Reminders for NAPDE

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Reminders on calculus

$$\int_\Omega -\Delta u v = \int_\Omega \nabla u \cdot \nabla v - \underbrace{\int_{\Gamma_D} \nabla u \cdot \mathbf{n} v}_{=0 \text{ if } v|_{\Gamma_D}=0}$$

$$\int_\Omega \text{div} u = \int_{\partial\Omega} u \cdot \mathbf{n}$$

Weak Formulations

Elliptic equations

$$\begin{cases} -\text{div}(\mu \nabla u) + \mathbf{b} \cdot \nabla u + \sigma u = f & \text{in } \Omega \quad g \in L^2(\Gamma_N) \\ u = 0 & \text{on } \Gamma_D \quad \partial\Omega = \Gamma_D \cup \Gamma_N \\ \mu \nabla u \cdot \mathbf{n} = g & \text{on } \Gamma_N \quad {\Gamma_D}^{\text{o}} \cap {\Gamma_N}^{\text{o}} = \varnothing \end{cases}$$

$$\Downarrow$$

$$\underbrace{\int_\Omega \mu \nabla u \cdot \nabla v + \int_\Omega \mathbf{b} \cdot \nabla u v + \int_\Omega \sigma u v}_{=:a(u,v)} = \int_\Omega f v + \underbrace{\int_{\Gamma_D} \mu \nabla u \cdot \mathbf{n} v}_{=0 \text{ if } v|_{\Gamma_D}=0} + \int_{\Gamma_N} \underbrace{\mu \nabla u \cdot \mathbf{n}}_{=g} v$$

$$\Downarrow$$

$$\begin{cases} \text{find } u \in V & V = \left\{ v \in H^1(\Omega), v|_{\Gamma_D} = 0 \right\} =: H^1_{\Gamma_D}(\Omega) \\ a(u,v) = \langle F, v \rangle & \forall\, v \in V \end{cases}$$

Parabolic equations

$$\begin{cases} \frac{\partial u}{\partial t} - \nu \frac{\partial^2 u}{\partial x^2} = f & 0 < x < d, t > 0 \\ u(x,0) = u_0(x) & 0 < x < d \\ u(0,t) = u(d,t) = 0 & t > 0 \end{cases}$$

$$\Downarrow$$

$$\int_\Omega \frac{\partial u(t)}{\partial t} v \, d\Omega + a(u(t), v) = \int_\Omega f(t) v \, d\Omega \quad \forall\, v \in V$$