

# Numerical Analysis for Partial Differential Equations

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# 1 Boundary Value Problems

## 1.1 Weak Formulation

Let's consider a problem

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega \\ +\text{B.C.} & \text{on } \partial\Omega \end{cases} \quad (1.1)$$

- $\Omega$ : open bounded domain in  $\mathbb{R}^d$ , with  $d = 2, 3$
- $\partial\Omega$ : boundary of  $\Omega$
- $f$ : given
- B.C. accordingly to  $\mathcal{L}$
- $\mathcal{L}$ : 2<sup>nd</sup> order operator, like:

$$(1) \quad \mathcal{L}u = -\text{div}(\mu \nabla u) + \mathbf{b} \cdot \nabla u + \sigma u \quad (\text{non-conservative form})$$

$$(2) \quad \mathcal{L}u = -\text{div}(\mu \nabla u) + \text{div}(\mathbf{b}u) + \sigma u \quad (\text{conservative form})$$

- $\mu \in L^\infty(\Omega)$ ,  $\mu(\mathbf{x}) \geq \mu_0 > 0$  uniformly bounded from below
- $\mathbf{b} \in (L^\infty(\Omega))^d$  transport term
- $\sigma \in L^2(\Omega)$  reaction term
- $f \in L^2(\Omega)$  can be less regular

## General elliptic problems

Consider

$$\begin{cases} -\text{div}(\mu \nabla u) + \mathbf{b} \cdot \nabla u + \sigma u = f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_D \\ \mu \nabla u \cdot \mathbf{n} = g & \text{on } \Gamma_N \end{cases} \quad \begin{matrix} g \in L^2(\Gamma_N) \\ \partial\Omega = \Gamma_D \cup \Gamma_N \\ \Gamma_D^\circ \cap \Gamma_N^\circ = \emptyset \end{matrix} \quad (1.2)$$

Suppose that  $f \in L^2(\Omega)$  and  $\mu, \sigma \in L^\infty(\Omega)$ . Also suppose that  $\exists \mu_0 > 0$  s.t.  $\mu(\mathbf{x}) \geq \mu_0$ , and  $\sigma(\mathbf{x}) \geq 0$  a.e. on  $\Omega$ . Then, given a test function  $v$ , we multiply the equation by  $v$ , and integrate on the domain  $\Omega$

$$\int_{\Omega} [-\text{div}(\mu \nabla u) + \mathbf{b} \cdot \nabla u + \sigma u] v = \int_{\Omega} f v$$

By applying Green's formula

$$\underbrace{\int_{\Omega} \mu \nabla u \cdot \nabla v + \int_{\Omega} \mathbf{b} \cdot \nabla u v + \int_{\Omega} \sigma u v}_{=: a(u, v)} = \int_{\Omega} f v + \underbrace{\int_{\Gamma_D} \mu \nabla u \cdot \mathbf{n} v}_{=0 \text{ if } v|_{\Gamma_D}=0} + \int_{\Gamma_N} \underbrace{\mu \nabla u \cdot \mathbf{n} v}_{=: g}$$

So the weak formulation of the problem is

$$\begin{cases} \text{Find } u \in V & V = \{v \in H^1(\Omega), v|_{\Gamma_D} = 0\} =: H_{\Gamma_D}^1(\Omega) \\ a(u, v) = \langle F, v \rangle & \forall v \in V \end{cases} \quad (1.3)$$

where  $a : V \times V \rightarrow \mathbb{R}$  is a bilinear form and  $F : V \rightarrow \mathbb{R}$  is a linear form s.t.  $\langle F, v \rangle \equiv F(v) = \int_{\Omega} f v + \int_{\Gamma_N} g v$ .

**Theorem 1.1** (Lax-Milgram)

Assume that

- $V$  Hilbert space with  $\|\cdot\|$  and inner product  $(\cdot, \cdot)$
- $F \in V^* : |F(v)| \leq \|F\|_{V^*} \|v\| \quad \forall v \in V$
- $a$  continuous:  $\exists M > 0 : |a(u, v)| \leq M \|u\| \|v\| \quad \forall u, v \in V$
- $a$  coercive:  $\exists \alpha > 0 : a(v, v) \geq \alpha \|v\|^2 \quad \forall v \in V$

Then, there exists a unique solution  $u$  of 1.3

Moreover

$$\alpha \|u\|^2 \leq a(u, u) = F(u) \leq \|F\|_{V^*} \|u\|$$

where  $\alpha$  is the coercivity constant. Hence

$$\|u\| \leq \frac{\|F\|_{V^*}}{\alpha} \rightarrow \text{stability/continuous dependence on data}$$

But what if some of the assumptions of Lax-Milgram (in particular coercivity) are not satisfied? We need a slightly more general problem to formulate Nečas theorem:

$$\begin{cases} \text{find } u \in V \\ a(u, w) = \langle F, w \rangle \quad \forall w \in W \end{cases} \quad (1.4)$$

They belong to different spaces:  $W$  for the test function,  $V$  the solutions

**Theorem 1.2** (Nečas)

Assume that  $F \in W^*$ . Consider the following conditions:

- $a$  continuous:  $\exists M > 0 : |a(u, w)| \leq M \|u\|_V \|w\|_W \quad \forall u \in V, w \in W$
- inf – sup condition:  $\exists \alpha > 0 : \forall v \in V \quad \sup_{w \in W \setminus \{0\}} \frac{a(v, w)}{\|w\|_W} \geq \alpha \|v\|_V$
- $\forall w \in W, w \neq 0, \exists v \in V : a(v, w) \neq 0$

These conditions are necessary and sufficient for the existence and uniqueness of a solution of 1.4, for any  $F \in W^*$ . Moreover

$$\|u\|_V \leq \frac{1}{\alpha} \|F\|_{W^*}$$

When  $W = V$  Lax-Milgram provides necessary and sufficient conditions for existence and uniqueness of solutions.

Going back to

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega \\ \text{+B.C.} & \text{on } \partial\Omega \end{cases}$$

What could be our choice of  $V$ ? Given that

$$u \in V : a(u, v) = F(v) \quad \forall v \in V$$

and

$$a(u, v) = \int_{\Omega} \mu \underbrace{\nabla u \nabla v}_{\nabla u, \nabla v \in L^2} + \int_{\Omega} b \underbrace{\nabla u v}_{\in L^1} + \int_{\Omega} \sigma \underbrace{uv}_{\in L^1}$$

We want to choose  $v$  in order to have all of these integrable

$$\Rightarrow V = \left\{ v \in L^2(\Omega), \nabla u \in [L^2(\Omega)]^d, v|_{\Gamma_D} = 0 \right\} = V_{\Gamma_D}$$

Knowing that a Sobolev space

$$H^1 = \left\{ v \in L^2(\Omega), \nabla u \in [L^2(\Omega)]^d \right\}$$

we can say  $V_{\Gamma_D} = \{v \in H^1(\Omega) : v|_{\Gamma_D} = 0\}$ , and if  $\Gamma_D = \partial\Omega$ , then  $V_{\Gamma_D} = H_0^1$

## 1.2 Approximation

Recall for a moment the weak formulation of a generic elliptic problem

$$\begin{cases} \text{Find } u \in V \\ a(u, v) = \langle F, v \rangle \quad \forall v \in V \end{cases} \quad (1.5)$$

with  $V$  being an appropriate Hilbert space, subset of  $H^1()$ ,  $a(\cdot, \cdot)$  being a continuous and coercive bilinear form from  $V \times V \rightarrow \mathbb{R}$ ,  $F(\cdot)$  being a continuous linear functional from  $V \rightarrow \mathbb{R}$ .

Let  $V_h \subset V$  be a family of spaces that depends on a parameter  $h > 0$ , such that  $\dim V_h = N_h < \infty$ . We can rewrite the weak formulation

$$\begin{cases} \text{Find } u_h \in V_h \\ a(u_h, v_h) = \langle F, v_h \rangle \quad \forall v_h \in V_h \end{cases} \quad (1.6)$$

and is called a **Galerkin problem**. Denoting with  $\{\varphi_j, j = 1, 2, \dots, N_h\}$  a basis of  $V_h$ , it is sufficient that the (1.6) is verified for each function of the basis. Also we need that

$$a(u_h, \varphi_i) = F(\varphi_i) \quad i = 1, 2, \dots, N_h$$

Since  $u_h \in V_h$

$$u_h(\mathbf{x}) = \sum_{j=1}^{N_h} u_j \varphi_j(\mathbf{x})$$

where  $u_j$  are unknown coefficients. Then

$$\sum_{j=1}^{N_h} u_j a(\varphi_j, \varphi_i) = F(\varphi_i)$$

We denote by  $A$  the matrix made by  $a_{ij} = a(\varphi_j, \varphi_i)$  and  $\mathbf{f}$  the vector of  $F(\varphi_i) = f_i$  components. If we denote the vector  $\mathbf{u}$  made by the unknown coefficients  $u_h$ .

$$A\mathbf{u} = \mathbf{f} \quad (1.7)$$

### Theorem 1.3

The stiffness matrix  $A$  associated to the Galerkin discretization of an elliptic problem, whose bilinear form is coercive is positive definite.

**Proof.** Recall that a matrix  $B \in \mathbb{R}^{n \times n}$  is said to be positive definite if

$$\mathbf{v}^T B \mathbf{v} \geq 0 \quad \forall \mathbf{v} \in \mathbb{R}^n$$

and

$$\mathbf{v}^T B \mathbf{v} = 0 \Leftrightarrow \mathbf{v} = \mathbf{0}$$

The correspondence

$$\mathbf{v} = (v_i) \in \mathbb{R}^{N_h} \longrightarrow v_h(x) = \sum_{j=1}^{N_h} v_j \varphi_j \in V_h$$

defines a bijection between  $V_h$  and  $\mathbb{R}^{N_h}$ . Given a generic vector  $\mathbf{v} = (v_i)$  of  $\mathbb{R}^{N_h}$ , thanks to the bilinearity and coercivity of  $a$  we obtain

$$\begin{aligned} \mathbf{v}^T A \mathbf{v} &= \sum_{j=1}^{N_h} \sum_{i=1}^{N_h} v_i a_{ij} v_j \\ &= \sum_{j=1}^{N_h} \sum_{i=1}^{N_h} v_i a(\varphi_j, \varphi_i) v_j \\ &= \sum_{j=1}^{N_h} \sum_{i=1}^{N_h} a(v_j \varphi_j, v_i \varphi_i) \\ &= a \left( \sum_{j=1}^{N_h} v_j \varphi_j, \sum_{i=1}^{N_h} v_i \varphi_i \right) \\ &= a(v_h, v_h) \geq \alpha \|v_h\|_V^2 \geq 0 \end{aligned}$$

Moreover, if  $\mathbf{v}^T A \mathbf{v} = 0$ , then  $\|v_h\|_V^2 = 0$ . ★

## Existence and uniqueness

### Corollary 1.1

The solution of the Galerkin problem (1.6) exists and is unique.

To prove this we can prove that the solution to (1.7) exists and is unique. The matrix  $A$  is invertible as the unique solution of  $A \mathbf{u} = \mathbf{0}$  is the null solution, meaning that  $A$  is definite positive.

## Stability

### Corollary 1.2

The Galerkin method is stable, uniformly with respect to  $h$ , by virtue of the following upper bound for the solution

$$\|u_h\|_V \leq \frac{1}{\alpha} \|F\|_{V^*}$$

The stability of the method guarantees that the norm  $\|u_h\|_V$  of the discrete solution remains bounded for  $h \rightarrow 0$ . Equivalently it guarantees that  $\|u_h - w_h\|_V \leq \frac{1}{\alpha} \|F - G\|_{V^*}$  with  $u_h$  and  $w_h$  being numerical solution corresponding to different data  $F$  and  $G$ .

## Convergence

**Lemma 1.1** (Galerkin orthogonality)

The solution  $u_h$  of the Galerkin method satisfies

$$a(u - u_h, v_h) = 0 \quad \forall v_h \in V_h \quad (1.8)$$

**Proof.** Since  $V_h \subset V$ , the exact solution  $u$  satisfies the weak problem (1.5) for each element  $v = v_h \in V_h$ , hence we have

$$a(u, v_h) = F(v_h) \quad \forall v_h \in V_h \quad (1.9)$$

By subtracting side by side (1.6) from (1.9), we obtain

$$a(u, v_h) - a(u_h, v_h) = 0 \quad \forall v_h \in V_h$$

from which the claim follows. ★

Also this can be generalized in the cases in which  $a(\cdot, \cdot)$  is not symmetric. Consider the value taken by the bilinear form when both its arguments are  $u - u_h$ . If  $v_h$  is an arbitrary element of  $V_h$  we obtain

$$a(u - u_h, u - u_h) = a(u - u_h, u - v_h) + a(u - u_h, v_h - u_h)$$

The last term is null by (1.8). Moreover

$$|a(u - u_h, u - v_h)| \leq M \|u - u_h\|_V \|u - v_h\|_V$$

having exploited the continuity of the bilinear form. Also by the coercivity

$$a(u - u_h, u - u_h) \geq \alpha \|u - u_h\|_V^2$$

hence

$$\|u - u_h\|_V \leq \frac{M}{\alpha} \|u - v_h\|_V \quad \forall v_h \in V_h$$

Such inequality holds for all functions  $v_h \in V_h$  and therefore we find

$$\underbrace{\|u - u_h\|_V}_{\text{Galerkin error}} \leq \frac{M}{\alpha} \underbrace{\inf_{w_h \in V_h} \|u - w_h\|_V}_{\text{Best Approximation Error}} \quad (1.10)$$

In order for the method to converge, it is sufficient that, for  $h \rightarrow 0$  the space  $V_h$  tends to saturate the entire space  $V$ .

$$\lim_{h \rightarrow 0} \inf_{v_h \in V_h} \|v - v_h\|_V = 0 \quad \forall v \in V \quad (1.11)$$

In that case the Galerkin method is convergent and it can be written that

$$\lim_{h \rightarrow 0} \|u - u_h\|_V = 0 \Leftrightarrow \text{convergence}$$

This space  $V_h$  must be chosen carefully to satisfy the saturation property (1.11).

### 1.3 Finite Element Method 1D

Let us suppose that  $\Omega$  is an interval  $(a, b)$ . How to create an approximation of the space  $H^1(a, b)$  that depend on a parameter  $h$ . Consider a partition  $\mathcal{T}_h$  in  $N + 1$  subintervals  $K_j = [x_{j-1}, x_j]$ , having width  $h_j = x_j - x_{j-1}$  with

$$a = x_0 < x_1 < \dots < x_{N-1} < x_N = b \quad (1.12)$$

and set  $h = \max_j h_j$ .

Since the functions of  $H^1(a, b)$  are continuous on  $[a, b]$ , it is possible to create the family of spaces

$$X_h^r = \{v_h \in C^0(\overline{\Omega}) : v_h|_{K_j} \in \mathbb{P}_r \ \forall K_j \in \mathcal{T}_h\}, \quad r = 1, 2, \dots \quad (1.13)$$

having denoted by  $\mathbb{P}_r$  the space of polynomials with degree lower or equal to  $r$  in the variable  $x$ . All these spaces are subspaces of  $H^1(a, b)$  as they are constituted by differentiable functions except for at most a finite number of points (the vertices of the partition). It is convenient to select a basis for the  $X_h^r$  space that is *Lagrangian*.

#### The space $X_h^1$

The space is constituted by the functions of the partition (1.12). Since only a straight line can pass through different points, the degrees of freedom (DOF, the number of values we need to assign to the basis to define the functions) of the functions will be equal to the number  $N + 2$  of vertices of the partition. It follows naturally that  $\{\varphi_i\}, i = 0, 1, \dots, N, N + 1$ . In this case the basis functions are characterized by the following properties

$$\varphi_i \in X_h^1 \text{ s.t } \varphi_i(x_j) = \delta_{ij}, \quad i, j = 0, 1, \dots, N, N + 1$$

where  $\delta_{ij}$  is the Kronecker delta. So we have our basis function that have value 1 in the node  $x_j$  and 0 elsewhere.

The formula for the basis function is then given by

$$\varphi_i(x) = \begin{cases} \frac{x - x_{i-1}}{x_i - x_{i-1}} & \text{for } x_{i-1} \leq x \leq x_i \\ \frac{x - x_{i+1}}{x_{i+1} - x_i} & \text{for } x_i \leq x \leq x_{i+1} \\ 0 & \text{otherwise} \end{cases} \quad (1.14)$$

### 1.4 Finite Element Method 2D

Now we can extend the FEM for multi-dimensional problems. For simplicity we will consider  $\Omega \subset \mathbb{R}^2$  with polygonal shapes  $\mathcal{T}_h$ . In this case the partition is called a triangulation. We can define the discretized domain

$$\Omega_h = \text{int} \left( \bigcup_{K \in \mathcal{T}_h} K \right)$$