Reminders for NAPDE

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Reminders on calculus

$$\int_\Omega -\Delta u v = \int_\Omega \nabla u \cdot \nabla v - \underbrace{\int_{\partial\Omega} \nabla u \cdot \mathbf{n} v}_{=0 \text{ if } v|_{\partial\Omega}=0}$$

$$\int_\Omega \mathrm{div} u = \int_{\partial\Omega} u \cdot \mathbf{n}$$

$$-\int_\Omega \mathbf{v} \cdot \nabla p = \int_\Omega p \mathrm{div} \mathbf{v} + \int_{\partial\Omega} \mathbf{v} \cdot \mathbf{n} p$$

$$\int_\Omega \frac{\partial}{\partial x} u^2 = \frac{1}{2} \int_{\partial\Omega} u \cdot \mathbf{n}$$

Lifting Operators

If u on $\partial\Omega$ is non-null, we need to solve this problem, otherwise we cannot use test functions that vanish at the boundary. To do so, given $u = g$ on $\partial\Omega$ we use a lifting operator $Rg \in H^1(\Omega) : Rg|_{\partial\Omega} = g$, and modify our solution such that $u = \overset{\circ}{u} + Rg$, so the function $\overset{\circ}{u}$ has the properties we need. We then look for bilinear formulation such as $a(\overset{\circ}{u}, v)$ and add to the right-hand side $-a(Rg, v)$.

Weak Formulations

Elliptic equations

$$\begin{cases} -\mathrm{div}(\mu \nabla u) + \mathbf{b} \cdot \nabla u + \sigma u = f & \text{in } \Omega \quad g \in L^2(\Gamma_N) \\ u = 0 & \text{on } \Gamma_D \quad \partial\Omega = \Gamma_D \cup \Gamma_N \\ \mu \nabla u \cdot \mathbf{n} = g & \text{on } \Gamma_N \quad {\Gamma_D}^{\circ} \cap {\Gamma_N}^{\circ} = \varnothing \end{cases}$$

$$\Downarrow$$

$$\underbrace{\int_\Omega \mu \nabla u \cdot \nabla v + \int_\Omega \mathbf{b} \cdot \nabla u v + \int_\Omega \sigma u v}_{=:a(u,v)} = \int_\Omega f v + \underbrace{\int_{\Gamma_D} \mu \nabla u \cdot \mathbf{n} v}_{=0 \text{ if } v|_{\Gamma_D}=0} + \int_{\Gamma_N} \underbrace{\mu \nabla u \cdot \mathbf{n}}_{=g} v$$

$$\Downarrow$$

$$\begin{cases} \text{find } u \in V & V = \left\{ v \in H^1(\Omega), v|_{\Gamma_D} = 0 \right\} =: H^1_{\Gamma_D}(\Omega) \\ a(u, v) = \langle F, v \rangle & \forall\, v \in V \end{cases}$$

Parabolic equations

$$\begin{cases} \frac{\partial u}{\partial t} - \nu \frac{\partial^2 u}{\partial x^2} = f & 0 < x < d, t > 0 \\ u(x,0) = u_0(x) & 0 < x < d \\ u(0,t) = u(d,t) = 0 & t > 0 \end{cases}$$

$$\Downarrow$$

$$\int_\Omega \frac{\partial u(t)}{\partial t} v \, d\Omega + a(u(t), v) = \int_\Omega f(t) v \, d\Omega \quad \forall \, v \in V$$

$$\Downarrow$$

for each $t > 0$, we need to find $u_h(t) \in V_h$ s.t.

$$\int_\Omega \frac{\partial u_h(t)}{\partial t} v_h \, d\Omega + a(u_h(t), v_h) = \int_\Omega f(t) v_h \, d\Omega \quad \forall \, v_h \in V_h$$

Discontinuous Galerkin

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

$$\Downarrow$$

$$\sum_{\mathcal{K} \in \mathcal{T}_h} \int_\mathcal{K} \nabla u \cdot \nabla v - \sum_{\mathcal{K} \in \mathcal{T}_h} \int_{\partial\Omega\mathcal{K}} \nabla u \cdot \mathbf{n}_\mathcal{K} v = \int_\Omega f v$$

$$\Downarrow$$

$$\sum_{\mathcal{K} \in \mathcal{T}_h} \int_{\partial\Omega\mathcal{K}} u \cdot \mathbf{n}_\mathcal{K} v = \sum_{F \in \mathcal{F}} \int_F \{\!\{u\}\!\} \cdot [\![v]\!] + \sum_{F \in \mathcal{F}'_h} \int_F [\![u]\!] \, \{\!\{v\}\!\} \qquad \text{Magic formula}$$

$$\Downarrow$$

$$\sum_{\mathcal{K} \in \mathcal{T}_h} \int_\mathcal{K} \nabla u \nabla v - \sum_{F \in \mathcal{F}_h} \int_F \{\!\{\nabla u\}\!\} \cdot [\![v]\!] - \theta \sum_{F \in \mathcal{F}_h} \int_F [\![u]\!] \cdot \{\!\{\nabla_h v\}\!\} + \sum_{F \in \mathcal{F}_h} \int_F \gamma \, [\![u]\!] \cdot [\![v]\!] = \int_\Omega f v$$

Numerical formulation

Continuous Galerkin

Space $V_h = \left\{ v_h \in \mathcal{C}^0(\overline{\Omega}) : v_h|_\mathcal{K} \in \mathbb{P}^r(\mathcal{K}) \; \forall \, \mathcal{K} \in \mathcal{T}_h, v_h|_{\Gamma_D} = 0 \right\}$

$$\text{find } u_h \in V_h : a(u_h, v_h) = F(v_h) \quad \forall \, v_h \in V_h$$

GLS

$$\text{find } u_h \in V_h : a(u_h, v_h) + \sum_{\mathcal{K} \in \mathcal{T}_h} \int_\Omega \mathcal{L} u_h \tau_\mathcal{K} \mathcal{L} v_h = \int_\Omega f v_h + \sum_{\mathcal{K} \in \mathcal{T}_h} \int_\Omega f \tau_\mathcal{K} \mathcal{L} v_h \qquad \forall \, v_h \in V_h$$

Discontinuous Galerkin

Space $V_h^p = \left\{ v_h \in L^2(\Omega) : v_h|_{\mathcal{K}} \in \mathbb{P}^{p_{\mathcal{K}}}(\mathcal{K})\ \forall \mathcal{K} \in \mathcal{T}_h \right\} \not\subseteq H_0^1(\Omega)$

$$\text{find } u_h \in V_h^p : \mathcal{A}(u_h, v_h) = \int_\Omega f v_h \quad \forall\, v_h \in V_h^p$$

where

$$\begin{aligned}\mathcal{A}(u,v) = &\sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{K}} \nabla u \nabla v - \sum_{F\in\mathcal{F}_h} \int_F \{\!\{\nabla_h u\}\!\} \cdot [\![v]\!] \\ &-\theta \sum_{F\in\mathcal{F}_h} \int_F [\![u]\!] \cdot \{\!\{\nabla_h v\}\!\} + \sum_{F\in\mathcal{F}_h} \int_F \gamma\, [\![u]\!] \cdot [\![v]\!]\end{aligned}$$

- $\theta = 1$ Symmetric interior penalty
- $\theta = -1$ Non-symmetric interior penalty
- $\theta = 0$ Incomplete interior penalty
- $\gamma = \alpha \frac{p^2}{h}$

In the case of Dirichlet B.C. $u = g$ on $\partial\Omega$ modify r.h.s. and introduce the set of boundary faces $\mathcal{F}_h^B$

$$\int_\Omega f v - \theta \sum_{F\in\mathcal{F}_h^B} \int_F g \nabla_h v \cdot n + \sum_{F\in\mathcal{F}_h^B} \int_F \gamma g v$$

In case of Neumann B.C. $\nabla u \cdot n = g$ on $\partial\Omega$ we introduce the set of interior faces $\mathcal{F}_h'$

$$\begin{aligned}\mathcal{A}(u,v) = &\sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{K}} \nabla u \nabla v - \sum_{F\in\mathcal{F}_h'} \int_F \{\!\{\nabla_h u\}\!\} \cdot [\![v]\!] \\ &-\theta \sum_{F\in\mathcal{F}_h'} \int_F [\![u]\!] \cdot \{\!\{\nabla_h v\}\!\} + \sum_{F\in\mathcal{F}_h'} \int_F \gamma\, [\![u]\!] \cdot [\![v]\!]\end{aligned}$$

and

$$\int_\Omega f v + \sum_{F\in\mathcal{F}_h^B} \int_F g v$$

SEM

Space $\mathbb{Q}_N(I) = \left\{ v(x) = \sum_{k=0}^N a_k x^k,\ a_k \in \mathbb{R} \right\}$
In one dimension $\mathbb{Q}_N = \mathbb{P}_N$
Space $V_N = \{v_N \in \mathbb{Q}_N : v_N|_{\Gamma_D} = g\}$
If the domain is meshed $V_N = \left\{ v_N \in \mathcal{C}^0(\overline{\Omega}) : v_N|_{\mathcal{K}} \mathrm{o} \boldsymbol{\varphi}_k \in \mathbb{Q}_N(\hat{\mathcal{K}})\ \forall\, \mathcal{K} \in \mathcal{T}_h \right\}$

$$\text{find } u_N \in V_N : a(u_N, v_N) = (F, v_N)_{L^2(\Omega)} \quad \forall\, v_N \in V_N$$

Galerkin-NI

Space $V_N = \{v_N \in \mathbb{P}_N(\Omega) : v|_{\Gamma_D} = g\}$

$$\text{find } u_N \in V_N : a_N(u_N, v_N) = f_N(v_N) \quad \forall\, v_N \in V_N$$

SEM-NI

Space $X_\delta = \left\{ v_\delta \in \mathcal{C}^0(\Omega) : v_\delta|_{\mathcal{K}} = \hat{v}_\delta \,\mathrm{o}\, \mathbf{F}_{\mathcal{K}}^{-1},\ \text{with } \hat{v}_\delta \in \mathbb{Q}_p(\hat{\mathcal{K}})\ \forall\, \mathcal{K} \in \mathcal{T}_h \right\}$

$$\text{find } u_\delta \in X_\delta : a_\delta(u_\delta, v_\delta) = F_\delta(v_\delta) \quad \forall\, v_\delta \in X_\delta$$

Stability

Continuous Galerkin

$$\|u_h\| \leq \frac{\|F\|_{V'}}{\alpha}$$

Stabilized Galerkin

$$\|u_h\|^2_{GLS} \leq C\|f\|^2_{L^2(\Omega)}$$

Also $\tau_{\mathcal{K}}$

$$\tau_{\mathcal{K}}(\mathbf{x}) = \delta\frac{h_{\mathcal{K}}}{|\mathbf{b}(\mathbf{x})|} \quad \tau_{\mathcal{K}}(\mathbf{x}) = \frac{h_{\mathcal{K}}}{2|\mathbf{b}(\mathbf{x})|}\varepsilon(\mathbb{P}e_{\mathcal{K}})$$

Parabolic equations

Stability of θ-method for $\theta < \frac{1}{2}$

$$\exists\, c > 0 : \Delta t \leq ch^2 \quad \forall\, h > 0$$

or even

$$\Delta t \leq \frac{2}{(1-2\theta)\lambda_h^{N_h}}$$

where $\lambda_h^{N_h}$ is the largest eigenvalue of the bilinear form.

Discontinuous Galerkin

Introduce broken Sobolev space

$$H^s(\mathcal{T}_h) = \left\{v \in L^2(\Omega) : v|_{\mathcal{K}} \in H^s(\mathcal{K})\ \forall\, \mathcal{K} \in \mathcal{T}_h\right\}$$

and the norms

$$\|v\|^2_{H^s(\mathcal{T}_h)} = \sum_{\mathcal{K}\in\mathcal{T}_h} \|v\|^2_{H^s(\mathcal{K})} \qquad \|v\|^2_{L^2(\mathcal{F}_h)} = \sum_{F\in\mathcal{F}} \|v\|^2_{L^2(F)}$$

$$\|v\|^2_{DG} = \|\nabla_h v\|^2_{L^2(\Omega)} + \|\sqrt{\gamma}\,[\![v]\!]\|^2_{L^2(\mathcal{F}_h)} \qquad |\!|\!|v|\!|\!|^2_{DG} = \|v\|^2_{DG} + \left\|\frac{1}{\sqrt{\gamma}}\,\{\!\!\{\nabla_h v\}\!\!\}\right\|^2_{L^2(\mathcal{F}_h)}$$

And some key properties to stability

- Continuity on $H^2(\mathcal{T}_h) \times V_h^p$

$$|\mathcal{A}(u, v_h)| \lesssim |\!|\!|u|\!|\!|_{DG}\,\|v_h\|_{DG} \quad \forall\, u \in H^2(\mathcal{T}_h), \forall\, v_h \in V_h^p$$

- Coercivity on $V_h^p \times V_h^p$

$$\mathcal{A}(v_h, v_h) \gtrsim \|v\|^2_{DG} \quad \forall\, v_h \in V_h^p$$

- Strong-consistency (Galerkin orthogonality):

$$\mathcal{A}(u, v_h) = \int_\Omega f v_h \ \ \forall\, v_h \in V_h^p \Rightarrow \mathcal{A}(u - u_h, v_h) = 0 \quad \forall\, v_h \in V_h^p$$

- Approximation. Let $\prod_h^p u \in V_h^p$ be a suitable approximation of u, then

$$|\!|\!|u - \textstyle\prod_h^p u|\!|\!|_{DG} \lesssim \frac{h^{\min(p,s)}}{p^{s-\frac{1}{2}}}\|u\|_{H^{s+1}(\mathcal{T}_h)}$$

  If $p \geq s$

$$|\!|\!|u - \textstyle\prod_h^p u|\!|\!|_{DG} \lesssim \left(\frac{h}{p}\right)^s p^{\frac{1}{2}}\|u\|_{H^{s+1}(\mathcal{T}_h)}$$

Spectral Methods

Strang Lemma

$$\|v_h\|_V \leq \frac{1}{\alpha^*} \sup_{v_h \in V_h \setminus \{0\}} \frac{F_h(v_h)}{\|v_h\|_V}$$

Convergence rates

Galerkin

Ceà Lemma

$$\|u - u_h\| \leq \frac{M}{\alpha} \inf_{v_h \in V_H} \|u - v_h\|$$

$$\|u - \textstyle\prod_h^r u\| \leq Ch^r \, |u|_{H^{r+1}(\Omega)}$$

Stabilized Galerkin

If

$$\mathbb{P}e_{\mathcal{K}}(\mathbf{x}) = \frac{|\mathbf{b}(\mathbf{x})| h_{\mathcal{K}}}{2\mu} > 1 \quad \forall\, \mathbf{x} \in \mathcal{K}$$

then

$$\|u - u_h\|_{GLS} \leq Ch^{r+\frac{1}{2}} \, |u|_{H^{r+1}(\Omega)}$$

Parabolic equations

$$\left\{ \|u(t) - u_h(t)\|^2_{L^2(\Omega)} + \alpha \int_0^t \|\nabla u(s) - \nabla u_h(s)\|^2_{L^2(\Omega)} \, ds \right\}^{\frac{1}{2}}$$

$$\leq Ch^r \left\{ |u_0|^2_{H^r(\Omega)} + \int_0^t |u(s)|^2_{H^{r+1}(Omega)} \, ds + \int_0^t \left| \frac{\partial u(s)}{\partial s} \right|^2_{H^{r+1}(\Omega)} ds \right\}^{\frac{1}{2}}$$

Galerkin-NI

$$\left\|u - u_N^{\mathrm{GNI}}\right\|_{H^1(\Omega)} \leq C(s) \left(\frac{1}{N}\right)^s \left(\|u\|_{H^{s+1}(\Omega)} + \|f\|_{H^s(\Omega)} \right)$$

SEM-NI

$$\|u - u_\delta\|_{H^1(\Omega)} \leq C(s) \left(h^{\min(p,s)} \left(\frac{1}{p}\right)^s \|u\|_{H^{s+1}} + h^{\min(p,r)} \left(\frac{1}{p}\right)^r \|f\|_{H^r(\Omega)} \right)$$

Discontinuous Galerkin

Interpolation error

$$\left|\left|\left| u - \textstyle\prod_h^p u \right|\right|\right|_{DG} \lesssim \frac{h^{\min(p,s)}}{p^{s-\frac{1}{2}}} \|u\|_{H^{s+1}(\mathcal{T}_h)}$$

And, since $\|u - u_h\|_{DG} \lesssim \left|\left|\left| u - \prod_h^p u \right|\right|\right|_{DG}$:
General (if α large enough for SIP and NIP):

$$\|u - u_h\|_{DG} \lesssim \frac{h^{\min(p,s)}}{p^{s-\frac{1}{2}}} \|u\|_{H^{s+1}(\mathcal{T}_h)}$$

L^2 norm (if Ω is a convex domain):

- SIP $\theta = 1$

$$\|u - u_h\|_{L^2(\Omega)} \lesssim \frac{h^{\min(p,s)+1}}{p^{s+\frac{1}{2}}} \|u\|_{H^{s+1}(\mathcal{T}_h)}$$

- NIP $\theta = -1$ and IIP $\theta = 0$

$$\|u - u_h\|_{L^2(\Omega)} \lesssim \frac{h^{\min(p,s)}}{p^{s-\frac{1}{2}}} \|u\|_{H^{s+1}(\mathcal{T}_h)}$$

Navier-Stokes

In case of inf-sup (LBB) condition satisfied by V and Q

$$\|\mathbf{u} - \mathbf{u}_h\|_V + \|p - p_h\|_Q \leq C(\alpha_h, \beta_h, \gamma, \delta) \left\{ \inf_{\mathbf{v}_h \in V_h} \|\mathbf{u} - \mathbf{v}_h\|_V + \inf_{q_h \in Q_h} \|p - q_h\|_Q \right\}$$

where

- α_h is the coercivity constant on the subspace V_h of divergence free velocities
- β_h is the LBB constant
- γ is the continuity constant of $a(\cdot, \cdot)$
- δ is the continuity constant of $b(\cdot, \cdot)$

In case of Taylor-Hoods elements

$$\|\mathbf{u} - \mathbf{u}_h\|_V + \|p - p_h\|_Q \leq Ch(\|\mathbf{u}\|_{H^{k+1}} + \|p\|_{H^k})$$

Code implementation

CG-FEM

- Matrix A;

  $$A_{ij} = \int_\Omega \nabla\varphi_j \nabla\varphi_i$$

  Loop on all the elements and compute locally (elements with $\hat{\cdot}$ are computed on the reference element):

  $$A_{loc_{ij}} = \det(\mathbf{B}_\mathcal{K}) \int_{\hat{\mathcal{K}}} \hat{\nabla}\hat{\varphi}_j^T \mathbf{B}_\mathcal{K}^{-1} \mathbf{B}_\mathcal{K}^{-1} \hat{\nabla}\hat{\varphi}_i = \frac{\det(\mathbf{B})}{2} \hat{\nabla}\hat{\varphi}_j^T \mathbf{B}_\mathcal{K}^{-1} \mathbf{B}_\mathcal{K}^{-T} \hat{\nabla}\hat{\varphi}_i$$

  Can be implemented as

```
function [K_loc]=C_lap_loc(Grad,w_2D,nln,BJ)
K_loc=zeros(nln,nln);
for i=1:nln
    for j=1:nln
        for k=1:length(w_2D)
            Binv = inv(BJ(:,:,k));   % inverse
            Jdet = det(BJ(:,:,k));   % determinant
            K_loc(i,j) = K_loc(i,j) + (Jdet.*w_2D(k)) .* ( (Grad(k,:,i)
                         * Binv) * (Grad(k,:,j) * Binv )');
        end
    end
end
```

  or (use this if P1)

```
for i=1:nln
    for j=1:nln
            Binv = inv(BJ(:,:,1));   % inverse
            Jdet = det(BJ(:,:,1));   % determinant
            K_loc(i,j) = K_loc(i,j) + 0.5 * Jdet
            *  Grad(1,:,i) * Binv * Binv' * Grad(1,:,j)';
    end
end
```

- Mass matrix M:

$$M_{ij} = \int_{\Omega} \varphi_j, \varphi_i$$

Loop on all the elements and calculate the local mass matrix

$$M_{loc_{ij}} = \det(\mathbf{B}_{\mathcal{K}}) \int_{\hat{\mathcal{K}}} \hat{\varphi}_j \hat{\varphi}_i$$

Can be implemented as

```
function [M_loc]=C_mass_loc(dphiq,w_2D,nln,BJ)
M_loc=zeros(nln,nln);
for i=1:nln
    for j=1:nln
        for k=1:length(w_2D)
            Jdet = det(BJ(:,:,k));      % determinant
            M_loc(i,j) = M_loc(i,j) + (Jdet.*w_2D(k))
                         .* dphiq(1,k,i).* dphiq(1,k,j);
        end
    end
end
```

- Transport matrix T

Can be implemented as

```
function [ADV_loc]=C_adv_loc(Grad,dphiq,beta,w_2D,nln,BJ)
ADV_loc=sparse(nln,nln);
for i=1:nln
    for j=1:nln
        for k=1:length(w_2D)
            Binv=inv(BJ(:,:,k));    % inverse
            Jdet=det(BJ(:,:,k));    % determinant
            ADV_loc(i,j) = ADV_loc(i,j)+(Jdet.*w_2D(k)).* dphiq(1,k,i)
                           *( (beta)*(Grad(k,:,j) * Binv )');
        end
    end
end
```

- Right-hand side $\mathbf{b}$:

$$b_i = \int_{\Omega} f \varphi_i$$

which is computed

```
function [f]=C_loc_rhs2D(force,dphiq,BJ,w_2D,pphys_2D,nln,mu)
f = zeros(nln,1);
x = pphys_2D(:,1);
y = pphys_2D(:,2);
F = eval(force);
for s = 1:nln
    for k = 1:length(w_2D)
        Jdet = det(BJ(:,:,k));  % determinant
        f(s) = f(s) + w_2D(k)*Jdet*F(k)*dphiq(1,k,s);
    end
end
```