Advanced Partial Differential Equations
Exams

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Exercise 1
For $a, \gamma \in \mathbb{R}$, consider the Cauchy problem for the wave equation

$$\begin{cases} u_{tt} - 4u_{xx} = 0 & (x,t) \in \mathbb{R} \times (0,\infty) \\ u(x,0) = e^{-x^2} + \gamma e^{-(x-a)^2} & x \in \mathbb{R} \\ u_t(x,0) = 0 & x \in \mathbb{R} \end{cases}$$

Show that the mass M of the solution is constant, then find the couples (a, γ) such that:

- $M = 0$.
- The solution $u(x,1)$ at $t = 1$ consists of only two "bumps".

To show that the mass M of the solution is constant we need to define such a mass, and then check its behavior over time.
The mass M of the solution is defined as $M(t) := \int_{\mathbb{R}} u(x,t)\,dx$.

$$M''(t) = \frac{\partial^2}{\partial^2 t} \int_{\mathbb{R}} u(x,t)\,dx = \int_{\mathbb{R}} u_{tt}\,dx = \int_{\mathbb{R}} 4u_{xx}\,dx = 4 \underbrace{\int_{\mathbb{R}} (u_x)_x\,dx}_{\text{div. form=0}} = 0$$

So $M(t) = A + Bt$. But $M'(t) = B$, and also $M'(t) = \int_{\mathbb{R}} u_t(x,t)\,dx$. Since $M'(t)$ is constant we take $M'(0) = \int_{\mathbb{R}} u_t(x,0)\,dx = 0 \Rightarrow B = 0$.
Then we conclude that $M(t) = A$ is constant too, and $M(0) = \int_{\mathbb{R}} e^{-x^2} + \gamma e^{-(x-a)^2}\,dx$

$$\int_{\mathbb{R}} e^{-x^2} + \underbrace{\gamma e^{-(x-a)^2}}_{\substack{x-a=y \\ dx=dy}}\,dx = (1+\gamma)\int_{\mathbb{R}} e^{-y^2}\,dy = \sqrt{\pi}(1+\gamma)$$

After that we need to show that $M = 0$, so $\sqrt{\pi}(1+\gamma) = 0 \iff \gamma = -1$
Then we look for the values of a such that the solution consists of two "bumps".

Remark 1
For a hyperbolic equation, we know that the solution $u(x,t) = \frac{1}{2}(g(x+ct) + g(x-ct))$

In this case we have $u(x,t) = \frac{1}{2}(\overbrace{g(x+2t)}^{\substack{\text{c is 2} \\ \text{bc } 4u_{xx}}} + g(x-2t))$, which becomes

$$u_{a,\gamma}(x,t) = \frac{1}{2}\left(e^{-(x+2t)^2} + \gamma e^{-(x-a+2t)^2} + e^{-(x-2t)^2} + \gamma e^{-(x-a-2t)^2}\right)$$

that, for $\gamma = -1$ and $t = 1$

$$u_{a,\gamma}(x,1) = \frac{1}{2}\left(e^{-(x+2)^2} - e^{-(x-a+2)^2} + e^{-(x-2)^2} - e^{-(x-a-2)^2}\right)$$

We can see that this solution has four "bumps" in $x = -2, x = 2, x = a-2, x = a+2$. To obtain two bumps we manipulate a and see that

- $-2 = a - 2 \Rightarrow a = 0 \Rightarrow u_{0,\gamma}(x,1) = \frac{1}{2}(e^{-(x+2)^2} - e^{-(x+2)^2} + e^{-(x-2)^2} - e^{-(x-2)^2}) = 0$

- $-2 = a + 2 \Rightarrow a = -4 \Rightarrow u_{-4,\gamma}(x,1) = \frac{1}{2}(e^{-(x+2)^2} - e^{-(x+6)^2} + e^{-(x-2)^2} - e^{-(x+2)^2}) = \frac{1}{2}(-e^{-(x+6)^2} + e^{-(x-2)^2})$
- $2 = a - 2 \Rightarrow a = 4 \Rightarrow u_{-4,\gamma}(x,1) = \frac{1}{2}(e^{-(x+2)^2} - e^{-(x-2)^2} + e^{-(x-2)^2} - e^{-(x-6)^2}) = \frac{1}{2}(-e^{-(x-6)^2} + e^{-(x+2)^2})$
- $-2 = a - 2 \Rightarrow a = 0 \Rightarrow u_{0,\gamma}(x,1) = \frac{1}{2}(e^{-(x+2)^2} - e^{-(x+2)^2} + e^{-(x-2)^2} - e^{-(x-2)^2}) = 0$

We can see that the solution has only two "bumps" in the cases $a = \pm 4$, so we conclude that the desired couples of (a, γ) are

$$\begin{cases} \gamma = -1 \\ a = -4 \end{cases} \vee \begin{cases} \gamma = -1 \\ a = 4 \end{cases}$$

Exercise 2
Let $\Omega \subset \mathbb{R}^n (n \geq 2)$ be a bounded smooth domain, let a be a measurable function in Ω. Consider the problem

$$\begin{cases} -\Delta u = a(x)u^3 & \Omega \\ u = 0 & \partial\Omega \end{cases} \tag{P}$$

Under which assumptions on the space dimension n can we write a variational formulation of problem (P) in $H_0^1(\Omega)$? For each of these dimensions find the most general assumptions on a that allow to write the variational formulation. Finally, write the variational formulation.

First, a quick reminder on Sobolev embedding, which will be very useful a.e. in this document

Remark 2
Let $\Omega \subseteq \mathbb{R}^n$ open with $\partial\Omega \in \text{Lip}$, $s \geq 0$,

$$H^s(\Omega) \subset \begin{cases} L^p(\Omega) & \forall\, 2 \leq p \leq 2^* & \text{if } n > 2s \\ L^p(\Omega) & \forall\, 2 \leq p < \infty & \text{if } n = 2s \\ C^0(\bar{\Omega}) & & \text{if } n < 2s \end{cases}$$

Increasing s increases the regularity, while increasing n decreases it.
The exponent 2^* is called critical exponent and is defined as $2^* := \frac{2n}{n-2s}$.
If $Omega$ is bounded, all these embeddings are compact except $H^s(\Omega) \subset L^{2^*}$ when $n > s$.

Since we want to know the variational formulation in H_0^1 we have $s = 1$ and need to check $n = 2, n \geq 3$. Remember a variational formulation makes sense if $\int_\Omega fv < \infty$.

$n = 2$. In this case we have $u^3, v \in H_0^1(\Omega)$, so by Sobolev embedding we know $u^3, v \in L^p(\Omega)$ for $2 \leq p < \infty$.

$$\left| \int_\Omega a(x)u^3 v \right| dx \leq \int_\Omega |a(x)||u^3||v|\, dx \underset{Holder}{\leq} \left(\int_\Omega |a(x)|^r \right)^{\frac{1}{r}} \left(\int_\Omega |u^3|^p \right)^{\frac{1}{p}} \left(\int_\Omega |v|^q \right)^{\frac{1}{q}} < \infty.$$

To use Holder inequality we need to find r, p, q such that $\frac{1}{r} + \frac{1}{p} + \frac{1}{q} = 1$. We see that,

$$\frac{1}{r} + \frac{1}{p} + \frac{1}{q} = 1 \iff a(x) \in L^r(\Omega) \qquad \text{with } r > 1$$

$n \geq 3$. In this case we have $u^3, v \in H_0^1(\Omega)$, so by Sobolev embedding we know $u^3, v \in L^p(\Omega)$ for $2 \leq p \leq 2^*$. We proceed as before, using Holder inequality, but decide to use $p = \frac{2^*}{3}$ and $q = \frac{1}{2^*}$.

$$\left| \int_\Omega a(x)u^3 v \right| dx \leq \int_\Omega |a(x)||u^3||v|\, dx \underset{Holder}{\leq} \left(\int_\Omega |a(x)|^r \right)^{\frac{1}{r}} \left(\int_\Omega |u^3|^{2^*} \right)^{\frac{3}{2^*}} \left(\int_\Omega |v|^{2^*} \right)^{\frac{1}{2^*}} < \infty.$$

In this case Holder inequality gives us

$$\frac{1}{r} + \frac{3}{2^*} + \frac{1}{2^*} = 1 \iff \frac{1}{r} = 1 - \frac{4}{2^*} \iff r = \frac{2^* - 4}{2^*}$$

Substituting $2^* = \frac{2n}{n-2}$ we get $r = \frac{n}{-n+4}$. Since $r > 0$ we need $n < 4$. In this case we have $a(x) \in L^3(\Omega)$ for $n = 3$, but also $a(x) \in L^\infty(\Omega)$ for $n = 4$.

At this point we can write the weak formulation of the problem. We multiply the equation by a test function $v \in H_0^1(\Omega)$ and obtain

$$\int_\Omega -\Delta u v \, dx = \int_\Omega a(x) u^3 v \, dx \qquad \forall v \in H_0^1(\Omega)$$

We integrate by parts the left hand side and obtain

$$\int_\Omega \nabla u \nabla v \, dx = \int_\Omega a(x) u^3 v \, dx \qquad \forall v \in H_0^1(\Omega)$$

This is the weak formulation of the problem. This is well posed if

Dimension	Assumptions on $a(x)$
$n = 2$	$a \in L^r(\Omega)$, $r > 1$
$n = 3$	$a \in L^3(\Omega)$
$n = 4$	$a \in L^\infty(\Omega)$
$n \geq 5$	No variational formulation

Exercise 3
Let $\Omega \subset \mathbb{R}^n$ be a bounded open set of class C^1, and let u be a sufficiently regular solution of the problem

$$\begin{cases} u_t - \Delta u = 0 & \Omega \times (0, \infty) \\ u = 0 & \partial\Omega \times (0, \infty) \\ u(x, 0) = \alpha(x) & x \in \Omega \end{cases}$$

Study monotonicity/boundedness properties of the energy $E_u(t) = \int_\Omega |\nabla u|^2 \, dx$.

The energy functional is defined as $E_u(t) = \int_\Omega |\nabla u|^2 \, dx = \|\nabla u\|^2_{L^2(\Omega)}$. We want to study its behavior over time, so we need to compute its derivative with respect to time.

$$\begin{aligned} \frac{d}{dt} E_u(t) &= \frac{d}{dt} \int_\Omega |\nabla u|^2 \, dx = \int_\Omega \frac{d}{dt} |\nabla u|^2 \, dx = \int_\Omega 2 \nabla u \cdot \nabla u_t \, dx = \\ &= \int_{\partial\Omega} 2 \underbrace{u \cdot \nu}_{=0} u_t - \int_\Omega 2 \Delta u u_t \, dx = -\int_\Omega 2 (\Delta u)^2 \, dx \leq 0. \end{aligned} \quad \Rightarrow E \text{ is non-increasing}$$

We see that the energy is non-increasing, since we obtain a positive quantity with a negative sign. Now we want to study the boundedness of the energy. We start by multiplying the equation by u

$$\int_\Omega u_t u \, dx - \int_\Omega \Delta u u \, dx = 0$$

Integrating by parts the second term we obtain

$$\int_\Omega u_t u \, dx - \int_{\partial\Omega} u \nabla u \cdot \nu \, dS + \int_\Omega (\nabla u)^2 \, dx = 0$$

Since $u = 0$ on the boundary we have

$$\int_\Omega u_t u \, dx = -\int_\Omega (\nabla u)^2 \, dx$$

We can rewrite the energy as

$$E_u(t) = \int_\Omega |\nabla u|^2 \, dx = -\int_\Omega u_t u \, dx = -\frac{1}{2} \int_\Omega (u^2)_t \, dx$$

Since the energy is non-increasing, we have that $E_u(t) \leq E_u(0) \forall t \geq 0$, so we have

$$E_u(t) = -\frac{1}{2} \int_\Omega (u^2)_t \, dx \leq -\frac{1}{2} \int_\Omega (u(x, 0)^2)_t \, dx = -\frac{1}{2} \int_\Omega \alpha(x)^2 \, dx$$

Since $\alpha(x)$ is bounded (is a function in H_0^1) we have that the energy is bounded too.

Exercise 4
Let $(X, \|\cdot\|)$ be a Banach space, and let $u \in C^1([0,T];X)$. Using the following abstract version of the *Fundamental Theorem of Calculus*:

$$\int_0^T u'(t)\,dt = u(T) - u(0)$$

prove that $\Lambda_{u'} = (\Lambda_u)' \in \mathcal{D}(0,T;X)$ where

$$\Lambda_f(\varphi) = \int_0^T \varphi(t) f(t)\,dt \qquad \forall f \in L^1(0,T;X)$$

By the definition of distributional derivative we have

$$(\Lambda_u(\varphi))' = -\Lambda_u(\varphi')\forall\varphi \in \mathcal{D}(0,T)$$

where

$$\Lambda_u(\varphi)' = -\int_0^T \varphi'(t)u(t)\,dt$$

We can integrate by parts the above expression

$$(\Lambda_u(\varphi))' = -\int_0^T \varphi'(t)u(t)\,dt = \underbrace{-\varphi(t)u(t)|_0^T}_{=0} + \int_0^T \varphi(t)u'(t)\,dt = \int_0^T \varphi(t)u'(t)\,dt = \Lambda_{u'}(\varphi)$$

We have shown that $\Lambda_{u'} = (\Lambda_u)'$ in $\mathcal{D}(0,T;X)$.

1.2 July 2021

Exercise 1
Let $\Omega \subseteq \mathbb{R}^2$ be a bounded open set of class C^∞, let $f \in L^2(\Omega)$. Consider the Dirichlet problem

$$\begin{cases} 2\partial_x^2 u + 3\partial_y^2 u + 2\partial_{xy} u = f & \Omega \\ u = 0 & \partial\Omega \end{cases} \tag{P}$$

(1) Prove that (P) admits a unique solution $u \in H_0^1(\Omega)$.

(2) What is the minimum $m \in \mathbb{N}$ for which $f \in H^m(\Omega)$ implies $u \in H^5(\Omega)$?

We can rewrite the equation in the form $\operatorname{div}(A\nabla u)$ with $A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$. Let's check if A is the correct matrix.

$$\operatorname{div}(A\nabla u) = \operatorname{div}\left(\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} u_x \\ u_y \end{pmatrix}\right) = \operatorname{div}\left(\begin{pmatrix} 2u_x + u_y \\ u_x + 3u_y \end{pmatrix}\right) = 2u_{xx} + 3u_{yy} + 2u_{xy}$$

Our Hilbert triplet is $H_0^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega)$. We define $V = H_0^1(\Omega)$, $H = L^2(\Omega)$, $V' = H^{-1}(\Omega)$. We can now write the weak formulation of the problem. We multiply the equation by a test function $v \in V$ and obtain

$$\int_\Omega \operatorname{div}(A\nabla u) v \, dx = \int_\Omega f v \, dx \qquad \forall v \in V$$

We integrate by parts the left-hand side and obtain

$$\int_\Omega A\nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx \qquad \forall v \in V$$

This is the weak formulation of the problem. Now we use Lax-Milgram theorem to prove the existence and uniqueness of the solution. We need to check the coercivity and boundedness of the bilinear form. We have that the bilinear form is

$$a(u, v) = \int_\Omega A\nabla u \cdot \nabla v \, dx$$

A bilinear form is continuous if there exists a constant $C > 0$ such that

$$|a(u, v)| \le C\|u\|_V \|v\|_V$$

Remark 1
In $H_0^1(\Omega)$ we have the norm $\|u\|_V = \|\nabla u\|_{L^2}$

We write the bilinear form explicitly and bound it

$$|a(u, v)| = \left|\int_\Omega A\nabla u \cdot \nabla v \, dx\right| \le \int_\Omega |A||\nabla u||\nabla v| \, dx \le |A|\|\nabla u\|_{L^2}\|\nabla v\|_{L^2} = |A|\|u\|_V \|v\|_V$$

We have that the bilinear form is continuous. We need to check the coercivity of the bilinear form. A bilinear form is coercive if there exists a constant $c > 0$ such that

$$a(u, u) \ge c\|u\|_V^2$$

We write the bilinear form explicitly

$$a(u, u) = \int_\Omega A\nabla u \cdot \nabla u \, dx = \int_\Omega |A||\nabla u|^2 \, dx = |A|\|\nabla u\|_{L^2}^2 \ge \|\nabla u\|_{L^2}^2 = \|u\|_V^2$$

We have that the bilinear form is coercive. By Lax-Milgram theorem we have that the problem admits a unique solution $u \in V$.
The second request is about the minimum m such that $f \in H^m(\Omega)$ implies $u \in H^5(\Omega)$.

Remark 2
We know that if $f \in H^m(\Omega)$ then $u \in H^{m+2}(\Omega)$.

We have that $f \in H^m(\Omega)$ implies $u \in H^{m+2}(\Omega)$, so we need $m+2 \geq 5 \Rightarrow m \geq 3$. The minimum m is 3.