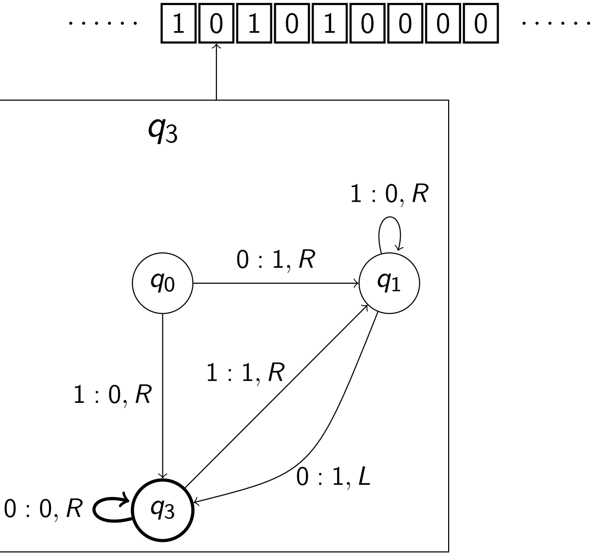


...... 1 1 0 0 1 0 0 0 0 ......
q_1
1 : 0, R
0 : 1, R
q_0
q_1
1 : 0, R
1 : 1, R
0 : 1, L
0 : 0, R
q_3

The Turing Machine



The Turing Machine



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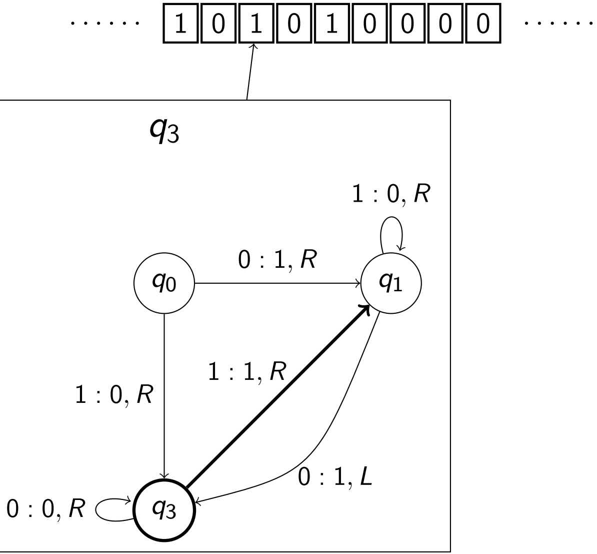

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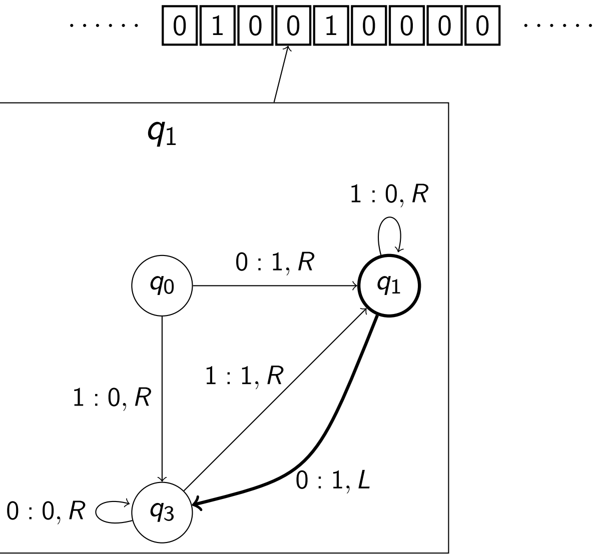

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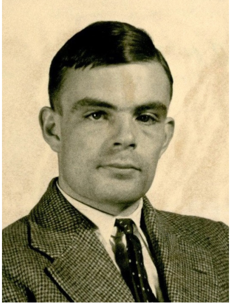

- $\mathcal{M} = \langle Q, q_0 \in Q, \Sigma, \delta, F \subseteq Q \rangle$
  - Q: finite set of states
  - Σ: alphabet
  - $\delta : Q \times \Sigma \to Q \times \Sigma \cup \{R, L\}$ transition rule
- Decision functions: $f(x \in \{0,1\}^\star) \in \{0,1\}$
- **Other MODELS**: Church's Lambda Calculus