Advanced Partial Differential Equations Exams

Andrea Bonifacio

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1 Exams 2021/22

1.1 June 2021

Exercise 1
For $a, \gamma \in \mathbb{R}$, consider the Cauchy problem for the wave equation

$$\begin{cases} u_{tt} - 4u_{xx} = 0 & (x,t) \in \mathbb{R} \times (0, \infty) \\ u(x,0) = e^{-x^2} + \gamma e^{-(x-a)^2} & x \in \mathbb{R} \\ u_t(x,0) = 0 & x \in \mathbb{R} \end{cases}$$

Show that the mass M of the solution is constant, then find the couples (a, γ) such that:

- $M = 0$.
- The solution $u(x,1)$ at $t = 1$ consists of only two "bumps".

To show that the mass M of the solution is constant we need to define such a mass, and then check its behavior over time.
The mass M of the solution is defined as $M(t) := \int_{\mathbb{R}} u(x,t)\, dx$.

$$M''(t) = \frac{\partial^2}{\partial^2 t} \int_{\mathbb{R}} u(x,t)\, dx = \int_{\mathbb{R}} u_{tt}\, dx = \int_{\mathbb{R}} 4u_{xx}\, dx = 4 \underbrace{\int_{\mathbb{R}} (u_x)_x\, dx}_{\text{div. form=0}} = 0$$

So $M(t) = A + Bt$. But $M'(t) = B$, and also $M'(t) = \int_{\mathbb{R}} u_t(x,t)\, dx$. Since $M'(t)$ is constant we take $M'(0) = \int_{\mathbb{R}} u_t(x,0)\, dx = 0 \Rightarrow B = 0$.
Then we conclude that $M(t) = A$ is constant too, and $M(0) = \int_{\mathbb{R}} e^{-x^2} + \gamma e^{-(x-a)^2}\, dx$

$$\int_{\mathbb{R}} e^{-x^2} + \underbrace{\gamma e^{-(x-a)^2}}_{\substack{x-a=y \\ dx=dy}}\, dx = (1+\gamma) \int_{\mathbb{R}} e^{-y^2}\, dy = \sqrt{\pi}(1+\gamma)$$

After that we need to show that $M = 0$, so $\sqrt{\pi}(1+\gamma) = 0 \iff \gamma = -1$
Then we look for the values of a such that the solution consists of two "bumps".

Remark 1
For a hyperbolic equation, we know that the solution $u(x,t) = \frac{1}{2}(g(x+ct) + g(x-ct))$

In this case we have $u(x,t) = \frac{1}{2}(\overbrace{g(x+2t)}^{\substack{\text{c is 2} \\ \text{bc } 4u_{xx}}} + g(x-2t))$, which becomes

$$u_{a,\gamma}(x,t) = \frac{1}{2}\left(e^{-(x+2t)^2} + \gamma e^{-(x-a+2t)^2} + e^{-(x-2t)^2} + \gamma e^{-(x-a-2t)^2}\right)$$

that, for $\gamma = -1$ and $t = 1$

$$u_{a,\gamma}(x,1) = \frac{1}{2}\left(e^{-(x+2)^2} - e^{-(x-a+2)^2} + e^{-(x-2)^2} - e^{-(x-a-2)^2}\right)$$

We can see that this solution has four "bumps" in $x = -2, x = 2, x = a-2, x = a+2$. To obtain two bumps we manipulate a and see that

- $-2 = a - 2 \Rightarrow a = 0 \Rightarrow u_{0,\gamma}(x,1) = \frac{1}{2}(e^{-(x+2)^2} - e^{-(x+2)^2} + e^{-(x-2)^2} - e^{-(x-2)^2}) = 0$

- $-2 = a + 2 \Rightarrow a = -4 \Rightarrow u_{-4,\gamma}(x, 1) = \frac{1}{2}(e^{-(x+2)^2} - e^{-(x+6)^2} + e^{-(x-2)^2} - e^{-(x+2)^2}) = \frac{1}{2}(-e^{-(x+6)^2} + e^{-(x-2)^2})$
- $2 = a - 2 \Rightarrow a = 4 \Rightarrow u_{-4,\gamma}(x, 1) = \frac{1}{2}(e^{-(x+2)^2} - e^{-(x-2)^2} + e^{-(x-2)^2} - e^{-(x-6)^2}) = \frac{1}{2}(-e^{-(x-6)^2} + e^{-(x+2)^2})$
- $-2 = a - 2 \Rightarrow a = 0 \Rightarrow u_{0,\gamma}(x, 1) = \frac{1}{2}(e^{-(x+2)^2} - e^{-(x+2)^2} + e^{-(x-2)^2} - e^{-(x-2)^2}) = 0$

We can see that the solution has only two "bumps" in the cases $a = \pm 4$, so we conclude that the desired couples of (a, γ) are

$$\begin{cases} \gamma = -1 \\ a = -4 \end{cases} \vee \begin{cases} \gamma = -1 \\ a = 4 \end{cases}$$

Exercise 2
Let $\Omega \subset \mathbb{R}^n (n \geq 2)$ be a bounded smooth domain, let a be a measurable function in Ω. Consider the problem

$$\begin{cases} -\Delta u = a(x)u^3 & \Omega \\ u = 0 & \partial\Omega \end{cases} \tag{P}$$

Under which assumptions on the space dimension n can we write a variational formulation of problem (P) in $H_0^1(\Omega)$? For each of these dimensions find the most general assumptions on a that allow to write the variational formulation. Finally, write the variational formulation.

First, a quick reminder on Sobolev embedding, which will be very useful a.e. in this document

Remark 2
Let $\Omega \subseteq \mathbb{R}^n$ open with $\partial\Omega \in \text{Lip}$, $s \geq 0$,

$$H^s(\Omega) \subset \begin{cases} L^p(\Omega) & \forall\, 2 \leq p \leq 2^* & \text{if } n > 2s \\ L^p(\Omega) & \forall\, 2 \leq p < \infty & \text{if } n = 2s \\ C^0(\bar{\Omega}) & & \text{if } n < 2s \end{cases}$$

Increasing s increases the regularity, while increasing n decreases it.
The exponent 2^* is called critical exponent and is defined as $2^* := \frac{2n}{n-2s}$.
If $Omega$ is bounded, all these embeddings are compact except $H^s(\Omega) \subset L^{2^*}$ when $n > s$.

Since we want to know the variational formulation in H_0^1 we have $s = 1$ and need to check $n = 2, n \geq 3$. Remember a variational formulation makes sense if $\int_\Omega fv < \infty$.

$n = 2$. In this case we have $u^3, v \in H_0^1(\Omega)$, so by Sobolev embedding we know $u^3, v \in L^p(\Omega)$ for $2 \leq p < \infty$.

$$\left|\int_\Omega a(x)u^3v\right| dx \leq \int_\Omega |a(x)||u^3||v|\, dx \underset{Holder}{\leq} \left(\int_\Omega |a(x)|^r\right)^{\frac{1}{r}} \left(\int_\Omega |u^3|^p\right)^{\frac{1}{p}} \left(\int_\Omega |v|^q\right)^{\frac{1}{q}} < \infty.$$

To use Holder inequality we need to find r, p, q such that $\frac{1}{r} + \frac{1}{p} + \frac{1}{q} = 1$. We see that,

$$\frac{1}{r} + \frac{1}{p} + \frac{1}{q} = 1 \iff a(x) \in L^r(\Omega) \qquad \text{with } r > 1$$

$n \geq 3$. In this case we have $u^3, v \in H_0^1(\Omega)$, so by Sobolev embedding we know $u^3, v \in L^p(\Omega)$ for $2 \leq p \leq 2^*$. We proceed as before, using Holder inequality, but decide to use $p = \frac{2^*}{3}$ and $q = \frac{1}{2^*}$.

$$\left|\int_\Omega a(x)u^3v\right| dx \leq \int_\Omega |a(x)||u^3||v|\, dx \underset{Holder}{\leq} \left(\int_\Omega |a(x)|^r\right)^{\frac{1}{r}} \left(\int_\Omega |u^3|^{2^*}\right)^{\frac{3}{2^*}} \left(\int_\Omega |v|^{2^*}\right)^{\frac{1}{2^*}} < \infty.$$

In this case Holder inequality gives us

$$\frac{1}{r} + \frac{3}{2^*} + \frac{1}{2^*} = 1 \iff \frac{1}{r} = 1 - \frac{4}{2^*} \iff r = \frac{2^* - 4}{2^*}$$

Substituting $2^* = \frac{2n}{n-2}$ we get $r = \frac{n}{-n+4}$. Since $r > 0$ we need $n < 4$. In this case we have $a(x) \in L^3(\Omega)$ for $n = 3$, but also $a(x) \in L^\infty(\Omega)$ for $n = 4$.

At this point we can write the weak formulation of the problem. We multiply the equation by a test function $v \in H_0^1(\Omega)$ and obtain

$$\int_\Omega -\Delta u v \, dx = \int_\Omega a(x) u^3 v \, dx \qquad \forall v \in H_0^1(\Omega)$$

We integrate by parts the left hand side and obtain

$$\int_\Omega \nabla u \nabla v \, dx = \int_\Omega a(x) u^3 v \, dx \qquad \forall v \in H_0^1(\Omega)$$

This is the weak formulation of the problem. This is well posed if

Dimension	Assumptions on $a(x)$
$n = 2$	$a \in L^r(\Omega)$, $r > 1$
$n = 3$	$a \in L^3(\Omega)$
$n = 4$	$a \in L^\infty(\Omega)$
$n \geq 5$	No variational formulation

Exercise 3
Let $\Omega \subset \mathbb{R}^n$ be a bounded open set of class C^1, and let u be a sufficiently regular solution of the problem

$$\begin{cases} u_t - \Delta u = 0 & \Omega \times (0, \infty) \\ u = 0 & \partial\Omega \times (0, \infty) \\ u(x,0) = \alpha(x) & x \in \Omega \end{cases}$$

Study monotonicity/boundedness properties of the energy $E_u(t) = \int_\Omega |\nabla u|^2 \, dx$.

The energy functional is defined as $E_u(t) = \int_\Omega |\nabla u|^2 \, dx = \|\nabla u\|^2_{L^2(\Omega)}$. We want to study its behavior over time, so we need to compute its derivative with respect to time.

$$\begin{aligned} \frac{d}{dt} E_u(t) = \frac{d}{dt} \int_\Omega |\nabla u|^2 \, dx = \int_\Omega \frac{d}{dt} |\nabla u|^2 \, dx = \int_\Omega 2 \nabla u \cdot \nabla u_t \, dx = \\ = \int_{\partial\Omega} 2 \underbrace{u \cdot \nu}_{=0} u_t - \int_\Omega 2 \Delta u u_t \, dx = -\int_\Omega 2 (\Delta u)^2 \, dx \le 0. \end{aligned} \quad \Rightarrow E \text{ is non-increasing}$$

We see that the energy is non-increasing, since we obtain a positive quantity with a negative sign. Now we want to study the boundedness of the energy. We start by multiplying the equation by u

$$\int_\Omega u_t u \, dx - \int_\Omega \Delta u u \, dx = 0$$

Integrating by parts the second term we obtain

$$\int_\Omega u_t u \, dx - \int_{\partial\Omega} u \nabla u \cdot \nu \, dS + \int_\Omega (\nabla u)^2 \, dx = 0$$

Since $u = 0$ on the boundary we have

$$\int_\Omega u_t u \, dx = -\int_\Omega (\nabla u)^2 \, dx$$

We can rewrite the energy as

$$E_u(t) = \int_\Omega |\nabla u|^2 \, dx = -\int_\Omega u_t u \, dx = -\frac{1}{2} \int_\Omega (u^2)_t \, dx$$

Since the energy is non-increasing, we have that $E_u(t) \le E_u(0) \forall t \ge 0$, so we have

$$E_u(t) = -\frac{1}{2} \int_\Omega (u^2)_t \, dx \le -\frac{1}{2} \int_\Omega (u(x,0)^2)_t \, dx = -\frac{1}{2} \int_\Omega \alpha(x)^2 \, dx$$

Since $\alpha(x)$ is bounded (is a function in H^1_0) we have that the energy is bounded too.

Exercise 4
Let $(X, \|\cdot\|)$ be a Banach space, and let $u \in C^1([0,T];X)$. Using the following abstract version of the *Fundamental Theorem of Calculus*:

$$\int_0^T u'(t)\,dt = u(T) - u(0)$$

prove that $\Lambda_{u'} = (\Lambda_u)' \in \mathcal{D}(0,T;X)$ where

$$\Lambda_f(\varphi) = \int_0^T \varphi(t) f(t)\,dt \qquad \forall f \in L^1(0,T;X)$$

By the definition of distributional derivative we have

$$(\Lambda_u(\varphi))' = -\Lambda_u(\varphi')\forall\varphi \in \mathcal{D}(0,T)$$

where

$$\Lambda_u(\varphi)' = -\int_0^T \varphi'(t)u(t)\,dt$$

We can integrate by parts the above expression

$$(\Lambda_u(\varphi))' = -\int_0^T \varphi'(t)u(t)\,dt = \underbrace{-\varphi(t)u(t)|_0^T}_{=0} + \int_0^T \varphi(t)u'(t)\,dt = \int_0^T \varphi(t)u'(t)\,dt = \Lambda_{u'}(\varphi)$$

We have shown that $\Lambda_{u'} = (\Lambda_u)'$ in $\mathcal{D}(0,T;X)$.

1.2 July 2021

Exercise 1
Let $\Omega \subseteq \mathbb{R}^2$ be a bounded open set of class C^∞, let $f \in L^2(\Omega)$. Consider the Dirichlet problem

$$\begin{cases} 2\partial_x^2 u + 3\partial_y^2 u + 2\partial_{xy} u = f & \Omega \\ u = 0 & \partial\Omega \end{cases} \tag{P}$$

(1) Prove that (P) admits a unique solution $u \in H_0^1(\Omega)$.

(2) What is the minimum $m \in \mathbb{N}$ for which $f \in H^m(\Omega)$ implies $u \in H^5(\Omega)$?

We can rewrite the equation in the form $\operatorname{div}(A\nabla u)$ with $A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$. Let's check if A is the correct matrix.

$$\operatorname{div}(A\nabla u) = \operatorname{div}\left(\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} u_x \\ u_y \end{pmatrix}\right) = \operatorname{div}\left(\begin{pmatrix} 2u_x + u_y \\ u_x + 3u_y \end{pmatrix}\right) = 2u_{xx} + 3u_{yy} + 2u_{xy}$$

Our Hilbert triplet is $H_0^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega)$. We define $V = H_0^1(\Omega)$, $H = L^2(\Omega)$, $V' = H^{-1}(\Omega)$. We can now write the weak formulation of the problem. We multiply the equation by a test function $v \in V$ and obtain

$$\int_\Omega \operatorname{div}(A\nabla u) v \, dx = \int_\Omega f v \, dx \qquad \forall v \in V$$

We integrate by parts the left-hand side and obtain

$$\int_\Omega A\nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx \qquad \forall v \in V$$

This is the weak formulation of the problem. Now we use Lax-Milgram theorem to prove the existence and uniqueness of the solution. We need to check the coercivity and boundedness of the bilinear form. We have that the bilinear form is

$$a(u, v) = \int_\Omega A\nabla u \cdot \nabla v \, dx$$

A bilinear form is continuous if there exists a constant $C > 0$ such that

$$|a(u, v)| \leq C\|u\|_V \|v\|_V$$

Remark 1
In $H_0^1(\Omega)$ we have the norm $\|u\|_V = \|\nabla u\|_{L^2}$

We write the bilinear form explicitly and bound it

$$|a(u, v)| = \left|\int_\Omega A\nabla u \cdot \nabla v \, dx\right| \leq \int_\Omega |A||\nabla u||\nabla v| \, dx \leq |A|\|\nabla u\|_{L^2}\|\nabla v\|_{L^2} = |A|\|u\|_V\|v\|_V$$

We have that the bilinear form is continuous. We need to check the coercivity of the bilinear form. A bilinear form is coercive if there exists a constant $c > 0$ such that

$$a(u, u) \geq c\|u\|_V^2$$

We write the bilinear form explicitly

$$a(u, u) = \int_\Omega A\nabla u \cdot \nabla u \, dx = \int_\Omega |A||\nabla u|^2 \, dx = |A|\|\nabla u\|_{L^2}^2 \geq \|\nabla u\|_{L^2}^2 = \|u\|_V^2$$

We have that the bilinear form is coercive. By Lax-Milgram theorem we have that the problem admits a unique solution $u \in V$.
The second request is about the minimum m such that $f \in H^m(\Omega)$ implies $u \in H^5(\Omega)$.

Remark 2
We know that if $f \in H^m(\Omega)$ then $u \in H^{m+2}(\Omega)$.

We have that $f \in H^m(\Omega)$ implies $u \in H^{m+2}(\Omega)$, so we need $m+2 \geq 5 \Rightarrow m \geq 3$. The minimum m is 3.

Exercise 2
Find solitary waves for the problem

$$\begin{cases} u_t - u_{xx} - u_x^2 = 0 & \mathbb{R} \times (0, \infty) \\ u(x,0) = g(x) & x \in \mathbb{R} \end{cases}$$

Moreover, discuss mass and momentum conservation for general solutions $u \in S(\mathbb{R})$ of (P).

Quick reminder about the solitary waves for parabolic equations.

Remark 3
In the case of a parabolic equation, we have that the solution $u(x,t) = g(x+ct)$ where c is the speed of the wave.

We are working with solution of the form $u(x,t) = g(x+ct)$, so we substitute this solution in the equation and obtain

$$cg'(x+ct) - g''(x+ct) - (g'(x+ct))^2 = 0 \Rightarrow cg'(x+ct) - g''(x+ct) = (g'(x+ct))^2$$

We perform a change of variable $s = x + ct$ and obtain

$$cg'(s) - g''(s) = (g'(s))^2$$

At this point we are working with an ODE, so we can solve it. We start by defining $y(s) = g'(s)$ and obtain

$$cy(s) - y'(s) = y(s)^2 \Rightarrow y'(s) = y(s)^2 - cy(s)$$

To solve this we introduce

$$z(s) = \frac{1}{y(s)} \Rightarrow z'(s) = -\frac{y'(s)}{y(s)^2}$$

We substitute $y'(s)$ and obtain

$$z'(s) = -\frac{cy(s) - y(s)^2}{y(s)^2} = -c\frac{1}{y(s)} + 1 \Rightarrow z'(s) + cz(s) = 1$$

Solving this ODE we obtain

$$\begin{aligned} z(s) = e^{-cs}\left(k + \int_0^s e^{ct}\,dt\right) = e^{-cs}\left(k + \frac{e^{ct}}{c}\Big|_0^s\right) = e^{-cs}\left(k + \frac{e^{cs}-1}{c}\right) = ke^{-cs} + \frac{1}{c} - \frac{e^{-cs}}{c} = \\ = e^{-cs}\left(k - \frac{1}{c}\right) + \frac{1}{c} = k_0 e^{-cs} + \frac{1}{c} \end{aligned}$$

At this point we use the definition of $z(s)$ and obtain

$$y(s) = \frac{1}{z(s)} = \frac{1}{k_0 e^{-cs} + \frac{1}{c}} = \frac{ce^{cs}}{ck_0 + e^{cs}} = \frac{ce^{cs}}{k_1 + e^{cs}}$$

We have found the solution for $g'(s)$, so we can integrate it to find $g(s)$

$$g(s) = \int_0^s \frac{ce^{cs}}{k_1 + e^{cs}}\,ds = \log(k_1 + e^{cs}) + k_2$$

We have found the solution for $g(s) = \log(k_1 + e^{cs}) + k_2$.

Now we can discuss mass and momentum conservation for general solutions $u \in S(\mathbb{R})$ of (P). We start by defining the mass and momentum of the solution

$$M(t) = \int_{\mathbb{R}} u(x,t)\,dx$$

$$\mathcal{M}(t) = \int_{\mathbb{R}} u(x,t)^2\,dx$$

We compute the derivative of the mass

$$M'(t) = \frac{d}{dt}\int_{\mathbb{R}} u(x,t)\,dx = \int_{\mathbb{R}} u_t(x,t)\,dx = \int_{\mathbb{R}} u_{xx}(x,t) + u_x(x,t)^2\,dx = \int_{\mathbb{R}} \underbrace{(u_x)_x}_{\text{div. form}=0} + u_x^2\,dx = \int_{\mathbb{R}} u_x^2\,dx \geq 0$$

We do not have mass conservation, since mass is not constant over time.
We compute the derivative of the momentum

$$\begin{aligned}\mathcal{M}'(t) = \frac{d}{dt}\int_{\mathbb{R}} u(x,t)^2\,dx = \int_{\mathbb{R}} 2u(x,t)u_t(x,t)\,dx = \int_{\mathbb{R}} 2uu_{xx} + 2u(x,t)u_x^2\,dx = \\ = \int_{\mathbb{R}} 2uu_{xx} + \int_{\mathbb{R}} 2uu_x^2\,dx = 2\left(\cancel{uu_x|_{\mathbb{R}}} - \int_{\mathbb{R}} u_x^2\,dx\right) + \int_{\mathbb{R}} 2uu_x^2\,dx = \\ = 2\int_r eal(u-1)u_x^2\,dx\end{aligned}$$

As we can see, the momentum is not conserved either.

Exercise 3
By using the Helmoltz-Weyl theorem and the variational formulation of the Stokes problem, explain how to derive the role of pressure.

Remark 4
We introduce three spaces:

- $\boldsymbol{G}_1 \coloneqq \{f \in \boldsymbol{L}^2(\Omega) \mid \nabla \cdot f = 0, \gamma_\nu f = 0\}$
- $\boldsymbol{G}_2 \coloneqq \{f \in \boldsymbol{L}^2(\Omega) \mid \nabla \cdot f = 0, \exists g \in H^1(\Omega) \text{ s.t. } f = \nabla g\}$
- $\boldsymbol{G}_3 \coloneqq \{f \in \boldsymbol{L}^2(\Omega) \mid \exists g \in H^1_0(\Omega) \text{ s.t. } f = \nabla g\}$

We also introduce the space $\boldsymbol{V} \coloneqq \{f \in \boldsymbol{L}^2(\Omega) \mid \nabla \cdot f = 0\}$ which is the space of divergence-free functions. We know that $\boldsymbol{V}$ is dense in $\boldsymbol{G}_1$.
A famous result by Helmoltz and Weyl states that the spaces $\boldsymbol{G}_1, \boldsymbol{G}_2, \boldsymbol{G}_3$ are mutually orthogonal in $\boldsymbol{L}^2(\Omega)$ and that $\boldsymbol{L}^2(\Omega) = \boldsymbol{G}_1 \oplus \boldsymbol{G}_2 \oplus \boldsymbol{G}_3$.

We start by writing the strong formulation of the Stokes problem with $f \in \boldsymbol{L}^2(\Omega)$

$$\begin{cases} -\eta\Delta u + \nabla p = f & \Omega \\ \nabla \cdot u = 0 & \Omega \\ u = 0 & \partial\Omega \end{cases} \tag{S}$$

Then we multiply the equation by a test function $v \in \boldsymbol{V}$ to obtain the weak formulation

$$\int_\Omega -\eta \nabla u : \nabla v + \int_\Omega p \nabla \cdot v = \int_\Omega f v \qquad \forall v \in \boldsymbol{V}$$

By the Helmoltz-Weyl theorem we know that $\nabla p \in \boldsymbol{G}_2 \oplus \boldsymbol{G}_3$, so when we multiply the equation by a test function $v \in V$ we have

$$\int_\Omega \nabla p \cdot v = 0$$

since $\boldsymbol{V}$ is dense in $\boldsymbol{G}_1$ and $\boldsymbol{G_1}$ is orthogonal to $\boldsymbol{G}_2 \oplus \boldsymbol{G}_3$. We can now write the variational formulation of the Stokes problem

$$\int_{\mathbb{R}} -\eta \nabla u : \nabla v = \int_\Omega f v \qquad \forall v \in \boldsymbol{V}$$

Now we observe that for every $f \in \boldsymbol{L^2}$ the function $v \mapsto \int_\Omega f v$ is a bounded linear functional on $\boldsymbol{V}$. Then, by Lax-Milgram corollary we obtain

$$\forall f \in \boldsymbol{L}^2 \quad \exists! u \in \boldsymbol{V} \text{ s.t. } \int_\Omega -\eta \nabla u : \nabla v = \int_\Omega f v \qquad \forall v \in \boldsymbol{V}$$

Also, thanks to elliptic regularity we have that $u \in \boldsymbol{H}^2(\Omega)$, so we have

$$\forall f \in \boldsymbol{L}^2 \quad \exists! u \in \boldsymbol{H}^2 \cap \boldsymbol{V} \text{ s.t. } \int_\Omega -\eta \nabla u : \nabla v = \int_\Omega f v \qquad \forall v \in \boldsymbol{V}$$

Since $\boldsymbol{V}$ is dense in $\boldsymbol{G}_1$ we rewrite it as

$$\forall f \in \boldsymbol{L}^2 \quad \exists! u \in \boldsymbol{H}^2 \cap \boldsymbol{V} \text{ s.t. } \int_\Omega (\eta \Delta u + f) v = 0 \qquad \forall v \in \boldsymbol{G}_1$$

As for ∇p, this means that $(\eta\Delta u + f) \in \boldsymbol{G}_2 \oplus \boldsymbol{G}_3$. Thanks to this finding we can write

$$\exists! p \in \boldsymbol{H}^1/\mathbb{R} \text{ s.t. } -\nabla p = \eta\Delta u + f$$

where the space $\boldsymbol{H}^1/\mathbb{R}$ is the space of functions in $\boldsymbol{H}^1$ up to a constant.
So we have $\underbrace{-\eta\Delta u}_{\in \boldsymbol{G}_1 \oplus \boldsymbol{G}_2} + \underbrace{\nabla p}_{\in \boldsymbol{G}_2 \oplus \boldsymbol{G}_3} = \underbrace{f}_{\in \boldsymbol{G}_1 \oplus \boldsymbol{G}_2 \oplus \boldsymbol{G}_3} \in \boldsymbol{L}^2$. This means that the role of the pressure is to satisfy the equation projected on $\boldsymbol{G}_2$.

1.3 September 2021

Exercise 1
Find solitary waves for the problem

$$\begin{cases} u_t - u_{xxx} = 0 & \mathbb{R} \times (0, \infty) \\ u(x,0) = g(x) & x \in \mathbb{R} \end{cases} \tag{P}$$

Moreover, discuss mass and momentum conservation for general solutions $u \in S(\mathbb{R})$ of (P).

We start by finding the solitary waves for the problem. We know that the solution is of the form $u(x,t) = g(x+ct)$, so we substitute this solution in the equation and obtain

$$cg'(x+ct) - g'''(x+ct) = 0 \Rightarrow cg'(x+ct) = g'''(x+ct)$$

We perform a change of variable $s = x + ct$ and obtain

$$cg'(s) = g'''(s)$$

At this point we are working with an ODE, so we can solve it. We start by defining $p(l)$ as the characteristic polynomial of the ODE

$$p(l) = l^3 - cl = l(l^2 - c) = 0$$

and now study the behavior of the roots of the polynomial when $c > 0$, $c = 0$, $c < 0$.

$c > 0$. We have three real roots $l = 0, \sqrt{c}, -\sqrt{c}$. The general solution is

$$g(s) = k_1 + k_2 e^{\sqrt{c}s} + k_3 e^{-\sqrt{c}s}$$

$c = 0$. We have a triple root $l = 0$. The general solution is

$$g(s) = k_1 + k_2 s + k_3 s^2$$

$c < 0$. We have a complex conjugate pair of roots $l = 0, \pm i\sqrt{-c}$. The general solution is

$$g(s) = k_1 + k_2 \cos\big(\sqrt{-c}s\big) + k_3 \sin\big(\sqrt{-c}s\big)$$

We have found the solution for $g(s)$. Now we can discuss mass and momentum conservation. Defining the mass and momentum of the solution as

$$M(t) = \int_{\mathbb{R}} u(x,t)\,dx$$

$$\mathcal{M}(t) = \int_{\mathbb{R}} u(x,t)^2\,dx$$

Starting from the mass, we take its derivative

$$M'(t) = \frac{d}{dt}\int_{\mathbb{R}} u(x,t)\,dx = \int_{\mathbb{R}} u_t(x,t)\,dx = \int_{\mathbb{R}} u_{xxx}(x,t)\,dx = \int_{\mathbb{R}} \underbrace{(u_x x)_x}_{\text{div. form}=0}\,dx = 0$$

We have mass conservation, since mass is constant over time.
We compute the derivative of the momentum

$$\mathcal{M}'(t) = \frac{d}{dt}\int_{\mathbb{R}} u(x,t)^2\,dx = \int_{\mathbb{R}} 2u(x,t)u_t(x,t)\,dx = \int_{\mathbb{R}} 2uu_{xxx}\,dx = 2\left(\cancel{uu_{xx}|_{\mathbb{R}}} - \int_{\mathbb{R}} u_x u_{xx}\,dx\right) =$$

$$= -\int_{\mathbb{R}} \underbrace{(u_x^2)_x}_{\text{div. form}=0}\,dx = 0$$

We have also momentum conservation.

Exercise 2
Let $\Omega := B(0,1) \subset \mathbb{R}^n$ with $n \geq 2$, and let $f \in H^3(\Omega)$. Justify or confute the following statements:

(1) one can surely conclude that $f \in C(\Omega)$;

(2) one can surely conclude that $\gamma_2(f) \in H^{1/2}(\partial\Omega)$;

(3) one can surely exclude that $\gamma_0(f) \in H^1(\partial\Omega)$;

(4) if $n = 8$, then $f \in L^{15/2}(\Omega)$.

We start by recalling the Sobolev embeddings with $2s = 6$

$$
\begin{array}{lll}
H^3(\Omega) \subset C(\Omega) & & \text{if } n < 6 \\
H^3(\Omega) \subset L^p(\Omega) & \forall 2 \leq p < \infty & \text{if } n = 6 \\
H^3(\Omega) \subset L^p(\Omega) & \forall 2 \leq p \leq \dfrac{2n}{n-6} & \text{if } n > 6
\end{array}
$$

Then we check the statements

(1) In this case we have that $f \in C(\Omega)$ if $n < 6$. Since $n \geq 2$ we can surely conclude that $f \in C(\Omega)$.

(2) In this case we recall that $\gamma_j(f) \in H^{s-j-1/2}(\partial\Omega)$. In the case of $f \in H^3(\Omega)$ we have that $\gamma_2(f) \in H^{3-2-1/2}(\partial\Omega) = H^{1/2}(\partial\Omega)$. We can surely conclude that $\gamma_2(f) \in H^{1/2}(\partial\Omega)$.

(3) In this case we proceed as before, but with $j = 0$. We have that $\gamma_0(f) \in H^{3-0-1/2}(\partial\Omega) = H^{5/2}(\partial\Omega)$. Since $H^{5/2} \subset H^1$ we cannot surely exclude that $\gamma_0(f) \in H^1(\partial\Omega)$.

(4) In this case we have $n = 8$ so we need to check if $f \in L^p(\Omega)$ with $2 \leq p \leq 2^*$. The critical exponent is $p = \frac{2 \cdot 8}{8-6} = 8$. Since $15/2 < 8$ we can surely conclude that $f \in L^{15/2}(\Omega)$.

Exercise 3
Let $\ell > 0$ and consider the eigenvalue problem

$$\begin{cases} \Delta^2 u + \lambda u_{xx} = 0 & (0,\pi)\times(-\ell,\ell) \\ u = \Delta u = 0 & \partial\left[(0,\pi)\times(-\ell,\ell)\right] \end{cases} \tag{P}$$

Prove that $\lambda = 1$ is not an eigenvalue of (P). For which values of ℓ is the least eigenvalue double?

For this problem we know that the eigenvalues of the problem are of the form

$$\lambda_{m,n} = m^2 + \frac{n^2\pi^2}{\ell^2} \qquad m,n \in \mathbb{N} \iff \begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \text{has a non-trivial solution}$$

The space of eigenfunctions is given by

$$EF = \left\{ u_{m,n}(x,y) = \sin(mx)\sin\left(\frac{n\pi y}{\ell}\right) \mid m,n \in \mathbb{N} \right\}$$

Now we can rewrite the Laplacian operator as

$$\Delta u_{m,n} = -\lambda_{m,n} u_{m,n}$$

While the two derivatives are

$$u_x = -m\cos(mx)\sin\left(\frac{n\pi y}{\ell}\right) \qquad u_{xx} = -m^2\sin(mx)\sin\left(\frac{n\pi y}{\ell}\right)$$

so $u_{xx} = -m^2 u_{m,n}$. As for the bi-Laplacian operator we have

$$\Delta^2 u_{m,n} = -\Delta\lambda_{m,n} u_{m,n} = -\lambda_{m,n}^2 u_{m,n}$$

We can now substitute the derivatives in the equation and obtain

$$-\lambda_{m,n}^2 u_{m,n} + \lambda m^2 u_{m,n} = 0 \Rightarrow \lambda = \frac{\lambda_{m,n}^2}{m^2}$$

We have obtained an explicit expression for λ

$$\lambda = \frac{1}{m^2}\left(m^2 + \frac{n^2\pi^2}{\ell^2}\right)^2$$

Putting this expression equal to 1 we obtain

$$m^2 = \left(m^2 + \frac{n^2\pi^2}{\ell^2}\right)^2 \Rightarrow m = m^2 + \frac{n^2\pi^2}{\ell^2} \Rightarrow m - m^2 = \frac{n^2\pi^2}{\ell^2}$$

Then, since $m,n \in \mathbb{N}$ we have

$$n^2 = \frac{\pi^2}{\ell^2}\left(m - m^2\right) > 0 \Rightarrow \left(m - m^2\right) > 0 \Rightarrow m(m-1) < 0 \Rightarrow m \in (0,1)$$

Since $m \in \mathbb{N}$ we have that this is impossible, so $\lambda = 1$ is not an eigenvalue of (P).
Now we want to find the values of ℓ for which the least eigenvalue is double. To do so we fix $n = 1$, since we know that the smallest eigenvalue is $\lambda_{1,1} = \left(1 + \frac{\pi^2}{\ell^2}\right)^2$, and obtain

$$\lambda_{m,n}^2 \geq \lambda_{m,1}^2 = \left(m^2 + \frac{\pi^2}{\ell^2}\right)^2 \qquad \forall m \in \mathbb{N}$$

Since we are looking for the least eigenvalue, we are looking at a minimization problem.

$$\min_{m\in\mathbb{N}} \lambda_{m,1}^2 = \min_{m\in\mathbb{N}} \left(m + \frac{\pi^2}{\ell^2 m}\right)^2 = \mu^*$$

This is the same as minimizing the function $f(x) = (x + \frac{a^2}{x})^2$ with $a = \frac{\pi}{\ell}$. What we know about f?

- $f : [1, \infty) \to \mathbb{R}$
- $f \in C^\infty$
- $f'(x) = \geq 0 \iff x \geq a$
- $\lim_{x\to\infty} f(x) = \infty$

If $a > 1, a \in \mathbb{N}$ means that μ^* can have multiplicity greater than 1. We choose $1 < a < 2$ so that $(\ell < \pi < 2\ell)$ and $\mu^* = \min\left\{\lambda_{1,1}^2, \lambda_{2,1}^2\right\}$. We define $\mu_{m,n} = \lambda_{m,n}^2$ and we have

$$\mu_{1,1} = \left(1 + \frac{\pi^2}{\ell^2}\right)^2 \qquad \mu_{2,1} = \left(2 + \frac{\pi^2}{2\ell^2}\right)^2.$$

To have them ordered $\mu_{1,1} < \mu_{2,1} \iff \pi \leq \ell\sqrt{2}$. Then we have three cases:

- $\ell \leq \pi < \ell\sqrt{2} \Rightarrow \mu^* = \mu_{1,1}$ which is simple;
- $\ell\sqrt{2} < \pi < 2\ell \Rightarrow \mu^* = \mu_{2,1}$ which is simple;
- $\pi = \ell\sqrt{2} \Rightarrow \mu^* = \mu_{1,1} = \mu_{2,1}$ which is double.

So, the value of ℓ for which the least eigenvalue is double is $\ell = \frac{\pi}{\sqrt{2}}$.

1.4 January 2022

Exercise 1
For the Korteweg-de Vries equation

$$\begin{cases} u_t + u_{xxx} + 6uu_x = 0 & \mathbb{R} \times (0, \infty) \\ u(x,0) = g(x) & x \in \mathbb{R} \end{cases} \tag{P}$$

prove that the "energy" $E(t) = \int_{\mathbb{R}} (u_x^2 - 2u^3)\, dx$ is conserved for general solutions $u \in S(\mathbb{R})$ of (P).

We start by computing the derivative of the energy

$$E'(t) = \frac{d}{dt} \int_{\mathbb{R}} (u_x^2 - 2u^3)\, dx = \int_{\mathbb{R}} 2u_x u_{xt} - 6u^2 u_t\, dx = \int_{\mathbb{R}} 2u_x u_{xt}\, dx - 6 \int_{\mathbb{R}} u^2 u_t\, dx$$

Dividing the problem in two parts we have

$$\int_{\mathbb{R}} 2u_x u_{xt}\, dx = \cancel{2u_x u_t\big|_{\mathbb{R}}} - \int_{\mathbb{R}} 2u_{xx} u_t\, dx = -\int_{\mathbb{R}} 2u_{xx}(-u_{xxx} - 6uu_x)\, dx =$$

$$= \int_{\mathbb{R}} 2u_{xx}u_{xxx} + 12u \underbrace{u_x u_{xx}}_{=\left(\frac{u_x^2}{2}\right)_x} dx = \int_{\mathbb{R}} \underbrace{(u_{xx}^2)_x}_{=0} + \cancel{12u\frac{u_x^2}{2}\Big|_{\mathbb{R}}} - 12 \int_{\mathbb{R}} u_x \frac{u_x^2}{2}\, dx = -6 \int_{\mathbb{R}} u_x^3\, dx$$

For the second part we have

$$6 \int_{\mathbb{R}} u^2 u_t\, dx = 6 \int_{\mathbb{R}} u^2(-u_{xxx} - 6uu_x)\, dx = -\int_{\mathbb{R}} 6u^2 u_{xxx} -$$

$$= \cancel{-6u^2 u_{xx}\big|_{\mathbb{R}}} + 6 \int_{\mathbb{R}} 2uu_xu_{xx} - 9 \int_{\mathbb{R}} \underbrace{(u^4)_x}_{=0} dx = 12 \int_{\mathbb{R}} uu_xu_{xx}\, dx = 12 \int_{\mathbb{R}} u \left(\frac{u^2}{2}\right)_x dx = \cancel{6uu_x^2\big|_{\mathbb{R}}} - 6 \int_{\mathbb{R}} u_x u$$

To recap, we have

$$E'(t) = \int_{\mathbb{R}} 2u_x u_{xt}\, dx - 6 \int_{\mathbb{R}} u^2 u_t\, dx$$

$$\int_{\mathbb{R}} 2u_x u_{xt}\, dx = -6 \int_{\mathbb{R}} u_x^3\, dx$$

$$\int_{\mathbb{R}} 6u^2 u_t\, dx = -6 \int_{\mathbb{R}} u_x^3\, dx$$

We can now substitute these results in the derivative of the energy

$$E'(t) = -6 \int_{\mathbb{R}} u_x^3\, dx + 6 \int_{\mathbb{R}} u_x^3\, dx = 0$$

We have that the energy is conserved for general solutions $u \in S(\mathbb{R})$ of (P).

Exercise 2
Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of class C^1, and let $f \in L^2(0,T;L^2(\Omega))$. Moreover, let $u_0 \in H^1(\Omega)$ and $u_1 \in L^2(\Omega)$. Prove that, for every $\gamma > 0$, the Galerkin method can be applied to the problem

$$\begin{cases} u_{tt} - \Delta u + \gamma u = f & \Omega \times (0,T) \\ \partial_\nu u = 0 & \partial\Omega \times (0,T) \\ u = u_0 & \Omega \times \{0\} \\ u_t = u_1 & \Omega \times \{0\} \end{cases} \tag{P}$$

which, therefore, admits a unique solution.

We start by writing the weak formulation of the problem, choosing adequate function spaces. We define the spaces

$$V = H^1(\Omega) \subseteq H = L^2(\Omega) \qquad V' = \big(H^1(\Omega)\big)'$$

We also need to introduce the space of weakly continuous functions over $[0,T]$.

Remark 1
Let H be a Hilbert space. The space of weakly continuous functions over $[0,T]$ is defined as

$$C^0_w([0,T];H) = \left\{ u \in L^\infty(0,T;H) \mid \lim_{t\to t_0} (u(t) - u(t_0), v)_H = 0, \quad \forall t_0 \in [0,T], \forall v \in H \right\}$$

We can now write the weak formulation of the problem (P), by multiplying the equation by a test function $v \in V$ and integrating over Ω

$$\begin{aligned} \int_\Omega f(t) v\, dx &= \int_\Omega u_{tt} v\, dx + \int_\Omega (-\Delta u + \gamma u) v\, dx = \\ &= \int_\Omega u_{tt} v\, dx - \cancel{\int_{\partial\Omega} \partial_\nu u v\, d\sigma} + \int_\Omega [\nabla u \cdot \nabla v + \gamma u v]\, dx = \; = \int_\Omega u_{tt} v\, dx + \int_\Omega \underbrace{\nabla u \cdot \nabla v + \gamma u v}_{B(u,v)}\, dx \end{aligned}$$

To write it more compactly we have

$$\frac{d^2}{dt^2}(u,v)_{L^2} + B(u,v) = (f,v)_{L^2} \qquad \forall v \in V$$

Another important abstract result is the following

Remark 2
If $B(u,v)$ is continuous and coercive, $u_0 \in V$ and $u_1 \in H$ and $f \in L^2(0,T;V')$, then we know that

$$\exists! u \in C^0_w([0,T];V) \cap C^0([0,T];H) \text{ with } u_t \in C^0_w([0,T];H), \quad u_{tt} \in L^2(0,T;V') \text{ for (P).}$$

So, we need to check if $B(u,v)$ is continuous and coercive. We start by checking the continuity of $B(u,v)$

$$\begin{aligned} |B(u,v)| \le \int_\Omega |\nabla u \cdot \nabla v| + |\gamma||u||v| &\overset{\text{CS + H}}{\le} \underbrace{\|\nabla u\|_{L^2}}_{\le \|u\|_V} \|\nabla v\|_{L^2} + |\gamma| \|u\|_{L^2} \|v\|_{L^2} \le \\ &\le (1 + |\gamma|) \|u\|_V \|v\|_V \le C \|u\|_V \|v\|_V \end{aligned}$$

We have that $B(u,v)$ is continuous. We now check if it is coercive

$$B(u,u) = \int_{\Omega} |\nabla u|^2 + |\gamma||u|^2 = (1 + |\gamma|) \int_{\mathbb{R}} |\nabla u|^2 + |u|^2 \geq (1 + |\gamma|)\|u\|_V^2$$

We need that $\alpha \geq 0$ to obtain $B(u,u) \geq \min\{\alpha, 1\}\|u\|_V^2$. Since $\gamma > 0$ by hypothesis, we have that $B(u,v)$ is coercive. We can now apply the abstract result and conclude that the solution of (P) exists and is unique.

Exercise 3
Let $\Omega := B(0,1) \subset \mathbb{R}^n$, and let

$$f(x) := \frac{e^{|x|} - 1}{|x|^{\alpha}}, \quad \text{with } \alpha > 0$$

Find the values of α for which $f \in H^1(\Omega)$.