

Numerical Analysis for Partial Differential Equations

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1 Boundary Value Problems

Let's consider a problem

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega \\ +\text{B.C.} & \text{on } \partial\Omega \end{cases} \quad (1.1)$$

- Ω : open bounded domain in \mathbb{R}^d , with $d = 2, 3$
- $\partial\Omega$: boundary of Ω
- f : given
- B.C. accordingly to \mathcal{L}
- \mathcal{L} : 2nd order operator, like:

$$(1) \quad \mathcal{L}u = -\text{div}(\mu \nabla u) + \mathbf{b} \cdot \nabla u + \sigma u \quad (\text{non-conservative form})$$

$$(2) \quad \mathcal{L}u = -\text{div}(\mu \nabla u) + \text{div}(\mathbf{b}u) + \sigma u \quad (\text{conservative form})$$

- $\mu \in L^\infty(\Omega)$, $\mu(\mathbf{x}) \geq \mu_0 > 0$ uniformly bounded from below
- $\mathbf{b} \in (L^\infty(\Omega))^d$ transport term
- $\sigma \in L^2(\Omega)$ reaction term
- $f \in L^2(\Omega)$ can be less regular

General elliptic problems

Consider

$$\begin{cases} -\text{div}(\mu \nabla u) + \mathbf{b} \cdot \nabla u + \sigma u = f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_D \\ \mu \nabla u \cdot \mathbf{n} = g & \text{on } \Gamma_N \end{cases} \quad \begin{matrix} g \in L^2(\Gamma_N) \\ \partial\Omega = \Gamma_D \cup \Gamma_N \\ \Gamma_D^\circ \cap \Gamma_N^\circ = \emptyset \end{matrix} \quad (1.2)$$

Suppose that $f \in L^2(\Omega)$ and $\mu, \sigma \in L^\infty(\Omega)$. Also suppose that $\exists \mu_0 > 0$ s.t. $\mu(\mathbf{x}) \geq \mu_0$, and $\sigma(\mathbf{x}) \geq 0$ a.e. on Ω . Then, given a test function v , we multiply the equation by v , and integrate on the domain Ω

$$\int_{\Omega} [-\text{div}(\mu \nabla u) + \mathbf{b} \cdot \nabla u + \sigma u] v = \int_{\Omega} f v$$

By applying Green's formula

$$\underbrace{\int_{\Omega} \mu \nabla u \cdot \nabla v + \int_{\Omega} \mathbf{b} \cdot \nabla u v + \int_{\Omega} \sigma u v}_{=: a(u, v)} = \int_{\Omega} f v + \underbrace{\int_{\Gamma_D} \mu \nabla u \cdot \mathbf{n} v}_{=0 \text{ if } v|_{\Gamma_D}=0} + \int_{\Gamma_N} \underbrace{\mu \nabla u \cdot \mathbf{n} v}_{=g}$$

So the weak formulation of the problem is

$$\begin{cases} \text{Find } u \in V & V = \{v \in H^1(\Omega), v|_{\Gamma_D} = 0\} =: H_{\Gamma_D}^1(\Omega) \\ a(u, v) = \langle F, v \rangle & \forall v \in V \end{cases} \quad (1.3)$$

where $a : V \times V \rightarrow \mathbb{R}$ is a bilinear form and $F : V \rightarrow \mathbb{R}$ is a linear form s.t. $\langle F, v \rangle \equiv F(v) = \int_{\Omega} f v + \int_{\Gamma_N} g v$.

Theorem 1.1 (Lax-Milgram)

Assume that

- V Hilbert space with $\|\cdot\|$ and inner product (\cdot, \cdot)
- $F \in V^* : |F(v)| \leq \|F\|_{V^*} \|v\| \quad \forall v \in V$
- a continuous: $\exists M > 0 : |a(u, v)| \leq M \|u\| \|v\| \quad \forall u, v \in V$
- a coercive: $\exists \alpha > 0 : a(v, v) \geq \alpha \|v\|^2 \quad \forall v \in V$

Then, there exists a unique solution u of 1.3

Moreover

$$\alpha \|u\|^2 \leq a(u, u) = F(u) \leq \|F\|_{V^*} \|u\|$$

where α is the coercivity constant. Hence

$$\|u\| \leq \frac{\|F\|_{V^*}}{\alpha} \rightarrow \text{stability/continuous dependence on data}$$

But what if some of the assumptions of Lax-Milgram (in particular coercivity) are not satisfied? We need a slightly more general problem to formulate Nečas theorem:

$$\begin{cases} \text{Find } u \in V \\ a(u, w) = \langle F, w \rangle \quad \forall w \in W \end{cases} \quad (1.4)$$

They belong to different spaces: W for the test function, V the solutions

Theorem 1.2

Assume that $F \in W^*$. Consider the following conditions:

- a continuous: $\exists M > 0 : |a(u, w)| \leq M \|u\|_V \|w\|_W \quad \forall u \in V, w \in W$
- inf – sup condition: $\exists \alpha > 0 : \forall v \in V \quad \sup_{w \in W \setminus \{0\}} \frac{a(v, w)}{\|w\|_W} \geq \alpha \|v\|_V$
- $\forall w \in W, w \neq 0, \exists v \in V : a(v, w) \neq 0$

These conditions are necessary and sufficient for the existence and uniqueness of a solution of 1.4, for any $F \in W^*$. Moreover

$$\|u\|_V \leq \frac{1}{\alpha} \|F\|_{W^*}$$