# Notes from Real and Functional Analysis

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# 1 Lecture 12/09/2022

## Element of set theory

Let $X$ be a set. Then

$$\mathcal{P}(X) = \{Y \mid Y \subseteq X\} \qquad \qquad \textbf{(Power Set)}$$

Let $I \subseteq \mathbb{R}$ be a set of indexes. A family of sets induced by $I$ is

$$\{E_i\}_{i \in I}, \quad E_i \subseteq X \qquad \qquad \textbf{(Family/Collection)}$$

If $I = \mathbb{N}$ is called a

$$\{E_n\}_{n \in \mathbb{N}} \qquad \qquad \textbf{(Sequence)}$$

**Definition 1.1**
$\{E_n\} \subseteq \mathcal{P}(X)$ is monotone increasing (resp. decreasing) if

$$E_n \subseteq E_{n+1} \, \forall n \qquad (\text{resp. } E_n \supseteq E_{n+1} \, \forall n)$$

and is written as

$$\{E_n\} \nearrow \qquad (\text{resp. } \{E_n\} \searrow)$$

Given a family of sets $\{E_i\}_{i \in I} \subseteq \mathcal{P}(X)$, will be often considered

$$\bigcup_{i \in I} E_i = \{x \in X : \exists \, i \in I \, s.t. \, x \in E_i\}$$

$$\bigcap_{i \in I} E_i = \{x \in X : x \in E_i, \forall i \in I\}$$

$\{E_i\}$ is said to be **disjoint** if $E_i \cap E_j = \varnothing \, \forall i \neq j$.

Examples:

$$[a, b] = \bigcap_{n=1}^{\infty} (a - \frac{1}{n}, b + \frac{1}{n})$$

$$(a, b) = \bigcup_{n=1}^{\infty} [a + \frac{1}{n}, b - \frac{1}{n}]$$

**Definition 1.2**
$\{E_n\} \subseteq \mathcal{P}(X)$. We define

$$\limsup_{n} E_n := \bigcap_{k=1}^{\infty} (\bigcup_{n=k}^{\infty} E_n)$$

$$\liminf_{n} E_n := \bigcup_{k=1}^{\infty} (\bigcap_{n=k}^{\infty} E_n)$$

If these two sets are equal, then

$$\lim_{n} E_n = \limsup_{n} E_n = \liminf_{n} E_n$$

**Proposition 1.1**
Some limits are:

- $\limsup_n E_n = \{x \in X : x \in E_n \text{ for } \infty - \text{many indexes } n\}$

- $\liminf_n E_n = \{x \in X : x \in E_n \text{ for all but finitely many indexes } n\}$

- $\liminf_n E_n \subseteq \limsup_n E_n$

- $(\liminf_n E_n)^C = \limsup_n E_n^C$

## Definition 1.3

We can define:

$$x \in \limsup_n E_n \quad \Leftrightarrow \quad x \in \bigcap_{k=1}^{\infty} \left( \bigcup_{n=k}^{\infty} E_n \right)$$

$$\Leftrightarrow \quad \forall k \in \mathbb{N} : \bigcup_{n=k}^{\infty} E_n$$

$$\Leftrightarrow \quad \forall k \in \mathbb{N} \, \exists n_k \geq k \, s.t. \, x \in E_{n_k}$$

So $x \in \limsup_n E_n \Rightarrow \exists m_1 = n_1 \, s.t. \, x \in E_{n_1}$

$\exists m_2 := n_{m_1+1} \geq m_1 + 1 \, s.t. \, x \in E_{n_2}$

$\vdots$

$\exists m_k := n_{m_{k-1}+1} \geq m_{k-1} + 1 \, s.t. \, x \in E_{n_k}$

$\vdots$

$x \in E_{m_1}, \ldots, E_{m_k}, \ldots$

On the other hand, assume that $x \in E_n$ for $\infty$-many indexes. We claim that $\forall k \in \mathbb{N} \, \exists n_k \geq k \, s.t. \, x \in E_{n_k} \Leftrightarrow x \in \limsup_n E_n$. If that claim is not true, then $\exists \bar{k} \, s.t. \, x \notin E_n \, \forall n > \bar{k} \Rightarrow x$ belongs at most to $E_1, \ldots, E_{\bar{k}}$, a contradiction. ★

## Definition 1.4

$\{E_i\}_{i \in I}$ is a **covering** of $X$ if

$$X \subseteq \bigcup_{i \in I} E_i$$

A subfamily of $E_i$ that is still a covering is called a **subcovering**

## Definition 1.5

Let $E \subseteq X$. The function $\chi_E : X \to \mathbb{R}$

$$\chi_E(x) := \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \in X \setminus E \end{cases}$$

is called **characteristic function** of $E$

Let $E_1, E_2$ be sets:

$$\chi_{E_1 \cap E_2} = \chi_{E_1} \cdot \chi_{E_2}$$

$$\chi_{E_1 \cup E_2} = \chi_{E_1} + \chi_{E_2} - \chi_{E_1 \cap E_2}$$

$$\{E_n\} \subseteq \mathcal{P}(X), \text{ disjoint}, E = \bigcup_{n=1}^{\infty} E_n \Rightarrow \mathcal{X}_{\mathcal{E}} = \sum_{n=1}^{\infty} \chi_{E_n}$$

$$\{E_n\} \subseteq \mathcal{P}, P = \liminf_n E_n, Q = \limsup_n E_n \Rightarrow \chi_P = \liminf \chi_{E_n}, \chi_Q = \limsup_n \chi_{E_n}$$

Recall that $\limsup_n a_n = \lim_{k \to \infty} \sup_{n \geq k} a_n$ and $\liminf_n a_n = \lim_{k \to \infty} \inf_{n \geq k} a_n$

Let's also check that $\chi_Q = \limsup_n \chi_{E_n}$

$$x \in \limsup_n E_n \quad \Leftrightarrow \quad \chi_Q(x) = 1$$

$$\Leftrightarrow \quad \forall k \in \mathbb{N} \, \exists n_k \geq k \, s.t. \, x \in E_{n_k}$$

If we fix $k$ then
$$\sup_{n \geq k} \chi_{E_n}(x) = \chi_{E_{n_k}}(x) = 1$$
$$\lim_{k \to \infty} \sup_{n \geq k} \chi_{E_n}(x) = \limsup_n \chi_{E_n}(x) = 1$$

Let now $x \notin \limsup E_n \Leftrightarrow \chi_Q(x) = 0$. Then $x$ belongs at most to finitely many $E_n \Rightarrow \exists \bar{k}$ s.t. $x \notin E_n, \forall n \geq \bar{k}$

If $k \geq \bar{k}$, then $\sup_{n \geq k} \chi_{E_n}(x) = 0 \Rightarrow \lim_{k \to \infty} \sup_{n \geq k} \chi_{E_n}(x) = \limsup_n \chi_{E_n}(x) = 0$

### Relations

Given $X, Y$ sets, is called a **relation** of $X$ and $Y$ a subset of $X \times Y$

$$R \subseteq X + Y \quad R = \{(x, y) : x \in X, y \in Y\}$$

$$(x, y) \in R \Leftrightarrow xRy$$

$$X = \{0, 1, 2, 3\} \quad R = \{(0, 1), (1, 2), (2, 1)\} \text{ is a relation in } X$$

**Definition 1.6**
A **function** from $X$ to $Y$ is a relation $R$ s.t. for any element $x$ of $X$ $\exists !$ element $y$ of $Y$ s.t. $xRy$

**Definition 1.7**
$R$ on $X$ is an **equivalence relation** if

(1) $xRx \; \forall \, x \in X$ ($R$ is **reflexive**)

(2) $xRy \Rightarrow yRx$ ($R$ is **symmetric**)

(3) $xRy, yRz \Rightarrow xRz$ ($R$ is **transitive**)

If $R$ is an equivalence relation, the set $E_X := \{y \in X \, : \, yRx\}, \; x \in X$ is called the **equivalence class** of $X$

**Definition 1.8**
$\frac{X}{R} := \{E_X \, : \, x \in X\}$ is the **quotient set**

Ex: $X = \mathbb{Z}$, let's say that $nRm$ if $n - m$ is even. This is an equivalence relation.

$$E_n = \{\ldots, n - 4, n - 2, n, n + 2, n + 4, \ldots\}$$

in this case if $n$ is even, $E_n = \{\text{even numbers}\}$ and if $n$ is odd, $E_n = \{\text{odd numbers}\}$

## Measure theory

**Definition 1.9**
A family $\mathcal{M} \subseteq \mathcal{P}(X)$ is called a $\sigma$**-algebra** if

(1) $X \in \mathcal{M}$

(2) $E \in \mathcal{M} \Rightarrow E^C = X \setminus E \in \mathcal{M}$

(3) If $E = \bigcup_{n \in \mathbb{N}}$ and $E_n \in \mathcal{M} \, \forall n$, then $E \in \mathcal{M}$

If $\mathcal{M}$ is a $\sigma$-algebra, $(X, \mathcal{M})$ is called **measurable space** and the sets in $\mathcal{M}$ are called **measurable**. Ex:

- $(X, \mathcal{P}(X))$ is a measurable space

- Let $X$ be a set, then $\{\varnothing, X\}$ is a $\sigma$-algebra

**Remark 1.1**

$\sigma$ is often used to denote the closure w.r.t. countably many operators. If we replace the countable unions with finite unions in the definition of $\sigma$-algebra, we obtain an **algebra**.

Some **basic properties** of a measurable space $(X, \mathcal{M})$:

(1) $\varnothing \in \mathcal{M}$: $\varnothing = X^C$ and $X \in \mathcal{M}$

(2) $\mathcal{M}$ is an algebra, and $E_1, \ldots, E_n \in \mathcal{M}$

$$E_1 \cup \ldots \cup E_n = E_1 \cup \ldots \cup E_n \cup \underbrace{\varnothing}_{\in \mathcal{M}} \cup \varnothing \ldots \in \mathcal{M}$$

(3) $E_n \in \mathcal{M}, \bigcap_{n \in \mathbb{N}} E_n \in \mathcal{M}$

$$\bigcap_{n \in \mathbb{N}} E_n = \underbrace{\left( \bigcup_{n \in \mathbb{N}} \underbrace{E_n^C}_{\in \mathcal{M}} \right)^C}_{\in \mathcal{M}} \qquad (\mathcal{M} \text{ is also closed under finite intersection})$$

- $E, F \in \mathcal{M} \Rightarrow E \setminus F \in \mathcal{M} = E \setminus F = E \cap F^C \in \mathcal{M}$

- If $\Omega \subset X$, then the **restriction** of $\mathcal{M}$ to $\Omega$, written as

$$\mathcal{M}|_\Omega := \{F \subseteq \Omega : F = E \cap \Omega, \text{ with } E \in \mathcal{M}\}$$

  is a $\sigma$-algebra on $\Omega$

**Theorem 1.1**

$\mathcal{S} \subseteq \mathcal{P}(X)$. Then it is well defined the smallest $\sigma$-algebra containing $\mathcal{S}$, the $\sigma$-algebra generated by $\mathcal{S} := \sigma_0(\mathcal{S})$:

- $\mathcal{S} \subseteq \sigma_0(\mathcal{S})$ and thus is a $\sigma$-algebra

- $\forall \sigma(\mathcal{M})$ s.t. $\mathcal{M} \supseteq \mathcal{S}$, we have $\mathcal{M} \supseteq \sigma_0(\mathcal{S})$

*Proof idea.*

$$\mathcal{V} = \{\mathcal{M} \subseteq \mathcal{P}(X) : \mathcal{M} \text{ is a } \sigma\text{-algebra and } \mathcal{S} \subseteq \mathcal{M}\} \neq \varnothing \text{ since } \mathcal{P}(X) \in \mathcal{V}$$

We define $\sigma_0(\mathcal{S}) = \bigcap \{\mathcal{M} : \mathcal{M} \in \mathcal{V}\}$, so it can be proved that this is the desired $\sigma$-algebra $\quad\bigstar$

**Borel sets**

Given $(X, d)$ metric space, the $\sigma$-algebra generated by the open sets is called **Borel** $\sigma$-algebra, written as $\mathcal{B}(X)$. The sets in $\mathcal{B}(X)$ are called **Borel sets**. The following sets are Borel sets:

- open sets

- closed sets

- countable intersections of open sets: $G_\sigma$ sets

- countable unions of closed sets: $F_\sigma$ sets

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**Remark 1.2**
$\mathcal{B}(\mathbb{R})$ can be equivalently defined as the $\sigma$-algebra generated by

$$\{(a,b) : \ a, b \in \mathbb{R}, a < b\}$$

$$\{(-\infty, b) : \ b \in \mathbb{R}\}$$

$$\{(a, +\infty) : \ a \in \mathbb{R}\}$$

$$\{[a,b) : \ a, b \in \mathbb{R}, a < b\}$$

$$\vdots$$

# 2 Lecture 14/09/2022

<u>Question</u>: What is $\mathcal{B}(\mathbb{R})$? Is $\mathcal{B}(\mathbb{R}) \neq \mathcal{P}(\mathbb{R})$? No.

**Definition 2.1**
$(X, \mathcal{M})$ measurable space. A function $\mu : \ \mathcal{M} \to [0, +\infty]$ is called a **positive measure** if $\mu(\varnothing) = 0$ and if $\mu$ is countably additive, that is

$$\forall \{E_n\} \subseteq \mathcal{M} \quad \text{disjoint}$$

we have that

$$\mu\left(\bigcup_{n=1}^{\infty}\right) = \sum_{n=1}^{\infty} \mu(E_n) \qquad\qquad \sigma\text{-additivity}$$

**Remark 2.1**
a set $A$ is countable if $\exists f \ 1 - 1$ s.t. $f : A \to \mathbb{N}$ Examples: $\mathbb{Z}, \mathbb{Q}$ are countable, while $\mathbb{R}$ is not, also $(0,1)$ is uncountable.

We always assume that $\exists E \neq \varnothing, E \in \mathcal{M}$ s.t. $\mu(E) \neq \infty$.

If $(X, \mathcal{M})$ is a measurable space, and $\mu$ is a measure on it, then $(X, \mathcal{M}, \mu)$ is a measure space.

Then:

(1) $\mu$ is **finitely additive**:

$$\forall E, F \in \mathcal{M}, \text{ with } E \cap F \neq \varnothing \Rightarrow \mu(E \cup F) = \mu(E) + \mu(F)$$

(2) the **excision property**

$$\forall E, f \in \mathcal{M}, \text{ with } E \subset F \text{ and } \mu(E) < +\infty \Rightarrow \mu(F \setminus E) = \mu(F) - \mu(E)$$

(3) **monotonicity**
$$\forall E, F \in \mathcal{M}, \text{ with } E \subset F \Rightarrow \mu(E) \leq \mu(F)$$

(4) if $\Omega \in \mathcal{M}$ then $(\Omega, \mathcal{M}|_\Omega, \mu|_{\mathcal{M}|_\Omega})$ is a measure space

***Proof.*** (1) $E_1 = E, E_2 = F, E_3 = \ldots = E_n = \ldots = \varnothing$ This is a disjoint sequence $\Rightarrow$ by $\sigma$-additivity.

$$\mu(E \cup F) = \mu\left(\bigcup_n E_n\right) = \sum_n \mu(E_n) = \mu(E) + \mu(F) + \underbrace{\mu(E_k)}_{=\mu(\varnothing)}$$

(2) $E \subset F$, so $F = E \cup (F \setminus E)$ and this is disjoint $\overset{(i)}{\Rightarrow} \mu(F) = \mu(E) + \mu(F \setminus E)$, and since $\mu(E) < \infty$, the property follows.

(3) $E \subset F \Rightarrow \mu(F) = \mu(E) + \underbrace{\mu(F \setminus E)}_{\geq 0} \geq \mu(E)$

(4)

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## Definition 2.2
$(X, \mathcal{M}, \mu)$ measure space.

- If $\mu(X) < +\infty$, we say that $\mu$ is **finite**.

- If $\mu(X) = +\infty$, and $\exists \ \{E_n\} \subset \mathcal{M}$ s.t. $X = \bigcup_n E_n$ and each $E_n$ has finite measure, then we say that $\mu$ is $\sigma$-finite.

- If $\mu(X) = 1$ we say that $\mu$ is a **probability measure**.

Some examples:

- Trivial Measure: $(X, \mathcal{M})$ measurable space. $\mu : \mathcal{M} \to [0, \infty]$ defined by $\mu(E) = 0 \quad \forall \, E \in \mathcal{M}$

- Counting Measure: $(X, \mathcal{P}(X))$ measurable space. We define

$$\mu_C : \mathcal{P}(X) \to [0, \infty], \quad \mu_C(E) = \begin{cases} n & \text{if } E \text{ has } n \text{ elements} \\ \infty & \text{if } E \text{ has } \infty\text{-many elements} \end{cases}$$

- Dirac Measure: $(X, \mathcal{P}(X))$ measurable space, $t \in X$. We define

$$\delta_t : \mathcal{P}(X) \to [0, +\infty], \quad \delta_t(E) = \begin{cases} 1 & \text{if } t \in E \\ 0 & \text{otherwise} \end{cases}$$

Continuity of the measure along monotone sequences
$(X, \mathcal{M}, \mu)$ measure space

(1) $\{E_i\} \subset \mathcal{M}$, $E_i \subseteq E_{i+1} \ \forall i$ and let

$$E = \bigcup_{i=1}^{\infty} E_i = \lim_i E_i$$

Then:
$$\mu(E) = \lim_i \mu(E_i)$$

(2) $\{E_i\} \subset \mathcal{M}$, $E_{i+1} \subseteq E_i \ \forall i$ and let $E = \bigcap_{i=1}^{\infty} E_i = \lim_i E_i$.

**Proof.** (1) if $\exists \ i$ s.t. $\mu(E_i) = +\infty$, then is trivial. Assume then that every $E_i$ has a finite measure, so that $E = \bigcup_{i=1}^{\infty} E_i = \bigcup_{i=0}^{\infty}(E_{i+1} \setminus E_i)$ with $E_0 = \varnothing$.

So, by $\sigma$-additivity
$$\mu(E) = \mu\left(\bigcup_{i=0}^{\infty}(E_{i+1} \setminus E_i)\right) =$$

$$= \sum_{i=0}^{\infty} \mu(E_{i+1} \setminus E_i) \overset{(excision)}{=} \sum_{i=0}^{\infty} \left( \mu(E_{i+1} - \mu(E_i)) \right) =$$

$$\overset{(telescopic\ series)}{=} \lim_n \mu(E_n) - \underbrace{\mu(E_0)}_{=0} = \lim_n \mu(E_n)$$

(2) For simplicity, suppose $\tau = 1$, and define $F_k = E_i \setminus E_k$

$$\{E_k\} \searrow \Rightarrow \{F_k\} \nearrow$$

$$\mu(E_i) = \mu(E_k) + \mu(F_k) \text{ and } \bigcup_k F_k = E_i \setminus \left( \bigcap_k E_k \right)$$

$$\mu(E_i) = \mu(\bigcup_k F_k) + \underbrace{\mu(\bigcap_k E_k)}_{\mu(E)} =$$

$$\overset{(i)}{=} \lim_k \mu(F_k) + \mu(E) = \lim_k \left( \mu(E_i) - \mu(E_k) \right) + \mu(E)$$

Since $\mu(E_i) < \infty$ we can subtract it from both sides

$$0 = -\lim_k \mu(E_k) + \mu(E)$$

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Counterexample: given $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu_C)$ measure space. Let $E_n = \{n, n+1, n+2, \ldots\}$. In this case $\mu_C(E_n) = +\infty$, $E_{n+1} \subseteq E_n \forall\, n$, but $\bigcap_n E_n = \varnothing \Rightarrow \mu\left( \bigcap_n E_n \right) = 0$

**Theorem 2.1** ($\sigma$-subadditivity of the measure)
$(X, \mathcal{M}, \mu)$ is a measure space. $\forall \{E_n\} \subseteq \mathcal{M}$ (not necessarily disjoint): $\mu\left( \bigcup_n E_n \right) \leq \sum_n \mu(E_n)$

***Proof.*** $E_1, E_2 \in \mathcal{M}$ and also $E_1 \cup E_2 = E_1 \cup (E_2 \setminus E_1)$ disjoint sets.

$$\mu(E_1 \cup E_2) = \mu(\underbrace{E_2 \setminus E_1}_{\subseteq E_2}) \overset{(monotonicity)}{\leq} \mu(E_1) + \mu(E_2)$$

that means that we have the subadditivity for finitely many sets.

$$A = \bigcup_{n=1}^{\infty} E_n, \quad A_k = \bigcup_{n=1}^{k} E_n$$

$$\{A_k\} \nearrow,\ A_{k+1} \supseteq A_k,\ \lim_k A_k = A$$

$$\mu\left( \bigcup_{n=1}^{\infty} E_n \right) \overset{(continuity)}{=} \lim_k \mu(A_k) = \lim_k \mu\left( \bigcup_{n=1}^{k} E_n \right) \leq$$

$$\leq \lim_k \sum_{n=1}^{k} \mu(E_n) = \sum_{n=1}^{\infty} \mu(E_n)$$

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Exercise: $(X, \mathcal{M})$ measurable space. $\mu : \mathcal{M} \to [0, +\infty]$ s.t. $\mu$ is finitely additive, $\sigma$-subadditive and $\mu(\varnothing) = 0 \Rightarrow \mu$ is $\sigma$-additive, and hence is a measure.

Exercise: the Borel-Cantelli lemma states that, given $(X, \mathcal{M}, \mu)$ and $\{E_n\} \subseteq \mathcal{M}$. Then

$$\sum_{n=0}^{\infty} \mu(E_n) < \infty \Rightarrow \mu(\limsup_n E_n) = 0$$

It can be phrased as:

> If the series of the probability of the events $E_n$ is convergent, then the probability that $\infty$-many events occur is $0$

***Proof.*** The thesis is:

$$\mu(\limsup_n E_n) = \mu\left(\bigcap_{n=1}^{\infty} \underbrace{\bigcup_{k \geq n} E_k}_{A_n := \bigcup_{k \geq n} E_k}\right)$$

Is it true that $\{A_n\} \searrow$? Yes.

$$A_{n+1} = \bigcup_{k \geq n+1} E_k \subseteq \bigcup_{k \geq n} E_k = A_n$$

Does some $A_n$ have a finite measure?

$$\mu(A_n) = \mu\left(\bigcup_{k \geq n} E_k\right) \leq \sum_{k \geq n} \mu(E_k) < \infty$$

by assumption. Therefore, we can use the continuity along decreasing sequences:

$$\mu(\limsup_n E_n) = \lim_n \mu(A_n) = \lim_n \mu\left(\bigcup_{k \geq n} E_k\right) \overset{\sigma-sub.}{\leq} \lim_n \sum_{k=n}^{\infty} \mu(E_k) = 0$$

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**Sets of $0$ measure**

$(X, \mathcal{M}, \mu)$ measure space.

- $N \subseteq X$ is a set of 0 measure if $N \in \mathcal{M}$ and $\mu(N) = 0$

- $E \subseteq X$ is called **negligible set** if $\exists\, N \in \mathcal{M}$ with 0 measure s.t. $E \subseteq N$ ($E$ does not necessarily stays in $\mathcal{M}$)

**Definition 2.3**
$(X, \mathcal{M}, \mu)$ measure space s.t. every negligible set is measurable (and hence of 0 measure), then $(X, \mathcal{M}, \mu)$ is said to be a **complete measure space**.

A measure space may not be complete. However, let

$$\overline{\mathcal{M}} := \{E \subseteq X : \exists\, F, G \in \mathcal{M} \text{ with } F \subseteq E \subseteq G \text{ and } \mu(G \setminus F) = 0\}$$

Clearly $\mathcal{M} \subseteq \overline{\mathcal{M}}$. For $E \in \overline{\mathcal{M}}$, take $F$ and $G$ as above and let $\bar{\mu}(E) = \bar{\mu}(F)$ then $\bar{\mu}|_{\mathcal{M}} = \mu$, and moreover:

**Theorem 2.2**
$(X, \mathcal{M}, \mu)$ is a complete measure space. Let's observe that $\bar{\mu}$ is well defined: let $E \subseteq X$ and $F_1, F_2, G_1, G_2$ s.t. $F_i \subset E \subset G_i \quad i = 1, 2$. Then $\mu(G_i \setminus F_i) = 0$. Now we have to check that $\mu(F_1) = \mu(F_2)$.

Since
$$F_1 \setminus F_2 \subseteq E \setminus F_2 \subseteq G_2 \setminus F_2$$

and $G_2 \setminus F_2$ has $0$ measure $\Rightarrow \mu(F_1 \setminus F_2) = 0$. Then $F_1 = (F_1 \setminus F_2) \cup (F_1 \cap F_2) \Rightarrow \mu(F_1) = \mu(F_1 \cap F_2)$. In the same way, $\mu(F_2) = \mu(F_1 \cap F_2)$

# 3  Lecture 15/09/2022

The elements of $\overline{\mathcal{M}}$ are sets of the type $E \cup N$, with $E \in \mathcal{M}$ and $\bar{\mu}(N) = 0$.

**Outer measure**

We wish to define a measure $\lambda$ "on $\mathbb{R}$" with the following properties:

(1) $\lambda((a, b)) = b - a$

(2) $\lambda(E + t)^\dagger = \lambda(E)$ for every measurable set $E \subset \mathbb{R}$ and $t \in \mathbb{R}$

It would be nice to define such a measure on $\mathcal{P}(\mathbb{R})$. In such case, note that $\lambda(\{x\}) = 0, \forall x \in \mathbb{R}$ But then

**Theorem 3.1** (Ulam)
The only measure on $\mathcal{P}(\mathbb{R})$ s.t. $\lambda(\{x\}) = 0 \quad \forall x$ is the trivial measure. Thus, a measure satisfying the two properties of the outer measure cannot be defined on $\mathcal{P}(\mathbb{R})$

We'll learn in what follows how to create a measure space on $\mathbb{R}$, with a $\sigma$-algebra including all the Borel sets, and a measure satisfying properties of the outer measure. This is the so called **Lebesgue measure**.

**Definition 3.1**
Given a set $X$. An **outer measure** is a function $\mu^* : \mathcal{P}(\mathbb{R}) \to [0, +\infty]$ s.t.

- $\mu^*(\varnothing) = 0$

- $\mu^*(A) \leq \mu^*(B)$ if $A \subseteq B$ (Monotonicity)

- $\mu^*(\bigcup_{n=1}^\infty E_n) \leq \sum_{n=1}^\infty \mu^*(E_n)$ ($\sigma$-subadditivity)

The common way to define an outer measure is to start with a family of elementary sets $\mathcal{E}$ on which a notion of measure is defined (e.g. intervals on $\mathbb{R}$, rectangles on $\mathbb{R}^2, \ldots$) and then to approximate arbitrary sets from outside by **countable** unions of members of $\mathcal{E}$.

**Proposition 3.1**
Let $\mathcal{E} \subset \mathcal{P}(\mathbb{R})$ and $\rho : \mathcal{E} \to [0, +\infty]$ be such that $\varnothing \in \mathcal{E}, X \in \mathcal{E}$ and $\rho(\varnothing) = 0$. For any $A \in \mathcal{P}(X)$, let

$$\mu^*(A) := \inf \left\{ \sum_{n=1}^\infty \rho(E_n) : E_n \in \mathcal{E} \text{ and } A \subset \bigcup_{n=1}^\infty E_n \right\}$$

Then $\mu^*$ is an outer measure, the outer measure generated by $(\mathcal{E}, \rho)$.

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$^\dagger \{x \in \mathbb{R} : x = y + t, \text{ with } y \in E\}$

**Proof.** $\forall\, A \subset X \,\exists\, \{E_n\} \subset \mathcal{E}$ s.t. $A \subset \bigcup_n E_n$ : take $E_n = X\,\forall\, n$ then $\mu^*$ is well defined. Obviously, $\mu^*(\varnothing) = 0$ (with $E_n = \varnothing \quad \forall\, n$), and $\mu^*(A) \leq \mu^*(B)$ for $A \subset B$ (any covering of $B$ with elements of $\mathcal{E}$ is also a covering of $A$.)

We have to prove the $\sigma$-subadditivity. Let $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{P}(X)$ and $\varepsilon > 0$. For each $n, \exists \left\{ E_{n_j} \right\}_{j\in\mathbb{N}} \in \mathcal{E}$ s.t. $A_n \subset \bigcup_{i=1}^{\infty} E_{n_j}$ and $\sum_{j=1}^{\infty} \rho(E_{n_j}) \leq \mu^*(A_n) + \frac{\varepsilon}{2^n}$. But then, if $A = \bigcup_{n=1}^{\infty} A_n$, we have that $A \subset \bigcup_{n,j\in\mathbb{N}^2} E_{n_j}$ and

$$\mu^*(A) \leq \sum_{n,j} \rho(E_{n_j}) \leq \sum_{n} \left( \mu^*(A_n) + \frac{\varepsilon}{2^n} \right) = \sum_{n} \mu^*(A_n) + \varepsilon$$

Since $\varepsilon$ is arbitrary, we are done. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ ★

Ex:

(1) $X \in \mathbb{R}, \mathcal{E} = \{(a,b) : a \leq b, a, b \in \mathbb{R}\}$ family of open intervals:

$$\rho((a,b)) = b - a$$

(2) $X = \mathbb{R}^n, \mathcal{E} = \{(a_1, b_1) \times \ldots \times (a_n, b_n) : a_i \leq b_i, a_i, b_i \in \mathbb{R}\}$:

$$\rho((a_1, b_1) \times \ldots \times (a_n, b_n)) = (b_1 - a_1) \cdot \ldots \cdot (b_n - a_n)$$

**Remark 3.1**

$E \in \mathcal{E} \Rightarrow \mu^*(E) = \rho(E)$.

In examples 1 and 2, we have in fact $\mu^*((a,b)) = b - a, \mu^*((a_1, b_1) \times \ldots \times (a_n, b_n)) = \prod_{i=1}^{n}(b_i - a_i)$

To pass from the outer measure to a measure there is a condition

**Definition 3.2** (Caratheodory condition)

If $\mu^*$ is an outer measure on $X$, a set $A \subset X$ is called $\mu^*$-**measurable** if

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^C) \quad \forall\, E \subset X$$

**Remark 3.2**

If $E$ is a "nice" set containing $A$, then the above equality says that the outer measure of $A$, $\mu^*(E \cap A)$, is equal to $\mu^*(E) - \mu^*(E \cap A^C)$, which can be thought as an "inner measure". So basically we are saying that $A$ is measurable if the outer and inner measure coincide. (Like the definition of Riemann integration with lower and upper sum)

**Remark 3.3**

$\mu^*$ is subadditive by def $\Rightarrow \mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^C) \quad \forall\, E, A \subset X$. So, to prove that a set is $\mu^*$-measurable it is enough to prove the reverse inequality, $\forall\, E \subset X$. In fact, if $\mu^*(E) = +\infty$, then $+\infty \geq \mu^*(E \cap A) + \mu^*(E \cap A^C)$, and hence $A$ is $\mu^*$-measurable iff

$$\mu^*(E) \geq \mu(E \cap A) + \mu^*(E \cap A^C) \quad \forall\, E \subset X \text{ with } \mu^*(E) < +\infty$$

Their relevance to the notion of $\mu^*$-measurability is clarified by the following

**Theorem 3.2** (Caratheodory)

If $\mu^*$ is an outer measure on $X$, the family

$$\mathcal{M} = \{A \subseteq X : A \text{ is } \mu^*\text{-measurable}\}$$

is a $\sigma$-algebra and $\mu^*|_{\mathcal{M}}$ is a complete measure.

**Lemma 3.1**
If $A \subset X$ and $\mu^*(A) = 0$, then $A$ is $\mu^*$-measurable.

***Proof.*** Let $E \subset X$ with $\mu^*(E) < +\infty$. Then

$$\mu^*(E) \geq \mu^*(E) + \mu^*(A) \overset{\ddagger}{\geq} \mu^*(E \cap A) + \mu^*(E \cap A^C)$$

This implies that A is $\mu^*$-measurable. ★

To sum up: $X$ set, $(\mathcal{E}, \rho)$ elementary and measurable sets, so $\mu^*$ is an outer measure. Then given $\mu^*$ and the Caratheodory condition, we have $(X, \mathcal{M}, \mu)$ that is a complete measure space.

**Remark 3.4**
So far we did not prove that $\mathcal{E} \subseteq \mathcal{M}$. We will do it in a particular case.

**Lebesgue measure**

- $X = \mathbb{R}$, $\mathcal{E}$ family of open intervals, $\rho((a,b)) = b - a = \lambda((a,b))$, the complete measure space is $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$ with $\mathcal{L}(\mathbb{R})$ the Lebesgue-measurable sets on $\mathbb{R}$ and $\lambda$ the Lebesgue measure on $\mathbb{R}$.

- $X = \mathbb{R}^n$, $\mathcal{E} = \{\prod_{k=1}^{n} (a_k, b_k) : a_k \leq b_k \quad \forall\, k = 1, \ldots, n\}$, $\rho\left(\prod_{k=1}^{n}(a_k, b_k)\right) = \prod_{k=1}^{n}(b_k - a_k)$ and this is a complete measure space $(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^n), \lambda_n)$

# 4 Lecture 21/09/2022

**Lebesgue measure**

$\mathcal{E} =$ family of open intervals (a,b), $a, b \in \mathbb{R}^*, a < b$. $\rho =$ lenght $l$. $\rho((a,b)) = b - a$.
<u>Notations:</u> open interval $I$ with lenght $l(I)$

**Outer measure**

$E \subset \mathbb{R}$. The outer measure of $E$ is

$$\lambda^*(E) = \inf \left\{ \sum_{n=1}^{+\infty} l(I_n) \,|\, I_n \text{ is an open interval}, E \subset \bigcup_{n=1}^{\infty} I_n \right\}$$

**Caratheodory condition (CC)**

$A \subset \mathbb{R}$ is $\lambda^*$-measurable if

$$\lambda^*(E) = \lambda^*(E \cap A) + \lambda^*(E \cap A^C) \qquad \forall\, E \subset \mathbb{R}$$

$$\{A \subset \mathbb{R} : A \text{ is } \lambda^*\text{-measurable}\} =: \mathcal{L}(\mathbb{R}) \qquad\qquad \text{(Lebesgue } \sigma\text{-algebra)}$$

$$\lambda := \lambda^*|_{\mathcal{L}(\mathbb{R})} \qquad\qquad \text{(Lebesgue measure on } \mathbb{R})$$

Then, $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$ is a complete measure space. In particular, $\lambda^*(A) = 0 \Rightarrow A \in \mathcal{L}(\mathbb{R})$ and $\lambda(A) = 0$.

---

$\ddagger E \cap A^C \subseteq E$ and $E \cap A \subseteq A$ + monotonicity

**Remark 4.1** (CC-Criterion for measurability)

To check that $A$ is $\lambda^*$-measurable, it is sufficient to check that

$$\lambda^* \geq \lambda^*(E \cap A) + \lambda^*(E \cap A^C)$$

for every $E \subset \mathbb{R}$ rith $\lambda^*(E) < +\infty$

**Proposition 4.1**

Any countable set is measurable, with 0 Lebesgue measure.

**Proof.** Let $a \in \mathbb{R}$,

$$\{a\} \subseteq (a - \varepsilon, a + \varepsilon), \forall\, \varepsilon > 0 \overset{\text{by def.}}{\Rightarrow} \lambda^*(\{a\}) \leq 2\varepsilon \overset{\lim}{\Rightarrow} \lambda^*(\{a\}) = 0$$

$\{a\}$ is measurable with $\lambda(\{a\}) = 0, \forall\, a \in \mathbb{R}$. Now if a set $A$ is countable, $A = \{a_n\}_{n \in \mathbb{N}} = \bigcup_n \{a_n\}$ (disjoint) $\Rightarrow \lambda(A) \underset{\sigma-add}{=} \sum_n \lambda(\{a_n\}) = 0$ &#9733;

**Remark 4.2**

$\lambda(\mathbb{Q} = 0)$. $\mathbb{Q}$ is dense on $\mathbb{R}$, $\bar{\mathbb{Q}} = \mathbb{R}$. In general, measure theoretical info and topological info cannot be compared.

**Proposition 4.2**

$\mathcal{B}(\mathbb{R}) \subseteq \mathcal{L}(\mathbb{R})$

**Remark 4.3**

So far we didn't prove the fact that open intervals are $\mathcal{L}$-measurable.

**Proof.** We know that $\mathcal{B}(\mathbb{R})$ is generated by $\{(a, +\infty) : a \in \mathbb{R}\}$. Then, we can directly show that $(a, +\infty) \in \mathcal{L}(\mathbb{R}) \quad \forall\, a \in \mathbb{R}$. Let $a \in \mathbb{R}$ be fixed. We use the criterion for measurability and we check that

$$\lambda^*(E) \geq \lambda^* \underbrace{(E \cap (a, +\infty))}_{=:E_1} + \lambda^* \underbrace{(E \cap (-\infty, a])}_{=:E_2} \quad \forall\, E \subset \mathbb{R}, \lambda^* < +\infty$$

Since $\lambda^*(E) < +\infty$, $\exists$ a countable union $\bigcup_n I_n \supset E$, where $I_n$ is an open interval $\forall\, n$ and

$$\sum_n l(I_n) \leq \lambda^*(E) + \varepsilon$$

Let $I_n^1 := I_n \cap E_1, I_n^2 := I_n \cap (-\infty, a + \frac{\varepsilon}{2^n})$. These are open intervals:

$$E_1 \subset \bigcup_n I_n^1 \qquad E_2 \subset_n I_n^2 \qquad\qquad \text{countable unions}$$

and moreover

$$l(I_n) \geq l(I_n^1) + l(I_n^2) - \frac{\varepsilon}{2^n}$$

By definition of $\lambda^*$, $\lambda^*(E_1) \leq \sum_n l(I_n^1)$ and $\lambda^*(E_2) \leq \sum_n l(I_n^2)$, therefore

$$\lambda^*(E_1) + \lambda^*(E_2) \leq \sum_n l(I_n^1) + \sum_n l(I_n^2) \leq \sum_n \left( l(I_n) + \frac{\varepsilon}{2^n} \right) = \left( \sum_n l(I_n) \right) + \varepsilon \leq \lambda^*(E) + 2\varepsilon$$

Since $\varepsilon$ was arbitrarily chosen, we have

$$\lambda^*(E) \geq \lambda^*(E_1) + \lambda^*(E_2)$$

which is the thesis. &#9733;

So, the Lebesgue measure measures all the open, closed $G_\delta$, $F_\delta$ sets. Clearly

$$\lambda((a,b)) = b - a$$

One can also show that $\lambda$ is invariant under translation.

Questions: $\mathcal{B}(\mathbb{R}) \subseteq \mathcal{L}(\mathbb{R}) \subseteq \mathcal{P}(\mathbb{R})$, is it a strict inclusion or not?

- By Ulam's theorem, if a measure is such that $\lambda(\{a\}) = 0, \forall\, a$ and all the sets in $\mathcal{P}(\mathbb{R})$ are measurable, then $\lambda \equiv 0$. This and the fact that $\lambda((a,b)) \neq 0$ simply that $\mathcal{L}(\mathbb{R}) \subsetneq [‡]\mathcal{P}(\mathbb{R})$ : $\exists$ non-measurable sets called Vitali sets. Every measurable set with positive measure contains a Vitali set. (Explanation)

- $\mathcal{B}(\mathbb{R}) \subsetneq \mathcal{L}(\mathbb{R})$. The construction of a $\mathcal{L}$-measurable se which is not a Borel set will be done during exercise classes.

The relation between $\mathcal{B}(\mathbb{R})$ and $\mathcal{L}(\mathbb{R})$ is clarified by

**Theorem 4.1** (Regularity of $\lambda$)

The following sentences are equivalent:

(1) $E \in \mathcal{L}(\mathbb{R})$

(2) $\forall\, \varepsilon > 0 \exists\, A \supset E$, $A$ open s.t.
$$\lambda(A \setminus E) < \varepsilon$$

(3) $\exists\, G \supset E$, $G$ of class $G_\delta$, s.t.
$$\lambda(G \setminus E) = 0$$

(4) $\exists\, C \subset E$, $C$ closed, s.t.
$$\lambda(E \setminus C) = 0$$

(5) $\exists\, F \subset E$, $F$ of class $F_\delta$, s.t.
$$\lambda(E \setminus F) = 0$$

__Consequence:__ $E \in \mathcal{L}(\mathbb{R}) \Rightarrow E = F \cup N$, where $F$ is of class $F_\delta$, and $\lambda(N) = 0$.

*Partial proof.* For simplicity, we will consider only sets with finite measure.

(1) $\Rightarrow$ (2) $E \in \mathcal{L}(\mathbb{R})$. By definition of $\lambda^*$, $\forall\, \varepsilon > 0 \exists\, \bigcup_n I_n \supset E$ s.t. each $I_n$ is an open interval, and
$$\lambda(E) = \lambda^*(E) \geq \sum_n l(I_n) - \varepsilon$$

We define $A = \bigcup_n I_n$, which is open. Also $A \supset E$ and

$$\lambda(A) = \lambda\left(\bigcup_n I_n\right) \overset{\sigma-\text{sub.}}{\leq} \sum_n l(I_n) \leq \lambda(E) + \varepsilon$$

Then, by excision
$$\lambda(A \setminus E) = \lambda(A) - \lambda(E) \leq \varepsilon$$

(2) $\Rightarrow$ (3) Define, for every $K \in \mathbb{N}$, an open set $A_k$ s.t. $A_k \supset E$ and $\lambda(A_k \setminus E) < \frac{1}{k}$. Let $A = \bigcap_k A_k$. This is a $G_\delta$ set, it contains $E$ (since each $A_k$ contains $E$) and

$$\lambda(A \setminus E) \underset{(A \subset A_k\, \forall\, k)}{\leq} \lambda(A_k \setminus E) < \frac{1}{k} \Rightarrow \lambda(A \setminus E) = 0 \quad \forall\, k$$

---
[‡]I had no choice

(3) $\Rightarrow$ (1)] If $E \subset \mathbb{R}$ and $\exists\, G \supset E$, with $G$ of class $G_\delta$, s.t. $\lambda(G \setminus E) = 0$, then

$$E = G \setminus (G \setminus E) \text{ is measurable}$$

since $G$ is a Borel set and $(G \setminus E)$ has 0 measure, then both are in $\mathcal{L}$

$\bigstar$

**Remark 4.4**

Any countable set has 0 measure. he inverse is false. An example is given by the **Cantor set**.

Let $T_0 = [0, 1]$. Then we define $T_{n+1}$ stating from $T_n$ in the following way: given $T_n$, finite union of closed disjoint intervals of lenght $l_n (\frac{1}{3})^n$, $T_{n+1}$ is obtained by removing from each interval of $T_n$, the open central subinterval of lenght $\frac{l_n}{3}$.

The Cantor set is $T := \bigcap_{k=0}^{+\infty}$. It can be proved that $T$ is compact, $\lambda(T) = 0$ and $T$ is uncountable.

If, instead of removing intervals of size $\frac{1}{3}, \frac{1}{9}, \ldots, \frac{1}{3^k}$, we remove sets of size $\left(\frac{\varepsilon}{3}\right)^k$, with $\varepsilon \in (0, 1)$, we obtain the **generalized Cantor set** (or **fat Cantor set**) $T_\varepsilon$. $T_\varepsilon$ is uncountable, compact and has no interior points (it contains no intervals). However, $\lambda(T_\varepsilon) = \frac{3(1-\varepsilon)}{3-2\varepsilon} > 0$

**Remark 4.5**

We worked on $\mathbb{R}$, but everything can be adapted to $\mathbb{R}^n$

**Measurable functions and integration**

**Definition 4.1**

$f : X \to Y$, then it is well defined the counterimage

$$f^{-1} : \mathcal{P}(Y) \to \mathcal{P}(Y)$$

$$E \to f^{-1}(E) = \{x \in X : f(x) \in E\}$$

**Definition 4.2**

$(X, \mathcal{M}), (Y, \mathcal{N})$ measurable spaces. $f : X \to Y$ is called **measurable** or $(\mathcal{M}, \mathcal{N})$-measurable if

$$f^{-1}(E) \in \mathcal{M} \text{ for every } E \in \mathcal{M}$$

so, the counterimage of measurable sets in $Y$ is a measurable set on $X$.

# 5 Lecture 22/09/2022

To check if a function is measurable or not, it is often sed the following proposition

**Proposition 5.1**

$(X, \mathcal{M}), (Y, \mathcal{N})$ measurable spaces. Let $\mathcal{F} \subseteq \mathcal{P}(Y)$ be s.t. $\mathcal{N} = \sigma_0(\mathcal{F})$. Then

$$f : X \to Y \text{ is } (\mathcal{M}, \mathcal{N}) - \text{measurable} \Leftrightarrow f^{-1}(E) \in \mathcal{M} \text{ for every } E \in \mathcal{F}$$

We will mainly focus on 2 situations:

(1) $((X, \mathcal{M}))$ is a measurable space obtained by means of an outer measure.
Ex: $(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^n))$, $(Y, d_y)$ metric space $\to (Y, \mathcal{B}(Y))$.

If $X \to Y$ is (Lebesgue) measurable $\Leftrightarrow (\mathcal{M}, \mathcal{B}(Y))$ is measurable

(2) $(X, d_X), (Y, d_Y)$ are metric spaces $\longrightarrow (X, \mathcal{B}(X)), (Y, \mathcal{B}(Y))$
$f : X \to Y$ is Borel measurable $\Leftrightarrow (\mathcal{M}(X), \mathcal{B}(Y))$-measurable.

**Remark 5.1**
$f$ is Lebesgue measurable if the continuity of the Borel set is a Lebesgue-measurable set.

**Proposition 5.2**
There are two parts:

(1) $(X, d_X), (Y, d_Y)$ metric spaces. If $f : X \to Y$ is continuous, then is Borel measurable

(2) $(Y, d_Y)$ metric space. If $f : \mathbb{R}^n \to Y$ is continuous, then it is a Lebesgue measure.

*Proof.* The proof is divided in:

(1) $f$ is continuous $\Leftrightarrow f^{-1}(A)$ is open $\forall A \subset Y$ open $\Rightarrow f^{-1}(A) \in \mathcal{B}(Y) \ \forall A \subset Y$ open Since $\mathcal{B}(Y) = \sigma_0$ (open sets) by proposition (1) this implies that $f$ is Borel measurable

(2) $f$ is continuous $\overset{(1)}{\Rightarrow} f$ is Borel measurable. $f^{-1}(A) \in \mathcal{B}(\mathbb{R}^n) \subseteq \mathcal{L}(\mathbb{R}^n) \forall A \in \mathcal{B}(Y)$. Namely $f$ is Lebesgue measurable

$\bigstar$

**Proposition 5.3**
$(X, \mathcal{M})$ measurable space, $(X, d_Y), (Y, d_Y)$ metric spaces. If $f : X \to Y$ is $(\mathcal{M}, \mathcal{B}(Y))$-measurable and $g : Y \to Z$ is continuous $\Rightarrow g \circ f : x \to Z$ is $(\mathcal{M}, \mathcal{B}(Y))$-measurable

**Proposition 5.4**
$(X, \mathcal{M})$ measurable space , $u, v : X \to \mathbb{R}$ measurable functions. Let $\Phi : \mathbb{R}^2 \to Y$ be continuous where $(Y, d_Y)$ is a metric space. Then $h : X \to Y$ defined by $h(x) = \Phi(u(x), v(x))$ is $(\mathcal{M}, \mathcal{B}(Y))$-measurable.

Consequence: $u, v$ measurable $\Rightarrow u + v$ is measurable.

*Proof.* Define $f : X \to \mathbb{R}^2$, $f(x) = u(x), v(x)$. By definition $h = \Phi \circ f$ by proposition (3) if we show that $f$ is $(\mathcal{M}, \mathcal{B}(\mathbb{R}^2))$-measurable, then $h$ is measurable. It can be proved that

$$\mathcal{B}(\mathbb{R}^2) = \sigma_0(\{\underbrace{(a_1, b_1) \times (a_2, b_2)}_{\text{open rectangle}} : a, b \in \mathbb{R}\})$$

Thanks to proposition (1), to check that $f$ is measurable. We can simply check that $f^{-1}(\mathbb{R} \in \mathcal{M})$ $\forall$ open rectangle in $\mathbb{R}^2$ and $R = I \times J$, with $I$ and $J$ open intervals:

$$F^{-1}(\mathbb{R}) = \{x \in X : (u(x), v(x)) \in \mathbb{R}\}$$
$$\Updownarrow$$
$$u(x) \in I \text{ and } v(x) \in J$$
$$= \{x \in X : u(x) \in I\} \cap \{x \in X : v(x) \in J\}$$
$$= \underbrace{u^{-1}(I)}_{\in \mathcal{M}} \cap \underbrace{v^{-1}(J)}_{\in \mathcal{M}} \in \mathcal{M}$$
$$\text{since both } u, v \text{ are measurable}$$

This completes the proof $\bigstar$

Consequences: by proposition 3 and 4, if $u$ and $v$ are measurable, then also $u + v, u \cdot v$. Other measurable functions include $u^+ = \max\{u, 0\}, u^- = -\min\{u, 0\}, |u| = u^+ + u^-, u^2, \ldots$
　　Recall that $u = u^+ - u^-$

**Remark 5.2**

$u^+$ is measurable since $u^+ = g \circ u$, where:

$$g(x) = \begin{cases} x & \text{where } x \geq 0 \\ 0 & \text{where } x < 0 \end{cases}$$

Most of the times we will work with functions $f : X \to \mathbb{R}$ or $f : X \to \underbrace{\overline{\mathbb{R}}}_{\mathbb{R} \cup \{\pm\infty\}}$ $(X, \mathcal{M})$

measurable space, then such a function $f$ is measurable iff

$$f^{-1}((a, +\infty)]^\dagger) \in \mathcal{M} \quad \forall a \in \mathbb{R}$$

or equivalently

$$f^{-1}([a, +\infty)]) \in \mathcal{M} \quad \forall a \in \mathbb{R}$$

Let now $\{f_n\}$ be a sequence of measurable functions from $X$ to $\overline{\mathbb{R}}$. Then we define

$$(\inf_n f_n)(x) = \inf_n f_n(x)$$

$$(\sup_n f_n)(x) = \sup_n f_n(x)$$

$$(\liminf_n f_n)(x) = \liminf_n f_n(x)$$

$$(\limsup_n f_n)(x) = \limsup_n f_n(x)$$

$$(\lim_n f_n)(x) = \lim_n f_n(x) \quad \text{if the limit exists}$$

**Proposition 5.5**

$(X, \mathcal{M})$ measurable space, $f_n : X \to \overline{\mathbb{R}}$ measurable, then

$$\sup_n f_n \quad \inf_n f_n \quad \liminf_n f_n \quad \limsup_n f_n$$

are measurable, in particular if $\lim_n f_n$ is well defined, then $f$ is measurable

***Proof.*** $(\sup f_n)^{-1}((a, \infty]) = \{x \in X : \quad\quad \sup f_n(x) > a \quad\quad\quad \}$
$$\Updownarrow$$
$$\exists \text{ some indexes } n \text{ s.t. } f_n(x) > a$$

$$\bigcup_n \{x \in X : f_n(x) > a\} = \bigcup_n \underbrace{f_n^{-1}((a, +\infty])}_{\in \mathcal{M}}$$

Then $(\sup f_n)^{-1}((a, \infty])$ is measurable, since it is the countable union of measurable sets.

Now we check that the $\limsup_n f_n$ is measurable

$$\limsup_n f_n(x) = \lim_n \underbrace{(\sup_{k > n} f_k(x))}_{\text{is decreasing on } n} = \inf_n (\sup_{k \geq n} f_k(x))$$

If we write $g_n(x) = \sup_{k \geq n} f_k(x)$, then

- $g_n$ is measurable, by what we proved previously

- $\limsup_n f_n = \inf_n g_n$ is measurable

★

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$^\dagger$We use ) if $f$ takes values in $\mathbb{R}$ and ] if $f$ takes values in $\overline{\mathbb{R}}$

## Simple functions

**Definition 5.1**

$(X, \mathcal{M})$ measurable space. A measurable function $s : X \to \overline{\mathbb{R}}$ is said to be simple if $s(X)$ is a finite set.

$$s(X) = \{a_1 \ldots, a_n\} \text{ for some } n \in \mathbb{N}, a_i \neq a_j$$

Then

$$s(x) = \sum_{n=1} a_n \chi_{E_n}(x)$$

where $E_n$ is a measurable set, $E_n = \{x \in X : s(X) = a_n\}$, and $E_i \cap E_j = \varnothing$ for $i \neq j$, and $\bigcup_{n=1}^{N} E_n = X$.

<u>Particular case</u>: if s:$\mathbb{R} \to \overline{\mathbb{R}}$, and each $E_n$ is a finite union of intervals, then $s$ is said to be a STEP function.

<u>Goal</u>: to approximate arbitrary measurable functions with simple functions.

**Theorem 5.1**

$(X, \mathcal{M})$ measurable space, $f : X \to [0, \infty]$ measurable. Then $\exists$ a sequence $\{s_n\}$ of simple functions s.t.

$$0 \leq s_1 \leq \ldots \leq s_n \leq \ldots \leq f \quad \text{(pointwise)} \atop \forall\, x \in X$$

and $s_n(x) \to f(x) \quad \forall\, x \in X$ as $n \to \infty$.

Moreover if f is bounded then $s_n \to f$ uniformly on $X$ as $n \to \infty$

*proof - for f bounded.* Fix $n \in \mathbb{N}$ and divide $[0, n)$ in $n \cdot 2^n$ intervals called $I_j = [a_j, b_j)$ with lenght $\frac{1}{2^n}$

Let $E_0 = f^{-1}([n, +\infty))$, $E_j = f^{-1}([a_j, b_j))$ for $j = 1, \ldots, n \cdot 2^n$

We let $\begin{array}{l} s_n(x) = a_j \quad \text{for } x \in E_j \\ s_n(x) = n \quad \text{for } x \in E_0 \end{array}$

Namely we define the simple function $s_n$ as

$$s_n(x) = n\chi_{E_0}(X) + \sum_{j=1}^{n \cdot 2^n} a_j \chi_{E_j}(x)$$

Then $s_n \leq s_{n+1}$ by contradiction, and, since $f$ is bounded, $E_0 = \varnothing$ for $n$ sufficiently large $(n > \sup f)$.

Then any $x \in X$ stays in $f^{-1}([a_j, b_j))$ for some $j$

$$\Rightarrow \quad \underset{\overset{\shortparallel}{s_n(x)}}{a_j} \quad \leq f(x) < b_j$$

$$\Rightarrow 0 \leq f(x) - s_n(x) < b_j - a_j = \frac{1}{2^n}$$

$$\Rightarrow \sup_{x \in X} |f(x) - s_n x| < \frac{1}{2^n} \to 0 \text{ as } n \to \infty$$

Namely, $s_n \to f$ uniformly on X. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ ★

# 6 Lecture 29/09/2022

**Remark 6.1**

On the relation between $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ and $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$ $\qquad$ ($\lambda$ = Lebesgue measure)

$(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ is not complete. In fact, $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$ is the completion of $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$.

Note that, $\forall E \in \mathcal{L}(\mathbb{R}) \exists$ a $G_\delta -$ set $A$ and an $F_\delta -$ set $B$ s.t.

$$A \supset E \text{ and } \lambda(A \setminus E) = 0$$
$$B \subset E \text{ and } \lambda(E \setminus B) = 0$$

$(X, \mathcal{M}, \mu)$ complete measure space.

Let $P(x)$ be a proposition depending on $x \in X$. We say that $P(x)$ is true $(\mu-)$ almost everywhere if

$$\mu \left( \{ x \in X : P(x) \text{ is false} \} \right) = 0$$

$P(x)$ is true $\underset{(\mu - \text{a.e.})}{\text{a.e.}}$ on $X$.

Ex: $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$, $f(x) = x^2$. Then $f(x) > 0$ a.e. on $\mathbb{R}$ (for a.e. $x$):

$$\{ f(x) \leq 0 \} = \{ 0 \}, \text{ and } \lambda(\{0\}) = 0$$

$(\mathbb{R}, \mathcal{P}(\mathbb{R}), \mu_C)$ with $\mu_C$ counting measure. Then it is not true that $f(x) > 0$ $\mu_C$-a.e.

$$\mu_C \left( \{0\} \right) = 1$$

It will be useful to consider sequences converging a.e.:

$$f_n \to f \qquad \text{a.e. on } X$$

if $\mu \left( \{ x \in X : \lim_n f_n(x) \neq f(x), \text{ or does not exist} \} \right) = 0$

**Proposition 6.1**

$(X, \mathcal{M}, \mu)$ complete measure space.

(1) $f : X \to \mathbb{R}$ is measurable, and $g = f$ a.e. on $X$, then $g$ is measurable

(2) $f_n \to f$ a.e. on $X$, $f_n : X \to \mathbb{R}$ measurable for all $n$, then $f$ is measurable

**Integration of non-negative functions**

Notation:

$$\{ x \in X : f(x) \geq 0 \} = \{ f \geq 0 \}$$
$$\{ x \in X : f(x) > 0 \} = \{ f > 0 \}$$
$$\vdots$$

$(X, \mathcal{M}, \mu)$ complete measure space. We consider measurable functions $f : X \to [0, +\infty]$

Convention: we define

$$a + \infty = +\infty \quad \forall a \in \mathbb{R}$$
$$a \cdot (+\infty) = \begin{cases} +\infty & \text{if } a \neq 0, a > 0 \\ 0 & \text{if } a = 0 \end{cases}$$

With this convention, $+$ and $\cdot$ of measurable functions are measurable functions.

**Definition 6.1**

Let $s : X \to [0, +\infty]$ be a measurable simple function,

$$s(x) = \sum_{n=1}^{m} a_n \chi_{D_n}(\bar{x})$$

where $D_1, \ldots, D_m$ are measurable, disjoint, and $\bigcup_{n=1}^{m} D_n = X$. Let also $E \in \mathcal{M}$. Then we define

$$\int_E s \, d\mu := \sum_{n=1}^{m} a_n \mu(D_n \cap E)$$

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**Remark 6.2**

Given a simple function $s$:

$$s : [a,b] \to \mathbb{R}, \lambda = \mu \Rightarrow \int_E s \, d\mu \text{ is the area under the curve}$$

**Remark 6.3**

There are several points:

- In the definition we have already used the convention $\mu(D_n \cap E = +\infty)$ for some $n$

- $E \in \mathcal{M} \Rightarrow \chi_E$ is a simple function

$$\chi_E(x) = 1 \cdot \chi_E + 0 \cdot \chi_{X \setminus E}(x)$$

  In this case

$$\int_X \chi_E \, d\mu = 1 \cdot \mu(E) + 0 \cdot \mu(X \setminus E) = \mu(E)$$

- $s\chi_E = \sum_{n=1}^m a_n \chi_{D_n \cap E} \Rightarrow \int_E s \, d\mu = \int_X s\chi_E \, d\mu$

**Definition 6.2**

$f : X \to [0, +\infty]$ measurable, $E \in \mathcal{M}$. The **Lebesgue integral** of $f$ on $E$, with respect to (w.r.t.) $\mu$, is

$$\int_E f \, d\mu = \sup \left\{ \int_E d\mu \, \middle| \, \begin{array}{l} s \text{ is simple} \\ 0 \leq s \leq f \end{array} \right\}$$

(1) If $f$ is simple, the definitions are consistent

(2) Also for $f$ measurable: $\int_E f \, d\mu = \int_X f\chi_E \, d\mu$

(3) $(\mathbb{N}, \mathbb{N}, \mu_C)$. $f : \mathbb{N} \to \mathbb{R}$ is a sequence $\{a_n\}_{n \in \mathbb{N}}$

$$\int_{\mathbb{N}} \{a_n\} \, d\mu_C = \sum_{n=0}^{\infty} a_n$$

Basic Properties.

Let $f, g : X \to [0, \infty]$ measurable. $E, F \in \mathcal{M}$, $\alpha \geq 0$. Then:

(1) $\mu(E) = 0 \Rightarrow \int_E f \, d\mu = 0$

(2) $f \leq g$ on $E \Rightarrow \int_E f \, d\mu \leq \int_E g \, d\mu$

(3) $E \subset F \Rightarrow \int_E f \, d\mu \leq \int_F f \, d\mu$

(4) $\alpha \geq o \Rightarrow \int_E \alpha f \, d\mu = \alpha \int_E d \, d\mu$

**Remark 6.4**

$([0,1], \mathcal{L}([0,1]), \lambda)$

Consider $\chi_{\mathbb{Q}}$, it is the Dirichlet function on $[0,1]$. This is not Riemann integrable.

However, $\int_{[0,1]} \chi_{\mathbb{Q}} \, d\lambda = \lambda(\mathbb{Q} \cap [0,1]) = 0$

**Theorem 6.1** (Chebychev's inequality)

$f : X \to [0, \infty]$ measurable, $c > 0$. Then

$$\mu(\{f \geq c\}) \leq \frac{1}{c} \int \{f \geq c\} f \, d\mu \leq \frac{1}{c} \int_X f \, d\mu$$

***Proof.***

$$\int_X f \, d\mu \overset{X \supset \{f \geq c\}}{\geq} \int_{\{f \geq c\}} f \, d\mu \geq \int_{\{f \geq c\}} c \, d\mu = c \int_{\{f \geq c\}} d\mu = c\mu\left(\{f \geq c\}\right)$$

★

**Theorem 6.2**
$s : X \to [0, \infty]$ simple. Define $\varphi : \mathcal{M} \to [0, \infty]$
$\varphi(E) = \int_E s \, d\mu$
$\Rightarrow \varphi$ is a measure.

***Proof.*** $\mu(\varnothing) = 0 \Rightarrow \varphi(\varnothing) = 0$ by definition.
**Definition 6.3** (sigma additivity)
$\{E_n \subset \mathcal{M}\}$ disjoint, and let $E = \bigcup_{n=1}^{\infty} E_n \Rightarrow s = \sum_{k=1}^{m} a_k \chi_{D_k} \ D_k \in \mathcal{M}$

Then, by definition and since $\mu$ is a measure and $E \cap D_k = \bigcup_n (E_n \cap D_k)$

$$\varphi(E) = \sum_{k=1}^{m} a_k \mu(D_k \cap E) = \sum_{k=1}^{\infty} a_k \sum_{n=1}^{\infty} \mu(E_n \cap D_k) =$$

$$\sum_{n=1}^{\infty} \left( \sum_{k=1}^{m} a_k \mu(E_n \cap D_k) \right) = \sum_{n=1}^{\infty} \int_{E_n} s \, d\mu = \sum_{n=1}^{\infty} \varphi(E_n)$$

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**Theorem 6.3** (Vanishing Lemma)
$f : X \to [0, \infty]$ measurable. $E \subset X$ measurable

$$\int_E f \, d\mu = 0 \Leftrightarrow f = 0 \text{ a.e. on } E$$

***Proof.*** $\Leftarrow$ easy
$\Rightarrow$ Consider $E \cap \{f > 0\} = \bigcup_{n=1}^{\infty} \underbrace{\left( E_n \{f \geq \frac{1}{n}\} \right)}_{=:E_n}$
Then $\{E_n\}$ is an increasing sequence. By Chebyshev

$$\mu(E_n) \leq \frac{1}{\frac{1}{n}} \int_E f \, d\mu = 0 \ \forall n \Rightarrow \mu(E_n) = 0 \quad \forall n$$

$\mu(E) \cup \{f > 0\} \overset{\text{continuity}}{=} \lim_n \mu(E_n) = 0$, namely $f = 0$ a.e. on $E$

★

The $\int$ does not see sets with 0 measure.
**Definition 6.4**
If $f : X \to [0, \infty]$ is measurable, and $\int_X f \, d\mu < \infty$ then we say that $f$ is integrable.

**Theorem 6.4** (Monotone Convergence - Beppo Levi)
$f_n : X \to [0, \infty]$ measurable. Suppose that

- $f_n(x) \leq f_{n+1}(x)$ for a.e. $x \in X$ for every $n$

- $f_n \to f$ a.e. on $X$

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Then
$$\int_X f \, d\mu = \lim_n \int_X f_n \, d\mu$$

**Proof.** Part 1.

Assume that the two hypotesis hold everywhere. First, if $f$ is measurable
$$\int_X f_n \, d\mu \nearrow \ \Rightarrow \exists \, \alpha = \lim_n \int_X f_n \, d\mu$$

Also, $f_n \le f$ everywhere $\Rightarrow \int_X f_n \, d\mu \le \int_X f \, d\mu \quad \forall \, n$

$$\Rightarrow \alpha \le \int_X f \, d\mu$$

We want to show that also $\ge$ is true. Let $s$ be a simple function s.t. $0 \le s \le f$ and $c \in (0,1)$
Let $E_n = \{f_n \ge cs\} \in \mathcal{M}$

- $E_n \in E_{n+1} \ \forall \, n$ :
  if $x \in E_n$, then $f_n(x) \ge cs(x) \Rightarrow f_{n+1}(x) \ge cs(x)$
  $\Rightarrow f_{n+1}(x) \ge f_n(x) \ge cs(x) \Rightarrow x \in E_{n+1}$

- Moreover, $X = \bigcup_{n=1}^{\infty} E_n$. Indeed:
  - if $f(x) = 0$, then $s(x) = 0 \Rightarrow f_1(x) = 0 = cs(x)$, $x \in E_1$
  - if $f(x) > 0$, then $cs(x) < f(x) = \lim_n f_n(x)$ since $s \le f$ and $c < 1$
  $\Rightarrow cs(x) < f_n(x)$ for $n$ sufficiently large, namely $x \in E_n$ for $n$ sufficiently large.

Therefore,
$$\alpha \ge \int_X f_n \, d\mu \ge \int_{E_n} f_n \, d\mu \ge c \int_{E_n} s \, d\mu = c\varphi(E_n)$$

$\forall n, \ \forall 0 \le s \le f, \forall c \in [0,1] \quad \varphi(E_n) = \int_{E_n} s \, d\mu$. $\varphi$ is a measure, and $\{E_n\} \nearrow$
Therefore, taking the lim when $n \to \infty$ by continuity

$$\alpha \ge \lim_n c \int_{E_n} s \, d\mu = c \int_X s \, d\mu \quad \forall c \in [0,1]$$

Take the limit when $c \to 1^- : \ \alpha \ge \int_X s \, d\mu \quad \forall \, 0 \le s \le f$
Take the sup over s: $\alpha \ge \int_X f \, d\mu$. We proved both inequalities, so the thesis holds.

Part 2.

Note that $\{x \in X : \text{assumpions of the theorem do not hold}\}$ is a set of zero measure. Take
$F$. $X = E \cup F$ since we have the assumpion on $E$ and $\mu(F) = 0$.
Then, by the Vanishing Lemma, since $(f - f\chi_E) = 0$ a.e. and $(f_n - f_n\chi_E) = 0$ we have that

$$\int_X f \, d\mu = \int_E d \, d\mu = \lim_n \int_E f_n \, d\mu = \lim_n \int_X f_n \, d\mu$$

★

# 7 Lecture 05/10/2022

**Theorem 7.1** (Monotone Convergence or Beppo Levi's theorem)
$f_n : X \to [0, +\infty]$ measurable. Suppose that

(1) $f_n(x) \le f_{n+1}(x)$ for a.e. $x \in X$, for every $n$

(2) $f_n \to f$ a.e. on $X$

Then

$$\int_X f \, d\mu = \lim_n \int_X f_n \, d\mu$$

**Corollary 7.1** (Monotone convergence for series)
$f_n : X \to [0, +\infty]$ measurable, then

$$\int_X \left( \sum_{n=0}^{\infty} f_n \right) d\mu = \sum_{n=0}^{\infty} \int_X f_n \, d\mu$$

**Theorem 7.2** (Approximation with simple functions)
Given $(X, \mathcal{M})$ measure space, $f : X \to [0, +\infty]$ measurable, then $\exists$ a sequence $\{s_n\}$ of simple functions s.t.

$$0 \leq s_1 \leq \ldots \leq s_n \leq \ldots \leq f \qquad \text{pointwise } \forall \, x \in X$$

and

$$s_n(x) \to f(x) \qquad \forall \, x \in X \text{ as } n \to \infty$$

Moreover, if $f$ is bounded, then $s_n \to f$ uniformly on $X$ as $n \to \infty$.

**Remark 7.1**
There is also

$$\int_X f \, d\mu = \sup \left\{ \int_X s \, d\mu \;\middle|\; \begin{array}{l} s \text{ is simple} \\ 0 \leq s \leq f \end{array} \right\}$$

But let $\{s_n\}$ be the sequence given by the simple approximation theorem. By monotone convergence

$$\int_X f \, d\mu = \lim_n \int_X s_n \, d\mu$$

Ex: $f, g : X \to [0, +\infty]$. Then

$$\int_X (f + g) \, d\mu = \int_X f \, d\mu + \int_X g \, d\mu$$

**Lemma 7.1** (Fatou's Lemma)
Given $f_n \to [0, +\infty]$ measurable $\forall \, n$. Then

$$\int_X (\liminf_n f_n) \, d\mu \leq \liminf_n \int_X f_n \, d\mu$$

In particular, if $f_n \to f$ a.e. on $X$, then

$$\int_X f \, d\mu \leq \liminf_n \int_X f_n d\mu$$

**Proof.** Given that $(\liminf_n f_n)(x) = \lim_n (\underbrace{\inf_{k \geq n} f_k(x)}_{= g_n(x)})$. Now, for every $x \in X$, $\{g_n(x)\} \nearrow$

$$g_{n+1}(x) = \inf_{k \geq n+1} f_k(x) \geq \inf_{k \geq n} f_k(x) = g_n(x)$$

Also, $g_n \geq 0$ on $X$. Thus, by monotone convergence

$$\int_X \liminf_n f_n \, d\mu = \int_X \lim_n g_n \, d\mu = \lim_n \int_X g_n \, d\mu = \liminf \int_X g_n \, d\mu$$

Now, note that $g_n(x) = \inf_{k \geq n} f_k(x) \leq f_n(x) \leq \liminf_n \int_X f_n \, d\mu$ ★

**Theorem 7.3** ($\sigma$-additivity of $\int$)

Given $(X, \mathcal{M}, \mu)$ measurable space, $\phi : X \to [0, +\infty]$. Define $\nu(E) = \int_E \phi \, d\mu$, with $E \in \mathcal{M}$. $\nu : \mathcal{M} \to [0, +\infty]$ is a measure. Moreover, let $f : X \to [0, +\infty]$ measurable

$$\int_X f \, d\nu = \int_X f\phi \, d\nu \qquad\qquad *$$

**Proof.** $\underline{\nu \text{ is a measure}}$:

$\nu(\varnothing) = 0$, since $\mu(\varnothing) = 0$. Now, let $E = \bigcup_{n=1}^\infty E_k$, $\{E_k\}$ disjoint. Then

$$\nu(E) = \int_X \phi\chi_E \, d\mu = \int_X \phi \sum_n \chi_{E_n} \, d\mu \underset{\substack{\text{monot. conv.} \\ \text{for } \sum}}{=} \sum_n \int_X \phi\chi_{E_n} \, d\mu = \sum_n \int_X \phi \, d\mu = \sum_n \nu(E_n)$$

We have proven $\sigma$ additivity, so $\nu$ is a measure.

$\underline{(*) \text{ holds}}$: Let $E \in \mathcal{M}$. Then

$$\int_X \chi_E \, d\nu = \int_E 1 \, d\nu = \nu(E) = \int_E = \int_E \phi \, d\mu = \int_X \phi\chi_E \, d\mu$$

This shows that $(*)$ holds for $\chi_E$, $\forall\, E$. Then it holds for simple functions.

Let now f be any measurable function, positive. Then we can take $\{s_n\}$ given by the simple approximation theorem

$$\int_X f \, d\nu \overset{\text{monot}}{=} \lim_n \int_X s_n \, d\nu = \lim_n \int_X s_n \phi \, d\mu \overset{\text{monot}}{=} \int_X f\phi \, d\mu$$

which is $(*)$ ★

**Remark 7.2**

$X, \mathcal{M}, \mu$ complete measure space. Then, by the vanishing lemma, it is not difficult to deduce that

$$f = g \text{ a.e. on } X \Leftrightarrow \int_E d \, d\mu = \int_E g \, d\mu \qquad \forall\, E \in \mathcal{M}$$

The $\int$ does not see differences of sets with 0 measure. As a consequence, in all the theorems, it is sufficient to assume that the assumptions are satisfid a.e.

## Integrals for real valued functions

$X, \mathcal{M}, \mu$ complete measure space.

$f : X \to \overline{\mathbb{R}} = [-\infty, \infty]$ measurable. Recall $f = f^+ - f^-$ where $f^+ = \max\{f, 0\}$, $f^- = -\min\{f, 0\}$ and $|f| = f^+ + f^-$. Note that both are positive and measurable.

**Definition 7.1**

we say that $f : X \to \overline{\mathbb{R}}$ measurable is integrable on $X$ if

$$\int_X |f| \, d\mu < \infty$$

If $f$ is integrable, we define $\int_X f \, d\mu = \int_X f^+ \, d\mu + \int_X f^- \, d\mu$

The set of integrable functions is denoted by

$$\mathcal{L}^1(X, \mathcal{M}, \mu) := \{f : X \to \overline{\mathbb{R}} \text{ integrable}\}$$

$$\mathcal{L}^1(X, \mathcal{M}, \mu) = \mathcal{L}^1(X) = \mathcal{L}^1$$

If $E \in \mathcal{M}$, we define

$$\int_E f \, d\mu = \int_X f\chi_E \, d\mu$$

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**Remark 7.3**

$f \in \mathcal{L}^1(X) \Rightarrow \int_X f \, d\mu \in \mathbb{R}$. Indeed $0 \leq f^{\pm} \leq |f|$

$$\Rightarrow 0 \leq \int_X f^+ \, d\mu, \ \int_X f^- \, d\mu \leq \int_X |f| \, d\mu < \infty$$

$$\Rightarrow \int_X f \, d\mu = \int_X f^+ \, d\mu - \int_X f^- \, d\mu \in \mathbb{R}$$

**Proposition 7.1**

$f : X \to \overline{\mathbb{R}}$ measurable. Then

(1) $f \in \mathcal{L}^1 \Leftrightarrow |f| \in \mathcal{L}^1 \Leftrightarrow$ both $f^+, \ f^- \in \mathcal{L}^1$

(2) $f \in \mathcal{L}^1$, then

$$\left| \int_X f \, d\mu \right| \leq \int_X |f| \, d\mu \qquad\qquad \text{(triangle inequality)}$$

***Proof.*** Of the second part.

$$\left| \int_X f \, d\mu \right| = \left| \int_X f^+ \, d\mu + \int_X f^- \, d\mu \right| \leq \int_X f^+ \, d\mu + \int_X f^- \, d\mu = \int_X |f| \, d\mu$$

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**Proposition 7.2**

$\mathcal{L}^1(X, \mathcal{M}, \mu)$ is a vector space, and $f, g \in \mathcal{L}^1$, $\alpha \in \mathbb{R}$

$$\Rightarrow \int_X (\alpha f + g) \, d\mu = \alpha \int_X f \, d\mu + \int_X g \, d\mu$$

by linearity of the integrals.

Many results can be extended from non negative functions to general functions.

**Theorem 7.4**

$(X, \mathcal{M}, \mu)$ complete measure space. $f, g \in \mathcal{L}^1$. Then

$$f = g \text{ a.e. on } X \Leftrightarrow \int_X |f - g| \, d\mu = 0 \Leftrightarrow \int_E f \, d\mu = \int_E g \, d\mu \qquad \forall \, E \in \mathcal{M}$$

The most relevant theorem for convergence is the following

**Theorem 7.5** (Dominated convergence theorem)

$\{f_n\}$ sequence of measurable functions $X \to \overline{\mathbb{R}}$. Suppose that

(1) $f_n \to f$ a.e. on $X$

(2) $\exists g : X \to \overline{\mathbb{R}}, \ g \in \mathcal{L}^1(X)$, such that $|f_n(x) \leq g(x)|$ a.e. on $X \ \forall \, n \in \mathbb{N}$

Then $f \in \mathcal{L}^1$ and

$$\lim_n \int_X |f_n - f| \, d\mu = 0 \qquad \left( \Rightarrow \int_X f \, d\mu = \lim_n \int_X f_n \, d\mu \right)$$

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**Proof.** Note that $f_n \in \mathcal{L}^1 \; \forall \; n$, since $|f_n| \leq g$ and we have the monotonicity of $\int$ for non negative functions

$$|f_n(x)| \leq g(x) \quad n \to \infty \qquad |f(x)| \leq g(x) \text{ a.e. on } X$$

$$\Rightarrow f \in \mathcal{L}^1(X)$$

Now, consider $\phi_n = 2g - |f_n - f|$. We have

$$|f_n - f| \leq |f_n| + |f| \leq 2g \quad \text{a.e. on } X \quad \phi_n \geq 0 \quad \text{a.e. on } X$$

We can use Fatou's lemma:

$$\int_X \underbrace{(\liminf_n \phi_n)}_{= \; 2g \text{ a.e.}} d\mu \leq \liminf_n \int_X \phi_n \, d\mu = \liminf_n \int_X (2g - |f_n - f|) \, d\mu =$$

$$\overset{\Vert}{\underset{\int_X 2g \, d\mu}{}}$$

$$= \int_X 2g \, d\mu + \liminf_n (-\int_X |f_n - f| \, d\mu) = \int_X 2g \, d\mu - \limsup_n \int_X |f_n - f| \, d\mu$$

Subtracting $\int_X 2g \, d\mu$ from both sides

$$0 \leq -\limsup_n \int_X |f_n - f| \, d\mu \Rightarrow 0 \leq \liminf_n \int_X |f_n - f| \, d\mu \leq \limsup_n \int_X |f_n - f| \, d\mu \leq 0$$

<div align="right">★</div>

**Remark 7.4**
If $\mu(X) < +\infty$, and $\exists M > 0$ s.t. $|f_n| \leq M$ a.e. on $X, \forall \; n$, then we can take $g = M$ as dominating function.

Comments on the relation between Riemann and Lebesgue integrals

Let $f : I \subset \mathbb{R} \to \mathbb{R}$, $I$ interval, be bounded. Assume also that $I$ is closed and bounded.

**Theorem 7.6**
Let $f$ be Riemann-integrable on $I$ ($f \in R(I)$). Then

$$f \in \mathcal{L}^1(I, \mathcal{L}(I), \lambda)$$

and

$$\int_I f \, d\lambda = \int_I f(x) \, dx$$

**Theorem 7.7**
$f \in R(I) \Leftrightarrow f$ is continuous on $x$, for a.e. $x \in I$.

Ex: $\chi_{\mathbb{Q}}$ on $[0,1]$ is not Riemann integrable, because it is discontinuous at any point. Note that, instead, $\chi_{\mathbb{Q}} = 0$ a.e. on $[0,1] \Rightarrow \int_{[0,1]} \chi_{\mathbb{Q}} \, d\lambda = 0$

# 8 Lecture 06/10/2022

Let $f \notin R(I)$. Is it true that $\exists g \in R(I)$ s.t. $g = f$ a.e. on $I$? No.

For instance, consider $T_{\mathcal{E}}$, the generalized Cantor set ($\lambda(T_{\mathcal{E}}) = 0$) and then consider $\chi_{T_{\mathcal{E}}}$. In general, $\chi_A$ is discontinuous on $\delta A$. But $T_{\mathcal{E}}$ has no interior parts $\Rightarrow T_{\mathcal{E}} = \delta T_{\mathcal{E}} \Rightarrow \chi_{T_{\mathcal{E}}}$ is discontinuous on $T_{\mathcal{E}}$, which has positive measure $\Rightarrow$ by the last theorem, $\chi_{T_\varepsilon}$ is not $R(I)$

Clearly
$$\int_{[0,1]} \chi_{T_{\mathcal{E}}} d\lambda = \lambda(T_{\mathcal{E}})$$

so $\chi_{T_{\mathcal{E}}} \in \mathcal{L}^1([0,1])$.

If $g = \chi_{T_{\mathcal{E}}}$ a.e., then $g$ is discontinuous at almost every part of $T_{\mathcal{E}} \Rightarrow g$ is discontinuous on a set of positive measure $\Rightarrow g \notin R(I)$. So, the Lebesgue integral is a true extension of the Riemann one.

Regarding generalized integrals we have

**Theorem 8.1**

$-\infty \leq a < b \leq +\infty, \quad f \in R^g([a,b])$ where

$$R^g([a,b]) = \{\text{Riemann-int functions on } [a,b] \text{ in the generalized sense}\}$$

Then, $f$ is $([a,b], \mathcal{L}([a,b]))$-measurable. Moreover

(1) $f \geq 0$ on $[a,b] \Rightarrow f \in \mathcal{L}^1([a,b])$

(2) $|f| \in R^g([a,b]) \Rightarrow f \in \mathcal{L}^1([a,b])$

and in both cases

$$\int_{[a,b]} f d\lambda = \int_a^b f(x) dx$$

If $f$ is in $R^g([a,b])$, but $|f| \notin R^g([a,b])$, then the two notions of $\int$ are not really related

Ex: $f(x) = \frac{\sin x}{x}, x \in [1, \infty]$

$$\int_1^\infty |f(x)| dx = +\infty \Rightarrow f \notin \mathcal{L}^1([1, +\infty])$$

. But on the other hand

$$\int_1^\infty \frac{\sin x}{x} dx = \lim_{\omega \to \infty} \int_1^\omega \frac{\sin x}{x} dx = \frac{\pi}{2}$$

**Proposition 8.1**

$(X, \mathcal{M}, \mu)$ complete measure space. Let $\{f_n\} \subseteq \mathcal{L}^1(X, \mathcal{M}, \mu)$.

Suppose that $\sum_{n=1}^\infty \int_X |f_n| d\mu < \infty$ Then the series $\sum_{n=1}^\infty f_n$ converges a.e. on $X$, it is in $\mathcal{L}^1(X)$ and

$$\int_X \left( \sum_{n=1}^\infty f_n \right) d\mu = \sum_{n=1}^\infty \int_X f_n \, d\mu$$

**Spaces of integrable functions**

$(X, \mathcal{M}, \mu)$ complete measure space.

$$\mathcal{L}^1 = \left\{ f : X \to \overline{\mathbb{R}} : \text{ f is integrable} \right\}$$

$\mathcal{L}^1$ is a vector space. On $\mathcal{L}^1$ we can introduce $d : \mathcal{L}^1 \times \mathcal{L}^1 \to [0, +\infty)$ defined by

$$d_1(f, g) = \int_X |f - g|$$

It is immediate to check that

$$d_1(f, g) = d_1(g, f) \qquad \text{(symmetry)}$$

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$$d_1(f,g) \le d_1(f,h) + d_1(h,g) \quad \forall f,g,h \in \mathcal{L}^1(X) \qquad \text{(triangular inequality)}$$

However, $d_1$ is not a distance on $\mathcal{L}^1(X)$, since

$$d_1(f,g) = 0 \Rightarrow f = g \quad \text{a.e on } X \qquad \text{(pseudo-distance)}$$

To overcome this problem, we introduce an equivalent relation in $\mathcal{L}^1(X)$: we say that

$$f \sim g \Leftrightarrow f = g \quad \text{a.e. on } X$$

If $f \in \mathcal{L}^1(X)$, we can consider the equivalence class

$$[f] = \left\{ g \in \mathcal{L}^1(X) : g = f \text{ a.e on } X \right\}$$

We define

$$L^1(X) = \frac{\mathcal{L}^1(X)}{\sim} = \{[f] : f \in \mathcal{L}^1(X)\}$$

$L^1(X)$ is a vector space, and on $L^1(X)$ the function $d_1$ is a distance:

$$d_1([f],[g]) = 0 \Leftrightarrow \int_X |[f] - [g]| \, d\mu = 0 \Leftrightarrow [f] = [g] \text{ a.e. } \Leftrightarrow f = g \text{ a.e.}$$

To simplify the notations, the elements of $L^1(X)$ are called functions, and one writes $f \in L^1(X)$. With this, we means that we choose a representative in $[f]$, and f denotes both the representative and the equivalence class. The representative can be arbitrarily modified on any set with 0 measure.

Another relevant space of measurable functions is the space of **essentially bounded** functions.

**Definition 8.1**

$f : X \to \overline{\mathbb{R}}$ measurable is called essentially bounded if $\exists\, M > 0$ s.t.

$$\mu(\{x \in X : |f(x)| \ge M\}) = 0$$

Ex:

$$f(x) = \begin{cases} 1 & x > 0 \\ +\infty & x = 0 \\ 0 & x < 0 \end{cases}$$

For $M > 1$, $\lambda(\{x \in \mathbb{R} : |f(x)| > M\}) = \lambda(\{0\}) = 0 \Rightarrow f$ is essentially bounded.

If $f$ is essentially bounded, it is well defined the **essential supremum** of $f$.

$$\operatorname*{ess\,sup}_X f := \inf \{M > 0 \text{ s.t. } f \le M \text{ a.e. on } X\} = \inf \{M > 0 \text{ s.t. } \mu(\{f \ge M\}) = 0\}$$

It can also be defined an essential inf.

**Remark 8.1**

Note that, by def of inf, $\forall\, \varepsilon > 0$ we have

$$f \le \left(\operatorname*{ess\,sup}_X f\right) + \varepsilon \qquad \text{a.e. on } X$$

We define

$$L^\infty(X, \mathcal{M}, \mu) = \frac{\mathcal{L}^\infty(X, \mathcal{M}, \mu)}{\sim}$$

$L^\infty(X)$ is a vector space, and it is also a metric space for $d_\infty(f,g) = \operatorname*{ess\,sup}_X |f - g|$

**Relation between different types of convergence**

$\{f_n\}$ sequence of measurable functions $X \to \overline{\mathbb{R}}$

- $f_n \to f$ pointwise (everywhere) on $X$ if $f_n(x) \overset{n\to\infty}{\to} f(x) \ \forall \, x \in X$

- $f_n \to f$ uniformly on $X$ if $\sup_{x \in X} |f_n(x) - f(x)| \overset{n\to\infty}{\to} 0$

- $f_n \to f$ a.e. on $X$ if

$$\mu\left(\left\{x \in X : \lim_n f_n(x) \neq f(x) \text{ or does not exist}\right\}\right) = 0$$
$$\Updownarrow$$
$$f_n(x) \to f(x) \text{ for a.e } x \in X$$

- <u>Convergence in $L^1(X)$</u>: $f_n \to f$ in $L^1(X)$ if

$$\int_X |f_n - f| \, d\mu \overset{n\to\infty}{\to} 0$$
$$\overset{||}{d_1(f_n, f)}$$

- <u>Convergence in measure/probability</u>: $f_n \to f$ in measure if $\forall \, \alpha 0$

$$\lim_{n\to\infty} \mu\left(\{|f_n - f| \geq \alpha\}\right) = 0$$

<u>Basic facts</u>: uniformly convergence $\underset{\not\Leftarrow}{\Rightarrow}$ pointwise $\underset{\not\Leftarrow}{\Rightarrow}$ a.e. convergence.

<u>Ex:</u> $f_n(x) = \exp\{-nx\}, x \in [0,1]$

$$f(x) = 0, \quad g(x) = \begin{cases} 0 & x \in (0,1] \\ 1 & x = 0 \end{cases}$$

Then $f_n \to g$ pointwise, $g = f$ a.e. $\Rightarrow f_n \to f$ a.e. on $[0,1]$. But $f(0) \neq g(0) \Rightarrow f_n \to f$ pointwise.

$f_n \not\to g$ uniformly on $[0,1]$ $\left|\begin{array}{l} f_n \in \mathcal{C}([0,1]) \\ f_n \to g \Rightarrow g \in \mathcal{C}([0,1]) \end{array}\right.$

a.e. $\not\Rightarrow$ uniform, but not all is lost...

**Theorem 8.2** (Egorov)

Let $\mu(X) < +\infty$, and suppose that $f_n \to f$ a.e. on $X$. Then, $\forall \, \varepsilon > 0, \exists X_\varepsilon \subset X$, measurable, s.t.

$$\mu(X \setminus X_\varepsilon) < \varepsilon$$

and $f_n \to f$ uniformly on $X_\varepsilon$

<u>Ex:</u> in an example $f_n \to 0$ a.e., $f_n \to 0$ uniformly on $[0,1]$, but $f_n \to 0$ uniformly on $[\varepsilon, 1]$.

Regarding a.e. convergence and in measure convergence there is the following theorem

**Theorem 8.3**

If $\mu(X) < +\infty$ and $f_n \to f$ a.e. on $X \Rightarrow f_n \to f$ is measure on $X$

**Proof.** Let $\alpha > 0$. We want to show that $\forall \, \varepsilon > 0 \ \exists \bar{n} \in \mathbb{N}$ s.t.

$$n > \bar{n} \Rightarrow \mu(\{|f_n - f| \geq \alpha\}) < \varepsilon$$

$f_n \to f$ a.e. on $X$, $\mu(X) < +\infty \overset{\text{Egorov}}{\Rightarrow} \exists \, X_\varepsilon \subseteq X$ s.t. $\mu(X \setminus X_\varepsilon) < \varepsilon$ and $f_n \to f$ uniformly on $X_\varepsilon \Leftrightarrow \sup_{X_\varepsilon} |f_n - f| \overset{n\to\infty}{\to} 0$.

In particular, this means that $\exists\, \bar{n} \in \mathbb{N}$ s.t. $n > \bar{n} \Rightarrow |f_n - f| < \alpha$ on $X_\varepsilon$.
Therefore

$$\{|f_n - f| \geq \alpha\} \cap X_\varepsilon = \varnothing \Rightarrow \{|f_n - f| \geq \alpha\} \subseteq X \setminus X_\varepsilon \qquad \text{for } n > \bar{n}$$

By monotonicity of $\mu$, we deduce that

$$\mu\left(\{|f_n - f| \geq \alpha\}\right) \leq \mu(X \setminus X_\varepsilon) < \varepsilon \quad \text{for } n > \bar{n}$$

Namely, $f_n \to f$ in measure. ★

### Remark 8.2
$\mu(X) < +\infty$ is essential

For example, in $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$ consider

$$f_n(x) = \chi_{[n, n+1)}(x)$$

$f_n(x) \to 0$ for every $x \in \mathbb{R}$. However, $\lambda(\{|f_n| \geq \frac{1}{2}\}) = \lambda([n, n+1)) = 1$ not $0$

# 9 Lecture 12/10/2022

### Remark 9.1
Convergence in measure $\Rightarrow$ a.e convergence?
No, not even if $\mu(X) < +\infty$.
Consider $\chi_{n,k} = \chi_{[\frac{k-1}{n}, \frac{k}{n}]}$ with $n \in \mathbb{N}, k = 1, \ldots, n$

$$\begin{aligned}
\chi_{1,1}(x) &= \chi_{[0,1]}(x) \\
\chi_{2,1}(x) &= \chi_{[0, \frac{1}{2}]}(x) \quad \chi_{2,2}(x) = \chi_{[\frac{1}{2}, 1]}(x) \\
\chi_{3,1}(x) &= \chi_{[0, \frac{1}{3}]}(x) \quad \chi_{3,2}(x) = \chi_{[\frac{1}{3}, \frac{2}{3}]}(x) \quad \chi_{3,3}(x) = \chi_{[\frac{2}{3}, 1]}(x)
\end{aligned}$$

For $n$ fixed and $k$ variable, we move the $\chi$ from the left to right. When the $\chi$ reaches 1, we switch $n$, and $\chi$ reappear from the left, being thinner.

$$\int_{[0,1]} \chi_{n,k}\, d\lambda = \frac{1}{n} \quad \int_{[0,1]} \chi_{n+1,k}\, d\lambda = \frac{1}{n+1}$$

We can order the elements of $\chi_{n,k}$ in a sequence, letting $f_p = \chi_{n,k}$ for $p = 1 + 2 + \ldots + (n-1) + k$. We will prove that $\{f_p\}$ converges in measure, but not a.e.

This is the **typewriter sequence** $(p(n, k))$. For every $x \in [0, 1]$ there are $\infty$ many indexes s.t. $f_p(x) = 1$ and $\infty$ many indexes s.t. $f_p(x) = 0$, meaning that $\nexists \lim_{p \to \infty} f_p(x)$ $f_p \not\to 0$ a.e. on $[0, 1]$.

But we do have convergence in measure to 0: $\alpha \in (0, 1)$

$$\lambda\left(\{|f_{p(n,k)}| \geq \alpha\}\right) = \lambda\left(\left[\frac{k-1}{n}, \frac{k}{n}\right]\right) = \frac{1}{n} \to 0 \text{ as } \underset{p \to \infty}{\overset{n \to \infty}{\Updownarrow}}$$

### Remark 9.2
So, $f_p \not\to 0$ a.e. on $[0, 1]$. But consider $\{f_{p(n,1)} : n \in \mathbb{N}\}$. This is a subsequence and, by definition

$$f_{p(n,1)}(x) = \chi_{n,1}(x) = \chi_{[0, \frac{0}{n}]}(x)$$

For this subsequence, we have $f_{p(n,1)}(x) \to 0$ as $n \to \infty \, \forall x \in (0, 1]$, then a.e. on $[0, 1]$
This is not random!

**Proposition 9.1**
If $\mu(X) < \infty$ and $f_n \to f$ in measure, then $\exists$ a subsequence $\{f_{n_k}\}$ s.t. $f_{n_k} \to f$ a.e. on $X$.

Now we analize the relation between convergence in $L^1(X)$ and the other convergences.

**Theorem 9.1**
$\{f_n\} \subset L^1(X), f \in L^1(X)$. If $f_n \to f$ in $L^1(X)$ then $f_n \to f$ in measure on $X$

**Proof.** By contradiction. Suppose that $f_n \not\to f$ in measure on X: $\exists \, \bar\alpha > 0$ s.t.

$$\limsup_{n \to \infty} \mu(\{|f_n - f| \geq \bar\alpha\}) > 0$$

$\Rightarrow \exists \, \bar\varepsilon$ and a subsequence $\{f_{n_k}\}$ s.t.

$$\mu(\{|f_{n_k} - f| \geq \bar\alpha\}) > \bar\varepsilon$$

Consider then

$$d_1(f_{n_k}, f) = \int_X |f_{n_k} - f| \, d\mu \overset{\text{monot.} \int}{\geq} \int_{\{|f_{n_k} - f| \geq \bar\alpha\}} |f_{n_k} - f| \, d\mu \geq$$

$$\geq \int_{\{|f_{n_k} - f| \geq \bar\alpha\}} 1 \, d\mu = \bar\alpha \, \mu(\{|f_{n_k} - f| \geq \bar\alpha\}) > \bar\alpha \, \bar\varepsilon$$

But, by assumption, $d_1(f_n, f) \to 0$

$$\Rightarrow d_1(f_{n_k}, f) \to 0$$

Contradiction. ★

**Remark 9.3**
The convergence in measure doesn't imply the convergence in $L^1$.
For example, consider
$$f_n(x) = n\chi_{[0, \frac{1}{n}]}(x)$$

$\underbrace{\mu(\{|f_n| \geq \alpha\})}_{= \frac{1}{n}} \to 0$ for every $\alpha$

On the other hand
$$\int_{[0,1]} n\chi_{[0,\frac{1}{n}]} \, d\lambda = \int_{[0,\frac{1}{n}]} n \, d\lambda = n\frac{1}{n} = 1$$

$f_n \not\to 0$ in $L^1$

Convergence a.e. $\not\Rightarrow$ convergence in $L^1$:
Use the same example above, $f_n \to 0$ a.e. on $[0,1] \not\Rightarrow f_n \to 0$ in $L^1$
Convergence in $L^1 \not\Rightarrow$ convergence a.e.:
Consider the typewriter sequence: $d_1(f_{p(n,k)}, 0) \to 0$ when $p \to \infty$
But we don't have a.e. convergence.
However, recall the dominated convergence theorem: (DOM)

$$f_n \to f \text{ a.e. } + \exists \text{ of a dominating function } \Rightarrow d(f_n, f) \to 0$$

It is also possible to show a reverse DOM:
If $f_n \to f$ in $L^1(X)$, then $\exists$ a subsequence $\{f_{n_k}\}$ and $w \in L^1(X)$ s.t.

(1) $f_{n_k} \to f$ a.e. on X

(2) $|f_{n_k}(x)| \leq w(x)$ for a.e. $x \in X$

# Derivatives of measures

$(X, \mathcal{M}, \mu)$ measure space, $\phi : X \to [0, \infty]$ measurable.

We learned that $\nu : \mathcal{M} \to [0, \infty]$ by

$$\nu(E) = \int_E \phi \, d\mu \text{ is a measure on } (X, \mathcal{M})$$

If the equation above holds, then we say that $\phi$ is the Radon Nykodym derivative of $\nu$ with respect to $\mu$ and we write

$$\phi = \frac{d\nu}{d\mu}$$

**Definition 9.1**

$\mu, \nu$ measures on $(X, \mathcal{M})$. We say that $\nu$ is absolutely continuous with respect to $\mu$, $\nu << \mu$ if

$$\mu(E) = 0 \Rightarrow \nu(E) = 0$$

**Lemma 9.1**

There is a necessary condition:

$$\exists \, \frac{d\nu}{d\mu} \Rightarrow \nu << \mu$$

***Proof.***

$$\nu(E) = \int_E \left( \frac{d\nu}{d\mu} \right) d\mu = 0$$

if $\mu(E) = 0$ by basic properties of $\int$       ★

**Theorem 9.2** (Radon Nykodim Theorem)

$(X, \mathcal{M})$ measurable space, $\mu, \nu$ measures.

If $\nu << \mu$ and moreover $\mu$ is $\sigma$-finite, then $\phi : X \to [0, \infty]$ measurable s.t.

$$\phi = \frac{d\nu}{d\mu} \quad \text{namely } \nu(E) = \int_E \phi \, d\mu \quad \forall \, E \in \mathcal{M}$$

**Remark 9.4**

If $\mu$ is not sigma finite the theorem may fail.

In $([0, 1], \mathcal{L}([0, 1]))$ consider the counting measure $\mu = \mu_C$ and the Lebesgue measure $\nu = \lambda$

$\nu << \mu$ since $\mu(E) = 0 \Leftrightarrow E = \varnothing \Rightarrow \lambda(E) = \nu(E) = 0$

But we can check that $\nexists \, \phi : [0, 1] \to [0, \infty]$ measurable s.t. $\lambda(E) = \int_E \phi \, d\mu_C$

Check by contradiction: assume that $\phi$ does exist, and take $x_0 \in [0, 1]$

$$0 = \lambda(\{x_0\}) = \int_{\{x_0\}} \phi \, d\mu_C = \phi(x_0) \overbrace{\mu_C(\{x_0\})}^{=1} = \phi(x_0)$$

$\Rightarrow \phi(x_0) = 0 \, \forall \, x_0 \in [0, 1]$.

But then $1 = \lambda([0, 1]) = \int_{[0,1]} 0 \, d\mu_C = 0$. Contradiction

Note that $\mu_C([0, 1]) = \infty$ and $([0, 1], \mathcal{L}([0, 1]), \mu_C)$ is not $\sigma$-finite ($[0, 1]$ is uncountable)

# Product Measure

$(X, \mathcal{M}, \mu), (Y, \mathcal{N}, \nu)$ measure spaces. The goal is to define a measure space on $X \times Y$

**Definition 9.2**

We call **measurable rectangle** in $X \times Y$ a set of type $A \times B$ where $A \in \mathcal{M}, B \in \mathcal{N}$

$$R = \{A \times B \subset X \times Y \text{ s.t. } A \in \mathcal{M}, B \in \mathcal{N}\}$$

We define the product $\sigma$-algebra $\mathcal{M} \otimes \mathcal{N}$ as $\sigma_0(R)$.
This is a $\sigma$-algebra in $X \times Y$

**Definition 9.3**

Let $E \subset X \times Y$. For $\bar{x} \in X$ and $\bar{y} \in Y$ we define

$$
\begin{aligned}
E_{\bar{x}} &= \{y \in Y : (\bar{x}, y) \in E\} \subseteq Y && \bar{x}\text{-section of } E \\
E_{\bar{y}} &= \{x \in X : (x, \bar{y}) \in E\} \subseteq X && \bar{y}\text{-section of } E
\end{aligned}
$$

**Proposition 9.2**

$(X, \mathcal{M}), (Y, \mathcal{N})$ measurable spaces. $E \in \mathcal{M} \otimes \mathcal{N}$
Then $E_x \in \mathcal{M}$ and $E_y \in \mathcal{N} \Rightarrow$ we can define

$$
\begin{aligned}
\varphi : \quad & X \to [0, \infty] && \psi : \quad Y \to [0, \infty] \\
& x \mapsto \nu(E_x) && \qquad y \mapsto \mu(E_y)
\end{aligned}
$$

**Theorem 9.3**

If $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are $\sigma$ finite spaces, then:

(1) $\varphi$ is $\mathcal{M}$-measurable and $\psi$ is $\mathcal{N}$-measurable

(2) we have that $\int_X \nu(E_x) \, d\mu = \int_Y \mu(E_y) \, d\nu$

Using the fact that $\mu$ and $\nu$ are measures, and that $\int$ of non negative function is a measure, we deduce the following

**Theorem 9.4** (Iterated integrals for characteristic functions)
$\mu \otimes \nu : \mathcal{M} \otimes \mathcal{N} \to \mathbb{R}$ defined by

$$(\mu \otimes \nu)(E) = \int_X \nu(E_x) \, d\mu = \int_Y \mu(E_y) \, d\nu$$

is a measure, the product measure.

**Remark 9.5** (On the completion of product measure spaces)
$(X, \mathcal{M}, \mu), (Y, \mathcal{N}, \nu)$ complete measures spaces.

In general it is not true that $(X \times Y, \mathcal{M} \otimes \mathcal{N}, \mu \otimes \nu)$ is complete.

Example: $X = Y = \mathbb{R}, \mathcal{M} = \mathcal{N} = \mathcal{L}(\mathbb{R}), \mu = \nu = \lambda$.

Given $A$ non meas. set , $A \subseteq [0, 1], B = \{y_0\}, E = A \times B$. If $E$ were measurable, then its sections must be measurable. But $E_{y_0} = A$ which is not measurable.

However, $E$ is negligible:

$$E \subseteq [0, 1] \times \{y_0\}, \text{ and } (\lambda \otimes \lambda)([0, 1] \times \{y_0\}) = 0$$

Then $(\lambda \otimes \lambda)$ is not complete

$$\Rightarrow (\mathbb{R}^2, \mathcal{L}(\mathbb{R}) \otimes \mathcal{L}(\mathbb{R}), \lambda \otimes \lambda) \neq (\mathbb{R}^2, \mathcal{L}(\mathbb{R}^2), \lambda_2)$$

**Theorem 9.5**

Let $\lambda_n$ be the Lebesgue measure in $\mathbb{R}^n$. If $n = K + m$, then $(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^n), \lambda_n)$ is the completion of $(\mathbb{R}^k \times \mathbb{R}^m, \mathcal{L}(\mathbb{R}^k) \otimes \mathcal{L}(\mathbb{R}^m), \lambda_k \otimes \lambda_m)$

# 10 Lecture 13/10/2022

**Integration on product spaces**

$(X, \mathcal{M}, \mu), (Y, \mathcal{N}, \nu)$ measure spaces. $f : X \times Y \to \overline{\mathbb{R}}$ measurable.

If $f \geq 0$, then

$$\iint_{X \times Y} f \, d\mu \otimes d\nu$$

Goal: obtain a formula of iterated integral like the one in Analysis 2.

$\forall \, \bar{x} \in X$ and $\bar{y} \in Y$, we define

$$
\begin{array}{ll}
f_{\bar{x}} : \; Y \to \overline{\mathbb{R}} & f_{\bar{y}} : \; X \to \overline{\mathbb{R}} \\
\quad y \mapsto f(\bar{x}, y) & \quad x \mapsto f(x, \bar{y})
\end{array}
$$

**Proposition 10.1**
If $f$ is measurable $\Rightarrow f_{\bar{x}}$ is $(\mathcal{N}, \mathcal{B}(\mathbb{R}))$-measurable and $f_{\bar{y}}$ is $(\mathcal{M}, \mathcal{B}(\overline{\mathbb{R}}))$-measurable. Then we can consider

$$\varphi : X \to \overline{\mathbb{R}} \quad \varphi(x) = \int_Y f_x \, d\nu = \int_Y f(x,y) \underbrace{d\nu(y)}_{dy}$$

$$\psi : Y \to \overline{\mathbb{R}} \quad \psi(y) = \int_X f_y \, d\mu = \int_X f(x,y) \, d\mu(x)$$

<u>Questions</u>: what is the solution of $\iint_{X \times Y}$, $\varphi$ and $\psi$?

**Theorem 10.1** (Tonelli's theorem)
$(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ complete measure spaces and $\sigma$-finite.
Suppose that $f$ is $(\mathcal{M} \otimes \mathcal{N}, \mathcal{B}(\overline{\mathbb{R}}))$-measurable and that $f > 0$ a.e. on $X \times Y$. Then $\psi$ and $\varphi$ are measurable and

$$\iint_{X \times Y} f \, d\mu \otimes d\nu = \int_X \varphi(x) \, d\mu(x) = \int_Y \psi(y) \, d\nu(y) \qquad \text{Integration formula}$$

Equally holds also if one of the integrals is $\infty$.

$$\int_X \varphi(x) \, d\mu(x) = \int_X \left( \int_Y f(x,y) \, d\nu(y) \right) d\mu(x)$$

$$\int_Y \psi(y) \, d\nu(y) = \int_Y \left( \int_X f(x,y) \, d\mu(x) \right) d\nu(y)$$

**Remark 10.1**
The double integral can be reduced to single integrals, iterated. Moreover we can always change the order of the integrals For sign changing functions the situation is more involved.

**Theorem 10.2** (Fubini's theorem)
$(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ complete measure spaces and $\sigma$-finite. If $f \in L^1(X \times Y)$, then $\psi$ and $\varphi$ defined above are measurable, the integration formula holds, and all the integrals are finite.

<u>Question</u>: how to check if $f \in L^1(X \times Y)$? Typically, to check that $f \in L^1(X \times Y)$ one uses Tonelli:

$$f \in L^1(X \times Y) \Leftrightarrow \iint_{X \times Y} |f| \, d\mu \otimes d\nu$$

We use Tonelli to check that this is finite. If $\iint_{X \times Y} |f| d\mu \otimes d\nu < \infty$ then we can apply Fubini for $\iint_{X \times Y} f \, d\mu \otimes d\nu$

**Remark 10.2**
the proof of Fubini's and Tonelli's theorems is based for the iterated integrals for characteristic functions. Note that

$$(\mu \otimes \nu)(E) = \begin{cases} \displaystyle\int_X \varphi(x)\,d\mu(x) = \int_X \left( \int_Y f(x,y)\,d\nu(y) \right) d\mu(x) \\[2mm] \displaystyle\int_Y \psi(y)\,d\nu(y) = \int_Y \left( \int_X f(x,y)\,d\mu(x) \right) d\nu(y) \end{cases}$$

**Remark 10.3**
Sometimes double integrals are very useful to compute single integrals.

Ex: $\int_{-\infty}^{+\infty} e^{-x^2} = \sqrt{\pi}$

# 11 Lecture 19/10/2022

## The first fundamental theorem of calculus

Consider $f \in L^1([a,b])$. We can define the **integral function**

$$F(x) = \int_{[a,b]} f\,d\lambda = \int_a^b f(t)\,dt, \quad x \in [a,b]$$

If $f \in \mathcal{C}([a,b])$, then $F$ is differentiable on $[a,b]$, and $F'(x) = f(x)$
What happens if $f \in L^1([a,b])$?

**Definition 11.1**
Given $f \in L^1([a,b])$. We say that $x \in [a,b]$ is a **Lebesgue point** for $f$ if

$$\lim_{h \to 0} \frac{1}{h} \int_x^{x+h} |f(t) - f(x)|\,dt = 0$$

If $x = a$ or $x = b$, this is the left/right lim.

**Remark 11.1**
A point $x$ is called a Lebesgue point for $f$ if $f$ 'does not oscillate too much' close to $x$:

- $f\ \mathcal{C}([a,b]) \to$ every $x \in [a,b]$ is a Lebesgue point.

- 
$$f(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$$

$$\lim_{h \to 0} \frac{1}{h} \int_0^h |f(t) - f(0)|\,dt = \lim_{h \to 0} \frac{1}{|h|} \int_0^h |0 - 1|\,dt = 0$$

**Theorem 11.1** (Lebesgue)
If $f \in L^1([a.b])$ then a.e. $x \in [a,b]$ is a Lebesgue point for $f$

**Remark 11.2**
In the definition of Lebesgue point, the pointwise values of $f$ are relevant

$$f = g \in L^1 \Leftrightarrow f = g \text{ a.e.}$$

Then the Lebesgue point of $f$ could be different from the one of $g$. This is not a big problem if $f = g$ a.e. on $[a,b] \Rightarrow f = g \in [a,b] \setminus N$ where $\lambda(N) = 0$; $x$ is a Lebesgue point for $f$, $\forall x \in [a,b] \setminus M$, $\lambda(M) = 0$
$\Rightarrow x$ is a Lebesgue point for $g$, $\forall x \in [a,b] \setminus (M \cup N)$
$[a,b] \setminus (M \cup N)$ is a set of full measure of Lebesgue points for $f$ and $g$.

To speak about Lebesgue points, one has to choose a specific representative $f \in L^1([a, b])$. If you change representative, you obtain the same set of Lebesgue points up to sets with 0-measure.

**Theorem 11.2** (First fundamental theorem of calculus)
Given $f \in L^1([a, b])$, $F(x) = \int_a^x f(t)\, dt$
Then $f$ is differentiable a.e. on $[a, b]$ and $F'(x) = f(x)$ a.e. in $[a, b]$

**_Proof._** Let $x \in [a, b]$ for any Lebesgue point for $f$ (a.e. $x \in [a, b]$ is fine). Consider

$$\left| \frac{F(x + h) - F(x)}{h} - f(x) \right| = \left| \frac{1}{h} \int_x^{x+h} (f(t) - f(x))\, dt \right| \leq \frac{1}{h} \int_x^{x+h} |f(t) - f(x)|\, dt \to 0$$

Since $x$ is a Lebesgue point. ★

**Definition 11.2**
Given $f : I \to \mathbb{R}$ is called **absolutely continuous** in $I$, $f \in AC(I)$, if $\forall\, \varepsilon > 0\; \exists\, \delta > 0$ s.t.

$$\bigcup_{k=1}^n [a_x, b_x] \in I \text{ with disjoint interiors}$$

$$\lambda(\bigcup_{k=1}^n [a_x, b_x]) = \sum_{k=1}^n (b_x - a_x) < \delta$$

$$\Rightarrow \sum_{k=1}^n |f(b_k) - f(a_k)| < \varepsilon$$

**Remark 11.3**
$f$ is uniformly continuous on $[a, b]$ if $\forall\, \varepsilon > 0\; \exists\, \delta > 0$ s.t.

$$|t - \tau| < \delta \Rightarrow |f(t) - f(\tau)| < \varepsilon$$

An absolutely continuous function is also uniformly continuous.
But the converse is false.

- If $f$ is Lipschitz on $[a, b] \Rightarrow f \in AC([a, b])$

Recall that $f \in \text{Lip}([a, b])$ if $\exists\, L > 0$ s.t.

$$|f(x) - f(y)| \leq L|x - y| \qquad \forall x, y \in [a, b]$$

<u>Check</u>: For any $\varepsilon > 0$, and consider

$$\sum_{k=1}^n |f(b_k) - f(a_k)| \leq \sum_{k=1}^n L(b_k - a_k) = L \sum_{k=1}^n (b_k - a_k)$$

If we take $\delta = \delta(\varepsilon) = \frac{\varepsilon}{L}$, then

$$\sum_{k=1}^n (b_x, a_x) < \delta \Rightarrow \sum_{k=1}^n |f(b_k) - f(a_k)| \leq L \sum_{k=1}^n (b_k - a_k)$$

★

$$\text{Lip}([a, b]) \subsetneq AC([a, b]) \subsetneq UC([a, b])$$

**Theorem 11.3** (Regularity of integral functions)
Given $f \in L^1([a,b])$, $F(x) = \int_a^x f(t)\,dt$, then $F \in \mathrm{AC}([a,b])$

To prove the theorem we need the

**Theorem 11.4** (Absolute continuity of the integral)
Given $f \in L^1(X, \mathcal{M}, \mu)$. Then $\forall\, \varepsilon > 0 \,\exists\, \delta > 0$ s.t.

$$\begin{matrix} E \in \mathcal{M} \\ \mu(E) < \delta \end{matrix} \Rightarrow \int_E |f|\,d\mu < \varepsilon$$

**Proof.** We fix $\varepsilon > 0$. Let $F_n := \{|f| < n\}$, $n \in \mathbb{N}$. Also $F_n \in \mathcal{M}\, \forall\, n$, $F_n \subseteq F_{n+1}$ and

$$\bigcup_{n=1}^{\infty} F_n = \{|f| < \infty\} =: F$$

$f \in L^1 \Rightarrow |f|$ is finite a.e.: $\mu(X \setminus F) = 0$. Therefore:

$$\int_X |f|\,d\mu = \int_{X \setminus F} |f|\,d\mu + \int_F |f|\,d\mu = \lim_{n \to \infty} \int_{F_n} |f|\,d\mu$$

$$\lim_{n \to \infty} \int_X |f|\,(\chi_{F_n})\,d\mu = 0$$

$\forall\, \varepsilon > 0 \,\exists\, \bar{n} \in \mathbb{N}$ s.t.

$$n > \bar{n} \Rightarrow \left| \int_X |f| \chi_{F_n^C}\,d\mu \right| < \frac{\varepsilon}{2}$$

Now, fix $\varepsilon > 0$, and take $n > \bar{n}(\varepsilon)$. If $E \in \mathcal{M}$, then

$$\int_E |f|\,d\mu = \int_{E \cap F_n} |f|\,d\mu + \int_{E \cap F_n^C} |f|\,d\mu \leq n \int_E 1\,d\mu + \int_{F_n^C} |f|\,d\mu$$

If we suppose that $\mu(E) < \frac{\varepsilon}{2n} =: \delta(\varepsilon)$, we deduce that

$$n \int_E 1\,d\mu = n\mu(E) < \frac{\varepsilon}{2}$$

Also, since $n > \bar{n}$

$$\int_{F_n^C} |f|\,d\mu < \frac{\varepsilon}{2}$$

$$\Rightarrow \int_E |f|\,d\mu < \varepsilon$$

★

*Regularity of integral functions.* Let $\varepsilon > 0$, and $\delta = \delta(\varepsilon) > 0$ be the value given by the absolute continuity of $\int |f|\,d\mu$. Take

$$E = \bigcup_{k=1}^{n} [a_k, b_k] \qquad E \subseteq [a, b]$$

If $\lambda(E) < \delta$, then

$$\sum_{k=1}^{n} |F(b_k) - F(a_k)| = \sum_{k=1}^{n} \left| \int_{a_k}^{b_k} f(t)\,dt \right| \leq \sum_{k=1}^{n} \int_{a_k}^{b_k} |f(t)|\,dt = \int_E |f|\,d\lambda < \varepsilon$$

by absolute continuity of $\int$

★

**Remark 11.4**

$\sqrt{x}$ is AC($[0, 1]$), but is not Lip($[0, 1]$).

$$\sqrt{x} = \int_0^x \frac{1}{2\sqrt{t}} \, dt$$

$\Rightarrow \sqrt{x}$ is the $\int$ function of a $L^1$ function

$\Rightarrow \sqrt{x} \in$ AC($[0, 1]$)

To sum up: the $\int$ function of a ($L^1$ function is AC, it is differentiable a.e., and

$$F(x) - F(a) = \int_a^x F'(t) \, dt \qquad\qquad \text{FC}$$

Suppose $G$ is differentiable a.e. on $[a, b]$ and FC holds for $G$:

$$G(x) - G(a) = \int_a^x G'(t) \, dt$$

What can we say about $G$?

**Remark 11.5**

If $G \in \mathcal{C}^1([a, b]) \Rightarrow$ FC holds.

If FC holds, then $G' \in L^1([a, b])$ (necessary condition)

Is the necessary condition also sufficient? In general not. Take $v(x)$, the Vital Cantor function: $v \in \mathcal{C}([0, 1]), v(0) = 0, v(1) = 1$. $v$ is differentiable a.e. on $[0, 1]$ but the calculus formula doesn't hold!

**Remark 11.6**

A function which is differentiable a.e. on an interval can behave very badly

**Theorem 11.5**

$G \in$ AC($[a, b]$). Then $G$ is differentiable a.e. on $[a, b]$, $G' \in L^1([a, b])$, and FC holds.

**Remark 11.7**

These theorems say that AC function are precisely the ones for which FC holds:

- $G \in$ AC $\Rightarrow$ FC holds.

- If FC holds, then $G' \in L^1([a, b])$

$\Rightarrow \int_a^x G'(t) \, dt \in$ AC

$\Rightarrow G(x) - G(a) = \int_a^x G'(t) \, dt \in$ AC

**Remark 11.8**

$v \in$ UC($[0, 1]$) by continuity and Heine Cantor, but $v \notin$ AC($[0, 1]$) because FC does not hold.

The proof of the second fundamental theorem of calculus is divided into two steps.

**Lemma 11.1**

The second fundamental theorem hold under the additional assumption that $G$ is monotone.

Second step: to get rid of the monotonicity.

For step 2, is it useful to give the

**Definition 11.3**

$[a, b] \subset \mathbb{R}$. Let

$$\mathcal{P}_{[a,b]} := \{(x_0, x_1, \ldots, x_n) : n \in \mathbb{N} \text{ and } a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$$

For $P \in \mathcal{P}_{[a,b]}$ and $f : [a, b] \to \overline{\mathbb{R}}$, define

$$v_a^b(f, P) := \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|$$

The total variation of $f$ on $[a, b]$ is

$$V_a^b(f) := \sup_{P \in \mathcal{P}_{[a,b]}} v_a^b(f, P) = \sup\{\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| : n \in \mathbb{N}, a = x_0 < x_1 < \cdots < x_n = b\}$$

If $V_a^b(f) < \infty$, we say that $f$ is a function with bounded variation, $f \in \mathrm{BV}\,([a, b])$

# 12 Lecture 20/10/2022

<u>Goal</u>: The $2^{nd}$ fundamental theorem of calculus. Given $G \in \mathrm{AC}([a, b])$. Then $G$ is differentiable a.e. on $[a, b]$, $G' \in L^1([a, b])$, and (FC) holds. Example and comments:

- If $f$ is bounded and monotone $\Rightarrow f \in \mathrm{BV}$

$$V_a^b(f) = |f(b) - f(a)|$$

  Note that $f$ may not be continuous

$$f(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases} \Rightarrow f \in \mathrm{BV}([-1, 1])$$

- $f \in \mathrm{BV}([a, b]) \Rightarrow f$ is bounded. Indeed

$$\sup_{x \in [a,b]} |f(x)| \leq |f(x)| + V_a^b(f) \stackrel{f \in \mathrm{BV}}{<} +\infty$$

- $f$ is continuous on $[a, b]$, or even if $f$ is differentiable everywhere in $[a, b] \not\Rightarrow f \in \mathrm{BV}([a, b])$

$$f(x) = \begin{cases} x^2 \cos \frac{2\pi}{x^2} & x \in (0, 1] \\ 0 & x = 0 \end{cases}$$

  It is continuous in $[0, 1]$, but $f \notin \mathrm{BV}([0, 1])$

- $f \in \mathrm{BV}([a, b]) \cap \mathrm{UC}([a, b]) \not\Rightarrow f \in \mathrm{AC}([a, b])$

$$\begin{aligned} &v \text{ a Vitali-Cantor function} \\ &v \text{ is bounded and monotone } \Rightarrow v \in \mathrm{BV}([0, 1]) \\ &v \in \mathrm{UC}([0, 1]) \end{aligned}$$

  But $v \notin \mathrm{AC}([0, 1])$

- If $f \in \mathrm{BV}([a, b]) \Rightarrow f$ is differentiable a.e. on $[a, b]$, and $f' \in L^1([a, b])$

We can now come back to the proof of Lemma 1 of the last lesson.

Preliminary result: $A \in \mathbb{R}$ open. Then

$$A = \bigcup_{n=1}^{\infty} (a_n, b_n) \text{ disjoint}$$

any open set of $\mathbb{R}$ is the (at most) countable union of open disjoint intervals.

Preliminary result (equivalent definition for AC): $f \in \mathrm{AC}([a,b]) \Leftrightarrow \forall \, \varepsilon > 0 \exists \, \delta > 0$ depending on $\varepsilon$ s.t.

$$\forall \bigcup_{n=1}^{\infty} [a_n, b_n], \quad [a_n, b_n] \text{ have disjoint interiors}$$

$$\sum_{n=1}^{\infty} (b_n - a_n) < \delta \Rightarrow \sum_{n=1}^{\infty} |f(b_n) - f(a_n)| < \varepsilon$$

***Proof***. We defined $\lambda$ starting from two properties

- invariance under translations

- $\lambda((x,y)) = y - x \; \forall \, a \leq < y \leq b$

Now, $G$ is monotone, say $G$ increasing (if $G \searrow$, take $-G$). We can repeat the construction of $\lambda$ in order to obtain a measure $\mu$ s.t.

- $\mu$ is invariant under translations

- $\mu((x,y)) = \underbrace{G(y) - G(x)}_{\geq 0} \; \forall \, a \leq x < y \leq b$ (for $\lambda$, take $G(t) = t$)

It can be proved that we obtain a measure on $(\mathbb{R}, \mathcal{L}(\mathbb{R}))$, complete.

On $(\mathbb{R}, \mathcal{L}(\mathbb{R}))$ we have two measures: $\lambda$ and $\mu$.

Idea: We take these measures on $([a,b], \mathcal{L}([a,b]))$, and we want to show that $\exists \frac{d\mu}{d\Lambda}$ (Radon-Nikodym)

We can check the hypothesis of the Radon-Nikodym theorem:

- $\lambda$ is $\sigma$-finite: $\lambda([a,b]) = b - a < +\infty$

- $\mu << \lambda$: $E \in \mathcal{L}([a,b])$, $\lambda(E) = 0 \Rightarrow \mu(E) = 0$

Assume $\lambda(E) = 0$. $G$ is $\mathrm{AC}([a,b])$: then $\forall \, \varepsilon > 0 \; \exists \, \delta = \delta(\varepsilon) > 0$ s.t.

$$\forall \bigcup_{n=1}^{\infty} [a_n, b_n], \quad [a_n, b_n] \text{ have disjoint interiors}$$

$$\lambda \left( \bigcup_{n=1}^{\infty} [a_n, b_n] \right) < \delta \Rightarrow \sum_{n=1}^{\infty} |G(b_n) - G(a_n)| < \varepsilon$$

Take this $\delta$. By regularity of $\lambda$, $\exists \, A$ open set of $[a,b]$ s.t. $A \supset E$ and $\lambda(A) < \delta$

$$A \text{ is open} \Rightarrow A = \left( \bigcup_{n=1}^{\infty} I_n^{\dagger} \right), \text{ disjoint}$$

---

$^{\dagger}$open intervals $= (x_n, y_n)$

it is a countable union of open intervals (maybe two of them contains $a$ or $b$)

$$\lambda(A) < \delta \Leftrightarrow \sum_{n=1}^{\infty}(y_n - x_n) < \delta$$

But then, since $\mu$ is a measure it is countably additive

$$\mu(E) \leq \mu(A) = \sum_n \mu(I_n) = \sum_n G(y_n) - G(x_n) \qquad \underset{\substack{\text{by the choice of } \delta \\ \text{and the fact that} \\ G \in \text{AC}}}{<} \qquad \varepsilon$$

We proved that
$$\lambda(E) = 0 \Rightarrow \forall\, \varepsilon > 0 : \mu(E) < \varepsilon \Rightarrow \mu(E) = 0$$

So $\mu << \lambda$. We can apply Radon Nikodym $\exists\, \phi : [a,b] \to [0,\infty]$ s.t.

$$G(x) - G(a) = \int_a^x \phi \, d\lambda$$

Since $G$ is bounded, then $\phi \in L^1([a,b])$

$$G(x) = G(a) + \int_a^x \phi(t) \, dt$$

By the first fundamental theorem of calculus, this is differentiable a.e.

$$\Rightarrow G'(x) = \phi(x) \text{ a.e. on } [a,b]$$

$$\Rightarrow G'(x) = G(a) + \int_a^x G'(t) \, dt$$

★

Now we want to get rid of the additional assumption (monotonicity).
<u>Preliminary result</u>: $f \in \text{BV}([a,b])$. Then

$$\varphi(x) = V_a^x(f), \quad \forall\, x \in [a,b]$$

is an increasing function.

**Proof.** By $a \leq x < y \leq b$. Then

$$V_a^y(f) = V_a^x(f) + \underbrace{V_x^y(f)}_{\geq 0} \geq V_a^x(f)$$

★

<u>Preliminary result</u>: If $G \in \text{AC}([a,b])$, then $G \in \text{BV}([a,b])$, and moreover

$$\varphi(x) = V_a^x(G) \text{ is in AC}([a,b])$$

*Proof of the second fundamental theorem of calculus in the general case.* $G \in \text{AC}([a,b])$

We want to write $G = G_1 + G_2$ where $G_1 \nearrow$ and $G_2 \searrow$, both AC.

Then the second fundamental theorem holds for $G_1$ and $G_2$ so it holds for $G$ by linearity of the integral.

We pose:

$$G_1(x) = \frac{G(x) + V_a^x(G)}{2}$$

$$G_2(x) = \frac{G(x) - V_a^x(G)}{2}$$

Clearly, $G_1 + G_2 = G$, $G_1, G_2$ are AC, by the last preliminary result.

$\underline{G_1 \nearrow}$: Let $a \le x < y \le b$

$$|G(y) - G(x)| \le V_x^y(G)$$

Therefore,

$$G_1(y) - G_1(x) = \frac{1}{2}( \underbrace{G(y) - G(x)}_{\substack{\ge -|G(y) - G(x)| \\ \ge -V_x^y(G)}} + V_a^y(G) + V_a^x(G)) \ge \frac{1}{2}(-V_x^y(G) + V_x^y(G)) = 0$$

So $G_1$ is decreasing. In an analogue way, we can prove that $G_2$ is decreasing. ★

# Functional analysis

Normed spaces and Banach spaces

**Definition 12.1**

Given $X$ vector space, a norm on $X$ is a function $\|\cdot\| : X \to [0, \infty)$ s.t.

- $\|x\| = 0 \Leftrightarrow x = 0$

- $\forall\, \alpha \in \mathbb{R}, \forall\, x \in X :$

$$\|\alpha x\| = |\alpha| \|x\| \text{ (positive homogeneity)} \qquad \text{(positive homogeneity)}$$

- $\forall\, x, y \in X:$

$$\|x + y\| \le \|x\| + \|y\| \qquad \text{(triangle inequality)}$$

Then, $(X, \|\cdot\|)$ is called a **normed space**

<u>Ex:</u> $|\|x\| - \|y\|| \le \|x - y\| \;\forall\, x, y \in X$

**Proposition 12.1**

$(X, \|\cdot\|)$ normed space. Then $(X, d)$ is a metric space for

$$d(x, y) = \|x - y\|$$

**Remark 12.1**

Normed space $\underset{\nleftarrow}{\rightarrow}$ metric space

<u>Examples:</u>

- $\mathbb{R}^N, \|x\|_p := \left(\sum_{i=1}^N |x_i|^p\right)^{\frac{1}{p}} \;\forall p \in [1, +\infty)$

$$\|x\|_\infty := \max_{i=1,\dots,N} |x_i|$$

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- $\mathcal{C}^0([a,b])$

$$\|f\|_\infty := \max_{x \in [a,b]} |f(x)|$$

- $L^1(X, \mathcal{M}, \mu)$

$$\|f\|_1 := \int_X |f| \, d\mu$$

This is a norm in $L^1$, but not on $\mathcal{L}^1$ ($\int_x |f| \, d\mu = 0 \Rightarrow f = 0$ a.e.)

- $L^\infty(X, \mathcal{M}, \mu)$

$$\|f\|_\infty := \operatorname*{ess\,sup}_{[a,b]} |f|$$

$(X, \|\cdot\|)$ normed space $\to (X, d)$ metric space $\to$ convergent sequences on $X$: $\{x_n\} \subset X$ is convergent in $X$ iff

$$d(x_n, x) \to 0 \Leftrightarrow \|x_n - x\| \to 0 \text{ as } n \to \infty$$

<u>Ex</u>: $x_n \to x$ in $X$, then $\|x_n\| \to \|x\|$ (the norm is a continuous function on $X$)

**Definition 12.2**
$\{x_n\}$ is a **Cauchy sequence** in $(X, \|\cdot\|)$ if $\forall \, \varepsilon > 0 \; \exists \, \bar{n} \in \mathbb{N}$ s.t.

$$n, m > \bar{n} \Rightarrow \|n_n - x_m\| < \varepsilon$$

**Definition 12.3**
$(X, \|\cdot\|)$ is called a **Banach space** if $(X, d)$ is complete, namely if any Cauchy sequence in $(X, d)$ is convergent.

If $(X, \|\cdot\|)$ is a normed space, we can speak about series in $X$. Let $\{x_n\} \subset X$ and $s_n = x_0 + x_1 + \ldots + x_n$, then $\sum_{n=0}^{+\infty} x_n = \{s_n\}$.

Then $\sum x_n$ is convergent if $\{s_n\}$ is convergent. If $\sum x_n$ is convergent, we write

$$s = \sum_{n=0}^{+\infty} x_n \Leftrightarrow s_n \to s$$

For numerical series

$$\sum_{n=1}^{\infty} |a_n| < +\infty \Rightarrow \sum a_n \text{ is convergent}$$

In general, in normed spaces

$$\sum_{n=1}^{\infty} \|x_n\| < +\infty \not\Rightarrow \sum_{n=1}^{\infty} x_n \text{ is convergent}$$

**Characterization**

$(X, \|\cdot\|)$ is a Banach space $\Leftrightarrow$ every series s.t. $\sum \|x_n\| < +\infty$ is also s.t. $\sum x_n$ is convergent

# 13 Lecture 26/10/2022

$(X, \|\cdot\|) \to (X, d) \to$ open sets, closed sets, bounded sets....

In $\mathbb{R}^n$ we are used to work with $\|\cdot\|_2$, but we could have many different norms.

**Definition 13.1**

Let $\|\cdot\|$ and $\|\cdot\|_2$ be two norms on the same vector space $X$. We say that these norms are **equivalent** if $\exists\, m, M > 0$ s.t.

$$m\|x\| \le \|x\|_2 \le M\|x\| \quad \forall\, x \in X$$

It can be proved that if two norms are equivalent they lead to different metric spaces, but to the same open sets, closed sets, convergent sequences, compact sets ...

**Theorem 13.1**

If $X$ is any finite dimension vector space, then all the norms on $X$ are equivalent.

**Remark 13.1**

This is why in $\mathbb{R}^n$ usually one does not specify the choice of the norm. One choose the Euclidean norm, since it comes from a scalar product. (ref. Hilbert spaces)

Preliminary fact: The set $S_1 = \{s \in \mathbb{R}^n : \|x\|_1 = 1\}$ is compact in $(\mathbb{R}^n, d)$

***Proof.*** We show that any norm is equivalent to $\|\cdot\|_1$

$$x = \sum_{i=1}^n x_i e_i \qquad \{e_i\}_{i=1,\dots,n} \text{ canonical basis}$$

Let's introduce the norm star

$$\|x\|_* = \left\|\sum_{i=1}^n x_i e_i\right\|_* \le \sum_{i=1}^n \|x_i e_i\|_* = \sum_{i=1}^n |x_i|\|e_i\|_* \le \left(\max_{1\le i\le n}\|e_i\|_*\right)\sum_{i=1}^n |x_i| = M\|x\|_1$$

We proved that $\exists\, M > 0$ s.t.
$$\|x\|_* \le M\|x\|_1 \quad \forall\, x \in X \tag{1}$$

Note that this proves that $\varphi(x) = \|x\|_*$ is continuous in $(X, d)$. Indeed

$$x_n \to x \Leftrightarrow d_1(x_n, x) \to 0$$

then

$$|\varphi(x_n) - \varphi(x)| = |\|x_n\|_* - \|x\|| \le \|x_n - x\|_* \overset{(1)}{\le} M\|x_n - x\|_1 \to 0$$

Therefore, by the Weierstrass theorem, $\exists$ a minimum point $x_0 \in S_1$ s.t.

$$\varphi(x) \ge \varphi(x_0) = m \quad \forall\, x \in S_1$$

(recall that $S_1$ is compact)
$$\|x\|_* \ge m \quad \forall\, x \in S_1$$

We claim that $m > 0$. If $m = 0$ then $\|x_0\|_* = 0 \Rightarrow x_0 = 0$ that is impossible, since $x_0 \in S_1$.

Thus $m > 0$. Let now $y \in \mathbb{R}^n, y \ne 0$. Then

$$\frac{y}{\|y\|_1} \in S_1 \Rightarrow \left\|\frac{y}{\|y\|_1}\right\|_* \ge m \Rightarrow \frac{1}{\|y\|_1}\|y\|_* \ge m \Rightarrow \|y\|_* m \ge m\|y\|_1 \quad \forall\, y \in \mathbb{R}^n$$

★

If $\dim X = +\infty$, then there are many non-equivalent norms.

Ex: In $\mathcal{C}^0([a,b])$, we can define $\|\cdot\|_\infty$ and $\|f\|_1 = \int_a^b |f(t)|\, dt$.

This is a norm in $\mathcal{C}^0$, but these norms are not equivalent.

**Separability**

$(X, d)$ metric space.

**Definition 13.2**

We say that $X$ is separable if $\exists \, A \in X$ which is dense $(\bar{A} = X)$ and countable

In $\mathbb{R}^n$, $\mathbb{Q}^n$ which is dense and countable. In $\infty - \dim$ we can have separable spaces or not. For instance, $(L^\infty, \|\cdot\|_\infty)$ is not separable. Instead $(\mathcal{C}^0([a,b]), \|\cdot\|_\infty)$ is a separable space.

*Sketch of the proof.* We will use the **Stone-Weierstrass theorem**.

The set of polynomials is dense on $\mathcal{C}^0([a,b])$ and is an uncountable set. However it can be proved that the set of polynomials with coefficients in $\mathbb{Q}$ is dense in the set of all polynomials

Moreover this set is countable. Then, by Stone-Weierstrass this is a countable dense set in $\mathcal{C}^0([a,b])$ ★

**Remark 13.2**

One can show that $\mathcal{C}^0(K)$ is separable whenever $K$ is a compact set of a metric space $(X, d)$

**Compactness**

In finite dimension (in $\mathbb{R}^n$), one has that

$$E \subset X \text{ is compact} \iff E \text{ is closed and bounded}$$

If $\dim X = \infty$, then only '$\Rightarrow$' is true. In finite dimension, we know that the closed unit ball is compact

$$\bar{B}_1(0) = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$$

What happens now if $(X, \|\cdot\|)$ is on $\infty - \dim$ normed space?

**Theorem 13.2** (Riesz's theorem)

$X$ normed space, $\dim X = +\infty \Rightarrow \bar{B}_1(0)$ is not compact

**Remark 13.3**

It is well known that if $E \in \mathbb{R}^n$ is compact, then $\forall \; \{x_n\} \in E \; \exists \; \{x_{n_k}\}$ subsequence s.t. $x_{n_k} \to x \in E$. This proposition is much harder to prove in $\infty - \dim$.

The proof of the Riesz's theorem is based on the Riesz's **quasi-orthogonality lemma**.

**Lemma 13.1** (Riesz Quasi-Orthogonality Lemma)

Let $X$ be a normed space, $E \subsetneq X$ a closed subspace. Then $\forall \, \varepsilon \in (0, 1) \; \exists \, x \in X$ s.t.

$$\|x\| = 1 \text{ and } \operatorname{dist}(x, E) = \inf_{y \in E} \|x - y\| \geq 1 - \varepsilon$$

**Proof.** Of the Riesz's Theorem. Assume that $\bar{B}_1(0)$ is compact, and $X$ has infinite dimension.

$\exists$ a sequence $\{E_n\}$ of finite dimensional subspaces (hence closed) of $X$ s.t.

$$E_{n-1} \subset E_n \text{ and } E_{n-1} \neq E_n$$

$E_{n-1}$ is a proper closed subspace of $E_n$, $\forall \, n$

We can apply the Riesz Lemma with $X = E_n$, $E = E_{n-1}$, $\varepsilon = \frac{1}{2}$. Then $\forall \, n \; \exists \, u_n \in E_n$ s.t. $\|u_n\| = 1$ and $\operatorname{dist}(u_n, E_{n-1}) \geq \frac{1}{2} \quad \forall \, n$

Therefore, we have a sequence $\{u_n\}$ with the following properties

$$\|u_n\| = 1 \qquad \forall \, n$$

$$\|u_n - u_m\| \geq \frac{1}{2} \qquad \forall \, n \neq m$$

$\Rightarrow$ this sequence cannot have any convergent subsequence. But then $\bar{B}_1(0) \supseteq \{u_n\}$, this implies that $\bar{B}_1(0)$ is not compact. Contradiction.

(In any $(X, \|\cdot\|)$ normed space, if $E$ is compact, then $\forall \, \{x_n\} \subset E \; \exists \, \{x_{n_k}\}$ s.t. $x_{n_k} \to x \in E$) ★

# 14    Lecture 27/10/2022

$(X, d)$ metric space.

**Definition 14.1**
$E \subset X$ is compact if for any open covering $\{A_i\}_{i \in I}$ has a finite subcover.

**Definition 14.2**
$E \subset X$ is sequentially compact if $\forall \{x_n\} \subset E$ there exists $\{x_{n_k}\}$ subsequence convergent to some limit $x \in E$

Well known fact: if $(X, d)$ is a metric space, then $E$ is compact $\Leftrightarrow$ $E$ is sequentially compact.

**Theorem 14.1** (Riesz Theorem)
$X$ normed space, $\dim X = \infty \Leftrightarrow \bar{B}_1(0)$ is not compact.

**Lemma 14.1** (Riesz quasi orthogonality Lemma)
$X$ normed space, $E \subsetneq X$ closed subspace. Then $\forall \, \varepsilon \in (0, 1) \; \exists \, x \in X$ s.t.

$$\|x\| = 1 \text{ and } \mathrm{dist}(x, E) = \inf_{y \in E} \|x - y\| \geq 1 - \varepsilon$$

**Remark 14.1** 
- $E \in X$ closed. Then $\mathrm{dist}(x, E) = 0 \Leftrightarrow x \in E$

- By definition of infimum, if $d = \mathrm{dist}(x, E)$, then $\forall \rho > 0 \; \exists \, z \in E$ s.t.

$$\|x - z\| < (1 + \rho)d$$

***Proof.*** Let $y \in X \setminus E$, and $d := \mathrm{dist}(y, E) > 0$, since $E$ is closed.
$\forall \, \rho > 0 \; \exists z \in E$ s.t.

$$\|y - z\| \leq (1 + \rho)d = \frac{d}{1 - \varepsilon} \tag{1}$$

since we choose $\rho$ s.t. $1 + \rho = \frac{1}{1 - \varepsilon}$. Now we set $x = \frac{y - z}{\|y - z\|}$.
Clearly $\|x\| = 1$. Moreover, $\forall \, u \in E$, we have that

$$\|x - u\| = \left\| \frac{y - z}{\|y - z\|} - u \right\| = \left\| \frac{y - z - \|y - z\|u}{\|y - z\|} \right\| = \frac{1}{\|y - z\|} \|y - (z + \|y - z\|u)\| =$$

$$= \frac{1}{\|y - z\|} \|y - w\| \geq \frac{1}{\|y - z\|} \mathrm{dist}(y, E) \overset{(1)}{\geq} \frac{1 - \varepsilon}{d} d = 1 - \varepsilon$$

Since this is true $\forall \, u \in E$, we deduce that

$$\mathrm{dist}(x, E) \geq 1 - \varepsilon$$

★

## Compactness on $\mathcal{C}^0([a, b])$

**Definition 14.3**
$\{f_n\}$ sequence in $\mathcal{C}^0([a, b])$. We say that $\{f_n\}$ is uniformly equicontinuous in $[a, b]$ if $\forall \, \varepsilon > 0 \; \exists \, \delta > 0$ depending only on $\varepsilon$ s.t.

$$|t - \tau| < \delta \Rightarrow \|f_n(t) - f_n(\tau)\| < \varepsilon \qquad \forall \, n$$

**Remark 14.2**
With respect to the uniform continuity, in this case $\delta$ does not depend on $f$. $\delta$ is the same for all the $f_n$ s

**Theorem 14.2**

$\{f_n\} \subseteq \mathcal{C}^0([a,b])$. Suppose that:

- $\{f_n\}$ is uniformly equicontinuous

- $\{f_n\}$ is bounded: $\exists\, M > 0$ s.t. $\|f_n\|_\infty < M \qquad \forall\, n$

Then $\exists$ a subsequence $\{f_{n_k}\}$ and $f \in \mathcal{C}^0([a,b])$ s.t. $f_{n_k} \to f$ uniformly.

Lebesgue spaces.

$(X, \mathcal{M}, \mu)$ measure space, $p \in [1, \infty]$. We defined $L^1(X)$ and $L^\infty(X)$. In a similar way, we define $L^p(X)\ \forall\, p \in [1, \infty]$

$$\mathcal{L}^p(X, \mathcal{M}, \mu) := \{f : X \to \overline{\mathbb{R}} \text{ measurable s.t. } \int_X |f|^p \, d\mu < \infty\}$$

On $\mathcal{L}^p$ we introduce the equivalent relation

$$f \sim g \text{ in } \mathcal{L}^p \Leftrightarrow f = g \text{ a.e. on } X$$

and define

$$L^p(X, \mathcal{M}, \mu) := \frac{\mathcal{L}^p(X, \mathcal{M}, \mu)}{\sim}$$

We want to show that this is a normed space with

$$\|f\|_p := \begin{cases} \left(\int_X |f|^p \, d\mu\right)^{\frac{1}{p}} & p \in [1, \infty) \\ \operatorname*{ess\,sup}_X |f| & p = \infty \end{cases}$$

The fact that $L^p$ is a vector space is easy to prove. The only non trivial part is that $f, g \in L^p \Rightarrow f + g \in L^p$.

This comes directly from the

**Lemma 14.2**

$p \in [1, \infty)$, $a, b \geq 0$. Then

$$(a + b)^p \leq 2^{p-1} (a^p + b^p)$$

$f, g \in L^p, \ p \in [1, \infty)$

$$\int_X |f + g|^p \, d\mu \leq \int_X (|f| + |g|)^p \, d\mu \leq 2^{p-1} \int_X (|f|^p + |g|^p) \, d\mu$$

$$= 2^{p-1} \int_X |f|^p \, d\mu + 2^{p-1} \int_X |g|^p \, d\mu < \infty$$

$L^p$ is a vector space, $\forall\, p \in [1, \infty)$.

$f, g \in L^\infty$. Then a.e.

$$\Rightarrow |f + g| \leq |f| + |g| \leq \|f\|_\infty + \|g\|_\infty < \infty \Rightarrow f + g \in L^\infty$$

$L^\infty$ is a vector space.

**Remark 14.3**

$l^p := L^p(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu_c)$. $l^p$ is a particular case of $L^p$

$$l^p = \{x = (x^{(k)})_{k \in \mathbb{N}} : \sum_{k=1}^\infty |x^{(k)}|^p < \infty\} \quad \|x\|_p = \left(\sum_{k=1}^\infty |x^{(k)}|^p\right)^{\frac{1}{p}} \quad p \in [1, \infty)$$

$$l^\infty = \{x = (x^{(k)})_{k \in \mathbb{N}} : \sup_{k \in \mathbb{N}} |x^{(k)}| < \infty\} \quad \|x\|_\infty = \sup_{k \in \mathbb{N}} |x^{(k)}|$$

Now we prove that $\|.\|_p$ is actually a norm in $L^p$. We will concentrate on $p < \infty$ ($p = \infty$ is the easy case)

Properties 1 and 2 of the norm are immediate to check:

(1) $\|f\|_p = 0 \Leftrightarrow \int_X |f|^p \, d\mu = 0 \Leftrightarrow f = 0$ a.e. on $X \Leftrightarrow f = 0 \in L^p$

(2) Obvious, by linearity

(3) About triangle inequality? We need some preliminaries

**Theorem 14.3** (Young's Inequality)
Let $p \in (1, \infty)$, $a, b \geq 0$. We say that $q$ is the conjugate exponent of p if

$$\frac{1}{p} + \frac{1}{q} = 1 \Leftrightarrow q = \frac{p}{p-1}$$

Then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

**Remark 14.4**
$p \in (1, \infty) \Rightarrow q \in (1, \infty)$. Moreover, we say that $1$ and $\infty$ are conjugate

**Proof.** $\varphi(x) = e^x$ is convex:

$$\varphi((1-t)x + ty) \leq (1-t)\varphi(x) + t\varphi(y) \qquad \forall x, y \in \mathbb{R} \quad \forall \, t \in [0,1]$$

If $a = 0$ or $b = 0$, then the thesis holds.
If $a, b > 0$

$$ab = e^{\log a} e^{\log b} = e^{\log a^{\frac{p}{p}}} e^{\log b^{\frac{q}{q}}} = e^{\frac{1}{p} \log a^p} e^{\frac{1}{q} \log b^q}$$

Since $\varphi$ is convex

$$\frac{1}{p} e^{\log a^p} + \frac{1}{q} e^{\log b^q} = \frac{1}{p} a^p + \frac{1}{q} b^q$$

$x = \log a^p$, $y = \log b^q$ $\qquad 1 - t = \frac{1}{p}, \; t = \frac{1}{q}$ $\qquad\qquad$ ★

**Theorem 14.4**
$(X, \mathcal{M}, \mu)$ measure space. $f, g$ measurable functions. $p, q \in [1, \infty]$ conjugate exponents.
Then

$$\|fg\|_1 \leq \|f\|_p \|g\|_q$$

**Proof.** Case $p, q \in (1, \infty)$. Obvious if $\|f\|_p \|g\|_q = \infty$.
If $\|f\|_p \|g\|_q = 0 \Rightarrow$ either $f = 0$ a.e. on $X$ or $g = 0$ a.e. on X $\Rightarrow fg = 0$ a.e. on $X \Rightarrow \|fg\|_1 = 0$.
Let then $\|f\|_p, \|g\|_p \in (0, \infty)$.
For $x \in X$, we set

$$a := \frac{|f(x)|}{\|f\|_p}, \; b := \frac{|g(x)|}{\|g\|_q}$$

and use Young:

$$\frac{|f(x)g(x)|}{\|f\|_p \|g\|_q} \leq \frac{1}{p} \frac{|f(x)|^p}{\|f\|_p^p} + \frac{1}{q} \frac{|g(x)|^q}{\|g\|_q^q}$$

$\forall \, x \in X$. By integrating, we obtain

$$\frac{1}{\|f\|_p \|g\|_q} \int_X |fg| \, d\mu \leq \frac{1}{p\|f\|_p^p} \int_X |f|^p \, d\mu + \frac{1}{q\|g\|_q^q} \int_X |g|^q \, d\mu = \frac{1}{p} + \frac{1}{q} = 1$$

$$\Rightarrow \|fg\| \leq \|f\|_p \|g\|_q$$

Case $p = 1$, $q = \infty$. Exercise $\qquad\qquad$ ★

**Theorem 14.5** (Minkowski Inequality)
$f, g \in L^p(X, \mathcal{M}, \mu)$, $p \in [1, \infty]$. Then

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p$$

***Proof.*** $p \in (1, \infty)$

$$\|f + g\|_p^p = \int_X |f + g|^p \, d\mu = \int_X |f + g||f + g|^{p-1} \, d\mu$$

$$\leq \int_X (|f| + |g|) |f + g|^{p-1} \, d\mu = \int_X |f||f + g|^{p-1} \, d\mu + \int_X |g||f + g|^{p-1} \, d\mu$$

Using Holder with $p$, $q = \frac{p}{p-1}$

$$\leq \|f\|_p \left( \int_X \left( |f + g|^{p-1} \right)^{\frac{p}{p-1}} \, d\mu \right)^{\frac{p-1}{p}} + \|g\|_p \left( \int_X \left( |f + g|^{p-1} \right)^{\frac{p}{p-1}} \, d\mu \right)^{\frac{p-1}{p}}$$

$$= \|f\|_p \|f + g\|_p^{p-1} + \|g\|_p \|f + g\|_p^{p-1}$$

We divide left hand side and right hand side by $\|f + g\|_p^{p-1}$:

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p$$

★

# 15 Lecture 09/11/2022

We introduced $L^p(X, \mathcal{M}, \mu)$, and we proved that this is a normed space with

$$\|f\|_p := \begin{cases} \left( \int_X |f|^p \, d\mu \right)^{\frac{1}{p}} & \text{if } p \in [1, +\infty) \\ \operatorname*{ess\,sup}_X |f| & \text{if } p = +\infty \end{cases}$$

## Inclusion of $L^p$ spaces

**Theorem 15.1**
Suppose that $\mu(X) < +\infty$. Then

$$1 \leq p \leq q \leq \infty \Rightarrow L^q(X) \subseteq L^p(X)$$

Meaning that any $f \in L^q$ is also in $L^p$. More precisely, $\exists\, C > 0$ depending on $\mu(X), p, q$ s.t.

$$\|f\|_p \leq \|f\|_q \quad f \in L^q(X)$$

***Proof.*** If $q = +\infty$
$f \in L^\infty(X)$: then $|f(x)| \leq \operatorname*{ess\,sup}_X |f| = \|f\|_\infty$ for a.e. $x \in X$, say $\forall\, x \in X \setminus A$, with $\mu(A) = 0$. Then

$$\int_X |f|^p \, d\mu = \int_{X \setminus A} |f|^p \, d\mu \leq \|f\|_\infty^p \int_{X \setminus A} 1 \, d\mu = \|f\|_\infty^p \underbrace{\mu(X)}_{=\mu(X \setminus A)}$$

If $q < +\infty$

Then $\frac{q}{p} > 1$, and we can use Hölder$\left(\frac{q}{p}, \left(\frac{q}{p}\right)'\right)$, where $\left(\frac{q}{p}\right)' = \frac{\frac{q}{p}}{\frac{q}{p}-1} = \frac{q}{q-p}$

$$\|f\|_p^p = \int_X |f|^p \, d\mu \overset{\text{Hölder}}{\leq} \left(\int_X \left(|f|^p\right)^{\frac{q}{p}} d\mu\right)^{\frac{p}{q}} \cdot \left(\int_X 1 \, d\mu\right)^{\frac{q-p}{p}} = \left(\int_X |f|^q \, d\mu\right)^{\frac{p}{q}} \cdot (\mu(X))^{\frac{q-p}{p}}$$

$$\Rightarrow \|f\|_p \leq \mu(X)? \frac{q-p}{qp} \|f\|_q$$

★

The assumption $\mu(X) < \infty$ is essential. For example, in $X = [1, \infty]$

$$\frac{1}{x} \in L^2([1, \infty]) \Leftrightarrow \int_1^\infty \frac{dx}{x^2} < \infty$$

$$\frac{1}{x} \notin L^1([1, \infty]) \Leftrightarrow \int_1^\infty \frac{dx}{x} = \infty$$

In particular, the previous theorem is false for $l^p$-spaces

$$l^p = L^p(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu_C)$$

$$1 \leq p \leq q \leq \infty \Rightarrow l^p \subseteq l^q, \text{ and } \exists \, C > 0 \text{ s.t. } \|x\|_q \leq C\|x\|_p \quad \forall \, x \in l^p$$

Without assumptions on $\mu(X)$, in general one has the interpolation inequality.

**Theorem 15.2**
$(X, \mathcal{M}, \mu)$ measure space. Let $1 \leq p \leq q \leq \infty$. If $f \in L^p(X) \cap L^q(X)$, then

$$f \in L^r(X) \quad \forall \, r \in (p, q)$$

and moreover

$$\|f\|_r \leq \|f\|_p^\alpha \|f\|_q^{1-\alpha}$$

where $\alpha$ is such that $\frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}$

***Proof.*** For exercise. Use Holder

★

## Completeness and Separability

**Theorem 15.3**
For $1 \leq p \leq \infty$, $L^p(X, \mathcal{M}, \mu)$ is a Banach space (with reference to $\|\cdot\|_p$)

***Proof.***

$p < \infty$ By using the characterization of completeness with the series, we want to show that if $\{f_n\} \subseteq L^p(X)$, and $\sum_{k=1}^\infty \|f_k\|_p < \infty \Rightarrow \sum_{k=1}^\infty f_k$ is convergent in $L^p$, namely $s_n = \sum_{k=1}^n f_k$ has a limit in $L^p$: $\|s_n - s\|_p \to 0$ as $n \to \infty$.

Let then $\{f_n\} \subseteq L^p(X)$ s.t.

$$\sum_{k=1}^\infty \|f_k\|_p = M < \infty$$

Define

$$g_n(x) = \sum_{k=1}^n |f_k(x)|$$

By Minkowski, $\|g_n\|_p \le \sum_{k=1}^n \|f_k\|_p \le M < \infty$. Moreover, for every $x \in X$ fixed, $\{g_n(x)\}$ is increasing $\Rightarrow g_n(x) \to g(x)$ as $n \to \infty$, $\forall \, x \in X$

$$\int_X |g|^p \, d\mu \stackrel{\text{Monot conv}}{=} \lim_n \int_X |g_n|^p \le M^p < \infty \Rightarrow g \in L^p(X)$$

$\Rightarrow |g|^p$ is finite a.e.:

$$\sum_{k=1}^\infty |f_k(x)| < \infty \text{ for a.e. } x \in X$$

$$\Rightarrow \sum_{k=1}^\infty f_k(x) \text{ is convergent a.e. to a limit } s(x)$$

Thus, we proved that $s_n(x) = \sum_{k=1}^n f_k(x) \to s(x)$ a.e. in $X$. Namely $|s_n - s|^p \to 0$ a.e. in $X$. To find a dominating function for $|s_n - s|^p$, we start by observing that

$$|s_n(x)| = \left| \sum_{k=1}^n f_k(x) \right| \le \sum_{k=1}^n |f_k(x)| = g_n(x) \le g(x) \text{ for a.e. } x \in X$$

Therefore

$$|s_n - s|^p \le 2^{p-1}(|s_n|^p + |s|^p) \le 2^{p-1}(g^p + g^p) = 2^p g^p \in L^1(X)$$

By the dominated convergence theorem

$$\int_X |s_n - s|^p \, d\mu \to 0 \Leftrightarrow \|s_n - s\|_p \to 0$$

Thus $L^p$ is complete.

$p = \infty$ exercise ★

To speak about separability, we give a

**Definition 15.1**
$g : \Omega \subset \mathbb{R}^n \to \mathbb{R}$. The support of $g$ is

$$\text{supp}\, g = \{x \in \Omega : \bar{g}(x) \ne 0\}$$

Also
$$\mathcal{C}_C^0 = \{f \in \mathcal{C}^0(\Omega) : \text{supp}\, f \text{ is compact in } \Omega\} = \mathcal{C}_O^0(\Omega) = \mathcal{C}_C(\Omega)$$

**Theorem 15.4**
$\Omega \in \mathcal{L}(\mathbb{R}), \lambda(\mathbb{R}) < +\infty$. Let also $f : \mathbb{R} \to \mathbb{R}$ measurable, s.t. $f \equiv 0$ in $\Omega^C$.
    Then $\forall \, \varepsilon > 0 \exists \, g \in \mathcal{C}_C^0(\mathbb{R})$ s.t.

$$\lambda\left(\{x \in \mathbb{R} : g(x) \ne f(x)\}\right) < \varepsilon$$

and
$$\sup_{\mathbb{R}} |g| \le \sup_{\mathbb{R}} |f|$$

**Definition 15.2**
Given $s$ simple function $= \sum_{k=1}^n a_k \chi_{E_k}$, where $E_1, \ldots, E_n$ are $\mathcal{L}$-measurable sets, $a_1, \ldots, a_n \in \mathbb{R}$.
$$E_1 \cup E_2 \cup \ldots \cup E_n = \mathbb{R}$$

We consider
$$\bar{\mathcal{S}}(\mathbb{R}) = \{s \text{ simple in } \mathbb{R} \text{ s.t. } \lambda((\{s \ne 0\}) < +\infty)\}$$

What does it mean for a simple function to be in $L^p(\mathbb{R})$?

$$\int_{\mathbb{R}} |s|^p \, d\mu = \sum_{k=1}^{n} a_k^p \lambda(E_k) < +\infty \qquad\qquad 1 \le p \le +\infty$$

iff $s \equiv 0$ outside a set of finite measure $\Leftrightarrow s \in \bar{\mathcal{S}}(\mathbb{R})$.
$\bar{\mathcal{S}}(\mathbb{R})$ is the set of integrable simple functions.

**Theorem 15.5**
$\bar{\mathcal{S}}(\mathbb{R})$ is dense in $L^p$, $\forall \, p \in (1, +\infty)$

Ma posso chiederti perchè hai deciso di scrivere s di r barrato e non tilde s di r?

**Proof.** $f \in L^p(\mathbb{R}), f \ge 0$ a.e. in $\mathbb{R}$.
We want to show that $\exists \, \{s_n\} \subseteq \bar{\mathcal{S}}(\mathbb{R})$ s.t. $\|s_n - f\|_p \to 0$.
By the simple approximation theorem, $\exists \, \{s_n\}$ of simple functions s.t. $\{s_n(x)\}$ is increasing, for every $x$, and $s_n \to f$ pointwise in $\mathbb{R}$.
Since $|s_n|^p \le f^p \Rightarrow s_n \in L^p$ for every n $\Rightarrow \{s_n\} \subseteq \bar{\mathcal{S}}(\mathbb{R})$. Moreover

$$|s_n - f|^p \to 0 \qquad \text{a.e. in } \mathbb{R}$$

$$|s_n - f|^p \le 2^{p-1}(|s_n|^p + |f|^p) \le 2^p |f|^p \in L^1$$

$\Rightarrow$ by dominated convergence

$$\int_{\mathbb{R}} |s_n - f|^p \, d\lambda \to 0 \text{ , namely } \|s_n - f\|_p \to 0$$

If $f$ is sign changing, then $f = f^+ - f^-$ and argue as before on $f^+$ and $f^-$     ★

**Theorem 15.6**
$\forall p \in [1, \infty)$, the space $L^p(\mathbb{R})$ is separable.

**Proof.** <u>sketch</u>

- Step 1: $\mathcal{C}_C^0(\mathbb{R})$ is dense in $L^p(\mathbb{R})$, $\forall \ \le p \le \infty$.

  Take $s \in \tilde{\mathcal{S}}(\mathbb{R})$. Then, by Lusin theorem, $\exists \{f_n\} \subseteq \mathcal{C}_C^0(\mathbb{R})$ s.t. $\|f_n - s\|_p \to 0$. Then, since any $f \in L^p$ can be approximated by simple integrable functions, we have that $f$ can be approximated by functions in $\mathcal{C}_C^0(\mathbb{R})$.

- Step 2:

  By Stone Weierstrass, the set of polinomials $\mathcal{P}(\mathbb{R})$ is dense in $\mathcal{C}_C^0(\mathbb{R})$ with the $\|\cdot\|_\infty$ norm. Since we work with functions with compact support, this implies that $\mathcal{P}(\mathbb{R})$ is dense in $\mathcal{C}_C^0(\mathbb{R})$ also with respect to $\|\cdot\|_p$

  $$\int_{-M}^{M} |f - p_n|^p \, d\lambda \le \|f - p_n\|_\infty^p 2M \to 0$$

  if $\|f - p_n\|_\infty \to 0, \Rightarrow \mathcal{P}(\mathbb{R})$ is dense in $L^p(\mathbb{R})$.
  $\tilde{\mathcal{P}}(\mathbb{R}) = \{$ polynomials with rational coefficients $\}$. This is countable, and is dense in $(\mathcal{P}(\mathbb{R})), \|\cdot\|_p$. $\Rightarrow$ is dense in $L^p$

      ★

What about $L^\infty(\mathbb{R})$? In this case $\mathcal{C}(\mathbb{R})$ are not dense in $L^\infty(\mathbb{R})$. For example, consider

$$f(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

If $g \in L^\infty$ s.t. $\|g - f\|_\infty < \frac{1}{3}$, then $g$ cannot be continuous. Assume by contradiction that $\exists\, g \in \mathcal{C}(\mathbb{R})$ s.t. $\|g - f\|_\infty < \frac{1}{3}$. Then

$$\operatorname{ess\,sup}_{\mathbb{R}} |g(x) - f(x)| < \frac{1}{3}$$

In particular, $g(x) < \frac{1}{3}\ \forall\, x < 0$

$$\Rightarrow \lim_{x \to 0^-} g(x) \leq \frac{1}{3}$$

On the other hand, $g(x) > \frac{2}{3}\quad \forall\, x > 0$

$$\Rightarrow g(0) = \lim_{x \to 0^+} g(x) \geq \frac{2}{3}$$

# 16  Lecture 10/11/2022

Quick recap about the 'delirium' on the separability
    The thing that you need to know, in $\to L^p(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$, are:

(1) $L^p$ is separable $\forall\, p \in [1, \infty)$

(2) $\tilde{S}(\mathbb{R})$ is dense in $L^p(\mathbb{R})\ \forall\, p \in [1, \infty)$, namely $\forall p \in L^p(\mathbb{R})$ and $\forall\, \varepsilon > 0\ \exists\, s \in \tilde{S}(\mathbb{R})$ s.t.

$$\|f - s\|_p < \varepsilon$$

(3) $\mathcal{C}_C^0(\mathbb{R})$ is dense in $L^p$, namely $\forall p \in L^p(\mathbb{R})$ and $\forall\, \varepsilon > 0\ \exists\, g \in \mathcal{C}_C^0(\mathbb{R})$ s.t.

$$\|f - g\|_p < \varepsilon$$

Everything remains true if you replace $\mathbb{R}$ with $X$ open or closed, or with $X \in L(\mathbb{R}^n)$, and consider $(X, L(X), \lambda)$.
    What happens for $L^\infty(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$?
    $\mathcal{C}(\mathbb{R})$ is not dense in $L^\infty$.
    By the simple approximation theorem, we have that simple functions are dense in $L^\infty$.

**Theorem 16.1**
$L^\infty(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$ is not separable.

***Proof.*** $\{\chi_{[-\alpha, \alpha]} : \alpha > 0\} \subseteq L^\infty(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)\ \chi_\alpha = \chi_{[-\alpha, \alpha]}$
    This is an uncountable family of functions. $\|\chi_\alpha - \chi_{\alpha'}\|\ \forall\, \alpha \neq \alpha'$

$$|\chi_\alpha(x) - \chi_{\alpha'}(x)| = \begin{cases} 0 & if\, x \in [-\alpha, \alpha] \cup (\alpha', \infty) \cup (-\infty, -\alpha') \\ 1 & if\, x \in (\alpha, \alpha'] \cup [-\alpha', \alpha) \end{cases}$$

In particular, $B_{\frac{1}{2}}(\chi_\alpha) \cap B_{\frac{1}{2}}(\chi_{\alpha'}) = \varnothing\ \forall\, \alpha \neq \alpha'$
    Assume by contradiction that $L^\infty(\mathbb{R})$ is separable: $\exists Z \subset L^\infty$ which is countable and dense. In particular, $\forall\, f \subset L^\infty\ \exists\, g \in Z$ s.t.

$$\|g - f\|_\infty < \frac{1}{2}$$

Therefore, $\forall\, \alpha\, \exists g_\alpha \in B_{\frac{1}{2}}(\chi_\alpha) \cap Z$. But $B_{\frac{1}{2}(\chi_\alpha) \cap B_{\{\frac{1}{2}}(\chi_{\alpha'})} = \varnothing$

$$\Rightarrow \alpha \neq \alpha', \text{ we have } g_\alpha \neq g_{\alpha'}$$

$Z \supseteq \{g_\alpha : \alpha > 0\}$, which is uncountable. This is not possible, since $Z$ is countable. $\qquad\bigstar$

**Remark 16.1**
The same is true if $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$ is swapped with $(X, \mathcal{L}(X), \lambda)$, $X$ is open or closed on $\mathbb{R}$ or $\mathbb{R}^n$

## Linear operators

$(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ normed spaces.
**Definition 16.1**
$T : D(T) \subseteq X \to Y$ is a **linear operator** (or map) if

$$T(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 T(x_1) + \alpha_2 T(x_2) \quad \forall\, x_1, x_2, \in D(T) \quad \forall\, \alpha_1, \alpha_2 \in \mathbb{R}$$

$D(T)$ is a linear subspace of $X$, and is called the domain of T. When $D(T) = X$ and $Y = \mathbb{R}$, $T$ is called linear functional.

**Definition 16.2**
A linear operator $T : D(T) \subseteq X \to Y$ is bounded if $D(T) = X$ and $\exists\, M > 0$ s.t.

$$\|T_X\|_Y \leq M|x|_X \forall\, x \in X$$

Recall that $T$ is continuous in $x_0 \in X$ iff

$$\forall\, \{x_n\} \subset X, x_n \xrightarrow{X} x_0 \Rightarrow Tx_n \xrightarrow{Y} Tx_0$$

Ex:

- $L : \mathbb{R}^n \to \mathbb{R}$ is a linear functional . Then $\exists\, y \in \mathbb{R}^n$ s.t.

$$Lx = <y, x> = (y, x) = y \cdot x$$

  In particular, then $L$ is continuous on $\mathbb{R}^n$ and bounded:

$$|L_X| < |<y, x>| \overset{\overset{\text{Cauchy-Schwarz}}{}}{\leq} \|y\|\|x\| \qquad \forall\, x \in \mathbb{R}^n$$

  So $L$ is bounded with $M = \|y\|$.

- Linear operators in $\infty$-dim may not be defined everywhere, and many may not be continuous: $(X, \|\cdot\|_X) = (Y, \|\cdot\|_Y) = (\mathcal{C}([0,1]), \|\cdot\|_\infty)$.
  Consider
$$\frac{d}{dx} : \mathcal{C}'([0,1]) \subseteq X \to Y \quad \frac{d}{dx}(\alpha f + \beta g) = \alpha \frac{d}{dx} f + \beta \frac{d}{dx} g$$
$$f \mapsto f'$$

  This is not continuous or bounded. For example, take $f_n(x) = \frac{1}{n} \sin 2\pi nx$. $\|f_n\|_\infty \to 0$ but $\|f_n'\|_\infty = 1$

  In this case $f_n \to 0 \nRightarrow \frac{d}{dx} f_n \to 0$, then $\frac{d}{dx}$ is not bounded as well.

- Let $(X, \|\cdot\|_X)$ be a normed space. If $\dim X = 0$, is it possible to find linear functionals which are not bounded? Yes.

**Definition 16.3**
A subset $\{qualcosa\}$ is called **Hamel basis** of $X$ if

$$\|e_i\|_X = 1 \ \forall \ i$$

and if every $x \in X$ can be written in a unique way as

$$x = \sum_{k=1}^n x_k e_{i_k}, \quad x_k \in \mathbb{R}, \ n \in \mathbb{N}$$

Every $x$ can be written uniquely as a finite linear combination of element of the basis. If $\dim X = \infty$ is not immediate that the Hamel basis exists. This can be proved using the axiom of choice. (Zorn's lemma).

Any normed space has a Hamel basis $\dim X = \infty \Rightarrow \{e_i\}_{i \in I}$ has $\infty$ many elements.

Let then $(X, \|\cdot\|_X)$ be $\infty$ dim, with Hamel basis $\{e_i\}_{i \in I}$. $I$ is infinite $\Rightarrow I \supseteq \mathbb{N}$.

We define $L : X \to \mathbb{R}$ in the following way

$$Le_0 = 0 \qquad Le_1 = 1 \quad \ldots \quad Le_n = n \quad \ldots$$
$$Le_i = 0 \quad \forall \ i \in I \setminus \mathbb{N}$$

Then, for $x \in X$ we set

$$Lx = L\left(\sum_{k=1}^n x_k e_{i_k}\right) = \sum_{k=1}^n x_k Le_{i_k}$$

$L$ is linear by contradiction, and it is not bounded:

$$|Le_n| = n \to \infty \quad \|e_n\|_X = 1 \ \forall n$$
$$\frac{|Le_n|}{\|e_n\|_X} \to \infty \Rightarrow L \text{ L is not bounded}$$

**Remark 16.2**
In practice, Hamel basis are hard to use. They differ from Hilbertian basis.

For linear operators, boundedness and continuity are equivalent.

**Theorem 16.2**
$T : X \to Y$ linear map. Then the following are equivalent

(1) $T$ is continuous in $0 \in X$

(2) $T$ is continuous everywhere in $X$

(3) $T$ is bounded

**Remark 16.3**
$T$ linear $\Rightarrow T0 = 0$. Indeed

$$T0 = T(0x) = 0Tx = 0$$

***Proof.***     • (2) $\Rightarrow$ (1) obvious.

    • (1) $\Rightarrow$ (3) Suppose by contradiction that $T$ is not bounded.

Then $\exists \{x_n\} \subset X$, $x_n \neq 0$, s.t.

$$\frac{\|Tx_n\|_Y}{\|x_n\|_X} \geq n \quad \forall n$$

Define

$$z_n := \frac{x_n}{n\|x_n\|_X}$$

Then $\|z_n\|_X = \frac{1}{n\|x_n\|}\|x_n\|_X \to 0$, namely $z_n \to 0$ in $X \Rightarrow$ ($T$ is continuous in $0$) $Tz_n \to T0 = 0$. However,

$$\|Tz_n\|_Y = \left\| T\left(\frac{x_n}{n\|x_n\|_X}\right) \right\| = \frac{1}{n\|x_n\|_X}\|Tx_n\|_Y \geq 1 \ \forall \ n$$

Contradiction.

- $(2) \Rightarrow (2)$ We observe that $\|Tx_1 - Tx_2\|_Y = \|T(x_1 - x_2)\|_Y \leq M\|x_1 - x_2\|_X \ \forall \ x_1 \ x_2 \in X$
Then, let $x \in X$ and let $x_n \to x$ in $X$: $\|x_n - x\|_X \to 0$. But then

$$\|Tx_n - Tx\|_Y \leq M\|x_n - x\|_X \to 0$$

namely $Tx_n \to Tx$ in $Y$. This is the continuity.

★

**Definition 16.4**
The set of linear operators $T : X \to Y$ which are also bounded (continuous) is denoted by $\mathcal{L}(X,Y)$
If $Y = X$, one simply writes $\mathcal{L}(X)$

This is a vector space. $\forall T, S \in \mathcal{L}(X,Y)$, $\forall \ \alpha, \beta \in \mathbb{R}$ :

$$(\alpha T + \beta S)(x) = \alpha Tx + \beta Sx \qquad \in \mathcal{L}(X,Y)$$

We can also introduce a norm:

$$\|T\|_{\mathcal{L}(X,Y)} = \|T\|_{\mathcal{L}} := \sup_{\|x\|_X \leq 1} \|Tx\|_Y$$

Also,

$$\|T\|_{\mathcal{L}(X,Y)} = \sup_{\|x\|_X = 1} \|Tx\|_Y = \sup_{x \neq 0} \frac{\|Tx\|_Y}{\|x\|_X} = \inf M > 0 \text{ s.t. } \|Tx\|_Y \leq M\|x\|_X \quad \forall \ x \in X$$

**Theorem 16.3**
$X$ normed space, $Y$ Banach space. Then $(\mathcal{L}(X,Y), \|\cdot\|_{\mathcal{L}(X,Y)})$ is a Banach space.

**Proof.** Let $\{T_n\}$ be a Cauchy sequence in $\mathcal{L}(X,Y)$. We want to show that $\exists T \in \mathcal{L}(X,Y)$ s.t.

$$\|T_n - T\|_{\mathcal{L}} \to 0$$

$\{T_n\}$ cauchy: $\forall \ \varepsilon > 0 \ \exists \bar{n} \in \mathbb{N}$ s.t.

$$n, m > \bar{n} \Rightarrow \|T_n - T_m\|_{\mathcal{L}} < \varepsilon$$

Consider then $\{T_n x\}$, $x \in X$

$$\|T_n x - T_m x\|_Y = \|(T_n - T_m)x\|_Y \leq \|T_n - T_m\|_Y \|x\|_X \leq \varepsilon\|x\|_X \tag{*}$$

This means that $\{T_n x\}$ is a Cauchy sequence in $Y$, which is complete: then $\forall \ x \in X \ \exists$ a vector $y_x \in Y$ s.t. $T_n x \to y_x$ in $Y$.

Define
$$T : X \to Y \qquad x \mapsto y_x = Tx$$

$T$ is linear: indeed, $\forall\, x_1,\, x_2 \in X$ and $\alpha_1,\, \alpha_2 \in \mathbb{R}$:

$$T(\alpha_1 x_1 + \alpha_2 x_2) = \lim_{n \to \infty} T_n(\alpha_1 x_1 + \alpha_2 x_2) = \lim_{n \to \infty}(\alpha_1 T_n x_1 + \alpha_2 T_n x_2) = \alpha_1 Tx_1 + \alpha_2 Tx_2$$

So $T$ is linear. It remains to show that $T$ is bounded, and that $\|T_n - T\|_{\mathcal{L}} \to 0$. To show that $T$ is bounded, note that, by (*), $\forall\, \varepsilon > 0\ \exists\, \bar{n}$ s.t.

$$n, m > \bar{n} \Rightarrow \|T_n x - T_m x\|_Y \leq \varepsilon \|x\|_X \quad \forall\, x$$

Take the limit for $m \to \infty$:
$$\|T_n x - Tx\|_Y \leq \varepsilon \|x\|_X$$

But then, since $T_n$ is bounded,

$$\|Tx\|_Y = \|Tx \pm T_n x\|_Y \leq \|T_n x\|_Y + \|Tx - T_n x\|_Y \leq M_n \|x\|_X + \varepsilon \|x\|_X = (M_n + \varepsilon)\|x\|_X$$

and $T$ is bounded. To show that $\|T_n - T\|_{\mathcal{L}} \to 0$, observe that $\forall\, \varepsilon > 0\ \exists\, \bar{n}$ s.t. $n > \bar{n}$

$$\|T_n x - Tx\|_Y \leq \varepsilon \|x\|_X \Leftrightarrow \frac{\|(T_n - T)x\|_Y}{\|x\|_X} \leq \varepsilon \quad \forall\, x \in X \setminus 0 \overset{\text{take sup over } x \neq 0}{\Rightarrow} \|T_n - T\|_{\mathcal{L}} < \varepsilon$$

namely, $T_n \to T$ in $\mathcal{L}$ ★

# 17 Lecture 16/11/2022

Let $T$ be a linear operator from $X$ to $Y$.

**Definition 17.1**
The **kernel** of $T$ is the set

$$\ker(T) = \{x \in X : Tx = 0\} \subset X$$

This is a vector subspace of $X$.

$T$ is injective $\Leftrightarrow \ker(T) = \{0\}$. If $T$ is continuous, $\ker(T)$ is closed

$$\ker(T) = T^{-1}(\{0\})$$

**Definition 17.2**
$X$, $Y$ normed spaces. $X$ and $Y$ are isomorphic if $\exists\, T \in \mathcal{L}(X, Y)$ bijective, and such that $T^{-1} \in \mathcal{L}(X, Y)$

**Definition 17.3**
$T \in \mathcal{L}(X, Y)$ is an isometry if
$$\|Tx\|_Y = \|x\|_X \quad \forall\, x \in X$$

**Definition 17.4**
If $X \subseteq Y$ is a vector subspace, and $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are normed space, then we can consider
$$J : \ X \to Y \ \ (\text{inclusion map})$$
$$x \mapsto x$$

If $J \in \mathcal{L}(X, Y)$ (namely, if $\exists\, M > 0$ s.t. $\|x\|_Y \leq M\|x\|_X\ \forall\, x \in X$), then we say that $J$ is an embedding of $X$ into $Y$, and we write $X \hookrightarrow Y$

<u>Ex</u>: $\mu(X) < \infty$, $1 \leq p < q \leq \infty$

$$L^q(X) \hookrightarrow L^p(X) \qquad\qquad\qquad (\text{inclusion of } L^p \text{ spaces})$$

# Some fundamentals theorems on linear operators

**Definition 17.5**

$(X, d)$ metric space. $A \subset X$. $x \in X$ is an **adherence point** of $A$ if $\forall \, r > 0 : B_r(x) \cap A \neq \varnothing$

$$\bar{A} = \{x \in X : x \text{ is an adherence point of } A\} = A \cup \partial A$$

**Definition 17.6**

$A \subset X$ is dense in $X$ if $\bar{A} = X$.

For example, $\mathbb{Q}$ is dense in $\mathbb{R}$, and $(a, b)$ is dense in $[a, b]$.

**Definition 17.7**

$A \subset X$ is nowhere dense if the interior of the closure of $A$ is empty, namely

$$\text{int}(\bar{A}) = \bar{A}^\circ = \varnothing$$

Ex: $\{\bar{x}\}^\circ = \{x\}^\circ = \varnothing$

$\mathbb{Z} \subset \mathbb{R}$: $\bar{\mathbb{Z}}^\circ = \mathbb{Z}^\circ = \varnothing$

$\mathbb{Q}$ is not nowhere dense: $(\bar{\mathbb{Q}})^\circ = (\mathbb{R})^\circ = \mathbb{R}$

**Definition 17.8**

$A \subset X$ is called **of first category** (or **meager set**) in $X$ if $A$ is the (at most) countable union of nowhere dense sets.

Ex: $\mathbb{Q}$ is of first category in $\mathbb{R}$: countable union of nowhere dense sets

$$\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\}$$

**Definition 17.9**

$A \subset X$ is of second category if it is not of first category.

**Theorem 17.1** (Baire category theory)

$(X, d)$ complete metric space. Then

- $\{U_n\}_{n=0}^\infty$ is a sequence of open and dense sets in $X \Rightarrow \cap_{n=0}^\infty U_n$ is dense in $X$.

- $X$ is of second category in itself: $X$ cannot be the countable union of nowhere dense sets.

Preliminaries:

- $A \subset X$ is dense $\Leftrightarrow \forall \, W \subset X$, $W$ open, $W \neq \varnothing$, we have that $A \cap W \neq \varnothing$

- $A$ is nowhere dense $\Leftrightarrow \left(\bar{A}\right)^C$ is open and dense

***Proof.*** Here's the proof of the two parts of the theorem:

(a) Thanks to the first preliminary, we show that $\forall W \subset X$ open and non empty we have $(\cap_n U_n) \cap W \neq \varnothing$

$$U_0 \text{ is open and dense: } \overset{1^{st}\text{prel.}}{\Rightarrow} \underbrace{U_0 \cap W}_{\text{is open}} \neq \varnothing$$

$$\Rightarrow \text{ it contains an open ball}$$

$$\Rightarrow (U_0 \cap W) \supset B_{r_0}(x_0) \text{ for some } x_0 \in X \text{ and } r_0 > 0$$

For $n > 0$, we choose $x_n \in X$ and $r_n > 0$ inductively in the following way: we have

$$U_n \cap B_{r_{n-1}}(x_{n-1}) \neq \varnothing \qquad\qquad (1^{st} \text{ prel. } + U_n \text{ is dense})$$

$$\Rightarrow \overline{B_{r_n}(x_n)} \subset (U_n \cap B_{r_{n-1}}(x_{n-1}))$$

all these balls
are included in
$B_{r_0}(x_0)$

with $x_n \in X$ and $0 < r_n < \frac{1}{2^n}$

By the condition on $r_n$, we see that

$$x_n, x_m \in B_{r_N}(x_N) \quad \forall\, n, m > N$$

$\Rightarrow \{x_n\}$ is a Cauchy sequence in $X$

$$d(x_n, x_m) \leq \frac{1}{2^N} \quad \forall\, n, m > N$$

$X$ is complete: $x_n \xrightarrow{d} x \in X$ Since

$$x_n \in B_{r_N}(x_N) \qquad\qquad\qquad\qquad\qquad\qquad \forall\, n > N$$
$$\Rightarrow x = \lim_n x_n \in \overline{B_{r_N}(x_N)} \subset (U_n \cap B_{r_0}(x_0)) \subset (U_N \cap W) \quad \forall\, n \in \mathbb{N}$$

$$\Rightarrow x_n \in \bigcap_n (U_n \cap W) = \left(\bigcap_n U_n\right) \cap W$$

This means that $\bigcap_n U_n$ is dense.

(b) It follows from (a):

If $\{E_n\}$ is a sequence of nowhere dense sets in $X$, then, by the second preliminary $\{(E_n)^C\}$ is a sequence of open and dense sets. By (a)

$$\bigcap_n (\overline{E_n})^C \neq \varnothing$$

$$\Rightarrow \bigcup_n E_n \subset \bigcup_n \overline{E_n} = X \setminus \left(\overbrace{\bigcap_n (\overline{E_n})^C}^{= \varnothing}\right) \neq X$$

$\bigstar$

Ex: $(X, \|\cdot\|) \; \infty - \dim$ Banach space. $\{e_i\}_{i \in I}$ Hamel basis.
  Then $I$ is uncountable.

**Theorem 17.2** (Banach Steinhaus)
$X$ Banach space, $Y$ normed space, $\mathcal{F} \subseteq \mathcal{L}(X, Y)$ family. Suppose that $\mathcal{F}$ is pointwise bounded:

$$\forall\, x \in X \quad \exists\, M_x > 0 \text{ s.t. } \sup_{T \in \mathcal{F}} \|Tx\|_Y \leq M_x \qquad\qquad \text{(PB)}$$

Then $\mathcal{F}$ is uniformly bounded:

$$\exists\, M \geq 0 \text{ s.t. } \sup_{T \in \mathcal{F}} \|T\|_{\mathcal{L}(X,Y)} \leq M \qquad\qquad \text{(UB)}$$

**Proof.** $\forall\, n \in \mathbb{N}$, let

$$C_n := \{x \in X : \|Tx\|_Y \leq n \quad \forall\, T \in \mathcal{F}\} = \cap_{T \in \mathcal{F}}\{x \in X : \|Tx\|_Y \leq n\}$$

$C_n$ is a closed set $\forall\, n$, since $T$ is continuous. (also $\varphi : X \to \mathbb{R}$ $\varphi(x) = \|Tx\|_Y$ is continuous)

By (PB), every $x \in X$ stays in some $C_n$: $X = \cup_{n=1}^{\infty} C_n$. Since $X$ is Banach, by the Baire theorem it is necessary that $\exists n_0 \in \mathbb{N}$ s.t. $C_{n_0}{}^{\mathrm{o}} \neq \varnothing \Rightarrow$ a ball $B_r(x_0) \subset C_{n_0}$: then

$$n_0 \geq \|T(x_0 + rz)\|_Y \overset{\text{linearity}}{=} \|Tx_0 + rTz\|_Y \overset{\text{triangle ineq}}{\leq} r\|Tz\|_Y - \|Tx_0\|_Y \quad \forall\, T \in \mathcal{F} \,\forall\, z \in \overline{B_1}(0)$$

To sum up: $\forall\, T \in \mathcal{F}$, $\forall\, z \in \overline{B_1}(0)$ we have

$$r\|Tz\|_Y - \|Tx_0\|_Y \leq n_0 \Rightarrow \|Tz\|_Y \leq \frac{1}{r}(n_0 + M_{x_0})$$

We take sup over $T \in \mathcal{F}$:

$$\sup_{T \in \mathcal{F}} \|T\|_{\mathcal{L}(X,Y)} \leq \frac{1}{r}\left(n_0 + M_{x_0}\right) =: M$$

★

## Corollary 17.1
$X$ Banach space, $Y$ normed space. $\{T_n\} \subseteq \mathcal{L}(X,Y)$ s.t. $\{T_n x\}$ have a limit, denoted by $Tx$, $\forall\, x \in X$ (pointwise convergence). Then $T \in \mathcal{L}(X,Y)$

**Proof.** $T$ is linear:
$$\begin{array}{ccc}
T_n(\alpha_1 x_1 + \alpha_2 x_2) & = & \alpha_1 T_n x_1 + \alpha_2 T_n x_2 \\
\downarrow & & \downarrow \\
T(\alpha_1 x_1 + \alpha_2 x_2) & = & \alpha_1 T x_1 + \alpha_2 T x_2
\end{array}$$

Now we observe that we have (PB): if $\{T_n x\}$ is convergent $\Rightarrow \{T_n x\}$ is bounded $\Rightarrow$ by Banach Steinhaus, $\{T_n\}$ is uniformly bounded:

$$\exists M > 0 \text{ s.t. } \sup_n \|T_n\|_{\mathcal{L}(X,Y)} \leq M$$

Therefore, $\forall\, x \in X$:

$$\|Tx\|_Y = \left\|\lim_n (T_n x)\right\|_Y = \lim_n \|T_n x\|_Y \leq \lim_n \|T_n\|_{\mathcal{L}} \|x\|_X \leq \lim_n M\|x\|_X = M\|x\|_X$$

Thus, $T$ is bounded: $T \in \mathcal{L}(X,Y)$ ★

# 18 Lecture 17/11/2022

Let $X, Y$ be normed spaces.

**Definition 18.1**
$T : X \to Y$ is called **open map** if, $\forall\, A \subset X$ open, the set $T(A) \subset Y$ is open.

**Remark 18.1**
Recall that $T$ is continuous on $X$ if $T^{-1}(O)$ is open on $X$, $\forall\, O$ open in $Y$.

Ex: $f(x)$ : constant is continuous, but not open. $f((a,b)) = \{\text{const}\}$

**Theorem 18.1** (Open map theorem)
$X, Y$ Banach spaces. $T \in \mathcal{L}(X,Y)$ is surjective. Then $T$ is an open map.

## Corollary 18.1

$X, Y$ Banach spaces, $T \in \mathcal{L}(X, Y)$ is bijective. Then $T$ is an isomorphism: $T^{-1} \in \mathcal{L}(X, Y)$

**Proof.**     • $T : Y \to X$ is linear. (Exercise. Hint: Use $T^{-1} \circ T = \text{Id} +$ linearity of $T$)

• We want now to check that $T^{-1}$ is continuous on $Y$: $(T^{-1})^{-1}(O)$ is open in $Y$, $\forall\, O$ open in $X$. We know that $T$ is an open map thanks to the open map theorem.

$$(T^{-1})^{-1}(O) = \left\{ y \in Y, T^{-1}(y) \in O \right\} = \left\{ y \in Y, T^{-1}(y) = x, \text{ for some } x \in O \right\} =$$

$$= \{ y \in Y, y = Tx, \text{ for some } x \in O \} = T(O) \text{ is open}$$

Since $T$ is an open map, $\forall\, O \subset X$, open.

★

## Corollary 18.2

$X$ vector space, $\|\cdot\|, \|\cdot\|_*$ norms on $X$. Assume $(X, \|\cdot\|), (X, \|\cdot\|_*)$ are Banach spaces. Assume that $\exists\, C_1 > 0$ s.t.

$$\|x\|_* \leq C_1 \|x\| \quad \forall\, x \in X$$

Then $\|\cdot\|$ and $\|\cdot\|_*$ are equivalent, namely $\exists\, C_2 > 0$ s.t.

$$\|x\| \leq C_2 \|x\|_*$$

**Proof.** Consider

$$I : \quad (X, \|\cdot\|) \quad \to (X, \|\cdot\|_*)$$
$$x \quad \mapsto x$$

By assumption, $I$ is bounded: $\exists\, C_1 > 0$ s.t.

$$\|Ix\|_* = \|x\|_* \leq C_2 \|x\|$$

$I$ is bijective.

Thus, by the corollary before

$$I^{-1} = I \in \mathcal{L}((X, \|\cdot\|_*), (X, \|\cdot\|))$$

namely $\exists\, C_2 > 0$ s.t.

$$\|Ix\| \leq C_2 \|x\|_*$$
$$\overset{\shortparallel}{\|x\|}$$

★

## Definition 18.2

$T : D(T) \subset X \to Y$ linear operator. We say that $T$ is **closed** if $\forall\, \{x_n\} \subset D(t)$.

$$\left. \begin{array}{ll} x_n \to x & \text{in } X \\ Tx_n \to y & \text{in } Y \end{array} \right\} \Rightarrow x \in D(T) \text{ and } Tx = y$$

<u>Ex</u>: $X = Y = \mathcal{C}^0([0, 1])$ with the supremum norm.

$$T = \frac{d}{dx}$$

$T$ is not continuous. But it is closed: it can be proved that if $\{f_n\} \subset \mathcal{C}^1([0, 1])$ is s.t.

$$\left. \begin{array}{ll} f_n \to f & \text{uniformly} \\ f_n' \to g & \text{uniformly} \end{array} \right\} \Rightarrow f \text{ is } \mathcal{C}^1([0, 1]) \text{ and } f' = g$$

<u>Ex</u>: $T \in \mathcal{L}(X, Y) \Rightarrow T$ is closed

**Remark 18.2**

$T$ is a closed operator $\Leftrightarrow$ the graph of $T$ is closed.

$$\text{graph}(T) = \{(x, Tx) : x \in X\}$$

**Theorem 18.2** (Closed graph theorem)

$X, Y$ Banach spaces.

$\quad T : X \to Y$ linear closed operator $(D(T) = X)$.

$\quad$ Then $T \in \mathcal{L}(X, Y)$.

**Remark 18.3**

In general it is easier to prove that an operator is closed, rather than it it continuous.

*Proof.* Define on $X$ the graph-norm of $T$

$$\|x\|_* = \|x\|_X + \|Tx\|_Y$$

Then is a norm on $X$. If $\{x_n\} \in X$ is a Cauchy sequence for $\|\cdot\|_*$, then $\{x_n\}$ is a Cauchy sequence in $(X, \|\cdot\|_X)$ and $\{Tx_n\}$ is a Cauchy sequence on $(Y, \|\cdot\|_Y)$

$$\Rightarrow \left. \begin{array}{ll} x_n \to x & \text{in } X \\ Tx_n \to y & \text{in } Y \end{array} \right\} \text{ since } T \text{ is closed, we deduce that } y = Tx$$

Thus

$$\|x_n - x\|_X + \|Tx_n - Tx\| \to 0$$

This proves that $(X, \|\cdot\|_*)$ is a Banach space. Also, we know that

$$\|x\|_X \le \|x\|_X + \|Tx\|_Y = \|x\|_*$$

By the last corollary of the open map theorem, $\exists\, C_2$ s.t.

$$\|x\|_* \le C_2 \|x_X\|$$

$$\|Tx\|_Y \le \|x\|_* \le C_2 \|x\|_X \quad \forall\, x \in X$$

This means that $T$ is bounded. ★

## Dual spaces

$X$ normed space:

$$X^* = \mathcal{L}(X, \mathbb{R}) \text{ is called } \textbf{dual space of } X$$

$X$ normed space, $Y$ Banach space $\Rightarrow \mathcal{L}(X, Y)$ is a Banach space with $\|\cdot\|_{\mathcal{L}}$.

$\quad$ Since $\mathbb{R}$ is a Banach space, the dual space $X^*$ is a Banach space with

$$\|L\|_* = \sup_{\|x\|_X \le 1} |Lx|$$

Ex:

- In $\mathbb{R}^n$, only linear functional is separated by a scalar product:

$$L : \mathbb{R}^n \to \mathbb{R} \text{ is linear } \Rightarrow \exists!\, y \in \mathbb{R}^n \text{ s.t. } Lx = <y, x>$$

  It can be proved that

$$L \in (\mathbb{R}^n)^* \mapsto y \in \mathbb{R}^n$$

  is an isometric isomorphism

$$(\mathbb{R}^n)^* \approx \mathbb{R}^n$$

  Then $X^*$ is very complicated.

- Dual of $L^p$?

  $(X, \mathcal{M}, \mu)$ measure space. $p \in [1, \infty]$, $p'$ conjugate exponent.

$$\frac{1}{p} + \frac{1}{p'} = 1 \iff \begin{cases} p' = \dfrac{p}{p-1} & p \in (1, \infty) \\ p' = \infty & p = 1 \\ p' = 1 & p = \infty \end{cases}$$

  For $g \in L^{p'}(X)$, define $L_g : L^p(X) \to \mathbb{R}$ by

$$l_g f := \int_X fg \, d\mu \qquad \forall \, f \in L^p(\Omega)$$

  This is well defined, by the Holder inequality:

$$\left| \int_X fg \, d\mu \right| \le \int_X |fg| \, d\mu = \|fg\|_1 \le \|g\|_{p'} \|f\|_p \qquad \qquad *$$

  If $g \in L^{p'}$, this shows that $L_g f \in \mathbb{R} \quad \forall \, f \in L^p$

**Proposition 18.1**
If $p \in [1, \infty]$ then $L_g \in (L^p(X)^*)$. Moreover,

- if $p > 1$, then $\|L_g\|_* = \|g\|_{p'}$

- if $p = 1$ then $\|L_g\|_* = \|g\|_\infty$ with more assumptions (they are satisfied in $(X, \mathcal{L}(X), \lambda)$)

**Remark 18.4**
We are saying that $L^{p'}$ can be identified with a subspace of the dual space $(L^p)^*$ and this identification is an isometry.

  Question: are there functional in $(L^p)^*$?

***Proof.*** (of the proposition)

- Case $p = \infty$ ex

- Case $p = 1$ but difficult it's ok if you don't do it

- Case $p \in (1, \infty)$

  $L_g$ is clearly linear, by linearity of $\int$, indeed: $\forall \, \alpha \, \beta \in \mathbb{R}$, $f_1 \, f_2 \in L^p(X)$. Then

$$L_g(\alpha f_1 + \beta f_2) = \int_X g(\alpha f_1 + \beta f_2) \, d\mu = \alpha \int_X g f_1 \, d\mu + \beta \int_X g f_2 \, d\mu = \alpha L_g f_1 + \beta L_g f_2$$

  We want to show now that $L_g$ is bounded. We proved in (*) that

$$|L_g f| \le \|g\|_{p'} \|f\|_p \quad \forall \, f \in L^p(\Omega)$$

  This shows that $L_g$ is bounded, with norm $\|L_g\|_* \le \|g\|_{p'}$ (remember that $\|T\|_{\mathcal{L}} = \inf\{M > 0 : \|Tx\|_Y \le M\|x\|_X \quad \forall x \in X\}$)

  We want to show that $\|L_g\|_* = \|g\|_{p'}$. If $\|L_g\|_* < \|g\|_{p'}$, then $\exists \, M < \|g\|_{p'}$ s.t.

$$|L_g f| \le M\|f\|_p \quad \forall \, f \in L^p$$

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We rule out this possibility by choosing an explicit $\tilde{f} \in L^p$ s.t.

$$\left| L_g \tilde{f} \right| = \|g\|_{p'} \left\| \tilde{f} \right\|_p$$

We take

$$\tilde{f} = \frac{|g|^{p'-1}}{\|g\|_{p'}^{p'-1}} \frac{g}{|g|}$$

Now,

$$\left\| \tilde{f} \right\|_p^p = \int_X \left| \tilde{f} \right|^p d\mu = \int_X \frac{|g|^{p(p'-1)}}{\|g\|_p^{p(p'-1)}} d\mu = (*)$$

$(p')' = p \Rightarrow p = \frac{p'}{p'-1} \Rightarrow p(p'-1) = p'$

$$(*) = \frac{1}{\|g\|_{p'}^{p'}} \int_X |g|^{p'} d\mu = \frac{\|g\|_{p'}^{p'}}{\|g\|_{p'}^{p'}} = 1$$

$$\left| L_g \tilde{f} \right| = \left| \int_X \frac{|g|^{p'-1}}{\|g\|_{p'}^{p'-1}} |g| \, d\mu \right| = \left| \int_X \frac{|g|^{p'}}{\|g\|_{p'}^{p'-1}} d\mu \right| = \frac{1}{\|g\|_{p'}^{p'-1}} \|g\|_{p'}^{p'} = \|g\|_{p'} = \|g\|_{p'} \left\| \tilde{f} \right\|_p$$

<div align="right">★</div>

**Hahn Banach**

**Definition 18.3**
$X$ vector space. A map $p : X \to \mathbb{R}$ is called sublinear functional if

- $p(\alpha x) = \alpha p(x) \qquad \forall\, x \in X, \, \alpha > 0$

- $p(x + y) \leq p(x) + p(y) \qquad \forall x, y \in X$

**Theorem 18.3** (Hahn Banach)
$X$ real vector space, $p : X \to \mathbb{R}$ sublinear functional. $Y$ subspace of $X$ and suppose that $\exists\, f : Y \to \mathbb{R}$ linear on $Y$ s.t.
$$f(y) \leq p(y) \quad \forall\, y \in Y$$

Then $\exists$ a linear functional $F : X \to \mathbb{R}$ s.t.

$$F(y) = f(y) \quad \forall\, y \in Y \qquad\qquad\qquad F \text{ is an extension of } f$$

Moreover,
$$F(x) \leq p(x) \quad \forall\, x \in X$$

# 19 Lecture 23/11/2022

**Theorem 19.1** (Hahn-Banach regarding continuous extension)
$X$ (real) normed space. $Y$ subspace of $X$, $f \in Y^* = \mathcal{L}(Y, \mathbb{R})$
    Then $\exists\, F \in X^* = \mathcal{L}(X, \mathbb{R})$ s.t.

$$F(y) = f(y) \qquad \forall\, y \in Y$$
$$\|F\|_{X^*} = \|f\|_{Y^*}$$

***Proof.*** Define $p : X \to \mathbb{R}$, $p(x) = \|f\|_{Y^*}\|x\|_X$ $\forall$ $x \in X$. Then $p$ is sublinear (from the properties of $\|\cdot\|_X$).

Moreover, $f(y) \leq |f(y)| \leq \|f\|_{Y^*}\|y\|_X = p(y)$ $\forall$ $y \in Y$. Then, by Hahn-Banach theorem (general version), $\exists$ $F : X \to \mathbb{R}$ s.t. $F$ is an extension of $f$ and $F(x) \leq p(x)$ $\forall$ $x \in X$.

Now, if $F(x) \geq 0$

$$|F(x)| = F(x) \leq p(x) = \|f\|_{Y^*}\|x\|_X$$

If $F(x) < 0$

$$|F(x)| = -F(x) = F(-x) \leq p(-x) = \|f\|_{Y^*}\|-x\|_X = \|f\|_{Y^*}\|x\|_X$$

$\forall$ $x \in X$

$$|F(x)| \leq \|f\|_{Y^*}\|x\|_X$$

namely, $F \in X^*$ (it is bounded), and

$$\|F\|_{X^*} \leq \|f\|_{Y^*}$$

Also, $\|F\|_{X^*} \geq \|f\|_{Y^*}$ since $F$ extends $f$:

$$\|F\|_{X^*} = \sup_{\|x\|_X \leq 1} |F(x)| \geq \sup_{\|y\|_Y \leq 1} |F(y)| = \sup_{\|y\|_X \leq 1, y \in Y} |f(y)| = \|f\|_{Y^*}$$

<div align="right">★</div>

Consequence 1

**Theorem 19.2**

$(L^\infty(X))^*$ 'strictly contains' $L^1(X)$

***Proof.*** We must show that $\exists$ $L \in (L^\infty(X))^*$ s.t. $\nexists$ $g \in L^1(X)$ s.t.

$$Lf = \int_X fg\,d\mu \quad \forall f \in L^\infty(X)$$

For simplicity, we consider $(X, \mathcal{M}, \mu) = ([-1,1], \mathcal{L}([-1,1]), \lambda)$. Let $Y$ be the subspace of $L^\infty([-1,1])$ of the bounded continuous functions $\mathcal{C}^0([-1,1])$. On $Y$ we define

$$\Lambda f = f(0) \quad \forall f \in Y$$

We can do it since $f \in \mathcal{C}^0([-1,1])$ (for elements in $L^\infty$ we cannot speak about pointwise values!).

$\Lambda$ is linear:

$$\Lambda(\alpha f + \beta g) = \alpha \Lambda f + \beta \Lambda g$$

Moreover, $\Lambda$ is in $Y^*$:

$$|\Lambda f| = |f(0)| < \max_{[-1,1]} |f| = \|f\|_\infty$$

This proves that $\Lambda \in Y^*$, $\|\Lambda\|_{Y^*} \leq 1$. By Hahn-Banach, $\exists$ $L \in (L^\infty(X))^*$ which is an extension of $\Lambda$, and is s.t.

$$\|L\|_{(L^\infty)^*}$$

Can we have

$$Lf = \int_{-1}^1 fg\,d\mu \quad \text{for some } g \in L^1(X)?$$

Suppose by contradiction that this is true, take

$$f_n \in \mathcal{C}^0([-1,1])$$

defined in this way:
$$f_n(x) = \varphi(nx)$$
where $\varphi$ is continuous, supp $\varphi \subseteq \left[\frac{-1}{2} \frac{1}{2}\right]$
$$\varphi(0) = 1, \varphi(nx) = 0 \quad \forall \, x \text{ s.t. } |nx| > \frac{1}{2} \Leftrightarrow |x| > \frac{1}{2n}$$

By contradiction,
$$\sup f_n \subseteq \left[-\frac{1}{2n}, \frac{1}{2n}\right] \Rightarrow f_n(x) \to 0$$

Therefore, if $g \in L^1([-1,1])$ is s.t.
$$\int_{-1}^{1} f_n g \, d\lambda = L f_n$$

Then, on one side
$$\int_{-1}^{1} f_n g \, d\mu = L f_n = f_n(0) = 1 \quad \forall \, n \tag{1}$$

But on the other side

- $f_n(x)g(x) \to 0$ a.e. in $[-1,1]$

- $|f_n(x)g(x)| \le g(x) \in L^1([-1,1])$

$$\overset{\text{DOM}}{\Rightarrow} \int_{-1}^{1} f_n g \, d\lambda \to 0 \tag{2}$$

But (1) and (2) are in contradiction. In conclusion, there is no $g \in L^!([-1,1])$ s.t.
$$\int_{-1}^{1} f g \, d\lambda = L f \quad \forall \, f \in L^\infty([-1,1])$$

★

Other consequences of the Hahn-Banach theorem
**Corollary 19.1**
$X$ (real) normed space, $x_0 \in X \setminus \{0\}$. Then $\exists \, L_{x_0} \in X^*$ s.t.
$$\|L_{x_0}\|_{X^*} = 1 \text{ and } L_{x_0}(x_0) = \|x_0\|$$

***Proof.*** Take $Y = \{\lambda x_0 : \lambda\}$ cose

This is linear and continuous on $Y \Rightarrow$ by Hahn-Banach (continuous extension) $\exists \, \tilde{L}_0 \in X^*$ s.t. $\tilde{L}_0$ extends $L_0$ and
$$\left\|\tilde{L}_0\right\|_{X^*} = \|L_0\|_{Y^*} = \sup_{\substack{\lambda x_0 \in Y \\ \|\lambda x_0\| = 1}} |L_0(\lambda x_0)| = \sup |\lambda| \|x_0\|_X| = 1$$

Thus $\tilde{L}_0$ is precisely the desired functional.
$$\tilde{L}_0(x_0) = L_0 = \|x_0\|_X$$
and
$$\left\|\tilde{L}_0\right\|_{X^*} = 1$$

★

**Corollary 19.2** (The bounded linear functionals separate points)
If $x, y \in X$ and $Lx = Ly \ \forall \ L \in X^* \Rightarrow x = y$ (if $x \neq y, \exists \ L \in X^*$ s.t. $Lx \neq Ly$)

**Proof.** Assume $x - y \neq 0$. Then, by the previous corollary, $\exists \ L \in X^*$ s.t.

$$\|L\|_{X^*} \text{ and } L(x-y) = \|x-y\|_X \Rightarrow Lx - Ly = L(x-y) = \|x-y\|_X \neq 0$$

$\bigstar$

**Corollary 19.3**
$X$ normed space, $Y$ closed subspace of $X$, $x_0 \in X \setminus Y$.
   Then $\exists \ L \in X^*$ s.t. $L|_Y = 0$ and $L_{x_0} \neq 0$

# Reflexive spaces

$X$ Banach space, $X^*$ dual space.
<u>Notation:</u> $L \in X^* : Lx = L(x) = <L, x> = \underset{X^*}{<} L, x \underset{X}{>}$
   $(X^*)^*$ dual space of $X^*$ is called the **bidual** of $X$, denoted by $X^*$

$$X^{**} = \mathcal{L}(X^*, \mathbb{R})$$

We can describe many elements of $X^{**}$ in the following way: for $x \in X$, define

$$\begin{aligned} \Lambda : \ & X^* \to \mathbb{R} \\ & L \mapsto Lx = \underset{X^*}{<} L, x \underset{X}{>} \end{aligned}$$

($\Lambda_x$ evaluates functionals in $X^*$ in the point $x$).
   $\Lambda_x$ is linear:

$$\Lambda_x(\alpha L_1 + \beta L_2) = (\alpha L_1 + \beta L_2)(x) = \alpha L_1 x + \beta L_2 x = \alpha \Lambda_x L_1 + \beta \Lambda_x L_2$$

Moreover, it is bounded

$$|\Lambda_x(L)| = |Lx| \underset{L \in X^*}{\leq} \|L\|_{X^*} \|x\|_X \quad \forall \ L \in X^*$$

Moreover,

$$\|\Lambda_x\|_{\mathcal{L}(X^*, \mathbb{R})} = \sup_{L \neq 0} \frac{|\Lambda_x L|}{\|L\|_X^*}$$

We claim that $\|\Lambda_x\|_{\mathcal{L}} = \|x\|_X$. Indeed, by the first corollary of Hahn-Banach, given any $x \in X \setminus \{0\} \ \exists \ Lx \in X^*$

$$\exists \ L_x \in X^* \text{ s.t } |L_x x| = \|x\|_X, \text{ and } \|L_x\|_{X^*} = 1$$

$$\Rightarrow \sup_{L \neq 0} \frac{|\Lambda_x L|}{\|L\|_{X^*}} = \sup_{L \neq 0} \frac{|Lx|}{\|L\|_{X^*}} \geq \frac{|L_x x|}{\|L_x\|_{X^*}} = \|x\|_X$$

$$\Rightarrow \|\Lambda_x\|_{X^{**}} = \|x\|_X$$

**Theorem 19.3**
$\exists$ a map

$$\begin{aligned} \tau : \ & X \to X^{**} \\ & x \mapsto \Lambda_x \end{aligned} \qquad \text{(Canonical Map)}$$

which is linear, continuous and an isometry. Namely, the canonical map is an isometric isomorphism from $X$ into $\tau(X) \subseteq X^{**}$

Question: are there other elements in $X^{**}$?

**Definition 19.1**

If the canonical map is surjective, then we say that $X$ is **reflexive**, $X \cong X^{**}$. Otherwise, $\tau(X)$ will be a strict close subspace of $X$.

**Remark 19.1**

$X$ reflexive $\not\Leftarrow$ $X$ and $X^{**}$ are isometrically isomorphic.

**Theorem 19.4**

$X$ reflexive space. Then every closed subspace of $X$ is reflexive.

**Theorem 19.5**

$X$ Banach.

$$X \text{ reflexive } \Leftrightarrow X^* \text{ reflexive}$$

**Theorem 19.6**

$X$ Banach.

- If $X^*$ is separable $\Rightarrow X$ is separable

- If $X$ is separable and reflexive $\Rightarrow X^*$ is separable

To show that a space is reflexive, it is convenient to introduce the following notion.

**Definition 19.2**

$X$ Banach space. $X$ is called **uniformly convex** if $\forall \, \varepsilon > 0 \, \exists \, \delta > 0$ s.t.

$$\forall \, x, y \in X \text{ with } \|x\| \leq 1, \|y\| \leq 1, \|x - y\| > \varepsilon$$

then we have

$$\left\| \frac{x+y}{2} \right\| < 1 - \delta$$

This is a quantitative version of the strict convexity.

**Definition 19.3**

$C \subset X$ is convex $\Leftrightarrow \forall \, x, y \in C : \frac{x+y}{2} \in C$
$C \subset X$ is **strictly convex** $\Leftrightarrow \forall \, x, y \in C : \frac{x+y}{2} \in C^{\mathrm{o}}$

Roughly speaking, $X$ is uniformly convex if $\overline{B_1(0)}$ is strictly convex in a quantitative way.

**Theorem 19.7** (Milman-Pettis)

Every uniformly convex Banach space is reflexive.

# 20  Lecture 24/11/2022

Recap on reflexivity:

$X$ Banach space. $X^{**} = (X^*)^*$ is the **bidual space**, $\mathcal{L}(X^*, \mathbb{R})$

$$\forall \, x \in X \, \exists \, \Lambda_x : X^* \to \mathbb{R} \text{ defined by } \Lambda_x(L) = Lx \quad \forall \, L \in X^*$$

We proved that $\Lambda_x \in X^{**}$. Thus we can define the **canonical map**:

$$\tau : \begin{array}{l} X \to X^{**} \\ x \mapsto \Lambda_x \end{array} \qquad \text{(Canonical Map)}$$

We stated that $\tau$ is an isometric isomorphism from $X$ into $\tau(X)$. This is true but for our purpose it's even too much, and it's difficult to prove in details. However, we can prove a slightly weaker result

## Theorem 20.1

$\tau$ is linear, continuous, and is an isometry

$$\|\tau(x)\|_{X^{**}} = \|x\|_X \quad \forall \, x \in X$$

Moreover, $\tau$ is injective. If $\tau$ is also surjective, it is an isometric isomorphism between $X$ and $X^{**}$

**Proof.** There are two parts:

- $\tau$ is linear and continuous: exercise.

  $\tau$ is an isometry: $\|\tau(x)\|_{X^{**}} = \|\Lambda_x\|_{X^{**}} = \|x\|_X$

  $\tau$ is injective: $x \neq y \Rightarrow \tau(x) \neq \tau(y)$?

  $x \neq y \Rightarrow$ by the second corollary to Hahn-Banach $\exists \, L \in X^*$ s.t. $Lx \neq Ly$.

  $$\underset{X^{**}}{<} \tau(x), L \underset{X^*}{>} = \Lambda_x(L) = Lx \neq Ly = \Lambda_y(L) = \underset{X^{**}}{<} \tau(x), L \underset{X^*}{>}$$

  Then, $\tau(x) \neq \tau(y)$ and $\tau$ is injective.

- Let now $\tau$ be surjective. Then $\tau \in \mathcal{L}(x, X^{**})$ and is bijective $\Rightarrow$ by a corollary of the open map theorem, $\tau^{-1} \in \mathcal{L}(X^{**}, X)$

★

## Definition 20.1

$X$ is reflexive if $\tau$ is surjective. In this case, $\tau$ is an isometric isomorphism between $X$ and $X^{**}$

We formally mentioned that

## Theorem 20.2

If $(X, \|\cdot\|)$ is uniformly convex $\Rightarrow (X, \|\cdot\|)$ is reflexive.

## Remark 20.1

## Proposition 20.1

If $(X, \|\cdot\|)$ is uniformly convex $\Rightarrow \overline{B_1(0)}$ is strictly convex

**Proof.** Is it true that if $x, y \in \overline{B_1(0)}$, then $\frac{x+y}{2} \in B_1(0)$? Since $(X, \|\cdot\|)$ is uniformly convex, we know that ($\|x - y\| =: \overline{\varepsilon} > 0$)

$$\forall \, \overline{\varepsilon} > 0 \quad \exists \, \overline{\delta} > 0 \text{ s.t. } \|x\| \leq 1 \, \|y\| \leq \, \|x - y\| > \frac{\overline{\varepsilon}}{2} \Rightarrow \left\| \frac{x+y}{2} \right\| < 1 - \overline{delta}$$

In particular,

$$\frac{x+y}{2} < 1 - \overline{\delta} < 1 \Rightarrow \frac{x+y}{2} \in B_1(0)$$

★

Consequence: $(\mathbb{R}^2, \|\cdot\|_1)$ and $(\mathbb{R}^2, \|\cdot\|_\infty)$ are not uniformly convex.

## Proposition 20.2

$(\mathbb{R}^2, \|\cdot\|_2)$ is uniformly convex

***Proof.*** Suppose by contradiction that this is false: $\exists \, \bar{\varepsilon} > 0$ and $\{x_n\}, \{y_n\} \subset \overline{B_1(0)}$ s.t.

$$\|x_n - y_n\| > \bar{\varepsilon}, \text{ but } \left\|\frac{x_n + y_n}{2}\right\| \geq 1 \tag{*}$$

$\overline{B_1(0)}$ is compact (since we are in $\mathbb{R}^2$) $\Rightarrow$ UTS $x_n \to \bar{x}$, $y_n \to \bar{y}$ as $n \to \infty$. Taking the limit in (*), we deduce that $\bar{x}, \bar{y} \in \overline{B_1(0)}$

$$\|\bar{x} - \bar{y}\| \geq \bar{\varepsilon} \text{ , and } \left\|\frac{\bar{x} + \bar{y}}{2}\right\| \geq 1$$

This is not possible, since $\overline{B_1(0)}$ is strictly convex. ★

**Theorem 20.3**
$(X, \mathcal{M}, \mu)$ complete measure space. Then $L^p(X)$ is reflexive $\forall \, p \in (1, \infty)$

***Proof.*** $(L^p(X), \|\cdot\|_p)$ is uniformly convex $\forall \, p \in (1, \infty)$ (Clarkson inequalities) ★

$L^1(X) and L^\infty(X)$ are not uniformly convex, and not reflexive.

# Dual space of $L^p$

**Theorem 20.4** (Riesz representation theorem)
$(X, \mathcal{M}, \mu)$ complete measure space, $p \in (1, \infty)$. Then

$$\forall \, L \in (L^p(X))^* \quad \exists! \, g \in L^{p'}(X)$$

with $p'$ conjugate exponent s.t. $L = L_g$, namely

$$Lf = \int_X fg \, d\mu \qquad \forall \, f \in L^p(X)$$

Moreover $\|L_g\|_{(L^p)^*} = \|g\|_{p'}$
    Thus: $T : g \in L^{p'} \mapsto L_g \in (L^p)^*$ is an isometric isomorphism.

***Proof.*** $1 < p < \infty$. Consider $T : L^{p'} \to (L^p)^*$ with $g \mapsto Tg :< Tg, f >= \int_X fg \, d\mu$ (namely $Tg = L_g$). We already know that

$$\|Tg\|_* = \|L_g\|_* = \|g\|_{p'}$$

$T$ is injective: for exercise.
    $T$ is surjective. Indeed, let $F := T(L^{p'}) \subseteq (L^p)^*$ subspace. Since $T$ is an isometry and $L^{p'}$ is complete, it can be shown that $T(L^{p'})$ is also complete $\Rightarrow T(L^{p'})$ is closed.
    If by contradiction $F \neq (L^p)^*$, then we can apply corollary 3 to Hahn Banach ($X = (L^p)^*$), $Y = F, x_0 = \lambda$:

$$\exists h \in (L^p)^{**} \text{ s.t. } < h, \lambda > \neq 0 \text{ and } h|_F = 0 :< h, Tg >= 0 \quad \forall g \in L^{p'} \tag{1}$$

But $L^p$ is reflexive ($1 < p < \infty$), then $h \in L^p \setminus \{0\}$:

$$< h, Tg >= (Tg)h = \int_X hg \, d\mu$$

Therefore, (1) tells us that

$$\int_X hg \, d\mu = 0 \quad \forall \, g \in L^{p'}(X)$$

Take $g = |h|^{p-2}h$. Therefore

$$0 = \int_X hg \, d\mu = \int_X h|h|^{p-2}h \, d\mu = \int_X |h|^p \, d\mu \Rightarrow h = 0 \in L^p$$

which is the desired contradiction.

$T$ is an isomorphism: for exercise. $\bigstar$

$$(L^p)^* = L^{p'}$$

**Remark 20.2**

$p = 1$. one can prove the

**Theorem 20.5**

$(X; \mathcal{M}, \mu)$ complete measure space, $\sigma$-finite. Then $\forall\, L \in (L^1(X))^*$    $\exists! g \in L^\infty(X)$ s.t. $L = L_g$:

$$Lf = \int_X fg \, d\mu \quad \forall f \in L^1(X)$$

Moreover, the map $L \in (L^1)^* \mapsto g \in L^\infty$ is an isometric isomorphism.

Recall that $(X, \mathcal{M}, \mu)$ is finite if $\mu(X) < \infty$.

**Definition 20.2**

$(X, \mathcal{M}, \mu)$ is $\sigma$-finite if either $\mu(X) < \infty$, or $X = \sum_{n=1}^\infty X_n$, where $\mu(X_n) < \infty$

# Weak Convergence

We know that $\overline{B_1(0)}$ is never compact in $\infty$ dimension. This is a problem is in proving convergence of sequences. a way to approach this issue consists in weakening the notion of convergence.

**Definition 20.3**

$X$ Banach space. $\{x_n\} \subset X$ sequence, $x \in X$. We say that $x_n$ tends to $x$ weakly (in $X$) as $n \to \infty$, $x_n \rightharpoonup x$ in $X$, if

$$Lx_n \to Lx \qquad \forall\, L \in X^*$$

**Remark 20.3**

Assume that $x_n \to x$ in $X$, namely $\|x_n - x\|_X \to 0$. If $f : X \to \mathbb{R}$ is continuous, then

$$\lim_{n \to \infty} f(x_n) = f(x)$$

In particular, this is true if $f = L \in X^*$:

$$x_n \to x \in X \Rightarrow Lx_n \to Lx \qquad \forall\, L \in X^*$$
$$x_n \rightharpoonup x \in X$$
$$x_n \to x \in X \Rightarrow x_n \rightharpoonup x \text{ weakly } \in X$$
$$\nLeftarrow$$

**Remark 20.4**

We will be interested in weak convergence in $L^p$. If $p \in [1, \infty)$, then $f_n \rightharpoonup f$ weakly in $L^p(X) \iff \int_X f_n g \, d, d\mu \to \int_X fg \, d\mu \; \forall\, g \in L^{p'}$

Similarily, in $l^p = L^P(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu_c)$, $x_n \rightharpoonup x$ weakly in $l^p \iff \sum_{k=1}^\infty x_n^{(k)} y^{(k)} \to \sum_{k=1}^\infty x^{(k)} y^{(k)}$ $\forall\, y \in l^{p'}$

**Proposition 20.3**

The weak limit is unique (if it exists)

**Proof.** By contradiction, suppose that $\exists \{x_n\} \subset X$ s.t. $x_n \rightharpoonup x_1$, $x_n \rightharpoonup x_2$ weakly in $X$, $x_1 \neq x_2$. Then

$$
\begin{aligned}
Lx_n &\to Lx_1 &&\forall\, L \in X^* \\
Lx_n &\to Lx_2 &&\forall\, L \in X^* \\
&\Rightarrow Lx_1 = Lx_2 &&\forall\, L \in X^*
\end{aligned}
$$

By Hahn Banach (corollary 2), this implies $x_1 = x_2$, a contradiction. ★

### Proposition 20.4
If $x_n \rightharpoonup x$ weakly in $X$, then $\{x_n\}$ is bounded, and

$$\|x\| \leq \liminf_{n\to\infty} \|x_n\| \qquad\qquad \text{weak lower semi continuity of } \|\cdot\|$$

**Proof.** • $\{x_n\}$ is bounded

$x_n \rightharpoonup x$ weakly $\Rightarrow \{Lx_n\}$ is bounded in $\mathbb{R}$, $\forall\, L \in X^*$. Consider $\Lambda_n \in X^{**}$ def by

$$\Lambda_n L = Lx_n \quad \forall\, L \in X^*$$

$\forall\, L \in X^* \exists\, M_L > 0$ s.t.

$$|\Lambda_n L| = |Lx_n| \leq M_L \qquad \forall\, n \qquad\qquad\text{PB}$$

(pointwise boundedness of $\{\Lambda_n\} \subset X^{**}$)

$$\Lambda_n : X^* = \mathcal{L}(X, \mathbb{R}) \to \mathbb{R}$$

By Banach Steinhaus, $\{\Lambda_n\}$ is uniformly bounded:

$$\sup_n \|\Lambda_n\|_{\mathcal{L}(X^*,\mathbb{R})} \leq M$$

Moreover, by Hahn Banach, $\forall\, n \in \mathbb{N}$ $\exists L_n \in X^*$ s.t. $\|L_n\|_* = 1$ and $L_n x_n = \|x_n\|$. Therefore

$$\|x_n\| = |L_n x_n| = |\Lambda_n L_n| \leq \|\Lambda_n\|_{\mathcal{L}} \|L_n\|_* \leq M \quad \forall n \in \mathbb{N}$$

• $x_n \rightharpoonup x$ weakly.

By corollary 1 of Hahn Banach, $\exists L_x \in x^*$ s.t. $\|L_x\|_* = 1$ and $L_x x = \|x\|$. Then

$$\|x\| = |L_x x| = \lim_n |L_x x_n| = \liminf_n |L_x x_n| \leq \liminf_n \|L_x\|_* \|x_n\|_X = \liminf_n \|x_n\|_X$$

★

### Proposition 20.5
$x_n \rightharpoonup x$ in $X$ weakly, and $L_n \to L$ (strongly) in $X^*$. Then

$$L_n X_n \to Lx \quad \text{in } \mathbb{R}$$

### Proposition 20.6
$X, Y$ Banach, $T \in \mathcal{L}(X, Y)$

$$x_n \rightharpoonup x \text{ weakly } \Rightarrow Tx_n \rightharpoonup Tx \text{ weakly}$$

# 21 Lecture 30/11/2022

We introduced the weak convergence. $X$ banach space. $x_n \subset X$ converges weakly to $x$, $x_n \rightharpoonup x$ weakly in $X$, if

$$Lx_n \to Lx \text{ in } \mathbb{R}, \quad \forall\, L \in X^* = \mathcal{L}(X, \mathbb{R})$$

Recall that:

• $x_n \to x$ strongly in $X$, namely $\|x_n - x\|_X \to 0$

# 22 Lecture 01/12/2022